EQUIVALENT DEFINITIONS OF SUPERSTABILITY IN TAME ABSTRACT ELEMENTARY CLASSES

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Abstract. In the context of abstract elementary classes (AECs) with a monster model, several possible definitions of superstability have appeared in the literature. Among them are no long splitting chains, uniqueness of limit models, and solvability. Under the assumption that the class is tame and stable, we show that (asymptotically) no long splitting chains implies solvability and uniqueness of limit models implies no long splitting chains. Using known implications, we can then conclude that all the previously-mentioned definitions (and more) are equivalent:

Corollary 0.1. Let $\mathbf{K}$ be a tame AEC with a monster model. Assume that $\mathbf{K}$ is stable in a proper class of cardinals. The following are equivalent:

(1) For all high-enough λ, $\mathbf{K}$ has no long splitting chains.
(2) For all high-enough λ, there exists a good λ-frame on a skeleton of $\mathbf{K}_\lambda$.
(3) For all high-enough λ, $\mathbf{K}$ has a unique limit model of cardinality λ.
(4) For all high-enough λ, $\mathbf{K}$ has a superlimit model of cardinality λ.
(5) For all high-enough λ, the union of any increasing chain of λ-saturated models is λ-saturated.
(6) There exists μ such that for all high-enough λ, $\mathbf{K}$ is (λ, μ)-solvable.

This gives evidence that there is a clear notion of superstability in the framework of tame AECs with a monster model.

Contents

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1. Introduction

In the context of classification theory for abstract elementary classes (AECs), a notion analogous to the first-order notion of *stability* exists: it is defined as one might expect[1] (by counting Galois types). However it has been unclear what a parallel notion to superstability might be. Recall that for first-order theories we have:

Fact 1.1. Let T be a first-order complete theory. The following are equivalent:

(1) T is stable in every cardinal $\lambda \geq 2^{|T|}$.
(2) For all infinite cardinals λ, the union of an increasing chain of λ-saturated models is λ-saturated.
(3) $\kappa(T) = \aleph_0$ and T is stable.
(4) T has a saturated model of cardinality λ for every $\lambda \geq 2^{|T|}$.
(5) T is stable and $\mathrm{D}^n[\bar{x} = \bar{x}, L(T), \infty] < \infty$.
(6) There does not exists a set of formulas $\Phi = \{\varphi_n(\bar{x}; \bar{y}_n) \mid n < \omega\}$ such that Φ can be used to code the structure $(\omega^{\leq \omega}, <, <_{lex})$

(1) $\implies$ (2) and (1) $\iff$ (ℓ) for $\ell \in \{3, 4, 5, 6\}$ all appear in Shelah's book [She90]. Albert and Grossberg [AG90, Theorem 13.2] established (2) $\implies$ (6).

In the last 30 years, in the context of classification theory for non elementary classes, several notions that generalize that of first-order superstablity have been considered. See papers by Grossberg, Shelah, VanDieren, Vasey and Villaveces: [GS86, Gro88], [She99], [SV99], [Van06, Van13], [GVV16], [Vas16b, Vas16a].

In [She99, p. 267] Shelah states that part of the program of classification theory for AECs is to show that all the various notions of first-order saturation (limit, superlimit, or model-homogeneous, see Section 2.2) are equivalent under the assumption of superstablity. A possible definition of superstability is *solvability* (see Definition 2.18), which appears in the introduction to [She09a] and is hailed as a true counterpart to first-order superstability. Full justification is delayed to [She] but [She09a, Chapter IV] already uses it. Other definitions of superstability analogous to the ones in Fact 1.1 can also be formulated. The main result of this paper is to show that, at least in tame AECs with a monster model, several definitions of superstability that previously appeared in the literature are equivalent (see the preliminaries for precise definitions of some of the concepts appearing below). Many of the implications have already been proven in earlier papers, but here we complete the loop by proving two theorems. Before stating them, some notation will be helpful:

Notation 1.2 (4.24.(5) in [Bal09])**.** Given a fixed AEC $\mathbf{K}$, set $H_1 := \beth_{\left(2^{\mathrm{LS}(\mathbf{K})}\right)^+}$.

Theorem 3.18. Let $\mathbf{K}$ be an $\mathrm{LS}(\mathbf{K})$-tame AEC with a monster model. There exists $\chi < H_1$ such that for any $\mu \geq \chi$, if $\mathbf{K}$ is stable in μ, there is a saturated model of cardinality μ, and every limit model of cardinality μ is χ-saturated, then $\mathbf{K}$ has no long splitting chains in μ.

[1]A justification for the definition is Fact 2.4, showing that it is equivalent (under tameness) to failure of the order property.

Theorem 4.9. Let $\mathbf{K}$ be an $\mathrm{LS}(\mathbf{K})$-tame AEC with a monster model. There exists $\chi < H_1$ such that for any $\mu \geq \chi$, if $\mathbf{K}$ is stable in μ and has no long splitting chains in μ then $\mathbf{K}$ is uniformly (μ', μ')-solvable, where $\mu' := (\beth_{\omega+2}(\mu))^+$.

These two theorems prove (3) implies (1) and (1) implies (6) of our main corollary, proven in detail after the statement of Corollary 5.7.

Corollary 1.3 (Main Corollary)**.** Let $\mathbf{K}$ be a $\mathrm{LS}(\mathbf{K})$-tame AEC with a monster model. Assume that $\mathbf{K}$ is stable in some cardinal greater than or equal to $\mathrm{LS}(\mathbf{K})$. The following are equivalent:

(1) There exists $\mu_1 < H_1$ such that for every $\lambda \geq \mu_1$, $\mathbf{K}$ has no long splitting chains in λ.
(2) There exists $\mu_2 < H_1$ such that for every $\lambda \geq \mu_2$, there is a good λ-frame on a skeleton of $\mathbf{K}_\lambda$.
(3) There exists $\mu_3 < H_1$ such that for every $\lambda \geq \mu_3$, $\mathbf{K}$ has a unique limit model of cardinality λ.
(4) There exists $\mu_4 < H_1$ such that for every $\lambda \geq \mu_4$, $\mathbf{K}$ has a superlimit model of cardinality λ.
(5) There exists $\mu_5 < H_1$ such that for every $\lambda \geq \mu_5$, the union of any increasing chain of λ-saturated models is λ-saturated.
(6) There exists $\mu_6 < H_1$ such that for every $\lambda \geq \mu_6$, $\mathbf{K}$ is (λ, μ_6)-solvable.

Moreover any of the above conditions also imply:

(7) There exists $\mu_7 < H_1$ such that for every $\lambda \geq \mu_7$, $\mathbf{K}$ is stable in λ.

Remark 1.4. The main corollary has a global assumption of stability (in some cardinal). While stability is implied by some of the equivalent conditions (e.g. by (2) or (6)) other conditions may be vacuously true if stability fails (e.g. (1)). Thus in order to simplify the exposition we just require stability outright.

Remark 1.5. In the context of the main corollary, if $\mu_1 \geq \mathrm{LS}(\mathbf{K})$ is such that $\mathbf{K}$ is stable in μ_1 and has no long splitting chains in μ_1, then for any $\lambda \geq \mu_1$, $\mathbf{K}$ is stable in λ and has no long splitting chains in λ (see Fact 2.7). In other words, superstability defined in terms of no long splitting chains transfers up.

Remark 1.6. In (3), one can also require the following strong version of uniqueness of limit models: if $M_0, M_1, M_2 \in \mathbf{K}_\lambda$ and both M_1 and M_2 are limit over M_0, then $M_1 \cong_{M_0} M_2$ (i.e. the isomorphism fixes the base). This is implied by (2): see Fact 2.16.

At present, we do not know how to prove analogs to the last two properties of Fact 1.1. Further, it is open whether stability on a tail ((7) in the main corollary) implies any of the above definitions of superstability (more on this in Section 6).

Question 1.7. Let $\mathbf{K}$ be an $\mathrm{LS}(\mathbf{K})$-tame AEC with a monster model. If $\mathbf{K}$ is stable on a tail of cardinals, does there exists a cardinal $\mu \geq \mathrm{LS}(\mathbf{K})$ such that $\mathbf{K}$ is stable in μ and has no long splitting chains in μ?

Interestingly, the proof of Corollary 1.3 does not tell us that the threshold cardinals μ_ℓ above are equal. In fact, it uses tameness heavily to move from one cardinal to the next and uses e.g. that one equivalent definition holds below λ to prove that another definition holds at λ. Showing equivalence of these definitions cardinal by cardinals, or even just showing that we can take $\mu_1 = \mu_2 = \ldots = \mu_6$ seems much harder. We also show that we can ask only for each property to hold in *a single high-enough* cardinals below H_1 (but the cardinal may not be the same for each property, see Corollary 5.7).

Note also that, while the analogous result is known for stability (see Fact 2.4), we do not know whether no long splitting chains should hold below the Hanf number:

Question 1.8. Let $\mathbf{K}$ be a $\text{LS}(\mathbf{K})$-tame AEC with a monster model. Assume that there exists $\mu \geq \text{LS}(\mathbf{K})$ such that $\mathbf{K}$ is stable in μ and has no long splitting chains in μ. Is the least such μ below H_1?

In general, we suspect that the problem of computing the cardinals μ_ℓ could play a role similar to the computation of the first stability cardinal for a first-order theory (which led to the development of forking, see e.g. the introduction of [GIL02]).

We discuss earlier work. Shelah [She09a, Chapter II] introduced good λ-frames (a local axiomatization of first-order forking in a superstable theory, see more in Section 2.4) and attempts to develop a theory of superstability in this context. He proves for example the uniqueness of limit models (see Fact 2.16, so (2) implies (3) in the main theorem is due to Shelah) and (with strong assumptions, see below) the fact that the union of a chain (of length strictly less than λ^{++}) of saturated models of cardinality λ^+ is saturated [She09a, II.8]. From this he deduces the existence of a good λ^+-frame on the class of λ^+-saturated models of $\mathbf{K}$ and goes on to develop a theory of prime models, regular types, independent sequences, etc. in [She09a, Chapter III]. The main issue with Shelah's work is that it does not make any global model-theoretic hypotheses (such as tameness or even just amalgamation) and hence often relies on set-theoretic assumptions as well as strong local model-theoretic hypotheses (few models in several cardinals). For example, Shelah's construction of a good frame in the local setup [She09a, II.3.7] uses categoricity in two successive cardinals, few models in the next, as well as several diamond-like principles.

By making more global hypotheses, building a good frame becomes easier and can be done in ZFC (see [Vas16b] or [She09a, Chapter IV]). Recently, assuming a monster model and tameness (a locality property of types introduced by VanDieren and the first author, see Definition 2.1), progress have been made in the study of superstability defined in terms of no long splitting chains. Specifically, [Vas16b, 5.6] proved (1) implies (7). Partial progress in showing (1) implies (2) is made in [Vas16b] and [Vas16a] but the missing piece of the puzzle, that (1) implies (5), is proven in [BV]. From these results, it can be deduced that *(1) implies (2)-(5)* (see [BV, Theorem 7.1]). Implications between variants of (3), (4) and (5) are also straightforward (see Fact 2.10). Finally, (6) directly implies (4) from its definition (see Section 2.5).

Thus as noted before the main contributions of this paper are (3) implies (1) and (1) implies (6). In Theorem 5.6 it is shown that, assuming a monster model and tameness, solvability in *some* high-enough cardinal implies solvability in *all* high-enough cardinals. Note that Shelah asks (inspired by the analogous question for categoricity) in [She09a, Questions N.4.4] what the solvability spectrum can be (in an arbitrary AEC). Theorem 5.6 provides a partial answer under the additional assumptions of a monster model and tameness. The proof notices that a powerful results of Shelah and Villaveces [SV99] (deriving no long splitting chains from categoricity) can be adapted to our setup (see Theorem 5.1 and Corollary 5.3). Shelah also asks [She09a, Question N.4.5] about the superlimit spectrum. In our context, we can show that if there is a high-enough *stability* cardinals λ with a superlimit model, then $\mathbf{K}$ has a superlimit on a tail of cardinals (see Corollary 5.7). We do not know if the hypothesis that λ is a stability cardinal is needed (see Question 5.9).

The background required to read this paper is a solid knowledge of AECs (for example Chapters 4-12 of Baldwin's book [Bal09] or the upcoming [Gro]). We rely on the first ten sections of [Vas16a], as well as on the material in [Vas16c, BV].

At the beginning of Sections 3 and 4, we make *global* hypotheses that hold until the end of the section (unless said otherwise). This is to make the statements of several technical lemmas more readable. We will repeat the global hypotheses in the statement of major theorems.

Since this paper was first circulated (July 2015), several related results have been proven. VanDieren [Van16a, Van16b] gives some relationships between versions of (3) and (5) in a single cardinal (with (1) as a background assumption). This is done without assuming tameness, using very different technologies than in this paper. This work is applied to the tame context in [VV], showing for example that (1) implies (3) holds cardinal by cardinal. In the recent [Vasb] Question 1.7 is answered positively, showing that no long splitting chains follow from stability on a tail of cardinals.

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2. Preliminaries

We assume familiarity with a basic text on AECs such as [Bal09] or [Gro] and refer the reader to the preliminaries of [Vas16c] for more details and motivations on the concepts used in this paper.

We use $\mathbf{K}$ (boldface) to denote a class of models together with an ordering (written $\leq_{\mathbf{K}}$). We will often abuse notation and write for example $M \in \mathbf{K}$. When it becomes necessary to consider only a class of models without an ordering, we will write K (no boldface).

Throughout all this paper, $\mathbf{K}$ is a fixed AEC. Most of the time, $\mathbf{K}$ will have amalgamation, joint embedding, and arbitrarily large models. In this case we say that $\mathbf{K}$ *has a monster model*.

Shelah's program of classification theory for abstract elementary classes started in 1977 with a circulation of a draft of [She87] (a revised version is [She09a, Chapter I]). As a full classification theory seems impossible due to various counterexamples (e.g. [HS90]) and immense technical difficulties of addressing some of the main conjectures, all known non-trivial results are obtained under some additional model-theoretic or even set-theoretic assumptions on the family of classes we try to develop structure/non-structure results for. In July 2001, Grossberg and VanDieren circulated a draft of a paper titled "Morley Sequences in Abstract Elementary Classes" (a revised version was published as [GV06b]). In that paper, they introduced tameness as a useful assumption to prove upward stability results as well as existence of Morley sequence with respect to non-splitting in stable AECs.

Definition 2.1 (3.2 in [GV06b])**.** Let $\chi \geq \mathrm{LS}(\mathbf{K})$ be a cardinal. $\mathbf{K}$ is χ*-tame* if for any $M \in \mathbf{K}_{\geq \chi}$ and any $p \neq q$ in $\mathrm{gS}(M)$, there exists $M_0 \in \mathbf{K}_\chi$ such that $p \upharpoonright M_0 \neq q \upharpoonright M_0$. We similarly define $(< \chi)$-tame (when $\chi > \mathrm{LS}(\mathbf{K})$).

We say that $\mathbf{K}$ *is tame* provided there exists a cardinal χ such that[2] $\mathbf{K}$ is χ-tame.

Remark 2.2. If $\mathbf{K}$ is χ-tame for $\chi > \mathrm{LS}(\mathbf{K})$, the class $\mathbf{K}' := \mathbf{K}_{\geq \chi}$ will be an $\mathrm{LS}(\mathbf{K}')$-tame AEC. Hence we will often directly assume that $\mathbf{K}$ is $\mathrm{LS}(\mathbf{K})$-tame.

Remark 2.3. In [GV06c] and [GV06a] Grossberg and VanDieren established several cases of Shelah's categoricity conjecture (which is after 40 years still the best known open problem in the field of AECs). At the time, the main justification for the tameness assumption was that it appears in all known cases of structural results and it seems to be difficult to construct non-tame classes. In 2013, Will Boney [Bon14b] derived from the existence of a class of strongly compact cardinals that all AECs are tame. In a preprint from 2014, Lieberman and Rosický [LR] pointed out that this theorem of Boney follows from a 25 year old theorem of Makkai and Paré ([MP89, 5.5.1]). In a forthcoming paper Boney and Unger [BU] establish that if every AEC is tame then a proper class of large cardinals exists. Thus tameness (for all AECs) is a large cardinal axioms. We believe that this is evidence for the assertion that tameness is a new interesting model-theoretic property, a new dichotomy[3], that should follow (see [GV06a, Conjecture 1.5]) from categoricity in a "high-enough" cardinal.

We will use the equivalence between stability and the order property under tameness [Vas16c, Theorem 4.13]:

Fact 2.4. Assume that $\mathbf{K}$ is $\mathrm{LS}(\mathbf{K})$-tame and has a monster model. The following are equivalent:

(1) $\mathbf{K}$ is stable in some cardinal greater than or equal to $\mathrm{LS}(\mathbf{K})$.
(2) There exists $\mu < H_1$ such that $\mathbf{K}$ is stable in all $\lambda \geq \mathrm{LS}(\mathbf{K})$ such that $\lambda = \lambda^\mu$.
(3) $\mathbf{K}$ does not have the $\mathrm{LS}(\mathbf{K})$-order property.

2.1. **Superstability and no long splitting chains.** A definition of superstability analogous to $\kappa(T) = \aleph_0$ in first-order model theory has been studied in AECs (see [SV99, GVV16, Van06, Van13, Vas16b, Vas16a]). Since it is not immediately

[2]As opposed to [GV06b, 3.3], we do *not* require that $\chi < H_1$.

[3]Consider, for example, the statement that in a monster model for a first-order theory T, for every sufficiently long sequence $\mathbf{I}$ there exists a subsequence $\mathbf{J} \subseteq \mathbf{I}$ such that $\mathbf{J}$ is indiscernible. In general, this is a large cardinal axiom, but it is known to be true when T is on the good side of a dividing line (in this case stability). We believe that the situation for tameness is similar.

obvious what forking should be in that framework, the more rudimentary independence relation of λ-splitting is used in the definition. Since in AECs, types over models are much better behaved than types over sets, so it does not make sense in general to ask for every type to not split over a finite set[4]. Thus we require that every type over the union of a chain does not split over a model in the chain. For technical reasons (it is possible to prove that the condition follows from categoricity), we require the chain to be increasing with respect to universal extension. This rephrases (1) in Corollary 1.3:

Definition 2.5. Let $\lambda \geq \text{LS}(K)$. We say $\mathbf{K}$ has *no long splitting chains in* λ if for any limit $\delta < \lambda^+$, any increasing $\langle M_i : i < \delta \rangle$ in $\mathbf{K}_\lambda$ with M_{i+1} universal over M_i for all $i < \delta$, any $p \in \text{gS}(\bigcup_{i<\delta} M_i)$, there exists $i < \delta$ such that p does not λ-split over M_i.

Remark 2.6. The condition in Definition 2.5 first appears in [She99, Question 6.1]. In [Bal09, 15.1], it is written as[5] $\kappa(\mathbf{K}, \lambda) = \aleph_0$. We do not adopt this notation, since it blurs out the distinction between forking and splitting, and does not mention that only a certain type of chains are considered. A similar notation is in [Vas16a, 3.16]: $\mathbf{K}$ has no long splitting chains in λ if and only if $\kappa_1(\mathfrak{i}_{\lambda\text{-ns}}(\mathbf{K}_\lambda), <_{\text{univ}}) = \aleph_0$.

In tame AECs with a monster model, no long splitting chains transfers upward:

Fact 2.7 (10.10 in [Vas16a])**.** Let $\mathbf{K}$ be an AEC with a monster model and let $\text{LS}(\mathbf{K}) \leq \lambda \leq \mu$. If $\mathbf{K}$ is stable in λ and has no long splitting chains in λ, then $\mathbf{K}$ is stable in μ and has no long splitting chains in μ.

2.2. **Definitions of saturated.** The search for a good definition of "saturated" in AECs is central. We quickly review various possible notions and cite some basic facts about them, including basic implications.

Implicit in the definition of no long splitting chains is the notion of a *limit model*. It plays a central role in the study of AECs that do not necessarily have amalgamation [SV99] (their study in this context was continued in [Van06, Van13]). We use the notation and basic definitions from [GVV16]. The two basic facts about limit models (in an AEC with a monster model) are:

1. Existence: If $\mathbf{K}$ is stable in λ, then for every M and every limit $\delta < \lambda^+$ there is a (λ, δ)-limit over M.
2. Uniqueness: Any two limit models of the same length are isomorphic.

Uniqueness of limit models that are *not* of the same cofinality is a key concept which is equivalent to superstability in first-order model theory.

A second notion of saturation can be defined using Galois types (when $\mathbf{K}$ has a monster model): for $\lambda > \text{LS}(\mathbf{K})$, say M is λ-*saturated* if every type over a $\leq_{\mathbf{K}}$-substructure of M of size less than λ is realized inside M. We will write $\mathbf{K}^{\lambda\text{-sat}}$ for the class of λ-saturated models in $\mathbf{K}$.

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[4] But see [Vasa, C.14] where a notion of forking over set is constructed from categoricity in a universal class.

[5] Of course, the κ notation has a long history, appearing first in [She70].

A third notion of saturation appears in [She87, Definition 3.1.1][6]. The idea is to encode a generalization of the fact that a union of saturated models should be saturated.

Definition 2.8. Let $M \in \mathbf{K}$ and let $\lambda \geq \mathrm{LS}(\mathbf{K})$. M is called *superlimit in λ if:*

(1) $M \in \mathbf{K}_\lambda$.
(2) M is "properly universal": For any $N \in \mathbf{K}_\lambda$, there exists $f : N \to M$ such that $f[N] <_{\mathbf{K}} M$.
(3) Whenever $\langle M_i : i < \delta \rangle$ is an increasing chain in $\mathbf{K}_\lambda$, $\delta < \lambda^+$ and $M_i \cong M$ for all $i < \delta$, then $\bigcup_{i<\delta} M_i \cong M$.

Note that the superlimit model is unique. The proof is a straightforward back and forth argument which we omit.

Fact 2.9. If M and N are superlimit in λ, then $M \cong N$.

The following local implications between the three definitions are known:

Fact 2.10 (Local implications)**.** Assume that $\mathbf{K}$ has a monster model. Let $\lambda \geq \mathrm{LS}(\mathbf{K})$ be such that $\mathbf{K}$ is stable in λ.

(1) If $\chi \in [\mathrm{LS}(\mathbf{K})^+, \lambda]$ is regular, then any (λ, χ)-limit model is χ-saturated.
(2) If $\lambda > \mathrm{LS}(\mathbf{K})$ and λ is regular, then $M \in \mathbf{K}_\lambda$ is saturated if and only if M is (λ, λ)-limit.
(3) If $\lambda > \mathrm{LS}(\mathbf{K})$, then any two limit models of size λ are isomorphic if and only if every limit model of size λ is saturated.
(4) If $M \in \mathbf{K}_\lambda$ is superlimit, then for any limit $\delta < \lambda^+$, M is (λ, δ)-limit and (if $\lambda > \mathrm{LS}(\mathbf{K})$) saturated.
(5) Assume that $\lambda > \mathrm{LS}(\mathbf{K})$ and there exists a saturated model M of size λ. Then M is superlimit if and only if in $\mathbf{K}_\lambda$, the union of any increasing chain (of length strictly less than λ^+) of saturated models is saturated.

Proof. (1), (2), and (3) are straightforward from the basic facts about limit models and the uniqueness of saturated models. (4) is by [Dru13, Corollary 2.3.10] and the previous parts. (5) then follows. □

Remark 2.11. (3) is stated for λ regular in [Dru13, Corollary 2.3.12] but the argument above shows that it holds for any λ.

2.3. **Skeletons.** The notion of a skeleton was introduced in [Vas16a, Section 5] and is meant to be an axiomatization of a subclass of saturated models of an AEC. It is mentioned in (2) of the main corollary.

Recall the definition of an abstract class, due to the first author [Gro] (or see [Vas16c, 2.7]): it is a pair $\mathbf{K}' = (K', \leq_{\mathbf{K}'})$ so that K' is a class of τ-structures in a fixed vocabulary $\tau = \tau(\mathbf{K}')$, closed under isomorphisms, and $\leq_{\mathbf{K}'}$ is a partial order on K' which respects isomorphisms and extends the τ-substructure relation.

Definition 2.12 (5.3 in [Vas16a])**.** A *skeleton* of an abstract class $\mathbf{K}^*$ is an abstract class $\mathbf{K}'$ such that:

[6]We use the definition in [She09a, Definition N.2.4.4] which requires in addition that the model be universal.

(1) $K' \subseteq K^*$ and for $M, N \in \mathbf{K}'$, $M \leq_{\mathbf{K}'} N$ implies $M \leq_{\mathbf{K}^*} N$.
(2) $\mathbf{K}'$ is dense in $\mathbf{K}^*$: For any $M \in \mathbf{K}^*$, there exists $M' \in \mathbf{K}'$ such that $M \leq_{\mathbf{K}^*} M'$.
(3) If α is a (not necessarily limit) ordinal and $\langle M_i : i < \alpha \rangle$ is a strictly $\leq_{\mathbf{K}^*}$-increasing chain in $\mathbf{K}'$, then there exists $N \in \mathbf{K}'$ such that $M_i \leq_{\mathbf{K}'} N$ and[7] $M_i \neq N$ for all $i < \alpha$.

Example 2.13. Let $\lambda \geq \mathrm{LS}(\mathbf{K})$. Assume that $\mathbf{K}$ is stable in λ, has amalgamation and no maximal models in λ. Let K' be the class of limit models of size λ in $\mathbf{K}$. Then $(K', \leq_{\mathbf{K}})$ (or even K' ordered with "being equal or universal over") is a skeleton of $\mathbf{K}_\lambda$.

Remark 2.14. If $\mathbf{K}'$ is a skeleton of $\mathbf{K}_\lambda$ and $\mathbf{K}'$ itself generates an AEC, then $M \leq_{\mathbf{K}'} N$ if and only if $M, N \in \mathbf{K}'$ and $M \leq_{\mathbf{K}} N$. This is because of the third clause in the definition of a skeleton (used with $\alpha = 2$) and the coherence axiom.

We can define notions such as amalgamation and Galois types for any abstract class (see the preliminaries of [Vas16c]). The properties of a skeleton often correspond to properties of the original AEC:

Fact 2.15. Let $\lambda \geq \mathrm{LS}(\mathbf{K})$ and assume that $\mathbf{K}$ has amalgamation in λ. Let $\mathbf{K}'$ be a skeleton of $\mathbf{K}_\lambda$.

(1) For P standing for having no maximal models in λ, being stable in λ, or having joint embedding in λ, $\mathbf{K}$ has P if and only if $\mathbf{K}'$ has P.
(2) Assume that $\mathbf{K}$ has joint embedding in λ and for every limit $\delta < \lambda^+$ and every $N \in \mathbf{K}'$ there exists $N' \in \mathbf{K}'$ which is (λ, δ)-limit over N (in the sense of $\mathbf{K}'$).
   (a) Let $M, M_0 \in \mathbf{K}'$ and let $\delta < \lambda^+$ be a limit ordinal. Then M is (λ, δ)-limit over M_0 in the sense of $\mathbf{K}'$ if and only M is (λ, δ)-limit over M_0 in the sense of $\mathbf{K}$.
   (b) $\mathbf{K}'$ has no long splitting chains in λ if and only if $\mathbf{K}$ has no long splitting chains in λ.

Proof. (1) is by [Vas16a, 5.8]. As for (2a), (2b), note first that the hypotheses imply (by (1)) that $\mathbf{K}$ is stable in λ and has no maximal models in λ. In particular, limit models of size λ exist in $\mathbf{K}$.

Let us prove (2a). If M is (λ, δ)-limit over M_0 in the sense of $\mathbf{K}'$, then it is straightforward to check that the chain witnessing it will also witness that M is (λ, δ)-limit over M_0 in the sense of $\mathbf{K}$. For the converse, observe that by assumption there exists a (λ, δ)-limit M' over M_0 in the sense of $\mathbf{K}'$. Furthermore, by what has just been observed M' is also limit in the sense of $\mathbf{K}$, hence by uniqueness of limit models of the same length, $M' \cong_{M_0} M$. Therefore M is also (λ, δ)-limit over M_0 in the sense of $\mathbf{K}'$. The proof of (2b) is similar, see [Vas16a, 6.7]. $\square$

2.4. **Good frames.** Good frames are a local axiomatization of forking in a first-order superstable theories. They are introduced in [She09a, Chapter II]. We will use the definition from [Vas16a, 8.1] which is weaker and more general than Shelah's, as it does not require the existence of a superlimit (as in [JS13]). As opposed to

[7] Note that if α is limit this follows.

[Vas16a], we allow good frames that are *not* type-full: we only require the existence of a set of well-behaved basic types satisfying some density property (see [She09a, Chapter II] for more). Note however that Remark 5.8 says that in the context of the main theorem the existence of a good frame implies the existence of a *type-full* good frame (possibly over a different class).

In [Vas16a, 8.1], the underlying class of the good frame consists only of models of size λ. Thus when we say that there is a good λ-frame on a class $\mathbf{K}'$, we mean the underlying class of the good frame is $\mathbf{K}'$, and the axioms of good frames will require that $\mathbf{K}'$ generates a non-empty AEC with amalgamation in λ, joint embedding in λ, no maximal models in λ, and stability in λ.

The only facts that we will use about good frames are:

Fact 2.16. Let $\lambda \geq \mathrm{LS}(\mathbf{K})$. If there is a good λ-frame on a skeleton of $\mathbf{K}_\lambda$, then $\mathbf{K}$ has a unique limit model of size λ. Moreover, for any $M_0, M_1, M_2 \in \mathbf{K}_\lambda$, if both M_1 and M_2 are limit over M_0, then $M_1 \cong_{M_0} M_2$ (i.e. the isomorphism fixes M_0).

Proof. Let $\mathbf{K}'$ be the skeleton of $\mathbf{K}_\lambda$ which is the underlying class of the good λ-frame. By [She09a, II.4.8] (see [Bon14a, 9.2] for a detailed proof), $\mathbf{K}'$ has a unique limit model of size λ (and the moreover part holds for $\mathbf{K}'$). By Fact 2.15.(2a), this must also be the unique limit model of size λ in $\mathbf{K}$ (and the moreover part holds in $\mathbf{K}$ too). □

Fact 2.17. Assume that $\mathbf{K}$ has a monster model and is $\mathrm{LS}(\mathbf{K})$-tame. If $\mu < H_1$ is such that $\mathbf{K}$ is stable in μ and has no long splitting chains in μ, then there exists $\lambda_0 < H_1$ such that for all $\lambda \geq \lambda_0$, there is a good λ-frame on $\mathbf{K}_\lambda^{\lambda\text{-sat}}$. Moreover, $\mathbf{K}_\lambda^{\lambda\text{-sat}}$ is a skeleton of $\mathbf{K}_\lambda$, $\mathbf{K}$ is stable in λ, any $M \in \mathbf{K}_\lambda^{\lambda\text{-sat}}$ is superlimit, and the union of any increasing chain of λ-saturated models is λ-saturated.

Proof. First assume that $\mathbf{K}$ has no long splitting chains in $\mathrm{LS}(\mathbf{K})$ and is stable in $\mathrm{LS}(\mathbf{K})$. By [BV, 7.1], there exists $\lambda_0 < \beth_{(2^{\mu^+})^+}$ such that for any $\lambda \geq \lambda_0$, any increasing chain of λ-saturated models is λ-saturated and there is a good λ-frame on $\mathbf{K}_\lambda^{\lambda\text{-sat}}$. That any $M \in \mathbf{K}_\lambda^{\lambda\text{-sat}}$ is a superlimit (Fact 2.10.(5)) and $\mathbf{K}_\lambda^{\lambda\text{-sat}}$ is a skeleton of $\mathbf{K}_\lambda$ easily follows, and stability in λ is given (for example) by Fact 2.15.(1).

Now by [BV, 6.12], we more precisely have that if $\mathbf{K}$ has no long splitting chains in μ and is stable in μ (with $\mu \geq \mathrm{LS}(\mathbf{K})$) and $(< \mathrm{LS}(\mathbf{K}))$-tame (tameness being defined using types over sets), then the same conclusion holds with $\beth_{(2^{\mu^+})^+}$ replaced by H_1. Now the use of $(< \mathrm{LS}(\mathbf{K}))$-tameness is to derive that there exists $\chi < H_1$ so that $\mathbf{K}$ does not have a certain order property of length χ, but [BV] relies on an older version of [Vas16c] which proves Fact 2.4 assuming $(< \mathrm{LS}(\mathbf{K}))$-tameness instead of $\mathrm{LS}(\mathbf{K})$-tameness. In the current version of [Vas16c], it is shown that $\mathrm{LS}(\mathbf{K})$-tameness suffices, thus the arguments of [BV] go through assuming $\mathrm{LS}(\mathbf{K})$-tameness instead of $(< \mathrm{LS}(\mathbf{K}))$-tameness. □

2.5. **Solvability.** Solvability appears as a possible definition of superstability for AECs in [She09a, Chapter IV]. The definition uses Ehrenfeucht-Mostowski models and we assume the reader has some familiarity with them, see for example [Bal09, Section 6.2] or [She09a, IV.0.8].

Definition 2.18. Let $\mathrm{LS}(\mathbf{K}) \le \mu \le \lambda$.

(1) [She09a, IV.0.8] Let $\Upsilon_\mu[\mathbf{K}]$ be the set of Φ proper for linear orders (that is, Φ is a set $\{p_n : n < \omega\}$, where p_n is an n-variable quantifier-free type in a fixed vocabulary $\tau(\Phi)$ and the types in Φ can be used to generate a $\tau(\Phi)$-structure $\mathrm{EM}(I, \Phi)$ for each linear order I; that is, $\mathrm{EM}(I, \Phi)$ is the closure under the functions of $\tau(\Phi)$ of the universe of I and for any $i_0 < \ldots < i_{n-1}$ in I, $i_0 \ldots i_{n-1}$ realizes p_n) with:
   (a) $|\tau(\Phi)| \le \mu$.
   (b) If I is a linear order of cardinality λ, $\mathrm{EM}_{\tau(\mathbf{K})}(I, \Phi) \in \mathbf{K}_{\lambda + |\tau(\Phi)| + \mathrm{LS}(\mathbf{K})}$, where $\tau(\mathbf{K})$ is the vocabulary of $\mathbf{K}$ and $\mathrm{EM}_{\tau(\mathbf{K})}(I, \Phi)$ denotes the reduct of $\mathrm{EM}(I, \Phi)$ to $\tau(\mathbf{K})$. Here we are implicitly also assuming that $\tau(\mathbf{K}) \subseteq \tau(\Phi)$.
   (c) For $I \subseteq J$ linear orders, $\mathrm{EM}_{\tau(\mathbf{K})}(I, \Phi) \le_{\mathbf{K}} \mathrm{EM}_{\tau(\mathbf{K})}(J, \Phi)$.
   We call Φ as above an *EM blueprint*.
(2) [She09a, IV.1.4.(1)] We say that Φ *witnesses* (λ, μ)*-solvability* if:
   (a) $\Phi \in \Upsilon_\mu[\mathbf{K}]$.
   (b) If I is a linear order of size λ, then $\mathrm{EM}_{\tau(\mathbf{K})}(I, \Phi)$ is superlimit of size λ.
   $\mathbf{K}$ is (λ, μ)*-solvable* if there exists Φ witnessing (λ, μ)-solvability.
(3) $\mathbf{K}$ is *uniformly* (λ, μ)*-solvable* if there exists Φ such that for all $\lambda' \ge \lambda$, Φ witnesses (λ', μ)-solvability.

Remark 2.19. If $\mathbf{K}$ is uniformly (λ, μ)-solvable, then $\mathbf{K}$ is (λ', μ)-solvable for all $\lambda' \ge \lambda$.

Fact 2.20 (IV.0.9 in [She09a])**.** Let $\mathbf{K}$ be an AEC and let $\mu \ge \mathrm{LS}(\mathbf{K})$. Then $\mathbf{K}$ has arbitrarily large models if and only if $\Upsilon_\mu[\mathbf{K}] \ne \emptyset$.

We give some more manageable definitions of solvability ((3) is the one we will use). Shelah already mentions one of them on [She09a, p. 53] (but does not prove it is equivalent).

Lemma 2.21. Let $\mathrm{LS}(\mathbf{K}) \le \mu \le \lambda$. The following are equivalent.

(1) $\mathbf{K}$ is [uniformly] (λ, μ)-solvable.
(2) There exists $\tau' \supseteq \tau(\mathbf{K})$ with $|\tau'| \le \theta$ and $\psi \in \mathbb{L}_{\mu^+, \omega}(\tau')$ such that:
   (a) ψ has arbitrarily large models.
   (b) [For all $\lambda' \ge \lambda$], if $M \models \psi$ and $\|M\| = \lambda$ [$\|M\| = \lambda'$], then $M \upharpoonright \tau(\mathbf{K})$ is in $\mathbf{K}$ and is superlimit.
(3) There exists $\tau' \supseteq \tau(\mathbf{K})$ and an AEC $\mathbf{K}'$ with $\tau(\mathbf{K}') = \tau'$, $\mathrm{LS}(\mathbf{K}') \le \mu$ such that:
   (a) $\mathbf{K}'$ has arbitrarily large models.
   (b) [For all $\lambda' \ge \lambda$], if $M \in \mathbf{K}'$ and $\|M\| = \lambda$ [$\|M\| = \lambda'$], then $M \upharpoonright \tau(\mathbf{K})$ is in $\mathbf{K}$ and is superlimit.

Proof.

- <u>(1) implies (2)</u>: Let Φ witness (λ, μ)-solvability and write $\Phi = \{p_n \mid n < \omega\}$. Let $\tau' := \tau(\Phi) \cup \{P, <\}$, where $P, <$ are symbols for a unary predicate and a binary relation respectively. Let $\psi \in \mathbb{L}_{\mu^+, \omega}(\tau')$ say:

(1) $(P, <)$ is a linear order.
(2) For all $n < \omega$ and all $x_0 < \cdots < x_{n-1}$ in P, $x_0 \ldots x_{n-1}$ realizes p_n.
(3) For all y, there exists $n < \omega$, $x_0 < \cdots < x_{n-1}$ in P, and ρ an n-ary term of $\tau(\Phi)$ such that $y = \rho(x_0, \ldots, x_{n-1})$.

Then if $M \models \psi$, $M \upharpoonright \tau = \text{EM}_{\tau(\mathbf{K})}(P^M, \Phi)$. Conversely, if $M = \text{EM}_{\tau(\mathbf{K})}(I, \Phi)$, we can expand M to a τ'-structure M' by letting $(P^{M'}, <^{M'}) := (I, <)$. Thus ψ is as desired.

- (2) implies (3): Given τ' and ψ as given by (2), Let Ψ be a fragment of $\mathbb{L}_{\mu^+, \omega}(\tau')$ containing ψ of size θ and let $\mathbf{K}'$ be $\text{Mod}(\psi)$ ordered by $\preceq_\Psi$. Then $\mathbf{K}'$ is as desired for (3).
- (3) implies (1): Directly from Fact 2.20.

$\square$

3. Forking and averages in stable AECs

In the introduction to his book [She09a, p. 61], Shelah asserts (without proof) that in the first-order context solvability (see Section 2.5) is equivalent to superstability. We aim to give a proof (see Corollary 5.4) and actually show (assuming amalgamation, stability, and tameness) that solvability is equivalent to any of the definitions in the main theorem. First of all, if there exists μ such that $\mathbf{K}$ is (λ, μ)-solvable for all high-enough λ, then in particular $\mathbf{K}$ has a superlimit in all high-enough λ, so we obtain (4) in the main corollary. We work toward a converse. The proof is similar to that in [BGS99]: we aim to code saturated models using their characterization with average of sequences (the crucial result for this is Lemma 3.16). In this section, we use the theory of averages in AECs (as developed by Shelah in [She09b, Chapter V.A] and by Boney and the second author in [BV]) to give a new characterization of forking (Lemma 3.12). We also prove the key result for (5) implies (1) in the main corollary (Theorem 3.18). All throughout, we assume:

Hypothesis 3.1.

(1) $\mathbf{K}$ has a monster model.
(2) $\mathbf{K}$ is $\text{LS}(\mathbf{K})$-tame.
(3) $\mathbf{K}$ is stable in some cardinal greater than or equal to $\text{LS}(\mathbf{K})$.
(4) We work inside a big monster model $\mathfrak{C}$.

We set $\kappa := \text{LS}(\mathbf{K})^+$ and work in the setup of [BV, Section 5]. In particular we identify Galois types with the set of their restrictions to models of size $\text{LS}(\mathbf{K})$. We encourage the reader to have a copy of [BV] available, since we will cite from there freely and use basic notation and terminology (χ-convergent, χ-based, (χ_0, χ_1, χ_2)-Morley, $\text{Av}_\chi(\mathbf{I}/A)$ etc.) without even an explicit citation. We will say that $p \in \text{gS}^{<\kappa}(M)$ *does not syntactically split* over $M_0 \leq_{\mathbf{K}} M$ if it does not split in the syntactic sense of [BV, 5.7]. Note that several results from [BV] that we quote assume $(< \text{LS}(\mathbf{K}))$-tameness (defined in terms of Galois types over sets). However, as argued in the proof of Fact 2.17, $\text{LS}(\mathbf{K})$-tameness suffices.

We will define several other cardinals $\chi_0 < \chi_0' < \chi_1 < \chi_1' < \chi_2$ (see Notation 3.4, 3.9, and 3.10). The reader can simply see them as "high-enough" cardinals with reasonable closure properties. If χ_0 is chosen reasonably, we will have $\chi_2 < H_1$.

The letters $\mathbf{I}$, $\mathbf{J}$ will denote sequences of tuples of length strictly less than κ. We will use the same conventions as in [BV, Section 5]. Note that the sequences *may be indexed by arbitrary linear orders*.

By Facts 2.4 and [She99, I.4.5.(3)] (recalling that there is a global assumption of stability in this section), we have:

Fact 3.2. There exists $\chi_0 < H_1$ such that $\mathbf{K}$ does not have the $\mathrm{LS}(\mathbf{K})$-order property of length χ_0.

Another property of χ_0 is the following more precise version of Fact 2.4 (see [Vas16c] on how to translate Shelah's syntactic version to AECs):

Fact 3.3 (Theorem V.A.1.19 in [She09b])**.** If $\lambda = \lambda^{\chi_0}$, then $\mathbf{K}$ is stable in λ. In particular, $\mathbf{K}$ is stable in χ_0'.

The following notation will be convenient:

Notation 3.4. Let χ_0 be any regular cardinal such that $\chi_0 \geq 2^{\mathrm{LS}(\mathbf{K})}$ and $\mathbf{K}$ does not have the $\mathrm{LS}(\mathbf{K})$-order property of length χ_0^+. For a cardinal λ, let $\gamma(\lambda) := (2^{2^\lambda})^+$. We write $\chi_0' := \gamma(\chi_0)$.

Remark 3.5. By Fact 3.2, one can take $\chi_0 < H_1$. In that case also $\chi_0' < H_1$. For the sake of generality, we do *not* require that χ_0 be least with the property above.

Recall [BV, 5.21] that if $\mathbf{I}$ is a $(\chi_0^+, \chi_0^+, \gamma(\chi_0))$-Morley sequence, then $\mathbf{I}$ is χ-convergent. We want to use this to relate average and forking:

Definition 3.6. Let $M_0, M \in \mathbf{K}^{(\chi_0')^+\text{-sat}}$ be such that $M_0 \leq_{\mathbf{K}} M$. Let $p \in \mathrm{gS}(M)$. We say that p *does not fork over* M_0 if there exists $M_0' \in \mathbf{K}_{\chi_0'}$ such that $M_0' \leq_{\mathbf{K}} M_0$ and p does not χ_0'-split over M_0'.

We will use without comments:

Fact 3.7. Forking has the following properties:

(1) Invariance under isomorphisms and monotonicity: if $M_0 \leq_{\mathbf{K}} M_1 \leq_{\mathbf{K}} M_2$ are all $(\chi_0')^+$-saturated and $p \in \mathrm{gS}(M_2)$ does not fork over M_0, then $p \upharpoonright M_1$ does not fork over M_0 and p does not fork over M_1.
(2) Set local character: if $M \in \mathbf{K}^{(\chi_0')^+\text{-sat}}$ and $p \in \mathrm{gS}(M)$, there exists $M_0 \in \mathbf{K}^{(\chi_0')^+\text{-sat}}$ of size $(\chi_0')^+$ such that $M_0 \leq_{\mathbf{K}} M$ and p does not fork over M_0.
(3) Transitivity: Assume $M_0 \leq_{\mathbf{K}} M_1 \leq_{\mathbf{K}} M_2$ are all $(\chi_0')^+$-saturated and $p \in \mathrm{gS}(M_2)$. If p does not fork over M_1 and $p \upharpoonright M_1$ does not fork over M_0, then p does not fork over M_0.
(4) Uniqueness: If $M_0 \leq_{\mathbf{K}} M$ are all $(\chi_0')^+$-saturated and $p, q \in \mathrm{gS}(M)$ do not fork over M_0, then $p \upharpoonright M_0 = q \upharpoonright M_0$ implies $p = q$. Moreover p does not λ-split over M_0 for any $\lambda \geq (chi_0')^+$.
(5) Local extension over saturated models: If $M_0 \leq_{\mathbf{K}} M$ are both saturated, $\|M_0\| = \|M\| \geq (\chi_0')^+$, $p \in \mathrm{gS}(M_0)$, there exists $q \in \mathrm{gS}(M)$ such that q extends p and does not fork over M_0.

Proof. Use [Vas16a, 7.5]. The generator used is the one given by Proposition 7.4.(2) there. For the moreover part of uniqueness, use [BGKV16, 4.2] (and [BGKV16, 3.12]). □

Note that the extension property need not hold in general. However if the class has no long splitting chains we have:

Fact 3.8. If $\mathbf{K}$ is has no long splitting chains in χ_0', then:

(1) ([Vas16a, 8.9] or [Vas16b, 7.1]) Forking has:
   (a) The extension property: If $M_0 \leq_{\mathbf{K}} M$ are $(\chi_0')^+$-saturated and $p \in \mathrm{gS}(M_0)$, then there exists $q \in \mathrm{gS}(M)$ extending p and not forking over M_0.
   (b) The chain local character property: If $\langle M_i : i < \delta \rangle$ is an increasing chain of $(\chi_0')^+$-saturated models and $p \in \mathrm{gS}(\bigcup_{i<\delta} M_i)$, then there exists $i < \delta$ such that p does not fork over M_i.
(2) [BV, Lemma 6.9] For any $\lambda > (\chi_0')^+$, $\mathbf{K}^{\lambda\text{-sat}}$ is an AEC with $\mathrm{LS}(\mathbf{K}^{\lambda\text{-sat}}) = \lambda$.

For notational convenience, we "increase" χ_0:

Notation 3.9. Let $\chi_1 := (\chi_0')^{++}$. Let $\chi_1' := \gamma(\chi_1)$.

We obtain a characterization of forking that adds to those proven in [Vas16a, Section 9]. A form of it already appears in [She09a, Observation IV.4.6]. Again, we define more cardinal parameters:

Notation 3.10. Let $\chi_2 := \beth_\omega(\chi_0)$.

Remark 3.11. We have that $\chi_0 < \chi_0' < \chi_1 < \chi_1' < \chi_2$, and $\chi_2 < H_1$ if $\chi_0 < H_1$.

Lemma 3.12. Let M_0, M be χ_2-saturated with $M_0 \leq_{\mathbf{K}} M$. Let $p \in \mathrm{gS}(M)$. The following are equivalent:

(1) p does not fork over M_0.
(2) $p \upharpoonright M_0$ has a nonforking extension to $\mathrm{gS}(M)$ and there exists $M_0' \leq_{\mathbf{K}} M_0$ with $\|M_0'\| < \chi_2$ such that p does not syntactically split over M_0'.
(3) $p \upharpoonright M_0$ has a nonforking extension to $\mathrm{gS}(M)$ and there exists $\mu \in [\chi_0^+, \chi_2)$ and $\mathbf{I}$ a $(\mu, \mu, \gamma(\mu)^+)$-Morley sequence for p, with all the witnesses inside M_0, such that $\mathrm{Av}_{\gamma(\mu)}(\mathbf{I}/M) = p$.

Remark 3.13. When $\mathbf{K}$ is has no long splitting chains in χ_0', forking has the extension property (Fact 3.8) so the first part of (2) and (3) always hold. However in Theorem 3.18 we apply Lemma 3.12 in the strictly stable case (i.e. $\mathbf{K}$ may only be *stable* in χ_0' and not have no long splitting chains there).

We recall more definitions and facts before giving the proof of Lemma 3.12:

Fact 3.14 (V.A.1.12 in [She09b])**.** If $p \in \mathrm{gS}(M)$ and M is χ_0^+-saturated, there exists $M_0 \in \mathbf{K}_{\leq \chi_0}$ with $M_0 \leq_{\mathbf{K}} M$ such that p does not syntactically split over M_0.

Fact 3.15. Let $M_0 \leq_{\mathbf{K}} M$ be both $(\chi_1')^+$-saturated. Let $\mu := \|M_0\|$. Let $p \in \mathrm{gS}(M)$ and let $\mathbf{I}$ be a $(\mu^+, \mu^+, \gamma(\mu))$-Morley sequence for p over M_0 with all the witnesses inside M. Then if p does not syntactically split or does not fork over M_0, then $\mathrm{Av}_{\gamma(\mu)}(\mathbf{I}/M) = p$.

Proof. For syntactic splitting, this is [BV, 5.25]. The Lemma is actually more general and the proof of [BV, 6.9] shows that this also works for forking. $\square$

Proof of Lemma 3.12. Before starting, note that if $\mu < \chi_2$, then $\mathbf{K}$ is stable in $2^{\mu+\chi_0} < \chi_2$ by Fact 3.3. Thus there are unboundedly many stability cardinals below χ_2, so we have "enough space" to build Morley sequences.

- (1) implies (2): By Fact 3.14, we can find $M_0' \leq_{\mathbf{K}} M_0$ such that $p \upharpoonright M_0$ does not syntactically split over M_0' and $\|M_0'\| \leq \chi_1$. Taking M_0' bigger, we can assume M_0' is χ_1-saturated and $p \upharpoonright M_0$ does not fork over M_0'. Thus by transitivity p does not fork over M_0'. Let $\mathbf{I}$ be a $(\chi_1^+, (\chi_1')^+, (\chi_1')^+)$-Morley sequence for $p \upharpoonright M_0$ over M_0' inside M_0. By [BV, 5.21], $\mathbf{I}$ is χ_1'-convergent. By [BV, 5.20], $\mathbf{I}$ is χ_1'-based on M_0'. Note also that $\mathbf{I}$ is a $(\chi_1^+, (\chi_1')^+, (\chi_1')^+)$-Morley sequence for p over M_0' and by Fact 3.15, $\text{Av}_{\chi_1'}(\mathbf{I}/M_0) = p$ so as $\mathbf{I}$ is χ_1'-based on M_0', p does not syntactically split over M_0'.
- (2) implies (3): As in the proof of (1) implies (2) (except χ_1 could be bigger).
- (3) implies (2): By Fact [BV, 5.21], $\mathbf{I}$ is $\gamma(\mu)$-convergent. Pick any $\mathbf{J} \subseteq \mathbf{I}$ of length $\gamma(\mu)$ and use [BV, 5.10] to find $M_0' \leq_{\mathbf{K}} M_0$ of size $\gamma(\mu)$ such that $\mathbf{J}$ is $\gamma(\mu)$-based on M_0'. Since also $\mathbf{J}$ is $\gamma(\mu)$-convergent, we have that $\mathbf{I}$ is $\gamma(\mu)$-based on M_0'. Thus $\text{Av}_{\gamma(\mu)}(\mathbf{I}/M) = p$ does not syntactically split over M_0'.
- (2) implies (1): Without loss of generality, we can choose M_0' to be such that $p \upharpoonright M_0$ also does not fork over M_0'. Let $\mu := \|M_0'\| + \chi_0$. Build a $(\mu^+, \mu^+, \gamma(\mu))$-Morley sequence $\mathbf{I}$ for p over M_0' inside M_0. If q is the nonforking extension of $p \upharpoonright M_0$ to M, then $\mathbf{I}$ is also a Morley sequence for q over M_0' so by the proof of (1) implies (2) we must have $\text{Av}_{\gamma(\mu)}(\mathbf{I}/M) = q$, but also $\text{Av}_{\gamma(\mu)}(\mathbf{I}/M) = p$, since p does not syntactically split over M_0' (Fact 3.15). Thus $p = q$.

□

The next result is a version of [She90, III.3.10] in our context. It is implicit in the proof of [BV, 5.27].

Lemma 3.16. Let $M \in \mathbf{K}^{\chi_2\text{-sat}}$. Let $\lambda \geq \chi_2$ be such that $\mathbf{K}$ is stable in unboundedly many $\mu < \lambda$. The following are equivalent.

1. M is λ-saturated.
2. If $q \in \text{gS}(M)$ is not algebraic and does not syntactically split over $M_0 \leq_{\mathbf{K}} M$ with $\|M_0\| < \chi_2$, there exists a $((\|M_0\| + \chi_0)^+, (\|M_0\| + \chi_0)^+, \lambda)$-Morley sequence for p over M_0 inside M.

Proof. (1) implies (2) is trivial using saturation. Now assume (2). Let $p \in \text{gS}(N)$, $\|N\| < \lambda$, $N \leq_{\mathbf{K}} M$. We show that p is realized in M. Let $q \in \text{gS}(M)$ extend p. If q is algebraic, we are done so assume it is not. Let $M_0 \leq_{\mathbf{K}} M$ have size $(\chi_1')^+$ such that q does not fork over M_0. By Lemma 3.12, we can increase M_0 if necessary so that q does not syntactically split over M_0 and $\mu := \|M_0\| \geq \chi_0$. Now by (2), there exists a (μ^+, μ^+, λ)-Morley sequence $\mathbf{I}$ for q over M_0 inside M. Now by Fact 3.15, $\text{Av}_{\gamma(\mu)}(\mathbf{I}/M) = q$. Thus $\text{Av}_{\gamma(\mu)}(\mathbf{I}/N) = p$. By [BV, 5.6] and the hypothesis of stability in unboundedly many cardinals below λ, p is realized by an element of $\mathbf{I}$ and hence by an element of M. □

We end by showing that if high-enough limit models are sufficiently saturated, then no long splitting chains holds. A similar argument already appears in the proof [She09a, IV.4.10]. We start with a more local version,

Lemma 3.17. Let $\lambda \geq \chi_2$. Let $\delta < \lambda^+$ be a limit ordinal and let $\langle M_i : i < \delta \rangle$ be an increasing chain of saturated models in $\mathbf{K}_\lambda$. Let $M_\delta := \bigcup_{i<\delta} M_i$. If M_δ is χ_2-saturated, then for any $p \in \mathrm{gS}(M_\delta)$, there exists $i < \delta$ such that p does not fork over M_i.

Proof. Without loss of generality, δ is regular. If $\delta \geq \chi_2$, by set local character (Fact 3.7.(2)), there exists M_0' of size χ_1 such that p does not fork over M_0' and $M_0' \leq_{\mathbf{K}} M_\delta$, so pick $i < \delta$ such that $M_0' \leq_{\mathbf{K}} M_i$ and use monotonicity.

Now assume $\delta < \chi_2$. By assumption, we have that M_δ is χ_2-saturated. We also have that p does not fork over M_δ (by set local character) so by Lemma 3.12, there exists $\mu \in [\chi_0^+, \chi_2)$ and $\mathbf{I}$ a $(\mu, \mu, \gamma(\mu)^+)$-Morley sequence for p with all the witnesses inside M_δ such that $\mathrm{Av}_{\gamma(\mu)}(\mathbf{I}/M_\delta) = p$. Since M_δ is χ_2-saturated (and there are unboundedly many stability cardinals below χ_2), we can increase $\mathbf{I}$ if necessary to assume that $|\mathbf{I}| \geq \chi_2$. Write $\mathbf{I}_i := |M_i| \cap \mathbf{I}$. Since $\delta < \chi_2$, there must exists $i < \delta$ such that $|\mathbf{I}_i| \geq \chi_2$. Note that $\mathbf{I}_i$ is a (μ, μ, χ_2)-Morley sequence for p. Because $\mathbf{I}$ is $\gamma(\mu)$-convergent and $|\mathbf{I}_i| \geq \chi_2 > \gamma(\mu)$, $\mathrm{Av}_{\gamma(\mu)}(\mathbf{I}_i/M_\delta) = p$. Letting $M' \geq_{\mathbf{K}} M_\delta$ be a saturated model of size λ and using local extension over saturated models (Fact 3.7.(5), $p \upharpoonright M_i$ has a nonforking extension to $\mathrm{gS}(M')$ and hence to $\mathrm{gS}(M_\delta)$. By Lemma 3.12, p does not fork over M_i, as desired. $\square$

For the convenience of the reader, we repeat the hypotheses of the section in the statement of the next result.

Theorem 3.18. Assume that $\mathbf{K}$ has a monster model, is $\mathrm{LS}(\mathbf{K})$-tame, and stable in some cardinal greater than or equal to $\mathrm{LS}(\mathbf{K})$.

Let $\chi_0 \geq \mathrm{LS}(\mathbf{K})$ be such that $\mathbf{K}$ does not have the $\mathrm{LS}(\mathbf{K})$-order property of length χ_0, and let $\chi_2 := \beth_\omega(\chi_0)$. Let $\lambda \geq \chi_2$ be such that $\mathbf{K}$ is stable in λ and there exists a saturated model of cardinality λ. If every limit model of cardinality λ is χ_2-saturated, then $\mathbf{K}$ has no long splitting chains in λ.

Proof. Let $\mathbf{K}'$ be $K_\lambda^{\chi_2\text{-sat}}$ ordered by being equal or universal over. Note that, by stability in λ, $\mathbf{K}'$ is a skeleton of $\mathbf{K}_\lambda$ (see Definition 2.12). Moreover since every limit model of cardinality λ is χ_2-saturated, for any limit $\delta < \lambda^+$, one can build an increasing continuous chain $\langle M_i : i \leq \delta \rangle$ in $\mathbf{K}_\lambda$ such that for all $i \leq \delta$, M_i is χ_2-saturated and (when $i < \delta$) M_{i+1} is universal over M_i. Therefore limit models exist in $\mathbf{K}'$, so the assumptions of Fact 2.15.(2b) are satisfied. So it is enough to see that $\mathbf{K}'$ (not $\mathbf{K}$) has no long splitting chains in λ.

Let $\delta < \lambda^+$ be limit and let $\langle M_i : i < \delta \rangle$ be an increasing chain of models in $\mathbf{K}'$, with M_{i+1} universal over M_i for all $i < \delta$. Let $M_\delta := \bigcup_{i<\delta} M_i$. By assumption, M_δ is χ_2-saturated. By uniqueness of limit models of the same length, we can assume without loss of generality that M_{i+1} is saturated for all $i < \delta$.

Let $p \in \mathrm{gS}(M_\delta)$. By Lemma 3.17 (applied to $\langle M_{i+1} : i < \delta \rangle$), there exists $i < \delta$ such that p does not fork over M_i. By the moreover part of Fact 3.7.(4), p does not λ-split over M_i, as desired. $\square$

4. No long splitting chains implies solvability

From now on we assume no long splitting chains:

Hypothesis 4.1.

(1) Hypothesis 3.1, namely $\mathbf{K}$ is a $\mathrm{LS}(\mathbf{K})$-tame AEC with a monster model that is stable in some cardinal greater than or equal to $\mathrm{LS}(\mathbf{K})$. We work inside a big monster model $\mathfrak{C}$ and fix cardinals $\chi_0 < \chi_0' < \chi_1 < \chi_1' < \chi_2$ as defined in Notation 3.4, 3.9, and 3.10. Note that by Fact 3.3 $\mathbf{K}$ is stable in χ_0'.
(2) $\mathbf{K}$ has no long splitting chains in χ_0'.

In Notation 4.3, we will define another cardinal χ with $\chi_2 < \chi$. If $\chi_0 < H_1$, we will also have that $\chi < H_1$.

Note that no long splitting chains in χ_0' and stability in χ_0' implies (Fact 2.7) that $\mathbf{K}$ is stable in all $\lambda \geq \chi_0'$. Further, forking is well-behaved in the sense of Fact 3.8. This implies that Morley sequences are closed under unions (here we use that they are indexed by arbitrary linear orders, as opposed to just well-orderings):

Lemma 4.2. Let $\langle I_\alpha : \alpha < \delta \rangle$ be an increasing (with respect to substructure) sequence of linear orders and let $I_\delta := \bigcup_{\alpha<\delta} I_\alpha$. Let M_0, M be χ_2-saturated such that $M_0 \leq_{\mathbf{K}} M$. Let μ_0, μ_1, μ_2 be such that $\chi_2 < \mu_0 \leq \mu_1 \leq \mu_2$, $p \in \mathrm{gS}(M)$ and for $\alpha < \delta$, let $\mathbf{I}_\alpha := \langle a_i : i \in I_\alpha \rangle$ together with $\langle N_i^\alpha : i \in I_\alpha \rangle$ be (μ_0, μ_1, μ_2)-Morley for p over M_0, with $N_i^\alpha \leq_{\mathbf{K}} N_i^\beta \leq_{\mathbf{K}} M$ for all $\alpha \leq \beta < \delta$ and $i \in I_\alpha$. For $i \in I_\alpha$, let $N_i^\delta := \bigcup_{\beta \in [\alpha, \delta)} N_i^\beta$. Let $\mathbf{I}_\delta := \langle a_i : i \in I_\delta \rangle$.

If p does not fork over M_0, then $\mathbf{I}_\delta \frown \langle N_i^\delta : i \in I_\delta \rangle$ is (μ_0, μ_1, μ_2)-Morley for p over M_0.

Proof. By Lemma 3.12, p does not syntactically split over M_0. Therefore the only problematic clauses in [BV, Definition 5.14] are (4) and (7). Let's check (4): let $i \in I_\delta$. By hypothesis, $\bar{a}_i$ realizes $p \upharpoonright N_i^\alpha$ for all sufficiently high $\alpha < \delta$. By local character of forking, there exists $\alpha < \delta$ such that $\mathrm{gtp}(\bar{a}_i / N_i^\delta)$ does not fork over N_i^α. Since $\mathrm{gtp}(\bar{a}_i / N_i^\delta) \upharpoonright N_i^\alpha = p \upharpoonright N_i^\alpha$ and p does not fork over $M_0 \leq_{\mathbf{K}} N_i^\alpha$, we must have by uniqueness that $p \upharpoonright N_i^\delta = \mathrm{gtp}(\bar{a}_i / N_i^\delta)$. The proof of (7) is similar. $\square$

For convenience, we make χ_2 even bigger:

Notation 4.3. Let $\chi := \gamma(\chi_2)$ (recall from Notation 3.4 that $\gamma(\chi_2) = \left(2^{2^{\chi_2}}\right)^+$). A Morley sequence means a $(\chi_2^+, \chi_2^+, \chi)$-Morley sequence.

Remark 4.4. By Remark 3.11, we still have $\chi < H_1$ if $\chi_0 < H_1$.

We are finally in a position to prove solvability (in fact even uniform solvability). We will use condition (3) in Lemma 2.21.

Definition 4.5. We define a class of models K' and a binary relation $\leq_{\mathbf{K}'}$ on K' (and write $\mathbf{K}' := (K', \leq_{\mathbf{K}'})$) as follows.

- K' is a class of $\tau' := \tau(\mathbf{K}')$-structures, with:

$$\tau' := \tau(\mathbf{K}) \cup \{N_0, N, F, R\}$$

where:
- N_0 and R are binary relations symbols.
- N is a ternary relation symbol.
- F is a binary function symbol.

- A τ'-structure M is in K' if and only if:
  (1) $M \restriction \tau(\mathbf{K}) \in \mathbf{K}^{\chi\text{-sat}}$.
  (2) R^M is a linear ordering on $|M|$. We write I for this linear ordering.
  (3) For[8] all $a \in |M|$ and all $i \in I$, $N^M(a, i) \leq_{\mathbf{K}} M \restriction \tau(\mathbf{K})$ (where we see $N^M(a, i)$ as an $\tau(\mathbf{K})$-structure; in particular, $N^M(a, i) \in \mathbf{K}$; it will follow from (4b) that the $N^M(a, i)$'s are increasing with i), $N_0^M(a) \leq_{\mathbf{K}} N^M(a, i)$, and $N_0^M(a)$ is saturated of size χ_2.
  (4) There exists a map $a \mapsto p_a$ from $|M|$ onto the non-algebraic Galois types over $M \restriction \tau(\mathbf{K})$ such that for all $a \in |M|$:
    (a) p_a does not fork[9] over $N_0^M(a)$.
    (b) $\langle F^M(a, i) : i \in I\rangle \frown \langle N^M(a, i) : i \in I\rangle$ is a Morley sequence for p_a over $N_0^M(a)$.
- $M \leq_{K'} M'$ if and only if:
  (1) $M \subseteq M'$.
  (2) $M \restriction \tau(\mathbf{K}) \leq_{\mathbf{K}} M' \restriction \tau(\mathbf{K})$.
  (3) For all $a \in |M|$, $N_0^M(a) = N_0^{M'}(a)$.

We show in Lemma 4.7 that $\mathbf{K}'$ is an AEC, but first let us see that this suffices:

Lemma 4.6. Let $\lambda \geq \chi$.

(1) If $M \in \mathbf{K}_\lambda$ is saturated, then there exists an expansion M' of M to τ' such that $M' \in \mathbf{K}'$.
(2) If $M' \in \mathbf{K}'$ has size λ, then $M' \restriction \tau(\mathbf{K})$ is saturated.

Proof.

(1) Let $R^{M'}$ be a well-ordering of $|M|$ of type λ. Identify $|M|$ with λ. By stability, we can fix a bijection $p \mapsto a_p$ from $\mathrm{gS}(M)$ onto $|M|$. For each $p \in \mathrm{gS}(M)$ which is not algebraic, fix $N_p \leq_{\mathbf{K}} M$ saturated such that p does not fork over N_p and $\|N_p\| = \chi_2$. Then use saturation to build $\langle a_p^i : i < \lambda\rangle \frown \langle N_p^i : i < \lambda\rangle$ Morley for p over N_p (inside M). Let $N_0^{M'}(a_p) := N_p$, $N^{M'}(a_p, i) := N_p^i$, $F^{M'}(a, i) := a_p^i$. For p algebraic, pick $p_0 \in \mathrm{gS}(M)$ nonalgebraic and let $N_0^{M'}(a_p) := N_0^{M'}(a_{p_0})$, $N^{M'}(a_{p_0}) := N^{M'}(a_{p_0})$, $F^{M'}(a_p) := F^{M'}(a_{p_0})$.
(2) By Lemma 3.16.

□

Lemma 4.7. $\mathbf{K}'$ is an AEC with $\mathrm{LS}(\mathbf{K}') = \chi$.

[8]For a binary relation Q we write $Q(a)$ for $\{b \mid Q(a, b)\}$, similarly for a tertiary relation.

[9]Note that by Lemma 3.12 this also implies that it does not syntactically split over some $M_0' \leq_{\mathbf{K}} N_0^M(a)$ with $\|M_0'\| < \chi_2$.

Proof. It is straightforward to check that $\mathbf{K}'$ is an abstract class with coherence. Moreover:

- $\underline{\mathbf{K}' \text{ satisfies the chain axioms}}$: Let $\langle M_i : i < \delta \rangle$ be increasing in $\mathbf{K}'$. Let $M_\delta := \bigcup_{i<\delta} M_i$.
  - $M_0 \leq_{\mathbf{K}'} M_\delta$, and if $N \geq_{\mathbf{K}'} M_i$ for all $i < \delta$, then $N \geq_{\mathbf{K}'} M_\delta$: Straightforward.
  - $M_\delta \in \mathbf{K}'$: $M_\delta \upharpoonright \tau(\mathbf{K})$ is χ-saturated by Fact 3.8. Moreover, R^{M_δ} is clearly a linear ordering of M_δ. Write I_i for the linear ordering (M_i, R_i). Condition 3 in the definition of $\mathbf{K}'$ is also easily checked. We now check Condition 4.
    Let $a \in |M_\delta|$. Fix $i < \delta$ such that $a \in |M_i|$. Without loss of generality, $i = 0$. By hypothesis, for each $i < \delta$, there exists $p_a^i \in \mathrm{gS}(M_i \upharpoonright \tau(\mathbf{K}))$ not algebraic such that $\langle F^{M_i}(a,j) \mid j \in I_i \rangle \frown \langle N^{M_i}(a,j) \mid j \in I_i \rangle$ is a Morley sequence for p_a^i over $N_0^{M_i}(a) = N_0^{M_0}(a)$. Clearly, $p_a^i \upharpoonright N_0^{M_0}(a) = p_a^0 \upharpoonright N_0^{M_0}(a)$ for all $i < \delta$. Moreover by assumption p_a^i does not fork over $N_0^{M_0}$. Thus for all $i < j < \delta$, $p_a^j \upharpoonright M_i = p_a^i \upharpoonright M_i$. By extension and uniqueness, there exists $p_a \in \mathrm{gS}(M_\delta \upharpoonright \tau(\mathbf{K}))$ that does not fork over $N_0^{M_0}(a)$ and we have $p_a \upharpoonright M_i = p_a^i$ for all $i < \delta$. Now by Lemma 4.2, $\langle F^{M_\delta}(a,j) \mid j \in I_\delta \rangle \frown \langle N^{M_\delta}(a,j) \mid j \in I_\delta \rangle$ is a Morley sequence for p_a over $N_0^{M_0}(a)$.
    Moreover, the map $a \mapsto p_a$ is onto the nonalgebraic Galois types over $M_\delta \upharpoonright \tau(\mathbf{K})$: let $p \in \mathrm{gS}(M_\delta \upharpoonright \tau(\mathbf{K}))$ be nonalgebraic. Then there exists $i < \delta$ such that p does not fork over M_i. Let $a \in |M_i|$ be such that $\langle F^{M_i}(a,j) \mid j \in I_i \rangle \frown \langle N^{M_i}(a,j) \mid j \in I_i \rangle$ is a Morley sequence for $p \upharpoonright M_i$ over $N_0^{M_i}(a)$. It is easy to check it is also a Morley sequence for p over $N_0^{M_i}(a)$. By uniqueness of the nonforking extension, we get that the extended Morley sequence is also Morley for p, as desired.
- $\underline{\mathrm{LS}(\mathbf{K}') = \chi}$: An easy closure argument.

$\square$

Theorem 4.8. $\mathbf{K}$ is uniformly (χ, χ)-solvable.

Proof. By Lemma 4.7, $\mathbf{K}'$ is an AEC with $\mathrm{LS}(\mathbf{K}') = \chi$. Now combine Lemma 4.6 and Lemma 2.21. Note that by Fact 3.8, for each $\lambda \geq \chi$ there is a saturated models of size λ, and it is also a superlimit. $\square$

For the convenience of the reader, we give a more quotable version of Theorem 4.8. For the next results, we drop Hypothesis 4.1.

Theorem 4.9. Assume that $\mathbf{K}$ has a monster model, is $\mathrm{LS}(\mathbf{K})$-tame, and is stable in some cardinal greater than or equal to $\mathrm{LS}(\mathbf{K})$. There exists $\chi < H_1$ such that for any $\mu \geq \chi$, if $\mathbf{K}$ is stable in μ and has no long splitting chains in μ then $\mathbf{K}$ is uniformly (μ', μ')-solvable, where $\mu' := (\beth_{\omega+2}(\mu))^+$.

Proof. Hypothesis 3.1 holds. Let $\chi < H_1$ be such that $\mathbf{K}$ does not have the $\mathrm{LS}(\mathbf{K})$-order property of length χ (see Fact 3.2).

Let $\mu \geq \chi$ be such that $\mathbf{K}$ is stable in μ and has no long splitting chains in μ. We apply Theorem 4.8 by letting χ_0 in Notation 3.4 stand for μ here. By Fact

2.7, $\mathbf{K}$ is stable in μ_1 and has no long splitting chains in μ_1 for every $\mu_1 \geq \mu$, thus Hypothesis 4.1 holds. Moreover χ_2 in Notation 3.10 corresponds to $\beth_\omega(\mu)$ here, and χ in Notation 4.3 corresponds to μ' here. Thus the result follows from Theorem 4.8. □

Corollary 4.10. Assume that $\mathbf{K}$ has a monster model and is $\mathrm{LS}(\mathbf{K})$-tame. If there exists $\mu < H_1$ such that $\mathbf{K}$ is stable in μ and has no long splitting chains in μ, then there exists $\mu' < H_1$ such that $\mathbf{K}$ is uniformly (μ', μ')-solvable.

Proof. Let $\mu < H_1$ be such that $\mathbf{K}$ is stable in μ and has no long splitting chains in μ. Fix $\chi < H_1$ as given by Theorem 4.9. Without loss of generality, $\mu \leq \chi$. By Fact 2.7, $\mathbf{K}$ is stable in χ and has no long splitting chains in χ, so apply the conclusion of Theorem 4.9. □

5. Superstability below the Hanf number

In this section, we prove the main corollary. In fact, we prove a stronger version that instead of asking for the properties to hold on a tail asks for them to hold only in a single high-enough cardinal. Toward this end, we start by explaining why no long splitting chains follows from categoricity in a high-enough cardinal. In fact, categoricity can be replaced by solvability. All the ingredients for this result are contained in [SV99] but this specific form has not appeared in print before. Note also that Shelah states a similar result in [She99, 5.5] but his definition of superstability is different.

Theorem 5.1 (The ZFC Shelah-Villaveces theorem)**.** Let $\mathbf{K}$ be an AEC with arbitrarily large models and amalgamation[10] in $\mathrm{LS}(\mathbf{K})$. Let $\lambda > \mathrm{LS}(\mathbf{K})$ be such that $\mathbf{K}_{<\lambda}$ has no maximal models. If $\mathbf{K}$ is $(\lambda, \mathrm{LS}(\mathbf{K}))$-solvable, then $\mathbf{K}$ is stable in $\mathrm{LS}(\mathbf{K})$ and has no long splitting chains in $\mathrm{LS}(\mathbf{K})$.

Proof. Set $\mu := \mathrm{LS}(\mathbf{K})$. In the proof of [SV99, Theorem 2.2.1], in (c), ask that $\sigma = \chi$, where χ is the least cardinal such that $2^\chi > \mu$. The proof that (c) cannot happen goes through, and the rest only uses amalgamation in μ. Note that in [SV99] categoricity in λ is assumed but, as in many arguments involving categoricity and EM models, the full power of categoricity is not used. Rather, all that is used is that there is a unique (up to isomorphism) EM model of size λ, and that every model in $\mathbf{K}_{\leq\lambda}$ embeds into an EM model. Solvability in λ implies these two conditions (because the superlimit model is unique by Fact 2.9 and universal by definition). □

Remark 5.2. Instead of $(\lambda, \mathrm{LS}(\mathbf{K}))$-solvability, the weaker condition that any EM model of cardinality λ is universal suffices.

Corollary 5.3. Let $\mathbf{K}$ be an AEC with a monster model. Let $\lambda > \mathrm{LS}(\mathbf{K})$. If $\mathbf{K}$ is categorical in λ, then $\mathbf{K}$ is stable in μ and has no long splitting chains in μ for all $\mu \in [\mathrm{LS}(\mathbf{K}), \lambda)$.

Proof. By Theorem 5.1 applied to $\mathbf{K}_{\geq\mu}$ for each $\mu \in [\mathrm{LS}(\mathbf{K}), \lambda)$. Note that, since $\mathbf{K}$ has arbitrarily large models, categoricity in λ implies $(\lambda, \mathrm{LS}(\mathbf{K}))$-solvability. □

[10] In [SV99], this is replaced by the generalized continuum hypothesis (GCH).

We conclude that solvability is equivalent to superstability in the first-order case:

Corollary 5.4. Let T be a first-order theory and let $\mathbf{K}$ be the AEC of models of T ordered by elementary substructure. Let $\mu \geq |T|$. The following are equivalent:

(1) T is stable in all $\lambda \geq \mu$.
(2) $\mathbf{K}$ is (λ, μ)-solvable, for some $\lambda > \mu$.
(3) $\mathbf{K}$ is uniformly (μ, μ)-solvable.

Proof sketch. (3) implies (2) is trivial. (2) implies (1) is by Corollary 5.3 together with Fact 2.7). Finally, (1) implies (3) is as in the proof of Theorem 4.9. $\square$

We can also use the ZFC Shelah-Villaveces theorem to prove the following interesting result, showing that the solvability spectrum satisfies an analog of Shelah's categoricity conjecture in tame AECs (Shelah conjectures that this should hold in general, see Question 4.4 in the introduction to [She09a]). To simplify the statement, we introduce one more piece of notation:

Definition 5.5. For $\mathrm{LS}(\mathbf{K}) < \mu \leq \lambda$, $\mathbf{K}$ is $(\lambda, < \mu)$-*solvable* if there exists $\mu_0 \in [\mathrm{LS}(\mathbf{K}), \mu)$ such that $\mathbf{K}$ is (λ, μ_0)-solvable.

Theorem 5.6. Assume that $\mathbf{K}$ has a monster model and is $\mathrm{LS}(\mathbf{K})$-tame. There exists $\chi < H_1$ such that for any $\mu \geq \chi$, if $\mathbf{K}$ is (λ, μ)-solvable for *some* $\lambda > \mu$, then $\mathbf{K}$ is uniformly (μ', μ')-solvable, where $\mu' := (\beth_{\omega+2}(\mu))^+$.

In particular, let $\mu > \chi$ be of the form $\mu = \beth_\delta$ with δ divisible by $\omega \cdot \omega$ (for example, $\mu = H_1$). If $\mathbf{K}$ is $(\lambda, < \mu)$-solvable for *some* $\lambda \geq \mu$, then $\mathbf{K}$ is $(\lambda', < \mu)$-solvable for *all* $\lambda' \geq \mu$.

Proof. Let $\chi < H_1$ be as given by Theorem 4.9. Let $\mu \geq \chi$ and fix $\lambda > \mu$ such that $\mathbf{K}$ is solvable in λ. By Theorem 5.1, $\mathbf{K}$ is stable in μ and has no long splitting chains in μ. Now apply Theorem 4.9. The last paragraph easily follows from the first. $\square$

We are now ready to prove the more general version of the main corollary where the properties hold only in a single high-enough cardinal below H_1 (but the cardinal may be different for each property).

Corollary 5.7. Assume that $\mathbf{K}$ has a monster model, is $\mathrm{LS}(\mathbf{K})$-tame, and is stable in some cardinal greater than or equal to $\mathrm{LS}(\mathbf{K})$. Then there exists $\chi \in (\mathrm{LS}(\mathbf{K}), H_1)$ such that the following are equivalent:

$(1)^-$ For some $\lambda_1 \in [\chi, H_1)$, $\mathbf{K}$ is stable in λ_1 and has no long splitting chains in λ_1.
$(2)^-$ For some $\lambda_2 \in [\chi, H_1)$, there is a good λ_2-frame on a skeleton of $\mathbf{K}_{\lambda_2}$.
$(3)^-$ For some $\lambda_3 \in [\chi, H_1)$, $\mathbf{K}$ has a unique limit model of cardinality λ_3.
$(4)^-$ For some $\lambda_4 \in [\chi, H_1)$, $\mathbf{K}$ is stable in λ_4 and has a superlimit model of cardinality λ_4.
$(5)^-$ For some $\lambda_5 \in [\chi, H_1)$, the union of any increasing chain of λ_5-saturated models is λ_5-saturated.
$(6)^-$ For some $\lambda_6 \in [\chi, H_1)$, $\mathbf{K}$ is $(\lambda_6, < \lambda_6)$-solvable (see Definition 5.5).

Remark 5.8. In (2), we do *not* assume that the good frame is type-full (i.e. it may be that there exists some nonalgebraic types which are not basic, so fork over their domain). However if (1) holds, then the proof of (1) implies (2) (Fact 2.17) actually builds a *type-full* frame. Therefore, in the presence of tameness, the existence of a good frame implies the existence of a *type-full* good frame (in a potentially much higher cardinal, and over a different class).

Before proving Corollary 5.7, we explain why Corollary 1.3 follows:

Proof of Corollary 1.3. Let χ be as given by Corollary 5.7. By Fact 2.4, there exists unboundedly-many regular stability cardinal in (χ, H_1). This implies that for $\ell \in \{1,2,3,4,5,6\}$, (ℓ) (from Corollary 1.3) implies $(\ell)^-$ (from Corollary 5.7). Moreover $(1)^-$ implies both (1) and (7) by Fact 2.7. Since Corollary 5.7 tells us that $(\ell_1)^-$ is equivalent to $(\ell_2)^-$ for $\ell_1, \ell - 2 \in \{1,2,3,4,5,6\}$, it follows that (ℓ_1) is equivalent to (ℓ_2) as well, and (7) is implied by any of these conditions. $\square$

Proof of Corollary 5.7. By Fact 2.4, $\mathbf{K}$ does not have the $\text{LS}(\mathbf{K})$-order property. By Fact 3.2, there exists $\chi_0 < H_1$ such that $\mathbf{K}$ does not have the $\text{LS}(\mathbf{K})$-order property of length χ_0. Let $\chi := \beth_\omega (\chi_0 + \text{LS}(\mathbf{K}))$.

By Fact 2.17, $(1)^-$ implies $(2)^-$, $(4)^-$, and $(5)^-$. By Fact 2.16, $(2)^-$ implies $(3)^-$. By Theorem 4.9, $(1)^-$ implies $(6)^-$. Conversely (Theorem 5.1), $(6)^-$ implies $(1)^-$. It remains to show that $(\ell)^-$ implies $(1)^-$ for $\ell \in \{3,4,5\}$.

We have by Fact 2.10.(4) that $(4)^-$ implies $(3)^-$. However we do not quite know that $(5)^-$ implies $(3)^-$: $\mathbf{K}$ might not be stable in λ_5. Thus we consider the following weakening of $(3)^-$:

- $(3)^*$ For some $\lambda_3^* \in [\chi, H_1)$, $\mathbf{K}$ is stable in λ_3^*, has a saturated model of cardinality λ_3^*, and every limit model of cardinality λ_3^* is χ-saturated.

Clearly, $(3)^-$ implies $(3)^*$ (see Fact 2.10.(3)). Moreover also $(5)^-$ implies $(3)^*$: Indeed, let $\lambda_3^* \in [\lambda_5, H_1)$ be a regular stability cardinal. Then $\mathbf{K}$ has a saturated model of cardinality λ_3^*, and from $(5)^-$ it is easy to see that any limit model of cardinality λ_3^* is λ_5-saturated, hence χ-saturated.

It remains to prove that $(3)^*$ implies $(1)^-$. This is Theorem 3.18, where χ_2 there stands for χ here. $\square$

Question 5.9. Is stability in λ_4 needed in condition $(4)^-$ of Corollary 5.7? That is, can one replace the condition with:

- $(4)^{--}$ For some $\lambda_4 \in [\chi, \theta)$, $\mathbf{K}$ has a superlimit model of cardinality λ_4.

The answer is positive when $\mathbf{K}$ is an elementary class [She12, Claim 3.1].

6. Future work

While we managed to prove that some analogs of the conditions in Fact 1.1 are equivalent, much remains to be done.

For example, we do not know whether (7) in Corollary 1.3 implies any of the equivalent properties (1)-(6). This would be a useful tool to check that specific examples have e.g. no long splitting chains. It is conceivable, however, that (7) is weaker than the other properties. If this speculation is correct, then there would be no unique extension of first-order superstability to even tame AECs.

Another direction would be to make precise what the analog to (5) and (6) in 1.1 should be in tame AECs. One possible definition for (6) would be:

Definition 6.1. Let $\lambda, \mu > \mathrm{LS}(\mathbf{K})$. We say that $\mathbf{K}$ has the (λ, μ)*-tree property* provided there exists $\{p_n(\mathbf{x}; \mathbf{y}_n) \mid n < \omega\}$ Galois-types over models of size less than μ and $\{M_\eta \mid \eta \in {}^{\leq \omega}\lambda\}$ such that for all $n < \omega, \nu \in {}^{n}\lambda$ and every $\eta \in {}^{\omega}\lambda$:

$$\langle M_\eta, M_\nu \rangle \models p_n \iff \nu \text{ is an initial segment of } \eta.$$

We say that $\mathbf{K}$ has the *tree property* if it has it for all high-enough μ and all high-enough λ (where the "high-enough" quantifier on λ can depend on μ).

We can ask whether no long splitting chains (or any other reasonable definition of superstability) implies that $\mathbf{K}$ does not have the tree property, or at least obtain many models from the tree property as in [GS86]. This is conjectured in [She99] (see the remark after Claim 5.5 there).

As for the D-rank in (5), perhaps a simpler analog would be the U-rank defined in terms of $(< \kappa)$-satisfiability in [BG, 7.2] (another candidate for a rank is Lieberman's R-rank, see [Lie13]).

Definition 6.2. Let $\mathbf{K}$ be a $\mathrm{LS}(\mathbf{K})$-tame AEC with amalgamation. Let $\kappa > \mathrm{LS}(\mathbf{K})$ be least such that $\kappa = \beth_\kappa$ (for concreteness). We define a map U with domain a type over κ-saturated models and codomain an ordinal or ∞ inductively by, for $p \in \mathrm{gS}(M)$:

(1) Always, $U[p] \geq 0$.
(2) For α limit, $U[p] \geq \alpha$ if and only if $U[p] \geq \beta$ for all $\beta < \alpha$.
(3) $U[p] \geq \beta + 1$ if and only if there exists a κ-saturated $M' \geq_{\mathbf{K}} M$ with $\|M'\| = \|M\|$ and an extension $q \in \mathrm{gS}(M')$ of p such that q is not $(< \kappa)$-satisfiable over M and $U[q] \geq \beta$.
(4) $U[p] = \alpha$ if and only if $U[p] \geq \alpha$ and $U[p] \ngeq \alpha + 1$.
(5) $U[p] = \infty$ if and only if $U[p] \geq \alpha$ for all ordinals α.

By [BG, 7.9], no long splitting chains implies that the U-rank is bounded but we do not know how to prove the converse. Perhaps it is possible to show that $U = \infty$ implies the tree property.

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