INFINITARY STABILITY THEORY

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Abstract. We introduce a new device in the study of abstract elementary classes: Galois Morleyization, which consists in expanding the models of the class with a relation for every Galois type of length less than a fixed cardinal κ. We show:

Theorem 0.1 (The semantic-syntactic correspondence)**.** An AEC K is fully $(< \kappa)$-tame and type short if and only if Galois types are syntactic in the Galois Morleyization.

We use this idea to make progress on the stability theory of tame and type short abstract elementary classes. We also prove strong structural results on good frames, an axiomatization of forking introduced by Shelah. The main theorems are:

Theorem 0.2 (Superstability from categoricity)**.** Let K be a $(< \kappa)$-tame AEC with amalgamation and joint embedding. If $\kappa = \beth_\kappa > \text{LS}(K)$ and K is categorical in a $\lambda > \kappa$, then:

- K is stable in all cardinals $\geq \text{LS}(K)$.
- K is categorical in κ.
- There is a type-full good λ-frame with underlying class K.

Theorem 0.3 (A global independence notion from categoricity)**.** Let K be a fully tame and type short AEC with amalgamation. If K is categorical in unboundedly-many cardinals, then there exists $\lambda \geq \text{LS}(K)$ such that $K_{\geq \lambda}$ admits a global independence relation with the properties of forking in a superstable first-order theory.

Contents

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1. Introduction

This paper has three parts (each of which has an independent introduction). The main result of the paper is proven in part 3 as Theorem 20.1. Here, we state it quite loosely:

Theorem 1.1. If K is a fully tame and type short abstract elementary class with amalgamation and K is categorical in unboundedly many cardinals, then for some $\lambda \geq \mathrm{LS}(K)$, $K_{\geq\lambda}$ admits a global independence notion with the properties of forking in a superstable first-order theory.

In the first two parts, we move toward this main theorem but we do not hurry and enjoy the view, proving several new results on the way.

A word about the framework of abstract elementary classes[1]. This was introduced by Shelah in [She87a] and encompasses classes of models of an $L_{\lambda,\omega}$ sentence (in particular, classes of models of a first-order theory), and even classes of models of an $L_{\omega_1,\omega}(Q)$ sentence, where Q is the quantifier "there exists uncountably many". There are many other examples, see [Gro02]. One can ask how general the assumptions of our main theorem are. For amalgamation, tameness, and type shortness, this is discussed in [BGa, p. 16]. Categoricity in unboundedly-many cardinals is not actually necessary and we can replace it with "categoricity in a high-enough cardinal", or with a superstability-like assumption (but then we have to restrict ourselves to sufficiently saturated models). Note that by arguments in [MS90] (adapted to general AECs in [Bonc]), all those properties follow from large cardinals and categoricity.

As to how AECs compare to other non first-order framework like homogeneous model theory (see [She70]), note that there is an example (due to Marcus, see [Mar72]) of an $L_{\omega_1,\omega}$-axiomatizable class which is categorical in all uncountable cardinals but does not have an $\aleph_1$-sequentially-homogeneous model. For $n < \omega$, an example due to Hart and Shelah (see [HS90, BK09]) has amalgamation and no maximal models and is categorical in all $\aleph_k$ with $k \leq n$, but no higher. By [GV06b], the example cannot be $\aleph_k$-tame for $k < n$. However if κ is a strongly compact cardinal, the example will be fully $(< \kappa)$-tame and type short by the main result of [Bonc]. The discussion on p. 74 of [Bal09] gives more non-homogeneous examples.

In general, classes from homogeneous model theory or quasiminimal pregeometry classes (see [Kir10]) are special cases of AECs that are always fully $(< \aleph_0)$-tame and type short, while finitary AECs (see [HK06]) are in our opinion also very close to being fully $(< \aleph_0)$-tame and type short (see for example [HK06, Theorem 4.11]). In this paper we work with the much more general assumption of $(< \kappa)$-tameness and type shortness for a possibly uncountable κ.

This paper was written while working on a Ph.D. thesis under the direction of Rami Grossberg at Carnegie Mellon University and I would like to thank Professor Grossberg for his guidance and assistance in my research in general and in this work specifically. I also thank Will Boney

[1] See the next section for the precise definition of the terms used in this discussion.

and Alexei Kolesnikov for valuable discussions on the idea of thinking of Galois types as formulas.

2. Preliminaries

We review some of the basics of abstract elementary classes and fix some notation. The reader is advised to skim through this section quickly and go back to it as needed.

2.1. Set theoretic terminology.

Notation 2.1. When we say that $\mathcal{F}$ *is an interval of cardinals*, we mean that $\mathcal{F} = [\lambda, \theta)$, the set of cardinals μ such that $\lambda \leq \mu < \theta$. Here, $\lambda \leq \theta$ are (possibly finite) cardinals but we allow $\theta = \infty$.

Notation 2.2. For κ an infinite cardinal, let κ_r denote the least regular cardinal above κ. That is, κ_r is κ if κ is regular or κ^+ otherwise.

We will often use the following function:

Definition 2.3 (Hanf function)**.** For λ an infinite cardinal, define $h(\lambda) := \beth_{(2^\lambda)^+}$.

Definition 2.4 (Iterated Hanf function)**.** Let α be an ordinal, λ an infinite cardinal. Define inductively on α:

$$h_\alpha(\lambda) := \begin{cases} \lambda & \text{if } \alpha = 0 \\ h(h_\beta(\lambda)) & \text{if } \alpha = \beta + 1 \\ \sup_{\beta < \alpha} h_\beta(\lambda) & \text{if } \alpha \text{ is limit} \end{cases}$$

Note that $h_1(\lambda) = h(\lambda)$. Also for λ infinite, $\lambda = \beth_\lambda$ if and only if for all $\mu < \lambda$, $h(\mu) < \lambda$ if and only if $\lambda = h_\delta(\aleph_0)$, for δ a limit ordinal.

2.2. **Syntax.** Our notation is standard, but since we will work with infinitary objects and need to be quite precise, we review the basics. We will often work with the logic $L_{\kappa,\kappa}$, see [Dic75] for the definition and basic results.

Definition 2.5. An *infinitary language* is a language where we also allow relation and function symbols of infinite arity. For simplicity, we require the arity to be an ordinal. An infinitary language is $(< \kappa)$*-ary* if all its symbols have arity strictly less than κ. A *finitary language* is a $(< \aleph_0)$-ary language.

For L an infinitary language, ϕ an $L_{\kappa,\kappa}$-formula and $\bar{x}$ a sequence of variables, we write $\phi = \phi(\bar{x})$ to emphasize that the free variables of ϕ appear among $\bar{x}$ (recall that a $L_{\kappa,\kappa}$-formula must have fewer than κ-many free variables, but we allow $\ell(\bar{x}) \geq \kappa$). We use a similar notation for sets of formulas. When $\bar{a}$ is an element in some L-structure and $\phi(\bar{x}, \bar{y})$ is a formula, we often abuse notation and say that $\psi(\bar{x}) = \phi(\bar{x}, \bar{a})$ is a formula (again, we allow $\ell(\bar{a}) \geq \kappa$). We say $\phi(\bar{x}, \bar{a})$ is a *formula over A* if $\bar{a} \in {}^{<\infty}A$.

Definition 2.6. Let M be an L-structure and p be a set of formulas over M, $A \subseteq |M|$. p is *$(< \mu)$-consistent over A in M* (or *$(< \mu)$-satisfiable over A in M*) if for any $p_0 \subseteq p$ with $|p_0| < \mu$, there exists $\bar{b} \in {}^{\ell(p)}A$ such that $M \models p_0[\bar{b}]$ (that is, $M \models \phi[\bar{b}]$ for all $\phi \in p_0$). When $\mu = |p|^+$ or $A = |M|$, we omit them. When M is clear from context, we also omit it.

Definition 2.7. For ϕ a formula over a set, let $\mathrm{FV}(\phi)$ denote the set of free variables of ϕ (that is, the smallest sequence $\bar{x}$ such that $\phi = \phi(\bar{x})$), $\ell(\phi) := \ell(\mathrm{FV}(\phi))$, and $\mathrm{dom}(\phi)$ be the smallest set A such that ϕ is over A. Define similarly the meaning of $\mathrm{FV}(p)$, $\ell(p)$, and $\mathrm{dom}(p)$ on a set p of formulas.

Definition 2.8. For L an infinitary language, M an L-structure, $A \subseteq |M|$ and $\bar{b} \in {}^{<\infty}|M|$, and Δ a set of L-formulas (in some logic), let[2]

$$\mathrm{tp}_\Delta(\bar{b}/A; M) := \{\phi(\bar{x}; \bar{a}) \mid \phi(\bar{x}, \bar{y}) \in \Delta \text{ and } M \models \phi[\bar{b}, \bar{a}]\}$$

The two examples we will work with are $\Delta = L_{\kappa,\kappa}$, and $\Delta = qL_{\kappa,\kappa}$, the set of *quantifier-free* $L_{\kappa,\kappa}$-formulas. In fact, we will more often than not work with quantifier-free formulas and so we may forget to say a formula is quantifier-free if it is clear from context. When κ and L are also clear from context, we write $\mathrm{tp}(\bar{b}/A; M)$ for $\mathrm{tp}_{qL_{\kappa,\kappa}}(\bar{b}/A; M)$.

Definition 2.9. For M an L-structure, Δ a set of L-formulas, $A \subseteq |M|$, α an ordinal or ∞, let

$$\mathrm{S}_\Delta^{<\alpha}(A; M) := \{\mathrm{tp}_\Delta(\bar{b}/A; M) \mid \bar{b} \in {}^{<\alpha}|M|\}$$

Define similarly the variations for $\leq \alpha$, α, etc. When we just write S, we mean $\mathrm{S}^1_{qL_{\kappa,\kappa}}$, and similarly if we write only one of the two parameters.

[2]Of course, we have in mind a canonical sequence of variables $\bar{x}$ of order types $\ell(\bar{b})$ that should really be part of the notation but (as is customary) we always omit this detail.

2.3. **Abstract classes.** We review the definition of an abstract elementary class. Abstract elementary classes (AECs) were introduced by Shelah in [She87a]. The reader unfamiliar with AECs should consult [Gro02] for an introduction.

We first define more general objects that we will sometimes use. Abstract classes are already defined in [Gro], while we first heard about μ-abstract elementary classes in a seminar of Will Boney (presenting joint work with Rami Grossberg) at Carnegie Mellon University in Spring 2013. They should be introduced in [BGb]. We will mostly use them to deal with classes of saturated models of an AEC.

We also talk about an AEC in $\mathcal{F}$ for $\mathcal{F}$ an interval of cardinal. This is convenient when dealing with good frames and appears already (for $\mathcal{F} = \{\lambda\}$) in [JS13, Definition 1.0.3.2]. Confusingly, Shelah calls an AEC in λ a λ-AEC in [She09a, Definition II.1.18].

Definition 2.10. An *abstract class* (AC for short) is a pair $(K, \leq)$, where:

(1) K is a class of L-structure, for some fixed infinitary language L (that we will denote by $L(K)$).
(2) $\leq$ is a partial order (that is, a reflexive and transitive relation) on K.
(3) If $M \leq N$ are in K and $f : N \cong N'$, then $f[M] \leq N'$ and both are in K.
(4) If $M \leq N$, then $M \subseteq N$.

Remark 2.11. We do not always strictly distinguish between K and $(K, \leq)$.

Notation 2.12. For K an abstract class, $M, N \in K$, we write $M < N$ when $M \leq N$ and $M \neq N$.

Notation 2.13. For K an abstract class, $\mathcal{F}$ an interval of cardinals, we write $K_{\mathcal{F}} := \{M \in K \mid \|M\| \in \mathcal{F}\}$. When $\mathcal{F} = \{\lambda\}$, we write K_λ for $K_{\{\lambda\}}$. We also use notation like $K_{\geq \lambda}$, $K_{<\lambda}$, etc.

Definition 2.14. An abstract class K is *in* $\mathcal{F}$ if $K_{\mathcal{F}} = K$.

Definition 2.15. For K an abstract class, μ a cardinal, define the *Löwenheim-Skolem number* $\text{LS}_{<\mu}(K)$ of K to be the least cardinal (could be ∞) $\lambda \geq |L(K)|$ such that $\lambda^{<\mu} = \lambda$ and for any $M \in K$ and any $A \subseteq |M|$, there exists $M_0 \leq M$ containing A with $\|M_0\| \leq |A|^{<\mu} + \lambda$. Usually μ is clear from context, so we may omit it.

Definition 2.16. Let K be an abstract class and let R be a binary relation on K. A sequence $\langle M_i : i < \delta \rangle$ of elements of K is *R-increasing* if for all $i < j < \delta$, $M_i R M_j$. When $R = \leq$, we omit it. *Strictly increasing* means $<$-increasing. $\langle M_i : i < \delta \rangle$ is *continuous* if for all limit $i < \delta$, $M_i = \bigcup_{j<i} M_j$.

Definition 2.17. For K an abstract class, θ a cardinal or ∞, define the *Tarski-Vaught number* $\text{TV}_{<\theta}(K)$ to be the least infinite cardinal (could be ∞) κ such that whenever $\theta > \delta = \text{cf}(\delta) \geq \kappa$, $\langle M_i : i < \delta \rangle$ is increasing in K, and $\delta < \theta$, $M_\delta := \bigcup_{i<\delta} M_i$ is such that:

(1) $M_\delta \in K$.
(2) $M_0 \leq M_\delta$.
(3) For any $N \in K$, if $M_i \leq N$ for all $i < \delta$, then $M_\delta \leq N$.

When $\theta = \infty$, we omit it.

Definition 2.18. For μ a cardinal or ∞, $\mathcal{F} = [\lambda, \theta)$ an interval of cardinals, we say an abstract class K in $\mathcal{F}$ is a *μ-abstract elementary class* (μ-AEC for short) *in* $\mathcal{F}$ if:

(1) Coherence: If M_0, M_1, M_2 are in K, $M_0 \leq M_2$, $M_1 \leq M_2$, and $|M_0| \subseteq |M_1|$, then $M_0 \leq M_1$.
(2) $L(K)$ is $(< \mu)$-ary[3].
(3) $\text{TV}_{<\theta}(K) \leq \mu$.
(4) If $\mu < \infty$, $\text{LS}_{<\mu}(K) < \infty$.

When $\mu = \aleph_0$, we omit it and when $\mathcal{F} = [0, \infty)$, we omit it.

Remark 2.19. Thus an AEC is an AC on a finitary language that satisfies coherence, $\text{LS}(K) = \text{LS}_{<\aleph_0}(K) < \infty$, and where every increasing chain has a least upper bound. On the other hand, an ∞-AEC is simply an AC with coherence.

In any abstract class, we can define a notion of embedding:

Definition 2.20. Let K be an AC. We say a function $f : M \to N$ is a *K-embedding* if $M, N \in K$ and $f : M \cong f[M] \leq N$. For $A \subseteq |M|$, we write $f : M \underset{A}{\to} N$ to mean that f fixes A pointwise. Unless otherwise stated, when we write $f : M \to N$ we mean that f is an embedding.

2.4. **Properties of abstract classes.** For reference, we give the definition of the following classical properties of abstract classes:

Definition 2.21. Let K be an abstract class.

(1) K has *no maximal models* if for any $M \in K$, there exists $N \in K$ with $M < N$.

[3]See Definition 2.5.

(2) K has *joint embedding* if for any $M_\ell \in K$, $\ell = 1, 2$, there exists $N \in K$ and $f_\ell : M_\ell \to N$.
(3) K has *amalgamation* if for any $M_0 \leq M_\ell$ in K, $\ell = 1, 2$, there exists $N \in K$ and $f_\ell : M_\ell \xrightarrow[M_0]{} N$.
(4) For λ a cardinal, let $I(K)$ denote the number of models in K up to isomorphism. We write $I(K, \lambda)$ for $I(K_\lambda)$. We say K is *categorical* if $I(K) = 1$. We say K is *categorical in* λ if K_λ is categorical, i.e. $I(K, \lambda) = 1$.

We may use the next facts without explicit mention.

Fact 2.22. Let K be an abstract class in $\mathcal{F} = [\lambda, \theta)$.

(1) If K_μ has no maximal models for all $\mu \in \mathcal{F}$, then K has no maximal models.
(2) If K is categorical in $\mu \in \mathcal{F}$, then K_μ has joint embedding.
(3) If every model in K contains a model of size λ, K has amalgamation, and K_λ has joint embedding, then K has joint embedding.
(4) If K is an AEC in $\mathcal{F} = [\lambda, \mu]$, $K_{<\mu}$ has no maximal models and K_μ has joint embedding, then K has joint embedding.
(5) If K is an AC in $\mathcal{F} = [\lambda, \theta)$, θ limit or ∞, and for all $\mu < \theta$, $K_{[\lambda,\mu]} \neq K$, then if K has joint embedding, K has no maximal models.
(6) If K is an AEC in $\mathcal{F}$ with $\lambda = \mathrm{LS}(K)$ and K_μ has amalgamation for all $\mu \in \mathcal{F}$, then K has amalgamation.

Proof. All are straightforward except perhaps the last one which is [She09a, Conclusion I.2.12] (see also [Gro] for a full proof). $\square$

Fact 2.23 (Lemma 16.14 in [Bal09])**.** Assume K has amalgamation. say $M \sim N$ if and only if M and N embed into a common model. This is an equivalence relation and each equivalence class is a sub-AC (see Definition 8.1) of K with joint embedding and amalgamation, and will be an AEC if K is (with the same Löwenheim-Skolem number).

2.5. **Galois types.** Let K be an abstract class. We define here a semantic notion of types for K. This was first introduced in [She87b, Definition II.1.9]. While Galois types are usually only defined over models, here we define them over sets. This is not harder and is often notationally convenient[4]. Note however that Galois types over sets are

[4]For example, types over the empty sets are used here in the definition of the Galois Morleyization. They appear implicitly in the definition of the order property in [She99, Definition 4.3] and explicitly in [GV06a, Notation 1.9].

in general not too well-behaved. For example, they can sometimes fail to have an extension if their domain is not an amalgamation base.

Definition 2.24.

(1) Let K^3 be the set of triples of the form $(\bar{b}, A, N)$, where $N \in K$, $A \subseteq |N|$, and $\bar{b}$ is a sequence of elements from N.
(2) For $(\bar{b}_1, A_1, N_1), (\bar{b}_2, A_2, N_2) \in K^3$, we say $(\bar{b}_1, A_1, N_1) E_{\text{at}} (\bar{b}_2, A_2, N_2)$ if $A := A_1 = A_2$, and there exists $f_\ell : N_\ell \underset{A}{\to} N$ such that $f_1(\bar{b}_1) = f_2(\bar{b}_2)$.
(3) Note that E_{at} is a symmetric and reflexive relation on K^3. We let E be the transitive closure of E_{at}.
(4) For $(\bar{b}, A, N) \in K^3$, let $\operatorname{gtp}(\bar{b}/A; N) := [(\bar{b}, A, N)]_E$. We call such an equivalence class a *Galois type*.
(5) For $p = \operatorname{gtp}(\bar{b}/A; N)$ a Galois type, define[5] $\ell(p) := \ell(\bar{b})$ and $\operatorname{dom}(p) := A$.
(6) We say a Galois types $p = \operatorname{gtp}(\bar{b}/A; N)$ is *algebraic* if $\bar{b} \in {}^{\ell(\bar{b})}A$ (it is easy to check this does not depend on the choice of representatives). We mostly use this when $\ell(p) = 1$.

We can go on to define the restriction of a type (if $A_0 \subseteq \operatorname{dom}(p)$, $I \subseteq \ell(p)$, we will write $p^I \restriction A_0$ when the realizing sequence is restricted to I and the domain is restricted to A_0), the image of a type under an isomorphism, or what it means for a type to be realized, see . Just as in [She09a, Observation II.1.11.4], we have:

Fact 2.25. If K has amalgamation, then $E = E_{\text{at}}$.

Note that the proof goes through, even though we are considering types over sets and not only over models.

Definition 2.26.

(1) Let $N \in K$, $A \subseteq |N|$, and α be an ordinal. Define:

$$\operatorname{gS}^\alpha(A; N) := \{\operatorname{gtp}(\bar{b}/A; N) \mid \bar{b} \in {}^\alpha|N|\}$$

(2) For $M \in K$ and α an ordinal, let:

$$\operatorname{gS}^\alpha(M) := \bigcup_{N \geq M} \operatorname{gS}^\alpha(M; N)$$

[5]It is easy to check that this does not depend on the choice of representatives.

(3) For α an ordinal, let:

$$\mathrm{gS}^\alpha(\emptyset) := \bigcup_{N \in K} \mathrm{gS}^\alpha(\emptyset; N)$$

When $\alpha = 1$, we omit it. Similarly define $\mathrm{gS}^{<\alpha}$, where α is allowed to be ∞.

Next, we recall the definition of tameness, a locality property of types. Tameness was introduced by Grossberg and VanDieren in [GV06a] and used to get an upward stability transfer (and an upward categoricity transfer in [GV06b]). Later on, Boney showed in [Bonc] that it followed from large cardinals and also introduced a dual property he called *type shortness*.

Definition 2.27 (Definitions 3.1 and 3.3 in [Bonc])**.** Let K be an abstract class and let Γ be a class (possibly proper) of Galois types in K. Let κ be an infinite cardinal.

(1) K is $(<\kappa)$-*tame for* Γ if for any $p \neq q$ in Γ, if $A := \mathrm{dom}(p) = \mathrm{dom}(q)$, then there exists $A_0 \subseteq A$ such that $|A_0| < \kappa$ and $p \restriction A_0 \neq q \restriction A_0$.
(2) K is $(<\kappa)$-*type short for* Γ if for any $p \neq q$ in Γ, if $\alpha := \ell(p) = \ell(q)$, then there exists $I \subseteq \alpha$ such that $|I| < \kappa$ and $p^I \neq q^I$.
(3) κ-tame means $(<\kappa^+)$-tame, similarly for type short.
(4) We may just say "short" instead of "type short".
(5) Usually, Γ will be a class of types over models only, and we often specify it in words. For example, $(<\kappa)$-*short for types of length* α means $(<\kappa)$-short for $\bigcup_{M \in K} \mathrm{gS}^\alpha(M)$.
(6) We say K is $(<\kappa)$-tame if it is $(<\kappa)$-tame for types of length one.
(7) We say K is *fully* $(<\kappa)$-tame if it is $(<\kappa)$-tame for $\bigcup_{M \in K} \mathrm{gS}^{<\infty}(M)$, similarly for short.

We may use the following without comment:

Lemma 2.28. If K is $(<\kappa)$-tame and short for $\{\mathrm{gtp}(\bar{a}\bar{b}/M; N), \mathrm{gtp}(\bar{a}'\bar{b}/M; N')\}$, then K is $(<\kappa)$-short for $\{\mathrm{gtp}(\bar{a}/M\bar{b}; N), \mathrm{gtp}(\bar{a}'/M\bar{b}; N')\}$.

Proof sketch. Let $p := \mathrm{gtp}(\bar{a}/M\bar{b}; N)$, $q := \mathrm{gtp}(\bar{a}'/M\bar{b}; N')$. Assume $p^I \restriction B = q^I \restriction B$ for all small I and B. We want to see $p = q$. For this, it is enough to show that $\mathrm{gtp}(\bar{a}\bar{b}/M; N) = \mathrm{gtp}(\bar{a}\bar{b}'/M; N')$, which follows from the shortness hypothesis. $\square$

Finally, we discuss the natural notion of stability in this context. Our definition is slightly unusual: we define what it means for a *model* to be stable in a given cardinal, and get a local notion of stability that is equivalent (in AECs) to the usual notion if amalgamation holds, but behaves better if amalgamation fails.

Definition 2.29 (Stability)**.** Let α be a cardinal, μ be a cardinal. A model $N \in K$ is $(<\alpha)$-*stable in* μ if for all $A \subseteq |N|$ of size μ, $|\mathrm{gS}^{<\alpha}(A;N)| \leq \mu$. Here and below, α-stable means $(<(\alpha^+))$-stable. We say "stable" instead of "1-stable".

K is $(<\alpha)$-*stable in* μ if every $N \in K$ is $(<\alpha)$-stable in μ. K is $(<\alpha)$-*stable* if it is $(<\alpha)$-stable in unboundedly many cardinals.

Define similarly *syntactically stable* for syntactic types (in this paper, the quantifier-free $L_{\kappa,\kappa}$-types where κ is clear from context).

The next fact spells out the connection between stability for different lengths and tameness.

Fact 2.30. Let K be an AEC and let $\mu \geq \mathrm{LS}(K)$.

(1) [Bona, Theorem 3.1]: If K is stable in μ, K_μ has amalgamation, and $\mu^\alpha = \mu$, then K is α-stable in μ.
(2) [GV06a, Corollary 6.4][6]: If K has amalgamation, is μ-tame, and stable in μ, then K is stable in all λ such that $\lambda^\mu = \lambda$.
(3) If K has amalgamation, is μ-tame, and is stable in μ, then K is α-stable (in unboundedly many cardinals), for all cardinals α.

Proof of (3). Given cardinals $\lambda_0 \geq \mathrm{LS}(K)$ and α, let $\lambda := (\lambda_0)^{\alpha+\mu}$. Combining the first two statements gives us that K is α-stable in λ. $\square$

2.6. Some facts about abstract elementary classes.

Fact 2.31 (Lemma II.1.23 in [She09a])**.** Let K be an AEC in λ with $\lambda \geq |L(K)|$. Then there exists a unique AEC K' such that $(K')_\lambda = K$ and $\mathrm{LS}(K') = \lambda$.

Notation 2.32. Let K be an AEC in $\mathcal{F}$, $\lambda \geq \mathrm{LS}(K)$ and $\mathcal{F}' \supseteq \mathcal{F}$ be an interval of cardinals. Write $K_{\mathcal{F}'}$ for $(K^1)_{\mathcal{F}'}$, where K^1 is as described above for K_λ. Of course, we write $K_{\geq\lambda}$ instead of $K_{[\lambda,\infty)}$.

Corollary 2.33. For λ an infinite cardinal, there are at most[7] 2^{2^λ} AECs K with $\mathrm{LS}(K) = \lambda$.

[6] The result we want can easily be seen to follow from the proof there: see [Bal09, Theorem 12.10].

[7] In fact this is optimal, see [Kue08, Theorem 6.2].

Proof sketch. Every such AEC K is determined by K_λ. Now count the isomorphism types[8]. $\square$

The next result shows that AECs are still in some sense syntactic object.

Fact 2.34 (Lemma I.1.9[9] in [She09a])**.** Let K be an AEC with $L := L(K)$. There exists a (finitary) language $L_1 \supseteq L$, a first-order L_1-theory T_1 of size $\mathrm{LS}(K)$, and a set Γ of at most $2^{\mathrm{LS}(K)}$ quantifier-free L_1-types such that $K = \mathrm{PC}(T_1, \Gamma, L)$ (recall that $\mathrm{PC}(T_1, \Gamma, L)$ denotes the set of reducts to L of models of T_1 omitting Γ). Moreover, we can choose L_1, T_1, and Γ such that for $M^1, N^1 \in \mathrm{EC}(T_1, \Gamma) := \{M^1 \models T_1 \mid M^1 \text{ omits } \Gamma\}$, $M^1 \subseteq N^1$ implies $M^1 \restriction L \leq N^1 \restriction L$.

As a direct consequence, we get a low Hanf number for existence:

Fact 2.35 (Conclusion I.1.11 in [She09a])**.** Let K be an AEC. If for every $\mu < h(\mathrm{LS}(K))$ $K_{\geq \mu} \neq \emptyset$, then K has arbitrarily large models.

Finally, we define two different notions of saturation: one is category-theoretic, another uses Galois types. We state a classical result showing that they are the same notion under reasonable assumptions. Note that we again define the local notions (but our definitions are equivalent to the usual ones assuming amalgamation).

Definition 2.36. Let K be an abstract class, $M \in K$ and μ be an infinite cardinal.

(1) For $N \geq M$, M is *μ-saturated*[10] *in* N if for any $A \subseteq |M|$ of size less than μ, any $p \in \mathrm{gS}^{<\mu}(A; N)$ is realized in M. M is *μ-saturated* if it is μ-saturated in N for all $N \geq M$. When $\mu = \|M\|$, we omit it.
(2) For $N \geq M$, M is *μ-model-homogeneous in* N if for any $M_0 \leq M$ and $M_0 \leq M_0' \leq N$, if $\|M_0'\| < \mu$, then there exists $f : M_0' \xrightarrow[M_0]{} M$. M is *μ-model-homogeneous* if it is μ-model-homogeneous in N for all $N \geq M$. When $\mu = \|M\|$, we omit it.

Remark 2.37.

[8]The result also follows from Fact 2.34: there are essentially 2^λ-many theories of size λ, and 2^{2^λ} sets of quantifier-free types for each theory.

[9]Or see [Bal09, Theorem 4.15], or [Gro].

[10]Pedantically, we should really say “Galois-saturated” to differentiate this from being syntactically saturated. In this paper, it will always be clear which is meant.

(1) For K an AEC, we defined the two notions also when $\mu \leq \text{LS}(K)$. While this does not make much sense for μ-model-homogeneous, it *does* make sense for saturation and we will use this.
(2) We could similarly define what it means for a *set* to be saturated in a model, but we have no use for it.

Fact 2.38 (Lemma II.1.14 in [She09a])**.** Let K be an AEC with $K_{<\text{LS}(K)} = \emptyset$, $M \in K$ and $\mu > \text{LS}(K)$ be a cardinal. Assume K_μ has amalgamation.

(1) M is μ-saturated if and only if it is μ-model-homogeneous.
(2) M is μ-saturated if and only if for every $N \geq M$ and every $A \subseteq |M|$ of size less than μ, any $p \in \text{gS}(A; N)$ is realized in M (so it is enough to look at types of length 1).

Remark 2.39. As mentioned above, we may look at μ-saturated models when $\mu \leq \text{LS}(K)$ so it is still useful to keep the two notions separate. We will also look at model-homogeneous models in classes that are not AECs.

Part 1. **All fully short and tame AECs are syntactic**

3. Introduction

Abstract elementary classes are sometimes described as a purely semantic framework in which to do model theory. It has been shown, however, that AECs are closely connected with more syntactic objects. See for example Shelah's presentation theorem (Fact 2.34), or Kueker's [Kue08] showing that an AEC with Löwenheim-Skolem number λ is closed under L_{∞,λ^+}-elementary equivalence.

In 1990, Makkai and Shelah [MS90] studied the model theory of $L_{\kappa,\omega}$, for κ a strongly compact cardinal. They showed [MS90, Proposition 2.10] that Galois types (a purely semantic notion of types) and syntactic $L_{\kappa,\omega}$-types were really the same. This implies in particular that in a class of models of an $L_{\kappa,\omega}$-sentence, Galois types are determined by their restrictions to domains of size less than κ. This property was later isolated (from an argument in [She99]) by Grossberg and VanDieren [GV06a] and called tameness.

In this part, we show that in *any* tame abstract elementary class, Galois types are (in some sense) syntactic. The basic idea stems from the observation (appearing for example in [Bonc, p. 15] or [Lie11, p. 206]) that in a $(< \kappa)$-tame abstract elementary class, Galois types over domains of size less than κ play a role analogous to first-order formulas. We make this observation precise by expanding the language of such an AEC with a relation symbol for every Galois type of size less than κ, and looking at $L_{\kappa,\kappa}$-formulas in the expanded language (a similar construction, for $\kappa = \aleph_0$, is described in [Kan, Section 2.4], and an abstract axiomatization of the "formal types" that result appears in [Shea]). We can then prove that quantifier-free syntactic types correspond to Galois types in this expansion. This is not hard to believe: we are simply thinking of a type as the set of its small restrictions. Shelah's E_μ [She99, Definition 1.8] is another way to express the same idea.

More generally, we give a natural axiomatic definition for the *Morleyization* of an AEC and show that adding relation symbols for Galois types is one example (but by no means the only one). We believe Morleyizations could have further applications, as they are quite well behaved. Here we prove (Theorem 5.3) a presentation theorem for Morleyizations, and deduce a compactness theorem from large cardinals.

Thinking of small Galois types as formulas appears to give a convenient notation and a productive point of view: we are able to do syntactic argument inside a single model which are hard to describe in a purely semantic way. Often, we can directly quote results from the framework of stability theory inside a model (introduced in Rami Grossberg's 1981 master thesis and studied for example in [Gro91a, Gro91b] or [She87b, Chapter I], see [She09b, Chapter V.A] for a more recent version) or transparently adapt arguments from the first-order theory. We should also mention [Hay] which is not directly related to our work but shows that syntactic methods can be used in unexpected ways to prove semantic results (here a categoricity transfer for quasiminimal pregeometry classes). Last but not least, we point out chapter IV of [She09a] which also uses syntactic methods and studies their interaction with Ehrenfeucht-Mostowski models. This is one aspect we do not explore in this work but that we believe is worth investigating.

As an application of the above methods, we prove (Theorem 3.1):

Theorem 3.1. Assume K is a $(< \kappa)$-tame AEC with amalgamation, $\kappa \geq \text{LS}(K)$. The following are equivalent:

(1) K is stable in *some* cardinal greater than or equal to κ.
(2) K does not have the order property.
(3) There exists $\mu \leq \lambda_0 < \beth_{(2^\kappa)^+}$ such that K is stable in any $\lambda \geq \lambda_0$ with $\lambda^{<\mu} = \lambda$.

This gives the equivalence between no order property and stability in tame AECs and generalizes one direction of the stability spectrum theorem of homogeneous model theory ([She70, Theorem 4.4], see also [GL02, Corollary 3.11]). The proof is not new (it is essentially a straightforward generalization of Shelah's first-order proof, see [She90, Theorem 2.10]) and we are told by Rami Grossberg that proving such results was one of the reason tameness was introduced (in fact theorems with the same spirit appear in [GV06a]). However we believe it is challenging to give a transparent proof of the result using Galois types only. The reason is that the classical proof uses local types and it is not clear how to naturally define them semantically.

Note that our method has other applications: in part 2, we look at the *coheir* relation and show it is has some of the properties of a well-behaved independence notion (this is used in part 3 to get a global independence notion). In [BVb], we will also use syntactic methods to investigate chains of Galois-saturated models.

This part is organized as follows. In section 4, we introduce *Morleyizations*[11] of AECs and the main example: adding a symbol for each small Galois type. We then prove that Galois types correspond to syntactic type in the Morleyization under tameness. In section 5, we prove a compactness theorem for abstract elementary classes below a strongly compact cardinal (in the spirit of [Bonc]). In section 6, we investigate various order properties and prove Theorem 3.1.

4. The semantic-syntactic correspondence

4.1. Abstract Morleyizations and Galois Morleyization.

Definition 4.1. Let K be an AC with language $L := L(K)$, and let $\widehat{L}$ be an infinitary language extending L. An *abstract* $\widehat{L}$*-Morleyization* of K is a class $\widehat{K}$ of $\widehat{L}$-structures satisfying the following properties:

(1) The map $\widehat{M} \mapsto \widehat{M} \upharpoonright L$ is a bijection from $\widehat{K}$ onto K. For $M \in K$, we will write $\widehat{M}$ for the unique element of $\widehat{K}$ whose reduct is M.
(2) Invariance: For $M, N \in K$, if $f : M \cong N$, then $f : \widehat{M} \cong \widehat{N}$.
(3) Monotonicity: If $M \leq N$ are in K, then $\widehat{M} \subseteq \widehat{N}$.

We say an abstract Morleyization $\widehat{K}$ is $(< \kappa)$*-ary* if $\widehat{L}$ is $(< \kappa)$-ary. Usually, we drop the "abstract" and just talk about a Morleyization.

[11]The name was suggested by Rami Grossberg.

Example 4.2.

(1) For K an abstract class, K is an L-Morleyization of K itself. This is because $\leq$ must extend $\subseteq$.
(2) Let T be a complete first-order theory in a language L. Let $K := (\mathrm{Mod}(T), \preceq)$. It is common to expand L to $\widehat{L}$ by adding a relation symbol for every first-order L-formula. We then expand T (to $\widehat{T}$) and every model M of T in the expected way (to some $\widehat{M}$) and obtain a new theory in which every formula is equivalent to an atomic one (this is commonly called the *Morleyization* of the theory and is the reason for our choice of terminology). Then $\widehat{K} := \mathrm{Mod}(\widehat{T})$ is a $\widehat{L}$-Morleyization of K.
(3) Assume $\mathfrak{s} = (K, \mathop{\smile\!\!\!\!\!\!\;|}, S^{\mathrm{bs}})$ is a good λ-frame (see [She09a, Section II]). Let $\widehat{L}$ be an expansion of $L(K)$ where we add a relation symbol $R_{\mathop{\smile\!\!\!\!\!\!\;|}}$ of arity $\lambda + \lambda + 1$ to designate $\mathop{\smile\!\!\!\!\!\!\;|}$. Define a $\widehat{L}$-Morleyization $\widehat{K}$ of K by expanding any $N \in K$ to $\widehat{N}$ where $\left(R_{\mathop{\smile\!\!\!\!\!\!\;|}}\right)^{\widehat{N}}(\bar{M}\bar{M}'a)$ holds exactly when $\bar{M}$, $\bar{M}'$ are enumerations of type λ of models $M \leq M'$ of size λ and $a \underset{M}{\overset{N}{\mathop{\smile\!\!\!\!\!\!\;|}}} M'$. The invariance and monotonicity axioms of a good frame tell us that $\widehat{K}$ is indeed an $\widehat{L}$-Morleyization of K. Thus a good λ-frame can be seen as a particular kind of Morleyization of an AEC (the basic types are just those that do not fork over their domain).
(4) Let K be an abstract class with $L := L(K)$ and let κ be an infinite cardinal. Add a (κ)-ary predicate P to L, forming a language $\widehat{L}$. Expand each $M \in K$ to a $\widehat{L}$-structure by defining $P^{\widehat{M}}(\bar{a})$ to hold if and only if $\bar{a}$ is the universe of a $\leq$-submodel of M (this is more or less what Shelah does in [She09a, Definition IV.1.9.1]). Then $\widehat{K}$ is an $\widehat{L}$-Morleyization of K.
(5) Let K be an abstract class and let κ be an infinite cardinal. Define an expansion $\widehat{L}$ of $L(K)$ by adding a relation symbol R_p of arity $\ell(p)$ for each $p \in \mathrm{gS}^{<\kappa}(\emptyset)$. Expand each $N \in K$ to a $\widehat{L}$-structure $\widehat{N}$ by specifying that for each $\bar{a} \in \widehat{N}$, $R_p^{\widehat{N}}(\bar{a})$ holds exactly when $\mathrm{gtp}(\bar{a}/\emptyset; N) = p$. It is straightforward to check that $\widehat{K}$ is a $(< \kappa)$-ary $\widehat{L}$-Morleyization of K. We write $\widehat{K}^{<\kappa}$ and $\widehat{L}^{<\kappa}$ for $\widehat{K}$ and $\widehat{L}$ respectively. We call $\widehat{K}^{<\kappa}$ the $(< \kappa)$-*Galois Morleyization* of K.

Note that a Morleyization can naturally be made into an abstract class:

Definition 4.3. Let $(K, \leq)$ be an abstract class and let $\widehat{K}$ be a Morleyization of K. Define an ordering $\widehat{\leq}$ on $\widehat{K}$ by $\widehat{M} \widehat{\leq} \widehat{N}$ if and only if $M := \widehat{M} \upharpoonright L(K) \leq \widehat{N} \upharpoonright L(K) =: N$.

Remark 4.4. For simplicity, we will abuse notation and write $(\widehat{K}, \leq)$ rather than $(\widehat{K}, \widehat{\leq})$. As usual, when the ordering is clear from context we omit it.

The next propositions are easy but conceptually quite interesting[12].

Proposition 4.5. Let $(K, \leq)$ be an abstract class with $L := L(K)$. Let $\widehat{K}$ be an $\widehat{L}$-Morleyization of K.

(1) $(\widehat{K}, \leq)$ is an abstract class.
(2) If every chain in K has an upper bound, then every chain in $\widehat{K}$ has an upper bound.
(3) Galois types are the same in K and $\widehat{K}$: $\text{gtp}(\bar{a}_1/A; N_1) = \text{gtp}(\bar{a}_2/A; N_2)$ if and only if $\text{gtp}(\bar{a}_1/A; \widehat{N}_1) = \text{gtp}(\bar{a}_2/A; \widehat{N}_2)$.
(4) Assume K is a μ-AEC and $\widehat{K}$ is a $(< \mu)$-ary Morleyization of K. Then $(\widehat{K}, \leq)$ is a μ-AEC with $\text{LS}_{<\mu}(\widehat{K}) = \text{LS}_{<\mu}(K) + |\widehat{L}|^{<\mu}$.
(5) Let $L \subseteq \widehat{L}' \subseteq \widehat{L}$. Then $\widehat{K} \upharpoonright \widehat{L}' := \{\widehat{M} \upharpoonright \widehat{L}' \mid \widehat{M} \in \widehat{K}\}$ is an $\widehat{L}'$-Morleyization of K.
(6) If $\widehat{\widehat{K}}$ is a Morleyization[13] of $\widehat{K}$, then $\widehat{\widehat{K}}$ is a Morleyization of K.

Proof. All are straightforward. As an example, we show that if K is a μ-AEC, $\widehat{K}$ is a $(< \mu)$-ary Morleyization, and $\langle \widehat{M_i} : i < \delta \rangle$ is increasing in $\widehat{K}$ with $\text{cf}(\delta) \geq \mu$, then letting $M_\delta := \bigcup_{i<\delta} M_i$, we have that $\bigcup_{i<\delta} \widehat{M_i} = \widehat{M_\delta}$ (so in particular $\bigcup_{i<\delta} \widehat{M_i} \in \widehat{K}$). Let R be a relation symbol in $\widehat{L}$ of arity α. Let $\bar{a} \in {}^{\alpha}|\widehat{M}|$. Assume $\widehat{M_\delta} \models R[\bar{a}]$. We show $\bigcup_{i<\delta} \widehat{M_i} \models R[\bar{a}]$. The converse is done by replacing R by $\neg R$, and the proof with function symbols is similar. Since $\text{cf}(\delta) \geq \mu$ and $\widehat{L}$ is $(< \mu)$-ary, $\bar{a} \in {}^{\alpha}|M_j|$ for some $j < \delta$. Since $M_j \leq M_\delta$, the monotonicity axiom implies $\widehat{M_j} \subseteq \widehat{M_\delta}$. Thus $\widehat{M_j} \models R[\bar{a}]$, and this holds for all sufficiently large js. Thus by definition of the union, $\bigcup_{i<\delta} \widehat{M_i} \models R[\bar{a}]$. $\square$

[12] We believe it would be worthwhile to isolate the "category-theoretic essence" of Morleyizations (of course this is quite vague as stated) so that we can have a better understanding of the big picture.

[13] Where of course we think of $\widehat{K}$ as an abstract class with the ordering induced from K.

4.2. **Formulas and syntactic types.** From now on until the end of the section, we assume:

Hypothesis 4.6. K is an abstract class with $L := L(K)$, κ is an infinite cardinal, $\widehat{K}$ is a $(< \kappa)$-ary $\widehat{L}$-Morleyization of K.

We will adopt the conventions of Section 2.2. In particular when we talk about a formula we mean a (usually quantifier-free) $L_{\kappa,\kappa}$-formula. For $N \in K$, we write $\text{tp}(\bar{a}/A; \widehat{N})$ for $\text{tp}_{qL_{\kappa,\kappa}}(\bar{a}/A; \widehat{N})$, and $\text{S}^\alpha(A; \widehat{N})$ for $\text{S}^\alpha_{qL_{\kappa,\kappa}}(A; \widehat{N})$ (and similarly for other variations). We may write $N \models \phi[\bar{a}]$ for $\widehat{N} \models \phi[\bar{a}]$.

Remark 4.7. When κ is clear from context, we sometimes say that a set is *small* if it has cardinality strictly less than κ, or that a type is *small* if its domain and length are small.

Proposition 4.8. Let $\phi(\bar{x})$ be a quantifier-free formula, $M \in K$, and $\bar{a} \in M$. If $f : M \to N$, then $\widehat{M} \models \phi[\bar{a}]$ if and only if $\widehat{N} \models \phi[f(\bar{a})]$.

Proof. Directly from the invariance and monotonicity properties of Morleyizations. $\square$

In general, Galois and syntactic types (even in the Morleyization) are different. However, Galois types are always finer than syntactic types in the Morleyization:

Lemma 4.9. Let $N_1, N_2 \in K$, $A \subseteq |N_\ell|$ for $\ell = 1, 2$. Let $\bar{b}_\ell \in N_\ell$. If $\text{gtp}(\bar{b}_1/A; N_1) = \text{gtp}(\bar{b}_2/A; N_2)$, then $\text{tp}(\bar{b}_1/A; \widehat{N}_1) = \text{tp}(\bar{b}_2/A; \widehat{N}_2)$.

Proof. By transitivity of equality, it is enough to show that if $(\bar{b}_1, A, N_1)E_{\text{at}}(\bar{b}_2, A, N_2)$, then $\text{tp}(\bar{b}_1/A; \widehat{N}_1) = \text{tp}(\bar{b}_2/A; \widehat{N}_2)$. So assume $(\bar{b}_1, A, N_1)E_{\text{at}}(\bar{b}_2, A, N_2)$. Then there exists $N \in K$ and $f_\ell : N_\ell \underset{A}{\to} N$ such that $f_1(\bar{b}_1) = f_2(\bar{b}_2)$. Let $\phi(\bar{x})$ be a formula over A. Assume $\widehat{N}_1 \models \phi[\bar{b}_1]$. By Proposition 4.8, $\widehat{N} \models \phi[f_1(\bar{b}_1)]$, so $\widehat{N} \models \phi[f_2(\bar{b}_2)]$, so by Proposition 4.8 again, $\widehat{N}_2 \models \phi[\bar{b}_2]$. Replacing ϕ by $\neg\phi$, we get the converse, so $\text{tp}(\bar{b}_1/A; \widehat{N}_1) = \text{tp}(\bar{b}_2/A; \widehat{N}_2)$. $\square$

Note that this used that syntactic types were quantifier-free. We have justified the following definition:

Definition 4.10. For a Galois type p, let let p^s be the corresponding syntactic type *in the Morleyization*. That is, if $p = \text{gtp}(\bar{b}/A; N)$, then $p^s := \text{tp}(\bar{b}/A; \widehat{N})$.

Proposition 4.11. Let $N \in K$, $A \subseteq |N|$. Let α be an ordinal. The map $p \mapsto p^s$ from $\mathrm{gS}^\alpha(A; N)$ to $\mathrm{S}^\alpha(A; \widehat{N})$ is a surjection.

Proof. If $\mathrm{tp}(\bar{b}/A; \widehat{N}) = q \in \mathrm{S}^\alpha(A; \widehat{N})$, then by definition $\left(\mathrm{gtp}(\bar{b}/A; N)\right)^s = q$. □

Remark 4.12. To investigate formulas with quantifiers, we could define a different version of Galois types using isomorphisms rather than embeddings, and remove the monotonicity axiom from the definition of a Morleyization. As we have no use for it, we avoid this approach.

4.3. **On when Galois types are syntactic.** We have seen in Proposition 4.11 that $p \mapsto p^s$ is a surjection, so Galois types are always finer than syntactic type in the Morleyization. It is natural to ask when they are the same, i.e. when $p \mapsto p^s$ is a *bijection.* For $\widehat{K}$ a Morleyization of $\widehat{K}^{<\kappa}$ (see Example 4.2.(5), note that this will mostly be used when $\widehat{K} = \widehat{K}^{<\kappa}$), we characterize when this is the case using shortness and tameness (Definition 2.27).

Theorem 4.13. Assume $\widehat{K}$ is a Morleyization of $\widehat{K}^{<\kappa}$.

Let Γ be a family of Galois types. The following are equivalent:

(1) K is $(<\kappa)$-tame and short for Γ.
(2) The map $p \mapsto p^s$ is a bijection from Γ onto $\Gamma^s := \{p^s \mid p \in \Gamma\}$.

Proof.

- <u>(1) implies (2)</u>: By Lemma 4.9, the map $p \mapsto p^s$ with domain Γ is well-defined and it is clearly a surjection onto $\mathcal{F}^s$. It remains to see it is injective. Let $p, q \in \Gamma$ be distinct. If they do not have the same domain or the same length, then $p^s \neq q^s$, so assume that $A := \mathrm{dom}(p) = \mathrm{dom}(q)$ and $\alpha := \ell(p) = \ell(q)$. Say $p = \mathrm{gtp}(\bar{b}/A; N)$, $q = \mathrm{gtp}(\bar{b}'/A; N')$. By the tameness and shortness hypotheses, there exists $A_0 \subseteq A$ and $I \subseteq \alpha$ of size less than κ such that $p_0 := p^I \upharpoonright A_0 \neq q^I \upharpoonright A_0 =: q_0$. Let $\bar{a}_0$ be an enumeration of A_0, and let $\bar{b}_0 := \bar{b} \upharpoonright I$, $\bar{b}_0' := \bar{b}' \upharpoonright I$. Let $p_0' := \mathrm{gtp}(\bar{b}_0\bar{a}_0/\emptyset; N)$, and let $\phi := R_{p_0'}(\bar{x}_0, \bar{a}_0)$, where $\bar{x}_0$ is a sequence of variables of type I. Since $\bar{b}_0$ realizes p_0 in N, $\widehat{N} \models \phi[\bar{b}_0]$, and since $\bar{b}_0'$ realizes q_0 in N' and $q_0 \neq p_0$, $\widehat{N'} \models \neg\phi[\bar{b}_0']$. Thus $\phi(\bar{x}_0) \in p^s$, $\neg\phi(\bar{x}_0) \in q^s$. By definition, $\phi(\bar{x}_0) \notin q$ so $p^s \neq q^s$.

- (2) implies (1): Let $p, q \in \Gamma$ be distinct with domain A and length α. Say $p = \text{gtp}(\bar{b}/A; N)$, $q = \text{gtp}(\bar{b}'/A; N')$. By hypothesis, $p^s \neq q^s$ so there exists $\phi(\bar{x})$ over A such that (without loss of generality) $\phi(\bar{x}) \in p$ but $\neg\phi(\bar{x}) \in q$. Let $A_0 := \text{dom}(\phi)$, $\bar{x}_0 := \text{FV}(\phi)$ (note that A_0 and $\bar{x}_0$ have size strictly less than κ). Let $\bar{b}_0$, $\bar{b}_0'$ be the corresponding subsequences of $\bar{b}$ and $\bar{b}'$ respectively. Let $p_0 := \text{gtp}(\bar{b}_0/A_0; N)$, $q_0 := \text{gtp}(\bar{b}_0'/A_0; N')$. Then it is straightforward to check that $\phi \in p_0^s$, $\neg\phi \in q_0^s$, so $p_0^s \neq q_0^s$ and hence (by Lemma 4.9) $p_0 \neq q_0$. Thus A_0 and I witness tameness and shortness respectively.

□

Remark 4.14. The proof shows that (2) implies (1) is valid when $\widehat{K}$ is any Morleyization of K.

Remark 4.15. Since Galois types in K and $\widehat{K}^{<\kappa}$ are the same (Proposition 4.5), nothing interesting happens if we take the Galois Morleyization a second time, i.e. $\widehat{\widehat{K}^{<\kappa}}^{<\kappa}$ is essentially the same as $\widehat{K}^{<\kappa}$.

Corollary 4.16. Assume $\widehat{K}$ is a Morleyization of $\widehat{K}^{<\kappa}$.

(1) K is fully $(<\kappa)$-tame and short if and only if for any $M \in K$ the map $p \mapsto p^s$ from $\text{gS}^{<\infty}(M)$ to $\text{S}^{<\infty}(M)$ is a bijection.
(2) K is $(<\kappa)$-tame if and only if for any $M \in K$ the map $p \mapsto p^s$ from $\text{gS}(M)$ to $\text{S}(M)$ is a bijection.

Proof. By Theorem 4.13 applied to $\Gamma := \bigcup_{M \in K} \text{gS}^{<\infty}(M)$ and $\Gamma := \bigcup_{M \in K} \text{gS}(M)$ respectively. □

Remark 4.17. For $M \in K$, $p, q \in \text{gS}(M)$, say $pE_{<\kappa}q$ if and only if $p \restriction A_0 = q \restriction A_0$ for all $A_0 \subseteq |M|$ with $|A_0| < \kappa$. Of course, if K is $(<\kappa)$-tame, then $E_{<\kappa}$ is just equality. More generally, the proof of Theorem 4.13 shows that if $\widehat{K}$ is a Morleyization of $\widehat{K}^{<\kappa}$, then $pE_{<\kappa}q$ if and only if $p^s = q^s$. Thus in that case syntactic types in the Morleyization can be seen as $E_{<\kappa}$-equivalence classes of Galois types. Note that $E_{<\kappa}$ appears in the work of Shelah, see for example [She99, Definition 1.8].

5. Large cardinals

We prove a generalization of the compactness theorem in [Bonc, Theorem 4.9]. In our case, the syntax makes the statement be just like in the first-order case, the only difference is that $\aleph_0$ is replaced by κ. Recall:

Definition 5.1. Let κ be a regular cardinal.

(1) κ is *measurable* if there is a nonprincipal κ-complete ultrafilter.
(2) κ is *strongly compact* if any κ-complete filter can be extended to a κ-complete ultrafilter.

We start by proving a presentation theorem for Morleyizations. This does not use large cardinals per say, but is only useful if there exists an uncountable cardinal μ such that $\mu = \mu^{<\mu}$.

First, by the same proof as the presentation theorem for AECs (Fact 2.34):

Fact 5.2. Let μ be a regular cardinal. Let $\widehat{K}$ be a μ-AEC with $\widehat{L}(K) = \widehat{L}$. Then there exists a $(< \mu)$-ary language $L_1 \supseteq \widehat{L}$, an $(L_1)_{\mu,\mu}$-theory T_1 of size $\mathrm{LS}(\widehat{K})$, and a set Γ of $(< \mu)$-ary quantifier-free L_1-types such that $\widehat{K}_{\leq \mu} = \left(\mathrm{PC}(T_1, \Gamma, \widehat{L})\right)_{\leq \mu}$ (recall that $\mathrm{PC}(T_1, \Gamma, \widehat{L})$ denotes the set of reducts to $\widehat{L}$ of models of T_1 omitting Γ).

Moreover if $\widehat{K}$ is a $\widehat{L}$-Morleyization of an AEC K with $L := L(K)$, then we can choose L_1, T_1, and Γ such that in addition $K = \mathrm{PC}(T_1, \Gamma, L)$ and for $M^1, N^1 \in \mathrm{EC}(T_1, \Gamma) := \{M^1 \models T_1 \mid M^1 \text{ omits } \Gamma\}$, $M^1 \subseteq N^1$ implies $M^1 \restriction L \leq N^1 \restriction L$.

The reason we can only code the class up to μ is that it is not clear how to write larger models as directed systems of smaller submodels in μ-AECs. However we can get rid of this restriction if $\widehat{K}$ is a Morleyization:

Theorem 5.3 (The presentation theorem for Morleyizations)**.** Let K be an AEC with $L := L(K)$ and let $\widehat{K}$ be a $(< \mu)$-ary $\widehat{L}$-Morleyization of K. If $\mu = \mu^{<\mu} \geq \mathrm{LS}(K) + |\widehat{L}|$, then there exists a $(< \mu)$-ary language $L_1 \supseteq \widehat{L}$, an $(L_1)_{\mu,\mu}$-theory T_1 of size $\mathrm{LS}(\widehat{K}) \leq \mu$, and a set Γ of $(< \mu)$-ary quantifier-free L_1-types such that:

(1) $\widehat{K} = \mathrm{PC}(T_1, \Gamma, \widehat{L})$.
(2) For $M^1, N^1 \in \mathrm{EC}(T_1, \Gamma)$, $M^1 \subseteq N^1$ implies $M^1 \restriction L \leq N^1 \restriction L$.

Proof. Let L_1, T_1, and Γ be as given by the moreover part of Fact 5.2 for $\widehat{K}$. They immediately give the second part and we are left to check the first. Let $N^1 \in \mathrm{EC}(T_1, \Gamma)$. We first show that $N^1 \restriction \widehat{L} \in \widehat{K}$. Write $N^0 := N^1 \restriction \widehat{L}$, $N := N^1 \restriction L$. Note that $N \in K$, so what we really have to show is that $N^0 = \widehat{N}$. Assume for simplicity that $\widehat{L}$ is relational. Let R be a relation symbol of arity $\alpha < \mu$ and let $\bar{a} \in {}^{\alpha}|N|$. We want to see:

$$N^0 \models R[\bar{a}] \Leftrightarrow \widehat{N} \models R[\bar{a}]$$

Assume $N^0 \models R[\bar{a}]$. We show $\widehat{N} \models R[\bar{a}]$. The converse is obtained by replacing R by $\neg R$ in the proof below.

Since $\mu = \mu^{<\mu} \geq (\text{LS}(K) + |\widehat{L}|)^{<\mu} = \text{LS}(\widehat{K}) \geq |L_1|$, we can pick some $N_0^1 \preceq_{(L_1)_{\mu,\mu}} N^1$ of size $\leq \mu$ containing $\bar{a}$. Write $N_0^0 := N_0^1 \upharpoonright \widehat{L}$ and $N_0 := N_0^1 \upharpoonright L$.

Now:

(1) Since $N^1 \in \text{EC}(T_1, \Gamma)$, $N_0^1 \in \text{EC}(T_1, \Gamma)$. Therefore $N_0^0 \in \text{PC}(T_1, \Gamma, \widehat{L})$. Since $\|N_0^0\| \leq \mu$, $N_0^0 \in \widehat{K}$. Thus by definition of a Morleyization $N_0^0 = \widehat{N_0}$.
(2) By definition of T_1 and Γ, $N_0 = N_0^1 \upharpoonright L \leq N^1 \upharpoonright L = N$.
(3) Since $N_0^1 \subseteq N^1$, we have in particular $N_0^0 \subseteq N^0$. By assumption, $N^0 \models R[\bar{a}]$, so also $N_0^0 \models R[\bar{a}]$.

Thus by the monotonicity property of Morleyizations, $\widehat{N} \models R[\bar{a}]$, as desired. A picture is below (note that the diagram contains different kinds of embeddings):

$$\begin{tikzcd}
L_1 : & N_0^1 \arrow[r] & N^1 & \\
\widehat{L} : & N_0^0 \arrow[u] \arrow[r] \arrow[rr, dotted, bend left] & N^0 \arrow[u] & \widehat{N} \\
L : & N_0 \arrow[u] \arrow[r] & N \arrow[u] \arrow[ur] &
\end{tikzcd}$$

This shows that $N^0 = \widehat{N}$, and completes the proof that $\text{PC}(T_1, \Gamma, \widehat{L}) \subseteq \widehat{K}$. Conversely, let $\widehat{N} \in \widehat{K}$. Then $N \in K = \text{PC}(T_1, \Gamma, L)$, so $N = N^1 \upharpoonright L$ for some $N^1 \in \text{EC}(T_1, \Gamma)$. Thus $N^0 := N^1 \upharpoonright \widehat{L} \in \text{PC}(T_1, \Gamma, \widehat{L})$. By what was just argued, $N^0 \in \widehat{K}$. Since the reduct map is a bijection from K to $\widehat{K}$, $\widehat{N} = N^0 = N^1 \upharpoonright \widehat{L}$. In particular, $\widehat{N} \in \text{PC}(T_1, \Gamma, \widehat{L})$. $\square$

We are now ready to begin looking at ultraproducts. Until the end of the section, we make the following hypothesis:

Hypothesis 5.4. K is an abstract elementary class, $\kappa > \text{LS}(K)$ is a measurable cardinal, $\widehat{K}$ is a $(<\kappa)$-ary Morleyization of K.

Before stating Łoś' theorem, some notation is in order:

Notation 5.5. Assume $M_i \in K$ for $i \in I$ and U is a κ-complete ultrafilter on I.

For $\bar{f} \in {}^{\alpha}\left(\prod_{i \in I} M_i\right)$. Write $[\bar{f}]_U$ for $\langle [f_\beta]_U : \beta < \alpha \rangle$ and $\bar{f}(i)$ for $\langle f_\beta(i) : \beta < \alpha \rangle$.

Theorem 5.6 (Łoś' theorem for Morleyizations)**.** Let U be a κ-complete ultrafilter on an index set I. Let $\langle M_i \in K : i \in I \rangle$ be given. Let $N := \prod_{i \in I} M_i/U$. Let $\phi(\bar{x})$ be a formula with $\ell(\bar{x}) = \alpha < \kappa$. Let $\bar{f} \in {}^{\alpha}\left(\prod_{i \in I} M_i\right)$.

Then $\prod_{i \in I} M_i/U \models \phi([\bar{f}]_U)$ if and only if $\{i \in I \mid M_i \models \phi(\bar{f}(i))\} \in U$.

Proof sketch. The usual proof of Łoś' theorem goes through (see for example [Dic75, Theorem 3.3.1]), the base case is proven[14] using Theorem 5.3 exactly as in [Bonc, Theorem 4.3]. ☐

Corollary 5.7. Let U be a κ-complete ultrafilter on an index set I. Let $\langle M_i \in K : i \in I \rangle$ be given. If M_i is κ-saturated for all $i \in I$, then $\prod_{i \in I} M_i/U$ is κ-saturated.

Proof. Set $\widehat{K} := \widehat{K}^{<\kappa}$ and let $p \in \mathrm{gS}(N^-)$ be a small type with $N^- \leq N$. Say $p = \mathrm{gtp}(\bar{a}/N^-; N')$. Let $[\bar{g}]_U$ represent some enumeration of N^- in N, and let $q := \mathrm{gtp}(\bar{a}[\bar{g}]_U/\emptyset; N')$. Let $\psi(\bar{x}, \bar{y})$ be the formula $\exists \bar{x} R_q(\bar{x}, \bar{y})$. Now apply Theorem 5.6 to $\psi(\bar{x}, [\bar{g}]_U)$. ☐

Before stating the next corollary, we need a definition:

Definition 5.8. Let $M \in K$ and let $S(\bar{x})$ be a set of formulas over M. We say (M, S) is *consistent* if there exists $M' \geq M$ such that S is consistent in $\widehat{M'}$. For μ a cardinal, (M, S) is $(< \mu)$-*consistent* if for any $S_0 \subseteq S$ of size less than μ, (M, S_0) is consistent.

Corollary 5.9 (Compactness)**.** Assume κ is strongly compact. Let $M \in K$ and let $S(\bar{x})$ be a set of formulas over M.

Then (M, S) is consistent if and only if (M, S) is $(< \kappa)$-consistent.

Proof. By the usual proof of compactness from Łoś' theorem (see for example [Dic75, Theorem 3.3.6]). ☐

Finally, recall that a strongly compact gives us shortness. In fact, the proof of [Bonc, Theorem 4.5] shows:

[14] Note that it is enough to prove the base case when $\widehat{L}$ contains one more symbol than L.

Fact 5.10. Assume κ is strongly compact. Then K is fully $(<\kappa)$-tame and short for all Galois types (not only those over models).

Therefore if κ is strongly compact, then Galois types are syntactic in any Morleyization of $\widehat{K}^{<\kappa}$ (in the sense of Theorem 4.13). This generalizes [MS90, Proposition 2.10] which made this observation for $L_{\kappa,\omega}$. We could go on to adapt many results of [MS90] to our context. We leave this to future work.

6. Order properties and stability spectrum

Hypothesis 6.1.

(1) K is an abstract elementary class.
(2) κ is an infinite cardinal, $\widehat{L}$ is a $(<\kappa)$-ary language.
(3) $\widehat{K}$ is a class of $\widehat{L}$-structures.

We do not assume that $\widehat{K} = \widehat{K}^{<\kappa}$, or even that $\widehat{K}$ is an $\widehat{L}$-Morleyization of K.

6.1. Several order properties.

The next definition is a natural syntactic extension of the first-order order property. A related definitions appears already in [She72] and has been well studied (see for example [GS86, GS]).

Definition 6.2 (Syntactic order property)**.** Let β and μ be cardinals. A model $\widehat{M} \in \widehat{K}$ has the *syntactic β-order property of length μ* if there exists $\langle \bar{a}_i : i < \mu \rangle$ inside $\widehat{M}$ with $\ell(\bar{a}_i) = \beta$ for all $i < \mu$ and a quantifier-free formula $\phi(\bar{x}, \bar{y})$ such that for all $i, j < \mu$, $\widehat{M} \models \phi[\bar{a}_i, \bar{a}_j]$ if and only if $i < j$.

Let α be a cardinal. $\widehat{M}$ has the *syntactic $(<\alpha)$-order property of length μ* if it has the syntactic β-order property of length μ for some $\beta < \alpha$. $\widehat{M}$ has the *syntactic order property of length μ* if it has the syntactic $(<\kappa)$-order property of length μ.

$\widehat{K}$ has the syntactic β-order of length μ if some $\widehat{M} \in \widehat{K}$ has it. *$\widehat{K}$ has the syntactic order property* if it has the syntactic order property for every length.

Arguably the most natural semantic definition of the order property in AECs appears in [She99, Definition 4.3]. For simplicity, we have removed one parameter from the definition.

Definition 6.3. Let α and μ be cardinals. A model $M \in K$ has the *α-order property of length μ* if there exists $\langle \bar{a}_i : i < \mu \rangle$ inside M with $\ell(\bar{a}_i) = \alpha$ for all $i < \mu$, such that for any $i_0 < j_0 < \mu$ and $i_1 < j_1 < \mu$, $\text{gtp}(\bar{a}_{i_0}\bar{a}_{j_0}/\emptyset; N) \neq \text{gtp}(\bar{a}_{j_1}\bar{a}_{i_1}/\emptyset; N)$.

We define variations such as "K has the α-order property" as in Definition 6.2.

Notice this is more general than the syntactic order property, since α is not required to be less than κ. However, when $\widehat{K} = \widehat{K}^{<\kappa}$ and α is small, the two properties are equivalent. Notice that this does not use any tameness.

Proposition 6.4. Let α and μ be cardinals with $\alpha < \kappa$. Assume $\widehat{K}$ is an $\widehat{L}$-Morleyization of K and work inside a model N. The syntactic α-order property of length μ implies the α-order property of length μ.

Conversely, if $\widehat{K}$ is a Morleyization of $\widehat{K}^{<\kappa}$, letting $\chi := |\text{gS}^{\alpha+\alpha}(\emptyset)|$, if $\mu \to (\lambda)^2_\chi$, then the α-order property of length μ implies the syntactic α-order property of length λ.

In particular, if $\widehat{K}$ is a Morleyization of $\widehat{K}^{<\kappa}$, the α-order property and the syntactic α-order property are equivalent.

Proof. That the syntactic order property implies the semantic one is a consequence of invariance. For the converse, let $\langle \bar{a}_i : i < \mu \rangle$ witness the semantic order property. Use the assumption on the coloring $(i < j) \mapsto \text{gtp}(\bar{a}_i\bar{a}_j/\emptyset; N)$ to get that (without loss of generality), $\langle \bar{a}_i : i < \lambda \rangle$ is such that whenever $i < j$, $\text{gtp}(\bar{a}_i\bar{a}_j/\emptyset; N) = p \in \text{gS}^{\alpha+\alpha}(\emptyset)$. But then (since by assumption $\text{gtp}(\bar{a}_i\bar{a}_j/\emptyset; N) \neq \text{gtp}(\bar{a}_j\bar{a}_i/\emptyset; N)$), $\phi(\bar{x}, \bar{y}) := R_p(\bar{x}, \bar{y})$ witnesses the syntactic order property.

For the last sentence, we know already the syntactic α-order property implies the α-order property. Conversely, assume the α-order property holds. Let $\chi := |\text{gS}^{\alpha+\alpha}(\emptyset)|$. Fix an infinite cardinal $\lambda \geq \chi$. Let $\mu := (2^\lambda)^+$. By the Erdős-Rado theorem, $\mu \to (\lambda^+)^2_\lambda$, so $\mu \to (\lambda)^2_\chi$. By hypothesis, the α-order property of length μ holds, so the syntactic α-order property of length λ holds. Since λ was arbitrary, the syntactic α-order property holds. $\square$

We will see later that assuming some tameness (and $\widehat{K}$ is a Morleyization of $\widehat{K}^{<\kappa}$), the order property is actually equivalent to the syntactic order property (even for non-small αs).

In the next part, we heavily use the assumption of no syntactic order property *of length κ*. We now look at how it compares to the (long) order property. Note that Proposition 6.4 already tells us that (if $\widehat{K}$ is a Morleyization of $\widehat{K}^{<\kappa}$) the $(<\kappa)$-order property implies the syntactic order property of length κ. To get an equivalence, we will assume κ is a fixed point of the Beth function. Recall:

Fact 6.5. Let α be a cardinal. If K has the α-order property of length μ for all $\mu < h(\alpha + \mathrm{LS}(K))$, then K has the α-order property.

Proof. By the same proof as [She99, Claim 4.5.3]. $\square$

Corollary 6.6. Let $\widehat{K}$ be an $\widehat{L}$-Morleyization of K. Assume $\beth_\kappa = \kappa > \mathrm{LS}(K)$.

(1) If $\widehat{K}$ has the syntactic order property of length κ, then K has the $(<\kappa)$-order property.
(2) If $\widehat{K}$ is a Morleyization of $\widehat{K}^{<\kappa}$, then $\widehat{K}$ has the syntactic order property of length κ if and only if K has the $(<\kappa)$-order property.

Proof.

(1) For some $\alpha < \kappa$, $\widehat{K}$ has the syntactic α-order property of length κ, and thus by Proposition 6.4 the α-order property of length κ. By the hypotheses on κ, $h(|\alpha| + \mathrm{LS}(K)) < \kappa$, so by Fact 6.5, K has the α-order property.
(2) By the first part and Proposition 6.4.

$\square$

For completeness, we also discuss the following semantic variation of the syntactic order property of length κ, introduced in [BGa, Definition 4.2]:

Definition 6.7. For $\kappa > \mathrm{LS}(K)$, K has the *weak κ-order property* if there are $\alpha, \beta < \kappa$, $M \in K_{<\kappa}$, $N \geq M$, types $p \neq q \in \mathrm{gS}^{\alpha+\beta}(M)$, and sequences $\langle \bar{a}_i : i < \kappa \rangle$, $\langle \bar{b}_i : i < \kappa \rangle$ from N so that for all $i, j < \kappa$:

(1) $i \leq j$ implies $\mathrm{gtp}(\bar{a}_i \bar{b}_j / M; N) = p$.
(2) $i > j$ implies $\mathrm{gtp}(\bar{a}_i \bar{b}_j / M; N) = q$.

Lemma 6.8. Let $\kappa > \mathrm{LS}(K)$.

(1) If K has the $(<\kappa)$-order property, then K has the weak κ-order property.
(2) If K has the weak κ-order property, and $\widehat{K}$ is a Morleyization of $\widehat{K}^{<\kappa}$, then $\widehat{K}$ has the syntactic order property of length κ.

In particular, if $\kappa = \beth_\kappa$, then the weak κ-order property, the $(<\kappa)$-order property of length κ, and the $(<\kappa)$-order property are equivalent.

Proof.

(1) Assume K has the $(<\kappa)$-order property. To see the weak order property, let $\alpha < \kappa$ be such that K has the α-order property. Fix an $N \in K$ such that N has a long-enough α-order property. Pick any $M \in K_{<\kappa}$ with $M \leq N$. By using the Erdős-Rado theorem twice, we can assume we are given $\langle \bar{c}_i : i < \kappa \rangle$ such that whenever $i < j < \kappa$, $\operatorname{gtp}(\bar{c}_i \bar{c}_j / M; N) = p$, and $\operatorname{gtp}(\bar{c}_j \bar{c}_i / M; N) = q$, for some $p \neq q \in \operatorname{gS}(M)$.

For $l < \kappa$, let $j_l := 2l$, and $k_l := 2l + 1$. Then $j_l, k_l < \kappa$, and $l \leq l'$ implies $j_l < k_{l'}$, whereas $l > l'$ implies $j_l > k_{l'}$. Thus the sequences defined by $\bar{a}_l := \bar{c}_{j_l}$, $\bar{b}_l := \bar{c}_{k_l}$ are as required.

(2) Assume K has the weak κ-order property and let $M, N, p, q, \langle \bar{a}_i : i < \kappa \rangle$, $\langle \bar{b}_i : i < \kappa \rangle$ witness it. For $i < \kappa$, Let $\bar{c}_i := \bar{a}_i \bar{b}_i$ and $\phi(\bar{x}_1 \bar{x}_2; \bar{y}_1 \bar{y}_2) := R_p(\bar{y}_1, \bar{x}_2)$. This witnesses the syntactic order property of length κ in $\widehat{K}^{<\kappa}$.

$\square$

6.2. **Order property and stability.** We now want to relate stability in terms of the number of types (see Definition 2.29) and the order property and use this to find many stability cardinals.

Note that if $\widehat{K}$ is a Morleyization of $\widehat{K}^{<\kappa}$, stability will coincide with syntactic stability given enough tameness and shortness (see Theorem 4.13). In general, they could be different, but by invariance, stability always implies syntactic stability (and so syntactic unstability implies unstability). This contrasts with the situation with the order properties, where the syntactic and regular order property are equivalent without tameness (see Proposition 6.4).

The basic relationship between the order property and stability is given by:

Fact 6.9. If K has the α-order property and $\mu \geq |\alpha| + \operatorname{LS}(K)$, then K is not α-stable in μ. If in addition $\alpha < \kappa$ and $\widehat{K}$ is a Morleyization of $\widehat{K}^{<\kappa}$, then $\widehat{K}$ is not even syntactically α-stable in μ.

Proof. [She99, Claim 4.8.2] is the first sentence. The proof (see [BGKV, Fact 5.13]) generalizes (using the syntactic order property) to get the second sentence. $\square$

This shows that the order property implies unstability and we now work towards a syntactic converse. The key is:

Fact 6.10 (Theorem V.A.1.19 in [She09b]). Let $\widehat{N} \in \widehat{K}$. Let $\alpha < \kappa$. Let $\chi \geq (|\widehat{L}| + 2)^{<\kappa}$ be a cardinal. If $\widehat{N}$ does not have the syntactic $(<\kappa)$-order property of length χ^+, then whenever $\lambda = \lambda^\chi + \beth_2(\chi)$, $\widehat{N}$ is (syntactically) $(<\kappa)$-stable in λ.

Corollary 6.11. Assume $\widehat{K} = \widehat{K}^{<\kappa}$. The following are equivalent:

(1) $\widehat{K}$ is syntactically $(<\kappa)$-stable in *some* cardinal greater than or equal to $\mathrm{LS}(K) + \kappa$.
(2) K does not have the $(<\kappa)$-order property.
(3) There exists $\mu \leq \lambda_0 < h(\kappa + \mathrm{LS}(K))$ such that $\widehat{K}$ is syntactically $(<\kappa)$-stable in any $\lambda \geq \lambda_0$ with $\lambda^{<\mu} = \lambda$. In particular, $\widehat{K}$ is syntactically $(<\kappa)$-stable.

Proof. (3) clearly implies (1). (1) implies (2): If K has the $(<\kappa)$-order property, then by Fact 6.9 it cannot be syntactically $(<\kappa)$-stable in any cardinal above $\mathrm{LS}(K) + \kappa$ (note that α-unstability for $\alpha < \kappa$ implies $(<\kappa)$-unstability).

Finally (2) implies (3). Assume K does not have the $(<\kappa)$-order property. By the contrapositive of Fact 6.5, for each $\alpha < \kappa$, there exists $\mu_\alpha < h(|\alpha| + \mathrm{LS}(K)) \leq h(\kappa + \mathrm{LS}(K))$ such that K does not have the α-order property of length μ_α. Since $2^{<(\kappa+\mathrm{LS}(K)^+)} < h(\kappa + \mathrm{LS}(K))$, we can without loss of generality assume that $2^{<(\kappa+\mathrm{LS}(K)^+)} \leq \mu_\alpha$ for all $\alpha < \kappa$. Let $\chi := \sup_{\alpha<\kappa} \mu_\alpha$. Then K does not have the $(<\kappa)$-order property of length χ. Since $\mathrm{cf}(h(\kappa + \mathrm{LS}(K))) = (2^{\kappa+\mathrm{LS}(K)})^+ > \kappa$, $\chi < h(\kappa + \mathrm{LS}(K))$. Let $\mu := \chi^+$ and $\lambda_0 := \beth_2(\chi)$. It is easy to check that $\mu \leq \lambda_0 < h(\kappa + \mathrm{LS}(K))$. Finally, observe that by an easy calculation $|\widehat{L}^\kappa| \leq 2^{<(\kappa+\mathrm{LS}(K)^+)}$, so $\chi \geq (|\widehat{L}^\kappa| + 2)^{<\kappa}$. Now apply Fact 6.10. $\square$

Remark 6.12. Shelah [Sheb, Theorem 3.3] claims (without proof) a version of (1) implies (3).

Assuming $(<\kappa)$-tameness for types of length $<\kappa$, we can of course convert the above result to a statement about Galois types. To replace "$(<\kappa)$-stable" by just "stable" (and get away with only tameness for types of length one) we will use amalgamation together with Fact 2.30.

Theorem 6.13. Assume K has amalgamation and is $(<\kappa)$-tame. The following are equivalent:

(1) K is stable in *some* cardinal greater than or equal to $\mathrm{LS}(K)+\kappa$.
(2) K does not have the order property.
(3) K does not have the $(<\kappa)$-order property.
(4) There exists $\mu \leq \lambda_0 < h(\kappa + \mathrm{LS}(K))$ such that K is stable in any $\lambda \geq \lambda_0$ with $\lambda^{<\mu} = \lambda$.

In particular, K is stable if and only if K does not have the order property.

Proof. Clearly, (4) implies (1) and (2) implies (3). (1) implies (2): If K has the α-order property, then by Fact 6.9 it cannot be α-stable in any cardinal above $\mathrm{LS}(K) + \kappa + |\alpha|$. By Fact 2.30.(3), K is not stable in any cardinal greater than or equal to $\kappa + \mathrm{LS}(K)$, so (1) fails. Finally, (3) implies (4) by combining Corollary 6.11 and Corollary 4.16. $\square$

Part 2. **Coheir: Canonicity and superstability**

7. Introduction

Independence (or forking) is one of the central notion of modern classification theory. In first-order model theory, it was introduced by Shelah [She78] and is one of the main device of his book. One can ask whether there is such a notion in the nonelementary context. In homogeneous model theory, this was investigated in [HL02] for the superstable case and [BL03] for the simple and stable cases. Some of those results were later generalized by Hyttinen and Kesälä [HK06] to tame and $\aleph_0$-stable finitary abstract elementary classes[15]. What about general abstract elementary classes? There we believe the answer is still a work in progress. In Shelah's book [She09a], the central concept is that of a good frame (a local independence notion for types of singletons) and some conditions are given for their existence (see the introduction to [Vas] for more background). However, the question of whether there is a *global* independence notion is still left open.

In this part, we combine and generalize the results on global independence in AECs of [BGa], [BGKV], and [Vas]. Recall that [BGa] gave a definition of coheir in the AEC framework and showed that, assuming tameness and shortness, a stability hypothesis, *and* the extension property, coheir had the other properties of forking a stable first-order theory. Unfortunately it seems that the extension property is one of

[15] Note that by [HK06, Theorem 4.11], such classes are actually fully $(< \aleph_0)$-tame and short.

the hardest to prove in practice. Here, we reprove several of Boney and Grossberg's results without using the extension property (but we still cannot prove extension itself). In fact, using the syntactic correspondence introduced in the previous part (Theorem 4.13), there is very little to do: it suffices to quote from existing proofs, either from Makkai and Shelah's study of $L_{\kappa,\omega}$ [MS90] or even from a first-order result appearing in [Pil82]. Our results on coheir are summarized in Theorem 10.11. They will be used in part 3 as a stepping stone toward building global independence relation.

Now, the main result of [BGKV] is the canonicity of the coheir relation (see the introduction of the paper for background), but again this relies on it having the extension property. On the other hand, [Vas] showed that one *could* prove the extension property for types of length 1 by looking at an independence relation over sufficiently saturated models. Here, we show that the methods of [BGKV] still work without the extension property if we work over sufficiently saturated models. As a consequence, we obtain that any reasonable independence relation has to look like coheir over sufficiently saturated models (Theorem 12.15). This answers [BGKV, Question 7.2], and allows us to make progress on the canonicity of Shelah's good frames ([BGKV, Question 6.14] is partially answered by Corollary 12.18 which says that two good frames on a tame and short AEC must be the same over sufficiently-saturated models). Regarding the extension property, the canonicity result tells us that either coheir has it (over sufficiently saturated models) or *no* independence relation has it. In that sense it is a dividing line. In part 3 we show that coheir will have extension if the class satisfies a superstability-like assumption.

To prove the canonicity result, we unfortunately have to redo or at least restate a fair bit of the "independence calculus" parts of [BGKV]. We take this as an opportunity to give a general framework in which to study independence (Definition 9.5) that generalizes both the very local good frame of Shelah and the global independence relation inside a monster model of [BGKV]. This becomes very useful in part 3 where a large amount of precise independence calculus is used.

Finally, recall that the main result of [Vas] is the construction of a good frame (an independence notion for types of singletons) from categoricity in a cardinal with cofinality bigger than the tameness cardinal. We remove the cofinality restriction at the cost of looking at a bigger cardinal. As a nice bonus, we obtain that the independence relation of the frame is coheir. This is proven as Theorem 14.10 (the categoricity and stability transfers are also new):

Theorem 7.1. Let K be an AEC with joint embedding and amalgamation. Assume K is $(< \kappa)$-tame for $\kappa = \beth_\kappa > \text{LS}(K)$. If K is categorical in a $\lambda > \kappa$, then:

(1) K is stable in any $\mu \geq \text{LS}(K)$.
(2) K is categorical in κ.
(3) K has a type-full good $(\geq \chi)$-frame, where $\chi := \min(\lambda, \beth_{(2^\kappa)^+})$. Furthermore the non-forking relation of the frame is κ-coheir.

In addition, we give a general definition of superstability (closely related to those implicit in [Vas] and [GVV]) which already implies the existence of a good frame and follows from the hypothesis of the above theorem.

This part is organized as follows. In section 8, we review some classical results on saturated and universal models (framed in the language of abstract classes). In section 9, we define our framework to study independence. In section 10, we introduce coheir and prove some of its properties. In section 11, we look at relationship between properties of independence relations, as well as how to go from a good frame to an independence relation (all of it is done in [BGKV] so we mostly quote). In section 12, we prove the canonicity of coheir. In section 13, we recall how to transfer an independence relation to bigger models and in section 14 we define superstability and prove the above theorem.

8. Dense sub-abstract classes

We define what it means for an abstract class K^1 to be a *skeleton* of an abstract class K. The main examples are classes of saturated models with the usual ordering (or even universal or limit extension). Except perhaps for Lemma 8.6, the results of this section are either easy or well known, we simply put them in our general language. This will be used to study *weakly good independence relations* (an analog of good frames for the strictly stable case) in section 12 and define superstability in section 14.

Definition 8.1. For $(K, \leq)$ an abstract class, we say $(K^1, \trianglelefteq)$ is a sub-AC of K if $K^1 \subseteq K$ (as a class of models), $(K^1, \trianglelefteq)$ is an AC, and $M \trianglelefteq N$ implies $M \leq N$. We similarly define sub-μ-AEC, etc. When $\trianglelefteq = \leq \restriction K^1$, we omit it (or may abuse notation and write $(K^1, \leq)$).

Definition 8.2. For $(K, \leq)$ an abstract class, we say a set $S \subseteq K$ is *dense* in $(K, \leq)$ if it is dense in the sense of partial orders: for any $M \in K$, there exists $M' \in S$ with $M \leq M'$.

Definition 8.3. An abstract class $(K^1, \trianglelefteq)$ is a *skeleton* of $(K, \leq)$ if:

(1) $(K^1, \trianglelefteq)$ is a sub-AC of $(K, \leq)$.
(2) K^1 is dense in $(K, \leq)$.
(3) If $\langle M_i : i < \alpha \rangle$ is a $\trianglelefteq$-increasing chain in K^1 (α not necessarily limit) and there exists $N \in K^1$ such that $M_i < N$ for all $i < \delta$, then we can choose such an N with $M_i \triangleleft N$ for all $i < \delta$.

Remark 8.4. The term "skeleton" is inspired from the term "skeletal" in [Vas], although there "skeletal" is applied to frames. The intended philosophical meaning is the same: K^1 has enough information about K so that for many purposes we can work with K^1 rather than K.

Here are several ways to build skeletons:

Proposition 8.5. Let $(K, \leq)$ be an abstract class.

(1) If $(K^1, \trianglelefteq)$ is a skeleton of $(K, \leq)$ and $(K^2, \trianglelefteq')$ is a skeleton of $(K^1, \trianglelefteq)$, then $(K^2, \trianglelefteq')$ is a skeleton of $(K, \leq)$.
(2) If $(K^1, \leq)$ is a dense sub-AC of K, then $(K^1, \leq)$ is a skeleton of K.
(3) Assume $(K^1, \trianglelefteq)$ is a dense sub-AC of $(K, \leq)$ with no maximal models satisfying in addition: If $M_0 \leq M_1 \triangleleft M_2$ are in K^1, then $M_0 \triangleleft M_2$. Then $(K^1, \trianglelefteq)$ is a skeleton of $(K, \leq)$.

Proof. Straightforward. $\square$

The next lemma is a useful tool to find proper extensions in the skeleton of an AEC:

Lemma 8.6. Let $(K^1, \trianglelefteq)$ be a skeleton of the AC $(K, \leq)$. Assume every chain in $(K, \leq)$ has an upper bound. Let $M \leq N$ be in K^1.

(1) If $(K, \leq)$ has amalgamation, then there exists $N' \in K^1$ such that $M \trianglelefteq N'$ and $N \trianglelefteq N'$.
(2) If $(K, \leq)$ has joint embedding, then there exists $M' \cong M$ and $N' \in K^1$ such that $M' \trianglelefteq N'$ and $N \trianglelefteq N'$.

Proof. We prove (1) and it will be clear how to modify the proof to get (2). If N is not maximal (with respect to either of the orderings, it does not matter by definition of a skeleton), then by definition of a skeleton we can find $N' \in K^1$ such that $N \lhd N'$ and $M \lhd N'$, as needed.

Now assume N is maximal. We claim that $M \unlhd N$. Suppose not. Let $\lambda := \|N\|$. We build $\langle M_i : i \leq \lambda^+ \rangle$ a strictly increasing chain in $(K^1, \unlhd)$ with $M := M_0$.

This is enough. Then $\|M_{\lambda^+}\| \geq \lambda^+$. However by amalgamation and the fact that N is maximal, M_{λ^+} embeds into N, a contradiction.

This is possible. The case $i = 0$ has already been done. If $i < \lambda^+$ is limit, we can by assumption find a $<$-upper bound $M_i' \in K$ to the chain $\langle M_j : j < i \rangle$. By density, find $M_i'' \in K^1$ such that $M_i' \leq M_i''$. We have that $M_j < M_i''$ for all $j < i$. By definition of a skeleton, this means we can find $M_i \in K^1$ with $M_j \lhd M_i$ for all $j < i$. If $i = j + 1$ is successor, we consider two cases:

- If M_j is not maximal, let $M_i \in K^1$ be a $\lhd$-extension of M_j.
- If M_j is maximal, then by amalgamation and the fact both N and M_j are maximal, we must have $N \cong_M M_j$. However by assumption $M_0 \unlhd M_j$ so $M = M_0 \unlhd N$, a contradiction.

□

Thus we get that many properties transfer to skeletons.

Proposition 8.7. Let $(K, \leq)$ be an abstract class where every chain has an upper bound and let $(K^1, \unlhd)$ be a skeleton of K.

(1) $(K, \leq)$ has no maximal models if and only if $(K^1, \unlhd)$ has no maximal models.
(2) $(K, \leq)$ has joint embedding if and only if $(K^1, \unlhd)$ has joint embedding.
(3) $(K, \leq)$ has amalgamation if and only if $(K^1, \unlhd)$ has amalgamation.
(4) $(K, \leq)$ is α-stable in μ if and only if $(K^1, \unlhd)$ is α-stable in μ.

Proof.

(1) Directly from the definition.
(2) If $(K^1, \unlhd)$ has joint embedding, then by density $(K, \leq)$ has joint embedding. Conversely, assume $(K, \leq)$ has joint embedding. Let $M_\ell \in K^1$, $\ell = 1, 2$. By density, find $N \in K^1$ and $f_\ell : M_\ell \to N$ $\leq$-embeddings. By Lemma 8.6, there exists $N_1 \in K^1$ and $g_1 : M_1 \cong M_1'$ such that $N \unlhd N_1$, $M_1' \unlhd N_1$. By Lemma 8.6 again, there exists $N_2 \in K^1$ and $g_2 : M_2 \cong M_2'$ such that

$N_1 \trianglelefteq N_2$, $M_2' \trianglelefteq N_2$. Thus we also have $M_1' \trianglelefteq N_2$. It follows that $g_\ell : M_\ell \to N_2$ is a $\trianglelefteq$-embedding.
(3) Similar.
(4) Straightforward.

$\square$

Our main examples of skeletons will be composed of:

Definition 8.8. Let K be an abstract class, μ be a cardinal.

(1) Let $K^{\mu\text{-sat}}$ be the class of μ-saturated models in $K_{\geq \mu}$.
(2) Let $K^{\mu\text{-mh}}$ be the class of μ-model-homogeneous models in $K_{\geq \mu}$.
(3) For $M, N \in K$, say $M <_{\text{univ}} N$ (N is *universal over* M) if and only if whenever we have $N' \geq N$ such that $\|N'\| = \|N\|$ and $M \leq M' \leq N'$, then there exists $f : M' \xrightarrow[M]{} N$. Say $M \leq_{\text{univ}} N$ if and only if $M = N$ or $M <_{\text{univ}} N$.
(4) For $M, N \in K$, μ a cardinal and $\delta < \mu^+$, say $M <_{\delta,\mu} N$ (N is *(δ, μ)-limit over* M) if and only if $M, N \in K_\mu$, $M < N$, and there exists $\langle M_i : i \leq \delta \rangle$ $<_{\text{univ}}$-increasing continuous such that $M_0 = M$ and $M_\delta = N$ if $\delta > 0$. Say $M \leq_{\delta,\mu}$ if $M = N$ or $M <_{\delta,\mu} N$.

Remark 8.9. So for $M, N \in K_\mu$, $M <_{0,\mu} N$ if and only if $M < N$, while $M <_{1,\mu}$ if and only if $M <_{\text{univ}} N$.

Remark 8.10. Variations on $<_{\delta,\mu}$ already appear as [She99, Definition 2.1]. Our definition of a universal extension is slightly different from the usual one (see e.g. [Van06, Definition I.2.1.2]): first, we only work locally as usual (but if amalgamation holds this does not matter), and second we ask only for $\|M\| \leq \|M'\| \leq \|N\|$ rather than $\|M'\| = \|M\|$.

Lemma 8.11. Let $(K, \leq)$ be an abstract class, μ be an infinite cardinal.

(1) $K^{\mu\text{-sat}}$ and $K^{\mu\text{-mh}}$ are sub-ACs of K. If K is an ∞-AEC, then they also are.
(2) If $\text{TV}(K) \leq \mu$, then $\text{TV}(K^{\mu\text{-sat}}) + \text{TV}(K^{\mu\text{-mh}}) \leq \mu$.
(3) If K is an AEC, then $K^{\mu\text{-sat}}$ and $K^{\mu\text{-mh}}$ are dense sub-μ-AECs of K.
(4) Assume K has amalgamation. For $\delta < \mu^+$, $(K, \leq_{\text{univ}})$ and $(K_\mu, \leq_{\delta,\mu})$ are sub-ACs of K satisfying: $M_0 \leq M_1 \lhd M_2$ implies $M_0 \lhd M_2$.
(5) If K is an AEC in μ with no maximal models and amalgamation, K is stable in μ, then (for $\delta < \mu^+$) $(K, \leq_{\delta,\mu})$ is a skeleton of K with no maximal models.

Proof. The first three parts should be straightforward. See for example [Vas, Proposition 2.12] for the last two (everything is easy except perhaps the fact that $(K, \leq_{\delta,\mu})$ has no maximal models, due to Shelah). ☐

9. Independence relations

Before investigating the coheir independence relation, we would like to have a general framework in which to study independence. One such framework is Shelah's good frames [She09a, Section II.6]. Another is given by the definition of independence relation in [BGKV, Definition 3.1]. Both definitions describe a relation "p does not fork over M" for $p \in \mathrm{gS}(N)$, $M \leq N$.

In [BGKV], it is also shown how to "close" such a relation to obtain a relation "p does not fork over M" when p is a type over an arbitrary set. We find that starting with such a relation makes the statement of symmetry transparent, and hence makes many proofs easier. We also give a more general definition than [BGKV], as we do not assume that everything happens in a big homogeneous monster model, and we allow working inside ∞-abstract elementary classes rather than only abstract elementary classes. The later feature is convenient when working with classes of saturated models and Morleyizations.

Because we quote extensively from [She09a], which deals with frames, and also because it is sometimes convenient to "forget" the extension of the relation to arbitrary sets, we will still define frames and recall their relationship with independence relations over sets.

9.1. **Frames.** Shelah's definition of a pre-frame appears in [She09a, Definition III.0.2.1] and is meant to axiomatize the bare minimum of properties a relation must satisfy in order to be a meaningful independence notions.

We make several changes: we do not mention basic types (we have no use for them), so in Shelah's terminology our pre-frames will be *type-full*. In fact, it is notationally convenient for us to define our frame on every type, not just the nonalgebraic ones. The *disjointness* property (see Definition 9.9) tells us that the frame behaves trivially on the algebraic types. We do not require it (as it is not required in [BGKV, Definition 3.1]) but it will hold of all frames we consider.

We require that the class on which the independence relation operates has amalgamation, and we do not require that the base monotonicity property holds (this is to preserve the symmetry between right and left properties in the definition. Of course, all the frames we consider will have base monotonicity). Finally, we allow the size of the models to lie in an interval rather than just be restricted to a single cardinal as Shelah does. We also parametrize on the length of the types. This allows more flexibility and was already the approach favored in [Vas, BVa].

Definition 9.1. Let $\mathcal{F} = [\lambda, \theta)$ be an interval of cardinals with $\lambda < \theta$, $\alpha \le \theta$ be a cardinal or ∞.

A *type-full pre-*$(<\alpha, \mathcal{F})$*-frame* is a pair $\mathfrak{s} = (K, \mathop{\smile}\limits^{\vert})$, where:

(1) K is an ∞-abstract elementary class in $\mathcal{F}$ with amalgamation.
(2) $\mathop{\smile}\limits^{\vert}$ is a relation on quadruples of the form (M_0, A, M, N), where $M_0 \le M \le N$ are all in K, $A \subseteq |N|$ is such that $|A \backslash |M_0|| < \alpha$. We write $\mathop{\smile}\limits^{\vert}(M_0, A, M, N)$ or $A \mathop{\smile}\limits^{\vert}_{M_0}^{N} M$ instead of $(M_0, A, M, N) \in \mathop{\smile}\limits^{\vert}$.
(3) The following properties hold:
  (a) Invariance: If $f : N \cong N'$ and $A \mathop{\smile}\limits^{\vert}_{M_0}^{N} M$, then $f[A] \mathop{\smile}\limits^{\vert}_{f[M_0]}^{N'} f[M]$.
  (b) Monotonicity: Assume $A \mathop{\smile}\limits^{\vert}_{M_0}^{N} M$. Then:
    (i) Ambient monotonicity: If $N' \ge N$, then $A \mathop{\smile}\limits^{\vert}_{M_0}^{N'} M$. If $M \le N_0 \le N$ and $A \subseteq |N_0|$, then $A \mathop{\smile}\limits^{\vert}_{M_0}^{N_0} M$.
    (ii) Left and right monotonicity: If $A_0 \subseteq A$, $M_0 \le M' \le M$, then $A_0 \mathop{\smile}\limits^{\vert}_{M_0'}^{N} M'$.
  (c) Left normality: If $A \mathop{\smile}\limits^{\vert}_{M_0}^{N} M$, then $AM_0 \mathop{\smile}\limits^{\vert}_{M_0}^{N} M$.

When α or $\mathcal{F}$ are clear from context or irrelevant, we omit them and just say that $\mathfrak{s}$ is a pre-frame (or just a frame). We may omit the "type-full". A $(\le \alpha)$-frame is just a $(<(\alpha+1))$-frame. We might omit α when $\alpha = 2$ (i.e. $\mathfrak{s}$ is a (≤ 1)-frame) and we might talk of a λ-frame or a $(\ge \lambda)$-frame instead of a $\{\lambda\}$-frame or a $[\lambda, \infty)$-frame.

Notation 9.2. For $\mathfrak{s} = (K, \mathop{\smile}\limits^{\vert})$ a pre-$(<\alpha, \mathcal{F})$-frame with $\mathcal{F} = [\lambda, \theta)$, write[16] $K_{\mathfrak{s}} := K$, $\mathop{\smile}\limits^{\vert}_{\mathfrak{s}} := \mathop{\smile}\limits^{\vert}$, $\alpha_{\mathfrak{s}} := \alpha$, $\mathcal{F}_{\mathfrak{s}} = \mathcal{F}$, $\lambda_{\mathfrak{s}} := \lambda$, $\theta_{\mathfrak{s}} := \theta$.

[16]Really, α, $\mathcal{F}$, and θ should be part of the data of the frame but we usually ignore this detail.

Notation 9.3. For $\mathfrak{s} = (K, \perp)$ a pre-frame, we write $\perp(M_0, \bar{a}, M, N)$ or $\bar{a} \underset{M_0}{\overset{N}{\perp}} M$ for $\operatorname{ran}(\bar{a}) \underset{M_0}{\overset{N}{\perp}} M$ (similarly when other parameters are sequences). When $p \in \mathrm{gS}^{<\infty}(M)$, we say p *does not* $\mathfrak{s}$*-fork over* M_0 (or just *does not fork over* M_0 if $\mathfrak{s}$ is clear from context) if $\bar{a} \underset{M_0}{\overset{N}{\perp}} M$ whenever $p = \mathrm{gtp}(\bar{a}/M; N)$ (it is easy to check that this does not depend on the choice of representatives).

Remark 9.4. We can more or less go back and forth from our definition of pre-frame to Shelah's. We sketch how. From a pre-frame $\mathfrak{s}$ in our sense (with $K_{\mathfrak{s}}$ an AEC), we can let $S^{\mathrm{bs}}(M)$ be the set of nonalgebraic $p \in \mathrm{gS}(M)$ that do not $\mathfrak{s}$-fork over M. Then restricting $\underset{\mathfrak{s}}{\perp}$ to the basic types we obtain (assuming that $\mathfrak{s}$ has base monotonicity, see Definition 9.9) a pre-frame in Shelah's sense. From a pre-frame $(K, \perp, S^{\mathrm{bs}})$ in Shelah's sense (where K has amalgamation), we can extend $\perp$ by specifying that algebraic and basic types do not fork over their domains. We then get a pre-frame $\mathfrak{s}$ in our sense with base monotonicity and disjointness.

9.2. **Independence relations.** We now define an independence relation that also takes sets on the right hand side.

Definition 9.5 (Independence relation)**.** Let $\mathcal{F} = [\lambda, \theta)$ be an interval of cardinals with $\aleph_0 \leq \lambda < \theta$, $\alpha, \beta \leq \theta$ be cardinals or ∞. A $(< \alpha, \mathcal{F}, < \beta)$*-independence relation* is a pair $\mathfrak{i} = (K, \perp)$, where:

(1) K is an ∞-abstract elementary class in $\mathcal{F}$ with amalgamation.
(2) $\perp$ is a relation on quadruples of the form (M, A, B, N), where $M \leq N$ are all in K, $A \subseteq |N|$ is such that $|A \backslash |M|| < \alpha$ and $B \subseteq |N|$ is such that $|B \backslash |M|| < \beta$. We write $\perp(M, A, B, N)$ or $A \underset{M}{\overset{N}{\perp}} B$ instead of $(M, A, B, N) \in \perp$.
(3) The following properties hold:
   (a) Invariance: If $f : N \cong N'$ and $A \underset{M}{\overset{N}{\perp}} B$, then $f[A] \underset{f[M]}{\overset{N'}{\perp}} f[B]$.
   (b) Monotonicity: Assume $A \underset{M}{\overset{N}{\perp}} B$. Then:
      (i) Ambient monotonicity: If $N' \geq N$, then $A \underset{M}{\overset{N'}{\perp}} B$. If $M \leq N_0 \leq N$ and $A \cup B \subseteq |N_0|$, then $A \underset{M}{\overset{N_0}{\perp}} B$.

(ii) Left and right monotonicity: If $A_0 \subseteq A$, $B_0 \subseteq B$, then $A_0 \underset{M}{\overset{N}{\mathop{\smile\!\!\!\!\!\!\;|}}} B_0$.

(c) Left and right normality: If $A \underset{M}{\overset{N}{\mathop{\smile\!\!\!\!\!\!\;|}}} B$, then $AM \underset{M}{\overset{N}{\mathop{\smile\!\!\!\!\!\!\;|}}} BM$.

When $\beta = \theta$, we omit it. More generally, when α, β are clear from context or irrelevant, we omit them and just say that $\mathfrak{i}$ is an independent relation.

We adopt the same notational conventions as for pre-frames.

Remark 9.6. It seems that in every case of interest $\beta = \theta$. We did not make it part of the definition to avoid breaking symmetry. Note also that the case $\alpha > \lambda$ is of particular interest in part 3.

Before listing the properties independence relations and frames could have, we discuss how to go from one to the other. The cl operation is called the *minimal closure* in [BGKV, Definition 3.4].

Definition 9.7.

(1) Given a pre-frame $\mathfrak{s} := (K, \mathop{\smile\!\!\!\!\!\!\;|})$, let $\text{cl}(\mathfrak{s}) := (K, \overset{\text{cl}}{\mathop{\smile\!\!\!\!\!\!\;|}})$, where $\overset{\text{cl}}{\mathop{\smile\!\!\!\!\!\!\;|}}(M, A, B, N)$ if and only if $M \leq N$, $|B| < \theta_{\mathfrak{s}}$, and there exists $N' \geq N$, $M' \geq M$ containing B such that $\mathop{\smile\!\!\!\!\!\!\;|}(M, A, M', N')$.
(2) Given a $(< \alpha, \mathcal{F})$-independence relation $\mathfrak{i} = (K, \mathop{\smile\!\!\!\!\!\!\;|})$ let $\text{pre}(\mathfrak{i}) := (K, \overset{\text{pre}}{\mathop{\smile\!\!\!\!\!\!\;|}})$, where $\overset{\text{pre}}{\mathop{\smile\!\!\!\!\!\!\;|}}(M, A, M', N)$ if and only if $M \leq M' \leq N$ and $\mathop{\smile\!\!\!\!\!\!\;|}(M, A, M', N)$.

Proposition 9.8.

(1) If $\mathfrak{i}$ is a $(< \alpha, \mathcal{F})$-independence relation, then $\text{pre}(\mathfrak{i})$ is a pre $(< \alpha, \mathcal{F})$-frame.
(2) If $\mathfrak{s}$ is a pre-$(< \alpha, \mathcal{F})$-frame, then $\text{cl}(\mathfrak{s})$ is a $(< \alpha, \mathcal{F})$-independence relation and $\text{pre}(\text{cl}(\mathfrak{s})) = \mathfrak{s}$.

Next, we give a long list of properties that an independence relation may or may not have. Most are classical and already appear for example in [BGKV]. We give them here again both for the convenience of the reader and because their definition is sometimes slightly modified compared to [BGKV]. They will be discussed throughout this paper (for example, Section 11 discusses implications between the properties).

Definition 9.9 (Properties of independence relations)**.** Let $\mathfrak{i} := (K, \mathop{\smile\!\!\!\!\!\!\;|})$ be a $(< \alpha, \mathcal{F}, < \beta)$-independence relation.

- $\mathfrak{i}$ has *disjointness* if $A \overset{N}{\underset{M}{\mathop{\smile}\limits^{\mid}}} B$ implies $A \cap B \subseteq |M|$.
- $\mathfrak{i}$ has *symmetry* if $A \overset{N}{\underset{M}{\mathop{\smile}\limits^{\mid}}} B$ implies that for all $B_0 \subseteq B$ of size less than α and all $A_0 \subseteq A$ of size less than β, $B_0 \overset{N}{\underset{M}{\mathop{\smile}\limits^{\mid}}} A_0$.
- $\mathfrak{i}$ has *right full symmetry* if $A \overset{N}{\underset{M}{\mathop{\smile}\limits^{\mid}}} B$ implies that for all $B_0 \subseteq B$, if $B_0 \overset{N}{\underset{M}{\mathop{\smile}\limits^{\mid}}} M$, then there exists $N' \geq N$, $M' \geq M$ containing A such that $B_0 \overset{N'}{\underset{M}{\mathop{\smile}\limits^{\mid}}} M'$.
- $\mathfrak{i}$ has *right base monotonicity* if $A \overset{N}{\underset{M}{\mathop{\smile}\limits^{\mid}}} B$ and $M \leq M' \leq N$, $M' \subseteq B \cup |M|$ implies $A \overset{N}{\underset{M'}{\mathop{\smile}\limits^{\mid}}} B$.
- $\mathfrak{i}$ has *right existence* if $A \overset{N}{\underset{M}{\mathop{\smile}\limits^{\mid}}} M$ for any $A \subseteq |N|$ with $|A| < \alpha$.
- $\mathfrak{i}$ has *right uniqueness* if whenever $M_0 \leq M$, $q_\ell \in \mathrm{gS}^{<\alpha}(M)$, $q_1 \upharpoonright M_0 = q_2 \upharpoonright M_0$, and q_ℓ does not fork over M_0, then $q_1 = q_2$.
- $\mathfrak{i}$ has *right set-uniqueness* if whenever $M_0 \leq M_\ell \leq N$, $\ell = 1, 2$, $B \subseteq |M_\ell|$ for $\ell = 1, 2$, $q_\ell \in \mathrm{gS}^{<\alpha}(M_\ell; N)$, $q_1 \upharpoonright M_0 = q_2 \upharpoonright M_0$, and q_ℓ does not fork over M_0, then $q_1 \upharpoonright M_0 B = q_2 \upharpoonright M_0 B$.
- $\mathfrak{i}$ has *right extension* if whenever $p \in \mathrm{gS}^{<\infty}(M)$ does not fork over $M_0 \leq M$ and $N \geq M$ is in K, then there exists $q \in \mathrm{gS}^{<\infty}(N)$ extending p and not forking over M_0.
- $\mathfrak{i}$ has *right set-extension* if whenever $p \in \mathrm{gS}^{\infty}(MB; N)$ does not fork over M and $B \subseteq C \subseteq |N|$ with $|C| < \beta$, there exists $N' \geq N$ and $q \in \mathrm{gS}^{<\infty}(MC; N')$ extending p such that q does not fork over M.
- $\mathfrak{i}$ has *right independent amalgamation* if $\alpha > \lambda$, $\beta = \theta$ and whenever $M_0 \leq M_\ell$ are in K, $\ell = 1, 2$, there exists $N \in K$ and $f_\ell : M_\ell \underset{M_0}{\longrightarrow} N$ such that $f_1[M_1] \overset{N}{\underset{M_0}{\mathop{\smile}\limits^{\mid}}} f_2[M_2]$.
- $\mathfrak{i}$ has the *right* $(< \kappa)$*-witness property* if whenever $M \leq M' \leq N$, $||M'|\backslash|M|| < \beta$, $A \subseteq |N|$, and $A \overset{N}{\underset{M}{\mathop{\smile}\limits^{\mid}}} B_0$ for all $B_0 \subseteq |M'|$ of size less than κ, then $A \overset{N}{\underset{M}{\mathop{\smile}\limits^{\mid}}} M'$. $\mathfrak{i}$ has the *right* $(< \kappa)$*-set-witness property* if this is true when M' is allowed to be an arbitrary set.

- $\mathfrak{i}$ has *right transitivity* if whenever $M_0 \leq M_1 \leq N$, $A \overset{N}{\underset{M_0}{\mathop{\copy\noforkbox}\limits}} M_1$ and $A \overset{N}{\underset{M_1}{\mathop{\copy\noforkbox}\limits}} B$ implies $A \overset{N}{\underset{M_0}{\mathop{\copy\noforkbox}\limits}} B$. *Strong right transitivity* is the same property when we do not require $M_0 \leq M_1$.
- $\mathfrak{i}$ has *right full model-continuity* if K is an AEC in $\mathcal{F}$, $\alpha > \lambda$, $\beta = \theta$, and whenever $\langle M_i^\ell : i \leq \delta \rangle$ is increasing continuous with δ limit, $\ell \leq 3$, for all $i < \delta$, $M_i^0 \leq M_i^\ell \leq M_i^3$, $\ell = 1, 2$, $\|M_\delta^1\| < \alpha$, and $M_i^1 \overset{M_i^3}{\underset{M_i^0}{\mathop{\copy\noforkbox}\limits}} M_i^2$ for all $i < \delta$, then $M_\delta^1 \overset{M_\delta^3}{\underset{M_\delta^0}{\mathop{\copy\noforkbox}\limits}} M_\delta^2$.
- *Strong extension-existence* is a technical property used in the proof of canonicity, see Definition 12.9.

When $\mathfrak{s}$ is just a pre-frame and this makes sense, we define the same properties similarly.

Note that we have defined the right version of the asymmetric properties. One can define a left version by looking at the *dual independence relation.*

Definition 9.10. Let $\mathfrak{i} := (K, \mathop{\copy\noforkbox}\limits)$ be a $(< \alpha, \mathcal{F}, < \beta)$-independence relation. Define the *dual independence relation* $\mathfrak{i}^d := (K, \overset{d}{\mathop{\copy\noforkbox}\limits})$ by $\overset{d}{\mathop{\copy\noforkbox}\limits}(M, A, B, N)$ if and only if $\mathop{\copy\noforkbox}\limits(M, B, A, N)$.

The next proposition is straightforward:

Proposition 9.11.

(1) If $\mathfrak{i}$ is a $(< \alpha, \mathcal{F}, < \beta)$-independence relation, then $\mathfrak{i}^d$ is a $(< \beta, \mathcal{F}, < \alpha)$-independence relation and $\left(\mathfrak{i}^d\right)^d = \mathfrak{i}$.
(2) Let $\mathfrak{i}$ be a $(< \alpha, \mathcal{F}, < \alpha)$-independence relation. Then $\mathfrak{i}$ has symmetry if and only if $\mathfrak{i} = \mathfrak{i}^d$.

Definition 9.12. For P a property, we will say $\mathfrak{i}$ *has left* P if $\mathfrak{i}^d$ has right P, similarly if we swap left and right. When we omit left/right, we mean the right version of the property.

Definition 9.13 (Locality cardinals)**.** Let $\mathfrak{i} = (K, \mathop{\copy\noforkbox}\limits)$ be a $(< \alpha, \mathcal{F})$-independence relation, $\mathcal{F} = [\lambda, \theta)$. Let $\alpha_0 < \alpha$.

(1) Let $\bar{\kappa}_{\alpha_0}(\mathfrak{i})$ be the minimal cardinal $\mu \geq |\alpha_0|^+ + \lambda^+$ (could be ∞) such that for any $M \leq N$ in K, any $A \subseteq |N|$ with $|A| \leq \alpha_0$, there exists $M_0 \leq M$ in $K_{<\mu}$ with $A \overset{N}{\underset{M_0}{\mathop{\copy\noforkbox}\limits}} M$.

(2) For R a binary relation on K, Let $\kappa_{\alpha_0}(\mathfrak{i}, R)$ be the minimal cardinal $\mu \geq |\alpha_0|^+ + \aleph_0$ (could be ∞) such that for any regular $\delta \geq \mu$, any R-increasing (recall Definition 2.16) $\langle M_i : i < \delta \rangle$ in K, any N extending all the M_is, and any $A \subseteq |N|$ of size $\leq \alpha_0$, there exists $i < \delta$ such that $A \overset{N}{\underset{M_i}{\perp}} B_\delta$. Of course, we have set[17] $B_\delta := \bigcup_{i<\delta} |M_i|$. When $R = \leq$, we omit it.

When K is clear from context, we may write $\bar{\kappa}_\beta(\perp)$. For $\alpha_0 \leq \alpha$, we also let $\bar{\kappa}_{<\alpha_0}(\mathfrak{i}) := \sup_{\alpha_0' < \alpha_0} \kappa_{\alpha_0'}(\mathfrak{i})$. Similarly for $\kappa_{<\alpha_0}$.

We similarly define $\bar{\kappa}_{\alpha_0}(\mathfrak{s})$ and $\kappa_{\alpha_0}(\mathfrak{s})$ for $\mathfrak{s}$ a pre-frame (in the definition of $\kappa_{\alpha_0}(\mathfrak{s})$, we require in addition that B_δ be a model in K).

Instead of listing all the properties, we define what it means for an independence relation to be [fully] good. Being good is defined as in Shelah's definition of a good frame [She09a, Definition II.2.1]. Fully good is only relevant when the types are allowed to have length $\geq \lambda$, and asks for more continuity. This will be used in part 3.

Definition 9.14.

(1) A *good* $(< \alpha, \mathcal{F})$*-independence relation* $\mathfrak{i} = (K, \perp)$ is a $(< \alpha, \mathcal{F})$-independence relation satisfying:
   (a) K is an AEC in $\mathcal{F}$, $K \neq \emptyset$, K has no maximal models and joint embedding, K is stable in all cardinals in $\mathcal{F}$.
   (b) $\mathfrak{i}$ has disjointness, symmetry, base monotonicity, set-uniqueness, set-extension, and for all $\alpha_0 < \alpha$, $\kappa_{\alpha_0}(\mathfrak{i}) = |\alpha_0|^+ + \aleph_0$ and $\bar{\kappa}_{\alpha_0}(\mathfrak{i}) = |\alpha_0|^+ + \lambda_{\mathfrak{s}}^+$.

(2) An independence relation is *almost good* if it satisfies all the properties above except for symmetry, and set-uniqueness is replaced by uniqueness, set-extension by extension.

(3) A *type-full good* $(< \alpha, \mathcal{F})$*-frame* $\mathfrak{s} = (K, \perp)$ is a pre-$(< \alpha, \mathcal{F})$-frame satisfying the same conditions as a $(< \alpha, \mathcal{F})$-independence relation except we replace symmetry by full symmetry and set-uniqueness, set-extension by uniqueness, extension respectively. *Almost good* means we do not require full symmetry. We usually omit the "type-full".

When we add "fully", we require in addition that the frame/independence relation satisfies full model-continuity.

[17] Recall that K is only an ∞-abstract elementary class, so may not be closed under unions of chains of length δ.

Remark 9.15. Our definition of "almost" conflicts with that in [JS, Definition 2.3], but we do not rely on this paper. There type-full frames are simply called full (not to be confused with our definition of "fully").

Definition 9.16. An AEC K is *[fully] [almost]* $(<\alpha, \mathcal{F})$*-good* if there exists a [fully] [almost] $(<\alpha, \mathcal{F})$-good independence relation $\mathfrak{i}$ with $K_{\mathfrak{i}} = K$. When $\alpha = \infty$ and $\mathcal{F} = [\text{LS}(K), \infty)$, we omit them.

Finally, we define a notation for restricting independence relations to smaller domains:

Notation 9.17. Let $\mathfrak{i}$ be a $(<\alpha, \mathcal{F}, <\beta)$-independence relation.

(1) For $\alpha_0 \leq \alpha$, $\mathfrak{i}^{<\alpha_0, \beta_0}$ denotes the $(<\alpha_0, \mathcal{F}, <\beta_0)$ obtained by restricting the types to have length α_0 and the right hand side to have size less than β_0 (in the natural way). When $\beta_0 = \beta$, we omit it.
(2) When $\mathfrak{i}$ is a $(<\alpha, \mathcal{F})$-independence relation and $\mathcal{F}_0 \subseteq \mathcal{F}$ is an interval of cardinals, we let $\mathfrak{i}_{\mathcal{F}_0}$ be the restriction of $\mathfrak{i}$ to models of size in $\mathcal{F}_0$, and types of appropriate length (that is, if $\mathcal{F}_0 = [\lambda, \theta_0)$, $\mathfrak{i}_{\mathcal{F}_0} = \left(\mathfrak{i}^{<\min(\alpha, \theta_0)}\right)_{\mathcal{F}_0}$).
(3) For K^1 a sub-∞-AEC of $K_{\mathfrak{i}}$, let $\mathfrak{i} \upharpoonright K^1$ be the $(<\alpha, \mathcal{F}, <\beta)$-independence relation obtained by restricting the underlying class to K^1.

10. Coheir

We look at the natural generalization of coheir (introduced in [LP79] for first-order logic) to our context. A definition of coheir for classes of models in $L_{\kappa,\omega}$ was first introduced in [MS90] and later adapted to general AECs in [BGa]. We give a slightly more conceptual definition here and show that coheir has many of the properties of forking in a stable first-order theory.

Definition 10.1. Let N be a structure (in some possibly infinitary language), $A \subseteq |N|$, and p a set of formulas over N.

(1) p is a *heir over* A if for any formula $\phi(\bar{x}; \bar{b}) \in p$ over A, there exists $\bar{a} \in A$ such that $\phi(\bar{x}; \bar{a}) \in p \upharpoonright A$.
(2) p is a *coheir over* A *in* N if for any $\phi(\bar{x}) \in p$ there exists $\bar{a} \in A$ such that $N \models \phi[\bar{a}]$. When N is clear from context, we drop it.

Remark 10.2. Working in $N \in K$, let $\bar{c}$ be a permutation of $\bar{c}'$, and A, B be sets. Then $\text{tp}(\bar{c}/B; N)$ is a coheir over A if and only if $\text{tp}(\bar{c}'/B; N)$ is a coheir over A. Similarly for heir. Thus we can talk about $\text{tp}(C/B; N)$ being a heir/coheir over A without worrying about the enumeration of C.

Remark 10.3. When we work with the logic $qL_{\kappa,\kappa}$, we may talk about κ-coheir or κ-heir if κ is not clear from context.

We will mostly look at coheir, but the next proposition tells us how to express one in term of the other.

Proposition 10.4. $\text{tp}(\bar{a}/A\bar{b}; N)$ is a heir over A if and only if $\text{tp}(\bar{b}/A\bar{a}; N)$ is a coheir over A.

Proof. Straightforward. $\square$

We now make coheir into an independence relation.

Definition 10.5. Assume:

(1) κ is an infinite cardinal.
(2) K is a κ-AEC, $\widehat{K}$ is a $(<\kappa)$-ary $\widehat{L}$-Morleyization of K.
(3) If $M \leq N$ are in K, then $\widehat{M} \preceq_{\Sigma_1(\widehat{L}_{\kappa,\kappa})} \widehat{N}$.

Define $\mathbf{i}_{\kappa\text{-ch}}(K, \widehat{K}) := (K, \mathop{\smile\!\!\!\!\!\,|}\,)$ by $\mathop{\smile\!\!\!\!\!\,|}\,(M, A, B, N)$ if and only if $M \leq N$ are in K, $A \cup B \subseteq |N|$, and $\text{tp}_{q\widehat{L}_{\kappa,\kappa}}(A/MB; \widehat{N})$ is a coheir over M.

When $\widehat{K} = \widehat{K}^{<\kappa}$, we omit it.

Remark 10.6. In [She09b, Definition V.A.0.9], Shelah gives weaker relations than $\preceq_{\Sigma_1(\widehat{L}_{\kappa,\kappa})}$ in (3) that also suffice for our purpose, but we do not see the need to introduce them here.

Proposition 10.7. If $K_{<\text{LS}(K)} = \emptyset$ and K has amalgamation, then $\mathbf{i}_{\kappa\text{-ch}}(K, \widehat{K})$ is a $(<\infty, [\text{LS}(K), \infty))$-independence relation.

Proof. Straightforward. $\square$

In the definition above, the simplest case is when $\widehat{K} = K$, where we are simply studying syntactic coheir. The interesting case for us is $\mathbf{i}_{\kappa\text{-ch}}(K^{\kappa\text{-sat}})$, where K is an AEC. In this case, there is a more semantic definition:

Proposition 10.8. Let κ be an infinite cardinal and let K be a κ-AEC such that $K = K^{\kappa\text{-sat}}$ and K has amalgamation. Then:

(1) K satisfies (3) in Definition 10.5, with $\widehat{K}$ a Morleyization of $\widehat{K}^{<\kappa}$.

(2) $p \in \mathrm{gS}^{<\infty}(B; N)$ does not $\mathfrak{i}_{\kappa\text{-ch}}(K)$-fork over $M \leq N$ if and only if for any $I \subseteq \ell(p)$ and any $B_0 \subseteq B$, if $|I_0| + |B_0| < \kappa$, $p^I \upharpoonright B_0$ is realized in M.

Proof. Straightforward $\square$

This shows that in the relevant cases, our definition is equivalent to that of [BGa, Definition 3.2] (and to [BGKV, Definition 3.8], if we require that the right hand side contains the base). For completeness, we also compare our definition of heir to the semantic definition ([BGa, Definition 6.1]).

Proposition 10.9. Let κ be an infinite cardinal and let K be a κ-AEC with amalgamation such that $K = K^{\kappa\text{-sat}}$. Let $\widehat{K}$ be a $(<\kappa)$-ary $\widehat{L}$-Morleyization of $\widehat{K}^{<\kappa}$. Let $M_0 \leq M \leq N$.

Then $\mathrm{tp}_{q\widehat{L}_{\kappa,\kappa}}(\bar{a}/M; \widehat{N})$ is a heir over M_0 if and only if for all $(<\kappa)$-sized $I \subseteq \ell(\bar{a})$, $M_0^- \leq M$, $M_0^- \leq M^- \leq M$ (where we allow M_0^- to be empty), there is $f : M^- \xrightarrow[M_0^-]{} M_0$ such that $\mathrm{gtp}(\bar{a}/M; N)$ extends $f(\mathrm{gtp}((\bar{a} \upharpoonright I)/M^-; N))$.

Proof. Assume first $\mathrm{tp}_{q\widehat{L}_{\kappa,\kappa}}(\bar{a}/M; \widehat{N})$ is a heir over M_0 and let I, $M_0^- \leq M- \leq M$ be as given by the semantic definition. Let $p := \mathrm{gtp}((\bar{a} \upharpoonright I)/M^-; N)$. Let $\bar{b}_0$ be an enumeration of M_0^- and let $\bar{b}$ be an enumeration of $|M^-| \backslash |M_0^-|$. Let $q := \mathrm{gtp}((\bar{a} \upharpoonright I)\bar{b}_0\bar{b}/\emptyset; N)$. Consider the formula $\phi(\bar{x}; \bar{b}; \bar{b}_0) := R_q(\bar{x}; \bar{b}; \bar{b}_0)$. Then ϕ is in $\mathrm{tp}_{q\widehat{L}_{\kappa,\kappa}}(\bar{a}/M; \widehat{N})$. By the syntactic definition of heir, there is $\bar{c} \in M_0$ such that $\phi(\bar{x}; \bar{c}; \bar{b}_0)$ is in $\mathrm{tp}_{q\widehat{L}_{\kappa,\kappa}}(\bar{a}/M_0; \widehat{N})$. By definition of $\widehat{K}^{<\kappa}$ this means that $\mathrm{gtp}(\bar{a} \upharpoonright I\bar{b}\bar{b}_0/\emptyset; N) = \mathrm{gtp}((\bar{a} \upharpoonright I)\bar{c}\bar{b}_0/\emptyset)$. By definition of Galois types, saturation of M, and amalgamation, there is $f : M^- \xrightarrow[M_0^-]{} M_0$ witnessing the equality, which is as desired.

The converse is similar. $\square$

Remark 10.10. The above proof shows the advantage of using a syntactic notation rather than working purely semantically.

We now investigate the properties of coheir.

Theorem 10.11. Let $K, \widehat{K}$ satisfy the assumptions of Definition 10.5 and $K_{<\mathrm{LS}(K)} = \emptyset$. Let $\mathfrak{i} := \mathfrak{i}_{\kappa\text{-ch}}(K, \widehat{K})$.

(1) If K has amalgamation, $\mathfrak{i}$ is a $(<\infty, [\mathrm{LS}(K), \infty))$-independence relation with disjointness, base monotonicity, left and right existence, left and right $(<\kappa)$-set-witness property, and strong left transitivity.
(2) If $\widehat{K}$ does not have the syntactic $(<\kappa)$-order property of length κ, then:
   (a) $\mathfrak{i}$ has symmetry and strong right transitivity.
   (b) For all α, $\bar{\kappa}_\alpha(\mathfrak{i}) \leq \left((\alpha + \mathrm{LS}(K))^{<\kappa}\right)^+$.
   (c) $\mathfrak{i}$ has syntactic uniqueness: If $M_0 \leq M \leq N_\ell$ for $\ell = 1, 2$, $|M_0| \subseteq B \subseteq |M|$. $q_\ell \in \mathrm{S}^{<\infty}(B; \widehat{N_\ell})$, $q_1 \upharpoonright M_0 = q_2 \upharpoonright M_0$ and q_ℓ is a coheir over M_0 in $\widehat{N_\ell}$ for $\ell = 1, 2$, then $q_1 = q_2$.
   (d) Stability: For α an ordinal, $\widehat{K}$ is locally syntactically α-stable in all $\lambda \geq \mathrm{LS}(\widehat{K})$ such that $\lambda^{<\bar{\kappa}_\alpha(\mathfrak{i})} = \lambda$.
   (e) If $\widehat{K}$ is a Morleyization of $\widehat{K}^{<\kappa}$:
      (i) If K is $(<\kappa)$-tame and short for types of length less than α, then $\mathfrak{i}^{<\alpha}$ has uniqueness.
      (ii) If $\theta > \mathrm{LS}(K)$ and K is $(<\kappa)$-tame and short for types of length less than θ, then $\mathfrak{i}_{[\mathrm{LS}(K), \theta)}$ has set-uniqueness.

Proof. Observe that (except for the uniqueness properties), one can work in $\widehat{K}$ and prove the properties there using purely syntactic methods (so amalgamation is never needed). More specifically, (1) is straightforward. As for (2), symmetry is exactly as in[18] [Pil82, Proposition 3.1], strong right transitivity follows from strong left transitivity and symmetry, local character is as in the proof of $(B)_\mu$ in [MS90, Proposition 4.8] and syntactic uniqueness is as in the proof of (U) in [MS90, Proposition 4.8].

The proof of stability is as in the first-order case (and will not be used). To get the last uniqueness properties, use the translation between Galois and syntactic types (Theorem 4.13). $\square$

Remark 10.12. We can give localized version of some of the above results. For example in the statement of the symmetry property it is enough to assume M does not have the syntactic $(<\kappa)$-order property of length κ. We could also have been more precise and state the uniqueness property in terms of being $(<\kappa)$-tame and short for $\{q_1, q_2\}$, where q_1, q_2 are the two Galois types we are comparing.

[18]Note that a proof of symmetry of nonforking from no order property already appears in [She78], but Pillay's proof for coheir is the one we use here.

Remark 10.13. Taking κ to be a fixed point of the Beth function, we can use Theorem 10.11.(2d) to get another proof of the equivalence between (syntactic) stability and no order property.

Remark 10.14. The extension property seems to be more problematic. In [BGa], Boney and Grossberg simply assumed it (they also showed that it followed from κ being strongly compact [BGa, Theorem 8.2.1]. It is easy to show the same result using Corollary 5.9). It will finally be proven[19] in part 3, using a superstability-like assumption.

11. Some independence calculus

We investigate relationship between properties and how to go from a frame to an independence relation. Most of it appears already in [BGKV] and has a much longer history, described there. The only new contribution of this section is how to obtain the witness properties from tameness and shortness (Proposition 11.1.(6)). This will be used in part 3. It also partially answers [BGKV, Question 5.5].

Proposition 11.1. Let $\mathfrak{i} = (K, \perp)$ be a $(< \alpha, \mathcal{F})$-independence relation with base monotonicity.

(1) If $\mathfrak{i}$ has set-extension, then it has symmetry if and only if it has full symmetry.
(2) If $\mathfrak{i}$ has the $(< \kappa)$-set-witness property, then it has the $(< \kappa)$-witness property.
(3) If $\mathfrak{i}$ has strong transitivity, then it has transitivity.
(4) If $\mathfrak{i}$ has uniqueness and extension, then it has transitivity.
(5) If $\mathfrak{i}$ has uniqueness and extension, then it has set-uniqueness. If $\mathfrak{i}$ has set-extension, then it has extension. If $\mathfrak{i}$ has set-uniqueness, then it has uniqueness.
(6) If $\mathfrak{i}$ has set-extension and uniqueness, then:
    (a) If K is $(< \kappa)$-tame for types of length $< \alpha$, then K has the right $(< \kappa)$-witness property.
    (b) If K is $(< \kappa)$-tame and short for types of length $< \theta_{\mathfrak{i}}$, then K has the right $(< \kappa)$-set-witness property. More precisely, if the set on the right hand side has size less than β, it is enough to require $(< \kappa)$-tameness and shortness for types of length less than $(\alpha + \beta)$.

[19] A word of caution: In [HL02, Section 4], the authors give an example of an ω-stable class that does not have extension. However, the extension property they consider is *over all sets*, not only over models.

(7) For $\alpha > \lambda$, if $\mathfrak{i}$ has extension and existence, then it has independent amalgamation. Conversely, if $\mathfrak{i}$ has transitivity and independent amalgamation, then it has extension and existence. Moreover if $\mathfrak{i}$ has uniqueness and independent amalgamation, then it has transitivity.
(8) If for all $\alpha_0 < \alpha$, $\min(\kappa_{\alpha_0}(\mathfrak{i}), \bar{\kappa}_{\alpha_0}(\mathfrak{i})) < \infty$, then $\mathfrak{i}$ has existence.
(9) If K is a μ-AEC (or just $\mathrm{TV}_{<\theta}(K) \leq \mu$), then for $\alpha_0 < \alpha$, $\kappa_{\alpha_0}(\mathfrak{i}) \leq \bar{\kappa}_{\alpha_0}(\mathfrak{i}) + \mu$.
(10) If K is a stable AEC with no maximal models, $\theta = \infty$ and $\mathfrak{i}$ has right uniqueness, existence, and extension, then $\mathfrak{i}$ has full symmetry. In particular, if $\mathfrak{i}$ is good, except perhaps for symmetry, and $\theta_{\mathfrak{i}} = \infty$, then $\mathfrak{i}$ is good.

Proof.

(1) Easy.
(2) Trivial.
(3) Trivial.
(4) As in the proof of [She09a, Claim II.2.18].
(5) The last two sentences are trivial. For the first one, let $M_0 \leq M_\ell \leq N$, B, q_ℓ, be as in by the definition of right set-uniqueness. For $\ell = 1, 2$, let q'_ℓ be an extension of q_ℓ to N that does not fork over M_0. By assumption, $q'_1 \upharpoonright M_0 = q_1 \upharpoonright M_0 = q_2 \upharpoonright M_0 = q'_2 \upharpoonright M_0$, so by uniqueness $q'_1 = q'_2$ and so $q_1 \upharpoonright M_0 B = q_2 \upharpoonright M_0 B$.
(6) (a) Let $M \leq M' \leq N$ be in K, $A \subseteq |N|$ have size less than α. Assume $A \overset{N}{\underset{M}{\perp}} B_0$ for all $B_0 \subseteq |M'|$ of size less than κ. We want to show that $A \overset{N}{\underset{M}{\perp}} M'$. Let $\bar{a}$ be an enumeration of A, $p := \mathrm{gtp}(\bar{a}/M; N)$. Note that (taking $B_0 = \emptyset$ above) normality implies p does not fork over M. By extension, let $q \in \mathrm{gS}^{<\alpha}(M')$ be an extension of p that does not fork over M. Using amalgamation and some renaming, we can assume without loss of generality that q is realized in N. Let $p' := \mathrm{gtp}(\bar{a}/M'; N)$. We claim that $p' = q$, which is enough by invariance. By the tameness assumption, it is enough to check that $p' \upharpoonright B_0 = q \upharpoonright B_0$ for all $B_0 \subseteq |M'|$ of size less than κ. Fix such a B_0. By assumption, $p' \upharpoonright B_0$ does not fork over M_0. By set-extension, find $N' \geq N$, $p'' \in \mathrm{gS}^{<\alpha}(M'; N')$ an extension of $p' \upharpoonright |M| B_0$ that does not fork over M_0. By uniqueness, $q = p''$, so $q \upharpoonright B_0 = p'' \upharpoonright B_0 = p' \upharpoonright B_0$, as desired.

(b) Similar to before. In the end, we appeal to Lemma 2.28 instead of tameness.

(7) The first sentence is easy, since independent amalgamation is a particular case of extension and existence. Moreover to show existence it is enough by monotonicity to show it for types of models. The proof of transitivity from uniqueness and independent amalgamation is as in (4).
(8) By definition of the local character cardinals.
(9) If $\delta = \text{cf}(\delta) \geq \bar{\kappa}_{\alpha_0}(\mathfrak{i}) + \mu$ and $\langle M_i : i < \delta \rangle$ is increasing in K, then (if it has size less than $\theta_{\mathfrak{i}}$), $M_\delta := \bigcup_{i<\delta} M_i$ is in K by definition of μ and if $p \in \text{gS}^{\leq \alpha_0}(M_\delta)$, by definition of $\bar{\kappa}_{\alpha_0}$ there exists $N \leq M_\delta$ of size less than $\bar{\kappa}_{\alpha_0}(\mathfrak{i})$ such that p does not fork over N. Now use regularity of δ to find $i < \delta$ with $N \leq M_i$.
(10) As in [BGKV, Corollary 5.16].

$\square$

Next, we investigate what properties are preserved by the operations cl and pre. This was done already in [BGKV, Section 5.1], so we mostly cite from there.

Proposition 11.2.

(1) If $\mathfrak{i}$ is an independence relation and pre($\mathfrak{i}$) has extension, then cl(pre($\mathfrak{i}$)) $= \mathfrak{i}$ if and only if $\mathfrak{i}$ has set-extension.
(2) If $\mathfrak{s}$ is a pre-frame, then cl($\mathfrak{s}$) has symmetry if and only if $\mathfrak{s}$ has full symmetry.
(3) If $\mathfrak{s}$ is a pre-frame with extension, then cl($\mathfrak{s}$) has set-uniqueness if and only if $\mathfrak{i}$ has uniqueness.
(4) If $\mathfrak{s}$ is an [almost] good frame, then cl($\mathfrak{s}$) is a good [good except perhaps for symmetry] independence relation. If $\mathfrak{i}$ is an [almost] good independence relation, then pre($\mathfrak{i}$) is an [almost] good frame.
(5) If $\mathfrak{i}$ is an [almost] good independence relation, then cl(pre($\mathfrak{i}$)) is good [good except perhaps for symmetry, and if $\theta_{\mathfrak{i}} = \infty$, then it is good].

Proof.

(1) As in [BGKV, Lemma 5.3].
(2) Easy.
(3) As in [BGKV, Lemma 5.4.3] (or by Proposition 11.1.(5)).
(4) Check that all the properties transfer, using the previous parts.
(5) Easy. The last sentence is by Proposition 11.1.(10).

□

The next lemma clarifies the relationship between full model continuity and local character. It will be used in part 3.

Lemma 11.3. Let $\mathfrak{i} = (K, \mathop{\smile\!\!\!\!\!\!|}\,)$ be a $(< \alpha, \mathcal{F})$-independence relation, $\mathcal{F} = [\lambda, \theta)$ and $\lambda < \alpha \leq \theta$. Assume:

(1) K is an AEC with $\text{LS}(K) = \lambda$.
(2) $\mathfrak{i}$ has base monotonicity, transitivity, and left set-extension.
(3) $\mathfrak{i}$ has full model continuity.
(4) $\mu_0 \geq \lambda^+$ is a cardinal such that for all $\alpha_0 < \mu_0$, $\bar{\kappa}_{\alpha_0}(\mathfrak{i}) = \lambda^+ + |\alpha_0|^+$.

Then for all cardinal $\mu < \alpha$, $\bar{\kappa}_\mu(\mathfrak{i}) = \lambda^+ + \mu^+$.

Proof. By induction on μ. If $\mu < \mu_0$ this holds by hypothesis, so assume $\mu \geq \mu_0$. In particular, $\lambda < \mu$.

Let $M \leq N$ be in K and let $A \subseteq |N|$ have size μ. We want to find $M_0 \leq M$ such that $A \overset{N}{\underset{M_0}{\mathop{\smile\!\!\!\!\!\!|}}} M$ and $\|M_0\| \leq \mu$. Say $A = \{a_i : i < \mu\}$. Build increasing continuous $\langle M_i^0 : i \leq \mu \rangle$, $\langle N_i^0 : i \leq \mu \rangle$, $\langle N_i : i \leq \mu \rangle$ such that for all $i < \mu$:

(1) $M_i \leq M$, $N \leq N_i$.
(2) $N_i^0 \leq N_i$.
(3) $M_i \leq N_i^0$.
(4) $a_i \in |N_{i+1}^0|$.
(5) $\|N_i^0\| < \mu$.
(6) For $i > 0$, $N_i^0 \overset{N_i}{\underset{M_i}{\mathop{\smile\!\!\!\!\!\!|}}} M$.

This is possible. By induction on $i < \mu$. If i is limit, take unions. If $i = 0$, take $N_0 := N$, any $M_0 \leq M$ in $K_{<\mu}$, and any $N_0^0 \leq N$ in $K_{<\mu}$ with $M_0 \leq N_0^0$.

Assume now that $i = j + 1$ and M_j, N_j, N_j^0 have been defined. Since $\|N_j^0\| < \mu$, the induction hypothesis implies there exists $M_i \leq M$ in $K_{<\mu}$ such that $a_j N_j^0 \overset{N_j}{\underset{M_i}{\mathop{\smile\!\!\!\!\!\!|}}} M$, and without loss of generality $M_j \leq M_i$. By left set-existence, we can find $N_i \geq N_j$ and $N_i^0 \leq N_j$ in $K_{<\mu}$ such that $a_j \in |N_i^0|$, $M_i \leq N_i^0$, and $N_i^0 \overset{N_i}{\underset{M_i}{\mathop{\smile\!\!\!\!\!\!|}}} M$.

This is enough. $A \subseteq |N_\mu^0|$ and by full model continuity, $N_\mu^0 \overset{N_\mu}{\underset{M_\mu}{\mathop{\smile\!\!\!\!\!\!\;|}}} M$. Also, $M_\mu \leq N_\mu^0$ which is in $K_{\leq \mu}$, and by monotonicity, $A \overset{N}{\underset{M_\mu}{\mathop{\smile\!\!\!\!\!\!\;|}}} M$, as desired. $\square$

The next technical lemma is also used in part 3. Roughly, it gives conditions under which we can take the base model given by local character to be contained both in the right hand side and in the left hand side.

Lemma 11.4. Let $\mathfrak{i} = (K, \mathop{\smile\!\!\!\!\!\!\;|})$ be a $(< \alpha, \mathcal{F})$-independence relation, $\mathcal{F} = [\lambda, \theta)$, with $\alpha > \lambda$. Assume:

(1) K is an AEC with $\text{LS}(K) = \lambda$.
(2) $\mathfrak{i}$ has base monotonicity and transitivity.
(3) μ is a cardinal, $\lambda \leq \mu < \theta$.
(4) $\mathfrak{i}$ has the left $(< \kappa)$-witness property for $\kappa \leq \mu$ regular.
(5) $\bar{\kappa}_\mu(\mathfrak{i}) = \mu^+$.

Let $M^0 \leq M^\ell \leq N$ be in K, $\ell = 1, 2$ and assume $M^1 \overset{N}{\underset{M^0}{\mathop{\smile\!\!\!\!\!\!\;|}}} M^2$. Let $A \subseteq |M^1|$, be such that $|A| \leq \mu$. Then there exists $N^1 \leq M^1$ and $N^0 \leq M^0$ such that:

(1) $A \subseteq |N^1|$, $A \cap |M^0| \subseteq |N^0|$.
(2) $N^0 \leq N^1$ are in $K_{\leq \mu}$.
(3) $N^1 \overset{N}{\underset{N^0}{\mathop{\smile\!\!\!\!\!\!\;|}}} M^2$.

Proof. For $\ell = 0, 1$, we build $\langle N_i^\ell : i \leq \kappa \rangle$ increasing continuous in $K_{\leq \mu}$ such that for all $i < \kappa$ and $\ell = 0, 1$:

(1) $A \subseteq |N_0^1|$, $A \cap |M^0| \subseteq N_0^0$.
(2) $N_i^\ell \leq M^\ell$.
(3) $N_i^0 \leq N_i^1$.
(4) $N_i^1 \overset{N}{\underset{N_{i+1}^0}{\mathop{\smile\!\!\!\!\!\!\;|}}} M^2$.

This is possible. Pick any $N_0^0 \leq M^0$ in $K_{\leq \mu}$ containing $A \cap |M^0|$. Now fix $i < \kappa$ and assume inductively that $\langle N_j^0 : j \leq i \rangle$, $\langle N_j^1 : j < i \rangle$ have been built. If i is a limit, we take unions. Otherwise, pick any $N_i^1 \leq M^1$ in $K_{\leq \mu}$ that contains A, N_j^1 for all $j < i$ and N_i^0. Now use right transitivity and $\bar{\kappa}_\mu(\mathfrak{i}) = \mu^+$ to find $N^0_{i+1} \leq M^0$ such that $N^1_i \underset{N^0_{i+1}}{\overset{N}{\mathop{\perp}}} M^2$. By base monotonicity, we can assume without loss of generality that $N^0_i \leq N^0_{i+1}$.

<u>This is enough</u>. We claim that $N^\ell := N^\ell_\kappa$ are as required. By coherence, $N^0 \leq N^1$ and since $\kappa \leq \mu$ they are in $K_{\leq \mu}$. Since $A \subseteq |N^1_0|$, $A \subseteq |N^1|$. It remains to see $N^1 \underset{N^0}{\overset{N}{\mathop{\perp}}} M^2$. By the left witness property, it is enough to check it for every $B \subseteq |N^1|$ of size less than κ. Fix such a B. Since κ is regular, there exists $i < \kappa$ such that $B \subseteq |N^1_i|$. By assumption and monotonicity, $B \underset{N^0_{i+1}}{\overset{N}{\mathop{\perp}}} M^2$. By base monotonicity, $B \underset{N^0_\kappa}{\overset{N}{\mathop{\perp}}} M^2$, as needed. $\square$

12. Splitting and canonicity of weakly good frames

We now consider another independence relation. This was first defined in [She99, Definition 3.2].

Definition 12.1 (μ-nonsplitting)**.** Let K be an ∞-AEC with amalgamation.

(1) For μ an infinite cardinal, define $\mathfrak{s}_{\mu\text{-ns}}(K) := (K, \mathop{\perp})$ by $\bar{a} \underset{M_0}{\overset{N}{\mathop{\perp}}} M$ if and only if $M_0 \leq M \leq N$, $A \subseteq |N|$, and whenever $M_0 \leq N_\ell \leq M$, $N_\ell \in K_{\leq \mu}$, $\ell = 1, 2$, and $f : N_1 \cong_{M_0} N_2$, then $f(\text{gtp}(\bar{a}/N_1; N)) = \text{gtp}(\bar{a}/N_2; N)$.
(2) Define $\mathfrak{s}_{\text{ns}}(K)$ to have underlying AEC K and nonforking relation defined such that p does not $\mathfrak{s}_{\text{ns}}(K)$-fork if and only if p does not $\mathfrak{s}_{\mu\text{-ns}}(K)$-fork for all infinite cardinals μ.

Proposition 12.2. Assume K is an ∞-AEC in $\mathcal{F} = [\lambda, \theta)$ with amalgamation. Let $\mathfrak{s} := \mathfrak{s}_{\text{ns}}(K)$.

(1) $\mathfrak{s}$ is a pre-$(< \infty, \mathcal{F})$-frame.
(2) $\mathfrak{s}$ has base monotonicity, left and right existence, and the $(< \mu^+)$-witness property.
(3) If K is an AEC in $\mathcal{F}$ and is stable in λ, then $\bar{\kappa}_{<\omega}(\mathfrak{s}_{\lambda\text{-ns}}(K)) = \lambda^+$.
(4) If $\mathfrak{t}$ is a pre-$(< \infty, \mathcal{F})$-frame with uniqueness and $K_{\mathfrak{t}} = K$, then whenever a type does not $\mathfrak{t}$-fork over M, it does not $\mathfrak{s}$-fork over M.

(5) If K is λ-tame for types of length less than α, then a type of length less than α does not $\mathfrak{s}_{\lambda\text{-ns}}(K)$-fork over M if and only if it does not $\mathfrak{s}$-fork over M.

Proof.

(1) Easy.
(2) Easy.
(3) By [She99, Claim 3.3.1].
(4) By [BGKV, Lemma 4.2].
(5) Straightforward (or see [BGKV, Proposition 3.12]).

$\square$

Proposition 12.2.(4) tells us that any reasonable independence relation will be extended by nonsplitting. In a sense, nonsplitting can be seen as a *maximal* candidate for an independence relation[20]. We abstract this feature into a definition:

Definition 12.3. Let $\mathfrak{s}$ be a pre-frame and let $\beta \leq \alpha_{\mathfrak{s}}$. $\mathfrak{s}$ is $(< \beta)$-*weakly good* if:

(1) $\mathfrak{s}$ has base monotonicity.
(2) $\mathfrak{s}$ is extended by nonsplitting: Whenever a type does not $\mathfrak{s}$-fork over M, it does not $\mathfrak{s}_{\text{ns}}(K_{\mathfrak{s}})$-fork over M.
(3) $\beta \leq \lambda_{\mathfrak{s}}^+$ and $\bar{\kappa}_{<\beta}(\mathfrak{s}) = \lambda_{\mathfrak{s}}^+$.

We say an independence relation $\mathfrak{i}$ is $(< \beta)$-weakly good if $\text{pre}(\mathfrak{i})$ is. When $\beta = \alpha$, we omit it.

Remark 12.4. We allow $\beta = 1$ in the definition above, in which case the second condition holds for free. Also note that if $\beta \leq \beta' \leq \alpha_{\mathfrak{s}}$ and $\mathfrak{s}$ is $(< \beta')$-weakly good, then $\mathfrak{s}$ is $(< \beta)$-weakly good.

We think of *weakly good* as being an analog of *good* for the strictly stable case. The next proposition gives examples of weakly good frames.

Proposition 12.5.

(1) If $\mathfrak{s}$ is an almost good frame and $\beta \leq \min(\alpha_{\mathfrak{s}}, \lambda_{\mathfrak{s}}^+)$, then $\mathfrak{s}$ is $(< \beta)$-weakly good.
(2) Let κ be an infinite cardinal, K be a κ-AEC with amalgamation such that $K = K^{\kappa\text{-sat}}$ and K does not have the $(< \kappa)$-order

[20] In a sense one can make precise, κ-coheir is a *minimal* candidate for such an independence relation.

property of length κ. Let $\mathfrak{i} := \mathfrak{i}_{\kappa\text{-ch}}(K)$. For any α, if K is $(<\kappa)$-tame and short for types of length less than α, $\bar{\kappa}_{<\alpha}(\mathfrak{i}) \leq \lambda^+$, and[21] $\lambda = \lambda^{<\kappa}$, then $(\mathfrak{i}_{\geq \lambda})^{<\alpha}$ is weakly good.

(3) If $\mathcal{F} = [\lambda, \theta)$ is an interval of cardinals, K is an AEC in $\mathcal{F}$ with amalgamation and K is stable in λ, then $(\mathfrak{s}_{\lambda\text{-ns}}(K))^{<\omega}$ is weakly good.

Proof.

(1) By definition of a almost good frame and Proposition 12.2.(4).
(2) By the uniqueness and local character properties of coheir.
(3) By Proposition 12.2.(3).

□

Here are some properties of weakly good frames:

Proposition 12.6. Let $\mathfrak{s} = (K, \mathop{\smile}\limits^{\mid})$ be a weakly good $(<\alpha, \mathcal{F})$-frame, $\mathcal{F} = [\lambda, \theta)$.

(1) Weak extension: Let $M_0 <_{\text{univ}} M \leq N$ be in K, $\|M\| = \|N\|$. If $p \in \text{gS}^{<\alpha}(M)$ does not fork over M_0, then there exists $q \in \text{gS}^{<\alpha}(N)$ extending p and not forking over M_0. Moreover if $\ell(p) = 1$, then p is algebraic if and only if q is.
(2) Weak uniqueness: Let $M_0 <_{\text{univ}} M \leq N$ be in K and assume K is $(\leq \|M_0\|)$-tame for types of length less than α over models of size $\leq \|N\|$. If $p, q \in \text{gS}^{<\alpha}(N)$ does not fork over M_0 and $p \upharpoonright M = q \upharpoonright M$, then $p = q$.
(3) For any $\lambda < \mu < \theta$, $\left(\mathfrak{s} \upharpoonright K^{\lambda^+\text{-mh}}\right)_\mu$ has extension.
(4) If K is λ-tame for types of length less than α, then $\mathfrak{s} \upharpoonright K^{\lambda^+\text{-mh}}$ has uniqueness.
(5) If K is λ-tame, then $\mathfrak{s} \upharpoonright K^{\lambda^+\text{-mh}}$ has disjointness.
(6) If K is λ-tame, then $K^{\lambda^+\text{-mh}}$ is stable in all $\mu > \lambda$ such that $\mu^\lambda = \mu$.
(7) If K is λ-tame and $K = (K_0)^{\lambda_0\text{-sat}}$ for K_0 an AEC with amalgamation, $\text{LS}(K) \leq \lambda$ and $\lambda_0 \leq \lambda$, then K is stable in all $\mu > \lambda$ such that $\mu^\lambda = \mu$.

Proof.

(1) As in [Van06, Theorem I.4.10].

[21] In fact, it is enough to require that for all $M \in K$, $A \subseteq |M|$, there exists $M_0 \leq M$ containing A of size $\leq \lambda$.

(2) As in [Van06, Theorem I.4.12] (or see [Bal09, Theorem 12.7]).
(3) Let $p \in \mathrm{gS}^{<\alpha}(M)$, $M \in K_\mu^{\lambda^+\text{-mh}}$. By set local character, there exists $M_0 \leq M$ in K_λ such that p does not fork over M_0. Since M is λ^+-model homogeneous, $M_0 <_{\mathrm{univ}} M$. Now use weak extension.
(4) Similar.
(5) It is enough to prove disjointness for types of length 1 so assume $\alpha = 2$. Assume $a \overset{N}{\underset{M_0}{\mathop{\smile}\limits^{\mid}}} M$ (with $M_0 \leq M \leq N$ in $K^{\lambda^+\text{-mh}}$) and $a \in M$. We show $a \in M_0$. Note that by extension and uniqueness we also have transitivity. By set local character, find $M_0' \leq M_0$ in K_λ such that $a \overset{N}{\underset{M_0'}{\mathop{\smile}\limits^{\mid}}} M$. By weak extension, we can find $p \in \mathrm{gS}(M)$ extending $p_0 := \mathrm{gtp}(a/M_0'; N)$ and not forking over M_0'. By the moreover part of weak extension, p_0 will be algebraic if and only if p is. Since $a \in N$, we must have by uniqueness that p is algebraic so p_0 is algebraic, i.e. $a \in M_0' \leq M_0$.
(6) Imitate the usual argument from the first-order theory.
(7) Apply the previous part, using that if $\mu^\lambda = \mu$, then every $M \in K_\mu$ embeds into a λ^+-model-homogeneous model of size μ.

$\square$

Remark 12.7. For weak extension and weak uniqueness to hold, it suffices that the frame be (< 1)-weakly good.

Regarding the extension property, we can say slightly more for coheir:

Proposition 12.8. Let κ be an infinite cardinal, K be a κ-AEC with amalgamation such that $K = K^{\kappa\text{-sat}}$ and K does not have the $(< \kappa)$-order property of length κ. Let $\mathfrak{i} := \mathfrak{i}_{\kappa\text{-ch}}(K)$. Let α be an ordinal and let $\lambda^+ \geq \bar{\kappa}_{<\alpha}(\mathfrak{i})$. Let $\mathfrak{i}' := \left(\mathfrak{i} \upharpoonright K^{\lambda^+\text{-mh}}\right)^{<\alpha}$.

Let $\mu \geq \lambda^+$ and assume K is $(< \kappa)$-tame and short for types of length $\leq \mu$. Then $(\mathfrak{i}')_\mu$ has set-extension.

More precisely, if $M \leq N$ are in $K_{\mathfrak{i}'}$, $B \subseteq C \subseteq |N|$, $p \in \mathrm{gS}^{<\alpha}(MB; N)$ does not $\mathfrak{i}'$-fork over M, $|C| \leq \|M\|$ and K is $(< \kappa)$-tame and short for types of length less than $(\alpha + |B|^+)$, then there exists $N' \geq N$ and $q \in \mathrm{gS}^{<\alpha}(MC; N')$ extending p which does not $\mathfrak{i}'$-fork over M.

Proof. It is enough to prove the last sentence. Without loss of generality, $\|M\| = \|N\|$ and $C = N$. Let $\mu := \|N\|$. By Proposition 12.6, $(\mathfrak{i}')_\mu$ has extension, so let $q \in \mathrm{gS}^{<\alpha}(N)$ be an extension of $p \upharpoonright M$ that does not fork over M. We claim that $q \upharpoonright MB = p$. To see this, use syntactic uniqueness (Theorem 10.11.(2c)) together with the correspondence between Galois-syntactic types and Lemma 2.28. $\square$

Next, we turn our eye to proving a canonicity theorem. We give a parametrized version of the property (E_+) introduced in [BGKV, Definition 4.4]. This is the key property the paper used to prove canonicity. The definition is further complicated by the lack of a monster model.

Definition 12.9. Let $\mathfrak{i} = (K, \mathop{\smile\!\!\!\!\!\mid}\,)$ be a $(< \alpha, \mathcal{F})$-independence relation and let $K^1 \subseteq K$. For $\beta \leq \theta_{\mathfrak{i}}$, we say $\mathfrak{i}$ has $(< \beta)$-*strong extension-existence* if for any $M_0 \leq M$ with $M_0 \in K^1$, any $A \subseteq |M|$ with $|A| < \alpha$, and any $\bar{b} \in {}^{<\beta}|M|$, there exists $N \geq M$ with the following property:

Whenever $N', \widehat{N}'$ are such that:

(1) $M_0 \leq N' \leq \widehat{N}'$ are in K, $\bar{b} \in {}^{<\beta}|N'|$.
(2) $M \leq \widehat{N}'$.
(3) $N' \cong_{M_0 \bar{b}} N$

Then there exists $\bar{b}' \in N'$ so that $\text{gtp}(\bar{b}'/M_0; N') = \text{gtp}(\bar{b}/M_0; M)$ and $A \overset{\widehat{N}'}{\underset{M_0}{\mathop{\smile\!\!\!\!\!\mid}}} \bar{b}'$.

Fact 12.10. Let $\mathfrak{i} = (K, \mathop{\smile\!\!\!\!\!\mid}\,)$ be a $(< \alpha, [\lambda, \theta))$-independence relation and assume every chain of ordinal length less than θ in K has an upper bound. Let $\beta \leq \alpha$. If:

(1) $\mathfrak{i}$ has base monotonicity, symmetry, and transitivity.
(2) For every $\alpha_0 < \alpha$ there exists a regular $\mu < \theta$ such that:
 (a) $\kappa_{\alpha_0}(\mathfrak{i}) \leq \mu$.
 (b) For all $\lambda_1 \in [\lambda, \theta)$, $\left(\mathfrak{i}_{[\lambda_1, \lambda_1^+ + \mu + |\beta|)}\right)^{<\beta}$ has extension.

Then $\mathfrak{i}$ has $(< \beta)$-strong extension-existence. More precisely, for every $\lambda_1 \in [\lambda, \theta)$, $\mathfrak{i}_{[\lambda_1, \lambda_1^+ + \mu^+ + |\beta|)}$ has $(< \beta)$-strong extension-existence.

Proof. As in the proof of [BGKV, Corollary 4.13]. $\square$

We can now prove a canonicity theorem. The crucial result is:

Fact 12.11. Let K be an ∞-AEC in $\mathcal{F} = [\lambda, \theta)$. Let $\mathfrak{i}$, $\mathfrak{i}'$ be $(< \alpha, \mathcal{F})$-independence relations with underlying AEC K. If:

(1) $\mathfrak{i}'$ has base monotonicity, uniqueness, existence, and extension.

(2) $\mathfrak{i}$ has base monotonicity, uniqueness, the $(<\beta)$-witness property, and $(<\beta)$-strong extension-existence, for $\beta \le \theta$.

Then $\text{pre}(\mathfrak{i}) = \text{pre}(\mathfrak{i}')$.

Proof. As in the proof of [BGKV, Corollary 4.8]: it is enough to show that $\underset{\mathfrak{i}'}{\perp}(A, M, N_0, N)$ implies $\underset{\mathfrak{i}}{\perp}(A, M, N_0, N)$ (where of course $M \le N_0 \le N$). So assume $\underset{\mathfrak{i}'}{\perp}(A, M, N_0, N)$. By the $<\beta$-witness property for $\mathfrak{i}$, it is enough to show $\underset{\mathfrak{i}}{\perp}(A, M, B, N)$. Now proceed as in the proof of [BGKV, Lemma 4.7], using explicit splitting instead of splitting. $\square$

One can ask for when $\mathfrak{i} = \mathfrak{i}'$ rather than just $\text{pre}(\mathfrak{i}) = \text{pre}(\mathfrak{i}')$. There is an easy answer:

Lemma 12.12. Let $\mathfrak{i}$ and $\mathfrak{i}'$ be independence relations on the same ∞-AEC K. Assume $\mathfrak{i}$ and $\mathfrak{i}'$ have extension and $\text{pre}(\mathfrak{i}) = \text{pre}(\mathfrak{i}')$. If:

(1) $\mathfrak{i}$ and $\mathfrak{i}'$ have the $(<\kappa)$-set-witness property.
(2) $\mathfrak{i}$ and $\mathfrak{i}'$ both have set-extension (at least for when the starting set B has size less than κ).

Then $\mathfrak{i} = \mathfrak{i}'$.

Proof. Straightforward. $\square$

Lemma 12.13. Let K be an ∞-AEC in $\mathcal{F} = [\lambda, \theta)$ such that every chain has an upper bound. Let $\mathfrak{i}$ be a $(<\alpha, \mathcal{F})$-independence relation with underlying ∞-AEC K and let $\mathfrak{i}'$ be $(<\alpha_0, \lambda)$-independence relations, with underlying ∞-AEC K_λ and $\alpha_0 \le \alpha \le \lambda^+$. Let $\beta \le \alpha$. If:

(1) $\mathfrak{i}'$ has base monotonicity, uniqueness, existence, and extension.
(2) $\mathfrak{i}$ has base monotonicity, uniqueness, symmetry, and transitivity.
(3) $(\mathfrak{i}_\lambda)^{<\beta}$ has extension.
(4) $\mathfrak{i}$ has the $(<\beta)$-witness property.
(5) For every $\alpha' < \alpha$, there exists a regular $\mu \le \lambda$ such that $\kappa_{\alpha'}(\mathfrak{i}) \le \mu$.

Then $\text{pre}((\mathfrak{i}_\lambda)^{<\alpha_0}) = \text{pre}(\mathfrak{i}')$.

Proof. By Fact 12.10, $\mathfrak{i}_\lambda$ has $(<\beta)$-strong extension-existence and thus so does $(\mathfrak{i}_\lambda)^{<\alpha_0}$. Now apply Fact 12.11. $\square$

Lemma 12.14 (Canonicity of weakly good frames)**.** Assume:

(1) $\mathcal{F} = [\lambda, \theta)$ is an interval of cardinals.
(2) K is a λ^+-AEC in $\mathcal{F}$ such that every chain of length less than θ has an upper bound.
(3) $K^{\lambda^+\text{-mh}}$ is dense in K.
(4) For every $M_0 \leq M$ in K, if $M_0 \in K_\lambda$ and $M \in K^{\lambda^+\text{-mh}}$, there exists $M_0' \in K^{\lambda^+\text{-mh}}$ of size λ^+ with $M_0 \leq M_0' \leq M$.
(5) $\alpha_0 \leq \alpha \leq \lambda^+$, K is λ-tame for types of length less than α.
(6) $\mathfrak{i}$ is a weakly good $(< \alpha, \mathcal{F})$ frame with underlying class K. $\mathfrak{i}'$ is a weakly good $(< \alpha_0, \mathcal{F})$-frame with underlying class K.

If:

(1) $\mathfrak{i} \upharpoonright K^{\lambda^+\text{-mh}}$ has symmetry and transitivity.
(2) $\mathfrak{i}$ has the $(< \alpha)$-witness property.

Then $\text{pre}(\mathfrak{i}^{<\alpha_0} \upharpoonright K^{\lambda^+\text{-mh}}) = \text{pre}(\mathfrak{i}' \upharpoonright K^{\lambda^+\text{-mh}})$.

Proof. By Proposition 12.6, $\mathfrak{i} \upharpoonright K^{\lambda^+\text{-mh}}$ and $\mathfrak{i}' \upharpoonright K^{\lambda^+\text{-mh}}$ have existence, extension (for models of the same size), and uniqueness. Note that for every $\alpha' < \alpha$, Proposition 9 implies:

$$\kappa_{\alpha'}(\mathfrak{i}) \leq \bar{\kappa}_{\alpha'}(\mathfrak{i}) = \lambda^+$$

Also, $K^{\lambda^+\text{-sat}}$ is a λ^+-AEC in $\mathcal{F}$ such that every chain has an upper bound (since it is dense in K). Thus we can apply Lemma 12.13 with $(K, \mathfrak{i}, \mathfrak{i}', \lambda, \alpha_0, \alpha)$ here standing for $(K^{\lambda^+\text{-mh}}, \mathfrak{i} \upharpoonright K^{\lambda^+\text{-mh}}, \mathfrak{i}' \upharpoonright K^{\lambda^+\text{-mh}}, \lambda^+, \alpha_0, \alpha)$ there, and $\beta = \alpha$, $\mu = \lambda^+$. We get that $\text{pre}(\mathfrak{i}^{<\alpha_0} \upharpoonright K^{\lambda^+\text{-mh}})_{\lambda^+} = \text{pre}(\mathfrak{i}' \upharpoonright K^{\lambda^+\text{-mh}})_{\lambda^+}$. By (4), the $(< \alpha)$-witness property, and the local character axiom of weakly good frames, it is not hard to see that this implies $\text{pre}(\mathfrak{i}^{<\alpha_0} \upharpoonright K^{\lambda^+\text{-mh}}) = \text{pre}(\mathfrak{i}' \upharpoonright K^{\lambda^+\text{-mh}})$. $\square$

We can now prove the canonicity of coheir.

Theorem 12.15. Let K be an AEC with amalgamation. and let κ be an infinite cardinal. Assume that K does not have the $(< \kappa)$-order property of length κ. Let $\mathfrak{i} := \mathfrak{i}_{\kappa\text{-ch}}(K^{\kappa\text{-sat}})$, and let $\lambda \geq \kappa$.

Let $\mathfrak{i}'$ be a weakly good $(< \alpha, \geq \lambda)$-independence relation with underlying class $K^{\kappa'\text{-sat}}$ for a cardinal κ'. If:

(1) $\mu \geq \lambda$ is a cardinal.
(2) K is stable in μ.
(3) $\kappa \leq \beta$ is such that $\mathfrak{i}'$ has the left $(< \beta)$-set-witness property.
(4) $(\beta + \text{LS}(K))^{<\kappa} \leq \mu = \mu^{<\kappa}$.

(5) K is $(<\kappa)$-tame and short for types of length less than β.

Then $\text{pre}(\mathfrak{i}^{<\alpha} \upharpoonright K^{\mu^+\text{-sat}}) = \text{pre}(\mathfrak{i}' \upharpoonright K^{\mu^+\text{-sat}})$. Moreover, if $\mathfrak{i}'$ has right $(<\chi)$-witness property for $\chi \geq \kappa$, set-extension (at least for starting sets of size less than χ), and K is $(<\kappa)$-tame and short for types of length less than χ, then $\mathfrak{i}^{<\alpha} \upharpoonright K^{\mu^+\text{-sat}} = \mathfrak{i}' \upharpoonright K^{\mu^+\text{-sat}}$.

Proof. By Proposition 12.5, $(\mathfrak{i}_{\geq \mu})^{<\beta}$ is weakly good and by definition of coheir has the left and right $(<\beta)$-witness property. Letting $\beta_0 := \min(\alpha, \beta)$, it is enough to show $\text{pre}(\mathfrak{i}^{<\beta_0} \upharpoonright K^{\mu^+\text{-sat}}) = \text{pre}((\mathfrak{i}')^{<\beta_0} \upharpoonright K^{\mu^+\text{-sat}})$. We check that the hypotheses of Lemma 12.14 hold with $(\mathfrak{i}, \mathfrak{i}', \lambda, \theta, \alpha_0, \alpha)$ there standing for $(\mathfrak{i}^{<\beta} \upharpoonright K^{\mu\text{-sat}}, (\mathfrak{i}')^{<\beta_0} \upharpoonright K^{\mu\text{-sat}}, \mu, \infty, \beta_0, \beta)$ here. Note that $K^{\mu^+\text{-mh}} = K^{\mu^+\text{-sat}}$ by Fact 2.38. The density hypotheses hold by stability in μ. Also, coheir has symmetry and transitivity by Theorem 10.11. The other hypotheses are straightforward to check.

For the moreover part, use Lemma 12.12 together with Proposition 12.8. $\square$

Corollary 12.16. Let K be an AEC with amalgamation. Let $\mathfrak{i}'$ be a weakly good $(<\alpha, \geq \lambda)$-independence relation with underlying class $K^{\kappa'\text{-sat}}$ (for some cardinal $\kappa' \leq \lambda$) and $\lambda \geq \text{LS}(K)$. Assume $\mathfrak{i}'$ has the left $(<\beta)$-set-witness property for some cardinal $\beta \leq \alpha$.

If $\kappa \geq \beta + \lambda^+$ is such that:

(1) $\beth_\kappa = \kappa$.
(2) For some $\kappa_0 < \kappa$, K is κ_0-tame.
(3) K is $(<\kappa)$-tame for types of length less than κ.

Then for $\mu := \kappa^{<\kappa}$, $\text{pre}(\mathfrak{i}_{\kappa\text{-ch}}(K^{\mu^+\text{-sat}}))^{<\alpha} = \text{pre}(\mathfrak{i}' \upharpoonright K^{\mu^+\text{-sat}})$.

Proof. By Proposition 12.6.(7) (looking at $(\mathfrak{i})_{\geq \lambda+\kappa_0}$), K is stable in any $\mu \geq \lambda + \kappa_0$ such that $\mu^{\lambda+\kappa_0} = \mu$. In particular, K is stable in $\mu := \kappa^{<\kappa}$. By Corollary 6.6, K does not have the $(<\kappa)$-order property of length κ. Now apply Theorem 12.15. $\square$

Corollary 12.17 (Canonicity of good independence relations)**.** Let $\mathfrak{i}' = (K, \mathop{\smile\kern-0.6em\mid}\,)$ be a good $(<\alpha, \geq \lambda)$-independence relation. Assume $\kappa \geq \lambda^+$ is such that $\beth_\kappa = \kappa$ and K is $(<\kappa)$-tame and short for types of length less than $\kappa + |\alpha|$.

Then for $\mu := \kappa^{<\kappa}$, $\mathfrak{i}_{\kappa\text{-ch}}(K^{\mu^+\text{-sat}})^{<\alpha} = \mathfrak{i}' \upharpoonright K^{\mu^+\text{-sat}}$.

Proof. By a standard argument (see for example [Bonb, Theorem 3.2]), K is λ-tame. By Proposition 11.1.(6), $\mathfrak{i}'$ has the right $(<\kappa)$-set-witness property for sets of size less than α so by symmetry it has the left $(<\kappa)$-set-witness property. Thus we can apply Corollary 12.16 and we get $\text{pre}(\mathfrak{i}_{\kappa\text{-ch}}(K^{\mu^+\text{-sat}}))^{<\alpha} = \text{pre}(\mathfrak{i}' \upharpoonright K^{\mu^+\text{-sat}})$. Now by Proposition 12.8, coheir has enough set-extension to apply Lemma 12.12. $\square$

Corollary 12.18 (Canonicity of good frames)**.** If $\mathfrak{s}$, $\mathfrak{s}'$ are two type-full good $(<\alpha, \geq \lambda)$-frames with the same underlying AEC K, then whenever $\kappa \geq \lambda^+$ is such that $\kappa = \beth_\kappa$ and K is $(<\kappa)$-tame and short for types of length less than $\kappa + |\alpha|$, we have $\mathfrak{s} \upharpoonright K^{\mu^+\text{-sat}} = \mathfrak{s}' \upharpoonright K^{\mu^+\text{-sat}}$, where $\mu := \kappa^{<\kappa}$.

Proof. Apply Corollary 12.17 to $\text{cl}(\mathfrak{s})$, $\text{cl}(\mathfrak{s}')$. $\square$

13. Transferring an independence relation up

In [She09a, Section II.2], Shelah showed how to extend a good λ-frame to a pre-$(\geq \lambda)$-frame. Later, [Bonb] (with improvements in [BVa]) gave conditions under which all the properties transferred. Similar ideas are used in [Vas] to directly build a good frame. In this section we adapt Shelah's definition to our more general setup and abstract the essence of the proof of [Vas] to show the construction can be done even on weakly good frames.

Definition 13.1. Let $\mathcal{F} = [\lambda, \theta)$ be an interval of cardinals, $\theta' \geq \theta$ be a cardinal or ∞, $\mathcal{F}' := [\lambda, \theta')$. Let K^1 be an ∞-AEC in $\mathcal{F}$ with amalgamation and let K be an AEC in $\mathcal{F}'$ with $\text{LS}(K) = \lambda$. Assume K^1 is a dense sub-∞-AEC of $K_{\mathcal{F}}$.

Let $\mathfrak{i} = (K^1, \mathop{\smile}\limits^{\mid})$ be a $(<\alpha, \mathcal{F})$-independence relation with $\alpha \leq \theta$.

Define $\mathfrak{i}_{\mathcal{F}', K} = (K, \underset{K}{\mathop{\smile}\limits^{\mid}})$ by $\underset{K}{\mathop{\smile}\limits^{\mid}}(M, A, B, N)$ if and only if $M \leq N$ are in K and there exists $M_0 \leq M$ in K^1 such that for all $B_0 \subseteq B$ of size less than θ and all $N_0 \leq N$ in K^1 containing M_0, A, and B_0, $\mathop{\smile}\limits^{\mid}(M_0, A, B_0, N_0)$.

When $K_{\mathcal{F}} = K^1$, we just write $\mathfrak{i}_{\mathcal{F}'}$. As usual, we write $\mathfrak{i}_{\geq \lambda}$ when $\theta' = \infty$.

For $\mathfrak{s}$ a pre-frame, define $\mathfrak{s}_{\mathcal{F}', K}$ similarly.

Remark 13.2. In general, we do not claim that $\mathfrak{i}_{\mathcal{F}', K}$ is an independence relation. Nevertheless, we will abuse notation and use the restriction operations on it.

Remark 13.3. We could have given a more general definition without mentioning AECs, but this suffices for our purpose.

Proposition 13.4. Let K, K^1, $\mathcal{F}$, $\mathcal{F}'$, λ, be as in Definition 13.1. Assume K has amalgamation. Then:

(1) If $K_{\mathcal{F}} = K^1$, $\mathfrak{i}_{\mathcal{F}'}$ is an independence relation.
(2) $\mathfrak{i}_{\mathcal{F}',K} \upharpoonright K^{\theta\text{-sat}}$ is an independence relation.

Proof. As in [She09a, Claim II.2.11], using density and homogeneity in the second case. □

Finally, we look at what happens when $\mathfrak{i}$ is weakly good and superstable-like:

Lemma 13.5. Let K, K^1, $\mathcal{F}$, $\mathcal{F}'$, λ, θ, θ', α be as in Definition 13.1 and $\theta = \lambda^+$, $\alpha \le \lambda$. Assume K has amalgamation and is λ-tame for types of length less than α. Let $\mathfrak{i}' := \mathfrak{i}_{\mathcal{F}',K} \upharpoonright K^{\lambda^+\text{-sat}}$. If:

(1) $\mathfrak{i}$ is weakly good.
(2) There exists an ordering $\trianglelefteq$ such that $(K^1, \trianglelefteq)$ is a skeleton of K_λ and for all $\alpha_0 < \alpha$, $\kappa_{\alpha_0}(\mathfrak{i} \upharpoonright K^1, \trianglelefteq) = |\alpha_0|^+ + \aleph_0$.

Then:

(1) $\mathfrak{i}'$ is a $(<\alpha, [\lambda^+, \theta'))$-independence relation.
(2) $\mathfrak{i}'$ has base monotonicity, uniqueness, and disjointness.
(3) For all $\lambda^+ \le \mu < \theta'$, $(\mathfrak{i}')_\mu$ has extension.
(4) For all $\alpha_0 < \alpha$, $\kappa_{\alpha_0}(\mathfrak{i}') = |\alpha_0|^+ + \aleph_0$.
(5) If K is stable in λ and has no maximal models, then K is stable in all $\mu \ge \lambda$.
(6) If K is stable in λ, $\bar{\kappa}_{<\alpha}(\mathfrak{i}') = \lambda^{++}$.
(7) If K is stable in λ and has joint embedding and no maximal models, then for any $\lambda' \ge \lambda^+$ in $\mathcal{F}'$, $(\mathfrak{i}')_{\lambda'}$ is an almost good $(<\alpha, \lambda')$-independence relation, except that $K_{\lambda'}^{\lambda^+\text{-sat}}$ may not be an AEC.

Proof.

(1) By Proposition 13.4.
(2) As in the proof of Proposition 12.6.
(3) As above.
(4) This is a generalization of the proof of [Vas, Lemma 4.8] but we have to say slightly more so we give the details. Fix $\alpha_0 < \alpha$ and $\delta = \mathrm{cf}(\delta) \ge |\alpha_0|^+ + \aleph_0$. Let $\langle M_i : i < \delta \rangle$ be increasing in $K^{\lambda^+\text{-sat}}$ and write $M_\delta := \bigcup_{i<\delta} M_i$ (note that we do not claim $M_\delta \in K^{\lambda^+\text{-sat}}$. However, $M_\delta \in K$). Let $p \in \mathrm{gS}^{\alpha_0}(M_\delta)$. We want

to find $i < \delta$ such that p does not fork over M_i. There are two cases:

- Case 1: $\delta < \lambda^+$:
  We imitate the proof of [She09a, Claim II.2.11.5]. Assume the conclusion fails. Build $\langle N_i : i < \delta \rangle$ $\trianglelefteq$-increasing in K^1, $\langle N_i' : i < \delta \rangle$ $\leq$-increasing in K^1 such that for all $i < \delta$:
  (a) $N_i \leq M_i$.
  (b) $N_i \leq N_i' \leq M_\delta$.
  (c) $p \upharpoonright N_i'$ $\mathfrak{i}$-forks over N_i.
  (d) $\bigcup_{j<i}(|N_j'| \cap |M_j|) \subseteq |N_i|$.
  This is possible. Assume N_j and N_j' have been constructed for $j < i$. Choose $N_i \leq M_i$ satisfying (4d) so that $N_j \trianglelefteq N_i$ for all $j < i$ (This is possible: use that M_i is λ^+-model-homogeneous and that in skeletons of AECs, chains have upper bounds). By assumption, p $\mathfrak{i}'$-forks over M_i, and so by definition of forking there exists $N_i' \leq M_\delta$ in K^1 such that $p \upharpoonright N_i'$ forks over N_i. By monotonicity, we can of course assume $N_i' \geq N_i$, $N_i' \geq N_j'$ for all $j < i$.
  This is enough. Let $N_\delta := \bigcup_{i<\delta} N_i$, $N_\delta' := \bigcup_{i<\delta} N_i'$. By local character for $\mathfrak{i}$, there is $i < \delta$ such that $p \upharpoonright N_\delta$ does not fork over N_i. By (4b) and (4d), $N_\delta' \leq N_\delta$. Thus by monotonicity $p \upharpoonright N_i'$ does not $\mathfrak{i}$-fork over N_i, contradicting (4c).
- Case 2: $\delta \geq \lambda^+$: Assume the conclusion fails. Let $\beta := |\alpha_0|^+ + \aleph_0$. Note that β is regular and $\beta < \lambda^+$ (this is where we are using that $\alpha \leq \lambda$). As in the previous case (in fact it is easier), we can build $\langle N_i : i < \beta \rangle$ $\trianglelefteq$-increasing continuous in K^1 such that $N_i \leq M_\delta$ and $p \upharpoonright N_{i+1}$ $\mathfrak{i}$-forks over N_i. By local character in $\mathfrak{i}$, there exists $i < \beta$ such that $p \upharpoonright N_\beta$ does not $\mathfrak{i}$-fork over N_i, contradiction.

(5) Exactly as in the proof of [Vas, Theorem 5.6].
(6) Directly from the definition of $\mathfrak{i}'$ and the fact that (by stability) λ^+-saturated models are dense in K_{λ^+}.
(7) By the previous parts.

$\square$

We conclude by recalling that when types have length 1, *all* the properties of a good frame transfer:

Fact 13.6. Assume $\mathfrak{s} = (K, \perp)$ is an [almost] good λ-frame and $\mathcal{F}' = [\lambda, \theta')$ is an interval of cardinals with $\lambda < \theta'$.

If $K_{\mathcal{F}'}$ is λ-tame and has amalgamation, then $\mathfrak{s}_{\mathcal{F}'}$ is an [almost] good $\mathcal{F}'$-frame.

Proof. By [BVa, Corollary 6.9]. Note that if (as in our setup) $\mathfrak{s}$ is type-full, then $\mathfrak{s}_{\mathcal{F}'}$ will also be type-full by Lemma 13.5.(4). $\square$

14. Superstability

Shelah has pointed out [She09a, p. 19] that superstability in abstract elementary classes suffers from schizophrenia, i.e. there are several different possible definitions that turn out to be equivalent in elementary classes but not necessarily so in AECs. The existence of a good $(\geq \lambda)$-frame is a possible definition but it is very hard to check so one would like a definition that implies existence of a good frame but is more tractable.

Shelah claims in chapter IV of his book that solvability ([She09a, Definition 1.4]) is such a notion, but his justification is yet to appear (in [Shea]). On the other hand previous work (for example [She99, SV99, GVV]) all rely on a local character property for nonsplitting. This is even made into a definition of superstability in [Gro02, Definition 7.12]. In [Vas] we gave a similar condition and used it with tameness to build a good frame. We pointed out that categoricity in a cardinal of cofinality greater than the tameness cardinal implied the superstability condition.

We now aim to show the same conclusion under categoricity in a high-enough cardinal of arbitrary cofinality. We also generalize the definition of superstability implicit in [Vas]:

Definition 14.1 (Superstability)**.** An AEC K is $(\mu, K^1, \mathop{\smile\!\!\!\!\!\!\;|}, \trianglelefteq)$-*superstable* if:

(1) $\mu \geq \text{LS}(K)$ is a cardinal.
(2) $K_{\geq \mu}$ is nonempty and has amalgamation, joint embedding, and no maximal models.
(3) K is μ-tame.
(4) K is stable in μ.
(5) $(K^1, \trianglelefteq)$ is a skeleton of K_μ.
(6) $\mathfrak{i} := ((K^1, \leq), \mathop{\smile\!\!\!\!\!\!\;|})$ is a weakly good $(\leq 1, \mu)$-independence relation.
(7) $\kappa_1(\mathfrak{i}, \trianglelefteq) = \aleph_0$.

K is $(\mu, \mathfrak{i}, \trianglelefteq)$-*superstable* if $\mathfrak{i} = (K^1, \mathop{\smile\!\!\!\!\!\!\;|})$ and K is $(\mu, K^1, \mathop{\smile\!\!\!\!\!\!\;|}, \trianglelefteq)$-superstable.

When we omit a parameter, we mean some value of the parameter exists so that the class is superstable, e.g. K is μ-superstable means that it is $(\mu, K^1, \mathop{\smile\kern-0.9em\mid\kern0.5em}, \trianglelefteq)$-superstable for some K^1, $\mathop{\smile\kern-0.9em\mid\kern0.5em}$, and $\trianglelefteq$.

For technical reasons, we will also use the following version that uses coheir rather than nonsplitting.

Definition 14.2. We say K is κ-*strongly* $(\mu, \kappa', \trianglelefteq)$-superstable if:

(1) $\kappa \leq \kappa' \leq \mu$ are infinite cardinals.
(2) K is $(< \kappa)$-tame.
(3) K does not have the $(< \kappa)$-order property of length κ.
(4) K is $\left(\mu, \left(\mathfrak{i}_{\kappa\text{-ch}}(K^{\kappa'\text{-sat}})\right)^{\leq 1}_{\mu}, \trianglelefteq\right)$-superstable.
(5) $\bar{\kappa}_1(\mathfrak{i}_{\kappa\text{-ch}}(K^{\kappa'\text{-sat}})) \leq \mu^+$.

As before, we may omit some parameters.

The next lemma gives two usable necessary conditions that imply that a class is (strongly) superstable:

Lemma 14.3. Let K be an AEC with amalgamation, joint embedding, and no maximal model. Let $\mu \geq \text{LS}(K)$ be such that K is μ-tame and stable in μ.

(1) If for some $\delta < \mu^+$, $\kappa_1(\mathfrak{s}_{\mu\text{-ns}}(K_\mu), \leq_{\mu,\delta}) = \aleph_0$, then K is $(\mu, \mathfrak{i}, \leq_{\mu,\delta})$-superstable, where:
$$\mathfrak{i} := (\text{cl}(\mathfrak{s}_{\mu\text{-ns}}(K_\mu)))^{\leq 1}$$
(2) If κ is an infinite cardinal such that:
  (a) K is $(< \kappa)$-tame.
  (b) $\text{LS}(K)^{<\kappa} \leq \mu$.
  (c) K does not have the $(< \kappa)$-order property of length κ.
  (d) $K^{\kappa\text{-sat}}_\mu$ is dense in K_μ.
  (e) For some $\delta < \mu^+$, $\kappa_1(\mathfrak{i}_{\kappa\text{-ch}}(K^{\kappa\text{-sat}}_\mu), \leq_{\mu,\delta}) = \aleph_0$.
  Then K is κ-strongly $(\mu, \kappa, \leq_{\mu,\delta})$-superstable.

Proof.

(1) Easy.
(2) By Theorem 10.11, (using $\text{LS}(K)^{<\kappa} \leq \mu$), $\bar{\kappa}_1(\mathfrak{i}_{\kappa\text{-ch}}(K^{\kappa\text{-sat}})) \leq \mu^+$. By Proposition 12.5, coheir is weakly good. Setting $K^1 := K^{\kappa\text{-sat}}_\mu$, we have by assumption that K^1 is dense in K_μ.

□

The importance of superstability is demonstrated by the next result which is implicit in [Vas].

Theorem 14.4. Assume K is a $(\mu, \mathfrak{i})$-superstable AEC. Then:

(1) K is stable in all $\lambda \geq \mu$.
(2) Let $\lambda \geq \mu^+$ and let $\mathfrak{i}' := \mathfrak{i}_{\geq \mu, K_{\geq \mu}} \upharpoonright K_{\geq \lambda}^{\mu^+\text{-sat}}$.
  (a) $\mathfrak{i}'$ is an almost good independence relation, except for extension and that $K_{\geq \lambda}^{\mu^+\text{-sat}}$ may not be an AEC. Moreover, $(\mathfrak{i}')_\lambda$ has extension.
  (b) If in addition $\mathfrak{i}$ witnesses that K is κ-*strongly* μ-superstable, then $\mathfrak{i}' = \left(\mathfrak{i}_{\kappa\text{-ch}}(K^{\mu^+\text{-sat}})\right)_\lambda^{\leq 1}$. That is, the independence relation is κ-coheir.
  (c) If $\lambda \geq \theta \geq \mu^+$ is such that $K' := K_{\geq \lambda}^{\theta\text{-sat}}$ is an AEC with $\text{LS}(K') = \lambda$, then $\text{pre}(\mathfrak{i}' \upharpoonright K')$ is a good $(\geq \lambda)$-frame that will be κ-coheir if $\mathfrak{i}$ witnesses that K is κ-strongly μ-superstable.

Proof. Say K is $(\mu, \mathfrak{i}, \trianglelefteq)$-superstable with $\mathfrak{i} = (K^1, \mathop{\smile\!\!\!\!\!\mid}\,)$. We apply Lemma 13.5. Stability follows by (5) there and $\mathfrak{i}'$ is almost good by (7). To see (2b), use the second condition in the definition of strong superstability and the definition of $\mathfrak{i}'$.

For (2c), it is easy to check that $\text{pre}(\mathfrak{i}'_\lambda \upharpoonright K')$ is an almost good λ-frame. By Fact 13.6, $\text{pre}((\mathfrak{i}' \upharpoonright K')_{\geq \lambda}) = \text{pre}(\mathfrak{i}' \upharpoonright K')$ is an almost good $(\geq \lambda)$-frame. By Proposition 11.1.(10), it also has symmetry. $\square$

Corollary 14.5. Let T be a complete first-order theory and let $K := (\text{Mod}(T), \preceq)$. Then our definitions of superstability and strong superstability coincide with the classical definition. More precisely for all $\mu \geq |T|$, K is (strongly) μ-superstable if and only if T is stable in all $\lambda \geq \mu$.

Proof. Straightforward. $\square$

Corollary 14.6. If K is [κ-strongly] μ-superstable and $\mu' \geq \mu$, then K is [κ-strongly] μ'-superstable.

Proof. Say K is $(\mu, \mathfrak{i})$-superstable. By Theorem 14.4 (and Proposition 12.5), K is $(\mu', \mathfrak{i}')$-superstable, where $\mathfrak{i}' := \mathfrak{i}_{\geq \mu, K_{\geq \mu}} \upharpoonright K_{\geq \mu'}^{\mu^+\text{-sat}}$. Proceed similarly if K is κ-strongly μ-superstable. $\square$

Theorem 14.4.(2b) is the reason we introduced strong superstability. While it may seem like a detail, we are interested in extending our good frame to a frame for types longer than one element and using coheir to do so seems reasonable. Using the canonicity of coheir, we can show they are equivalent but we have to assume more tameness:

Theorem 14.7. Assume K is μ-superstable. Let $\kappa > \mu$ be such that $\kappa = \beth_\kappa$ and K is $(< \kappa)$-tame for types of length less than κ.

Then K is κ-strongly (μ', μ')-superstable, where $\mu' := (\kappa^{<\kappa})^+$.

In particular a fully tame AEC is strongly superstable if and only if it is superstable.

Proof. By Theorem 14.4, K has an almost good $(\geq \kappa)$-independence relation $\mathfrak{i}'$ for μ^+-saturated models. By Corollary 12.16 applied with $\lambda = \mu^+$, $\mathfrak{i}' \upharpoonright K^{\mu'\text{-sat}}$ is κ-coheir. It is now straightforward to check that the result follows. □

We conclude this section by showing categoricity implies strong superstability. We first recall several known consequences of categoricity.

Fact 14.8. Let K be an AEC with joint embedding and amalgamation. Assume K is categorical in a $\lambda > \mathrm{LS}(K)$. Then:

1. $K_{<\lambda}$ has no maximal models. Moreover if K has arbitrarily large models, then K has no maximal models.
2. K is stable in all $\mu \in [\mathrm{LS}(K), \lambda)$.
3. For $\mathrm{LS}(K) \leq \mu < \mathrm{cf}(\lambda)$, $\kappa_1(\mathfrak{s}_{\mu\text{-ns}}(K_\mu), \leq_{\mu,\omega}) = \aleph_0$.
4. Assume K does not have the weak κ-order property (see Definition 6.7) and $\kappa \leq \mu < \lambda$. Then $\kappa_1(\mathfrak{i}_{\kappa\text{-ch}}(K_\mu^{\kappa\text{-sat}}), \leq_{\mathrm{univ}}) = \aleph_0$.
5. If the model of size λ is μ-saturated for $\mu > \mathrm{LS}(K)$, then every member of $K_{\geq \chi}$ is μ-saturated, where $\chi := \min(\lambda, \sup_{\mu_0 < \mu} h(\mu_0))$.

Proof.

1. Use joint embedding to embed the model to be extended inside the model of size λ. Similarly use joint embedding for the moreover part.
2. Use Ehrenfeucht-Mostowski models (see for example the proof of [Bal09, Theorem 8.24]).
3. Straightforward, using categoricity and amalgamation.
4. By [She99, Lemma 6.3].
5. Apply the first part and [BGa, Theorem 6.5].
6. See (the proof of) [BGa, Theorem 5.4].

□

The following theorem is also implicit in [Vas]. It is really a simple consequence of Fact 14.8.(3).

Theorem 14.9. Let K be an AEC with arbitrarily large models, joint embedding and amalgamation. Assume K is μ-tame and categorical in a $\lambda > \text{LS}(K)$ with $\text{cf}(\lambda) > \mu$. Then K is μ-superstable.

Proof. By Fact 14.8.(2), (1), K has no maximal models and is stable in μ. Now apply Fact 14.8.(3) and Lemma 14.3. $\square$

We can now remove the restriction on the cofinality and get strong superstability. The price to pay is a potentially much bigger categoricity cardinal.

Theorem 14.10. Assume K is an AEC with joint embedding and amalgamation. Let $\kappa = \beth_\kappa > \text{LS}(K)$ and assume K is $(< \kappa)$-tame. If K is categorical in a $\lambda > \kappa$, then:

(1) K is κ-strongly κ-superstable.
(2) K is stable in all cardinals $\geq \text{LS}(K)$.
(3) The model of size λ is saturated.
(4) K is categorical in κ.
(5) K has a type-full good $(\geq \chi)$-frame, where $\chi := \min(\lambda, h(\kappa))$. Furthermore the non-forking relation of the frame is κ-coheir.

Proof. First observe that by Fact 2.35, K has arbitrarily large models. We proceed in several steps.

First, we show K is μ-superstable for some $\mu < \lambda$. If $\lambda = \kappa^+$, then this follows directly from Theorem 14.9 with $\mu = \kappa$, so assume $\lambda > \kappa^+$. We claim that K is κ-strongly μ-superstable for $\mu := \kappa^+$. We check the conditions of Lemma 14.3. As in the proof of Theorem 14.9, K has no maximal models and is stable in μ. Since $\kappa = \beth_\kappa$, this implies that K does not have the $(< \kappa)$-order property of length κ. Also, as $\kappa > \text{LS}(K)$ and is strong limit, $\text{LS}(K)^{<\kappa} = \kappa \leq \mu$. $K_\mu^{\kappa\text{-sat}}$ is dense in K_μ (use stability and the fact μ is regular). Finally, the local character condition holds by Fact 14.8.(4).

Second, we prove (2). By Fact 14.8.(2), K is stable everywhere below λ. By Theorem 14.4, K is stable in every $\mu' \geq \mu$. In particular, it is stable in and above λ, so (2) follows.

Third, we show (3). Since K is stable in λ, we can build a λ_0^+-saturated model of size λ for all $\lambda_0 < \lambda$. Thus the model of size λ is λ_0^+-saturated for all $\lambda_0 < \lambda$, and hence λ-saturated.

Fourth, we prove (4). Since the model of size λ is saturated, it is κ-saturated. By Fact 14.8.(5), every model of size $h(<\kappa) = \kappa$ is κ-saturated. By uniqueness of saturated models, K is categorical in κ.

Fifth, observe that since every model of size κ is saturated, $K_\kappa^{\kappa\text{-sat}}$ is dense in K_κ. This was the only missing property in the proof of strong μ-superstability above, so (1) holds.

Finally, we prove (5). We have seen that the model of size λ is saturated, thus μ^+-saturated. By Fact 14.8.(5), every model of size $\geq \chi$ is μ^+-saturated. Now use (1) with Theorem 14.4. $\square$

Remark 14.11. A stronger version of the categoricity transfer is proven in [MS90] assuming κ is a strongly compact cardinal.

Remark 14.12. If one just wants to get strong superstability from categoricity, we suspect it should be possible to replace the $\beth_\kappa = \kappa$ hypothesis by something more reasonable (maybe just asking for the categoricity cardinal to be above 2^κ). Since we are only interested in eventual behavior here, we leave this to future work.

Notice that the categoricity transfer above implies that if θ is regular and K is categorical in a $\lambda > \mu_\theta := h_\theta(\kappa)$, then K is categorical in μ_θ, which has cofinality θ. Of course, we can also use the fact that for a fixed μ_0, there are only set-many AECs with Löwenheim-Skolem number μ_0 and get that there *exists* a (very big) $\mu \geq \kappa$ depending only on κ such that if K is categorical above μ, then it is categorical in every fixed point of the Beth function above μ. As an application if K is categorical in $\lambda > \mu_{\kappa^+}$, then it is categorical in μ_{κ^+}, and hence by Theorem 14.9 is κ-superstable and so has a type-full good $(\geq h(\kappa))$-frame. This improves the bound on χ on Theorem 14.10 at the cost of a higher initial categoricity cardinal (and we also do not know that the nonforking relation is coheir).

Part 3. Building a global independence notion

15. Introduction

Some previous work on global independence notions has been discussed in the introduction to part 2. We should also mention [MS90] which studied coheir in $L_{\kappa,\omega}$ where κ is a strongly compact cardinal. It turns out many of the results can be adapted to fully $(<\kappa)$-tame and short abstract elementary classes (this is really the observation that lead to

[BGa]). Unfortunately Makkai and Shelah use the full strength of the strongly compact cardinal to deduce the extension property for coheir.

Recall that in part 2 we proved that under reasonable conditions coheir had all the properties of forking in a stable theory, except perhaps for extension. We showed that under a condition we called *strong superstability* (Definition 14.2), coheir induced a good frame: an independence notion for types of singleton that has all the properties of forking in a superstable theory (including the extension property). Thus in particular coheir has the extension property for types of singletons.

Here we continue the last section of part 2: we take the good frame built there and show that (under reasonable conditions) one can extend it to a global independence notion (which in particular has the extension property). More precisely, using the terminology of Definition 9.14, we prove:

Theorem 15.1. Let K be a fully $(< \kappa)$-tame and short abstract elementary class with joint embedding and amalgamation. If $\kappa = \beth_\kappa > \text{LS}(K)$ and K is categorical in a $\mu > \lambda_0 := \left(\kappa^{<\kappa}\right)^{+5}$, then $K_{\geq \lambda}$ is fully good, where $\lambda := \min(\mu, \beth_{\left(2^{\lambda_0}\right)^+})$.

A more readable but less precise statement is:

Corollary 15.2. Let K be a fully tame and short AEC with amalgamation. If K is categorical in unboundedly many cardinals, then there exists $\lambda \geq \text{LS}(K)$ such that $K_{\geq \lambda}$ is fully good.

We also get a similar result using only superstability instead of categoricity, see Theorem 20.1. This answers questions asked for example in [She99, Remark 4.9.1] or more recently in [BGKV, Question 7.1]. Just like in first-order, we expect such a global independence notion to have significant applications. As an example, recall that the main test question for AECs is:

Conjecture 15.3 (Shelah's categoricity conjecture)**.** If K is an AEC that is categorical in unboundedly many cardinals, then K is categorical on a tail of cardinals.

The conjecture already appears in [She90, Open problem D.3(a)]. Shelah claims in [She09a, Discussion III.12.40] that in ω-successful good frames, the categoricity conjecture follows from the weak continuum hypothesis. It is a byproduct of our methods that fully good AECs are (when restricted to sufficiently-saturated models) ω-successful. Thus we obtain several corollaries concerning Shelah's categoricity conjecture. Note that [She09a, Theorem IV.12] is stronger than any of those corollaries (since Shelah does not assume shortness) but we haven't checked Shelah's argument.

Recalling that shortness and amalgamation follow from categoricity and large cardinals (see [Bonc]), we obtain (again granting Discussion III.12.40 in Shelah's book) that Shelah's categoricity conjecture follows from weak GCH and unboundedly-many strongly compacts. We believe this shows how powerful a global independence relation can be.

We now outline our construction and how the paper is organized. All throughout we strongly rely on the results of [She09a]. In section 16, starting with a nice-enough good $(\leq 1, \lambda)$-frame, we show it has a technical condition Shelah calls *weakly successful* and this allows us to extend the frame to types of length λ (the details are in section 17). In the end we obtain a good $(\leq \lambda, \lambda)$-independence relation[22]. Quoting more of Shelah's work, we get that the independence relation also has a strong continuity property (that is, it is *fully* good rather than only good). This continuity property allows us to extend the independence relation to bigger models as in [Bonb] and obtain a fully good $(\leq \lambda, \geq \lambda)$-independence relation. This is done in section 18. With more independence calculus, we can also increase the length of the types of the independence relation and get the desired fully good $(< \infty, \geq \lambda)$-independence relation. This is section 19. In section 20, we prove the main theorem and conclude.

16. Domination

In this section, our aim is to take a sufficiently nice good λ-frame (for types of length 1) and show it can be extended to types of length $\leq \lambda$. To do this, we will give conditions under which a good λ-frame is *weakly successful*, a key technical property of [She09a, Chapter II], see Definition 16.4.

The hypotheses we will work with are:

Hypothesis 16.1.

(1) $\mathfrak{i} = (K, \perp)$ is a $(< \infty, [\mu, \infty))$-independence relation.
(2) $\mathfrak{s} := \text{pre}(\mathfrak{i}^{\leq 1})$ is a type-full good $[\mu, \infty)$-frame.
(3) $\lambda > \mu$ is a cardinal.

[22]Recall (Definition 9.5) that frames are defined only for types over models while independence relations are defined for types over sets.

(4) For all $n < \omega$:
   (a) $K^{\lambda^{+n}\text{-sat}}$ is an AEC with Löwenheim-Skolem number λ^{+n}.
   (b) $\kappa_{\lambda^{+n}}(\mathfrak{i}) = \lambda^{+n+1}$.
(5) $\mathfrak{i}$ has base monotonicity and uniqueness.
(6) $\mathfrak{i}$ has the left and right $(\leq \mu)$-set-witness properties.

Remark 16.2. We could have given more local hypotheses (e.g. by replacing ∞ by θ or only assuming (4) for n below some fixed $m < \omega$) and made some of the required properties more precise (this is part of what should be done to improve "short" to "diagonally tame" in the main theorem, see the discussion in section 20).

The key is that we assume there is *already* an independence notion for longer types. However, it is potentially quite weak compared to what we want. The next fact shows that our hypotheses are reasonable.

Fact 16.3. Assume K^0 is a fully $(< \kappa)$-tame and short κ-strongly μ_0-superstable AEC. Then for any $\mu \geq (\mu_0^{<\kappa})^{+2}$, Hypothesis 16.1 holds for $\lambda := \mu^+$, $K := (K^0)^{\mu\text{-sat}}$ and $\mathfrak{i} := \mathfrak{i}_{\kappa\text{-ch}}(K)$.

Proof. By results to appear in [BVb], for any $\mu' \geq \mu$, $K^{\mu'\text{-sat}}$ is an AEC with $\text{LS}(K^{\mu'\text{-sat}}) = \mu'$. By Theorem 14.4.(2c), κ-coheir induces a good $(\geq \mu)$-frame for μ-saturated models. The other conditions follow directly from the definition of strong superstability and the properties of coheir (Theorem 10.11). For example, the local character condition holds because $\mu^{<\kappa} = \mu$ implies $(\mu^{+n})^{<\kappa} = \mu^{+n}$ for any $n < \omega$. $\square$

The next technical property is of great importance in Chapter II and III of [She09a]. Our definition follows [JS13, Definition 4.1.5] (but as usual, we work only with type-full frames).

Definition 16.4. Let $\mathfrak{t} = (K_{\mathfrak{t}}, \perp)$ be a type-full good $\lambda_{\mathfrak{t}}$-frame.

(1) For $M_0 \leq M_\ell$ in K, $\ell = 1, 2$, an *amalgam of M_1 and M_2 over M_0* is a triple (f_1, f_2, N) such that $N \in K_{\mathfrak{t}}$ and $f_\ell : M_\ell \xrightarrow[M_0]{} N$.
(2) Let (f_1^x, f_2^x, N^x), $x = a, b$ be amalgams of M_1 and M_2 over M_0. We say (f_1^a, f_2^a, N^a) and (f_1^b, f_2^b, N^b) are *equivalent over M_0* if there exists $N_* \in K_{\mathfrak{t}}$ and $f^x : N^x \to N_*$ such that $f^b \circ f_1^b = f^a \circ f_1^a$ and $f^b \circ f_2^b = f^a \circ f_2^a$, namely, the following commutes:

$$\begin{tikzcd}
 & N^b \arrow[r, dotted, "f^b"] & N_* \\
M_1 \arrow[ur, "f_1^b"] \arrow[rr, "f_2^b"] & & N^a \arrow[u, dotted, "f^a"'] \\
M_0 \arrow[u] \arrow[r] & M_2 \arrow[uu, "f_1^a"] \arrow[ur, "f_2^a"'] &
\end{tikzcd}$$

Note that being "equivalent over M_0" is an equivalence relation ([JS13, Proposition 4.3]).

(3) Let $K^{3,\mathrm{uq}} = K_{\mathfrak{t}}^{3,\mathrm{uq}}$ be the set of triples (a, M, N) such that $M \le N$ are in K, $a \in |N| \backslash |M|$ and for any $M_1 \ge M$ in K, there exists a unique (up to equivalence over M_0) amalgam (f_1, f_2, N_1) of N and M_1 over M such that $\mathrm{gtp}(f_1(a)/f_2[M_1]; N_1)$ does not fork over M. We call the elements of $K^{3,\mathrm{uq}}$ *uniqueness triples*.

(4) $K^{3,\mathrm{uq}}$ has the *existence property* if for any $M \in K_{\mathfrak{t}}$ and any nonalgebraic $p \in \mathrm{gS}(M)$, one can write $p = \mathrm{gtp}(a/M; N)$ with $(a, M, N) \in K^{3,\mathrm{uq}}$. We also talk about the *existence property for uniqueness triples*.

(5) $\mathfrak{s}$ is *weakly successful* if $K^{3,\mathrm{uq}}$ has the existence property.

The uniqueness triples can be seen as describing a version of domination. They were introduced for the purpose of starting with a frame (for 1-types) and extending it to a nonforking notion for models. Now, since we already have an independence notion for longer types, we can follow [MS90, Definition 4.21] and give a more explicit version of domination that is exactly as in the first-order case.

Definition 16.5 (Domination)**.** Fix $N \in K$. For $M \le N$, $B, C \subseteq |N|$, B $\mathfrak{i}$-*dominates* C over M (in N) if for any $N' \ge N$ and any $D \subseteq |N'|$, $B \overset{N'}{\underset{M}{\perp}} D$ implies $B \cup C \overset{N'}{\underset{M}{\perp}} D$.

We say that B $\mathfrak{i}$-*model-dominates* C over M in N if for any $N' \ge N$ and any $M \le N_0' \le N'$, $B \overset{N'}{\underset{M}{\perp}} N_0'$ implies $B \cup C \overset{N'}{\underset{M}{\perp}} N_0'$.

Of course, when $\mathfrak{i}$ is clear from context, we omit it.

Model-domination turns out to be the technical variation we need, but of course if $\mathfrak{i}$ has set-existence, then it is equivalent to domination. We start with a few easy ambient monotonicity properties:

Lemma 16.6. Let $M \leq N$. Let $B, C \subseteq |N|$ and assume B [model-]dominates C over M in N. Then:

(1) If $N' \geq N$, then B [model-]dominates C over M in N'.
(2) If $M \leq N_0 \leq N$ contains $B \cup C$, then B [model-]dominates C over M in N_0.

Proof. We only do the proofs for the non-model variation but of course the model variation is completely similar.

(1) By definition of domination.
(2) Let $N' \geq N_0$ and $D \subseteq |N'|$ be given such that $B \overset{N'}{\underset{M}{\mathop{\smile}\limits^{\vert}}} D$. By amalgamation, there exists $N'' \geq N$ and $f : N' \xrightarrow[N_0]{} N''$. By invariance, $B \overset{N''}{\underset{M}{\mathop{\smile}\limits^{\vert}}} f[D]$. By definition of domination, $B \cup C \overset{f[N']}{\underset{M}{\mathop{\smile}\limits^{\vert}}} f[D]$. By invariance again, $B \cup C \overset{N'}{\underset{M}{\mathop{\smile}\limits^{\vert}}} D$, as desired.

$\square$

The next result is key for us: it ties domination with the notion of uniqueness triples:

Lemma 16.7. Assume $M_0 \leq M_1$ are in K_λ, and $a \in M_1$ model-dominates M_1 over M_0 (in M_1). Then $(a, M_0, M_1) \in K_{\mathfrak{s}_\lambda}^{3,\mathrm{uq}}$.

Proof. Let $M_2 \geq M_0$ be in K_λ. First, we need to show that there exists (b, M_2, N) such that $\mathrm{gtp}(b/M_2; N)$ extends $\mathrm{gtp}(a/M_0; M_1)$ and $\mathrm{gtp}(b/M_2; N)$ does not fork over M_0. This holds by the extension property of good frames.

Second, we need to show that any such amalgam is unique: Let (f_1^x, f_2^x, N^x), $x \in \{a, b\}$ be amalgams of M_1 and M_2 over M_0 such that $f_1^x(a) \overset{N^x}{\underset{M_0}{\mathop{\smile}\limits^{\vert}}} f_2^x[M_2]$.
We want to show that the two amalgams are equivalent: we want $N_* \in K_\lambda$ and $f^x : N^x \to N_*$ such that $f^b \circ f_1^b = f^a \circ f_1^a$ and $f^b \circ f_2^b = f^a \circ f_2^a$, namely, the following commutes:

$$\begin{tikzcd}
 & N^b \arrow[r, dotted, "f^b"] & N_* \\
M_1 \arrow[ur, "f_1^b"] \arrow[rr, "f_1^a"] & & N^a \arrow[u, dotted, "f^a"'] \\
M_0 \arrow[u] \arrow[r] & M_2 \arrow[uu, "f_2^b"] \arrow[ur, "f_2^a"'] &
\end{tikzcd}$$

For $x = a, b$, rename f_2^x to the identity to get amalgams $((f_1^x)', \mathrm{id}_{M_2}, (N^x)')$ of M_1 and M_2 over M_0. For $x = a, b$, the amalgams $((f_1^x)', \mathrm{id}_{M_2}, (N^x)')$ and (f_1^x, f_2^x, N^x) are equivalent over M_0, hence we can assume without loss of generality that the renaming has already been done and $f_2^x = \mathrm{id}_{M_2}$

Thus we know that $f_1^x(a) \overset{N^x}{\underset{M_0}{\downfree}} M_2$ for $x = a, b$. By domination, $f_1^x[M_1] \overset{N^x}{\underset{M_0}{\downfree}} M_2$. Let $\bar{M}_1$ be an enumeration of M_1. Using amalgamation, we can obtain the following diagram:

$$\begin{tikzcd}
N^a \arrow[r, dotted, "g^a"'] & N' \\
M_1 \arrow[u, "f_1^a"] \arrow[r, "f_1^b"'] & N^b \arrow[u, dotted, "g^b"']
\end{tikzcd}$$

This shows $\mathrm{gtp}(f_1^a(\bar{M}_1)/M_0; N^a) = \mathrm{gtp}(f_1^b(\bar{M}_1)/M_0; N^b)$. By uniqueness, $\mathrm{gtp}(f_1^a(\bar{M}_1)/M_2; N^a) = \mathrm{gtp}(f_1^b(\bar{M}_1)/M_2; N^b)$. Let N_* and $f^x : N^x \underset{M_2}{\longrightarrow} N_*$ witness the equality. Since $f_2^x = \mathrm{id}_{M_2}$, $f^b \circ f_2^b = f^b \upharpoonright M_2 = \mathrm{id}_{M_2} = f^a \circ f_2^a$. Moreover, $(f^b \circ f_1^b)(\bar{M}_1) = f^b(f_1^b(\bar{M}_1)) = f^a(f_2^a(\bar{M}_1))$ by definition, so $f^b \circ f_1^b = f^a \circ f_1^a$. This completes the proof. $\square$

Remark 16.8. The converse will hold if $\mathfrak{i}$ has left set-extension.

Remark 16.9. The relationship of uniqueness triples with domination is already mentioned in [JS13, Proposition 4.1.7], although the definition of domination there is different.

Thus to prove the existence property for uniqueness triples, it will be enough to imitate the proof of [MS90, Proposition 4.22]. We first show that we can work inside a monster model.

Lemma 16.10. Let $M \leq N$ and $B \subseteq |N|$. Let $\mathfrak{C} \geq N$ be $\|N\|^+$-model-homogeneous. Then B [model-]dominates N over M in $\mathfrak{C}$ if and only if for any $D \subseteq |\mathfrak{C}|$, $B \underset{M}{\overset{\mathfrak{C}}{\perp}} D$ implies $N \underset{M}{\overset{\mathfrak{C}}{\perp}} D$.

Proof. We prove the non-trivial direction for domination. The proof for model-domination is similar. Assume $\mathfrak{C}' \geq \mathfrak{C}$ and $D \subseteq |\mathfrak{C}'|$ is such that $B \underset{M}{\overset{\mathfrak{C}'}{\perp}} D$. We want to show that $N \underset{M}{\overset{\mathfrak{C}'}{\perp}} D$. Suppose not. Then we can use the $(\leq \mu)$-witness property to assume without loss of generality that $|D| \leq \mu$, and so we can find $N \leq N' \leq \mathfrak{C}'$ containing D with $\|N'\| = \|N\|$ and $B \underset{M}{\overset{N'}{\perp}} D$, $N \underset{M}{\overset{N'}{\not\perp}} D$. By homogeneity, find $f : N' \xrightarrow[N]{} \mathfrak{C}$. By invariance, $B \underset{M}{\overset{f[N']}{\perp}} f[D]$ but $N \underset{M}{\overset{f[N']}{\not\perp}} f[D]$. By monotonicity, $B \underset{M}{\overset{\mathfrak{C}}{\perp}} f[D]$ but $N \underset{M}{\overset{\mathfrak{C}}{\not\perp}} f[D]$, a contradiction. $\square$

Lemma 16.11 (Lemma 4.20 in [MS90])**.** Let $\langle M_i : i < \lambda^+ \rangle$, $\langle N_i : i < \lambda^+ \rangle$ be increasing in K_λ such that $M_i \leq N_i$ for all $i < \lambda^+$. Let $M_{\lambda^+} := \bigcup_{i<\lambda^+} M_i$, $N_{\lambda^+} := \bigcup_{i<\lambda^+} N_i$.

Then there exists $i < \lambda^+$ such that $N_i \underset{M_i}{\overset{N_{\lambda+}}{\perp}} M_{\lambda^+}$.

Proof. For each $i < \lambda^+$, let $j_i < \lambda^+$ be least such that $N_i \underset{M_{j_i}}{\overset{N_{\lambda+}}{\perp}} M_{\lambda^+}$ (exists since $\kappa_\lambda(\mathfrak{i}) = \lambda^+$). Let i^* be such that $j_i < i^*$ for all $i < i^*$ and $\text{cf}(i^*) \geq \mu^+$. By definition of j_i and base monotonicity we have that for all $i < i^*$, $N_i \underset{M_{i^*}}{\overset{N_{\lambda+}}{\perp}} M_{\lambda^+}$. By the left $(\leq \mu)$-set-witness property, $N_{i^*} \underset{M_{i^*}}{\overset{N_{\lambda+}}{\perp}} M_{\lambda^+}$. $\square$

Lemma 16.12 (Proposition 4.22 in [MS90])**.** Let $M \in K_\lambda$ be saturated. Let $\mathfrak{C} \geq M$ be saturated of size λ^+. Work inside $\mathfrak{C}$. Write $A \underset{M}{\perp} B$ for $A \underset{M}{\overset{\mathfrak{C}}{\perp}} B$.

- There exists a saturated $N \leq \mathfrak{C}$ in K_λ such that $M \leq N$, N contains a, and a model-dominates N over M (in $\mathfrak{C}$).

- In fact, if $M^* \leq M$ is in $K_{<\lambda}$, $a \underset{M^*}{\downarrow} M$, and $r \in \mathrm{gS}^{\leq\lambda}(M^* a; \mathfrak{C})$, then N can be chosen so that it realizes r.

Proof. Since $\bar{\kappa}_1(\mathfrak{s}) = \mu^+ \leq \lambda$, it suffices to prove the second part. Assume it fails.

Claim: For any saturated $M' \geq M$ in K_λ, if $a \underset{M}{\downarrow} M'$, then the second part fails with M' replacing M.

Proof of claim: By transitivity, $a \underset{M^*}{\downarrow} M'$. By uniqueness of saturated models, there exists $f : M' \cong_{M^*} M$, which we can extend to an automorphism of $\mathfrak{C}$. Thus we also have $f(a) \underset{M^*}{\downarrow} M$. By uniqueness, we can assume without loss of generality that f fixes a as well. Since the second part above is invariant under applying f^{-1}, the result follows.

We now construct increasing continuous chains $\langle M_i : i \leq \lambda^+ \rangle$, $\langle N_i : i \leq \lambda^+ \rangle$ such that for all $i < \lambda^+$:

(1) $M_0 = M$.
(2) $M_i \leq N_i$.
(3) $M_i \in K_\lambda$ is saturated.
(4) $a \underset{M_0}{\downarrow} M_i$.
(5) $N_i \underset{M_i}{\not\downarrow} M_{i+1}$.

This is enough: the sequences contradict Lemma 16.11. This is possible: take $M_0 = M$, and N_0 any saturated model of size λ containing M_0 and a and realizing r. At limits, take unions (we are using that $K^{\lambda\text{-sat}}$ is an AEC). Now assume everything up to i has been constructed. By the claim, the second part above fails for M_i, so in particular N_i cannot be model-dominated by a over M_i. Thus (implicitly using Lemma 16.10) there exists $M_i' \geq M_i$ with $a \underset{M_i}{\downarrow} M_i'$ and $N_i \underset{M_i}{\not\downarrow} M_i'$. By the witness property, we can assume without loss of generality that $\|M_i'\| \leq \lambda$, so using extension and transitivity, we can find $M_{i+1} \in K_\lambda$ saturated containing M_i' so that $a \underset{M_i}{\downarrow} M_{i+1}$. By monotonicity we still have $N_i \underset{M_i}{\not\downarrow} M_{i+1}$. Let $N_{i+1} \in K_\lambda$ be any saturated model containing N_i and M_{i+1}. $\square$

Theorem 16.13. $\mathfrak{s}_\lambda \upharpoonright K^{\lambda\text{-sat}}$ is a *weakly successful* type-full good λ-frame.

Proof. Since $\mathfrak{s}_\lambda$ is a type-full good frame, $\mathfrak{s}_\lambda \upharpoonright K^{\lambda\text{-sat}}$ also is. To show it is weakly successful, we want to prove the existence property for uniqueness triples. So let $M \in K_\lambda^{\lambda\text{-sat}}$ and $p \in \mathrm{gS}(M)$ be nonalgebraic. Say $p = \mathrm{gtp}(a/M; N')$. Let $\mathfrak{C}$ be a monster model with $N' \leq \mathfrak{C}$. By Lemma 16.12, there exists $N \leq \mathfrak{C}$ in $K_\lambda^{\lambda\text{-sat}}$ such that $M \leq N$, $a \in |N|$, and a dominates N over M in $\mathfrak{C}$. By Lemma 16.6, a dominates N over M in N. By Lemma 16.7, $(a, M, N) \in K^{3,\text{uq}}_{\mathfrak{s}_\lambda \upharpoonright K^{\lambda\text{-sat}}}$. Now, $p = \mathrm{gtp}(a/M; N') = \mathrm{gtp}(a/M; \mathfrak{C}) = \mathrm{gtp}(a/M; N)$, as desired. $\square$

The term "weakly successful" suggests that there must exist a definition of "successful". Indeed, this is the case:

Definition 16.14 (Definition 10.1.1 in [JS13])**.** A type-full good λ-frame $\mathfrak{t} = (K_\mathfrak{t}, \overset{\mathfrak{t}}{\smile})$ is *successful* if it is weakly successful and $\leq^{\mathrm{NF}}_{\lambda^+}$ has smoothness: whenever $\langle N_i : i \leq \delta \rangle$ is a $\leq^{\mathrm{NF}}_{\lambda^+}$-increasing continuous chain of saturated models in $(K_\mathfrak{t})_{\lambda^+}$, $N \in (K_\mathfrak{t})_{\lambda^+}$ is saturated and $i < \delta$ implies $N_i \leq^{\mathrm{NF}}_{\lambda^+} N$, then $N_\delta \leq^{\mathrm{NF}}_{\lambda^+} N$.

The only thing we need to know about the relation $\leq^{\mathrm{NF}}_{\lambda^+}$ is:

Fact 16.15 (Theorem 4.1 in [Jarb])**.** If $\mathfrak{t} = (K_\mathfrak{t}, \overset{\mathfrak{t}}{\smile})$ is a type-full good λ-frame, $(K_\mathfrak{t})_{[\lambda, \lambda^+]}$ has amalgamation and is λ-tame, then $\leq \upharpoonright (K_\mathfrak{t})^{\lambda^+\text{-sat}}_{\lambda^+} = \leq^{\mathrm{NF}}_{\lambda^+}$.

Corollary 16.16. $\mathfrak{s}_\lambda \upharpoonright K^{\lambda\text{-sat}}$ is a *successful* type-full good λ-frame.

Proof. By Theorem 16.13, $\mathfrak{s}_\lambda \upharpoonright K^{\lambda\text{-sat}}$ is weakly successful. To show it is successful, it is enough (by Fact 16.15), to see that $\leq$ has smoothness. But this holds since K is an AEC. $\square$

For a good λ-frame $\mathfrak{t}$, Shelah also defines a λ^+-frame $\mathfrak{t}^+$ ([She09a, Definition III.1.7]). He then goes on to show:

Fact 16.17 (Claim III.1.9 in [She09a])**.** If $\mathfrak{t}$ is a successful good λ-frame, then $\mathfrak{t}^+$ is a good λ^+-frame.

Note that in our case, it is easy to check that:

Fact 16.18. $(\mathfrak{s}_\lambda)^+ = \mathfrak{s}_{\lambda^+} \upharpoonright K^{\lambda^+\text{-sat}}$.

Definition 16.19 (Definition III.1.12 in [She09a])**.** Let $\mathfrak{t}$ be a pre-λ-frame.

(1) By induction on $n < \omega$, define $\mathfrak{t}^{+n}$ as follows:
  (a) $\mathfrak{t}^{+0} = \mathfrak{t}$.

(b) $\mathfrak{t}^{+(n+1)} = (\mathfrak{t}^{+n})^+$.

(2) By induction on $n < \omega$, define "$\mathfrak{t}$ is n-successful" as follows:
   (a) $\mathfrak{t}$ is 0-successful if and only if it is a good λ-frame.
   (b) $\mathfrak{t}$ is $(n+1)$-successful if and only if it is a successful good λ-frame and $\mathfrak{t}^+$ is n-successful.
(3) $\mathfrak{t}$ is ω-*successful* if it is n-successful for all $n < \omega$.

Thus by Fact 16.17, $\mathfrak{t}$ is 1-successful if and only if it is a successful good λ-frame. More generally a good λ-frame $\mathfrak{t}$ is n-successful if and only if $\mathfrak{t}^{+m}$ is a successful good λ^{+m}-frame for all $m < n$.

Theorem 16.20. $\mathfrak{s}_\lambda \restriction K^{\lambda\text{-sat}}$ is an ω-*successful* type-full good λ-frame.

Proof. By induction on $n < \omega$, simply observing that we can replace λ by λ^{+n} in Corollary 16.16. $\square$

Note that all the results of [She09a, Chapter III] will apply to such a frame.

17. A fully good long frame

Hypothesis 17.1. $\mathfrak{s} = (K, \mathop{\smile}\limits^{\vert})$ is a weakly successful type-full good λ-frame.

This is reasonable since the previous section showed us how to build such a frame. Our goal is to extend $\mathfrak{s}$ to obtain a fully good $(\leq \lambda, \lambda)$-independence relation.

Fact 17.2 (Conclusion II.6.34 in [She09a])**.** There exists a relation $\mathrm{NF} \subseteq {}^4K$ satisfying:

(1) $\mathrm{NF}(M_0, M_1, M_2, M_3)$ implies $M_0 \leq M_\ell \leq M_3$ are in K for $\ell = 1, 2$.
(2) $\mathrm{NF}(M_0, M_1, M_2, M_3)$ and $a \in |M_1| \backslash |M_2|$ implies $\mathrm{gtp}(a/M_2; M_3)$ does not $\mathfrak{s}$-fork over M_0.
(3) Invariance: NF is preserved under isomorphisms.
(4) Monotonicity: If $\mathrm{NF}(M_0, M_1, M_2, M_3)$:
   (a) If $M_0 \leq M_\ell' \leq M_\ell$ for $\ell = 1, 2$, then $\mathrm{NF}(M_0, M_1', M_2', M_3')$.
   (b) If $M_3' \leq M_3$ contains $|M_1| \cup |M_2|$, then $\mathrm{NF}(M_0, M_1, M_2, M_3')$.
   (c) If $M_3' \geq M_3$, then $\mathrm{NF}(M_0, M_1, M_2, M_3')$.
(5) Symmetry: $\mathrm{NF}(M_0, M_1, M_2, M_3)$ if and only if $\mathrm{NF}(M_0, M_2, M_1, M_3)$.
(6) Long right transitivity: If $\langle M_i : i \leq \alpha \rangle$, $\langle N_i : i \leq \alpha \rangle$ are increasing continuous and $\mathrm{NF}(M_i, N_i, M_{i+1}, N_{i+1})$ for all $i < \alpha$, then $\mathrm{NF}(M_0, N_0, M_\alpha, N_\alpha)$.

(7) Full existence: If $M_0 \leq M_\ell$, $\ell = 1, 2$, then for some $M_3 \in K$, $f_\ell : M_\ell \underset{M_0}{\longrightarrow} M_3$, we have $\text{NF}(M_0, f_1[M_1], f_2[M_2], M_3)$.

(8) Uniqueness: If $\text{NF}(M_0^\ell, M_1^\ell, M_2^\ell, M_3^\ell)$, $\ell = 1, 2$ and $f_i : M_i^1 \cong M_i^2$ for $i = 0, 1, 2$ and $f_0 \subseteq f_1$, $f_0 \subseteq f_2$, then $f_1 \cup f_2$ can be extended to $f_3 : M_3^1 \to M_4^2$, for some M_4^2 with $M_3^2 \leq M_4^2$.

Notation 17.3. We write $M_1 \overset{M_3}{\underset{M_0}{\perp}} M_2$ instead of $\text{NF}(M_0, M_1, M_2, M_3)$. If $\bar{a}$ is a sequence, we write $\bar{a} \overset{M_3}{\underset{M_0}{\perp}} M_2$ for $\text{ran}(\bar{a}) \overset{\text{NF}, M_3}{\underset{M_0}{\perp}} M_2$, and similarly if sequences appear at other places.

NF is only a relation for models in K and we would like to make it into a relation taking arbitrary sets of size less than or equal to λ on the left hand side.

We start by showing that uniqueness is really the same as the uniqueness property stated for coheir. We drop Hypothesis 17.1 for the next lemma.

Lemma 17.4. Let K be an AEC in λ and assume K has amalgamation. The following are equivalent for a relation $\text{NF} \subseteq {}^4 K$ satisfying (1), (3), (4) of Fact 17.2:

(1) Uniqueness.

(2) Uniqueness in the sense of frames: If $A \overset{N}{\underset{M_0}{\perp}} M_1$ and $A' \overset{N'}{\underset{M_0}{\perp}} M_1$ for models A and A', $\bar{a}$ and $\bar{a}'$ are enumerations of A and A' respectively, $p := \text{gtp}(\bar{a}/M_1; N)$, $q := \text{gtp}(\bar{a}'/M_1; N')$, and $p \restriction M_0 = q \restriction M_0$, then $p = q$.

Proof.

- <u>(1) implies (2)</u>: Since $p \restriction M_0 = q \restriction M_0$, there exists $N'' \geq N'$ and $f : N \underset{M_0}{\longrightarrow} N''$ such that $f(\bar{a}) = \bar{a}'$. Therefore by invariance, $\bar{a}' \overset{N''}{\underset{M_0}{\perp}} f[M_1]$. Let $f_0 := \text{id}_{M_0}$, $f_1 := f^{-1} \restriction f[M_1]$, $f_2 := \text{id}_{A'}$. By uniqueness, there exists $N''' \geq N''$, $g \supseteq f_1 \cup f_2$, $g : N'' \to N'''$. Consider the map $h := g \circ f : N \to N'''$. Then $g \restriction M_1 = \text{id}_{M_1}$ and $h(\bar{a}) = g(\bar{a}') = \bar{a}'$, so h witnesses $p = q$.

- (2) implies (1): By some renaming, it is enough to prove that whenever $M_2 \overset{N}{\underset{M_0}{\perp}} M_1$ and $M_2 \overset{N'}{\underset{M_0}{\perp}} M_1$, there exists $N'' \geq N'$ and $f : N' \xrightarrow[|M_1| \cup |M_2|]{} N''$. Let $\bar{a}$ be an enumeration of M_2. Let $p := \text{gtp}(\bar{a}/M_1; N)$, $q := \text{gtp}(\bar{a}/M_1; N')$. We have that $p \upharpoonright M_0 = \text{gtp}(\bar{a}/M_1; M_2) = q \upharpoonright M_0$. Thus $p = q$, so there exists $N'' \geq N'$ and $f : N \xrightarrow[M_1]{} N''$ such that $f(\bar{a}) = \bar{a}$. In other words, f fixes M_2, so is the desired map.

□

We now extend NF to take sets on the left hand side. This step is already made by Shelah in [She09a, Claim III.9.6], for singletons rather than arbitrary sets. We check that Shelah's proofs still work.

Definition 17.5. Define $\text{NF}'(M_0, A, M, N)$ to hold if and only if $M_0 \leq M \leq N$ are in K, $A \subseteq |N|$, and there exists $N' \geq N$, $N_A \geq M$ with $N_A \leq N'$ and $N_A \overset{N'}{\underset{M_0}{\perp}} M$. We abuse notation and also write $A \overset{N}{\underset{M_0}{\perp}} M$ instead of $\text{NF}'(M_0, A, M, N)$. We let $\mathfrak{t} := (K, \perp)$.

Proposition 17.6.

(1) If $M_0 \leq M_\ell \leq M_3$, $\ell = 1, 2$, then $\text{NF}(M_0, M_1, M_2, M_3)$ if and only if $\text{NF}^1(M_0, M_1, M_2, M_3)$.
(2) $\mathfrak{t}$ is a (type-full) pre-$(\leq \lambda, \lambda, \lambda)$-frame.
(3) $\mathfrak{t}$ has base monotonicity, full symmetry, uniqueness, existence, and extension.

Proof. Exactly as in [She09a, Claim III.9.6]. □

We now turn to local character. The key is:

Fact 17.7 (Claim III.1.17.2 in [She09a])**.** Given $\langle M_i : i \leq \delta \rangle$ increasing continuous, we can build $\langle N_i : i \leq \delta \rangle$ increasing continuous such that for all $i \leq j \leq \delta$, $N_i \overset{\text{NF}, N_j}{\underset{M_i}{\perp}} M_j$ and $M_\delta <_{\text{univ}} N_\delta$.

Lemma 17.8. For all $\alpha \leq \lambda$, $\kappa_\alpha(\mathfrak{t}) = |\alpha|^+ + \aleph_0$.

Proof. Let $\langle M_i : i \leq \delta + 1 \rangle$ be increasing continuous with $\delta = \text{cf}(\delta) > |\alpha|$. Let $A \subseteq |M_{\delta+1}|$ have size $\leq \alpha$. Let $\langle N_i : i \leq \delta \rangle$ be as given by Fact 17.7. By universality, we can assume without loss of generality that $M_{\delta+1} \leq N_\delta$. Thus $A \subseteq |N_\delta|$ and by the cofinality hypothesis, there exists $i < \delta$ such that $A \subseteq |N_i|$. In particular, $A \overset{N_\delta}{\underset{M_i}{\perp}} M_\delta$, so $A \overset{M_{\delta+1}}{\underset{M_i}{\perp}} M_\delta$, as needed. $\square$

Remark 17.9. In [JS13] (and later in [JS12, Jara, Jarb]), the authors have considered *semi-good* λ-frames, where the stability condition is replaced by almost stability ($|\mathrm{gS}(M)| \leq \lambda^+$ for all $M \in K_\lambda$), and an hypothesis called the conjugation property is often added. Many of the above results carry through in that setup but we do not know if Lemma 17.8 would also hold.

Finally, we get to the last property: disjointness. The situation is a bit muddy: At first glance, (2) in Fact 17.2 seems to give it to us for free (since we are assuming $\mathfrak{s}$ has disjointness), but unfortunately we are assuming $a \notin |M_2|$ there. We will use a trick similar to the proof of Lemma 12.6.(5). We make the additional hypothesis of categoricity in λ (this is reasonable since one can always restrict oneself to the class of λ-saturated models). Note that disjointness is never used in a crucial way here (but it is always nice to have, as it implies for example disjoint amalgamation when combined with extension).

Lemma 17.10. If K is categorical in λ, then $\mathfrak{t}$ has disjointness and $\mathfrak{t}^{\leq 1} = \mathfrak{s}$.

Proof. The last equation follows from the first and Fact 17.2.(2). Now it is of course enough to show that $\mathfrak{t}^{\leq 1}$ has disjointness, so assume $a \overset{N}{\underset{M_0}{\perp}} M$ and $a \in |M|$. We show $a \in |M_0|$. By stability, we can find $\langle N_i : i \leq \omega \rangle$ $<_{\text{univ}}$-increasing continuous in K. By categoricity, we can do this so that $N_\omega = M_0$. By local character and transitivity, there exists $i < \omega$ such that $a \overset{N}{\underset{M_i}{\perp}} M$. Note that $N_i <_{\text{univ}} N_{i+1} \leq N_\omega$, so $N_i <_{\text{univ}} N_\omega$ by definition of universality. By Proposition 12.2.(4), $\mathfrak{t}^{\leq 1}$ is weakly good. By weak extension (see Proposition 12.6), we can find $p \in \mathrm{gS}(M)$ extending $p_0 := \mathrm{gtp}(a/M_0; N)$ and not forking over N_i. By the moreover part of weak extension, p_0 will be algebraic if and only if p is. Since $a \in |M|$ and $a \overset{N}{\underset{N_i}{\perp}} M$, weak uniqueness implies that p is algebraic. Therefore p_0 is algebraic, so $a \in M_0$. $\square$

What about continuity for chains? The long right transitivity property seems to suggest we can say something, and indeed we can:

Fact 17.11. Assume $\lambda = \lambda_0^{+3}$ and there exists an ω-successful good λ_0-frame $\mathfrak{s}'$ such that $\mathfrak{s} = (\mathfrak{s}')^{+3}$.

Assume δ is a limit ordinal and $\langle M_i^\ell : i \leq \delta \rangle$ is increasing continuous in K_λ, $\ell \leq 3$. If $M_i^1 \overset{M_i^3}{\underset{M_i^0}{\perp}} M_i^2$ for each $i < \delta$, then $M_\delta^1 \overset{M_\delta^3}{\underset{M_\delta^0}{\perp}} M_\delta^2$.

Proof. By [She09a, Claim III.12.2], all the hypotheses at the beginning of each section of Chapter III in the book hold for $\mathfrak{s}$. Now apply Claim III.8.19 in the book. $\square$

Remark 17.12. Instead of assuming $\lambda_0^{+3} = \lambda$, we could (as Shelah does) axiomatize more and assume only that $\mathfrak{s}$ satisfies properties like existence of prime triples, weak orthogonality being orthogonality, etc. Note that the hypothesis of Fact 17.11 implies that $\mathfrak{s}$ is type-full and ω-successful.

We obtain:

Theorem 17.13.

(1) If K is categorical in λ, then $\mathfrak{t}$ is a good $(\leq \lambda, \lambda)$-frame.
(2) If $\lambda = \lambda_0^{+3}$ and there exists an ω-successful good λ_0-frame $\mathfrak{s}'$ such that $\mathfrak{s} = (\mathfrak{s}')^{+3}$, then $\mathfrak{t}$ is a fully good $(\leq \lambda, \lambda)$-frame.

Proof. $\mathfrak{t}$ is good by Proposition 17.6, Lemma 17.8, and Lemma 17.10. The second part follows from Fact 17.11 (note that by definition of the successor frame, K will be categorical in λ in that case). $\square$

Remark 17.14. If $\mathfrak{t}$ is [fully] good, $\text{cl}(\mathfrak{t})$ (see Definition 9.7) will be a [fully] good $(\leq \lambda, \lambda)$-independence relation by Proposition 11.2.

Remark 17.15. In [BVa, Corollary 6.10], it is shown that λ-tameness and amalgamation imply that a good λ-frame extends to a good $(< \infty, \lambda)$-frame. However, the definition of a good frame there is not the same as it does *not* assume that the frame is type-full. Thus the conclusion of Theorem 17.13 is much stronger.

18. Extending the base and right hand side

Hypothesis 18.1.

(1) $\mathfrak{i} = (K, \perp)$ is a fully good $(< \theta, \mathcal{F})$-independence relation, $\mathcal{F} = [\lambda, \theta)$.
(2) $\theta' \geq \theta$, $\mathcal{F}' := [\lambda, \theta')$.

(3) $K' := K_{\mathcal{F}'}$ has amalgamation and is $(<\theta)$-tame for types of length less than θ.

In this section, we give conditions under which $\mathfrak{i}$ becomes a fully good $(<\theta, \mathcal{F}')$-independence relation. In the next section, we will make the left hand side bigger and get a fully good $(<\theta', \mathcal{F}')$-independence relation. Of course, the main case for us is when $\theta' = \infty$ and $\theta = \lambda^+$.

Notation 18.2. Let $\mathfrak{i}' := \mathfrak{i}_{\mathcal{F}'}$ (recall Definition 13.1). Write $\mathfrak{s} := \text{pre}(\mathfrak{i})$, $\mathfrak{s}' := \text{pre}(\mathfrak{i}')$. We abuse notation and also denote $\underset{\mathfrak{i}'}{\perp}$ by $\perp$.

We want to investigate when the properties of $\mathfrak{i}$ carry over to $\mathfrak{i}'$.

Lemma 18.3.

(1) $\mathfrak{i}'$ is a $(<\theta, \mathcal{F}')$-independence relation.
(2) K' has joint embedding, no maximal models, and is stable in all cardinals.
(3) $\mathfrak{i}'$ has base monotonicity, transitivity, uniqueness, and disjointness.
(4) $\mathfrak{i}'$ has full model continuity.

Proof.

(1) By Proposition 13.4.
(2) By applying Fact 13.6 to $\mathfrak{s}^{\leq 1}$.
(3) See [She09a, Claim II.2.11] for base monotonicity and transitivity. Disjointness is straightforward from the definition of $\mathfrak{i}'$, and uniqueness follows from the tameness hypothesis and the definition of $\mathfrak{i}'$.
(4) Assume $\langle M_i^\ell : i \leq \delta \rangle$ is increasing continuous in K', $\ell \leq 3$, δ is regular, $M_i^0 \leq M_i^\ell \leq M_i^3$ for $\ell = 1, 2$, $i < \delta$, and $M_i^1 \overset{M_i^3}{\underset{M_i^0}{\perp}} M_i^2$ for all $i < \delta$. Let $N := M_\delta^3$. By ambient monotonicity, $M_i^1 \overset{N}{\underset{M_i^0}{\perp}} M_i^2$ for all $i < \delta$. We want to see that $M_\delta^1 \overset{N}{\underset{M_\delta^0}{\perp}} M_\delta^2$. Since $\|M_\delta^1\| < \theta$, M_δ^1 and M_δ^0 are in K. Thus it is enough to show that for all $M' \leq M_\delta^2$ in K with $M_\delta^0 \leq M'$, $M_\delta^1 \overset{N}{\underset{M_\delta^0}{\perp}} M'$. Fix such an M'. We consider two cases:
 - Case 1: $\delta < \theta$: Then we can find $\langle M_i' : i \leq \delta \rangle$ increasing continuous in K such that $M_\delta' = M'$ and for all $i < \delta$,

$M_0^\delta \leq M_i' \leq M_i^2$. By monotonicity, for all $i < \delta$, $M_i^1 \overset{N}{\underset{M_i^0}{\downarrow}} M_i'$. By full model continuity in K, $M_\delta^1 \overset{N}{\underset{M_\delta^0}{\downarrow}} M'$, as desired.

- Case 2: $\delta \geq \theta$: Since $M_\delta^0, M_\delta^1 \in K$, we can assume without loss of generality that $M_\delta^0 = M_0^0$, $M_\delta^1 = M_\delta^1$. Since δ is regular, there exists $i < \delta$ such that $M' \leq M_i^2$. By assumption, $M_0^1 \overset{N}{\underset{M_0^0}{\downarrow}} M_i^2$, so by monotonicity, $M_0^1 \overset{N}{\underset{M_0^0}{\downarrow}} M'$, as needed.

$\square$

We now turn to local character.

Lemma 18.4. Assume $\langle M_i : i \leq \delta \rangle$ is increasing continuous, $p \in \mathrm{gS}^\alpha(M_\delta)$, $\alpha < \theta$ and $\delta = \mathrm{cf}(\delta) > |\alpha|$.

(1) If $|\alpha|^+ < \theta$, then there exists $i < \delta$ such that p does not fork over M_i.
(2) If $\theta = \mu^+$, $|\alpha| = \mu$ and $\mathfrak{i}$ has the left $(< \mathrm{cf}(\mu))$-set-witness property, then there exists $i < \delta$ such that p does not fork over M_i.

Proof.

(1) As in the proof of Lemma 13.5.(4).
(2) Assume this fails. By the proof of Lemma 13.5.(4), we can find a counterexample with $\delta = \theta$ and $M_i \in K$ for all $i < \delta$, so assume we already have this counterexample. Say $p = \mathrm{gtp}(\bar{a}/M_\delta; N)$ and let $A := \mathrm{ran}(\bar{a})$. Write $A = \bigcup_{j < \mathrm{cf}(\lambda)} A_j$ with $\langle A_j : i < \mathrm{cf}(\mu) \rangle$ increasing continuous and $|A_j| < \mu$. By the first part for each $j < \mathrm{cf}(\mu)$ there exists $i_j < \delta$ such that $A_j \overset{N}{\underset{M_{i_j}}{\downarrow}} M_\delta$. Let $i := \sup_{j < \mathrm{cf}(\mu)} i_j$. We claim that $A \overset{N}{\underset{M_i}{\downarrow}} M_\delta$. By the $(< \mathrm{cf}(\mu))$-set-witness property and the definition of $\mathfrak{i}'$ (here we use that $M_i \in K$), it is enough to show this for all $B \subseteq A$ of size less than $\mathrm{cf}(\mu)$. But any such B is contained in an A_j, and so the result follows from base monotonicity.

$\square$

Lemma 18.5. Assume $\mathfrak{i}'$ has existence. Then $\mathfrak{i}'$ has right independent amalgamation.

Proof. As in for example [Bonb, Theorem 5.3], using full model continuity. $\square$

Putting everything together, we obtain:

Theorem 18.6. Assume either of the following hold:

(1) θ is a limit cardinal.
(2) $\theta = \mu^+$ and K is $(< \operatorname{cf}(\mu))$-tame and short for types of length less than θ

Then $\mathfrak{i}'$ is a fully almost good $(< \theta, \mathcal{F}')$-independence relation. If in addition $\theta' = \infty$, then $\operatorname{cl}(\operatorname{pre}(\mathfrak{i}'))$ is a fully good $(< \theta, \geq \lambda)$-independence relation.

Proof. The last sentence is by Proposition 11.2.(5). It remains to show that $\mathfrak{i}'$ is fully almost good. The basic properties are proven in Lemma 18.3. If we are in the second case, by Proposition 11.1.(6) and symmetry, K has the left $< (\operatorname{cf}(\mu))$-set-witness property. Thus by Lemma 18.4, for any $\alpha < \theta$, $\kappa_\alpha(\mathfrak{i}') = |\alpha|^+ + \aleph_0$. In particular, $\mathfrak{i}'$ has existence, and thus by the definition of $\mathfrak{i}'$ and transitivity in $\mathfrak{i}$, $\bar{\kappa}_\alpha(\mathfrak{i}') = |\alpha|^+ + \lambda^+$. Finally by Lemma 18.5, $\mathfrak{i}'$ has independent amalgamation and so by Proposition 11.1.(7), $\mathfrak{i}'$ has extension. $\square$

19. Extending the left hand side

We now enlarge the left hand side of the independence relation built in the previous section.

Hypothesis 19.1.

(1) $\mathfrak{i} = (K, \mathop{\smile\!\!\!\!|})$ is a fully good $(< \alpha, \mathcal{F})$-independence relation, $\mathcal{F} = [\lambda, \theta)$.
(2) α is a cardinal with $\lambda^+ \leq \alpha < \theta$.
(3) K is $(< \alpha)$-short for types of length less than θ.

Definition 19.2. Define $\mathfrak{i}^{<\theta} = (K, \overset{<\theta}{\mathop{\smile\!\!\!\!|}})$ by setting $\overset{<\theta}{\mathop{\smile\!\!\!\!|}}(M_0, A, B, N)$ if and only if $|A| < \theta$ and for all $A_0 \subseteq A$ of size less than α, $A_0 \overset{N}{\underset{M_0}{\mathop{\smile\!\!\!\!|}}} B$.

We similarly define $\mathfrak{s}^{<\theta}$ for $\mathfrak{s}$ a pre-$(< \alpha, \mathcal{F})$-frame.

Remark 19.3. This notation conflicts with the one introduced in [BVa, Definition 4.3], but the idea is the same: we extend the frame to have longer types. The difference is that $\mathfrak{i}^{<\theta}$ is type-full.

Notation 19.4. Write $\mathfrak{i}' := \mathfrak{i}^{<\theta}$. We abuse notation and also write $\mathop{\smile\hspace{-0.7em}\mid\hspace{0.3em}}$ for $\overset{<\theta}{\mathop{\smile\hspace{-0.7em}\mid\hspace{0.3em}}}$.

Lemma 19.5.

(1) $\mathfrak{i}'$ is a $(<\theta, \mathcal{F})$-independence relation.
(2) K has joint embedding, no maximal models, and is stable in all cardinals.
(3) $\mathfrak{i}'$ has base monotonicity, transitivity, disjointness, existence, symmetry, and uniqueness.

Proof.

(1) Straightforward.
(2) Because $\mathfrak{i}$ is good.
(3) Base monotonicity, transitivity, disjointness, existence, and symmetry are straightforward. Uniqueness is by the shortness hypothesis.

□

Lemma 19.6. Assume $\kappa < \alpha$ is a regular cardinal such that $\mathfrak{i}$ has the left $(<\kappa)$-witness property. Then $\mathfrak{i}'$ has full model continuity.

Proof. Let $\langle M_i^\ell : i \leq \delta \rangle$, $\ell \leq 3$ be increasing continuous in K such that $M_i^0 \leq M_i^\ell \leq M_i^3$, $\ell = 1, 2$, and $M_i^1 \underset{M_i^0}{\overset{M_i^3}{\mathop{\smile\hspace{-0.7em}\mid\hspace{0.3em}}}} M_i^2$. Without loss of generality, δ is regular. Let $N := M_\delta^3$. We want to show that $M_\delta^1 \underset{M_\delta^0}{\overset{N}{\mathop{\smile\hspace{-0.7em}\mid\hspace{0.3em}}}} M_\delta^2$. Let $A \subseteq |M_\delta^1|$. have size less than α. Write $\mu := |A|$. By monotonicity, assume without loss of generality that $\lambda + \kappa \leq \mu$. We show that $A \underset{M_\delta^0}{\overset{N}{\mathop{\smile\hspace{-0.7em}\mid\hspace{0.3em}}}} M_\delta^2$ which is enough by definition of $\mathfrak{i}'$. We consider two cases.

- <u>Case 1: $\delta > \mu$</u>: By local character in $\mathfrak{i}$ there exists $i < \delta$ such that $A \underset{M_i^2}{\overset{N}{\mathop{\smile\hspace{-0.7em}\mid\hspace{0.3em}}}} M_\delta^2$. By right transitivity, $A \underset{M_i^0}{\overset{N}{\mathop{\smile\hspace{-0.7em}\mid\hspace{0.3em}}}} M_\delta^2$, so by base monotonicity, $A \underset{M_\delta^0}{\overset{N}{\mathop{\smile\hspace{-0.7em}\mid\hspace{0.3em}}}} M_\delta^2$.

- Case 2: $\delta \leq \mu$: For $i \leq \delta$, let $A_i := A \cap |M_i^1|$. Build $\langle N_i : i \leq \delta \rangle$, $\langle N_i^0 : i \leq \delta \rangle$ increasing continuous in $K_{\leq \mu}$ such that for all $i < \delta$:
  (1) $A_i \subseteq |N_i|$.
  (2) $N_i \leq M_i^1$, $A \subseteq |N_i|$.
  (3) $N_i^0 \leq M_i^0$, $N_i^0 \leq N_i$.
  (4) $N_i \overset{N}{\underset{N_i^0}{\perp}} M_i^2$.

  This is possible. Fix $i \leq \delta$ and assume N_j, N_j^0 have already been constructed for $j < i$. If i is limit, take unions. Otherwise, recall that we are assuming $M_i^1 \overset{N}{\underset{M_i^0}{\perp}} M_i^2$. By Lemma 11.4 (with $A_i \cup \bigcup_{j<i} |N_j|$ standing for A there), we can find $N_i^0 \leq M_i^0$ and $N_i \leq M_i^1$ in $K_{\leq \mu}$ such that $N_i^0 \leq N_i$, $N_i \overset{N}{\underset{N_i^0}{\perp}} M_i^2$, $A_i \subseteq |N_i|$, $N_j \leq N_i$ for all $j < i$, and $N_j^0 \leq N_i^0$ for all $j < i$. Thus they are as desired.
  This is enough. Note that $A_\delta = A$, so $A \subseteq |N_\delta|$. By full model continuity in $\mathfrak{i}$, $N_\delta \overset{N}{\underset{N_\delta^0}{\perp}} M_\delta^2$. By monotonicity, $A \overset{N}{\underset{M_\delta^0}{\perp}} M_\delta^2$, as desired.

$\square$

Lemma 19.7. Assume $\mathfrak{i}'$ has full model continuity and K is λ-tame for types of length less than θ. Then $\mathfrak{i}'$ has extension.

Proof. We show $\mathfrak{i}'$ has independent amalgamation which is enough by Proposition 11.1.(7).

Let $M^0 \leq M^\ell$, $\ell = 1, 2$ be in K. We want to find $N \geq M^2$ in K, $f : M^1 \xrightarrow[M^0]{} N$ such that $f[M^1] \overset{N}{\underset{M^0}{\perp}} M^2$. Let $\lambda_\ell := \|M^\ell\|$ for $\ell \leq 2$.

Work by induction on λ_0. Assume we know the result when $\lambda_0 = \lambda_1 = \lambda_2$. Then we can work by induction on (λ_1, λ_2): if they are both λ_0, the result holds by assumption. If not, we can assume by symmetry that $\lambda_1 \leq \lambda_2$, write $M^2 = \bigcup_{i<\|M^2\|} M_i^2$, $\|M_i^2\| < \lambda_2$ for all $i < \lambda_2$, and do a directed system argument as in [Bonb, Theorem 5.3] (using full model continuity and the induction hypothesis).

Now assume that $\lambda_0 = \lambda_1 = \lambda_2$. If $\lambda_0 = \lambda$, we get the result by extension in $\mathfrak{i}$, so assume $\lambda_0 > \lambda$. Let $\mathfrak{C} \in K_{\lambda_0^+}$ be model-homogeneous (can be built using amalgamation and stability), and without loss of generality, $M_0 \leq M_\ell \leq \mathfrak{C}$ for $\ell = 1, 2$. For any $N \leq M^0$ in K_λ, we can find $f_N \in \mathrm{Aut}_N(\mathfrak{C})$ such that $f_N[M^1] \underset{N}{\overset{\mathfrak{C}}{\mathop{\perp\!\!\!\!\smile}}} M^2$ (we have implicitly extended $\mathop{\perp\!\!\!\!\smile}$ to ambient models in $K_{\lambda_0^+}$ here). Let $\mathbb{P} := \{N \in K_\lambda \mid N \leq M^0\}$, and let $M_N := f_N[M^1]$. For $N \leq N' \leq M^0$ in $\mathbb{P}$, let $f_{N,N'} := f_{N'} \circ f_N^{-1}$. It is easy to check that $\langle (M_N, M^2, \mathfrak{C}), f_{N,N'} : N \in \mathbb{P}\rangle$ is a directed system indexed by $\mathbb{P}$. Let $(M, M', \widehat{M})$ be the direct limit of the system, and for $N \in \mathbb{P}$, let $g_N : (M_N, M^2, \mathfrak{C}) \cong_N (M, M', \widehat{M})$ be the corresponding limit map. By full model continuity, $M \underset{M^0}{\overset{\widehat{M}}{\mathop{\perp\!\!\!\!\smile}}} M'$. Moreover, for any $N \in \mathbb{P}$, g_N witnesses that (for appropriate enumerations):

$$\mathrm{gtp}(M^2/N; \mathfrak{C}) = \mathrm{gtp}(M'/N; \widehat{M})$$

Therefore by tameness, $\mathrm{gtp}(M^2/M^0; \mathfrak{C}) = \mathrm{gtp}(M'/M^0; \widehat{M})$. By homogeneity, we can find $f : \widehat{M} \cong_{M^0} \mathfrak{C}$ such that $f[M'] = M^2$. Then by invariance, $f[M] \underset{M^0}{\overset{\mathfrak{C}}{\mathop{\perp\!\!\!\!\smile}}} M^2$. Finally, observe that for any $N \in \mathbb{P}$, $g_N \circ f_N$ witnesses that (for appropriate enumerations again):

$$\mathrm{gtp}(M^1/N; \mathfrak{C}) = \mathrm{gtp}(M_N/N; \mathfrak{C}) = \mathrm{gtp}(M/N; \widehat{M})$$

Thus by tameness, $\mathrm{gtp}(M^1/M^0; \mathfrak{C}) = \mathrm{gtp}(M/M^0; \widehat{M})$, and so by homogeneity there exists a witness $g : \mathfrak{C} \cong_{M^0} \widehat{M}$ such that $g[M^1] = M$. Therefore, $f \circ g \in \mathrm{Aut}_{M^0}(\mathfrak{C})$ sends M^1 to $f[M]$ and so witnesses the extension property for M_0, M_1, M_2.

$\square$

Local character is slightly tricky: we have to look at the closure:

Lemma 19.8. Assume $\mathfrak{i}'$ has full model continuity and extension and $\mathfrak{i}'' := \mathrm{cl}(\mathrm{pre}(\mathfrak{i}'))$ has symmetry. Then:

(1) $\mathfrak{i}' = \mathfrak{i}''$.
(2) For all $\theta_0 < \theta$, $\kappa_{\theta_0}(\mathfrak{i}') = |\theta_0|^+ + \aleph_0$ and $\bar{\kappa}_{\theta_0}(\mathfrak{i}') = |\theta_0|^+ + \lambda^+$.

Proof. Note that if $\mathfrak{i}''$ has symmetry, then it has left set-extension so $\mathfrak{i}'$ also does. Since $\mathfrak{i}'$ has symmetry, it also has right set-extension which means that actually $\mathfrak{i}' = \mathfrak{i}''$. Now let's prove the second part.

If $\theta_0 < \alpha$, this is true because $\mathfrak{i}$ is good. Assume now that $\alpha \leq \theta_0$. In particular, $\lambda < \theta_0$, and we might as well assume that θ_0 is a cardinal. It is enough to show that $\bar{\kappa}_{\theta_0}(\mathfrak{i}') = \theta_0^+$ since then it follows on general grounds that $\kappa_{\theta_0}(\mathfrak{i}') = \theta_0^+$. Since $\mathfrak{i}'$ has left set-extension, base monotonicity, transitivity, existence, and full monotonicity, we can apply Lemma 11.3 with (μ_0, α) there standing for (α, θ) here. $\square$

Putting everything together, we get:

Theorem 19.9. If:

(1) $\theta = \infty$.
(2) For some regular $\kappa < \alpha$, K is $(< \kappa)$-tame for types of length less than α.
(3) K is λ-tame for types of all lengths.

Then $\mathfrak{i}'$ is a fully good $(< \infty, \geq \lambda)$-independence relation.

Proof. Lemma 19.5 gives most of the properties of a good independence relation. By Proposition 11.1.(6) and symmetry, $\mathfrak{i}$ has the left $(< \kappa)$-witness property. By Lemma 19.6, $\mathfrak{i}'$ has full model continuity. By Lemma 19.7, $\mathfrak{i}'$ has extension.

Let $\mathfrak{i}'' := \text{cl}(\text{pre}(\mathfrak{i}'))$. By Proposition 11.1.(5), $\mathfrak{i}''$ has symmetry and thus by Lemma 19.8, $\mathfrak{i}'' = \mathfrak{i}'$ and has the local character properties. Because $\mathfrak{i}''$ has set-extension, $\mathfrak{i}'$ also does. $\square$

20. The main theorem

Recall (Definition 9.16) that an AEC K is fully good if there is a fully good independence relation with underlying class K. Intuitively, a fully good independence relation is one that satisfies all the properties of forking in a superstable first-order theory. The main theorem of this paper says that fully tame and short superstable classes are fully good, at least on a dense subclass of saturated models[23]:

Theorem 20.1. Let K be a fully $(< \kappa)$-tame and short abstract elementary class.

(1) If K is μ-superstable, $\kappa = \beth_\kappa > \mu$, and $\lambda := (\kappa^{<\kappa})^{+7}$, then $K^{\lambda\text{-sat}}$ is fully good.
(2) If K is κ-strongly μ-superstable and $\lambda := (\mu^{<\kappa})^{+6}$, then $K^{\lambda\text{-sat}}$ is fully good.

[23] The number 7 in (1) is possibly the largest natural number ever used in a statement about abstract elementary classes!

(3) If K has joint embedding and amalgamation, $\kappa = \beth_\kappa > \text{LS}(K)$, and K is categorical in a $\mu > \lambda_0 := (\kappa^{<\kappa})^{+5}$, then $K_{\geq \lambda}$ is fully good, where $\lambda := \min(\mu, h(\lambda_0))$.

Proof.

(1) By Theorem 14.7, K is κ-strongly $(\kappa^{<\kappa})^+$-superstable. Now apply (2).
(2) By Fact 16.3, Hypothesis 16.1 holds for $\mu' := (\mu^{<\kappa})^{+2}$, λ there standing for $(\mu')^+$ here, and $K' := K^{\mu'\text{-sat}}$. By Theorem 16.20, there is an ω-successful type-full good $(\mu')^+$-frame $\mathfrak{s}$ on $K^{(\mu')^+\text{-sat}}$. By Theorem 17.13 and Remark 17.14, $\mathfrak{s}^{+3}$ induces a fully good $(\leq \lambda, \lambda)$-independence relation $\mathfrak{i}$ on $K^{(\mu')^{+4}\text{-sat}} = K^{\lambda\text{-sat}}$. By Theorem 18.6, $\mathfrak{i}' := \text{cl}(\text{pre}(\mathfrak{i}_{\geq \lambda}))$ is a fully good $(\leq \lambda, \geq \lambda)$-independence relation on $K^{\lambda\text{-sat}}$. By Theorem 19.9, $(\mathfrak{i}')^{<\infty}$ is a fully good $(< \infty, \geq \lambda)$-independence relation on $K^{\lambda\text{-sat}}$. Thus $K^{\lambda\text{-sat}}$ is fully good.
(3) By Theorem 14.10, K is κ-strongly κ-superstable. By (2), $K^{\lambda_0^+\text{-sat}}$ is fully good. By Fact 14.8.(5), all the models in $K_{\geq \lambda}$ are λ_0^+-saturated, hence $K_{\geq \lambda}^{\lambda_0^+\text{-sat}} = K_{\geq \lambda}$ is fully good.

$\square$

We end by discussing the necessity of the hypotheses of the above theorem. It is easy to see that a fully good AEC is superstable. Moreover, the existence of a relation $\perp$ with disjointness and independent amalgamation directly implies disjoint amalgamation. An interesting question is whether there is a general framework in which to study independence without assuming amalgamation, but this is out of the scope of this paper. Now joint embedding is not really necessary but it doesn't do any harm, as we can always shrink an AEC with amalgamation so that it still has arbitrarily large models but also has joint embedding. In that setup this is analogous to assuming a first-order theory is complete.

Shortness is harder to justify. One can ask:

Question 20.2. Let K be a fully good AEC. Is K short?

If the answer is positive, we believe the proof to be nontrivial. We suspect however that the shortness hypothesis of our main theorem can be weakened to a condition that easily holds in all fully good classes. In fact, we propose the following:

Definition 20.3. An AEC K is *diagonally* $(<\kappa)$*-tame* if for any $\kappa' \geq \kappa$, K is $(<\kappa')$-tame for types of length less than κ'. K is *diagonally* κ*-tame* if it is diagonally $(<\kappa^+)$-tame. K is *diagonally tame* if it is weakly $(<\kappa)$-short for some κ.

Observe the following:

Proposition 20.4.

(1) If K is fully $(<\kappa)$-tame and short, then K is diagonally $(<\kappa)$-tame.
(2) If K is diagonally $(<\kappa)$-tame, then K is $(<\kappa)$-tame.
(3) If $\mathfrak{i}$ is an almost good $(<\infty, \geq \lambda)$-independence relation, then $K_{\mathfrak{i}}$ is diagonally λ-tame.

Proof. Should be clear. □

Note that the equivalence between superstability and strong superstability (Theorem 14.7) was proven under the assumption of weak shortness only. Thus we suspect the answer to the following should be positive:

Question 20.5. In Theorem 20.1, can "fully $(<\kappa)$-tame and short" be replaced by "diagonally $(<\kappa)$-tame?

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