Stability theory for concrete categories

Sebastien Vasey

Harvard University

May 14, 2020

What is stability (in model theory)?

Several ways to think about it:

- The study of universes with “good Ramsey theory”.

What is stability (in model theory)?

Several ways to think about it:

- The study of universes with “good Ramsey theory”.
- A generalized theory of field extensions.

What is stability (in model theory)?

Several ways to think about it:

- The study of universes with "good Ramsey theory".
- A generalized theory of field extensions.
- Existence of an axiomatic notion of "being independent", generalizing linear and algebraic independence.

A puzzle

In a party of six people, either three of them all know each other, or three of them all do not know each other.

A puzzle

In a party of six people, either three of them all know each other, or three of them all do not know each other. More formally and generally:

Theorem (Ramsey, 1930)

For any natural number k, there exists a natural number n such that:

$$n \to (k)_2$$

A puzzle

In a party of six people, either three of them all know each other, or three of them all do not know each other. More formally and generally:

Theorem (Ramsey, 1930)

For any natural number k, there exists a natural number n such that:

$$n \to (k)_2$$

The notation is due to Erdős and Rado. It means: for any set X with at least n elements and any coloring F of the unordered pairs from X in two colors, there exists $H \subseteq X$ with $|H| = k$ so that F is constant on the pairs from H (we call H a *homogeneous set* for F).

A puzzle

In a party of six people, either three of them all know each other, or three of them all do not know each other. More formally and generally:

Theorem (Ramsey, 1930)

For any natural number k, there exists a natural number n such that:

$$n \to (k)_2$$

The notation is due to Erdős and Rado. It means: for any set X with at least n elements and any coloring F of the unordered pairs from X in two colors, there exists $H \subseteq X$ with $|H| = k$ so that F is constant on the pairs from H (we call H a *homogeneous set* for F).

If $k = 3$, $n = 6$ suffices. If $k = 5$, the optimal value of n is not known.

An infinite variation on the puzzle

In an infinite party, infinitely-many people all know each other or infinitely-many all do not know each other.

More formally:

Theorem (Ramsey, 1930)

$$\aleph_0 \to (\aleph_0)_2$$

An infinite variation on the puzzle

In an infinite party, infinitely-many people all know each other or infinitely-many all do not know each other.

More formally:

Theorem (Ramsey, 1930)

$$\aleph_0 \to (\aleph_0)_2$$

Said differently, for any set X with $|X| \geq \aleph_0$ and any coloring F of the unordered pairs from X, there exists $H \subseteq X$ so that $|H| = \aleph_0$ and F is constant on the unordered pairs from H.

An infinite variation on the puzzle

In an infinite party, infinitely-many people all know each other or infinitely-many all do not know each other.

More formally:

Theorem (Ramsey, 1930)

$$\aleph_0 \to (\aleph_0)_2$$

Said differently, for any set X with $|X| \geq \aleph_0$ and any coloring F of the unordered pairs from X, there exists $H \subseteq X$ so that $|H| = \aleph_0$ and F is constant on the unordered pairs from H.

Warning: The theorem does *not* say that $|X| = |H|$: it does *not* rule out an uncountable party with where all friends/strangers groups (= homogeneous sets) are countable.

Ramsey's dream

For any infinite cardinal λ, in a party of λ people, there is a group of λ-many that all know each other or a group of λ-many that all do not know each other. That is:

$$\lambda \to (\lambda)_2$$

Ramsey's dream

For any infinite cardinal λ, in a party of λ people, there is a group of λ-many that all know each other or a group of λ-many that all do not know each other. That is:

$$\lambda \to (\lambda)_2$$

This is wrong for most cardinals λ.

Counterexamples to Ramsey's dream

Proposition (Sierpiński)

$|\mathbb{R}| \not\to (|\mathbb{R}|)_2$.

Counterexamples to Ramsey's dream

Proposition (Sierpiński)

$|\mathbb{R}| \not\to (|\mathbb{R}|)_2$.

Proposition (Erdős-Kakutani)

$$|\mathbb{R}| \not\to (3)_{\aleph_0}$$

Counterexamples to Ramsey's dream

Proposition (Sierpiński)

$|\mathbb{R}| \not\to (|\mathbb{R}|)_2$.

Proposition (Erdős-Kakutani)

$$|\mathbb{R}| \not\to (3)_{\aleph_0}$$

Proof.

Take $F(\{x, y\}) =$ some rational between x and y. A set H homogeneous for F cannot contain three elements! $\square$

Counterexamples to Ramsey's dream

Proposition (Sierpiński)

$|\mathbb{R}| \not\to (|\mathbb{R}|)_2$.

Proposition (Erdős-Kakutani)

$$|\mathbb{R}| \not\to (3)_{\aleph_0}$$

Proof.

Take $F(\{x, y\}) =$ some rational between x and y. A set H homogeneous for F cannot contain three elements! $\square$

In the reals, a countable set allows one to distinguish uncountably-many points. There are however many structures where this is not the case.

Ramsey's dream in the complex field

Proposition

If F is a coloring of the unordered pairs of complex numbers in two colors *such that* $F(\{f(x), f(y)\}) = F(\{x, y\})$ *for any field automorphism* f *of* $\mathbb{C}$, then F has a homogeneous set of cardinality $|\mathbb{C}|$.

Ramsey's dream in the complex field

Proposition

If F is a coloring of the unordered pairs of complex numbers in two colors *such that* $F(\{f(x), f(y)\}) = F(\{x, y\})$ *for any field automorphism* f *of* $\mathbb{C}$, then F has a homogeneous set of cardinality $|\mathbb{C}|$.

Proof.

Any transcendence basis for $\mathbb{C}$ does the job. $\square$

Ramsey's dream in the complex field

Proposition

If F is a coloring of the unordered pairs of complex numbers in two colors *such that* $F(\{f(x), f(y)\}) = F(\{x, y\})$ *for any field automorphism* f *of* $\mathbb{C}$, then F has a homogeneous set of cardinality $|\mathbb{C}|$.

Proof.

Any transcendence basis for $\mathbb{C}$ does the job. □

This proves $|\mathbb{C}| \to |\mathbb{C}|_2$ but "relativized to $\mathbb{C}$" (for colorings preserved by automorphisms).

Ramsey's dream in the complex field

Proposition

If F is a coloring of the unordered pairs of complex numbers in two colors *such that* $F(\{f(x), f(y)\}) = F(\{x, y\})$ *for any field automorphism* f *of* $\mathbb{C}$, then F has a homogeneous set of cardinality $|\mathbb{C}|$.

Proof.

Any transcendence basis for $\mathbb{C}$ does the job. $\square$

This proves $|\mathbb{C}| \to |\mathbb{C}|_2$ but "relativized to $\mathbb{C}$" (for colorings preserved by automorphisms).

Problem: Study this phenomenon.

One possible framework ("classical model theory"): study the models of a given list of axioms. Morphisms between the models are those preserving all formulas. Stability theory was developped there by Shelah in the 1970s.

One possible framework ("classical model theory"): study the models of a given list of axioms. Morphisms between the models are those preserving all formulas. Stability theory was developped there by Shelah in the 1970s.

A much more general (and harder) framework: work in any category $\mathbf{K}$ such that:

- $\mathbf{K}$ is concrete (there is a faithful $U : \mathbf{K} \to \mathbf{Set}$).
- All morphisms are concrete monomorphisms (injections).
- $\mathbf{K}$ has concrete directed colimits (also known as direct limits – basically closure under unions of increasing chains).
- (Smallness condition) Every object is a directed colimit of a fixed set of "small" subobjects.

One possible framework ("classical model theory"): study the models of a given list of axioms. Morphisms between the models are those preserving all formulas. Stability theory was developped there by Shelah in the 1970s.

A much more general (and harder) framework: work in any category $\mathbf{K}$ such that:

- $\mathbf{K}$ is concrete (there is a faithful $U : \mathbf{K} \to \mathbf{Set}$).
- All morphisms are concrete monomorphisms (injections).
- $\mathbf{K}$ has concrete directed colimits (also known as direct limits – basically closure under unions of increasing chains).
- (Smallness condition) Every object is a directed colimit of a fixed set of "small" subobjects.

This is essentially the definition of an *abstract elementary class (AEC)* (due to Shelah, late 1970s).

There are more general frameworks, including accessible categories (not in this talk).

Types

A category **K** has *amalgamation* if any diagram of the form $B \leftarrow A \rightarrow C$ can be completed to a commuting square (no universal property required – this is much weaker than pushouts).

Types

A category $\mathbf{K}$ has *amalgamation* if any diagram of the form $B \leftarrow A \rightarrow C$ can be completed to a commuting square (no universal property required – this is much weaker than pushouts).

Definition

Given a concrete category $\mathbf{K}$ with amalgamation and an object A of $\mathbf{K}$, a *type over* A is just a pair $(x, A \xrightarrow{f} B)$, with $x \in B$. Two types $(x, A \xrightarrow{f} B)$, $(y, A \xrightarrow{g} C)$ are considered *the same* if there exists maps h_1, h_2 so that $h_1(x) = h_2(y)$ and the following diagram commutes:

$$
\begin{tikzcd}
B \arrow[r, dotted, "h_1"] & D \\
A \arrow[u, "f"] \arrow[r, "g"] & C \arrow[u, dotted, "h_2"']
\end{tikzcd}
$$

Types in fields, linear orders, and graphs

Essentially, one can think of types over a fixed base A as the orbits of an automorphism group fixing A.

Types in fields, linear orders, and graphs

Essentially, one can think of types over a fixed base A as the orbits of an automorphism group fixing A.

For example in the category of fields, $e^{\frac{1}{3}}$ and $e^{\frac{1}{2}}$ have the same type over $\mathbb{Q}$ but not the same type over $\mathbb{Q}(e)$.

Types in fields, linear orders, and graphs

Essentially, one can think of types over a fixed base A as the orbits of an automorphism group fixing A.

For example in the category of fields, $e^{\frac{1}{3}}$ and $e^{\frac{1}{2}}$ have the same type over $\mathbb{Q}$ but not the same type over $\mathbb{Q}(e)$.

In the category of fields, there are at most $\max(|A|, \aleph_0)$ types over every object A (just one type for the transcendental element).

Types in fields, linear orders, and graphs

Essentially, one can think of types over a fixed base A as the orbits of an automorphism group fixing A.

For example in the category of fields, $e^{\frac{1}{3}}$ and $e^{\frac{1}{2}}$ have the same type over $\mathbb{Q}$ but not the same type over $\mathbb{Q}(e)$.

In the category of fields, there are at most $\max(|A|, \aleph_0)$ types over every object A (just one type for the transcendental element).

In the category of linear orders, there are $|\mathbb{R}|$ types over $\mathbb{Q}$. In general, types correspond to Dedekind cuts.

Types in fields, linear orders, and graphs

Essentially, one can think of types over a fixed base A as the orbits of an automorphism group fixing A.

For example in the category of fields, $e^{\frac{1}{3}}$ and $e^{\frac{1}{2}}$ have the same type over $\mathbb{Q}$ but not the same type over $\mathbb{Q}(e)$.

In the category of fields, there are at most $\max(|A|, \aleph_0)$ types over every object A (just one type for the transcendental element).

In the category of linear orders, there are $|\mathbb{R}|$ types over $\mathbb{Q}$. In general, types correspond to Dedekind cuts.

In the category of graphs with induced subgraph embeddings, there are at least $2^{|V(G)|}$ types over any graph G.

Definition (Stability)

A concrete category **K** is *stable in* λ if there are at most λ-many types over any object of cardinality λ. *Stable* means stable in an unbounded class of cardinals.

Definition (Stability)

A concrete category $\mathbf{K}$ is *stable in* λ if there are at most λ-many types over any object of cardinality λ. *Stable* means stable in an unbounded class of cardinals.

Examples

- The category of graphs with induced subgraph embeddings and the category of linear orders are unstable. The category of fields is stable (in all cardinals).

Definition (Stability)

A concrete category **K** is *stable in* λ if there are at most λ-many types over any object of cardinality λ. *Stable* means stable in an unbounded class of cardinals.

Examples

- The category of graphs with induced subgraph embeddings and the category of linear orders are unstable. The category of fields is stable (in all cardinals).
- (Eklof 1971, Mazari-Armida) The category of R-modules with embeddings is always stable, and stable in all cardinals if and only if R is Noetherian.

More examples of (un)stability

- The category of differential fields of characteristic zero is stable in all cardinals.

More examples of (un)stability

- The category of differential fields of characteristic zero is stable in all cardinals.
- The category of Banach spaces with linear isometries is unstable (c.f. Krivine-Maurey 1989).

More examples of (un)stability

- The category of differential fields of characteristic zero is stable in all cardinals.
- The category of Banach spaces with linear isometries is unstable (c.f. Krivine-Maurey 1989).
  [Why? Consider c_0 (bounded complex-valued sequences with sup norm). Let $f_n := \sum_{i \leq n} e_i$. Then $\|e_m + f_n\| = 2$ if and only if $m \leq n$. Now make up a similar example where things are indexed by rationals.]

More examples of (un)stability

- The category of differential fields of characteristic zero is stable in all cardinals.
- The category of Banach spaces with linear isometries is unstable (c.f. Krivine-Maurey 1989).
  [Why? Consider c_0 (bounded complex-valued sequences with sup norm). Let $f_n := \sum_{i \leq n} e_i$. Then $\|e_m + f_n\| = 2$ if and only if $m \leq n$. Now make up a similar example where things are indexed by rationals.]
- The category of Hilbert spaces with linear isometries is stable.

More examples of (un)stability

- The category of differential fields of characteristic zero is stable in all cardinals.
- The category of Banach spaces with linear isometries is unstable (c.f. Krivine-Maurey 1989).
  [Why? Consider c_0 (bounded complex-valued sequences with sup norm). Let $f_n := \sum_{i \leq n} e_i$. Then $\|e_m + f_n\| = 2$ if and only if $m \leq n$. Now make up a similar example where things are indexed by rationals.]
- The category of Hilbert spaces with linear isometries is stable.

Note the last two are technically not AECs (under any choice of concrete functor! Lieberman-Rosický-V, preprint).

Stability and order

Theorem (V. 2016, Boney)

A tame AEC $\mathbf{K}$ with amalgamation is stable if and only if it does not "code an order": any faithful functor $\mathbf{Lin} \xrightarrow{F} \mathbf{K}$ factors through the forgetful functor.

$$
\begin{tikzcd}
\mathbf{Lin} \arrow[r, "F"] \arrow[d, "U"] & \mathbf{K} \\
\mathbf{Set} \arrow[ur, dotted] &
\end{tikzcd}
$$

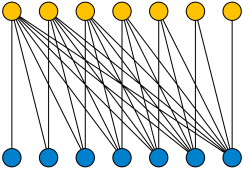
Order in graphs: an intermission

Graphs with induced subgraph embeddings are unstable, so they must have the order property: where is it?

Order in graphs: an intermission

Graphs with induced subgraph embeddings are unstable, so they must have the order property: where is it?

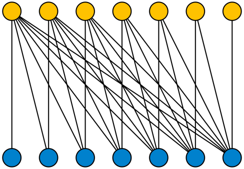
It is given by a half graph: for any linear ordering L, consider the bipartite graph on $L \sqcup L$ where we put an edge from i to j if only if $i \leq j$ (the picture below is for $L = \{1, 2, 3, 4, 5, 6, 7\}$):



Order in graphs: an intermission

Graphs with induced subgraph embeddings are unstable, so they must have the order property: where is it?

It is given by a half graph: for any linear ordering L, consider the bipartite graph on $L \sqcup L$ where we put an edge from i to j if only if $i \leq j$ (the picture below is for $L = \{1, 2, 3, 4, 5, 6, 7\}$):



Graphs omitting half graphs are studied in finite combinatorics too (Malliaris-Shelah, *Regularity lemmas for stable graphs*. TAMS 2014).

Ramsey's dream in stable AECs

Theorem (V.)

If **K** is an abstract elementary class with amalgamation and **K** is stable in λ, then:

$$\lambda^+ \xrightarrow{\mathbf{K}} \left(\lambda^+\right)_\lambda$$

Here λ^+ is the cardinal right after λ.

Ramsey's dream in stable AECs

Theorem (V.)

If **K** is an abstract elementary class with amalgamation and **K** is stable in λ, then:

$$\lambda^+ \xrightarrow{\mathbf{K}} (\lambda^+)_\lambda$$

Here λ^+ is the cardinal right after λ.

The partition notation means that given objects $A \to B$ in **K** with $|A| = \lambda$, $|B| = \lambda^+$, if F is a coloring of pairs from B in λ-many colors so that any two pairs with the same type over A have the same color, then we can find a homogeneous set for F of cardinality λ^+.

Shelah's eventual categoricity conjecture

Any two algebraically closed fields of the same characteristic and the same uncountable cardinality are isomorphic (because they have the same transcendence degree).

Shelah's eventual categoricity conjecture

Any two algebraically closed fields of the same characteristic and the same uncountable cardinality are isomorphic (because they have the same transcendence degree).

Thus it seems any AEC with a "perfect theory of dimension" should have unique objects of each high-enough cardinality. Morley (1965) proved a sort of converse for first-order classes, and Shelah proposed this should generalize:

Shelah's eventual categoricity conjecture

Any two algebraically closed fields of the same characteristic and the same uncountable cardinality are isomorphic (because they have the same transcendence degree).

Thus it seems any AEC with a "perfect theory of dimension" should have unique objects of each high-enough cardinality. Morley (1965) proved a sort of converse for first-order classes, and Shelah proposed this should generalize:

Conjecture (Shelah, late seventies)

An AEC with a single object of *some* high-enough cardinality has a single object in *all* high-enough cardinalities.

Shelah's eventual categoricity conjecture

Any two algebraically closed fields of the same characteristic and the same uncountable cardinality are isomorphic (because they have the same transcendence degree).

Thus it seems any AEC with a "perfect theory of dimension" should have unique objects of each high-enough cardinality. Morley (1965) proved a sort of converse for first-order classes, and Shelah proposed this should generalize:

Conjecture (Shelah, late seventies)

An AEC with a single object of *some* high-enough cardinality has a single object in *all* high-enough cardinalities.

The only known way to prove such statements is via stability theory.

Toward Shelah's eventual categoricity conjecture

Theorem (V. 2017)

Shelah's eventual categoricity conjecture is true for universal AECs.

Toward Shelah's eventual categoricity conjecture

Theorem (V. 2017)

Shelah's eventual categoricity conjecture is true for universal AECs.

Theorem (Shelah-V.)

Shelah's eventual categoricity conjecture is true for all AECs, assuming a large cardinal axiom (there exists a proper class of strongly compact cardinals).

Toward Shelah's eventual categoricity conjecture

Theorem (V. 2017)

Shelah's eventual categoricity conjecture is true for universal AECs.

Theorem (Shelah-V.)

Shelah's eventual categoricity conjecture is true for all AECs, assuming a large cardinal axiom (there exists a proper class of strongly compact cardinals).

Theorem (V. 2019)

Assuming the GCH, Shelah's eventual categoricity conjecture is true for AECs with amalgamation. In this case one can list all possibilities for the class of cardinals in which the category has a unique object.

Ramsey's dream for stable AECs

Theorem (V.)

If **K** is an abstract elementary class with amalgamation and **K** is stable in λ, then:

$$\lambda^+ \xrightarrow{\mathbf{K}} (\lambda^+)_\lambda$$

Ramsey's dream for stable AECs

Theorem (V.)

If $\mathbf{K}$ is an abstract elementary class with amalgamation and $\mathbf{K}$ is stable in λ, then:

$$\lambda^+ \xrightarrow{\mathbf{K}} \left(\lambda^+\right)_\lambda$$

Sketch of proof: we first fix a big homogeneous object $\mathfrak{C} \in \mathbf{K}$, with lots of automorphisms. Work inside $\mathfrak{C}$.

Ramsey's dream for stable AECs

Theorem (V.)

If **K** is an abstract elementary class with amalgamation and **K** is stable in λ, then:

$$\lambda^+ \xrightarrow{\mathbf{K}} (\lambda^+)_\lambda$$

Sketch of proof: we first fix a big homogeneous object $\mathfrak{C} \in \mathbf{K}$, with lots of automorphisms. Work inside $\mathfrak{C}$.

Fix distinct elements $(a_i)_{i<\lambda^+}$. We will find a subset $I \subseteq \lambda^+$, $|I| = \lambda^+$ such that for any $i_0 < \ldots < i_{n-1}$, $j_0 < \ldots < j_{n-1}$, $a_{i_0} \ldots a_{i_{n-1}}$ and $a_{j_0} \ldots a_{j_{n-1}}$ have the same type.

Further fix a chain $M_0 \to M_1 \to M_2 \to \ldots$ of length λ^+ such that for each $i < \lambda^+$:

Further fix a chain $M_0 \to M_1 \to M_2 \to \dots$ of length λ^+ such that for each $i < \lambda^+$:

- M_i has cardinality λ.
- $a_i \in M_{i+1}$.
- M_{i+1} realizes all types over M_i (and more).
- If i is a limit ordinal, $M_i = \bigcup_{j<i} M_j$.

Further fix a chain $M_0 \to M_1 \to M_2 \to \ldots$ of length λ^+ such that for each $i < \lambda^+$:

- M_i has cardinality λ.
- $a_i \in M_{i+1}$.
- M_{i+1} realizes all types over M_i (and more).
- If i is a limit ordinal, $M_i = \bigcup_{j<i} M_j$.

Definition

We say $\mathbf{tp}(a_i/M_i)$ *splits over* M_{i_0}, $i_0 < i$, if there is an automorphism f of $\mathfrak{C}$ such that:

- f fixes M_{i_0} pointwise.
- f fixes M_i setwise.
- a_i and $f(a_i)$ do *not* have the same type over M_i.

Further fix a chain $M_0 \to M_1 \to M_2 \to \dots$ of length λ^+ such that for each $i < \lambda^+$:

- M_i has cardinality λ.
- $a_i \in M_{i+1}$.
- M_{i+1} realizes all types over M_i (and more).
- If i is a limit ordinal, $M_i = \bigcup_{j<i} M_j$.

Definition

We say $\mathbf{tp}(a_i/M_i)$ *splits over* M_{i_0}, $i_0 < i$, if there is an automorphism f of $\mathfrak{C}$ such that:

- f fixes M_{i_0} pointwise.
- f fixes M_i setwise.
- a_i and $f(a_i)$ do *not* have the same type over M_i.

Intuitively, a type does *not* split over a base if it is "determined" by the restriction to that base. **Near-examples:** $\mathbf{tp}(\sqrt{3}/\mathbb{Q}(\sqrt{2}))$ does not split over $\mathbb{Q}$, but $\mathbf{tp}(1-\sqrt{2}/\mathbb{Q}(\sqrt{2}))$ splits over $\mathbb{Q}$.

Lemma

If $(\delta_k)_{k<\lambda}$ is a chain of ordinals below λ^+ and p is a type over $M_{\sup_{k<\lambda}\delta_k}$, then there is $k<\lambda$ such that $p \upharpoonright M_{\delta_{k+1}}$ does not split over M_{δ_k}.

Lemma

If $(\delta_k)_{k<\lambda}$ is a chain of ordinals below λ^+ and p is a type over $M_{\sup_{k<\lambda} \delta_k}$, then there is $k < \lambda$ such that $p \restriction M_{\delta_{k+1}}$ does not split over M_{δ_k}.

Proof idea.

If not, $p \restriction M_{\delta_{k+1}}$ splits over M_{δ_k} for all k. Let $\mu \leq \lambda$ be least such that $2^\mu > \lambda$. Copy these failures of splitting into a binary tree of height μ. We get 2^μ distinct types over a base of size $2^{<\mu} = \lambda$, contradicting stability. $\square$

Lemma

There exists $\alpha < \beta$ and a type p over M_β such that:

- p does not split over M_α.
- For any $\beta' \geq \beta$, there is $\gamma = \gamma_{\beta'} > \beta'$ such that $\mathbf{tp}(a_\gamma / M_{\beta'})$ extends p and does not split over M_α.

Lemma

There exists $\alpha < \beta$ and a type p over M_β such that:

- p does not split over M_α.
- For any $\beta' \geq \beta$, there is $\gamma = \gamma_{\beta'} > \beta'$ such that $\mathbf{tp}(a_\gamma / M_{\beta'})$ extends p and does not split over M_α.

After proving this lemma, we can define inductively $\beta_0 := \beta$, and $\beta_i := \gamma_{\left(\sup_{j<i} \beta_j\right)}$, and one can show $(a_{\beta_i})_{i<\lambda^+}$ is the desired homogeneous set.

Lemma

There exists $\alpha < \beta$ and a type p over M_β such that:

- p does not split over M_α.
- For any $\beta' \geq \beta$, there is $\gamma = \gamma_{\beta'} > \beta'$ such that $\mathbf{tp}(a_\gamma / M_{\beta'})$ extends p and does not split over M_α.

After proving this lemma, we can define inductively $\beta_0 := \beta$, and $\beta_i := \gamma_{\left(\sup_{j<i} \beta_j\right)}$, and one can show $(a_{\beta_i})_{i<\lambda^+}$ is the desired homogeneous set.

How do we prove the lemma?

Lemma

There exists $\alpha < \beta$ and a type p over M_β such that:

- p does not split over M_α.
- For any $\beta' \geq \beta$, there is $\gamma = \gamma_{\beta'} > \beta'$ such that $\mathbf{tp}(a_\gamma / M_{\beta'})$ extends p and does not split over M_α.

After proving this lemma, we can define inductively $\beta_0 := \beta$, and $\beta_i := \gamma_{\left(\sup_{j<i} \beta_j\right)}$, and one can show $(a_{\beta_i})_{i<\lambda^+}$ is the desired homogeneous set.

How do we prove the lemma? By contradiction! Suppose it fails.

Lemma

There exists $\alpha < \beta$ and a type p over M_β such that:

- p does not split over M_α.
- For any $\beta' \geq \beta$, there is $\gamma = \gamma_{\beta'} > \beta'$ such that $\mathbf{tp}(a_\gamma / M_{\beta'})$ extends p and does not split over M_α.

After proving this lemma, we can define inductively $\beta_0 := \beta$, and $\beta_i := \gamma_{\left(\sup_{j<i} \beta_j\right)}$, and one can show $(a_{\beta_i})_{i<\lambda^+}$ is the desired homogeneous set.

How do we prove the lemma? By contradiction! Suppose it fails.

Then for each β, for all types p over M_β, for all $\alpha < \beta$, there exists $\beta' = \beta'_{\alpha,\beta,p} > \beta$ such that for all $\gamma > \beta'$, either $\mathbf{tp}(a_\gamma / M_{\beta'})$ does not extend p or splits over M_α.

Lemma

There exists $\alpha < \beta$ and a type p over M_β such that:

- p does not split over M_α.
- For any $\beta' \geq \beta$, there is $\gamma = \gamma_{\beta'} > \beta'$ such that $\mathbf{tp}(a_\gamma / M_{\beta'})$ extends p and does not split over M_α.

After proving this lemma, we can define inductively $\beta_0 := \beta$, and $\beta_i := \gamma_{\left(\sup_{j<i} \beta_j\right)}$, and one can show $(a_{\beta_i})_{i<\lambda^+}$ is the desired homogeneous set.

How do we prove the lemma? By contradiction! Suppose it fails.

Then for each β, for all types p over M_β, for all $\alpha < \beta$, there exists $\beta' = \beta'_{\alpha,\beta,p} > \beta$ such that for all $\gamma > \beta'$, either $\mathbf{tp}(a_\gamma / M_{\beta'})$ does not extend p or splits over M_α.

Define inductively $\delta_0 := 0$, $\delta_i := \sup_{j<i} \sup_{\alpha<\delta_j, p \text{ type over } M_{\delta_j}} \beta'_{\alpha,\delta_j,p}$. Apply the previous lemma to $(\delta_i)_{i<\lambda}$ and get a contradiction.

Thank you!

Some references:

- Sebastien Vasey, *Accessible categories, set theory, and model theory: an invitation*, arXiv:1904.11307.
- Sebastien Vasey, *Shelah's eventual categoricity conjecture in universal classes: part II*, Selecta Mathematica **23** (2017), no. 2, 1469–1506.
- Michael Lieberman, Jiří Rosický, and Sebastien Vasey, *Forking independence from the categorical point of view*, Advances in Mathematics **346** (2019), 719–772.
- Sebastien Vasey, *The categoricity spectrum of large abstract elementary classes with amalgamation*, Selecta Mathematica **25** (2019), no. 5, 65 (51 pages).
- Saharon Shelah and Sebastien Vasey, *Categoricity and multidimensional diagrams*, arXiv:1805.0629.
- Michael Lieberman, Jiří Rosický, and Sebastien Vasey, *Cellular categories and stable independence*, arXiv:1904.05691.