SET-THEORETIC ASPECTS OF ACCESSIBLE CATEGORIES

MICHAEL LIEBERMAN, JIŘÍ ROSICKÝ, AND SEBASTIEN VASEY

Abstract. An accessible category is, roughly, a category with all sufficiently directed colimits, in which every object can be resolved as a directed system of "small" subobjects. Such categories admit a purely category-theoretic replacement for cardinality: the internal size. Generalizing results and methods from [LRVa], we examine set-theoretic problems related to internal sizes and prove several Löwenheim-Skolem theorems for accessible categories. For example, assuming the singular cardinal hypothesis, we show that a large accessible category has an object in all internal sizes of high-enough cofinality. We also introduce the notion of a *filtrable* accessible category—one in which any object can be represented as the colimit of a *chain* of strictly smaller objects—and examine the conditions under which an accessible category is filtrable.

Contents

Date: February 18, 2019
AMS 2010 Subject Classification: Primary 18C35. Secondary: 03C45, 03C48, 03C52, 03C55, 03C75, 03E05.
Key words and phrases. internal size, presentability rank, existence spectrum, accessibility spectrum, filtrations, singular cardinal hypothesis.
The second author is supported by the Grant agency of the Czech republic under the grant 19-00902S.

1. Introduction

Recent years have seen a burst of research activity investigating connections between accessible categories and abstract model theory and, increasingly, considering the way that set-theoretic questions subtly influence each area. Abstract model theory, which has always had the aim of generalizing—in a uniform way—fragments of the rich classification theory of first order logic to encompass the broader nonelementary classes of structures that abound in mathematics proper, is perhaps most closely identified with abstract elementary classes (AECs, [She87]), but also encompasses metric AECs (mAECs, [HH09]), compact abstract theories (cats, [BY05]), and a host of other proposed frameworks. While accessible categories appear in many areas that model theory fears to tread—homotopy theory, for example—they are, fundamentally, generalized categories of models, and the ambition to recover a portion of classification theory in this context has been present since the very beginning, [MP89, p. 6]. That these fields are connected has been evident for some time—the first recognition that AECs are special accessible categories came independently in [BR12] and [Lie11]—but it is only recently that a precise middle-ground has been identified: the μ-AECs of [BGL$^+$16].

While we recall the precise definition of μ-AEC below, we note that they are a natural generalization of AECs in which the ambient language is allowed to be μ-ary, one assumes closure only under μ-directed unions rather than unions of arbitrary chains, and the Löwenheim-Skolem-Tarski property is weakened accordingly. The motivations for this definition were largely model-theoretic—in a typical AEC, for example, the subclass of μ-saturated models is not an AEC, but does form a μ-AEC—but it turns out, remarkably, that μ-AECs are, up to equivalence, precisely the accessible categories all of whose morphisms are monomorphisms (Fact 2.12). This provides an immediate link between model- and category-theoretic analyses of problems in classification theory, a middle ground in which the tools of each discipline can be brought to bear (and, moreover, this forms the basis of a broader collection of correspondences between μ-AECs with additional properties—universality, admitting intersections—and accessible categories with added structure—locally multipresentable, locally polypresentable [LRV19b]).

Among other things, this link forces a careful consideration of how one should measure the size of an object: in μ-AECs, we can speak of the cardinality of the underlying set, but we also have a purely category-theoretic notion of *internal size*, which is defined—and more or less well-behaved—in any accessible category (see Definition 2.3). This is derived in straightforward fashion from the *presentability rank* of an object M, namely the least regular cardinal λ (if it exists) such that any morphism sending M into the colimit of a λ-directed system factors through a component of the system. In most cases, the presentability rank is a successor, and the internal size is then defined to be the predecessor of the presentability rank.

The latter notion generalizes, e.g. cardinality in sets (and more generally in AECs), density character in complete metric spaces, cardinality of orthonormal bases in Hilbert spaces, and minimal cardinality of a generator in classes of algebras (Example 2.4). In a sense, one upshot of [LRVa] is that internal size is the more suitable notion for classification theory, not least because eventual categoricity in power fails miserably, while eventual categoricity in internal size is still very much open. A related question is that of *LS-accessibility*: in an accessibly category $\mathcal{K}$, is it the case that there is an object of internal size λ for every sufficiently large λ? Under what ambient set-theoretic assumptions, or concrete category-theoretic assumptions on $\mathcal{K}$, does this hold? Broadly, approximations to LS-accessibility can be thought of as replacements for the Löwenheim-Skolem theorem in accessible categories. Notice that the analogous statement for cardinality fails miserably: for example, there are no Hilbert spaces whose cardinality has countable cofinality.

One broad aim of the present paper is to relate classical properties of a category (phrased in terms of limits and colimits) to the good behavior of internal sizes in this category. To properly frame these classical properties, we first recall the definition of an accessible category. For λ a regular cardinal, a category is λ*-accessible* if it has λ-directed colimits, has only a set (up to isomorphism) of λ-presentable objects (i.e. objects with presentability rank at most λ), and every object can be written as a λ-directed colimit of λ-presentable objects. A category is *accessible* if it is λ-accessible for some λ. Note that λ-accessible does not always imply λ'-accessible for $\lambda' > \lambda$ (see also Fact 2.9). If a given category has this property (i.e. it is accessible on a tail of regular cardinal), then we call it *well accessible*. In general, the class of cardinals λ such that a given category is λ-accessible (the *accessibility spectrum*) is a key measure of the complexity of the category. For example, accessible categories with directed colimits [BR12, 4.1] or μ-AECs with intersections [LRVa, 5.4] are both known to be well accessible while general μ-AECs need not be. In the present paper, we attempt to systematically relate the accessibility spectrum to the behavior of internal sizes. For example:

- We prove that in any well accessible category, high-enough presentability ranks have to be successors (Corollary 5.4). This holds more generally of categories where the accessibility spectrum is unbounded below weakly inaccessibles. In particular, we recover the known results that any accessible category with directed colimits [BR12, 4.2], any μ-AEC with intersections [LRVa, 5.5(1)], and—assuming the singular cardinal hypothesis (SCH)—any accessible category [LRVa, 3.11], has high-enough presentability ranks successor.
- We prove, assuming SCH, that in large accessible categories with all morphisms monos, for all high-enough cardinals λ, λ^+-accessibility implies existence of an object of internal size λ (Corollary 6.11). In this sense, the accessibility spectrum is contained in the existence spectrum. In particular, well accessible categories with all morphisms monos are LS-accessible.
- Assuming SCH, *any* large accessible category has objects of all internal sizes with high-enough cofinality. In particular, it is weakly LS-accessible (i.e. has objects of all high-enough *regular* internal sizes). This is Theorem 7.12.

Regarding the SCH assumption, we point out that we use a weaker version ("eventual SCH", see Definition 2.1(5)) which follows from the existence of a strongly compact cardinal [Jec03, 20.8]. Thus our conclusions follow from this large cardinal axiom. In reality, we work primarily in ZFC, obtain some local results depending on cardinal arithmetic, and then apply SCH to simplify the statements. Sometimes weaker assumptions than SCH suffice, but we do not yet know whether the conclusions above hold in ZFC itself. Unsurprisingly, dealing with successors of *regular* cardinals is often easier, and can sometimes be done in ZFC.

Another contribution of the present account is the following: throughout the model- and set-theoretic literature, one finds countless constructions that rely on the existence of filtrations, i.e. the fact that models can be realized as the union of a continuous increasing chain of models of strictly smaller size. In a λ-accessible category, on the other hand, one has that any object can be realized as the colimit of a (more general) λ-directed system of λ-presentable objects, but there is no guarantee that one can extract from this system a cofinal chain consisting of objects that are also small. We here introduce the notion of *well filtrable* accessible category (Definition 8.5), in which the internal size analog of this essential model-theoretic property holds, and show that certain well-behaved classes of accessible categories are well filtrable. We prove general results on existence of filtrations in arbitrary accessible categories (Corollary 8.10), but really aim to study accessible categories with directed colimits. Our main result is Theorem 9.10: assuming in addition that all morphisms are monos, such categories *are* well filtrable. This result improves on [Ros97, Lemma 1] (which established existence of filtrations only for object of regular internal sizes) and will be used in a forthcoming paper on forking independence [LRVb] (a follow-up to [LRV19a]).

The background required to read this paper is a familiarity with classical set theory (e.g. [Jec03, §1-8]) and basic category theory (along the lines of [AHS04]); we freely use results and terminology related to accessible categories, e.g. [AR94, MP89]. The notion of μ-AEC, whose definition we recall below, first appears in [BGL$^+$16]. In a sense, [LRVa] is also an essential prerequisite for this paper, but we nonetheless try to recall the most essential notions here to make the paper as self-contained as possible. We will also give the proof of a known result if it can be derived in a particularly straightforward way from results given here. The first few sections contain some basic model-theoretic results on μ-AECs that nevertheless have not appeared exactly in this form before. In particular, we reprove the presentation theorem for μ-AECs, fixing a mistake in [BGL$^+$16, §3] reported to us by Marcos Mazari-Armida. We would like to thank him again for his thorough reading of our earlier work.

2. Preliminaries

We start by recalling the necessary set-theoretic notation.

Definition 2.1. Let λ and μ be infinite cardinals with μ regular.

(1) For A a set, we write $[A]^{<\lambda}$ for the set of all subsets of A of cardinality strictly less than λ, and similarly define $[A]^{\lambda}$. For B a set, we write ${}^{B}A$ for the set of all functions from B to A, and let ${}^{<\lambda}A := \bigcup_{\alpha<\lambda} {}^{\alpha}A$.

(2) We write λ^- for the predecessor of the cardinal λ, defined as follows:

$$\lambda^- = \begin{cases} \theta & \text{if } \lambda = \theta^+ \\ \lambda & \text{if } \lambda \text{ is limit} \end{cases}$$

(3) We say that λ is *μ-closed* if $\theta^{<\mu} < \lambda$ for all $\theta < \lambda$.

(4) When we write a statement like "for all high-enough θ, ...", we mean "there exists a cardinal θ_0 such that for all $\theta \geq \theta_0$, ...".
(5) We say that SCH *holds at* λ if $\lambda^{\mathrm{cf}(\lambda)} = 2^{\mathrm{cf}(\lambda)} + \lambda^+$ (SCH stands for the *singular cardinal hypothesis* — note that the equation is always true for regular λ). We say that SCH *holds above* λ if SCH holds at θ for all cardinals $\theta \geq \lambda$. The *eventual singular cardinal hypothesis* (ESCH) is the statement "SCH holds at all high-enough θ", or more precisely "there exists θ_0 such that SCH holds above θ_0".

It is a result of Solovay (see [Jec03, 20.8]) that SCH holds above a strongly compact cardinal. Thus ESCH follows from this large cardinal axiom. We will assume ESCH in several results of the present paper.

The facts below are well-known to set theorists. We give the proofs for completeness.

Fact 2.2.

(1) If λ is a μ-closed cardinal (μ regular), then $\lambda = \lambda^{<\mu}$ if and only if λ has cofinality at least μ.
(2) If SCH holds above an infinite cardinal θ, then for every cardinal λ and every regular cardinal μ, $\lambda^{<\mu} \leq \lambda^+ + \sup_{\theta_0 < \theta} \theta_0^{<\mu}$. In particular, every cardinal strictly greater than $\sup_{\theta_0 < \theta} \theta_0^{<\mu}$ which is not the successor of a cardinal of cofinality strictly less than μ is μ-closed.

Proof.

(1) If $\lambda = \lambda^{<\mu}$, then λ has cofinality at least μ by König's theorem ($\lambda^{\mathrm{cf}(\lambda)} > \lambda$). Conversely, if λ has cofinality at least μ and is μ-closed then $\lambda^{<\mu} = \sum_{\alpha < \lambda} |{}^{<\mu}\alpha| = \lambda$.
(2) It suffices to prove the result when $\mu = \kappa^+$, for some infinite cardinal κ. We proceed by induction on λ. If $\lambda < \theta$ or $\lambda < 2^\kappa$, this is clear, so assume that $\lambda \geq \theta + 2^\kappa$. If there exists $\lambda_0 < \lambda$ such that $\lambda_0^\kappa \geq \lambda$, we can apply the induction hypothesis, so assume that $\lambda_0^\kappa < \lambda$ for all $\lambda_0 < \lambda$. By [Jec03, 5.20(iii)], either $\lambda^\kappa = \lambda$ or $\mathrm{cf}(\lambda) \leq \kappa$ and $\lambda^\kappa = \lambda^{\mathrm{cf}(\lambda)} = \lambda^+ + 2^{\mathrm{cf}(\lambda)} \leq \lambda^+ + 2^\kappa = \lambda^+$ (where the second equality is by the SCH hypothesis). In both cases, we have the desired result.

   For the "in particular" part, let $\lambda > \theta' := \sup_{\theta_0 < \theta} \theta_0^{<\mu}$ and assume that λ is not the successor of a cardinal of cofinality strictly less than μ. We prove by induction on λ that λ is μ-closed. If $\lambda = (\theta')^+$, then by assumption $\mathrm{cf}(\theta') \geq \mu$ and so $(\theta')^{<\mu} = \theta'$, hence λ is μ-closed. Assume now inductively that $\lambda > (\theta')^+$ and that the result holds below λ. If λ is a limit cardinal, then by the induction hypothesis there are cofinally-many μ-closed cardinals below it and this suffices to establish that λ is μ-closed. If λ is a successor cardinal, say $\lambda = \lambda_0^+$, then by assumption $\mathrm{cf}(\lambda_0) \geq \mu$. If λ_0 is μ-closed, we are done by the first part. If λ_0 is not μ-closed then by the induction hypothesis it must be the successor of a cardinal λ_{00} of cofinality strictly less than μ. Then by what has just been established, $\lambda_{00}^{<\mu} = \lambda_0$, so $\lambda_0^{<\mu} = \lambda_0$, as desired.

□

The ideas at the heart of the category-theory-enriched form of classification theory at work here, in [LRVa], and in [LRV19a], are the notions of *presentability rank* and *internal size*.

Definition 2.3. Let λ and μ be infinite cardinals, μ regular, and let $\mathcal{K}$ be a category.

(1) We say that a diagram $D : I \to \mathcal{K}$ is *μ-directed* if I is a μ-directed poset (that is, every subset of size strictly less than μ has an upper bound). A *μ-directed colimit* is just the colimit of a μ-directed diagram.
(2) We say that an object M in $\mathcal{K}$ is *μ-presentable* if the hom-functor
$$\mathrm{Hom}_{\mathcal{K}}(M, -) : \mathcal{K} \to \mathrm{Set}$$
preserves μ-directed colimits. Equivalently, M is μ-presentable if every morphism $f : M \to N$, with N a μ-directed colimit with cocone $\langle N_i \xrightarrow{f_i} N \mid i \in I\rangle$, the map f factors essentially uniquely through one of the f_i's.
(3) We say that M is *$(< \lambda)$-presentable* if it is θ-presentable for some regular $\theta < \lambda + \aleph_1$.
(4) The *presentability rank* of an object M in $\mathcal{K}$, denoted $\mathrm{r}_{\mathcal{K}}(M)$, is the smallest μ such that M is μ-presentable. We sometimes drop $\mathcal{K}$ from the notation if it is clear from context.
(5) The *internal size* of M in $\mathcal{K}$ is defined to be $|M|_{\mathcal{K}} = \mathrm{r}_{\mathcal{K}}(M)^-$. Again, we may drop $\mathcal{K}$ from the notation if it is clear from context.

Example 2.4. We recall that internal size corresponds to the natural notion of size in familiar categories:

- In the category of sets, the internal size of any infinite set is precisely its cardinality. In an AEC, too, the internal size of any sufficiently big model will be its cardinality (see [Lie11, 4.3] or Fact 2.13 here).
- In the category of complete metric spaces and contractions, the internal size of any infinite space is its density character (the minimal cardinality of a dense subset). This is true, as well, for sufficiently big models in a general metric AEC, [LR17, 3.1].
- In the category of Hilbert spaces and linear contractions, the internal size of any infinite dimensional space is the cardinality of its orthonormal basis.
- In the category of free algebras with exactly one ω-ary function, the internal size is the minimal cardinality of a generator. In fact, a similar characterization holds in any μ-AEC with a notion of generation (i.e. with intersections), see [LRVa, 5.7].

As the above examples indicate, the relationship between internal size and cardinality can be very delicate—particularly in a context as general as μ-AECs or, equivalently, accessible categories with monomorphisms (henceforth *monos*)—and seems to become tractable only under mild set- or category-theoretic assumptions. This is the substance of [LRVa], results of which we refine in the present paper.

We recall from [LRVa, 3.1, 3.3] an essential piece of terminology:

Definition 2.5. Let λ and μ be infinite cardinals, μ regular.

(1) A *$(\mu, < \lambda)$-system* in a category $\mathcal{K}$ is a μ-directed diagram consisting of $(< \lambda)$-presentable objects.

(2) We say that a $(\mu, < \lambda)$-system with colimit M is *proper* if the identity map on M does not factor through any object in the system.

The following two results on the relationship between presentability and directedness of systems are basic.

Fact 2.6. Let λ, μ, and θ be infinite cardinals, μ regular. Let $\mathcal{K}$ be a category with μ-directed colimits.

(1) [LRVa, 3.5] For λ a regular cardinal, the colimit of a (μ, λ)-system with θ objects is always $(\theta^+ + \lambda)$-presentable. In fact (for λ not necessarily regular), if $\mathrm{cf}(\lambda) > \theta$ and λ is not the successor of a singular cardinal, the colimit of a $(\mu, < \lambda)$-system with θ objects is always $(< (\theta^{++} + \lambda))$-presentable.
(2) [LRVa, 3.4] Let M be the colimit of a $(\mu, < \lambda)$-system.
   (a) If M is μ-presentable, then the system is not proper.
   (b) If the system is not proper, then M is $(< \lambda)$-presentable.

We hereby obtain a criterion for the existence of objects whose presentability rank is the successor of a regular cardinal (this is already implicit in the proof of [LRVa, 3.12]):

Corollary 2.7. Let μ be a regular cardinal. In a category with μ-directed colimits, the colimit of a proper (μ, μ^+)-system containing at most μ objects has presentability rank μ^+.

Proof. Let M be this colimit. By Fact 2.6(1), M is μ^+-presentable. By Fact 2.6(2), M is not μ-presentable. $\square$

The notion of a $(\mu, < \lambda)$-system also allows a more parameterized and compact rephrasing of the definition of an accessible category:

Definition 2.8. Let $\mathcal{K}$ be a category and λ and μ be infinite cardinals, μ regular.

(1) [LRVa, 3.6] We say that $\mathcal{K}$ is $(\mu, < \lambda)$-accessible if it has the following properties:
   (a) $\mathcal{K}$ has μ-directed colimits.
   (b) $\mathcal{K}$ contains a set of $(< \lambda)$-presentable objects, up to isomorphism.
   (c) Any object in $\mathcal{K}$ is the colimit of a $(\mu, < \lambda)$-system.
(2) We say that $\mathcal{K}$ is (μ, λ)-*accessible* if λ is regular and $\mathcal{K}$ is $(\mu, < \lambda^+)$-accessible. We say that $\mathcal{K}$ is μ-*accessible* if it is (μ, μ)-accessible (this corresponds to the usual definition from, e.g. [AR94, MP89]). We sometimes say *finitely accessible* instead of $\aleph_0$-accessible.
(3) [BR12, 2.1] We say that $\mathcal{K}$ is *well μ-accessible* if it is θ-accessible for each regular cardinal $\theta \geq \mu$. We say that $\mathcal{K}$ is *well accessible* if it is well μ_0-accessible for some regular cardinal μ_0.

We will use the following result, allowing us to change the index of accessibility of a category (see [MP89, 2.3.10] or [LRVa, 3.8]):

Fact 2.9. Let $\mathcal{K}$ be a $(\mu, < \lambda)$-accessible category. If θ is a μ-closed regular cardinal, then $\mathcal{K}$ is $(\theta, < (\lambda + \theta^+))$-accessible.

The following two definitions describe good behavior of the existence spectrum of an accessible category (LS-accessibility appears in [BR12, 2.4], weak LS-accessibility is introduced in [LRVa, A.1]):

Definition 2.10. An accessible category $\mathcal{K}$ is *LS-accessible* if it has objects of all high-enough successor presentability ranks. We say that $\mathcal{K}$ is *weakly LS-accessible* if it has objects of all high-enough presentability ranks that are successors of *regular* cardinals.

Similar to an accessible category, a μ-AEC is an abstract class of structures in which any model can be obtained by sufficiently highly directed colimits of small objects:

Definition 2.11 ([BGL$^+$16, §2])**.** Let μ be a regular cardinal.

(1) A *(μ-ary) abstract class* is a pair $\mathbf{K} = (K, \leq_{\mathbf{K}})$ such that K is a class of structures in a fixed μ-ary vocabulary $\tau = \tau(\mathbf{K})$, and $\leq_{\mathbf{K}}$ is a partial order on K that respects isomorphisms and extends the τ-substructure relation. For $M \in \mathbf{K}$ we write UM for the universe of M.
(2) An abstract class $\mathbf{K}$ is a *μ-abstract elementary class* (or *μ-AEC* for short) if it satisfies the following three axioms:
   (a) Coherence: for any $M_0, M_1, M_2 \in \mathbf{K}$, if $M_0 \subseteq M_1 \leq_{\mathbf{K}} M_2$ and $M_0 \leq_{\mathbf{K}} M_2$, then $M_0 \leq_{\mathbf{K}} M_1$.
   (b) Chain axioms: if $\langle M_i : i \in I \rangle$ is a μ-directed system in $\mathbf{K}$, then:
      (i) $M := \bigcup_{i \in I} M_i$ is in $\mathbf{K}$.
      (ii) $M_i \leq_{\mathbf{K}} M$ for all $i \in I$.
      (iii) If $M_i \leq_{\mathbf{K}} N$ for all $i \in I$, then $M \leq_{\mathbf{K}} N$.
   (c) Löwenheim-Skolem-Tarski (LST) axiom: there exists a cardinal $\lambda = \lambda^{<\mu} \geq |\tau(\mathbf{K})| + \mu$ such that for any $M \in \mathbf{K}$ and any $A \subseteq UM$, there exists $M_0 \in \mathbf{K}$ with $M_0 \leq_{\mathbf{K}} M$, $A \subseteq UM_0$, and $|UM_0| \leq |A|^{<\mu} + \lambda$. We write $\mathrm{LS}(\mathbf{K})$ for the least such λ.

   When $\mu = \aleph_0$, we omit it and call $\mathbf{K}$ an *abstract elementary class* (AEC for short).

One can see any μ-AEC $\mathbf{K}$ (or, indeed, any μ-ary abstract class) as a category in a natural way: a morphism between models M and N in $\mathbf{K}$ is a map $f : M \to N$ which induces an isomorphism from M onto $f[M]$, and such that $f[M] \leq_{\mathbf{K}} N$. We abuse notation slightly: we will still use boldface when referring to this category, i.e. we denote it by $\mathbf{K}$ and not $\mathcal{K}$, to emphasize the concreteness of the category.

As mentioned in the introduction, these classes are an ideal locus of interaction between abstract model theory and accessible categories:

Fact 2.12 ([BGL$^+$16, §4])**.** Any μ-AEC $\mathbf{K}$ with $\mathrm{LS}(\mathbf{K}) = \lambda$ is a λ^+-accessible category with μ-directed colimits and all morphisms monos. Conversely, any μ-accessible category $\mathcal{K}$ with all morphisms monos is equivalent (as a category) to a μ-AEC $\mathbf{K}$ with $\mathrm{LS}(\mathbf{K}) \leq \max(\mu, \nu)^{<\mu}$, where ν is the size of the full subcategory of $\mathcal{K}$ on the set of (representatives) of μ-presentable objects.

We finish with two facts on internal sizes in μ-AECs. The first describes the relationship between presentability and cardinality:

Fact 2.13. Let $\mathbf{K}$ be a μ-AEC and let $M \in \mathbf{K}$. Then:

(1) [LRVa, 4.5] $r_{\mathbf{K}}(M) \leq |UM|^+ + \mu$.
(2) If $\lambda > \mathrm{LS}(\mathbf{K})$ is a μ-closed cardinal such that M is $(< \lambda^+)$-presentable, then $|UM| < \lambda$.

Proof of (2). If λ is singular, we can replace λ by a regular $\lambda_0 \in [\mathrm{LS}(\mathbf{K})^+, \lambda)$ such that M is λ_0-presentable and λ_0 is μ-closed, so without loss of generality, λ is regular. Now apply [LRVa, 4.8, 4.9]. □

The second gives a sufficient condition for existence of objects of presentability rank the successor of a regular cardinal. The proof is very short given what has already been said, so we give it.

Fact 2.14 ([LRVa, 3.12])**.** Let λ and μ be infinite cardinals with μ regular. If $\mathcal{K}$ is a $(\mu, < \lambda)$-accessible category with all morphisms monos and $\mathcal{K}$ has an object that is not $(< \lambda)$-presentable, then $\mathcal{K}$ has a proper $(\mu, < \lambda)$-system with μ objects. In particular, if in addition $\lambda \leq \mu^{++}$, then $\mathcal{K}$ has an object of presentability rank μ^+.

Proof. Let N be an object of $\mathcal{K}$ that is not $(< \lambda)$-presentable. Since $\mathcal{K}$ is $(\mu, < \lambda)$-accessible, N is the colimit of a $(\mu, < \lambda)$-system $\langle N_i : i \in I \rangle$. Since N is not $(< \lambda)$-presentable, the system is proper (Fact 2.6(2)). Since I is μ-directed and all morphisms of $\mathcal{K}$ are monos, we can pick a strictly increasing chain $\langle i_k : k < \mu \rangle$ inside I such that $\langle M_{i_k} : k < \mu \rangle$ is still proper. This then gives the desired proper $(\mu, < \lambda)$-system with μ objects. The "in particular" part follows from Corollary 2.7. □

3. Directed systems and concrete posets

As observed in [BGL$^+$16, 4.1], if $\mathbf{K}$ is a μ-AEC then any $M \in \mathbf{K}$ is the μ-directed union of all its $\mathbf{K}$-substructures of cardinality at most $\mathrm{LS}(\mathbf{K})$. In an AEC (i.e. when $\mu = \aleph_0$), it is well known that furthermore one can write $M = \bigcup_{s \in [UM]^{<\mu}} M_s$, where each M_s has cardinality at most $\mathrm{LS}(\mathbf{K})$ and $s \subseteq UM_s$ for all s. In the proof of [BGL$^+$16, 3.2], it was asserted without proof that the corresponding statement was also true when $\mu > \aleph_0$. This was a key ingredient of the proof of the presentation theorem there. It was pointed out to us (by Marcos Mazari-Armida) that the proof for AECs does *not* generalize to μ-AECs: since we cannot take unions, there are problems at limit steps. Thus we in fact do not know whether the statement is still true for μ-AECs. In this section, we prove a weakening and in the next two sections reprove the presentation theorem and related axiomatizability results.

We will work in a more general setup than μ-AECs:

Definition 3.1. A *concrete poset* is a partially ordered set $\mathbb{P}$ together with a function $U : \mathbb{P} \to \mathrm{Set}$ such that whenever $M, N \in \mathbb{P}$, $M \leq N$ implies $UM \subseteq UN$.

Definition 3.2. For a concrete poset $(\mathbb{P}, U)$ and an infinite cardinal θ, an object M of $\mathbb{P}$ is *θ-closed* if whenever $A \subseteq UM$ is such that $|A| < \theta$, there exists $M_0 \leq M$ such that $|UM_0| < \theta$ and $A \subseteq UM_0$.

The following lemma asserts that, while we may not be able to resolve an object M of a sufficiently closed concrete poset by a system indexed by $[UM]^{<\theta}$, we can at least get a system indexed by a *cofinal* subset of $[UM]^{<\theta}$:

Lemma 3.3. Let $(\mathbb{P}, U)$ be a concrete poset, let θ be an infinite cardinal, and let $M \in \mathbb{P}$ be θ-closed. Let $\mathcal{F} \subseteq [UM]^{<\theta}$ be such that for any $s \in \mathcal{F}$, $|\mathcal{P}(s) \cap \mathcal{F}| < \mathrm{cf}(\theta)$. Then there exists $\mathcal{F}_0 \subseteq \mathcal{F}$ and $\langle M_s : s \in \mathcal{F}_0 \rangle$ such that $\mathcal{F}_0$ is cofinal in $\mathcal{F}$ and for any $s, t \in \mathcal{F}_0$:

(1) $M_s \leq M$.
(2) $|UM_s| < \theta$.
(3) $s \subseteq UM_s$.
(4) $s \subseteq t$ implies $UM_s \subseteq UM_t$.

Proof. For a (not necessarily cofinal) family $\mathcal{F}_0 \subseteq \mathcal{F}$, let us say that a sequence $\langle M_s : s \in \mathcal{F}_0 \rangle$ is *good* if it satisfies conditions (1) to (4) above. Order the set of good sequences by extension. It is clear that any chain of good sequences has an upper bound (given by their union). Moreover, the empty sequence is good. Thus by Zorn's lemma, there is a maximal good sequence $\langle M_s : s \in \mathcal{F}_0 \rangle$. We claim that $\mathcal{F}_0$ is cofinal in $\mathcal{F}$. If not, let $t \in \mathcal{F}$ be such that there is no $s \in \mathcal{F}_0$ with $t \subseteq s$. Let $A := t \cup \bigcup_{s \in \mathcal{F}_0 \cap \mathcal{P}(t)} UM_s$. Since $\mathrm{cf}(\theta) > |\mathcal{F}_0 \cap \mathcal{P}(t)|$ and $|UM_s| < \theta$ for all $s \in \mathcal{F}_0$, we must have that $|A| < \theta$. Since M is θ-closed, there exists $M_t \leq M$ with $A \subseteq UM_t$ and $|UM_t| < \theta$. We claim that $\langle M_s : s \in \mathcal{F}_0 \rangle \frown M_t$ is good, and this will contradict the maximality of $\langle M_s : s \in \mathcal{F}_0 \rangle$. Conditions (1) to (3) above are straightforward to check from the construction of M_t. To check (4), pick $s_0 \subseteq s_1$ in $\mathcal{F}_0 \cup \{t\}$. We check that $UM_{s_0} \subseteq UM_{s_1}$. If $s_0, s_1 \in \mathcal{F}_0$, we know by assumption that this holds. If $s_0 = t$, then because t witnessed that $\mathcal{F}_0$ was not cofinal in $\mathcal{F}$, we know that $s_1 \notin \mathcal{F}_0$, so $s_1 = t$, hence $UM_t = UM_{s_0} \subseteq UM_{s_1} = UM_t$. Finally, if $s_1 = t$ and $s_0 \neq t$, then by construction of A, $UM_{s_0} \subseteq A \subseteq UM_t = UM_{s_1}$, as desired. $\square$

As an application of Lemma 3.3, we study what happens if we weaken the Löwenheim-Skolem-Tarski (LST) axiom of μ-AECs to the "weak LST axiom": there exists a cardinal $\lambda \geq |\tau(\mathbf{K})| + \mu$ such that $\lambda = \lambda^{<\mu}$ and every object of $\mathbf{K}$ is λ^+-closed (i.e. for all $M \in \mathbf{K}$ and all $A \subseteq UM$ *of cardinality at most* λ, there exists $M_0 \in \mathbf{K}$ with $M_0 \leq_{\mathbf{K}} M$, $|UM| \leq \lambda$, and $A \subseteq UM$). It was shown in [BGL+16, 4.6] that such a weakening still implies the original LST axiom, but the proof did not give that the minimal λ satisfying the weak LST axiom should be the Löwenheim-Skolem-Tarski number. We prove this now, in the more general setup of coherent concrete posets with μ-directed unions.

Definition 3.4. A concrete poset $(\mathbb{P}, U)$ is *coherent* if whenever $M_0 \leq M_2$, $M_1 \leq M_2$, and $UM_0 \subseteq UM_1$, we have $M_0 \leq M_1$.

Definition 3.5. For $(\mathbb{P}, U)$ a concrete poset and μ a regular cardinal, $\mathrm{LS}_{<\mu}(\mathbb{P}, U)$ is the least cardinal $\lambda \geq \mu$ such that $\lambda = \lambda^{<\mu}$ and for any $M \in \mathbb{P}$, any $A \subseteq UM$, there exists $M_0 \leq M$ with $A \subseteq UM_0$ and $|UM_0| \leq |A|^{<\mu} + \lambda$.

Remark 3.6. If $(\mathbb{P}, U)$ is a concrete poset, μ is a regular cardinal, and $\lambda = \lambda^{<\mu} \geq \mathrm{LS}_{<\mu}(\mathbb{P}, U)$, then any element of $\mathbb{P}$ is λ^+-closed.

Definition 3.7. For μ a regular cardinal, a concrete poset $(\mathbb{P}, U)$ *has μ-directed unions* if whenever $\langle M_i : i \in I\rangle$ is μ-directed in $\mathbb{P}$, then there exists $M \in \mathbb{P}$ which is a least upper bound of $\langle M_i : i \in I\rangle$ and such that $UM = \bigcup_{i\in I} UM_i$.

Lemma 3.8. Let $(\mathbb{P}, U)$ be a coherent concrete poset with μ-directed unions and let λ be an infinite cardinal. If $\lambda = \lambda^{<\mu}$ and any element of $\mathbb{P}$ is λ^+-closed, then $\mathrm{LS}_{<\mu}(\mathbb{P}, U) \leq \lambda$.

Proof. Let $M \in \mathbb{P}$ and let $A \subseteq UM$. Apply Lemma 3.3 with $\mathcal{F} := [A]^{<\mu}$ and $\theta := \lambda^+$ (note that $2^{<\mu} \leq \lambda^{<\mu} = \lambda < \theta$, so the cardinal arithmetic condition there is satisfied). Let $\langle M_s : s \in \mathcal{F}_0\rangle$ be as given there. This is a μ-directed system by coherence, so let N be the least upper bound of the system. By construction, $N \leq M$ and $A \subseteq UN$. Moreover, $|UN| \leq |\mathcal{F}_0| \cdot \lambda \leq |A|^{<\mu} + \lambda$, as needed. □

Theorem 3.9. Let $\mathbf{K}$ be an abstract class satisfying the coherence and chain axioms of μ-AECs, and let λ be an infinite cardinal. If $\lambda = \lambda^{<\mu}$ and any element of $\mathbf{K}$ is λ^+-closed, then $\mathbf{K}$ is a μ-AEC with $\mathrm{LS}(\mathbf{K}) \leq \lambda$.

Proof. Fix $M \in \mathbf{K}$ and apply Lemma 3.8 to the concrete poset $(\mathbb{P}, U)$, where $\mathbb{P}$ is the set of $\leq_{\mathbf{K}}$-substructure of M (ordered by $\leq_{\mathbf{K}}$) and U is the universe functor. □

4. Presentation theorem and axiomatizability

We reprove here the presentation theorem for μ-AECs (and more generally for accessible categories with μ-directed colimits and all morphisms monos), in the form outlined and motivated in [LRV19b, §6] (there, additional assumptions on the existence of certain directed colimits had to be inserted to make the proof work). The idea is simple: any μ-accessible category is equivalent to the category of models of an $\mathbb{L}_{\infty,\mu}$-sentence, and we can Skolemize such a sentence to obtain the desired functor. We first state the three facts we will use. Recall that $\mathrm{Mod}(\phi)$ denotes the category of models of the sentence ϕ, with morphisms all homomorphisms (i.e. maps preserving functions and relations). See also [LRV19b, §4] for a summary of what is known on axiomatizability of accessible categories.

Fact 4.1 ([MP89, 3.2.3, 3.3.5, 4.3.2])**.** Any μ-accessible category is equivalent to $\mathrm{Mod}(\phi)$, for ϕ an $\mathbb{L}_{\infty,\mu}$-formula.

Fact 4.2 (Skolemization)**.** Let $\mu \leq \lambda$. If ϕ is an $\mathbb{L}_{\lambda,\mu}$-sentence in the vocabulary τ, there exists an expansion τ^+ of τ with function symbols and an $\mathbb{L}_{\lambda,\mu}$-sentence ϕ^+ such that:

(1) ϕ^+ is universal.
(2) $|\tau^+| \leq |\tau| + \mu$.
(3) The reduct map is a surjection from $\mathrm{Mod}(\phi^+)$ onto $\mathrm{Mod}(\phi)$. Thus it is an essentially surjective faithful functor from $\mathrm{Mod}(\phi^+)$ into $\mathrm{Mod}(\phi)$ which preserves μ-directed colimits.

The result below was stated for $\mu = \aleph_0$ in [LR16, 2.5], but the proof easily generalizes.

Fact 4.3 ([LR16, 2.5])**.** If $\mathcal{K}$ is an accessible category with μ-directed colimits and all morphisms monos, there exists a μ-accessible category $\mathcal{L}$ and a faithful essentially surjective functor $F : \mathcal{L} \to \mathcal{K}$ preserving μ-directed colimits. In fact, if $\mathcal{K}$ is λ-accessible, $\mathcal{L}$ is the free completion under μ-directed colimits of the full subcategory of $\mathcal{K}$ induced by its λ-presentable objects.

Theorem 4.4 (The presentation theorem for μ-AECs)**.** If $\mathcal{K}$ is an accessible category with μ-directed colimits and all morphisms monos, then there exists a μ-universal class $\mathcal{L}$ and an essentially surjective faithful functor $F : \mathcal{L} \to \mathcal{K}$ preserving μ-directed colimits.

Proof. By Fact 4.3, there exists a μ-accessible category $\mathcal{K}^1$ and a faithful essentially surjective functor $F^1 : \mathcal{K}^1 \to \mathcal{K}$. By Fact 4.1, $\mathcal{K}^1$ is equivalent to $\mathrm{Mod}(\phi)$, for some $\mathbb{L}_{\infty,\mu}$-sentence ϕ. Since all morphisms are monos, we may assume that non-equality is part of the vocabulary of ϕ. By Fact 4.2, we can find ϕ^+ a Skolemization of ϕ. In particular (since non-equality is part of the vocabulary), $\mathrm{Mod}(\phi^+)$ will be a μ-universal class, and $F^0 : \mathrm{Mod}(\phi^+) \to \mathrm{Mod}(\phi)$ is an essentially surjective faithful functor preserving μ-directed colimits. Set $\mathcal{L} := \mathrm{Mod}(\phi^+)$, $F := F^1 \circ F^0$. $\square$

Note that if we apply Theorem 4.4 to a μ-AEC $\mathbf{K}$, the functor is *not* directly given by a reduct from some expansion of $\mathbf{K}$ (we first have to pass through several equivalence of categories). Thus Theorem 4.4 does not immediately prove (as in [Bon14] for AECs) that μ-AECs are closed under sufficiently complete ultraproducts. For this, we will prove that a certain functorial expansion of the μ-AEC is axiomatizable by an infinitary logic (*without* passing to an equivalent category):

Definition 4.5. Let $\mathbf{K}$ be a μ-AEC. The *substructure functorial expansion* of $\mathbf{K}$ is the abstract class $\mathbf{K}^+$ defined as follows:

(1) $\tau(\mathbf{K}^+) = \tau(\mathbf{K}) \cup \{P\}$, where P is an $\mathrm{LS}(\mathbf{K})$-ary predicate.
(2) $M^+ \in \mathbf{K}^+$ if and only if $M^+ \upharpoonright \tau(\mathbf{K}) \in \mathbf{K}$ and for any $\bar{a} \in {}^{\mathrm{LS}(\mathbf{K})}M^+$, $P^{M^+}(\bar{a})$ holds if and only if $\mathrm{ran}(\bar{a}) \leq_{\mathbf{K}} M^+ \upharpoonright \tau(\mathbf{K})$, where we see $\mathrm{ran}(\bar{a})$ as a $\tau(\mathbf{K})$-structure.
(3) For $M^+, N^+ \in \mathbf{K}^+$, $M^+ \leq_{\mathbf{K}^+} N^+$ if and only if $M^+ \upharpoonright \tau(\mathbf{K}) \leq_{\mathbf{K}} N^+ \upharpoonright \tau(\mathbf{K})$.

The substructure expansion is "functorial" in the sense of [Vas16, 3.1]: the reduct functor gives an isomorphism of concrete categories. The substructure functorial expansion has the property of having very simple morphisms:

Theorem 4.6. Let $\mathbf{K}$ be a μ-AEC and let $\mathbf{K}^+$ be its substructure functorial expansion. If $M^+, N^+ \in \mathbf{K}^+$ are such that $M^+ \subseteq N^+$, then $M^+ \leq_{\mathbf{K}^+} N^+$.

Proof. For $M \in \mathbf{K}$, write M^+ for the expansion of M to $\mathbf{K}^+$. Let $M, N \in \mathbf{K}$ and assume that $M^+ \subseteq N^+$. We have to see that $M \leq_{\mathbf{K}} N$. For this, it is enough to show that for any $M_0 \leq_{\mathbf{K}} M$ of cardinality at most $\mathrm{LS}(\mathbf{K})$, we also have that $M_0 \leq_{\mathbf{K}} N$ (indeed, we can then take the $\mathrm{LS}(\mathbf{K})^+$-directed union of all such M_0's). So let $M_0 \leq_{\mathbf{K}} M$ have cardinality at most $\mathrm{LS}(\mathbf{K})$: we must show that $M_0 \leq_{\mathbf{K}} N$. Let $\bar{a}$ be an enumeration of M_0. We have that $M^+ \models P[\bar{a}]$ (where P is the additional predicate in $\tau(\mathbf{K})^+$), so $N^+ \models P[\bar{a}]$ (as M^+ is a substructure of N^+). This means that $M_0 \leq_{\mathbf{K}} N$, as desired. $\square$

The substructure functorial expansion of a μ-AEC can be axiomatized (a more complication variation of this, for AECs, is due to Baldwin and Boney [BB17, 3.9]). Since the ordering is trivial by the previous result, this shows that any μ-AEC is isomorphic (as a category) to the category of models of an $\mathbb{L}_{\infty,\infty}$ sentence, where the morphisms are injective homomorphisms.

Theorem 4.7. Let $\mathbf{K}$ be a μ-AEC and let $\mathbf{K}^+$ be its substructure functorial expansion. There is an $\mathbb{L}_{\left(2^{\mathrm{LS}(\mathbf{K})}\right)^+,\mathrm{LS}(\mathbf{K})^+}$ sentence ϕ such that $\mathbf{K}^+$ is the class of models of ϕ.

Proof. First note that for each $M_0 \in \mathbf{K}_{\leq \mathrm{LS}(\mathbf{K})}$, there is a sentence $\psi_{M_0}(\bar{x})$ of $\mathbb{L}_{\mathrm{LS}(\mathbf{K})^+,\mathrm{LS}(\mathbf{K})^+}$ coding its isomorphism type, i.e. whenever $M \models \phi[\bar{a}]$, then $\bar{a}$ is an enumeration of an isomorphic copy of M_0. Similarly, whenever M_0, M_1 are in $\mathbf{K}_{\leq \mathrm{LS}(\mathbf{K})}$ with $M_0 \leq_{\mathbf{K}} M_1$, there is $\psi_{M_0,M_1}(\bar{x}, \bar{y})$ that codes that $(\bar{x}, \bar{y})$ is isomorphic to (M_0, M_1) (so in particular $\bar{x} \leq_{\mathbf{K}} \bar{y}$). Let S be a complete set of members of $\mathbf{K}_{\leq \mathrm{LS}(\mathbf{K})}$ (i.e. any other model is isomorphic to it) and let T be a complete set of pairs (M_0, M_1), with each in $\mathbf{K}_{\leq \mathrm{LS}(\mathbf{K})}$, such that $M_0 \leq_{\mathbf{K}} M_1$. Now define the following:

$$\phi_1 = \forall \bar{x} \exists \bar{y} \left(\left(\bigvee_{M_0 \in S} \psi_{M_0}(\bar{y}) \right) \wedge \bar{x} \subseteq \bar{y} \wedge P(\bar{y}) \right)$$

$$\phi_2 = \forall \bar{x} \forall \bar{y} \left((\bar{x} \subseteq \bar{y} \wedge P(\bar{x}) \wedge P(\bar{y})) \rightarrow \bigvee_{(M_0,M_1) \in T} \psi_{M_0,M_1}(\bar{x}, \bar{y}) \right)$$

$$\phi = \phi_1 \wedge \phi_2$$

Where $\bar{x} \subseteq \bar{y}$ abbreviates the obvious formula. This works. First, any $M^+ \in \mathbf{K}^+$ satisfies ϕ_1 by the LST axiom and satisfies ϕ_2 by the coherence axiom. Conversely, assume that $M^+ \models \phi$ and let $M := M^+ \upharpoonright \tau(\mathbf{K})$. Consider the set:

$$I = \{M_0 \in \mathbf{K}_{\leq \mathrm{LS}(\mathbf{K})} \mid UM_0 \subseteq UM, P^{M^+}(\bar{M_0})\}$$

Where $\bar{M_0}$ refers to some enumeration of M_0. Then by construction of ϕ, I is a μ-directed system in $\mathbf{K}$ and $\bigcup I = M$, so $M \in \mathbf{K}$. Similarly, $P^{M^+}(\bar{a})$ holds if and only if $\mathrm{ran}(\bar{a}) \leq_{\mathbf{K}} M$, so $M^+ \in \mathbf{K}^+$. $\square$

Corollary 4.8. Any μ-AEC $\mathbf{K}$ is closed under $\left(2^{\mathrm{LS}(\mathbf{K})}\right)^+$-complete ultraproducts (in the sense that the appropriate generalization of [Bon14, 4.3] holds).

Proof. By Theorems 4.6 and 4.7, Łoś' theorem, and the fact that taking reducts commutes with ultraproducts. $\square$

5. On successor presentability ranks

We start our study of the existence spectrum of an accessible category $\mathcal{K}$: the set of regular cardinals λ such that $\mathcal{K}$ has an object of presentability rank λ. The goal is to say as much as possible by just looking at the accessibility spectrum: the set of cardinals λ such that $\mathcal{K}$ is λ-accessible.

In this section, we consider the question, first systematically investigated in [BR12], of whether the presentability rank of an object always has to be a successor (or, said differently, whether there can be objects of weakly inaccessible presentability rank). Assuming the accessibility spectrum is sufficiently large, we show there are *no* objects of weakly inaccessible presentability rank, and explain how this generalizes previous results.

The following easy lemma characterizes existence in terms of the accessibility spectrum. It will also be used in the next section:

Lemma 5.1. Let λ be a regular cardinal and let $\mathcal{K}$ be a category. The following are equivalent:

(1) $\mathcal{K}$ is $(\lambda, <\lambda)$-accessible.
(2) $\mathcal{K}$ is λ-accessible and has *no* objects of presentability rank λ.

Proof. Assume $\mathcal{K}$ is $(\lambda, <\lambda)$-accessible. By definition, $\mathcal{K}$ is clearly λ-accessible. If M is a λ-presentable object of $\mathcal{K}$, then it is a λ-directed colimit of $(<\lambda)$-presentable objects, hence by Fact 2.6 must itself be $(<\lambda)$-presentable.

Conversely, if $\mathcal{K}$ is λ-accessible and has no objects of presentability rank λ, then any (λ, λ)-system must be a $(\lambda, <\lambda)$-system, hence $\mathcal{K}$ is $(\lambda, <\lambda)$-accessible. □

The following new result gives a criterion for $(\lambda, <\lambda)$-accessibility when λ is weakly inaccessible.

Theorem 5.2. If λ is weakly inaccessible and $\mathcal{K}$ is $(\mu, <\lambda)$-accessible for unboundedly-many $\mu < \lambda$, then $\mathcal{K}$ is $(\lambda, <\lambda)$-accessible.

Proof. Let S be the set of all regular cardinals $\mu < \lambda$ such that $\mathcal{K}$ is $(\mu, <\lambda)$-accessible. Let M be an object of $\mathcal{K}$. For each $\mu \in S$, fix a μ-directed system $\langle M_i^\mu : i \in I_\mu \rangle$, with maps $\langle f_{i,j}^\mu : i \leq j \in I_\mu \rangle$ whose colimit is M (with colimit maps f_i^μ, $i \in I_\mu$). Let $I := \{(i, \mu) \mid \mu \in S, i \in I_\mu\}$. Order it by $(i, \mu_1) \leq (j, \mu_2)$ if and only if $\mu_1 \leq \mu_2$, and there exists a unique map $g : M_i^{\mu_1} \to M_j^{\mu_2}$ so that $f_i^{\mu_1} = f_j^{\mu_2} g$.

Observe that $(I, \leq)$ is a partial order. Also, if we fix $\mu \in S$ and $i \in I_\mu$, then $M_i^{\mu_1}$ is $(<\lambda)$-presentable, hence μ_1-presentable for some regular $\mu_1 \in S$, $\mu \leq \mu_1$. Thus for any $\mu_2 \in S$ with $\mu_1 \leq \mu_2$, there exists $j \in I_{\mu_2}$ such that $(i, \mu) \leq (j, \mu_2)$.

The last paragraph quickly implies that I is λ-directed, and so the diagram induced by I is the desired $(\lambda, <\lambda)$-system with colimit M. □

Corollary 5.3. If λ is weakly inaccessible and $\mathcal{K}$ is μ-accessible for unboundedly-many $\mu < \lambda$, then $\mathcal{K}$ does not have an object of presentability rank λ.

Proof. By Theorem 5.2, $\mathcal{K}$ is $(\lambda, <\lambda)$-accessible. Now apply Lemma 5.1. □

We obtain that any high-enough presentability rank in a well-accessible category (recall Definition 2.8(3)) must be a successor, which improves on [BR12, 4.2] and [LRVa, 5.5]:

Corollary 5.4. If $\mathcal{K}$ is a well μ-accessible category, then the presentability rank of any object that is not μ-presentable must be a successor.

Proof. Immediate from Corollary 5.3. □

We have also recovered [LRVa, 3.11]:

Corollary 5.5. Let μ be a regular cardinal and let $\lambda > \mu$ be a weakly inaccessible cardinal. If $\mathcal{K}$ is a $(\mu, < \lambda)$-accessible category and λ is μ-closed, then $\mathcal{K}$ has no object of presentability rank λ.

In particular, assuming ESCH, high-enough presentability ranks are successors in any accessible category.

Proof. Since λ is μ-closed and limit, there are unboundedly-many regular $\theta \in [\mu, \lambda)$ that are μ-closed. By Fact 2.9, for any such θ, $\mathcal{K}$ is $(\theta, < \lambda)$-accessible. By Theorem 5.2, $\mathcal{K}$ is $(\lambda, < \lambda)$-accessible, hence by Lemma 5.1 cannot have an object of presentability rank λ. □

6. The existence spectrum of a μ-AEC

We now refine a few results of [LRVa] concerning the existence spectrum of μ-AECs, especially [LRVa, 4.13].

We aim to study proper $(\lambda, < \lambda)$-systems—in the sense of Definition 2.5—and show that under certain conditions they do *not* exist. This will give conditions under which an object of presentability rank λ *does* exist:

Lemma 6.1. Let λ be a regular cardinal and let $\mathcal{K}$ be a λ-accessible category. If $\mathcal{K}$ has an object that is not $(< \lambda)$-presentable and $\mathcal{K}$ has no proper $(\lambda, < \lambda)$-systems, then $\mathcal{K}$ has an object of presentability rank λ.

Proof. By Lemma 5.1, it suffices to show that $\mathcal{K}$ is *not* $(\lambda, < \lambda)$-accessible. Suppose for a contradiction that $\mathcal{K}$ is $(\lambda, < \lambda)$-accessible. Let M be an object that is not $(< \lambda)$-presentable. Then M is the colimit of a $(\lambda, < \lambda)$-system, which must be proper because M is not $(< \lambda)$-presentable (see Fact 2.6), a contradiction to the assumption that there are no proper $(\lambda, < \lambda)$-systems. □

To help the reader, let us consider what a $(\lambda^+, < \lambda^+)$-system should be in an AEC $\mathbf{K}$ with $\lambda > \mathrm{LS}(\mathbf{K})$. Since internal sizes correspond to cardinalities in that context (see Fact 2.13 or simply [Lie11, 4.3]), such a system must be a λ^+-directed system consisting of object of cardinality strictly less than λ. Because it is "too directed," the system cannot be proper (i.e. its colimit will just be a member of the system). We attempt here to generalize such an argument to suitable μ-AECs. We will succeed when λ is μ-closed (Theorem 6.6 — notice that this is automatic when $\mu = \aleph_0$).

We will use the following key bound on the internal size of a subobject:

Lemma 6.2. If $\mathbf{K}$ is a μ-AEC, $M \leq_{\mathbf{K}} N$ are in $\mathbf{K}$, $\lambda > \mathrm{LS}(\mathbf{K})$ is a μ-closed cardinal, and N is $(< \lambda^+)$-presentable, then M is $(< \lambda^+)$-presentable.

Proof. By Fact 2.13, $|UN| < \lambda$. Of course, $|UM| \leq |UN|$, so $|UM| < \lambda$ By Fact 2.13 again, $r_{\mathbf{K}}(M) \leq |UM|^+ + \mu$, so $r_{\mathbf{K}}(M) \leq \lambda + \mu = \lambda$, so M is $(< \lambda^+)$-presentable, as desired. $\square$

We require an additional refinement, concerning systems in which bounded subsystems have small colimits:

Definition 6.3. A system $\langle M_i : i \in I \rangle$ in a given category is *boundedly* $(< \lambda)$-*presentable* if whenever $I_0 \subseteq I$ is bounded in I, the colimit of $\langle M_i : i \in I_0 \rangle$ is $(< \lambda)$-presentable (whenever it exists).

Lemma 6.4. Let $\mathbf{K}$ be a μ-AEC and let $\lambda > \mathrm{LS}(\mathbf{K})^+$ be such that λ^- is μ-closed. Then any system in $\mathbf{K}$ consisting of $(< \lambda)$-presentable objects is boundedly $(< \lambda)$-presentable.

Proof. Let $\langle M_i : i \in I \rangle$ be a system consisting of $(< \lambda)$-presentable objects. Let $I_0 \subseteq I$ be bounded in I, say by i, and such that the colimit M_{I_0} of the resulting system exists. We have that $M_{I_0} \leq_{\mathbf{K}} M_i$. Since λ^- is μ-closed and M_i is $(< \lambda)$-presentable, we can find $\lambda_0 \in [\mathrm{LS}(\mathbf{K}), \lambda)$ regular and μ-closed such that M_i is λ_0-presentable. By Lemma 6.2, M_{I_0} is also λ_0-presentable, hence $(< \lambda)$-presentable, as desired. $\square$

Using the bound of Lemma 6.2 again, we now show that for most successor cardinals θ, there are no proper θ-directed boundedly $(< \theta)$-presentable systems:

Lemma 6.5. Let $\mathbf{K}$ be a μ-AEC. If $\lambda > \mathrm{LS}(\mathbf{K})$ is μ-closed, then there are no proper λ^+-directed boundedly $(< \lambda^+)$-presentable systems.

Proof. Assume for a contradiction that $\langle M_i : i \in I \rangle$ is such a system, with colimit M. First, if λ is regular, then using λ^+-directedness and properness we can find a chain $I_0 \subseteq I$ of type λ such that $i < j$ in I_0 implies $M_i \neq M_j$. Then $\langle M_i : i \in I_0 \rangle$ is proper, so its colimit (union) M_{I_0} is not λ-presentable (Fact 2.6). However, I_0 is bounded as I is λ^+-directed, a contradiction to the hypothesis of bounded $(< \lambda^+)$-presentability.

Assume now that λ is singular. Let $\delta := \mathrm{cf}(\lambda)$ and write $\lambda = \sup_{\alpha < \delta} \lambda_\alpha$, with $\mathrm{LS}(\mathbf{K}) < \lambda_0$ and each λ_α regular and μ-closed. Let $I_\alpha := \{i \in I \mid M_i \text{ is } \lambda_\alpha\text{-presentable}\}$. Note that $I = \bigcup_{\alpha < \delta} I_\alpha$.

We claim that there exists $\alpha < \delta$ such that I_α is cofinal in I. Suppose not, and for each $\alpha < \delta$ pick $a_\alpha \in I$ such that a_α is not bounded by any element of I_α. Since I is δ^+-directed there exists a above all the a_α's, but then $a \in I_\alpha$ for some $\alpha < \delta$, contradicting the choice of a_α. Thus there is $\alpha < \delta$ such that I_α is cofinal in I. By renaming, we can assume without loss of generality that I_0 is already cofinal in I, hence I_α is cofinal in I for all $\alpha < \delta$. Note that I_0 must itself be λ^+-directed.

Now pick $\langle i_j : j < \lambda \rangle$ an increasing sequence in I_0 such that $\langle M_{i_j} : j < \lambda \rangle$ is strictly increasing (this is possible by properness of the system). For $k < \lambda$ of cofinality at least μ, let $N_k = \bigcup_{j<k} M_{i_j}$. Note that if $\mathrm{cf}(k) \geq \lambda_\alpha$, then by Fact 2.6, N_k is

not λ_α-presentable. Fix $\alpha < \delta$ such that $\lambda_0 < \lambda_\alpha$. We then have that $N^1 := N_{\lambda_\alpha}$ is not λ_α-presentable, but $N^2 := M_{i_{\lambda_0}}$ is λ_0-presentable. Since $N^1 \leq_{\mathbf{K}} N^2$, this contradicts Lemma 6.2. □

We have arrived at the main technical result of this section.

Theorem 6.6. Let $\mathbf{K}$ be a μ-AEC. If $\lambda > \mathrm{LS}(\mathbf{K})$ is μ-closed, then there are no proper $(\lambda^+, < \lambda^+)$-systems in $\mathbf{K}$.

Proof. By Lemma 6.4 (with λ^+ in place of λ) and Lemma 6.5. □

We get that, at least for successors of high-enough μ-closed cardinals (or under SCH, see below), the presentability rank spectrum contains the accessibility spectrum.

Corollary 6.7. Let $\mathbf{K}$ be a μ-AEC. If $\lambda > \mathrm{LS}(\mathbf{K})$ is a μ-closed cardinal such that $\mathbf{K}$ is λ^+-accessible and $\mathbf{K}$ has an object of cardinality at least λ, then $\mathbf{K}$ has an object of presentability rank λ^+.

Proof. Let $M \in \mathbf{K}$ have cardinality at least λ. Using Fact 2.13 together with the assumption that λ is μ-closed, M is not $(< \lambda^+)$-presentable. By Theorem 6.6, there are no proper $(\lambda^+, < \lambda^+)$-systems in $\mathbf{K}$. By Lemma 6.1, $\mathbf{K}$ has an object of presentability rank λ^+. □

We have in particular recovered [LRVa, 4.13]. This will later be further generalized to any accessible category (Theorem 7.11).

Corollary 6.8. Let $\mathbf{K}$ be a μ-AEC and let $\lambda = \lambda^{<\mu} \geq \mathrm{LS}(\mathbf{K})$. We have that $\mathbf{K}$ has an object of presentability rank λ^+ if at least one of the following conditions hold:

(1) $\lambda > \mathrm{LS}(\mathbf{K})$, λ is μ-closed, and $\mathbf{K}$ has an object of cardinality at least λ.
(2) λ is regular and $\mathbf{K}$ has an object of cardinality at least λ^+.

Proof. Since $\lambda = \lambda^{<\mu}$, λ^+ is μ-closed, so by Fact 2.9, $\mathcal{K}$ is λ^+-accessible. Now:

(1) This follows from Corollary 6.7.
(2) Since $\mathcal{K}$ has μ-directed colimits, it is also (λ, λ^+)-accessible. Since $\mathcal{K}$ has an object of cardinality at least λ^+ and λ^+ is μ-closed, Fact 2.13 (with λ^+ in place of λ) implies that this object is not λ^+-presentable. Now apply Fact 2.14.

□

Remark 6.9. The example of well-orderings ordered by initial segment given in [LRVa, 6.2] shows that, even when $\mu = \aleph_0$, we may not have an object of rank $\mathrm{LS}(\mathbf{K})^+$ if $\mathrm{LS}(\mathbf{K})$ is singular.

Under SCH, the statements simplify and we recover [LRVa, 4.15]:

Corollary 6.10. Let $\mathbf{K}$ be a large μ-AEC. If SCH holds above $\mathrm{LS}(\mathbf{K})$, then for every $\lambda > \mathrm{LS}(\mathbf{K})$ of cofinality at least μ, $\mathbf{K}$ has an object of presentability rank λ^+. In particular, $\mathbf{K}$ is weakly LS-accessible.

Proof. By Fact 2.2, $\lambda = \lambda^{<\mu}$. If λ is regular, we may apply Corollary 6.8(2). If λ is singular, then by Fact 2.2 it is μ-closed so one may apply Corollary 6.8(1). $\square$

Still under SCH, we obtain that the (successor) accessibility spectrum is eventually contained in the existence spectrum:

Corollary 6.11. Assume ESCH. Let $\mathcal{K}$ be a large category with all morphisms monos. For all high-enough successor cardinals θ, if $\mathcal{K}$ is θ-accessible, then $\mathcal{K}$ has an object of presentability rank θ.

Proof. By Fact 2.12, $\mathcal{K}$ is equivalent to a μ-AEC $\mathbf{K}$. By replacing $\mathbf{K}$ by a tail segment if necessary, we can assume without loss of generality that SCH holds above $\text{LS}(\mathbf{K})$. Pick $\theta > \text{LS}(\mathbf{K})^+$ successor such that $\mathbf{K}$ is θ-accessible. Write $\theta = \lambda^+$. If $\text{cf}(\lambda) \geq \mu$, Corollary 6.10 gives the result, so we may assume that $\text{cf}(\lambda) < \mu$. In this case, the SCH assumption implies that λ is μ-closed so we can apply Corollary 6.7. $\square$

Recall (Definition 2.10) that a category is *LS-accessible* if it has objects of all high-enough internal sizes.

Corollary 6.12. Assuming ESCH, any large well accessible category with all morphisms monos is LS-accessible.

Note that Corollary 6.12 can be seen as a joint generalization of [LR16, 2.7] (LS-accessibility of large accessible categories with directed colimits and all morphisms monos) and [LRVa, 5.9] (LS-accessibility of large μ-AECs with intersections): in both cases, the categories in question are well accessible with all morphisms monos (see Fact 2.9, [LRVa, 5.4]). Note however that the proof of LS-accessibility of large accessible categories with directed colimits and all morphisms monos does not assume SCH.

7. The existence spectrum of an accessible category

A downside of the previous section was the assumption that all morphisms were monos. In the present section, we look at what can be said for arbitrary accessible categories. The main tool is the fact that the inclusion functor $\mathcal{K}_{mono} \to \mathcal{K}$ is (in a sense we make precise) accessible, hence plays reasonably well with internal sizes. While the notion of an accessible functor appears already in [MP89, §2.4], we give here a more parameterized definition, in the style of Definition 2.8:

Definition 7.1. Let λ and μ be infinite cardinals, with μ regular. A functor $F : \mathcal{K} \to \mathcal{L}$ is $(\mu, < \lambda)$*-accessible* if it preserves μ-directed colimits and both $\mathcal{K}$ and $\mathcal{L}$ are $(\mu, < \lambda)$-accessible. We say that F is (μ, λ)*-accessible* if λ is regular and F is $(\mu, < \lambda^+)$-accessible. We say that F is μ*-accessible* precisely when it is (μ, μ)-accessible.

Fact 7.2 ([LRV19a, 6.2])**.** If $\mathcal{K}$ is a μ-accessible category, there there exists a cardinal $\lambda \geq \mu$ such that $\mathcal{K}_{mono}$ is (μ, λ)-accessible and moreover the inclusion functor F of $\mathcal{K}_{mono}$ into $\mathcal{K}$ is (μ, λ)-accessible.

The following properties, describing the interaction of an accessible functor with presentability, were first systematically investigated in [BR12, §3]:

Definition 7.3. A functor $F : \mathcal{K} \to \mathcal{L}$ *preserves λ-presentable objects* if whenever M is λ-presentable in $\mathcal{K}$, then $F(M)$ is λ-presentable (in $\mathcal{L}$). We say that F *reflects λ-presentable objects* if M is λ-presentable in $\mathcal{K}$ whenever $F(M)$ is λ-presentable $\mathcal{L}$. We also say that F *preserves λ-ranked objects* if whenever $r_{\mathcal{K}}(M) = \lambda$, then $r_{\mathcal{L}}(FM) = \lambda$. Similarly define what it means for F to *reflect λ-ranked objects*.

Of course, there is a simple test to determine when a functor preserves rank given information as to whether it preserves and reflects presentable objects:

Lemma 7.4. Let $F : \mathcal{K} \to \mathcal{L}$ be an accessible functor and let μ be a regular cardinal. If F preserves μ-presentable objects and reflects $(< \mu)$-presentable objects, then F preserves μ-ranked objects. Similarly, if F preserves $(< \mu)$-presentable objects and reflects μ-presentable objects, then F reflects μ-ranked objects.

Proof. We prove the first statement (the proof of the second is similar). Let M be an object of $\mathcal{K}$ such that $r(M) = \mu$. Then $r(FM) \leq \mu$ because F preserves μ-presentable objects, and if $r(FM) < \mu$, then $r(M) < \mu$ because F reflects $(< \mu)$-presentable objects, contradiction. Thus $r(M) = r(FM)$. □

For a functor to *reflect* λ-presentable objects, it enough that it is sufficiently accessible and that the functor *reflects split epimorphisms* (i.e. if Ff is a split epi, then f is a split epi). This was isolated in [BR12, 3.6]. We now proceed to mine the proof of this result to extract what can be said in our more parameterized setup:

Lemma 7.5. Let λ and μ be cardinals, μ regular. Let $F : \mathcal{K} \to \mathcal{L}$ be a functor reflecting split epimorphisms and preserving μ-directed colimits. Let $\langle M_i : i \in I \rangle$ be a $(\mu, < \lambda)$-system with colimit M. If $\langle M_i : i \in I \rangle$ is proper, then $\langle FM_i : i \in I \rangle$ is proper. If in addition F preserves $(< \lambda)$-presentable objects, then $\langle FM_i : i \in I \rangle$ is a $(\mu, < \lambda)$-system.

Proof. Since F preserves μ-directed colimits, the colimit of $\langle FM_i : i \in I \rangle$ is FM. Suppose that the identity map on FM factors through some FM_i, via a map $g : FM \to FM_i$. That is, $(Ff_i)g = \mathrm{id}_{FM}$, where $f_i : M_i \to M$ is a colimit map. Then Ff_i is a split epimorphism, hence f_i is a split epimorphism, i.e. $f_i g = \mathrm{id}_M$, so $\langle M_i : i \in I \rangle$ is not proper. The last sentence is immediate from the definition. □

Fact 7.6 ([BR12, 3.6])**.** Let λ be an uncountable cardinal, and let $F : \mathcal{K} \to \mathcal{L}$ be a functor reflecting split epimorphisms. If there exists a regular cardinal $\lambda_0 < \lambda$ such that F is $(\lambda_0, < \lambda)$-accessible and for all regular cardinals $\mu \in [\lambda_0, \lambda)$, $\mathcal{K}$ is $(\mu, < \lambda)$-accessible, then F reflects $(< \lambda)$-presentable objects.

Proof. Assume that FM is $(< \lambda)$-presentable. Pick a regular cardinal $\mu \in [\lambda_0, \lambda)$ such that FM is μ-presentable. By $(\mu, < \lambda)$-accessibility, M is the colimit of a $(\mu, < \lambda)$-system $\langle M_i : i \in I \rangle$. Assume for a contradiction that M is *not* $(< \lambda)$-presentable. Then $\langle M_i : i \in I \rangle$ must be proper by Fact 2.6(2). By Lemma 7.5, $\langle FM_i : i \in I \rangle$ is proper, hence its colimit FM cannot be μ-presentable by Fact 2.6(2), a contradiction. □

In passing, we can deduce the following powerful criterion for existence of an object of regular internal size. Notice that this is a generalization of Fact 2.14 (which is the special case of the identity functor).

Theorem 7.7. Let λ be a regular cardinal and let $F : \mathcal{K} \to \mathcal{L}$ be a (λ, λ^+)-accessible functor that preserves λ^+-presentable objects and reflects isomorphisms. If all morphisms in $\mathcal{K}$ are monos and $\mathcal{K}$ has an object that is not λ^+-presentable, then $\mathcal{L}$ has an object of presentability rank λ^+.

Proof. Since F reflects isomorphisms and all morphisms in $\mathcal{K}$ are monos, F reflects split epimorphisms. By Fact 2.14, $\mathcal{K}$ has a proper (λ, λ^+)-system $\langle M_i : i \in I \rangle$ with λ-many objects. By Lemma 7.5, $\langle FM_i : i \in I \rangle$ is a proper (λ, λ^+)-system. By Corollary 2.7, the colimit of this system in $\mathcal{L}$ has presentability rank λ^+. □

To *preserve* λ-presentable objects, a cardinal arithmetic assumption on λ suffices:

Fact 7.8. Let $F : \mathcal{K} \to \mathcal{L}$ be a (μ, λ_0)-accessible functor. Let $\lambda_1 \geq \lambda_0$ be a regular cardinal such that the image of any λ_0-presentable object is λ_1-presentable. If $\lambda > \lambda_1$ is such that λ^- is μ-closed, then F preserves $(< \lambda)$-presentable objects.

Proof. Similar to the proof of [AR94, 2.19]. □

We obtain the following cardinal arithmetic test for preservation and reflection of ranks:

Lemma 7.9. Let $F : \mathcal{K} \to \mathcal{L}$ be a (μ, λ_0)-accessible functor that preserves λ_0-presentable objects and reflects split epimorphisms.

If $\theta > \lambda_0$ is the successor of a μ-closed cardinal of cofinality at least μ, then F preserves and reflects θ-ranked objects.

Proof. Write $\theta = \lambda^+$, with λ a μ-closed cardinal of cofinality at least μ. Note that $\lambda^{<\mu} = \lambda$, since it has cofinality at least μ. Thus both λ and θ are μ-closed. This implies that F preserves θ-presentable objects and preserves $(< \theta)$-presentable objects (Fact 7.8). We also have that F reflects $(< \theta)$-presentable objects and F reflects θ-presentable objects (use Facts 2.9 and 7.6). Now apply Lemma 7.4. □

Using Corollary 6.8, we obtain the following existence spectrum result if the domain of the functor is a large μ-AEC:

Lemma 7.10. Let $\mathbf{K}$ be a μ-AEC, let $\lambda_0 > \mathrm{LS}(\mathbf{K})$ be regular, and let $F : \mathbf{K} \to \mathcal{L}$ be a (μ, λ_0)-accessible functor that preserves λ_0-presentable objects and reflects isomorphisms. Let $\lambda \geq \lambda_0$ be a μ-closed cardinal of cofinality at least μ. If $\mathbf{K}$ has an object of cardinality at least λ, then $\mathcal{L}$ has an object of presentability rank λ^+.

Proof. Since all morphisms of $\mathbf{K}$ are monos, F reflects split epimorphisms. Since λ is μ-closed and has cofinality at least μ, $\lambda = \lambda^{<\mu}$. By Corollary 6.8, $\mathbf{K}$ has an object M of presentability rank λ^+. By Lemma 7.9 (where θ there stand for λ^+ here), F preserves λ^+-ranked objects, so FM has presentability rank λ^+. □

Putting all the results together, we obtain an existence spectrum result for any large accessible category. This extends Corollary 6.8.

Theorem 7.11. Let $\mathcal{K}$ be a large μ-accessible category.

(1) For every high-enough regular λ such that $\lambda = \lambda^{<\mu}$, $\mathcal{K}$ has an object of presentability rank λ^+.
(2) There exists a regular cardinal μ' such that for every high-enough μ'-closed cardinal λ of cofinality at least μ', $\mathcal{K}$ has an object of presentability rank λ^+.

Proof.

(1) By Fact 7.2, there exists a cardinal λ_0 such that the inclusion functor F of $\mathcal{K}_{mono}$ into $\mathcal{K}$ is (μ, λ_0)-accessible. Of course, F also reflects isomorphisms. Let $\lambda > \lambda_0$ be regular such that $\lambda = \lambda^{<\mu}$. By Fact 7.8, F preserves λ^+-presentable objects and by Fact 2.9, F is (λ, λ^+)-accessible. Since $\mathcal{K}$ is large, $\mathcal{K}_{mono}$ is also large, so by Theorem 7.7, $\mathcal{K}$ has an object of presentability rank λ^+.
(2) As before, $\mathcal{K}_{mono}$ is an accessible category with all morphisms monos, so by Fact 2.12, it is equivalent to a μ'-AEC $\mathbf{K}^*$, for some regular cardinal μ'. Let $F : \mathbf{K}^* \to \mathcal{K}$ be the composition of the equivalence with the inclusion of $\mathcal{K}_{mono}$ into $\mathcal{K}$. Then F is (μ', λ_0)-accessible, for some regular cardinal $\lambda_0 > \mathrm{LS}(\mathbf{K})$, and F reflects isomorphisms. Taking λ_0 bigger if needed (and using Fact 7.8), we can assume without loss of generality that F also preserves λ_0-presentable objects. Let $\lambda \geq \lambda_0$ be a μ'-closed cardinal of cofinality at least μ'. Since $\mathbf{K}$ is large, Lemma 7.10 applies and so $\mathcal{L}$ has an object of presentability rank λ^+.

□

We obtain the main result of this section. This extends for example [LRVa, A.2] — weak LS-accessibility of large locally multipresentable categories — at the cost of ESCH:

Corollary 7.12. Assuming ESCH, any large accessible category has objects of all internal sizes of high-enough cofinality. In particular, any large accessible category is weakly LS-accessible.

Proof. Let $\mathcal{K}$ be a large μ-accessible category, and let μ' be as given by Theorem 7.11. Let λ be a high-enough cardinal of cofinality at least μ' (the proof will give how big we need to take it). If λ is a regular cardinal, then (by ESCH, see Fact 2.2) $\lambda = \lambda^{<\mu}$, so by Theorem 7.11, $\mathcal{K}$ has an object of presentability rank λ^+. Assume now that λ is a singular cardinal. Then λ is in particular a limit cardinal, so (by ESCH) λ is μ'-closed. By Theorem 7.11, $\mathcal{K}$ has an object of presentability rank λ^+. □

8. Filtrations

We consider conditions under which, in a general category, we can ensure than any object is not merely the colimit of an appropriately directed system of objects of strictly smaller internal size, but rather the colimit of a *chain* of such objects. The existence of such *filtrations* (sometimes also called *resolutions*) is crucial to a host of model-theoretic constructions, and should be of considerable use in the further development of classification theory at the present level of generality.

Definition 8.1. For μ a regular cardinal and λ an infinite cardinal, a $(\mu, < \lambda)$-*chain* (in a category $\mathcal{K}$) is a diagram $\langle M_i : i < \mu \rangle$ indexed by μ, all of whose objects are $(< \lambda)$-presentable. We call μ the *length* of the chain. A $(< \lambda)$-*chain* is a $(\mu, < \lambda)$-chain for some regular $\mu < \lambda$. For λ a regular cardinal, a λ-*chain* is a $(< \lambda^+)$-chain.

For θ a regular cardinal, we say that a chain $\langle M_i : i < \mu \rangle$ is θ-*smooth* if for every $i < \mu$ of cofinality at least θ, M_i is the colimit of $\langle M_j : j < i \rangle$.

Note that $(\mu, < \lambda)$-chains are $(\mu, < \lambda)$-systems in the sense of Definition 2.5. We will use the terminology of systems introduced in the preliminaries. The reader may also wonder why we are looking only at chains indexed by a regular cardinal. This is because any system $\langle M_i : i \in I \rangle$ indexed by a linear order I has a cofinal subsystem of the form $\langle M_{i_j} : i_j < \mu \rangle$, where μ is the cofinality of I.

The next definition is the object of study of this section:

Definition 8.2. Let $\mathcal{K}$ be a category. A *filtration* of an object M is a $(< r_{\mathcal{K}}(M))$-chain with colimit M. We call M *filtrable* if it (has a presentability rank and) has a filtration.

Observe that the length of a filtration is determined by the presentability rank:

Lemma 8.3. Let λ be a regular cardinal and let $\mathcal{K}$ be a category. If there exists a $(< \lambda)$-chain whose colimit is not $(< \lambda)$-presentable, then λ is a successor and the chain must have length $\operatorname{cf}(\lambda^-)$. In particular, any filtrable object M has successor presentability rank and any of its filtrations will have length $\operatorname{cf}(|M|_{\mathcal{K}})$.

Proof. Let $\mu < \lambda$ be a regular cardinal and let $\langle M_i : i < \mu \rangle$ be a $(\mu, < \lambda)$-chain in $\mathcal{K}$ with a colimit M that is not $(< \lambda)$-presentable. By Fact 2.6(2), the chain is proper. Next, assume for a contradiction that λ is weakly inaccessible. By Fact 2.6(1), M is $(< \lambda)$-presentable, a contradiction to the fact that it has presentability rank λ. This shows that λ is a successor. Let $\lambda_0 = \lambda^-$. We now have to see that $\operatorname{cf}(\lambda_0) = \mu$. We consider two cases depending on whether λ_0 is regular or singular:

- If λ_0 is regular, then $\langle M_i : i < \mu \rangle$ is a (μ, λ_0)-system. Since $\mu < \lambda$, we know that $\mu \leq \lambda_0$. If $\mu < \lambda_0$, then by Fact 2.6(1), M would be $(\mu^+ + \lambda_0)$-presentable, hence λ_0-presentable, contradicting that it has presentability rank λ. Thus $\mu = \lambda_0 = \operatorname{cf}(\lambda_0)$.
- If λ_0 is singular, then $\langle M_i : i < \mu \rangle$ is a $(\mu, < \lambda_0)$-system, and moreover (because μ is regular) $\mu \neq \lambda_0$, so $\mu^+ < \lambda_0$. If $\operatorname{cf}(\lambda_0) > \mu$, then there exists a regular $\lambda_1 < \lambda_0$ such that $\langle M_i : i < \mu \rangle$ is a (μ, λ_1)-system. By Fact 2.6(1), M is $(\mu^+ + \lambda_1)$-presentable, hence $(< \lambda)$-presentable, a contradiction.

  Thus $\operatorname{cf}(\lambda_0) \leq \mu$. If $\operatorname{cf}(\lambda_0) < \mu$, let $\langle \theta_\alpha : \alpha < \operatorname{cf}(\lambda_0) \rangle$ be an increasing chain of regular cardinals cofinal in λ_0. For all $i < \mu$, there exists $\alpha = \alpha_i < \operatorname{cf}(\lambda_0)$ such that M_i is θ_α-presentable. By cardinality consideration, there must exist $I \subseteq \mu$ of cardinality μ and $\alpha < \operatorname{cf}(\lambda_0)$ such that for all $i \in I$, M_i is θ_α-presentable. Since μ is regular, I is cofinal in μ, so M is still the colimit of $\langle M_i : i \in I \rangle$. The latter system is a (μ, θ_α)-system, hence (again by Fact 2.6(1)) M is $(\mu^+ + \theta_\alpha)$-presentable, so $(< \lambda)$-presentable, a contradiction. The only remaining possibility is that $\mathrm{cf}(\lambda_0) = \mu$, which is what we wanted to prove.

□

It follows that if the category has directed colimits, we can take the filtration to be smooth. More generally:

Lemma 8.4. Any filtration of an object M of presentability rank λ is boundedly $(< \lambda)$-presentable (Definition 6.3). In particular, if $\theta < \lambda$ is regular such that $\mathcal{K}$ has θ-directed colimits, then M has a θ-smooth filtration.

Proof. Let $\mu < \lambda$ be regular, and let $\langle M_i : i < \mu \rangle$ be a $(< \lambda)$-chain whose colimit is M. By Lemma 8.3, λ is a successor cardinal and $\mu = \mathrm{cf}(\lambda^-)$. Let $\delta < \mu$ be a limit ordinal. We will show that the colimit of $\langle M_i : i < \delta \rangle$ (assuming it exists) is $(< \lambda)$-presentable. The "in particular" part will then follow, since, when $\mathrm{cf}(\delta) \geq \theta$, it suffices to replace M_δ by the colimit of $\langle M_i : i < \delta \rangle$.

Note that $\delta < \mu \leq \lambda^-$, so $\delta^+ < \lambda$. By cofinality considerations, there exists a regular cardinal $\lambda_0 < \lambda$ such that for all $i < \delta$, M_i is λ_0-presentable. By Fact 2.6(1), we get that the colimit of $\langle M_i : i < \delta \rangle$ is $(\delta^+ + \lambda_0)$-presentable, hence $(< \lambda)$-presentable, as desired. □

Using the definition of presentability, it is also easy to generalize the well known facts that, for objects of regular cardinality (that is, of presentability rank the successor of a regular cardinal), any two smooth filtrations are the same on a *club*: a *closed unbounded* set of indices. See for example [BGL$^+$16, 6.11].

We give a name to categories where every object of a given presentability rank is filtrable:

Definition 8.5. For a regular cardinal λ, we say an accessible category $\mathcal{K}$ is λ-*filtrable* if any object of presentability rank λ is filtrable. For a regular cardinal μ, we say that $\mathcal{K}$ is *well μ-filtrable* if it is λ-filtrable for any regular $\lambda \geq \mu$. We say $\mathcal{K}$ is *well filtrable* if it is well μ-filtrable for some regular cardinal μ.

Similarly, we say that $\mathcal{K}$ is *almost λ-filtrable* if any object M of presentability rank λ is a retract of a filtrable λ-presentable object N (i.e. there exists a split epimorphisms from N to M). Define *almost well λ-filtrable* and *almost well filtrable* as expected.

Remark 8.6. The technical notion of being *almost* filtrable is included here because we do not know whether retracts of filtrable objects are filtrable. Of course, in categories where all morphisms are monos, retracts are just isomorphisms and so this technical distinction is irrelevant.

In the rest of this section, we give a a couple of easy examples. In the next section, we will use these examples to prove that any accessible category with directed colimits is almost well filtrable.

Lemma 8.7. Let $\mathbf{K}$ be a μ-AEC and let $M \in \mathbf{K}$. If $|UM| > \mathrm{LS}(\mathbf{K})$ and $|UM|$ is μ-closed, then M is filtrable. In particular, $\mathbf{K}$ is θ-filtrable for any $\theta > \mathrm{LS}(\mathbf{K})^+$ that is the successor of a μ-closed cardinal of cofinality at least μ.

Proof. Let $\lambda := |UM|$, and let $\delta := \text{cf}(\lambda)$. Using the cofinality assumption, find $\langle A_i : i < \delta \rangle$ an increasing continuous chain of subsets of UM such that $|A_i| < \lambda$ for all $i < \delta$ and $UM = \bigcup_{i<\delta} A_i$. Using the Löwenheim-Skolem-Tarski axiom, build $\langle M_i : i < \delta \rangle$ increasing in $\mathbf{K}$ such that for all $i < \delta$, $A_i \subseteq UM_i$, $M_i \leq_{\mathbf{K}} M$, and $|UM_i| < \lambda$. This is possible: assume $i < \delta$ and we are given $\langle M_j : j < i \rangle$. Let $A := A_i \cup \bigcup_{j<i} UM_j$. By cofinality considerations, $|A| < \lambda$. Since λ is μ-closed, there exists $M_i \leq_{\mathbf{K}} M$ with $|UM_i| < \lambda$ and $A \subseteq UM_i$. This completes the construction. We then have that $M = \bigcup_{i<\lambda} M_i$, and by Fact 2.13, M has presentability rank λ^+ while each M_i is $(< \lambda^+)$-presentable.

For the "in particular" part, let $\theta > \text{LS}(\mathbf{K})^+$ be the successor of a μ-closed cardinal λ of cofinality at least μ. Then $\lambda = \lambda^{<\mu}$ so θ is also μ-closed. Thus (by Fact 2.13) an object of $\mathbf{K}$ is θ-presentable if and only if it has cardinality λ, and the result follows. $\square$

We have in particular derived the following fact, well-known when stated in terms of cardinalities:

Theorem 8.8. Any AEC $\mathbf{K}$ is well $\text{LS}(\mathbf{K})^{++}$-filtrable.

Proof. By Facts 2.12 and 2.9, $\mathbf{K}$ is well $\text{LS}(\mathbf{K})^+$-accessible. By Corollary 5.4, presentability ranks greater than $\text{LS}(\mathbf{K})^+$ must be successors. Now apply Lemma 8.7 with $\mu = \aleph_0$. $\square$

Corollary 8.9. Any finitely accessible category with all morphisms monos is well filtrable.

Proof. By Fact 2.12, such a category is equivalent to an AEC, so this is a special case of Theorem 8.8. $\square$

Using the tools of Section 7, we can also obtain a result for any accessible category:

Corollary 8.10. For any accessible category $\mathcal{K}$, there exists a regular cardinal μ such that for all high-enough μ-closed cardinals λ of cofinality at least μ, $\mathcal{K}$ is λ^+-filtrable. In particular, assuming ESCH, $\mathcal{K}$ is filtrable in any cardinal that is the successor of a limit cardinal of high-enough cofinality, or the double successor of any cardinal of high-enough cofinality.

Proof. Similar to the proof of Theorem 7.11 and Corollary 7.12, using Lemma 8.7. $\square$

The next result will not be needed later, and uses some material from [LRVa] on μ-AECs with intersections (i.e. with a well-behaved closure operator). There, the internal size is just the minimal size of a generator, and the existence of filtrations follows immediately.

Theorem 8.11. Any μ-AEC with intersections is well μ^+-filtrable.

Proof. Let $\mathbf{K}$ be a μ-AEC, let $\lambda \geq \mu^+$ be a regular cardinal and let $M \in \mathbf{K}$ have presentability rank λ. By [LRVa, 5.7], pick $A \subseteq UM$ such that $M = \text{cl}^M(A)$ and $|A|^+ = \lambda$. Let $\delta := \text{cf}(|A|)$, and write $A = \bigcup_{i<\delta} A_i$, where $|A_i| < |A|$ and

$A_i \subseteq A_j$ for $i < j < \delta$. Let $M_i := \mathrm{cl}^M(A_i)$. By [LRVa, 5.7] again, each M_i is $(< \lambda)$-presentable, as desired. ☐

9. Filtrations and reflections

In this section, we investigate filtrations in accessible categories with directed colimits. The goal is to prove that these categories are well filtrable. One result in this direction is the second author's [Ros97, Lemma 1], which establishes that such categories are filtrable at successors of *regular* cardinals:

Fact 9.1. For a regular cardinal λ, any λ-accessible category with directed colimits is λ^+-filtrable.

The case of successors of singular cardinals seems much harder, but we succeed in proving it for finitely accessible categories (Corollary 9.11). For accessible categories with directed colimits, we prove only that they are almost well filtrable (Corollary 9.12). In each case, the argument relies on embedding such categories as reflective subcategories of nice categories of the kind considered at the end of the previous section, and pulling the filtration back along the reflection functor. Before proceeding, we recall:

Definition 9.2. A full subcategory $\mathcal{K}^*$ of a category $\mathcal{K}$ is *reflective* if the inclusion functor $i : \mathcal{K}^* \to \mathcal{K}$ has a left adjoint $F : \mathcal{K} \to \mathcal{K}^*$. In this case, we call F the *reflection functor*, or *reflector*.

Note that we assume reflective subcategories are full, as is now more or less customary. We will assume, too, that the reflector is the identity when restricted to the reflective subcategory.

Example 9.3. The category of complete metric spaces is a reflective subcategory of the category of all metric spaces: the reflector takes each metric space to its completion.

Lemma 9.4. Suppose $\mathcal{K}^*$ is a reflective subcategory of $\mathcal{K}$, with reflector F. If $\langle M_i : i \in I \rangle$ is a diagram in $\mathcal{K}$, then, whenever both colimits exist:

$$F(\mathrm{colim}_i^{\mathcal{K}} M_i) \cong \mathrm{colim}_i^{\mathcal{K}^*} F(M_i)$$

Proof. Since F is a left adjoint, it preserves arbitrary colimits. ☐

Definition 9.5. Let $\mathcal{K}^*$ be a reflective subcategory of $\mathcal{K}$, with reflector F. For an object M of $\mathcal{K}^*$, let $r_F(M)$ be the minimal regular cardinal λ such that there is $M_0 \in \mathcal{K}$ so that M is a retract of $F(M_0)$ and M_0 is λ-presentable in $\mathcal{K}$.

In the case of Example 9.3 above, where the reflector F corresponds to metric completion, r_F will give the successor of the density character. This generalizes to:

Lemma 9.6. Let λ be a regular cardinal. Assume $\mathcal{K}$ is λ-accessible and $\mathcal{K}^*$ is a reflective subcategory closed under λ-directed colimits inside $\mathcal{K}$. Let M be an object of $\mathcal{K}^*$.

(1) If M is λ-presentable in $\mathcal{K}$, then $r_F(M) \leq \lambda$.

(2) If $r_F(M) \leq \lambda$, then M is λ-presentable in $\mathcal{K}^*$.
(3) If M is λ-presentable in $\mathcal{K}^*$, then $r_F(M) \leq \lambda$.

Proof.

(1) Notice that $M = F(M)$, as we assume F to be the identity on $\mathcal{K}^*$.
(2) By diagram chase.
(3) Resolve M as a λ-directed colimit of λ-presentable objects M_i in $\mathcal{K}$. Since $M = F(M)$, M is the λ-directed colimit of the $F(M_i)$'s in $\mathcal{K}^*$. Since each M_i is λ-presentable in $\mathcal{K}$, $r_F(F(M_i)) \leq \lambda$, so $F(M_i)$ is λ-presentable in $\mathcal{K}^*$. Since M is λ-presentable in $\mathcal{K}^*$, it is a retract of some $F(M_i)$, as desired.

□

Lemma 9.7. Let μ be a regular cardinal, let $\mathcal{K}$ be a well μ-accessible category, let $\mathcal{K}^*$ be a reflective subcategory of $\mathcal{K}$ closed under μ-directed colimits inside $\mathcal{K}$, and let F be a reflector. If M is an object of $\mathcal{K}^*$ that is not $(< \mu)$-presentable in $\mathcal{K}^*$, then $r_{\mathcal{K}^*}(M) = r_F(M)$.

Proof. Let $\lambda := r_{\mathcal{K}^*}(M)$. We know that $\lambda \geq \mu$, since M is not $(< \mu)$-presentable. By Lemma 9.6(3), $r_F(M) \leq \lambda$. Let $\lambda_0 \in [\mu, \lambda]$ be a regular cardinal. By Lemma 9.6(2), if $r_F(M) \leq \lambda_0$, then M would be λ_0-presentable in $\mathcal{K}^*$, and hence $r_{\mathcal{K}^*}(M) \leq \lambda_0$. This is only possible for $\lambda_0 = \lambda$, hence $r_F(M) = \lambda$. □

Fact 9.8 ([BR12, 4.5])**.** If $\mathcal{K}^*$ is a (μ, λ)-accessible category, then $\mathcal{K}^*$ is a reflective subcategory of a μ-accessible category $\mathcal{K}$, and $\mathcal{K}^*$ is closed under λ-directed colimits inside $\mathcal{K}$.

We can now generalize our results on filtrations. The key lemma shows that if a reflexive subcategory is well filtrable, then the bigger category is (almost) well filtrable.

Lemma 9.9. Let $\mu < \lambda$ be regular cardinals, and let $\mathcal{K}^*$ be a well μ-accessible category, which is a reflective subcategory of a well λ-filtrable category $\mathcal{K}$, and further is closed under μ-directed colimits inside $\mathcal{K}$. Then $\mathcal{K}^*$ is almost well λ-filtrable.

Proof. Let $\theta \geq \lambda$ be a regular cardinal. Let M be an object of $\mathcal{K}^*$ of presentability rank θ. By Lemma 9.7, there is M^* which is θ-presentable in $\mathcal{K}$ and so that M is a retract of $F(M^*)$. Since $\mathcal{K}$ is θ-filtrable, one can pick a filtration $\langle M_i^* : i < \delta \rangle$ of M^* in $\mathcal{K}$. Now let $M_i := F(M_i^*)$. Then (by definition of r_F), $r_F(M_i) < \lambda$, so by Lemma 9.7) M_i is $(< \lambda)$-presentable in $\mathcal{K}^*$. Furthermore, $\langle M_i : i < \delta \rangle$ form a chain in $\mathcal{K}^*$, and (still in $\mathcal{K}^*$) $F(M^*)$ is the colimit of this chain (Lemma 9.4). □

Theorem 9.10. Any accessible category with directed colimits and all morphisms monos is well filtrable.

Proof. Let $\mathcal{K}^*$ be a μ-accessible category with directed colimits and all morphisms monos. By Fact 9.8, there exists a finitely accessible category $\mathcal{K}$ so that $\mathcal{K}^*$ is a reflective full subcategory of $\mathcal{K}$, closed under μ-directed colimits inside $\mathcal{K}$. We know that all the morphisms in $\mathcal{K}$ are monos, so by Corollary 8.9, $\mathcal{K}$ is well filtrable. By Fact 2.9, $\mathcal{K}^*$ is well μ-accessible. By Lemma 9.9, we deduce that $\mathcal{K}^*$ is almost well filtrable, but since all morphisms in $\mathcal{K}^*$ are monos, split epimorphisms are just isomorphism, so $\mathcal{K}^*$ is in fact well filtrable. $\square$

Corollary 9.11. Any finitely accessible category is well filtrable.

Proof. Let $\mathcal{K}$ be a finitely accessible category. Then $\mathcal{K}_{mono}$ is an accessible category with directed colimits and all morphisms monos (Fact 7.2). By Theorem 9.10, $\mathcal{K}_{mono}$ is well filtrable. Moreover the embedding of $\mathcal{K}_{mono}$ into $\mathcal{K}$ preserves directed colimits (Fact 7.2) and reflects split epimorphisms. Thus (Facts 7.6, 7.8), this embedding preserves and reflects presentability ranks starting at some cardinal. It immediately follows that $\mathcal{K}$ itself is well filtrable. $\square$

Corollary 9.12. Any accessible category with directed colimits is almost well filtrable.

Proof. Imitate the proof of Theorem 9.10: any accessible category with directed colimits is a reflective subcategory of a finitely accessible category, which is well filtrable by Corollary 9.11. $\square$

References

[AHS04] Jiří Adámek, Horst Herrlich, and George E. Strecker, *Abstract and concrete categories*, online edition ed., 2004, Available from `http://katmat.math.uni-bremen.de/acc/`.

[AR94] Jiří Adámek and Jiří Rosický, *Locally presentable and accessible categories*, London Math. Society Lecture Notes, Cambridge University Press, 1994.

[BB17] John T. Baldwin and Will Boney, *Hanf numbers and presentation theorems in AECs*, Beyond first order model theory (José Iovino, ed.), CRC Press, 2017, pp. 327–352.

[BGL+16] Will Boney, Rami Grossberg, Michael J. Lieberman, Jiří Rosický, and Sebastien Vasey, *μ-Abstract elementary classes and other generalizations*, The Journal of Pure and Applied Algebra **220** (2016), no. 9, 3048–3066.

[Bon14] Will Boney, *Tameness from large cardinal axioms*, The Journal of Symbolic Logic **79** (2014), no. 4, 1092–1119.

[BR12] Tibor Beke and Jiří Rosický, *Abstract elementary classes and accessible categories*, Annals of Pure and Applied Logic **163** (2012), 2008–2017.

[BY05] Itay Ben-Yaacov, *Uncountable dense categoricity in cats*, The Journal of Symbolic Logic **70** (2005), no. 3, 829–860.

[HH09] Åsa Hirvonen and Tapani Hyttinen, *Categoricity in homogeneous complete metric spaces*, Archive for Mathematical Logic **48** (2009), 269–322.

[Jec03] Thomas Jech, *Set theory*, 3rd ed., Springer-Verlag, 2003.

[Lie11] Michael J. Lieberman, *Category-theoretic aspects of abstract elementary classes*, Annals of Pure and Applied Logic **162** (2011), no. 11, 903–915.

[LR16] Michael J. Lieberman and Jiří Rosický, *Classification theory for accessible categories*, The Journal of Symbolic Logic **81** (2016), no. 1, 151–165.

[LR17] ———, *Metric abstract elementary classes as accessible categories*, The Journal of Symbolic Logic **82** (2017), no. 3, 1022–1040.

[LRVa] Michael J. Lieberman, Jiří Rosický, and Sebastien Vasey, *Internal sizes in μ-abstract elementary classes*, Journal of Pure and Applied Algebra, To appear. URL: `http://arxiv.org/abs/1708.06782v4`.

[LRVb] ———, *More on stable independence*, In preparation.

[LRV19a] ———, *Forking independence from the categorical point of view*, Advances in Mathematics **346** (2019), 719–772.

[LRV19b] ———, *Universal abstract elementary classes and locally multipresentable categories*, Proceedings of the American Mathematical Society **147** (2019), no. 3, 1283–1298.

[MP89] Michael Makkai and Robert Paré, *Accessible categories: The foundations of categorical model theory*, Contemporary Mathematics, vol. 104, American Mathematical Society, 1989.

[Ros97] Jiří Rosický, *Accessible categories, saturation and categoricity*, The Journal of Symbolic Logic **62** (1997), no. 3, 891–901.

[She87] Saharon Shelah, *Classification of non elementary classes II. Abstract elementary classes*, Classification Theory (Chicago, IL, 1985) (John T. Baldwin, ed.), Lecture Notes in Mathematics, vol. 1292, Springer-Verlag, 1987, pp. 419–497.

[Vas16] Sebastien Vasey, *Infinitary stability theory*, Archive for Mathematical Logic **55** (2016), 567–592.

E-mail address: lieberman@math.muni.cz

URL: http://www.math.muni.cz/~lieberman/

Department of Mathematics and Statistics, Faculty of Science, Masaryk University, Brno, Czech Republic

Institute of Mathematics, Faculty of Mechanical Engineering, Brno University of Technology, Brno, Czech Republic

E-mail address: rosicky@math.muni.cz

URL: http://www.math.muni.cz/~rosicky/

Department of Mathematics and Statistics, Faculty of Science, Masaryk University, Brno, Czech Republic

E-mail address: sebv@math.harvard.edu

URL: http://math.harvard.edu/~sebv/

Department of Mathematics, Harvard University, Cambridge, Massachusetts, USA