Stability theory for concrete categories

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Several ways to think about it:

- The study of universes with “good Ramsey theory”.
- A generalized theory of field extensions.
- Existence of an axiomatic notion of “being independent”, generalizing linear and algebraic independence.
- *(Not in this talk)* Cofibrant generation in abstract homotopy theory (“morphisms being generated by a small set”).

A puzzle

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The notation is due to Erdős and Rado. It means: for any set X with at least n elements and any coloring F of the unordered pairs from X in two colors, there exists $H \subseteq X$ with $|H| = k$ so that F is constant on the pairs from H (we call H a *homogeneous set* for F).

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If $k = 3$, $n = 6$ suffices. If $k = 5$, the optimal value of n is not known.

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Warning: The theorem does *not* say that $|X| = |H|$: it does *not* rule out an uncountable party with where all friends/strangers groups (= homogeneous sets) are countable.

Ramsey's dream

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This is wrong for most cardinals λ.

Counterexamples to Ramsey's dream

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Take $F(\{x, y\}) =$ some rational between x and y. A set H homogeneous for F cannot contain three elements! $\square$

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In the reals, a countable set allows one to distinguish uncountably-many points. There are however many structures where this is not the case.

Ramsey's dream in the complex field

Proposition

If F is a coloring of the unordered pairs of complex numbers in two colors *such that* $F(\{f(x), f(y)\}) = F(\{x, y\})$ *for any field automorphism* f *of* $\mathbb{C}$, then F has a homogeneous set of cardinality $|\mathbb{C}|$.

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Problem: Build a general framework to study this phenomenon.

Types

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Definition

Given a concrete category **K** with amalgamation and an object A of **K**, a *type over* A is just a pair $(x, A \xrightarrow{f} B)$, with $x \in B$. Two types $(x, A \xrightarrow{f} B)$, $(y, A \xrightarrow{g} C)$ are considered *the same* if there exists maps h_1, h_2 so that $h_1(x) = h_2(y)$ and the following diagram commutes:

$$\begin{tikzcd} B \arrow[r, dotted, "h_1"] & D \\ A \arrow[u, "f"] \arrow[r, "g"] & C \arrow[u, dotted, "h_2"'] \end{tikzcd}$$

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In the category of linear orders, there are $|\mathbb{R}|$ types over $\mathbb{Q}$. In general, types correspond to Dedekind cuts.

In the category of graphs with induced subgraph embeddings, there are at least $2^{|V(G)|}$ types over any graph G.

Definition (Stability)

A concrete category **K** is *stable in* λ if there are at most λ-many types over any object of cardinality λ. *Stable* means stable in an unbounded class of cardinals.

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- (Eklof 1971, Mazari-Armida) The category of R-modules with embeddings is always stable, and stable in all cardinals if and only if R is Noetherian.
- (Kucera and Mazari-Armida) The category of R-modules with pure embeddings is always stable, and stable in all cardinals if and only if R is pure-semisimple.

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  [Why? Consider c_0 (bounded complex-valued sequences with sup norm). Let $f_n := \sum_{i \leq n} e_i$. Then $\|e_m + f_n\| = 2$ if and only if $m \leq n$. Now make up a similar example where things are indexed by rationals.]

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- The category of Hilbert spaces with linear isometries is stable.

Stability and order

Theorem (V. 2016, Boney)

A tame AEC $\mathbf{K}$ with amalgamation is stable if and only if it does not "code an order": any faithful functor $\mathbf{Lin} \xrightarrow{F} \mathbf{K}$ factors through the forgetful functor.

$$\begin{tikzcd}
\mathbf{Lin} \arrow[r, "F"] \arrow[d, "U"] & \mathbf{K} \\
\mathbf{Set} \arrow[ur, dotted] &
\end{tikzcd}$$

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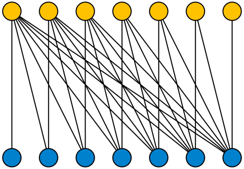
It is given by a half graph: for any linear ordering L, consider the bipartite graph on $L \sqcup L$ where we put an edge from i to j if only if $i \leq j$ (the picture below is for $L = \{1, 2, 3, 4, 5, 6, 7\}$):



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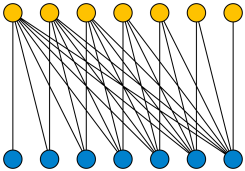

Graphs omitting half graphs are studied in finite combinatorics too (Malliaris-Shelah, *Regularity lemmas for stable graphs*. TAMS 2014).

Ramsey's dream in stable AECs

Theorem (V.)

If **K** is an abstract elementary class with amalgamation and **K** is stable in λ, then:

$$\lambda^+ \xrightarrow{\mathbf{K}} \left(\lambda^+\right)_\lambda$$

Here λ^+ is the cardinal right after λ.

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Here λ^+ is the cardinal right after λ.

The partition notation means that given objects $A \to B$ in **K** with $|A| = \lambda$, $|B| = \lambda^+$, if F is a coloring of pairs from B in λ-many colors so that any two pairs with the same type over A have the same color, then we can find a homogeneous set for F of cardinality λ^+.

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Definition (Shelah, late 1970s)

An abstract elementary class (AEC) is a concrete category **K** satisfying the following conditions:

- All morphisms are concrete monomorphisms (injections).
- **K** has concrete directed colimits (also known as direct limits – basically closure under unions of increasing chains).
- (Smallness condition) Every object is a directed colimit of a fixed set of "small" subobjects.

Examples of abstract elementary classes

All the categories mentioned before are AECs, except Banach and Hilbert spaces (under any concrete functor! Lieberman-Rosický-V., preprint).

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If T is any list of first-order axioms (things like $\forall x \exists y . xy = 1$), then the category of models (with morphisms the functions preserving all formulas) is an AEC.

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One goal of the research presented here is **to develop a general framework for the parts of model theory that are "category-theoretic".**

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The only known way to prove such statements is via stability theory.

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Theorem (V. 2019)

Assuming the GCH, Shelah's eventual categoricity conjecture is true for AECs with amalgamation. In this case one can list all possibilities for the class of cardinals in which the category has a unique object.

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Fix distinct elements $(a_i)_{i<\lambda^+}$. We will find a subset $I \subseteq \lambda^+$, $|I| = \lambda^+$ such that for any $i_0 < \ldots < i_{n-1}$, $j_0 < \ldots < j_{n-1}$, $a_{i_0} \ldots a_{i_{n-1}}$ and $a_{j_0} \ldots a_{j_{n-1}}$ have the same type.

Further fix a chain $M_0 \to M_1 \to M_2 \to \dots$ of length λ^+ such that for each $i < \lambda^+$:

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- M_i has cardinality λ.
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- M_{i+1} realizes all types over M_i (and more).
- If i is a limit ordinal, $M_i = \bigcup_{j<i} M_j$.

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Definition

We say $\mathbf{tp}(a_i/M_i)$ *splits over* M_{i_0}, $i_0 < i$, if there is an automorphism f of $\mathfrak{C}$ such that:

- f fixes M_{i_0} pointwise.
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Intuitively, a type does *not* split over a base if it is "determined" by the restriction to that base. **Near-examples:** $\mathbf{tp}(\sqrt{3}/\mathbb{Q}(\sqrt{2}))$ does not split over $\mathbb{Q}$, but $\mathbf{tp}(1-\sqrt{2}/\mathbb{Q}(\sqrt{2}))$ splits over $\mathbb{Q}$.

Lemma

If $(\delta_k)_{k<\lambda}$ is a chain of ordinals below λ^+ and p is a type over $M_{\sup_{k<\lambda} \delta_k}$, then there is $k < \lambda$ such that $p \upharpoonright M_{\delta_{k+1}}$ does not split over M_{δ_k}.

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Proof idea.

If not, $p \restriction M_{\delta_{k+1}}$ splits over M_{δ_k} for all k. Let $\mu \leq \lambda$ be least such that $2^\mu > \lambda$. Copy these failures of splitting into a binary tree of height μ. We get 2^μ distinct types over a base of size $2^{<\mu} = \lambda$, contradicting stability. $\square$

Lemma

There exists $\alpha < \beta$ and a type p over M_β such that:

- p does not split over M_α.
- For any $\beta' \geq \beta$, there is $\gamma = \gamma_{\beta'} > \beta'$ such that $\mathbf{tp}(a_\gamma / M_{\beta'})$ extends p and does not split over M_α.

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After proving this lemma, we can define inductively $\beta_0 := \beta$, and $\beta_i := \gamma_{\left(\sup_{j<i} \beta_j\right)}$, and one can show $(a_{\beta_i})_{i<\lambda^+}$ is the desired homogeneous set.

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Then for each β, for all types p over M_β, for all $\alpha < \beta$, there exists $\beta' = \beta'_{\alpha,\beta,p} > \beta$ such that for all $\gamma > \beta'$, either $\mathbf{tp}(a_\gamma / M_{\beta'})$ does not extend p or splits over M_α.

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After proving this lemma, we can define inductively $\beta_0 := \beta$, and $\beta_i := \gamma_{\left(\sup_{j<i} \beta_j\right)}$, and one can show $(a_{\beta_i})_{i<\lambda^+}$ is the desired homogeneous set.

How do we prove the lemma? By contradiction! Suppose it fails.

Then for each β, for all types p over M_β, for all $\alpha < \beta$, there exists $\beta' = \beta'_{\alpha,\beta,p} > \beta$ such that for all $\gamma > \beta'$, either $\mathbf{tp}(a_\gamma / M_{\beta'})$ does not extend p or splits over M_α.

Define inductively $\delta_0 := 0$, $\delta_i := \sup_{j<i} \sup_{\alpha<\delta_j, p \text{ type over } M_{\delta_j}} \beta'_{\alpha,\delta_j,p}$. Apply the previous lemma to $(\delta_i)_{i<\lambda}$ and get a contradiction.

Thank you!

Some references:

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