# CATEGORICITY AND INFINITARY LOGICS

WILL BONEY AND SEBASTIEN VASEY

Abstract. We use model-theoretic forcing to show:

**Theorem 0.1.** Let $K$ be an abstract elementary class categorical in unboundedly many cardinals. Then there exists a cardinal $\lambda$ such that whenever $M, N \in K$ have size at least $\lambda$, $M \leq N$ if and only if $M \preceq_{L_{\infty, \text{LS}(K)^+}} N$.

This fixes a gap in Shelah's proof of the following result:

**Theorem 0.2.** Let $K$ be an abstract elementary class categorical in unboundedly many cardinals. Then the class of $\lambda$ such that:
(1) $K$ is categorical in $\lambda$;
(2) $K$ has amalgamation in $\lambda$; and
(3) there is a good $\lambda$-frame with underlying class $K_\lambda$

is stationary.

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## 1. Introduction

A major driving force in the field of classification theory for non-elementary classes is Shelah's categoricity conjecture[1]:

**Conjecture 1.1** (Open problem D.(3a) in [She90])**.** If $L$ is a countable language and $\psi \in L_{\omega_1,\omega}$ is categorical in one $\mu \geq \beth_{\omega_1}$, then it is categorical in all $\mu \geq \beth_{\omega_1}$.

In the framework of abstract elementary classes (AECs), Shelah states an eventual version of the conjecture as:

**Conjecture 1.2** (Conjecture N.4.2 in [She09])**.** An AEC that is categorical in a high-enough cardinal is categorical on a tail of cardinals.

More precisely, there should exist a function $\lambda \mapsto \mu_\lambda$ such that if $K$ is an AEC categorical in *some* $\mu \geq \mu_{\text{LS}(K)}$, then it is categorical in *all* $\mu' \geq \mu_{\text{LS}(K)}$. There are several conjectures as to what the above function should be. Hart and Shelah [HS90] have shown that, as opposed to the first-order setup, $\mu_{\aleph_0} \geq \aleph_\omega$ and Shelah conjectures that this is essentially optimal in Conjecture N.4.3 of [She09]. Still he says it is probably more realistic to expect to prove $\mu_\lambda \leq \beth_{(2^\lambda)^+}$.

However if one is only interested in the *existence* of the map $\lambda \mapsto \mu_\lambda$, then a Hanf number argument (using the axiom of replacement, see [Bal09, Conclusion 15.13]) shows that categoricity in a high-enough cardinal implies categoricity in unboundedly many cardinals. Therefore for the purpose of establishing Conjecture 1.2 (sometimes also called Shelah's *eventual* categoricity conjecture), we can restrict ourselves to AECs categorical in unboundedly many cardinals[2]. In such AECs, several difficulties with dealing with categoricity in a single cardinal disappear. For example, assuming amalgamation and no maximal models, a major issue is to show that a class categorical in a $\lambda > \text{LS}(K)$ has a Galois-saturated model in $\lambda$. This is one reason why Shelah assumes that $\lambda$ is regular in [She99]. However if $K$ is also categorical in a $\lambda' > \lambda$, then $K$ will be Galois-stable in $\lambda$, hence the model of size $\lambda$ will automatically be Galois-saturated. A consequence of this result is:

[1]For a review of the literature on the conjecture, see the introduction of [Vasb].

[2]One can of course also ask what the Hanf number for unbounded categoricity is. That is, given $\lambda$, what is the least $\mu_0$ such that an AEC with Löwenheim-Skolem number $\lambda$ categorical in some $\mu \geq \mu_0$ is categorical in a unboundedly many cardinals. Even assuming large cardinals, we are not aware of any known explicit bound.

**Proposition 1.3.** Assume that $K$ is an AEC with amalgamation and no maximal models, categorical in unboundedly many cardinals. Let $\lambda \geq \text{LS}(K)^+$ be such that $K$ is categorical in $\lambda$. For any $M, N \in K_\lambda$, if $M \leq N$, then $M \preceq_{L_{\infty,\text{LS}(K)^+}} N$.

*Proof sketch.* Both $M$ and $N$ are Galois-saturated. By the Tarski-Vaught test, it suffices to show that if $\psi \in L_{\infty,\text{LS}(K)^+}$, $\bar{a} \in {}^{\text{LS}(K)}|M|$, and $N \models \exists \bar{x} \psi[\bar{x}, \bar{a}]$, then there exists $\bar{b} \in {}^{\text{LS}(K)}|M|$ such that $N \models \psi[\bar{b}, \bar{a}]$. So assume $N \models \exists \bar{x} \psi(\bar{x}, \bar{a})$. Fix $\bar{b}' \in {}^{\text{LS}(K)}|N|$ such that $N \models \psi[\bar{b}', \bar{a}]$. By saturation, there exists $\bar{b} \in {}^{\text{LS}(K)}|M|$ such that $\bar{b}'$ and $\bar{b}$ realize the same Galois type over the range of $\bar{a}$. In particular, there is an automorphism of $N$ fixing $\bar{a}$ taking $\bar{b}'$ to $\bar{b}$. But then by invariance we have that $N \models \psi[\bar{b}, \bar{a}]$, as desired. □

The proof of Proposition 1.3 uses amalgamation and no maximal models heavily, for example to obtain that $M$ and $N$ are Galois-saturated and to build the automorphism taking $\bar{b}'$ to $\bar{b}$. In this paper, we show that a variation of Proposition 1.3 is true *without* assuming amalgamation and no maximal models. The change to make is that we have to take $\lambda$ much bigger than $\text{LS}(K)^+$. In fact, we do not know of an explicit bound on $\lambda$ (see Question 3.16), only that a $\lambda$ satisfying the conclusion of Proposition 1.3 must exist. This is Theorem 0.1 of the abstract, proven here as Theorem 4.2. We restate it here in a slightly different way:

**Theorem 1.4.** Let $K$ be an AEC categorical in unboundedly many cardinals. For any $\theta$, there exists a cardinal $\mu_0(\theta)$ such that for any $\mu$ so that $K$ is categorical in $\mu \geq \mu_0(\theta)$, for any $M, N \in K_{\geq \mu}$, $M \leq N$ implies $M \preceq_{L_{\infty,\theta}} N$.

Theorem 1.4 has several applications. For one thing, we can use it to show that for an AEC $K$ categorical in unboundedly many cardinals, there exists $\lambda$ so that $K_{\geq \lambda}$ is nothing but the class of models of a complete $L_{(2^{\text{LS}(K)})^+,\text{LS}(K)^+}$-sentence, ordered by $L_{\infty,\text{LS}(K)^+}$-elementary substructure. This adds to a result of Kueker ([Kue08, Theorem 7.4]) which proved this (without the characterization of the ordering) assuming amalgamation and categoricity in a single cardinal of high-enough cofinality[3] .

More importantly, the condition that $M \leq N$ implies $M \preceq_{L_{\infty,\theta}} N$ (for an appropriate $\theta$) is crucial in Chapter IV of Shelah's book on AECs

[3]Kueker also shows [Kue08, Theorem 7.2] that $M \preceq_{\infty,\text{LS}(K)^+} N$ implies $M \leq N$, but we are interested in the converse here.

[She09]. There Shelah uses the condition to obtain amalgamation in a single cardinal from categoricity in a suitable cardinal (see [She09, Theorem IV.1.30]). In [She09, Claim IV.1.12], Shelah gives a short proof of the conclusion of Theorem 1.4 if $\mu = \mu^{<\theta}$. After much more work, this is improved to $\mu > 2^{<\theta}$ and $\text{cf}(\mu) \geq \theta > \text{LS}(K)$, see [She09, Conclusion IV.2.12]. In [She09, Conclusion IV.2.14], Shelah even claims to prove Theorem 1.4. An implicit conclusion of this work (the second part is stated explicitly as [Vasb, Fact 4.9]) is:

**Claim 1.5.** If $K$ is an AEC categorical in unboundedly many cardinal, then:

(1) $K$ is categorical in a stationary class of cardinals.
(2) There exists a categoricity cardinal $\lambda$ such that $K$ has amalgamation in $\lambda$. In fact, $K$ has a type-full good $\lambda$-frame in $\lambda$ (see [She09, Definition II.2.1]).

The second part of claim 1.5 has been used by the second author to prove Shelah's eventual categoricity conjecture for universal classes [Vasb][4].

*However*, we have identified a gap in Shelah's proof of [She09, Conclusion IV.2.14] (hence invalidating Claim 1.5): there Shelah uses model-theoretic forcing and proves [She09, Claim IV.2.13] that it is enough to have a certain downward Löwenheim-Skolem theorem for forcing. Shelah had previously established a downward Löwenheim-Skolem theorem *for* $L_{\infty,\theta}$, and claims without comments (in his proof of Conclusion IV.2.14) that it suffices to use it. However it is not clear that forcing and the classical $L_{\infty,\theta}$ satisfaction relation coincide, in fact this is essentially what Shelah wants to prove.

Here, we fix Shelah's gap by providing the needed downward Löwenheim-Skolem theorem for forcing (this is Theorem 3.14). For technical reasons explained in Section 3, our definition of forcing is slightly different than Shelah's so we have to go through Shelah's argument again. Note that, except for a few elementary facts about Ehrenfeucht-Mostowski models, the proof of Theorem 1.4 is completely self-contained and should be readable to anyone familiar with the basics of AECs, as presented for example in Chapter 4 of [Bal09].

[4]We believe that such claims, obtaining some amalgamation from categoricity, will be central in the resolution of Shelah's categoricity conjecture. In fact, it has been conjectured by Grossberg (see [Gro02, Conjecture 2.3]) that amalgamation should simply follow from categoricity in a high-enough cardinal.

For clarity, we point out once again that the main contribution of the paper is the downward Löwenheim-Skolem theorem for forcing. The idea of using model-theoretic forcing to prove Theorem 1.4 is due to Shelah and apart from the downward Löwenheim-Skolem theorem for forcing, arguments very similar to ours appear already in [She09, Section IV.2]. The paper is organized as follows. We state and prove an abstract version of the downward Löwenheim-Skolem theorem for forcing in Section 2. Section 3 applies it to forcing and proves Theorem 1.4. In Section 4, we conclude and provide a proof of Claim 1.5.

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## 2. Categoricity and the Löwenheim-Skolem theorem

In this section, we set the stage and prove an abstract downward Löwenheim-Skolem theorem in categorical classes.

We will use a weakened definition of abstract logic compared to e.g. [BFB85, Definition I.1.1.1]: we want the logic to be only over a specific vocabulary $\tau_0$. We still allow constant symbols in order to define a notion of free variables.

**Definition 2.1.** Let $\tau_0$ be a vocabulary.

(1) A vocabulary $\tau$ is said to be *over* $\tau_0$ if $\tau_0 \subseteq \tau$ and $\tau \backslash \tau_0$ contains only constant symbols.
(2) A *logic over* $\tau_0$ is a pair $(\mathcal{L}, \models_{\mathcal{L}})$ where $\mathcal{L}$ is a mapping defined on vocabularies $\tau$ such that $\mathcal{L}(\tau)$ is a class (called the class of $\mathcal{L}$-sentences of vocabulary $\tau$) and $\models_{\mathcal{L}}$ is a binary relation between structures and $\mathcal{L}$-sentences. Moreover the following properties hold:
    (a) If $\tau$ is *not* over $\tau_0$, then $\mathcal{L}(\tau) = \emptyset$.
    (b) If $\tau \subseteq \sigma$ are both over $\tau_0$, then $\mathcal{L}(\tau) \subseteq \mathcal{L}(\sigma)$.
    (c) If $M \models_{\mathcal{L}} \phi$, then $\phi \in \mathcal{L}(\tau(M))$.
    (d) Isomorphism property: If $M \models_{\mathcal{L}} \phi$ and $f : M \cong N$, then $N \models_{\mathcal{L}} \phi$.
    (e) Reduct property: If $\phi \in \mathcal{L}(\tau)$, $\tau \subseteq \tau(M)$, and $\tau$ and $\tau(M)$ are over $\tau_0$, then $M \models_{\mathcal{L}} \phi$ if and only if $M \restriction \tau \models_{\mathcal{L}} \phi$.
    (f) Renaming property: Let $\tau$ and $\sigma$ be over $\tau_0$. Let $\rho : \tau \to \sigma$ be a bijection fixing $\tau_0$. For each $\phi \in \mathcal{L}(\tau)$ there is a sentence $\phi^\rho \in \mathcal{L}(\sigma)$ such that for all $\tau$-structures $M$,

denoting by $M^\rho$ the "renamed" structure, we have $M \models_{\mathcal{L}} \phi$ if and only if $M^\rho \models_{\mathcal{L}} \phi^\rho$.

We often do not distinguish between $(\mathcal{L}, \models_{\mathcal{L}})$ and $\mathcal{L}$.

(3) Let $\theta$ be a cardinal. A logic $\mathcal{L}$ over $\tau_0$ is *$\theta$-small* if:
   (a) For every vocabulary $\tau$, $\mathcal{L}(\tau)$ is a *set* (so it is not a proper class).
   (b) For every vocabulary $\tau$ over $\tau_0$ and every $\phi \in \mathcal{L}(\tau)$ there exists $\tau' \subseteq \tau$ which is over $\tau_0$ so that $\phi \in \mathcal{L}(\tau')$ and $|\tau' \backslash \tau_0| < \theta$[5].

We say that $\mathcal{L}$ is *small* if it is $\theta$-small for some cardinal $\theta$.

**Remark 2.2.** In a logic $\mathcal{L}$ over $\tau_0$, we have a natural notion of formula. We write $\phi(\bar{x}) \in \mathcal{L}(\tau_0)$ (or just say that it is a formula) to mean that $\phi$ is a sentence in $\tau_0 \cup \{c_{x(i)} : i \in \text{dom}(\bar{x})\}$, where each $c_{x(i)}$ is a new constant symbol. We sometimes do not even mention $\bar{x}$ if it is clear from context. For $\phi(\bar{x})$ a formula, a $\tau_0$-structure $M$ and $\bar{a} \in {}^{\ell(\bar{x})}|M|$, we write $M \models_{\mathcal{L}} \phi[\bar{a}]$ to mean $(M, \bar{a}) \models_{\mathcal{L}} \phi$. Thus for example if $\mathcal{L}$ is $\theta$-small, then every formula has strictly less than $\theta$-many variables.

**Remark 2.3.** Assume the logic $\mathcal{L}$ over $\tau_0$ is $\theta$-small. Fix distinct constant symbols $\{c_i : i < \theta\}$ not in $\tau_0$. By smallness and the renaming property, for any vocabulary $\tau$ over $\tau_0$, any sentence $\phi \in \mathcal{L}(\tau)$ can be renamed to a sentence in $\mathcal{L}(\tau_0 \cup \{c_i : i \subseteq I\})$, where $I \subseteq \theta$ and $|I| < \theta$. In particular any sentence can be renamed to a sentence in $\mathcal{L}(\tau_0 \cup \{c_i : i < \theta\})$ so, up to elementary equivalence, there is only a set of $\mathcal{L}$-formulas.

**Definition 2.4.** Let $\mathcal{L}$ be a logic over $\tau_0$. For $\tau_0$-structures $M$ and $N$, we write $M \preceq_{\mathcal{L}} N$ if $|M| \subseteq |N|$ and for every formula $\phi(\bar{x})$ and $\bar{a} \in {}^{\ell(\bar{x})}|M|$, we have $M \models_{\mathcal{L}} \phi[\bar{a}]$ if and only if $N \models_{\mathcal{L}} \phi[\bar{a}]$.

We want to study the interplay of logics and classes of models ordered by a partial ordering. The definition of an *abstract class* was introduced by Grossberg in [Gro] and aims to capture the minimal set of requirements such a class should satisfy. We will work with an even weaker definition where closure under isomorphism is not required; indeed, the lack of closure under isomorphisms is crucial. The idea is to capture classes of models that are all contained in a big (perhaps proper-class-sized model). The extra generality is used in the proof of Theorem 3.14 (the class $K^{**}$ there is not closed under isomorphisms).

[5]This is analog to asking for an abstract logic to have occurrence number $\theta$, see [BFB85, Definition II.6.1.3].

**Definition 2.5.** A pair $(K, \leq)$ is an *ordered class of structures* (or just an *ordered class*) if:

(1) $K$ is a class of $\tau$-structures for a fixed vocabulary $\tau = \tau(K)$.
(2) $\leq$ is a reflexive and transitive relation on $K$.
(3) If $M \leq N$, then $M \subseteq N$.

We say that $(K, \leq)$ is an *abstract class* if, in addition, both $K$ and $\leq$ are closed under isomorphism. That is, whenever $M, N \in K$, $M \leq N$ and $f : N \cong N'$, we have that $f[M], N' \in K$ and $f[M] \leq N'$. We say that an ordered class $(K, \leq)$ is *coherent* if whenever $M_0, M_1, M_2 \in K$, $|M_0| \subseteq |M_1| \subseteq |M_2|$, $M_0 \leq M_2$ and $M_1 \leq M_2$, then $M_0 \leq M_1$.

**Remark 2.6.** We often do not distinguish between $K$ and the pair $(K, \leq)$.

**Remark 2.7.** In this section, we do not use that $M \leq N$ implies $M \subseteq N$ but only that it implies $|M| \subseteq |N|$. Hence we are very close to the framework of concrete categories. In fact, Theorem 2.17 gives a version of one of our results in this framework.

We are interested in when the ordering of an ordered class plays well with the logic. This is the purpose of the next definition.

**Definition 2.8.** Let $K$ be an ordered class and let $\mathcal{L}$ be a logic over $\tau(K)$. We say that $K$ is *monotonic for* $\mathcal{L}$ if whenever $M, N \in K$ are such that $M \leq N$, then for any formula $\phi(\bar{x})$ of $\mathcal{L}$ and any $\bar{a} \in {}^{<\theta}|M|$, if $M \models \phi[\bar{a}]$, then $N \models \phi[\bar{a}]$.

**Example 2.9.**

(1) If $K$ is a class of $\tau$-structures and $\mathcal{L}$ is a logic over $\tau(K)$ such that $M \preceq_{\mathcal{L}} N$ implies $M \subseteq N$, then $(K, \preceq_{\mathcal{L}})$ is monotonic for $\mathcal{L}$. For example if $T$ is a first-order theory and $K = \mathrm{Mod}(T)$, then $(K, \preceq)$ is monotonic for first-order logic (restricted to $\tau(K)$ in the natural way).
(2) Define a logic $\mathcal{L}$ whose formulas are cardinals (for any vocabulary) and so that $M \models_{\mathcal{L}} \lambda$ if and only if $\|M\| \geq \lambda$. Then any ordered class $K$ is monotonic for $\mathcal{L}$, but if $M, N \in K$, $M \leq N$ but $\|M\| \neq \|N\|$, then $M \not\preceq_{\mathcal{L}} N$.
(3) If $K$ is an ordered class and $(\mathcal{L}, \models_{\mathcal{L}})$ is a logic, then define $M \models^*_{\mathcal{L}} \phi$ to mean that, $N \models_{\mathcal{L}} \phi$ for all $N \in K$ such that $M \leq N$. Then $K$ is monotonic for $(\mathcal{L}, \models^*)$. The same is true if $\models^*$ is replaced by any reasonable notion of model-theoretic forcing, such as Definition 3.2).

Note that we do *not* assume that the logic is closed under negation (i.e. if $\phi$ is a formula there is not necessarily a formula equivalent to $\neg\phi$). Thus as exemplified above even if $K$ is monotonic for $\mathcal{L}$, we may not have that $M \leq N$ implies $M \preceq_{\mathcal{L}} N$. The goal of the section is to find hypotheses under which we can prove this. The following coherence criterion will be very useful:

**Proposition 2.10.** Let $K$ be an ordered class and let $\mathcal{L}$ be a logic over $\tau(K)$. Assume that $K$ is monotonic for $\mathcal{L}$.

(1) For $M, M', N \in K$, if $M \preceq_{\mathcal{L}} N$ and $M \leq M' \leq N$, then $M \preceq_{\mathcal{L}} M' \preceq_{\mathcal{L}} N$.
(2) If $M, N \in K$ and $M \leq N$, then $M \preceq_{\mathcal{L}} N$ if and only if there exists $M_0 \leq M$ such that $M_0 \preceq_{\mathcal{L}} N$.

*Proof.*

(1) We prove that $M \preceq_{\mathcal{L}} M'$ and the proof that $M' \preceq_{\mathcal{L}} N$ is similar. If $M \models \phi[\bar{a}]$, then $M' \models_{\mathcal{L}} \phi[\bar{a}]$ by monotonicity. Conversely if $M' \models_{\mathcal{L}} \phi[\bar{a}]$, then $N \models_{\mathcal{L}} \phi[\bar{a}]$ by monotonicity, so $M \models_{\mathcal{L}} \phi[\bar{a}]$ by elementarity.
(2) For the left to right direction, take $M_0 = M$. For the right to left, assume $M_0 \leq M$ is such that $M_0 \preceq_{\mathcal{L}} N$. Use the previous part to conclude that $M \preceq_{\mathcal{L}} N$.

$\square$

The next lemma is the key to all subsequent arguments. Formally, it should be stated and proven in Von Neumann-Bernays-Gödel set theory (see [Jec03, p. 70], note that it is a conservative extension of ZFC), or some other formalism that can handle proper classes.

**Lemma 2.11.** Let $K$ be an ordered class monotonic and let $\mathcal{L}$ be a small logic over $\tau(K)$. Assume that $K$ is monotonic for $\mathcal{L}$.

Let $\langle M_i : i \in \mathrm{OR}\rangle$ be a chain in $K$ such that for some $\mu$, for all $i$, if $\mathrm{cf}(i) \geq \mu$, then $|M_i| = \bigcup_{j<i} |M_j|$.

Then there exists $i \in \mathrm{OR}$ such that $M_i \preceq_{\mathcal{L}} M_{i+1}$.

*Proof.* Suppose not. By increasing $\mu$, we can assume that $\mu$ is regular, $\mu \geq |\mathcal{L}(\tau(K))|$, and $\mathcal{L}$ is a $\mu$-small logic. Thus for each $i \in \mathrm{OR}$ there exists $\phi_i \in \mathcal{L}(\tau(K))$ and $\bar{a}_i \in {}^{<\mu}|M_i|$ such that $M_{i+1} \models_{\mathcal{L}} \phi_i[\bar{a}_i]$ but $M_i \not\models_{\mathcal{L}} \phi_i[\bar{a}_i]$. For each $i \in \mathrm{OR}$, let $\alpha(i)$ be the least $\alpha \leq i$ such that $\bar{a}_i \in {}^{<\mu}|M_\alpha|$. Note that if $\mathrm{cf}(i) \geq \mu$, then $\alpha(i) < i$. Thus by Fodor's lemma for proper classes of cardinals, there exists a proper class $S \subseteq$

OR and an ordinal $\alpha_0$ such that $i \in S$ implies $\alpha(i) = \alpha_0$. By Remark 2.3, we can assume without loss of generality that $\{\phi_i : i \in \mathrm{OR}\}$ is a set. Since ${}^{<\mu}|M_{\alpha_0}|$ is also a set and $S$ is a proper class, there exists $i < j$ in $S$ such that $\bar{a}_i = \bar{a}_j$ and $\phi_i = \phi_j$. Since $M_{i+1} \models_{\mathcal{L}} \phi_i[\bar{a}_i]$, monotonicity implies $M_j \models \phi_i[\bar{a}_i]$, so $M_j \models \phi_j[\bar{a}_j]$, a contradiction. $\square$

**Remark 2.12.**

(1) Notice that the smallness of the logic is a necessary condition. Take $\mathcal{L}$ as described in Example 2.9.(2). Let $K$ be the AEC of all sets (with no structures), and for $i \in \mathrm{OR}$, let $M_i := \aleph_i$. Then $M_i \not\preceq_{\mathcal{L}} M_{i+1}$ for all $i \in \mathrm{OR}$.
(2) Surprisingly, this is not the first time that Fodor's lemma for a proper class of cardinals is used in the study of AECs: see the proof of [She, Theorem 3.17].
(3) We would prefer that there is some big cardinal that can take the place of OR in the above. However, at the very general level of logics and ordered classes, this does not seem reasonable. Still one can ask whether this is true of forcing:

**Question 2.13.** If $K$ is an AEC monotonic for $(L^*_{\infty,\theta}, \Vdash_{K^*})$ (see Definition 3.2), is there a cardinal $\lambda$ (depending on $\theta$ and $\mathrm{LS}(K)$) such that the chain in Lemma 2.11 is only required to be of length $\lambda$?

A positive answer would allow us to answer Question 3.16, which would give us explicit bounds on many theorems in this paper.

Lemma 2.11 suggests some additional hypotheses on the class $K$:

**Definition 2.14.** Let $\mu$ be an infinite cardinal. An ordered class $K$ is *$\mu$-nice* if:

(1) For any limit ordinal $\delta$, any chain $\langle M_i : i < \delta \rangle$ has an upper bound $M$.
(2) If in addition $\mathrm{cf}(\delta) \geq \mu$, then we can take $M$ with $|M| = \bigcup_{j<i} |M_j|$.

We say that $K$ is *nice* if it is $\mu$-nice for some $\mu$.

**Remark 2.15.** This is reminiscent of $\mu$-AECs (to be introduced in [BGL$^+$], see [Vasa, Definition 2.13]), where it is required that every $\mu$-directed system has its union inside $K$. As pointed out in [BGL$^+$], this is (in contrast to AECs where this leads to equivalent definitions) different from just requiring that every chain of cofinality at least $\mu$ has its union inside $K$. Here we will only need to deal with chains, and we also require that chains of low cofinality have an upper bound (but it need not be the union). Of course, there are other differences between $\mu$-AECs and $\mu$-nice ordered classes (the former are required to be closed under isomorphisms, coherent, and satisfy a Löwenheim-Skolem axiom). Still:

(1) Any AEC is $\aleph_0$-nice.
(2) Any $\mu$-AEC where every chain has an upper bound is $\mu$-nice.

Next we apply Lemma 2.11 to nice classes.

**Theorem 2.16.** Let $K$ be an ordered class. Let $\mathcal{L}$ be a small logic over $\tau(K)$. Assume that $K$ is monotonic for $\mathcal{L}$.

If $K$ is nice and nonempty, then there exists $M \in K$ such that for all $N \in K$, if $M \leq N$ then $M \preceq_{\mathcal{L}} N$.

*Proof.* Let $\mu$ be such that $K$ is $\mu$-nice. Suppose the conclusion fails. We build an increasing $\langle M_i \in K : i \in \mathrm{OR}\rangle$ such that for all $i \in \mathrm{OR}$:

(1) $M_i \npreceq_{\mathcal{L}} M_{i+1}$.
(2) If $\mathrm{cf}(i) \geq \mu$, then $|M_i| = \bigcup_{j<i} |M_j|$.

This is enough as this contradicts Lemma 2.11. This is possible: Take any $M_0 \in K$. At limits, use $\mu$-niceness. Given $M_i$, we know that there must exist $M_{i+1} \geq M_i$ such that $M_i \npreceq_{\mathcal{L}} M_{i+1}$, as desired. $\square$

Theorem 2.16 is essentially combinatorial: nothing about the logic is used, except that it is small and monotonic. The same argument gives a logic-free version in the setup of concrete categories (this will not be used so we omit the proof).

Recall that a category $C$ is *concrete* if there exists a faithful functor $U : C \to \mathrm{Set}$. For $M, N$ objects in a concrete category $C$, we write $M \leq_U N$ if there exists an arrow $f$ from $M$ to $N$ such that $U(f)$ is the inclusion map. We define $\mu$-nice for concrete categories in the expected way (this depends on the functor $U$).

**Theorem 2.17.** Let $C$ be a concrete category and let $U : C \to \mathrm{Set}$ be a functor. Let $F : C \to \mathrm{Set}$ be a subfunctor of $U$ (that is, for every objects $M$ and $N$, $F(M) \subseteq U(M)$ and $M \leq_U N$ implies $F(M) \subseteq F(N)$).

If $C$ nice and nonempty, then there exists an object $M$ such that for all objects $N$, if $M \leq_U N$, then $F(M) = F(N) \cap U(N)$.

**Remark 2.18.** We can deduce Theorem 2.16 from Theorem 2.17. Given $K$ a nice, nonempty, ordered class and $\mathcal{L}$ a small logic so that $K$

is monotonic for $\mathcal{L}$, let $\theta$ be a regular cardinal so that $\mathcal{L}$ is $\theta$-small and $K$ is $\theta$-nice. Let the objects of $C$ be the members of $K$ and say there is an arrow from $M$ to $N$ if and only if $M \leq N$. Let $U(M) := {}^{<\theta}|M| \times \{\phi(\bar{x}) \mid \phi \text{ is a formula in } \mathcal{L}\}$, and $F(M) := \{(\bar{a}, \phi) : M \models_{\mathcal{L}} \phi[\bar{a}]\}$.

We now return to abstract classes and study how an additional assumption of categoricity can improve Theorem 2.16. We will be interested in downward Löwenheim-Skolem numbers. In the intended applications, $R$ below will be $\preceq_{\mathcal{L}}$:

**Definition 2.19.** Let $K$ be an ordered class, $\mu$ be a cardinal, and $R$ be a reflexive and transitive relation on $K$. We let $\text{LS}_{R,\mu}(K)$ be the least $\mu_0 \geq |\tau(K)| + \aleph_0$ such that for every $M \in K_\mu$ and every $A \subseteq |M|$, there exists $M_0 \in K$ containing $A$ such that $M_0 \leq M$, $R(M_0, M)$, and $\|M_0\| \leq |A| + \mu_0$. When $R = \leq$, we omit it. We write $\text{LS}_R(K)$ for $\sup_\mu \text{LS}_{R,\mu}(K)$.

**Lemma 2.20.** Let $K$ be a coherent abstract class. Let $\mathcal{L}$ be a small logic over $\tau(K)$. Assume that $K$ is monotonic for $\mathcal{L}$.

If:

(1) $K$ is categorical in unboundedly many cardinals and $\text{LS}(K) < \infty$.
(2) $K^*$ is a nonempty nice ordered class with $K^* \subseteq K$.
(3) For any $M \in K^*$ and any $\lambda \geq \|M\|$, there exists $N \in K^*$ so that $M \leq N$ and $\|N\| = \lambda$.

Then then there exists a cardinal $\mu_0$ such that:

(1) For any $\mu \geq \mu_0$ such that $K$ is categorical in $\mu$, $\text{LS}_{\preceq_{\mathcal{L}},\mu}(K) \leq \mu_0$.
(2) If $K^* = K$, then for every $M, N \in K_{\geq \mu_0}$, if $M \leq N$, then $M \preceq_{\mathcal{L}} N$. In particular, $\text{LS}_{\preceq_{\mathcal{L}}}(K) \leq \mu_0$.

*Proof.* By Theorem 2.16, there exists $M \in K^*$ such that for all $N \in K^*$, if $M \leq N$ then $M \preceq_{\mathcal{L}} N$. Note that if $M' \in K^*$ is such that $M \leq M'$, then by Proposition 2.10, $M' \leq N$ implies $M' \preceq_{\mathcal{L}} N$. Therefore we can make $M$ bigger if necessary to assume without loss of generality that $\|M\| \geq \text{LS}(K)$. Let $\mu_0 := \|M\|$.

(1) $\mu \geq \mu_0$ be such that $K$ is categorical in $\mu$. Let $N \in K_\mu$ be arbitrary. Let $N' \geq M$ be from $K^*_\mu$. By categoricity, there exists $f : N' \cong N$. By definition of $M$, $M \preceq_{\mathcal{L}} N'$. Since $\mathcal{L}$ is a logic over $\tau(K)$, it is invariant under isomorphisms so $f[M] \preceq_{\mathcal{L}} N$. As $K$ is closed under isomorphisms and $M \leq N'$, $f[M] \leq N$. Now given $A \subseteq |N|$, simply pick any $N_0 \leq N$ containing $A$ and

$f[M]$ with $\|N_0\| \leq |A| + \mu_0$ (this is possible since $\text{LS}(K) \leq \mu_0$). By coherence, $f[M] \leq N_0$ so by Proposition 2.10, $N_0 \preceq_{\mathcal{L}} N$, as desired.

(2) By making $M$ larger we can assume without loss of generality that $K$ is categorical in $\mu_0$. Assume $M_1 \leq M_2$ are both in $K_{\geq \mu_0}$. Let $M_1' \in K_{\mu_0}$ be such that $M_1' \leq M_1$. We show that $M_1' \preceq_{\mathcal{L}} M_2$ which is enough by Proposition 2.10. Let $f : M_1' \cong M$ and extend it to $g : M_2 \cong N$ with $M \leq N$. By assumption, $M \preceq_{\mathcal{L}} N$. By invariance, $M_1' \preceq_{\mathcal{L}} M_2$, as desired.

□

The following is an immediate application of Lemma 2.20.

**Corollary 2.21.** Let $K$ be an AEC with no maximal models. Let $\mathcal{L}$ be a small logic over $\tau(K)$ and assume that $K$ is monotonic for $\mathcal{L}$. If $K$ is categorical in arbitrarily large cardinalities, then there is $\mu_0$ such that, for all $M, N \in K_{\geq \mu_0}$, we have:

$$M \leq N \implies M \preceq_{\mathcal{L}} N$$

If the AEC has maximal models, we can still use Lemma 2.20 with appropriate $K$ and $K^*$ to obtain a downward Löwenheim-Skolem theorem for model-theoretic forcing (see Theorem 3.14). This is done in the next section.

## 3. Forcing in AECs

In this section, we define an appropriate variation of model-theoretic forcing and use it to prove the main theorem 1.4. Our version of forcing will be defined only for formulas of a particular form:

**Definition 3.1.** The logic $L^*_{\mu,\theta}$ (where we allow $\mu$ and $\theta$ to be $\infty$) is defined as for $L_{\mu,\theta}$ except we require that formulas do not contain disjunctions or existential quantifiers. A *fragment* of $L^*_{\mu,\theta}$ is a set of $L^*_{\mu,\theta}$-formulas which is closed under renaming, subformulas, negations, finite conjunctions, and universal quantification over finitely many variables.

The use of fragments (which are by definition sets) is necessary to make use of the results about small logics from the previous section.

**Definition 3.2.** Let $K^*$ be an ordered class (see Definition 2.5). For $\phi$ an $L^*_{\infty,\theta}$ formula, $M \in K^*$, and $\bar{a} \in {}^{<\theta}|M|$, define $M \Vdash_{K^*} \phi[\bar{a}]$ by induction on $\phi$ as follows:

(1) If $\phi$ is atomic, $M \Vdash_{K^*} \phi[\bar{a}]$ if and only if $M \models \phi[\bar{a}]$.
(2) If $\phi = \neg\psi$, $M \Vdash_{K^*} \phi[\bar{a}]$ if and only if for no $N \geq M$ from $K^*$ do we have $N \Vdash_{K^*} \psi[\bar{a}]$.
(3) If $\phi = \wedge_{i<\alpha}\phi_i$, $M \Vdash_{K^*} \phi[\bar{a}]$ if and only if $M \Vdash_{K^*} \phi_i[\bar{a}]$ for all $i < \alpha$.
(4) If $\phi = \forall \bar{x}\psi(\bar{x})$, $M \Vdash_{K^*} \phi[\bar{a}]$ if and only if for all $N \geq M$ from $K^*$ and all $\bar{b} \in {}^{<\theta}|N|$, there exists $N' \geq N$ from $K^*$ such that $N' \Vdash_{K^*} \psi[\bar{b}, \bar{a}]$.

When $K^*$ is clear from context, we omit it and just write $M \Vdash \phi[\bar{a}]$.

**Remark 3.3.** In [She09], Shelah defines forcing in AECs[6] with a clause for the existential quantifier, not the universal one (see Definition I.4.3 or Definition IV.2.3.(2) in [She09]). Shelah defines $M \Vdash \exists \bar{x}\psi(\bar{x}, \bar{a})$ if and only if for any $N \geq M$ there exists $N' \geq N$ and $\bar{b} \in {}^{<\infty}|N'|$ such that $N' \Vdash \psi[\bar{b}, \bar{a}]$. Are there any direct implications between the two definitions? This seems difficult to establish because, while any formula in $L_{\mu,\theta}$ is classically equivalent to a formula in $L^*_{\mu,\theta}$, it may not be equivalent in the logic induced by forcing. Indeed, forcing "lacks the excluded middle,"[7] so turning universals into existentials is uninstructive. However, a key difference seems to be where the element $\bar{b}$ is quantified; in our definition, it is from the first extension, while it comes from the second extension ($N'$) in Shelah's definition.

Thus we believe that our definition is easier to deal with. For example, our proof of Theorem 3.25 avoids using Shelah's construction of linear orders with few automorphisms, see [She09, Claims IV.2.8.(1), IV.2.11.(1)]. Another difference is that Shelah defines forcing on $K^*_\mu$, where $K^*$ is the class of EM models of the AEC $K$ and $\mu$ is a solvability cardinal (see Definition 3.11 and Hypothesis 3.13). In order to be able to apply Lemma 2.20, we will work with forcing on the entire $K^*$, but will show that it is equivalent to working only in $K^*_\mu$, assuming some reasonable hypotheses (see Lemma 3.17). It is not clear to us how to prove such a result using Shelah's definition.

The following version of the universal clause will be very useful: the key is that it has only one quantification over models.

**Proposition 3.4.** *$M \Vdash_{K^*} \forall \bar{x}\psi(\bar{x}, \bar{a})$ if and only if for all $N \geq M$ and all $\bar{b} \in {}^{<\infty}|N|$, $N \nVdash_{K^*} \neg\psi[\bar{b}, \bar{a}]$.*

[6]Of course, model-theoretic forcing has a long history starting with Robinson. See [Hod06] for a presentation of many applications.

[7]This means that forcing $\phi$ and forcing $\neg\neg\phi$ are different. The first always implies the second and forcing $\neg\neg\phi$ is equivalent to forcing $\neg\neg\neg\neg\phi$.

*Proof.* Directly from the definition. $\square$

**Proposition 3.5.** Let $K^*$ be an ordered class.

(1) (Monotonicity) If $M \leq N$ are in $K^*$ and $M \Vdash \phi[\bar{a}]$, then $N \Vdash \phi[\bar{a}]$.
(2) (Consistency) If $M \Vdash \phi[\bar{a}]$, then $M \nVdash \neg\phi[\bar{a}]$. If $M \Vdash \neg\phi[\bar{a}]$, then $M \nVdash \phi[\bar{a}]$.
(3) (Decidability) For any $\phi$, $M \in K^*$, and $\bar{a} \in {}^{<\theta}|M|$, there exists $N \geq M$ in $K^*$ such that either $N \Vdash \phi[\bar{a}]$ or $N \Vdash \neg\phi[\bar{a}]$.

*Proof.* Straight from the definition. $\square$

**Definition 3.6.** Let $\Lambda$ be a fragment of $L^*_{\infty,\infty}$. Let $K^*$ be an ordered class. We define a logic over $\tau(K)$, $(\mathcal{L}_{K^*}(\Lambda), \models_{\mathcal{L}_{K^*}(\Lambda)})$ as follows:

(1) For a vocabulary $\tau$ over $\tau(K)$, the $\tau$-sentences are the sentences in $\Lambda$ (where we identify a formula $\phi(\bar{x})$ with free variables with a sentence with additional constant symbols).
(2) $M \models_{\mathcal{L}_{K^*}} \phi[\bar{a}]$ if and only if $M \Vdash_{K^*} \phi[\bar{a}]$.

**Remark 3.7.** If $\Lambda$ is a fragment of $L^*_{\infty,\infty}$, then $\mathcal{L}_{K^*}(\Lambda)$ is small. Also, $K^*$ is monotonic for $\mathcal{L}_{K^*}(\Lambda)$.

For what follows, we assume some familiarity with Ehrenfeucht-Mostowski models, see for example [Bal09, Section 6.2] or [She09, Definition IV.0.8]. Recall:

**Definition 3.8.** Let $K$ be an AEC. We say that $\Phi$ is an *EM blueprint for* $K$ if $\Phi$ is a set of quantifier-free types proper for linear orders in a vocabulary $\tau(\Phi) \supseteq \tau(K)$, and (writing $\mathrm{EM}(I,\Phi)$ for the $\tau(\Phi)$-structure generated by the linear order $I$ and $\Phi$ and $\mathrm{EM}_{\tau(K)}(I,\Phi)$ for $\mathrm{EM}(I,\Phi) \upharpoonright \tau(K)$):

(1) For any linear order $I$, $\mathrm{EM}_{\tau(K)}(I,\Phi) \in K$ and $\|\mathrm{EM}(I,\Phi)\| = |I| + |\tau'| + \mathrm{LS}(K)$.
(2) For any linear orders $I$, $J$, if $I \subseteq J$, then $\mathrm{EM}_{\tau(K)}(I,\Phi) \leq \mathrm{EM}_{\tau(K)}(J,\Phi)$.

From the presentation theorem, we get (see e.g. [She99, Claim 0.6]):

**Fact 3.9.** If an AEC $K$ has arbitrarily large models, then there exists an EM blueprint $\Phi$ for $K$ with $|\tau(\Phi)| = \mathrm{LS}(K)$.

**Definition 3.10.** Let $K$ be an AEC and let $\Phi$ be an EM blueprint for $K$. We write $K^*_\Phi$ (or just $K^*$ when $\Phi$ is clear from context) for the class of $M \in K$ such that there exists a linear order $I$ with $M \cong \mathrm{EM}_{\tau(K)}(I,\Phi)$.

If in addition $K$ is categorical in $\lambda \geq \mathrm{LS}(K)$, then in particular all the EM-models of size $\lambda$ must be isomorphic. In fact, the class of EM-models of size $\lambda$ will generate an AEC. This property is isolated by Shelah in [She09, Chapter IV] and called *solvability*. We define only the weaker notion of pseudo-solvability here (regular solvability asks in addition that the EM-model of size $\lambda$ be superlimit, so in particular every $M \in K_\lambda$ embeds into an EM-model of size $\lambda$).

**Definition 3.11** (Definition IV.1.4.(3) in [She09])**.** Let $K$ be an AEC and let $\Phi$ be an EM blueprint for $K$. We say that $(K, \Phi)$ is *pseudo* $(\lambda, \theta)$*-solvable* if:

(1) $\mathrm{LS}(K) \leq \theta \leq \lambda$. $|\Phi| \leq \theta$.
(2) $K^*_\Phi$ is categorical in $\lambda$.
(3) If $\delta < \lambda^+$ and $\langle M_i : i < \delta \rangle$ are increasing in $K^*_\Phi$ and have size $\lambda$, then $\bigcup_{i<\delta} M_i$ is in $K^*_\Phi$.

We say that $K$ is *pseudo* $(\lambda, \theta)$*-solvable* if $(K, \Phi)$ is pseudo $(\lambda, \theta)$-solvable for some EM blueprint $\Phi$.

**Remark 3.12.** If $K$ has arbitrarily large models, $\Phi$ is an EM blueprint for $K$, and $K$ is categorical in a $\lambda \geq \mathrm{LS}(K) + |\tau(\Phi)|$, then $(K, \Phi)$ is pseudo $(\lambda, |\tau(\Phi)| + \mathrm{LS}(K))$-solvable.

In the introduction to [She09, Chapter IV], Shelah claims that solvability is an appropriate notion of superstability for AECs. Solvability and it's variants essentially say that there is a categorical "core" or sub-AEC that has nice properties. In [GV, Theorem 0.1], it is shown to be equivalent to many other definitions of superstability when the class is tame and has amalgamation. Many consequences of categoricity can be proven (or at least slightly weaker variations) with only an assumption of solvability. This is the case for our results too. For the rest of this section, we assume:

**Hypothesis 3.13.**

(1) $K$ is an AEC with arbitrarily large models.
(2) $\Phi$ is a fixed EM blueprint for $K$ with $|\tau(\Phi)| = \mathrm{LS}(K)$.
(3) $(K, \Phi)$ is pseudo $(\mu, \mathrm{LS}(K))$-solvable for unboundedly many cardinals $\mu$.

Recall from Definition 3.10 that $K^* = K^*_\Phi$ denotes the class of models isomorphic to a $\Phi$-EM model. We write $\mathcal{L}(\Lambda)$ instead of $\mathcal{L}_{K^*}(\Lambda)$. When we say that $\mu$ is a pseudo-solvability cardinal we mean that $(K, \Phi)$ is pseudo $(\mu, \mathrm{LS}(K))$-solvable.

Using Lemma 2.20, we can prove a downward Löwenheim-Skolem theorem for forcing:

**Theorem 3.14.** Let $\Lambda$ be a fragment of $L^*_{\infty,\infty}$. Then there exists $\mu_0$ such that whenever $\mu$ is a pseudo-solvability cardinal and $\mu \geq \mu_0$, we have $\mathrm{LS}_{\preceq_{\mathcal{L}(\Lambda)},\mu}(K^*) \leq \mu_0$ (see Definition 2.19).

*Proof.* Note that we *cannot* apply Lemma 2.20 directly with $K, K^*$ there standing for $K^*, K^*$ here. Indeed, we do not know that the union of an increasing chain of Ehrenfeucht-Mostowski models is an Ehrenfeucht-Mostowski model, and in general we do not know that $K^*$ is nice (see Definition 2.14). However we can let $K^{**} := \{\mathrm{EM}_{\tau(K)}(\alpha, \Phi) \mid \alpha \in \mathrm{OR}\}$. Now observe:

(1) $K^*$ is a coherent abstract class which is categorical in unboundedly many cardinals and $\mathrm{LS}(K^*) = \mathrm{LS}(K)$.
(2) $K^{**} \subseteq K^*$, and $K^{**}$ is an ordered class of structures (but not an abstract class) so that:
   (a) $K^{**}$ is not empty and is $\aleph_0$-nice (it is isomorphic to the category of ordinals ordered by inclusion[8]).
   (b) If $M \in K^{**}$ and $\lambda > \|M\|$, then $N := \mathrm{EM}_{\tau(K)}(\lambda, \Phi) \geq M$ and $\|N\| = \lambda$.

Therefore we can apply Lemma 2.20 with $K, K^*$ there standing for $K^*, K^{**}$ here. $\square$

This justifies the next definition.

**Definition 3.15.** For $\Lambda$ a fragment of $L^*_{\infty,\infty}$, let $\mu_0(\Lambda)$ be the least cardinal $\mu_0 \geq \mathrm{LS}(K)$ such that whenever $\mu$ is a pseudo-solvability cardinal and $\mu \geq \mu_0$, $\mathrm{LS}_{\preceq_{\mathcal{L}(\Lambda)},\mu}(K^*) \leq \mu_0$.

Of course it would be nice if we could require only pseudo-solvability in one cardinal above a certain explicit threshold (e.g. $\beth_{(2^{\mathrm{LS}(K)})^+}$). At present, we do not know how to do this.

**Question 3.16.** Can one give an explicit bound for e.g. $\mu_0(L^*_{(2^\theta)^+,\theta})$?

We can use the downward Löwenheim-Skolem theorem for forcing to see that local and global forcing turn out to be the same, at least in high-enough pseudo-solvability cardinals:

[8]Importantly, it is not isomorphic to the category of ordinals with order-preserving maps.

**Lemma 3.17.** Let $\Lambda$ be a fragment of $L^*_{\infty,\theta}$. Let $\mu \geq \mu_0(\Lambda)$ be a pseudo-solvability cardinal.

Then for any $M \in K^*_\mu$, $\phi \in \Lambda$, and $\bar{a} \in {}^{<\theta}|M|$, we have that $M \Vdash_{K^*} \phi[\bar{a}]$ if and only if $M \Vdash_{K^*_\mu} \phi[\bar{a}]$.

*Proof.* Let $M \in K^*_\mu$ and let $\phi \in \Lambda$, $\bar{a} \in {}^{<\theta}|M|$. We prove by induction on $\phi$ that $M \Vdash_{K^*} \phi[\bar{a}]$ if and only if $M \Vdash_{K^*_\mu} \phi[\bar{a}]$.

If $\phi$ is atomic, this is clear. Similarly if $\phi$ is a conjunction. If $\phi = \neg\psi$, then the left to right direction holds. Conversely, assume $M \nVdash_{K^*} \phi[\bar{a}]$. Then there exists $N \geq M$ in $K^*$ such that $N \Vdash_{K^*} \psi[\bar{a}]$. Extending $N$ if necessary we can assume that $\|N\|$ is a pseudo-solvability cardinal. Since $\|N\| \geq \mu_0(\Lambda)$, there is $N_0 \in K^*_\mu$ with $M \leq N_0 \leq N$ such that $N_0 \leq_{\mathcal{L}(\Lambda)} N$. Thus, $N_0 \Vdash_{K^*} \psi[\bar{a}]$. By induction, $N_0 \Vdash_{K^*_\mu} \psi[\bar{a}]$, so by definition $M \nVdash_{K^*_\mu} \phi[\bar{a}]$.

Assume $\phi(\bar{y}) = \forall \bar{x} \psi(\bar{x}, \bar{y})$. For the left to right direction, assume $M \Vdash_{K^*} \phi[\bar{a}]$ and we want to see that $M \Vdash_{K^*_\mu} \phi[\bar{a}]$. So let $N \in K^*_\mu$ be such that $N \geq M$ and let $\bar{b} \in {}^{<\theta}|N|$. By assumption, there exists $N' \geq N$ in $K^*$ such that $N' \Vdash_{K^*} \psi[\bar{b}, \bar{a}]$. As $\mu \geq \mu_0(\theta)$, we can pick $N'_0 \leq N'$ such that $\|N'_0\| = \mu$, $N \leq N'_0$, and $N'_0 \preceq_{\mathcal{L}(\Lambda)} N'$. Then $N'_0 \Vdash_{K^*} \psi[\bar{b}, \bar{a}]$. By the induction hypothesis, $N'_0 \Vdash_{K^*_\mu} \psi[\bar{b}, \bar{a}]$, as needed.

For the right to left direction, assume $M \Vdash_{K^*_\mu} \phi[\bar{a}]$. Let $N \geq M$ be in $K^*$ and let $\bar{b} \in {}^{<\theta}|N|$. Let $\mu' \geq \|N\|$ be a pseudo-solvability cardinal. By definition of $\mu_0(\Lambda)$, there exists $N' \in K^*_{\mu'}$ such that $N \preceq_{\mathcal{L}(\Lambda)} N'$. Now, let $N'_0 \leq N'$ be such that $\|N'_0\| = \mu$, $\bar{b} \in {}^{<\theta}|N'_0|$, $M \leq N'_0$, and $N'_0 \preceq_{\mathcal{L}(\Lambda)} N'$. By assumption and Proposition 3.4, $N'_0 \nVdash_{K^*_\mu} \neg\psi[\bar{b}, \bar{a}]$. By induction, $N'_0 \nVdash_{K^*} \neg\psi[\bar{b}, \bar{a}]$. Therefore, by elementarity $N' \nVdash_{K^*} \neg\psi[\bar{b}, \bar{a}]$ and so $N \nVdash_{K^*} \neg\psi[\bar{b}, \bar{a}]$. Since $N$ and $\bar{b}$ were arbitrary, $M \Vdash_{K^*} \phi[\bar{a}]$, as desired. $\square$

**Remark 3.18.** In the preceding proof and several times later, we prove the induction stage for $\exists\psi$ by appealing to the induction hypothesis on $\neg\psi$. Since we have independently proved the induction stage for $\neg\psi$, this is valid.

We aim to show that for every high-enough pseudo-solvability cardinal $\mu$, $M \in K^*_\mu$, and $L^*_{\infty,\theta}$-formula $\phi$, $M \Vdash \phi[\bar{a}]$ if and only if $M \models \phi[\bar{a}]$ (we call a model with this property *generic*). This can be done via a counting argument when $\mu = \mu^{<\theta}$, but in general we will use two cardinals $\mu_1 < \mu_2$, where $\mu_1^{<\theta} < \mu_2$, together with Theorem 3.14 to reflect forcing downward. We first define what it means for a pair of models $(M, N)$ with $M \in K^*_{\mu_1}$ and $N \in K^*_{\mu_2}$ to be generic:

**Definition 3.19** (Definition IV.2.3.(3).(c) in [She09])**.** Let $\mathrm{LS}(K) \leq \mu_1 \leq \mu_2$. Let $M \in K_{\mu_1}$, $N \in K_{\mu_2}$ be such that $M \leq N$. $(M, N)$ is $\Lambda$-*generic* if for any $\phi \in \Lambda$, $\bar{a} \in {}^{<\infty}|M|$, $N \Vdash_{K^*_{\mu_2}} \phi[\bar{a}]$ if and only if $M \models \phi[\bar{a}]$.

$M$ is $\Lambda$-*generic* if $(M, M)$ is $\Lambda$-generic.

**Remark 3.20.** Shelah uses the opposite order (i.e. he requires $M \in K_{\mu_2}$, $N \in K_{\mu_1}$). We thought writing the models from left to right in increasing order of size was clearer.

Note that having a generic model is enough to prove that $K$-substructure implies elementary substructures (not in the forcing logic $\mathcal{L}(\Lambda)$ but in the actual $L_{\infty,\infty}$ sense).

**Lemma 3.21.** Assume every model in $K^*_\mu$ is $\Lambda$-generic. Then for any $M, N$ in $K^*_\mu$, $M \leq N$ implies $M \preceq_\Lambda N$.

*Proof.* Assume $M \leq N$ are in $K^*_\mu$. Assume $M \models \phi[\bar{a}]$, $\bar{a} \in {}^{<\infty}|M|$, $\phi \in \Lambda$. Then $M \Vdash_{K^*_\mu} \phi[\bar{a}]$, so by monotonicity $N \Vdash_{K^*_\mu} \phi[\bar{a}]$, hence by genericity $N \models \phi[\bar{a}]$. The converse is done by replacing $\phi$ with $\neg\phi$. $\square$

The next lemma is the analog of [She09, Claim 2.11.(5)].

**Lemma 3.22.** Let $\Lambda$ be a fragment of $L^*_{\infty,\theta}$ for $\theta$ a regular cardinal. Let $\mu_1 \leq \mu_2$ be such that:

(1) $\mu_1$ and $\mu_2$ are pseudo-solvability cardinals.
(2) $\mathrm{LS}(K) + |\Lambda| + \theta < \mu_1$.
(3) $\mu_1^{<\theta} \leq \mu_2$.

Let $M \in K^*_{\mu_1}$, $N \in K^*_{\mu_2}$ be such that $M \leq N$. Then there exists $M' \geq M$, $N' \geq M$ in $K^*_{\mu_1}$, $K^*_{\mu_2}$ respectively such that $M' \leq N'$ and $(M', N')$ is $\Lambda$-generic.

*Proof.* First we show (the important bit is that $\bar{b} \in {}^{<\theta}|M'|$):

Claim 1. There exists $N' \geq N$ in $K^*_{\mu_2}$, $M' \geq M$ in $K^*_{\mu_1}$ such that $M' \leq N'$ and for any $\psi(\bar{x}, \bar{y}) \in \Lambda$, any $\bar{a} \in {}^{\ell(\bar{y})}|M|$, and any $N'' \geq N'$ in $K^*_{\mu_2}$, either $N'' \Vdash \forall \bar{x} \psi(\bar{x}, \bar{a})$ or there exists $\bar{b} \in {}^{\ell(\bar{x})}|M'|$ and $N''' \geq N''$ in $K^*_{\mu_2}$ so that $N''' \Vdash \neg\psi(\bar{b}, \bar{a})$.

Proof of claim 1. Let $\psi \in \Lambda$. We show how to find $(M', N')$ satisfying the conclusion of the claim for $\psi$. Then since $|\Lambda| < \mu_1$, we can iterate the proof to get the full claim.

Suppose for a contradiction that the claim fails for $\psi$.

Build $\langle M_i : i \leq \theta^+ \rangle$, $\langle N_i : i \leq \theta^+ \rangle$ increasing continuous and $\langle \bar{a}_i : i < \theta^+ \rangle$, $\langle \bar{b}_i : i < \theta^+ \rangle$ such that for all $i < \theta^+$:

(1) $M_0 = M$, $N_0 = N$.
(2) $M_i \in K^*_{\mu_1}$, $N_i \in K^*_{\mu_2}$, $M_i \leq N_i$.
(3) $\bar{a}_i \in {}^{<\theta}|M_0|$.
(4) $\bar{b}_i \in {}^{<\theta}|M_{i+1}|$.
(5) $N_{i+1} \Vdash \neg\psi[\bar{b}_i, \bar{a}_i]$.
(6) For any $\bar{b} \in {}^{<\theta}|M_i|$ and any $N^* \geq N_{i+1}$ in $K^*_{\mu_2}$, $N^* \nVdash \neg\psi[\bar{b}, \bar{a}_i]$.

This is possible. The base case is already defined and at limits we take unions. For $j = i + 1$ a successor, we use that $(M_i, N_i)$ cannot satisfy the conclusion of the claim (where $(M_i, N_i)$ here stands for $(M', N')$ there). Therefore there exists $\bar{a}_i \in {}^{<\theta}|M_0|$ and $N'_{i+1} \geq N_i$ such that $N'_{i+1} \nVdash \forall \bar{x} \psi(\bar{x}, \bar{a}_i)$ and for any $\bar{b} \in {}^{<\theta}|M_i|$ and any $N^* \geq N'_{i+1}$, $N^* \nVdash \neg\psi[\bar{b}, \bar{a}_i]$ (we will choose $N_{i+1} \geq N'_{i+1}$, so this takes care of clause (6) by the monotonicity of forcing). Since $N'_{i+1} \nVdash \forall \bar{x} \psi(\bar{x}, \bar{a}_i)$, there must exist $N_{i+1} \geq N'_{i+1}$ in $K^*_{\mu_2}$ and $\bar{b}_i \in {}^{<\theta}|N_{i+1}|$ so that $N_{i+1} \Vdash \neg\psi[\bar{b}_i, \bar{a}_i]$. Now pick any $M_{i+1} \leq N_{i+1}$ in $K^*_{\mu_1}$ which contains $\bar{b}_i$ and $M_i$.

This is enough. For each $i < \theta^+$, let $\alpha(i)$ be the least $\alpha < \theta^+$ such that $\bar{a}_i \in {}^{<\theta}|M_i|$. When $\mathrm{cf}(i) \geq \theta$, we must have that $\alpha(i) < i$. Therefore by Fodor's lemma there exists a stationary set $S \subseteq \theta^+$ of points of cofinality $\theta$ and $\alpha_0 < \theta^+$ such that for any $i \in S$, $\alpha(i) = \alpha_0$, i.e. $\bar{b}_i \in {}^{<\theta}|M_{\alpha_0}|$. However we must have that $\bar{b}_i \notin {}^{<\theta}|M_i|$ (otherwise $N'_i := N_{i+1}$ would be a counterexample to (6)). This is a contradiction, so the claim must be true.

Claim 2. There exists $N' \geq N$ in $K^*_{\mu_2}$, $M' \geq M$ in $K^*_{\mu_1}$ such that for any $\psi(\bar{x}, \bar{y}) \in \Lambda$, any $\bar{a} \in {}^{\ell(\bar{y})}|M|$, either $N' \Vdash \forall \bar{x} \psi(\bar{x}, \bar{a})$ or there exists $\bar{b} \in {}^{\ell(\bar{x})}|M'|$ so that $N' \Vdash \neg\psi(\bar{b}, \bar{a})$.

Proof of claim 2. Let $(M', N')$ be as given by Claim 1.

Let $\psi \in \Lambda$. As before, we show how to find $(M'', N'')$ satisfying the conclusion of the claim for $\psi$. Then since $|\Lambda| < \mu_1$, we can iterate the proof to get the full claim.

Let $\chi := \mu_1^{<\theta}$. Note that $\chi \leq \mu_2$. Write $|M| = \{\bar{a}_i : i < \chi\}$. Build $\langle N_i : i \leq \chi \rangle$ increasing continuous such that for all $i < \chi$:

(1) $N_0 = N'$.
(2) $N_i \in K^*_{\mu_2}$.
(3) Either $N_{i+1} \Vdash \forall \bar{x} \psi(\bar{x}, \bar{a}_i)$, or there exists $\bar{b} \in {}^{\ell(\bar{x})}|M'|$ such that $N_{i+1} \Vdash \neg\psi[\bar{b}, \bar{a}_i]$.

This is possible. The base and limit cases are already given. Given $j = i + 1$, if $N_i \Vdash \forall \bar{x} \psi(\bar{x}, \bar{a}_i)$, take $N_{i+1} = N_i$. Otherwise by definition of $(M', N')$, there must exist $\bar{b} \in {}^{\ell(\bar{x})}|M'|$ and $N_{i+1} \geq N_i$ in $K^*_{\mu_2}$ such that $N_{i+1} \Vdash \neg\psi[\bar{b}, \bar{a}_i]$.

This is enough: take $M'' := M'$, $N'' := N_{\theta^+}$. They are as required by monotonicity of forcing.

Claim 3. There exists $N' \geq N$ in $K^*_{\mu_2}$, $M' \geq M$ in $K^*_{\mu_1}$ such that for any $\psi(\bar{x}, \bar{y}) \in \Lambda$, any $\bar{a} \in {}^{\ell(\bar{y})}|M'|$ (in the previous claim, $M'$ was $M$), either $N' \Vdash \forall \bar{x} \psi(\bar{x}, \bar{a})$ or there exists $\bar{b} \in {}^{\ell(\bar{x})}|M'|$ so that $N' \Vdash \neg\psi(\bar{b}, \bar{a})$.

Proof of claim 3. We build $\langle M_i : i \leq \theta^+ \rangle$, $\langle N_i : i \leq \theta^+ \rangle$ increasing continuous such that for all $i < \theta^+$:

(1) $M_0 = M$, $N_0 = N$.
(2) $M_i \in K^*_{\mu_1}$, $N_i \in K^*_{\mu_2}$, $M_i \leq N_i$.
(3) For any $\psi(\bar{x}, \bar{y}) \in \Lambda$ and any $\bar{a} \in {}^{\ell(\bar{y})}|M_i|$, either $N_{i+1} \Vdash \forall \bar{x} \psi(\bar{x}, \bar{a})$ or there exists $\bar{b} \in {}^{\ell(\bar{x})}|M_{i+1}|$ so that $N_{i+1} \Vdash \neg\psi(\bar{b}, \bar{a})$.

This is possible by Claim 2. This is enough: $M_{\theta^+}$, $N_{\theta^+}$ are as desired.

Now we claim that $(M', N')$ as given by Claim 3 is $\Lambda$-generic. Let $\phi \in \Lambda$, $\bar{a} \in {}^{<\theta}|N'|$. By induction on $\phi$ we prove that $N' \Vdash \phi[\bar{a}]$ if and only if $M' \models \phi[\bar{a}]$. When $\phi$ is atomic this is clear and if $\phi$ is a conjunction this follows straight from the induction hypothesis. If $\phi = \neg\psi$, note that (taking $\ell(\bar{x}) = 0$ in the statement of Claim 3), either $N' \Vdash \psi[\bar{a}]$ or $N' \Vdash \neg\psi[\bar{a}]$. Thus the result follows from the induction hypothesis too.

If $\phi = \forall \bar{x} \psi(\bar{x}, \bar{y})$, assume $M' \models \phi[\bar{a}]$. Then if $N' \nVdash \phi[\bar{a}]$, there must exist by assumption a $\bar{b} \in {}^{<\theta}|N'|$ such that $N' \Vdash \neg\psi[\bar{b}, \bar{a}]$. By induction this means that $M' \models \neg\psi[\bar{b}, \bar{a}]$ hence $M' \models \neg\phi[\bar{a}]$, contradiction. Similarly, if $M' \not\models \phi[\bar{a}]$, let $\bar{b} \in M'$ witness it and use the induction hypothesis to conclude $N' \Vdash \neg\psi[\bar{b}, \bar{a}]$. By Proposition 3.4, we cannot have $N' \Vdash \phi[\bar{a}]$. Thus, $N' \nVdash \phi[\bar{a}]$. $\square$

As in [She09, Claim IV.2.12], we get the following:

**Lemma 3.23.** Let $\Lambda$ be a fragment of $L^*_{\infty,\theta}$ for $\theta$ a regular cardinal. Let $\mu_1$ and $\mu_2$ be cardinals. Assume:

(1) $\mathrm{LS}(K) + |\Lambda| + \theta < \mu_1$.
(2) $\mu_1^{<\theta} \leq \mu_2$.
(3) $\mu_1$ and $\mu_2$ are pseudo-solvability cardinals.
(4) $\mathrm{LS}_{\mathcal{L}(\Lambda),\mu_2}(K^*) \leq \mu_1$.
(5) For $\ell = 1, 2$, $M \in K^*_{\mu_\ell}$, $\phi \in \Lambda$, $M \Vdash_{K^*_{\mu_\ell}} \phi[\bar{a}]$ if and only if $M \Vdash_{K^*} \phi[\bar{a}]$.

Then any model in $K^*_{\mu_1}$ is $\Lambda$-generic.

*Proof.* We build increasing continuous $\langle M_i : i \leq \theta^+ \rangle$, $\langle N_i : i \leq \theta^+ \rangle$ and increasing $\langle M'_i : i < \theta^+ \rangle$ such that for all $i < \theta^+$:

(1) $M_i, M'_i \in K^*_{\mu_1}$, $N_i \in K^*_{\mu_2}$.
(2) $M_i \leq M'_i \leq M_{i+1}$.
(3) $M'_i \leq N_i$.
(4) $(M_{i+1}, N_{i+1})$ is $\Lambda$-generic.
(5) $M'_i \preceq_{\mathcal{L}(\Lambda)} N_i$.

This is possible. Let $M_0 \leq N_0$ be arbitrary such that $M_0 \in K^*_{\mu_1}$, $N_0 \in K^*_{\mu_2}$. For $i$ limit, let $M_i = \bigcup_{j<i} M_j$, $N_i = \bigcup_{j<i} N_j$. Now given $M_i, N_i$, we use the downward Löwenheim-Skolem property to find $M'_i \leq N_i$ containing $M_i$ such that $M'_i \preceq_{\mathcal{L}(\Lambda)} N_i$ and $M_i \in K^*_{\mu_1}$. Then we use Lemma 3.22 to find $(M_{i+1}, N_{i+1})$ $\Lambda$-generic extending $(M'_i, N_i)$.

This is enough. We claim that $M_{\theta^+} \in K^*_{\mu_1}$ is $\Lambda$-generic (this suffices by pseudo-solvability in $\mu_1$). To see this, first observe that for any $\phi \in \Lambda$ and any $\bar{a} \in {}^{<\theta}|M_{\theta^+}|$, either $M_{\theta^+} \Vdash \phi[\bar{a}]$ or $M_{\theta^+} \Vdash \neg\phi[\bar{a}]$. This is because there exists $i < \theta^+$ such that $\bar{a} \in {}^{<\theta}|M_i|$, and by genericity either $M_{i+1} \Vdash \phi[\bar{a}]$ or $M_{i+1} \Vdash \neg\phi[\bar{a}]$, so by monotonicity the same holds for $M_{\theta^+}$.

Now we prove by induction on $\phi \in \Lambda$ that $M_{\theta^+} \Vdash \phi[\bar{a}]$ if and only if $M_{\theta^+} \models \phi[\bar{a}]$. For $\phi$ atomic, this holds by definition, and for $\phi$ a conjunction we use the induction hypothesis. If $\phi = \neg\psi$, assume $M_{\theta^+} \Vdash \phi[\bar{a}]$, then $M_{\theta^+} \not\Vdash \psi[\bar{a}]$ so by induction $M_{\theta^+} \models \phi[\bar{a}]$. Conversely, if $M_{\theta^+} \not\Vdash \phi[\bar{a}]$, then by the observation above $M_{\theta^+} \Vdash \psi[\bar{a}]$ so $M_{\theta^+} \models \psi[\bar{a}]$ by induction.

If $\phi(\bar{y}) = \forall \bar{x} \psi(\bar{x}, \bar{y})$, assume $M_{\theta^+} \not\models \phi[\bar{a}]$. Then there exists $\bar{b} \in {}^{<\theta}|M_{\theta^+}|$ such that $M_{\theta^+} \models \neg\psi[\bar{b}, \bar{a}]$. By induction, $M_{\theta^+} \Vdash \neg\psi[\bar{b}, \bar{a}]$, hence (by Proposition 3.4) $M_{\theta^+} \not\Vdash \phi[\bar{a}]$. Conversely, assume $M_{\theta^+} \not\Vdash \phi[\bar{a}]$. By the argument above, there exists $i < \theta^+$ such that $M_i \Vdash \neg\phi[\bar{a}]$. By monotonicity, $M'_i \Vdash \neg\phi[\bar{a}]$. By construction, $N_i \Vdash \neg\phi[\bar{a}]$. By monotonicity again, $N_{i+1} \Vdash \neg\phi[\bar{a}]$. By genericity, $M_{i+1} \models \neg\phi[\bar{a}]$. Thus there exists $\bar{b} \in {}^{<\theta}|M_{i+1}|$ such that $M_{i+1} \models \neg\psi[\bar{b}, \bar{a}]$. By genericity, $N_{i+1} \Vdash \neg\psi[\bar{b}, \bar{a}]$. Thus by elementarity $M'_{i+1} \Vdash \neg\psi[\bar{b}, \bar{a}]$, so by monotonicity also $M_{\theta^+} \Vdash \neg\psi[\bar{b}, \bar{a}]$. By induction, $M_{\theta^+} \models \neg\psi[\bar{b}, \bar{a}]$, so $M_{\theta^+} \models \neg\phi[\bar{a}]$, so $M_{\theta^+} \not\models \phi[\bar{a}]$. $\square$

We are almost ready to prove the main result of this section. The final point is that the set of $L_{\infty,\theta}$ is not a set so we cannot apply Theorem 3.14 with $\Lambda = L_{\infty,\theta}$. However $L_{(2^\theta)^+,\theta}$ *is* a set and the proof of Scott's isomorphism theorem gives:

**Fact 3.24** (Corollary 5.3.33 in [Dic75] or Claim IV.2.8.(2) in [She09])**.** If $M \preceq_{L^*_{(2^\theta)^+,\theta}} N$, then $M \preceq_{L_{\infty,\theta}} N$.

Recall that there is a global hypothesis of pseudo-solvability in unboundedly many cardinals in this section.

**Theorem 3.25.** For any $\theta$ there exists $\mu_0(\theta)$ such that if $\mu \geq \mu_0(\theta)$ is a pseudo-solvability cardinal and $M, N \in K^*_\mu$, then $M \leq N$ implies $M \preceq_{L_{\infty,\theta}} N$.

*Proof.* By replacing $\theta$ by $\theta^+$ if necessary, we can assume without loss of generality that $\theta$ is regular. Let $\Lambda := L^*_{(2^\theta)^+,\theta}$. Let $\mu_0(\theta) := \mu_0(\Lambda) + (\mathrm{LS}(K) + |\Lambda| + \theta)^+$, where $\mu_0(\Lambda)$ is as given by Definition 3.15. Its existence is justified by Theorem 3.14. Let $\mu \geq \mu_0(\theta)$ be a pseudo-solvability cardinal. By Lemma 3.17, forcing in $K^*_\mu$ and forcing in $K^*$ (for formulas in $\Lambda$) coincide. By Lemma 3.23 (with $\mu_1$ there standing for $\mu$ here and $\mu_2$ taken to be a pseudo-solvability cardinal such that $\mu_1^{<\theta} \leq \mu_2$), any model in $K^*_\mu$ is $\Lambda$-generic. By Lemma 3.21, $M \leq N$ implies $M \preceq_\Lambda N$ which by Fact 3.24 implies $M \preceq_{L_{\infty,\theta}} N$. $\square$

## 4. Final remarks

In [Kue08, Theorem 7.4], Kueker proved:

**Fact 4.1.** Let $K$ be an AEC with amalgamation, joint embedding, and no maximal models. Let $\kappa := \mathrm{LS}(K)$. Let $\lambda \geq \mathrm{LS}(K)$ be a cardinal such that $\mathrm{cf}(\lambda) > \kappa$. Then there is a complete $\phi \in L_{(2^\kappa)^+,\kappa^+}$ such that $K_{\geq\lambda} = (\mathrm{Mod}(\phi))_{\geq\lambda}$.

Assuming categoricity in unboundedly many cardinals, we can improve Kueker's result in many respects: we do not need to assume amalgamation, joint embedding, or no maximal models and can get rid of the cofinality assumption. Moreover we can say what the ordering on $K$ looks like. This is Theorem 0.1 from the abstract:

**Theorem 4.2.** If $K$ is an AEC categorical in unboudedly many cardinals with $\mathrm{LS}(K) = \kappa$, then there exists a complete $\phi \in L_{(2^\kappa)^+,\kappa^+}$ and $\lambda$ such that $K_{\geq \lambda} = (\mathrm{Mod}(\phi))_{\geq \lambda}$ and for $M, N \in K_{\geq \lambda}$, $M \leq N$ if and only if $M \preceq_{L_{\infty,\kappa^+}} N$.

*Proof.* Let the map $\theta \mapsto \mu_0(\theta)$ be as given by Theorem 3.25. Let $\lambda \geq \mu_0(\kappa^+) + \mathrm{LS}(K)$ be such that $K$ is categorical in $\lambda$. Note that $K$ is also pseudo-solvable in $\lambda$ so by Theorem 3.25, $M, N \in K_\lambda$ and $M \leq N$ implies $M \preceq_{L_{\infty,\kappa^+}} N$. Now if $M, N \in K_{\geq \lambda}$ and $M \leq N$, write $M$ and $N$ as a directed system of models in $K_\lambda$ indexed by the set ${}^\kappa\|N\|$. Then it is easy to check[9] that $M \preceq_{L_{\infty,\kappa^+}} N$.

Now by Shelah [She09, Claim IV.1.10.(4)] or Kueker [Kue08, Theorem 7.2.(a)]), we have that if $M \in K$ and $M \equiv_{L_{\infty,\kappa^+}} N$, then $N \in K$ (this does not need the categoricity assumption). So let $T$ be the complete $L_{\infty,\kappa^+}$-theory of any $M \in K_\lambda$ (it does not matter which by categoricity), and let $\phi \in L_{(2^\kappa)^+,\kappa^+}$ code it (see Fact 3.24). If $N \in K_{\geq \lambda}$, find $N_0 \leq N$ with $N_0 \in K_\lambda$. We know that $N_0 \preceq_{L_{\infty,\kappa^+}} N$ and $N_0 \models T$ so $N \models T$.

As before (see Shelah [She09, Claim IV.1.10.(3)] or Kueker [Kue08, Theorem 7.2.(b)]), if $M \preceq_{L_{\infty,\kappa^+}} N$ and $N \in K$, then $M \in K$ and $M \leq N$. This completes the proof. $\square$

4.1. **Shelah's amalgamation theorem.** In Chapter IV of his book on AECs [She09], Shelah proves, assuming that $M \leq N$ implies $M \preceq_{L_{\infty,\theta}} N$ for a suitable $\theta$, that categoricity in a high-enough cardinal implies amalgamation in a specific cardinal below the categoricity cardinal:

**Fact 4.3** (Shelah's amalgamation theorem)**.** Let $K$ be an AEC and let $\lambda, \mu$ be cardinals such that $\mathrm{LS}(K) < \lambda = \beth_\lambda < \mu$ and $\mathrm{cf}(\lambda) = \aleph_0$. Assume that there exists an EM blueprint $\Phi$ such that $(K, \Phi)$ is pseudo $(\mu, \mathrm{LS}(K))$-solvable (see Definition 3.11).

Assume further that for every $\theta \in [\mathrm{LS}(K), \lambda)$, and every $M, N \in K^*_\Phi$ (see Definition 3.10) of size $\mu$, $M \leq N$ implies $M \preceq_{L_{\infty,\theta}} N$. Then:

(1) ([She09, Theorem IV.1.30]) $K^*_\Phi$ has disjoint amalgamation in $\lambda$.
(2) ([She09, Theorem IV.4.10]) If in addition there exists an increasing sequence of cardinals $\langle \lambda_n : n < \omega \rangle$ such that:
   (a) $\sup_{n<\omega} \lambda_n = \lambda$.
   (b) For all $n < \omega$, $\lambda_n = \beth_{\lambda_n}$ and $\mathrm{cf}(\lambda_n) = \aleph_0$.
   
   Then there is a type-full good $\lambda$-frame with underlying class $(K^*_\Phi)_\lambda$.

[9]The argument in a more general setup will appear in [BGL+].

With similar methods, Shelah also proves a categoricity transfer. This is given by (the proof of) [She09, Observation IV.3.5]:

**Fact 4.4.** Let $K$ be an AEC and let $\langle \lambda_n : n \leq \omega \rangle$ be an increasing continuous sequence of cardinals such that for all $n < \omega$:

(1) $\text{LS}(K) < \lambda_\omega = \beth_{\lambda_\omega}$.
(2) $K$ is categorical in $\lambda_n$.
(3) For every $\theta < \lambda_n$, and every $M, N \in K_{\geq \lambda_{n+1}}$, if $M \leq N$, then $M \preceq_{L_{\infty,\theta}} N$.

Then $K$ is categorical in $\lambda_\omega$.

Combining these two facts with Theorem 4.2, we obtain Theorem 0.2 from the abstract. The argument already appears in [Vasb, Fact 4.9]:

**Theorem 4.5.** Let $K$ be an AEC categorical in unboundedly many cardinals. Set $S$ to be the class of cardinals $\lambda$ such that:

(1) $\text{LS}(K) < \lambda = \beth_\lambda$ and $\text{cf}(\lambda) = \aleph_0$;
(2) $K$ is categorical in $\lambda$; and
(3) there is a type-full good $\lambda$-frame with underlying class $K_\lambda$ (in particular, $K$ has amalgamation in $\lambda$).

Then $S$ is stationary.

*Proof.* For each cardinal $\theta$, let $\mu_0(\theta)$ be the least cardinal $\mu > \theta$ such that $K$ is categorical in $\mu$ and for each $M, N \in K_{\geq \mu}$, $M \leq N$ implies $M \preceq_{L_{\infty,\theta}} N$. Note that $\mu_0(\theta)$ exists by Theorem 4.2.

Let $C$ be a closed unbounded class of cardinals.

Build $\langle \lambda_i : i \leq \omega \cdot \omega \rangle$ and $\langle \lambda_i' : i \leq \omega \cdot \omega \rangle$ increasing continuous such that for all $i \leq \omega \cdot \omega$:

(1) $\lambda_i' \in C$.
(2) For all $j < i$, $\lambda_j < \lambda_i' \leq \lambda_i$.
(3) $\lambda_0 > \text{LS}(K)$
(4) $K$ is categorical in $\lambda_i$
(5) $\lambda_{i+1} > \mu_0(\beth_{\lambda_i})$.

This is possible: when $i$ is a zero or a successor, we pick $\lambda_i' \in C$ such that $\lambda_j < \lambda_i'$ for all $j < i$ (this is possible as $C$ is unbounded). Then we pick a categoricity cardinal $\lambda_i \geq \lambda_i'$ strictly above $\text{LS}(K)$ (and, if $i = j + 1$ also strictly above $\mu_0(\lambda_j)$). For $i$ limit, let $\lambda_i := \lambda_i' := \sup_{j<i} \lambda_j$. Note that $\lambda_i' \in C$ as $C$ is closed. Also, $\beth_{\lambda_i} = \lambda_i$ and $\text{cf}(\lambda_i) = \aleph_0$. Therefore by Fact 4.4, $K$ is categorical in $\lambda_i$. This is enough: let $\lambda := \lambda_{\omega \cdot \omega}$. Since

$C$ is club, $\lambda \in C$. Also, $\lambda$ is a limit of fixed points of the beth function of cofinality $\aleph_0$. Moreover $K$ is categorical (and hence pseudo solvable) in some $\mu \geq \mu_0(\lambda)$, so by Fact 4.3 we obtain that $\lambda \in S$. $\square$

## References


[Bal09] John T. Baldwin, *Categoricity*, University Lecture Series, vol. 50, American Mathematical Society, 2009.

[BFB85] Jon Barwise, Solomon Feferman, and Andreas Baudisch (eds.), *Model-theoretic logics*, Perspectives in mathematical logic, Springer-Verlag, 1985.

[BGL$^+$] Will Boney, Rami Grossberg, Michael Lieberman, Jiří Rosický, and Sebastien Vasey, *$\mu$-Abstract elementary classes*, In preparation.

[Dic75] M. A. Dickmann, *Large infinitary languages*, Studies in logic and the foundations of mathematics, vol. 83, North-Holland, 1975.

[Gro] Rami Grossberg, *A course in model theory I*, A book in preparation.

[Gro02] ———, *Classification theory for abstract elementary classes*, Contemporary Mathematics **302** (2002), 165–204.

[GV] Rami Grossberg and Sebastien Vasey, *Superstability in abstract elementary classes*, Preprint. URL: `http://arxiv.org/abs/1507.04223v2`.

[Hod06] Wilfrid Hodges, *Building models by games*, Dover, 2006.

[HS90] Bradd Hart and Saharon Shelah, *Categoricity over $P$ for first order $T$ or categoricity for $\phi \in L_{\omega_1,\omega}$ can stop at $\aleph_k$ while holding for $\aleph_0, \dots, \aleph_{k-1}$*, Israel Journal of Mathematics **70** (1990), 219–235.

[Jec03] Thomas Jech, *Set theory*, 3rd ed., Springer-Verlag, 2003.

[Kue08] David W. Kueker, *Abstract elementary classes and infinitary logics*, Annals of Pure and Applied Logic **156** (2008), 274–286.

[She] Saharon Shelah, *A.E.C. with not too many models*, Preprint. URL: `http://arxiv.org/abs/1302.4841v2`.

[She90] ———, *Classification theory and the number of non-isomorphic models*, 2nd ed., Studies in logic and the foundations of mathematics, vol. 92, North-Holland, 1990.

[She99] ———, *Categoricity for abstract classes with amalgamation*, Annals of Pure and Applied Logic **98** (1999), no. 1, 261–294.

[She09] ———, *Classification theory for abstract elementary classes*, Studies in Logic: Mathematical logic and foundations, vol. 18, College Publications, 2009.

[Vasa] Sebastien Vasey, *Infinitary stability theory*, Preprint. URL: `http://arxiv.org/abs/1412.3313v3`.

[Vasb] ———, *Shelah's eventual categoricity conjecture in universal classes*, Preprint. URL: `http://arxiv.org/abs/1506.07024v4`.



*E-mail address*: `wboney@math.harvard.edu`

*URL*: `http://math.harvard.edu/~wboney/`

Department of Mathematics, Harvard University, Cambridge, Massachusetts, USA

*E-mail address*: sebv@cmu.edu

*URL*: http://math.cmu.edu/~svasey/

Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, Pennsylvania, USA