FORKING AND SUPERSTABILITY IN TAME AECS

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ABSTRACT. We prove that any tame abstract elementary class categorical in a suitable cardinal admits a global good frame: A forking-like notion defined on all types of single elements. This gives the first known construction of a good frame in ZFC. We show that we already obtain a well-behaved independence relation assuming only a superstability-like hypothesis instead of categoricity. These methods are applied to obtain an upward stability transfer theorem from categoricity and tameness, as well as new conditions for uniqueness of limit models.

CONTENTS

1. INTRODUCTION

In 2009, Shelah published a two volume book [She09c, She09d] on classification theory for abstract elementary classes. The central new structural notion is that of a good λ-frame (for a given abstract elementary class (AEC) K): a generalization of first-order forking to types over models of size λ in K (see Section 2.4 below for the precise definition). The existence of a good frame shows that K is very

well-behaved at λ and the aim was to use this frame to deduce more on the structure of K above λ. Part of this program has already been accomplished through several hundreds of pages of hard work (see for example [She01], [She09b], [She09e], [JS13]). Among many other results, Shelah shows that good frames exist under strong categoricity assumptions and additional set-theoretic hypotheses:

Fact 1.1 (Theorem 3.7 in [She09b]). Assume $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$ and the weak diamond ideal in λ^+ is not λ^{++}-saturated.

Let K be an abstract elementary class with $\mathrm{LS}(K) \le \lambda$. Assume:

(1) K is categorical in λ and λ^+.
(2) $0 < I(\lambda^{++}, K) < \mu_{\mathrm{unif}}(\lambda^{++}, 2^{\lambda^+})$

Then K has a good λ-frame.

It is a major open problem whether the set-theoretic hypotheses in Fact 1.1 are necessary. In this paper, we show that if the class already has some *global* structure, then good frames are much easier to build. For example we prove (in ZFC):

Theorem 1.2. Let K be an abstract elementary class with amalgamation, joint embedding, and no maximal models. Assume K is categorical in a high-enough successor λ^+. Then K has a type-full good λ-frame.

By the main theorem of [She99], the hypotheses of Theorem 1.2 imply K is categorical in λ. On the other hand, we do not need any set-theoretic hypothesis and we do not need to know anything about the number of models in λ^{++}. Moreover, the frame Shelah constructs typically defines a notion of nonforking only for a restricted class of basic types (the minimal types). In our frame, nonforking is defined for every type. This is technically very convenient and closer to the first-order intuition. Of course, we pay for this luxury by assuming amalgamation and no maximal models.

Our proof relies on two key properties of AECs. The first one is tameness (a locality property of Galois types, see Definition 2.4), and in fact assuming it lets us relax the successor assumption in Theorem 1.2:

Theorem 1.3. Let K be an abstract elementary class with amalgamation, joint embedding, and no maximal models. Assume K is μ-tame and categorical in some cardinal λ such that $\mathrm{cf}(\lambda) > \mu$. Then K has a type-full good $\ge \lambda$-frame.

That is, not only do we obtain a good λ-frame, but we can also extend this frame to any model of size $\ge \lambda$ (this last step essentially follows from earlier work of Boney [Bonb]). Hence we obtain a global forking notion, although only defined for 1-types. A global forking notion for types of all lengths is obtained in [BG] (using stronger tameness hypotheses than ours) but the authors assume the extension property for coheir, and it is unclear when this holds, even assuming categoricity

everywhere. Thus our result partially answers [BGKV, Question 7.1]. We also obtain new theorems whose statements do not mention frames:

Corollary 1.4. Let K be an abstract elementary class with amalgamation, joint embedding, and no maximal models. Assume K is μ-tame and categorical in some cardinal λ such that $\mathrm{cf}(\lambda) > \mu$. Then K is stable everywhere.

Remark 1.5. Shelah already established in [She99] that categoricity in $\lambda > \mathrm{LS}(K)$ implies stability below λ. The first upward stability transfer for tame AECs appeared in [GV06b]. Later, [BKV06] gave some variations, showing for example $\aleph_0$-stability and a strong form of tameness implies stability everywhere. Our upward stability transfer improves on [BKV06, Corollary 4.7] which showed that categoricity in a successor λ implies stability in λ.

Corollary 1.6. Let K be an abstract elementary class with amalgamation, joint embedding, and no maximal models. Assume K is μ-tame and categorical in some cardinal λ such that $\mathrm{cf}(\lambda) > \mu$. Then K has a unique limit model in every $\lambda' \geq \lambda$.

Remark 1.7. This is also new and complements the conditions for uniqueness of limit models given in [She99], [Van06], and [GVV].

The second key property in our proof is a technical condition we call local character of μ-nonsplitting for $\vartriangleleft$-chains (see Definition 3.10). This follows from categoricity in a cardinal of cofinality larger than μ and we believe it is a good candidate for a definition of superstability, at least in the tame context. Under this hypothesis, we already obtain a forking notion that is well-behaved for μ^+-saturated base models and can prove the upward stability transfer given by Corollary 1.4. Local character of nonsplitting already played a key role in other papers such as [SV99], [Van06], and [GVV].

Even if only stability and tameness hold, we can still look at the length of the chains for which μ-nonsplitting has local character (analogous to the cardinal $\kappa(T)$ in the first-order context). Using GCH, we can generalize one direction of the first-order characterization of the stability spectrum (Theorem 7.6).

The paper is structured as follows: In Section 2, we review background in the theory of AECs and give the definition of good frames. In Section 3, we fix a cardinal μ and build a μ-frame-like object named a "poor man's frame". This is done by using the weak extension and uniqueness properties of nonsplitting isolated by VanDieren [Van02], together with the assumption of local character of nonsplitting. In Section 4, we show that some of the properties of our poor man's frame in μ lift to cardinals above μ (and in fact become better than they were in μ). This is done using the same methods as in [She09b, Section 2].

In Section 5, we show assuming tameness that the other properties of the poor man's frame lift as well and similarly become better, so that we obtain a good frame without symmetry (if we restrict ourselves to μ^+-saturated models and so,

assuming categoricity in the right cardinal, to all models). This uses the ideas from [Bonb]. Next in Section 6 we show how to get symmetry by using more tameness together with the order property (this is where we really use that we have structure properties holding globally and not only at a few cardinals). Finally, we put everything together in Section 7 (Theorem 1.2 is proven there as Theorem 7.4 and Theorem 1.3 as Theorem 7.3). In Section 8, we conclude.

At the beginning of Sections 3, 4, 5, and 6, we give hypotheses that are assumed to hold everywhere in those sections. We made an effort to show clearly how much of the structural properties (amalgamation, tameness, superstability, etc.) are used at each step, but our construction is new even for the case of a totally categorical AEC K with amalgamation, joint embedding, no maximal models, and $\mathrm{LS}(K)$-tameness. It might help the reader to keep this case in mind throughout.

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2. Preliminaries

2.1. **Abstract elementary classes.** We assume the reader is familiar with the definition of an abstract elementary class (AEC) and the basic related concepts. See [Gro02] for an introduction.

For the rest of this section, fix an AEC K. We denote the partial ordering on K by $\leq$, and write $M < N$ if $M \leq N$ and $M \neq N$. For R a binary relation on K and δ an ordinal, an R-*increasing chain* $(M_i)_{i<\delta}$ is a sequence of models in K such that for all $i < \delta$, if $i + 1 < \delta$ then $R(M_i, M_{i+1})$. The chain is *continuous* if it is $\leq$-increasing and for any limit $i < \delta$, $M_i = \bigcup_{j<i} M_j$. When we talk of an increasing chain, we mean a $\leq$-increasing chain. Strictly increasing means $<$-increasing.

For K an abstract elementary class and $\mathcal{F}$ an interval[1] of cardinals of the form $[\lambda, \theta)$, where $\theta > \lambda \geq \mathrm{LS}(K)$ is either a cardinal or ∞, let $K_{\mathcal{F}} := \{M \in K \mid \|M\| \in \mathcal{F}\}$. We write K_λ instead of $K_{\{\lambda\}}$, $K_{\geq\lambda}$ instead of $K_{[\lambda,\infty)}$ and $K_{\leq\lambda}$ instead of $K_{[\mathrm{LS}(K),\lambda]}$.

The following properties of AECs are classical:

Definition 2.1 (Amalgamation, joint embedding, no maximal models)**.** Let $\mathcal{F}$ be an interval of cardinals as above.

(1) $K_{\mathcal{F}}$ has *amalgamation* if for any $M_0 \leq M_\ell \in K_{\mathcal{F}}$, $\ell = 1, 2$ there exists $N \in K_{\mathcal{F}}$ and $f_\ell : M_\ell \xrightarrow[M_0]{} N$, $\ell = 1, 2$.

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[1]The definitions that follow make sense for an arbitrary set of cardinals $\mathcal{F}$, but the proofs of most of the facts below require that $\mathcal{F}$ is an interval.

(2) $K_{\mathcal{F}}$ has *joint embedding* if for any $M_\ell \in K_{\mathcal{F}}$, $\ell = 1, 2$ there exists $N \in K_{\mathcal{F}}$ and $f_\ell : M_\ell \to N$, $\ell = 1, 2$.

(3) $K_{\mathcal{F}}$ has *no maximal models* if for any $M \in K_{\mathcal{F}}$ there exists $N > M$ in $K_{\mathcal{F}}$.

Fact 2.2. Let $\mathcal{F}$ be an interval of cardinals as above.

(1) If K_μ has no maximal models for all $\mu \in \mathcal{F}$, then $K_{\mathcal{F}}$ has no maximal models.

(2) If K_μ has amalgamation for all $\mu \in \mathcal{F}$, then $K_{\mathcal{F}}$ has amalgamation.

Proof. No maximal models is straightforward and amalgamation is [She09a, Conclusion 2.12]. $\square$

Finally, we will also use:

Lemma 2.3. Let $\mathcal{F} = [\lambda, \theta)$ be an interval of cardinals as above. If $K_{\mathcal{F}}$ has amalgamation and K_λ has joint embedding, then $K_{\mathcal{F}}$ has joint embedding.

Proof sketch. Let $M_\ell \in K_{\mathcal{F}}$, $\ell = 1, 2$. Pick $M'_\ell \leq M_\ell$ of size λ. Use joint embedding on M'_1, M'_2, then use amalgamation. $\square$

2.2. **Galois types, stability, and tameness.** We assume familiarity with Galois types (see [Gro02, Section 6]). For $M \in K$, we write $S^\alpha(M)$ for the set of Galois types of sequences of length α over M. We will at one point also consider types over the empty set, which are defined analogously (see e.g. [She, Definition 1.4]). We write $S(M)$ for $S^1(M)$. We write $S^{\mathrm{na}}(M)$ for the set of *nonalgebraic* 1-types over M, that is:

$$S^{\mathrm{na}}(M) := \{\mathrm{gtp}(a/M; N) \mid a \in N \backslash M, M \leq N \in K\}$$

From now on, we will write $\mathrm{tp}(a/M; N)$ for $\mathrm{gtp}(a/M; N)$.

We briefly review the notion of tameness. Although it appears implicitly (for saturated models) in [She99], tameness as a property of AECs was first introduced in [GV06b] and used to prove a stability spectrum theorem. It was later used in [GV06a] to prove an upward categoricity transfer. Our definition follows [Bonc, Definition 3.1].

Definition 2.4 (Tameness). Let $\lambda > \kappa > \mathrm{LS}(K)$. Let α be a cardinal. We say that K is (κ, λ)-*tame for α-length types* if for any $M \in K_{\leq \lambda}$ and any $p, q \in S^\alpha(M)$, if $p \neq q$, then there exists $M_0 \leq M$ of size $\leq \kappa$ such that $p \upharpoonright M_0 \neq q \upharpoonright M_0$. We define similarly $(\kappa, < \lambda)$-tame, $(< \kappa, \lambda)$-tame, etc. When $\lambda = \infty$, we omit it. When $\alpha = 1$, we omit it. We say that K is *fully κ-tame* if it is κ-tame for all lengths.

We also recall that we can define a notion of stability:

Definition 2.5 (Stability). Let $\lambda \geq \mathrm{LS}(K)$ and α be cardinals. We say that K is α-*stable in* λ if for any $M \in K_\lambda$, $|S^\alpha(M)| \leq \lambda$.

We say that K is *stable* in λ if it is 1-stable in λ.

We say that K is α-*stable* if it is α-stable in λ for some $\lambda \geq \mathrm{LS}(K)$. We say that K is *stable* if it is 1-stable in λ for some $\lambda \geq \mathrm{LS}(K)$. We write "unstable" instead of "not stable".

We define similarly stability for $K_\mathcal{F}$. For example, K_λ is stable if and only if K is stable in λ.

Remark 2.6. If $\alpha < \beta$, and K is β-stable in λ, then K is α-stable in λ.

The following follows from [Bona, Theorem 1.1].

Fact 2.7. Let $\lambda \geq \mathrm{LS}(K)$. Let α be a cardinal. Assume K is stable in λ and $\lambda^\alpha = \lambda$. Then K is α-stable in λ.

2.3. **Universal and limit extensions.**

Definition 2.8 (Universal and limit extensions). For $M, N \in K$, we say that N is *universal over* M (written $M <_{\mathrm{univ}} N$) if and only if $M < N$ and for any $M' \geq M$ with $||M'|| = ||M||$, M' can be embedded inside N over M. We also write $N >_{\mathrm{univ}} M$ for $M <_{\mathrm{univ}} N$.

For $\mu \geq \mathrm{LS}(K)$ and $0 < \delta < \mu^+$ an ordinal, we say that N is *(μ,δ)-limit* over M (written $M <_{\mu,\delta} N$) if and only if $M \leq N \in K_\mu$ and there is a $<_{\mathrm{univ}}$-increasing chain $(M_i)_{i \leq \delta}$ with $M_0 = M$, $M_\delta = N$ and $M_\delta = \bigcup_{i<\delta} M_i$ if δ is limit. We also write $N >_{\mu,\delta} M$ for $M <_{\mu,\delta} N$.

We say that N is μ-*limit* over M if it is μ-limit over M for some limit δ. We say that N is δ-*limit* if it is $(||N||, \delta)$-limit over M for some M. We say that it is *limit* if it is δ-limit for some limit δ.

Definition 2.9. A model $N \in K$ is μ-*model-homogeneous* if for any $M \leq N$ with $||M|| < \mu$, we have $M <_{\mathrm{univ}} N$. N is *model-homogeneous* if it is $||N||$-model-homogeneous.

Fact 2.10. Let $\mu \geq \mathrm{LS}(K)$. Assume K_μ has amalgamation, no maximal models, and is stable. For any $M \in K_\mu$, there exists $N >_{\mathrm{univ}} M$ in K_μ. Therefore there is a model-homogeneous $N > M$ in K_{μ^+}.

Proof. The first part is by [She09b, Claim 1.16.1(a)]. The second part follows from iterating the first part μ^+ many times. $\square$

Remark 2.11. By [She01, Lemma 0.26], for $\mu > \mathrm{LS}(K)$, N is μ-model-homogeneous if and only if it is μ-saturated.

Proposition 2.12. Let $M_0, M_1, M_2 \in K_\mu$, $\mu \geq \mathrm{LS}(K)$ and $0 < \delta < \mu^+$. Then:

(1) $M_0 <_{\mu,\delta} M_1$ implies $M_0 <_{\mathrm{univ}} M_1$.
(2) $M_0 <_{\mathrm{univ}} M_1 \leq M_2$ implies $M_0 <_{\mathrm{univ}} M_2$.
(3) Assume K_μ has amalgamation. Then $M_0 \leq M_1 <_{\mu,\delta} M_2$ implies $M_0 <_{\mu,\delta} M_2$.
(4) Assume K_μ has amalgamation, no maximal models, and is stable. Then there exists M_0' such that $M_0 <_{\mu,\delta} M_0'$.
(5) Conversely, if for every $M_0 \in K_\mu$ there exists $M_0' \in K_\mu$ such that $M_0 <_{\mathrm{univ}} M_0'$, then K_μ has amalgamation, no maximal models, and is stable.

Proof.

(1) Fix $(N_i)_{i \leq \delta}$ witnessing that $M_0 <_{\mu,\delta} M_1$. Let $M_0' \geq M_0$ have size μ. Since $\delta > 0$, N_1 is well defined, and is universal over $N_0 = M_0$, hence M_0' can be embedded inside N_1 over M_0, and hence since $N_1 \leq M_1$ can be embedded inside M_1 over M_0.
(2) Let $M_0' \geq M_0$ have size μ. Since M_0' embed inside M_1 over M_0, it also embeds inside M_2 over M_0.
(3) Let $(N_i)_{i \leq \delta}$ witness $M_0 <_{\mu,\delta} M_1$. We show that $M_0 <_{\mathrm{univ}} N_1$. This is enough since then $M_0 \frown (N_i)_{0 < i \leq \delta}$ will witness that $M_0 <_{\mu,\delta}$. Let $M_0' \geq M_0$ have size μ. By amalgamation, find $M_1' \geq M_1$ and $h : M_0' \xrightarrow[M_0]{} M_1'$. Now use universality of M_2 over M_1 to find $g : M_1' \xrightarrow[M_1]{} M_2$. Let $f := g \circ h$. Then $f : M_0' \xrightarrow[M_0]{} M_2$, as desired.
(4) Iterate Fact 2.10 δ many times.
(5) Let $M_0 \in K_\mu$ and let $M_0' >_{\mathrm{univ}} M_0$ be in K_μ. M_0' witnesses that M_0 is not maximal in K_μ. Moreover, M_0 is an amalgamation base, since any two models of size μ extending M_0 can amalgamated over M_0 inside M_0'. Finally, all types over M_0 are realized in M_0' which has size μ, there can be at most μ many of them, so stability follows.

$\square$

We give orderings satisfying the conclusion of Proposition 2.12 a name:

Definition 2.13 (Good ordering). A *good ordering on K_μ* is a binary relation $\lhd$ on K_μ satisfying the following properties. For any $M_0, M_1, M_2 \in K_\mu$:

(1) $M_0 \lhd M_1$ implies $M_0 <_{\mathrm{univ}} M_1$.
(2) There exists $N_0 \rhd M_0$ in K_μ.
(3) $M_0 \leq M_1 \lhd M_2$ implies $M_0 \lhd M_2$.

Note that this implies that $\lhd$ is a strict partial ordering on K_μ extending $<$.

For $0 < \delta < \mu^+$, a model $M \in K_\mu$ is $(\delta, \lhd)$-*limit* if there exists a $\lhd$-increasing chain $(M_i)_{i<\delta}$ in K_μ such that $M = \bigcup_{i<\delta} M_i$. M is $\lhd$-*limit* if there exists a limit δ such that M is $(\delta, \lhd)$-limit.

Remark 2.14. Assume K_μ has amalgamation, no maximal models, and is stable. Then by Proposition 2.12, for any $0 < \delta < \mu^+$, $<_{\mu,\delta}$ is a good ordering on K_μ. Moreover, the existence of *any* good ordering on K_μ implies that $<_{\text{univ}}$ is a good ordering, and hence that K_μ has amalgamation, no maximal models, and is stable.

Let $\text{LS}(K) \le \mu < \lambda$. Even assuming stability everywhere, is is unclear whether there should be any μ^+-model-homogeneous model in λ (think for example of the case $\text{cf}(\lambda) = \omega$). The following tells us we can at least get an approximation to one:

Lemma 2.15. Let $\text{LS}(K) \le \mu^+ < \lambda$. Assume $K_{[\mu,\lambda)}$ has amalgamation, no maximal models, and is stable in μ' for unboundedly many $\mu < \mu' < \lambda$ (that is, for any $\mu < \mu' < \lambda$, there exists $\mu' \le \mu'' < \lambda$ such that $K_{\mu''}$ is stable).

For any $N_0 \in K_{[\mu,\lambda)}$, there exists $(N_i)_{i<\lambda}$ $<_{\text{univ}}$-increasing continuous in $K_{[\mu,\lambda)}$ and each N_{i+1}, is μ^+-model-homogeneous. Moreover any $M \ge N_0$ in $K_{[\mu,\lambda]}$ can be embedded inside $N := \bigcup_{i<\lambda} N_i$ over N_0.

Proof. We build $(N_i)_{i<\lambda}$ by induction. N_0 is already given and without loss of generality $||N_0|| \ge \mu^+$. Take unions at limits and for a given N_i, first take $N_i' \ge N_i$ such that $K_{||N_i'||}$ is stable, and iterate Fact 2.10 μ^+-many times to pick $N_{i+1} >_{\text{univ}} N_i'$ (and so by Lemma 2.12 also $N_{i+1} >_{\text{univ}} N_i$) in $K_{||N_i'||}$ that is also μ^+-model homogeneous.

Now given $M \ge N_0$ in $K_{[\mu,\lambda]}$, let $(M_i)_{i\le\lambda}$ be an increasing continuous resolution of M with $||M_i|| < \lambda$ for all $i < \lambda$ and $M_0 = N_0$. Inductively build $(f_i)_{i\le\lambda}$ an increasing continuous chain of K-embeddings such that for each $i \le \lambda$, $f_i : M_i \xrightarrow[M_0]{} N_i$. This is easy since $N_{i+1} >_{\text{univ}} N_i$ for all $i < \lambda$. Then f_λ embeds M into N. $\square$

2.4. **Good frames.** Good frames were first defined in [She09b]. The idea is to provide a localized (i.e. only for base models of a given size λ) axiomatization of a forking-like notion for (a "nice enough" set of) 1-types. Jarden and Shelah (in [JS13]) later gave a slightly more general definition, not assuming the existence of a superlimit model and dropping some of the redundant clauses. We will use a slight variation here: we assume the models come from $K_{\mathcal{F}}$, for $\mathcal{F}$ an interval, instead of just K_λ. We first adapt the definition of a pre-λ-frame from [She09e, Definition 0.2.1] to such an interval:

Definition 2.16 (Pre-Frame)**.** Let $\mathcal{F}$ be an interval of the form $[\lambda, \theta)$, where λ is a cardinal, and $\theta > \lambda$ is either a cardinal or ∞.

A *pre-$\mathcal{F}$-frame* is a triple $\mathfrak{s} = (K, \underset{\smile}{\cup}, S^{\text{bs}})$, where:

(1) K is an abstract elementary class with $\lambda \geq \mathrm{LS}(K)$, $K_\lambda \neq \emptyset$.

(2) S^{bs} is a function with domain $K_\mathcal{F}$ and if $M \in K_\mathcal{F}$, then $S^{\mathrm{bs}}(M) \subseteq S^{\mathrm{na}}(M)$.

(3) $\underset{}{\downarrow}$ is a relation on quadruples of the form (M_0, M_1, a, N), where $M_0 \leq M_1 \leq N$, $a \in N$, and M_0, M_1, N are all in $K_\mathcal{F}$. We write $\underset{}{\downarrow}(M_0, M_1, a, N)$ or $M_1 \underset{M_0}{\overset{N}{\downarrow}} a$ instead of $(M_0, M_1, a, N) \in \underset{}{\downarrow}$. We require $\underset{}{\downarrow}$ has the following properties:

  (a) Invariance: If $f : N \cong N'$ and $M_1 \underset{M_0}{\overset{N}{\downarrow}} a$, then $f[M_1] \underset{f[M_0]}{\overset{N'}{\downarrow}} f(a)$. If $p \in S^{\mathrm{bs}}(N)$, then $f(p) \in S^{\mathrm{bs}}(N')$.

  (b) Monotonicity: If $M_1 \underset{M_0}{\overset{N}{\downarrow}} a$, $M_0 \leq M_0' \leq M_1' \leq M_1 \leq N' \leq N \leq N''$ with $a \in N'$ and $N'' \in K_\mathcal{F}$, then $M_1' \underset{M_0'}{\overset{N'}{\downarrow}} a$ and $M_1' \underset{M_0'}{\overset{N''}{\downarrow}} a$.

We write λ-frame instead of $\{\lambda\}$-frame, $(\geq \lambda)$-frame instead of $[\lambda, \infty)$-frame. We sometimes drop the $\mathcal{F}$ when it is clear from context.

A pre-frame is *type-full* if $S^{\mathrm{bs}}(M) = S^{\mathrm{na}}(M)$ for all $M \in K_\mathcal{F}$.

For $\mathcal{F}' \subseteq \mathcal{F}$ an interval, we let $\mathfrak{s} \upharpoonright \mathcal{F}'$ denote the pre-$\mathcal{F}'$-frame defined in the obvious way by restricting the basic types and $\underset{}{\downarrow}$ to models in $K_{\mathcal{F}'}$. For $\lambda' \in \mathcal{F}$, we write $\mathfrak{s} \upharpoonright \lambda'$ instead of $\mathfrak{s} \upharpoonright \{\lambda'\}$.

Notation 2.17. If $\mathfrak{s} = (K, \underset{}{\downarrow}, S^{\mathrm{bs}})$ is a pre-$\mathcal{F}$-frame, we sometimes say "$\mathrm{tp}(a/M_1; N)$ does not fork over M_0" for $M_1 \underset{M_0}{\overset{N}{\downarrow}} a$. This makes sense by the monotonicity and invariance axioms.

Definition 2.18 (Good frame)**.** Let $\mathcal{F}$ be as above.

A *good $\mathcal{F}$-frame* is a pre-$\mathcal{F}$-frame $(K, \underset{}{\downarrow}, S^{\mathrm{bs}})$ satisfying in addition:

  (1) $K_\mathcal{F}$ has amalgamation, joint embedding, and no maximal model.

  (2) bs-Stability: $|S^{\mathrm{bs}}(M)| \leq ||M||$ for all $M \in K_\mathcal{F}$.

  (3) Density of basic types: If $M < N$ and $M, N \in K_\mathcal{F}$, then there is $a \in N$ such that $\mathrm{tp}(a/M; N) \in S^{\mathrm{bs}}(M)$.

  (4) Existence: $p \in S^{\mathrm{bs}}(M)$ if and only if p does not fork over M.

  (5) Extension: If $p \in S(N)$ does not fork over M, and $N' \geq N$ is in $K_\mathcal{F}$, then there is $q \in S(N')$ extending p that does not fork over M.

  (6) Uniqueness: If $p, q \in S(N)$ do not fork over M and $p \upharpoonright M = q \upharpoonright M$, then $p = q$.

  (7) Symmetry: If $M_2 \underset{M_0}{\overset{N}{\downarrow}} a_1$, $a_2 \in M_2$, and $\mathrm{tp}(a_2/M_0; N) \in S^{\mathrm{bs}}(M_0)$, then there is M_1 containing a_1 and $N' \geq N$ such that $M_1 \underset{M_0}{\overset{N'}{\downarrow}} a_2$.

(8) Local character: If δ is a limit ordinal, $(M_i)_{i \leq \delta}$ is an increasing chain in $K_{\mathcal{F}}$ with $M_\delta = \bigcup_{i<\delta} M_i$, and $p \in S^{\mathrm{bs}}(M_\delta)$, then there exists $i < \delta$ such that p does not fork over M_i.

(9) Continuity: If δ is a limit ordinal, $(M_i)_{i \leq \delta}$ is an increasing chain in $K_{\mathcal{F}}$ with $M_\delta = \bigcup_{i<\delta} M_i$, $p \in S(M_\delta)$ is so that $p \restriction M_i$ does not fork over M_0 for all $i < \delta$, then p does not fork over M_0.

(10) Transitivity[2]: If $M_0 \leq M_1 \leq M_2$, $p \in S(M_2)$ does not fork over M_1 and $p \restriction M_1$ does not fork over M_0, then p does not fork over M_0.

When we talk of a good $\mathcal{F}$-frame without a given property P, we mean that the frame is not required to satisfy P: it may or may not have it.

We say that K has a good $\mathcal{F}$-frame [without P] if there is a good $\mathcal{F}$-frame [without P] where K is the underlying AEC.

Remark 2.19. Using $\mathcal{F}$ instead of just λ is only a convenience: just like an abstract elementary class K is determined by $K_{\mathrm{LS}(K)}$, a good $\mathcal{F}$-frame without symmetry $\mathfrak{s}$ is determined by $(\mathfrak{s} \restriction \min(\mathcal{F}))$. See [She09b, Section 2] for more on this.

3. A POOR MAN'S FRAME FROM NONSPLITTING

Hypothesis 3.1.

(1) K is an abstract elementary class. $\mu \geq \mathrm{LS}(K)$ is a cardinal. $K_\mu \neq \emptyset$.
(2) K_μ has amalgamation.

In this section, we start our quest for a good frame. Recall the following generalization of first-order splitting from [She99, Definition 3.2]:

Definition 3.2. For $p \in S(N)$, we say that p μ-*splits over* M if $M \leq N$ and there exists $N_1, N_2 \in K_\mu$ so that $M \leq N_\ell \leq N$ for $\ell = 1, 2$, and there is $h : N_1 \cong_M N_2$ such that $h(p \restriction N_1) \neq p \restriction N_2$.

When μ is clear from context, we drop it.

Remark 3.3 (Monotonicity). If $p \in S(N)$ does not μ-split over M and $M \leq M' \leq N' \leq N$ are in K_μ, then $p \restriction N'$ does not μ-split over M'.

Remark 3.4. If $\mathfrak{s} = (K, \underset{\smile}{\cup}, S^{\mathrm{bs}})$ is a good μ-frame without symmetry, and $p \in S(N)$ does not fork over M, then p does not μ-split over M (this is an easy exercise in the use of the uniqueness property, see e.g. [BGKV, Lemma 4.2]). Thus splitting can be seen as a first approximation to a forking notion.

Our starting point will be the following extension and uniqueness properties of splitting, first isolated by VanDieren [Van02, Theorem II.7.9, Theorem II.7.11].

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[2]This actually follows from uniqueness and extension, see [She09b, Claim 2.18].

Intuitively, they tell us that the usual uniqueness and extension property of a nonforking notion hold provided we have enough room (concretely, the base model has to be "shifted" by a universal extension).

Fact 3.5. Let $M_0 <_{\text{univ}} M \leq N$ with $M_0, M, N \in K_\mu$. Then:

    (1) Weak uniqueness: If $p_\ell \in S(N)$ does not split over M_0, $\ell = 1, 2$, $p_1 \restriction M = p_2 \restriction M$, then $p_1 = p_2$.

    (2) Weak extension: If $p \in S(M)$ does not split over M_0, then there exists $q \in S(N)$ extending p that does not split over M_0. Moreover, q is nonalgebraic if p is.

Proof. See [Van06, Theorem I.4.12] for weak uniqueness. For weak extension, use universality to get $h : N \xrightarrow{M_0} M$. Further extend h to an isomorphism $\hat{h} : \widehat{N} \cong_{M_0} \widehat{M}$. So that $\widehat{M}$ contains a realization a of p. Let $a' := \hat{h}^{-1}(a)$, and let $q := \text{tp}(a/N; \widehat{N})$. The proof of [Van06, Theorem I.4.10] shows q is indeed an extension of p that does not split over M_0. In addition if q is algebraic, $a' \in N$ so $a = h(a') \in M$, so p is algebraic. $\square$

We will mostly use those two properties instead of the exact definition of splitting. However, they characterize splitting in the following sense:

Proposition 3.6. Assume K_μ has amalgamation, no maximal models, and is stable. Let $\mathfrak{s}$ be a type-full pre-μ-frame with underlying AEC K. The following are equivalent.

    (1) For all types $p \in S(N)$, $N \in K_\mu$ and $M \leq N$, if p does not fork over M, then for any $M <_{\text{univ}} M' \leq N$, p does not split over M'.

    (2) Nonforking satisfies weak uniqueness and weak extension (i.e. the conclusion of Fact 3.5 holds with "split" replaced by "fork").

Proof. Left as an exercise (not used). $\square$

We also obtain a weak transitivity property:

Proposition 3.7 (Weak transitivity of nonsplitting). Let $M_0 \leq M_1 <_{\text{univ}} M_1' \leq M_2$ in K_μ. Let $p \in S(M_2)$. If $p \restriction M_1'$ does not split over M_0 and p does not split over M_1, then p does not split over M_0.

Proof. By weak extension, find $q \in S(M_2)$ extending $p \restriction M_1'$ and not splitting over M_0. By monotonicity, q does not split over M_1. By weak uniqueness, $p = q$, as needed. $\square$

We now turn to building a forking notion that will satisfy true uniqueness and extension (see Definition 2.18) in K_μ. The idea is simple enough: we shift back the base model by a universal extension.

Definition 3.8 (Nonforking). For $M_0 \leq M \leq N$ in K_μ, say $p \in S(N)$ *does not fork over* (M_0, M) if:

(1) $M_0 <_{\text{univ}} M \leq N$.
(2) p does not split over M_0.

Say p *does not fork over* M if there exists M_0 so that p does not fork over (M_0, M).

When we say that a type p does not fork over (M_0, M), we think of M as the base, and M_0 as the particular witness to the nonforking. It would be nice if we could get rid of the witness entirely and get that nonforking satisfies extension and uniqueness, but uniqueness seems to depend on the particular witness.

If instead we define p does not fork* over M iff for all $M_0 <_{\text{univ}} M$ there exists $M_0 \leq M_0' <_{\text{univ}} M$ so that p does not fork over (M_0', M) then extension and uniqueness hold, but local character (assuming local character of nonsplitting) is problematic. Thus it seems we have to carry along the witness in our investigation.

Lemma 3.9 (Basic properties of nonforking). Below, all models are in K_μ.

(1) Monotonicity: If $p \in S(N)$ does not fork over (M_0, M), $M_0 \leq M_0' \leq M \leq M' \leq N' \leq N$ and $M_0' <_{\text{univ}} M'$, then $p \restriction N'$ does not fork over (M_0', M'). In particular, if $p \in S(N)$ does not fork over M and $M \leq M' \leq N' \leq N$, then $p \restriction N'$ does not fork over M'.
(2) Extension: If $p \in S(N)$ does not fork over (M_0, M) and $N' \geq N$, then there is $q \in S(N')$ extending p that does not fork over (M_0, M). If p is nonalgebraic, then q is nonalgebraic.
(3) Uniqueness: If $p_\ell \in S(N)$ does not fork over (M_0, M), $\ell = 1, 2$, and $p_1 \restriction M = p_2 \restriction M$, then $p_1 = p_2$.
(4) Transitivity: Let $M_1 \leq M_2 \leq M_3$ and let $p \in S(M_3)$. If $p \restriction M_2$ does not fork over (M_0, M_1) and p does not fork over (M_0', M_2) for $M_0 \leq M_0'$, then p does not fork over (M_0, M_1).
(5) Nonalgebraicity: If $p \in S(N)$ does not fork over M and $p \restriction M$ is not algebraic, then p is not algebraic.

Proof. Monotonicity follows directly from the definition (and Proposition 2.12.(2)), extension and uniqueness are just restatements of Fact 3.5, and transitivity is a restatement of Proposition 3.7. For nonalgebraicity, assume $p \restriction M$ is nonalgebraic. Then it has a nonalgebraic nonforking extension to N by extension, and this extension must be p by uniqueness, so the result follows. □

Assuming some local character for nonsplitting, we obtain the local character and continuity properties:

Definition 3.10. Let R be a binary relation on K_μ, and let κ be a regular cardinal. We say that μ-nonsplitting has κ-*local character for R-increasing chains* if for any

R-increasing $(M_i)_{i \leq \delta}$ with $\mathrm{cf}(\delta) \geq \kappa$, $M_\delta = \bigcup_{i < \delta} M_i$, and any $p \in S(M_\delta)$, there is $i < \delta$ so that p does not split over M_i.

Remark 3.11. If K_μ is stable, then by [GV06b, Fact 4.6], then μ-nonsplitting has μ^+-local character for $\leq$-increasing chains.

Lemma 3.12. Let $\lhd$ be a good ordering on K_μ, and let κ be a regular cardinal. Assume nonsplitting has κ-local character for $\lhd$-increasing chains. Then:

  (1) κ-local character for $\lhd$-increasing chains: If $(M_i)_{i \leq \delta}$ is a $\lhd$-increasing chain in K_μ with $\mathrm{cf}(\delta) \geq \kappa$, $M_\delta = \bigcup_{i < \delta} M_i$ and $p \in S(M_\delta)$, then there exists $i < \delta$ so that p does not fork over (M_i, M_{i+1}).
  (2) Continuity for $\lhd$-increasing chains: If $(M_i)_{i \leq \delta}$ is a $\lhd$-increasing chain in K_μ with $\mathrm{cf}(\delta) \geq \kappa$, $M_\delta = \bigcup_{i < \delta} M_i$ and $p \in S(M_\delta)$ such that $p \restriction M_i$ does not fork over M_0 for all $i < \delta$, then p does not fork over M_0. Moreover, if in addition $p \restriction M_i$ does not fork over (M_0', M_0) for all $i < \delta$ (i.e. the witness is always the same), then p does not fork over (M_0', M_0).
  (3) Existence over $(\geq \kappa, \lhd)$-limits: If $M \in K_\mu$ is $(\delta, \lhd)$-limit for some δ with $\mathrm{cf}(\delta) \geq \kappa$, then any $p \in S(M)$ does not fork over M. In fact, if $p_0, ..., p_{n-1} \in S(M)$, $n < \omega$, then there exists $M_0 <_{\mathrm{univ}} M$ such that p_i does not fork over (M_0, M) for all $i < n$.

Proof.

  (1) Follows from κ-local character of nonsplitting for $\lhd$-increasing chains.
  (2) By κ-local character, there exists $i < \delta$ so that p does not fork over (M_i, M_{i+1}). By assumption, there exists $M_0' <_{\mathrm{univ}} M_0$ so that $p \restriction M_{i+1}$ does not fork over (M_0', M_0). Since $M_0' \leq M_i$, we can apply transitivity to obtain that p does not fork over (M_0', M_0). The proof of the moreover part is similar.
  (3) By local character and monotonicity.

$\square$

Thus if nonsplitting has $\aleph_0$-local character for $\lhd$-increasing chains for some good ordering $\lhd$ and if all models in K_μ are $\lhd$-limit (e.g. if K_μ is categorical), then we are very close to having a good μ-frame without symmetry (and with weaker local character and continuity) but we must carry along the witness. In the next section, we use tameness to transfer our poor man's frame in μ to a better one above μ.

4. Going up without tameness

Hypothesis 4.1.

  (1) K is an abstract elementary class. $\mu \geq \mathrm{LS}(K)$ is a cardinal. $K_\mu \neq \emptyset$.

(2) $\lhd$ is a good ordering on K_μ. In particular (by Remark 2.14), K_μ has amalgamation, no maximal models, and is stable.

In [She09b, Section 2], Shelah showed how to extend a good μ-frame to all models in $K_{\geq\mu}$. The resulting object will in general not be a good frame, but several of the properties are nevertheless preserved. In this section, we apply the same procedure on our poor man's frame and show Shelah's arguments still go through (assuming the base models are μ^+-homogeneous).

Definition 4.2 (Nonforking in $K_{\geq\mu}$)**.** For $N \in K_{\geq\mu}$, $M_0 \leq M \in K_\mu$, $p \in S^{\mathrm{na}}(N)$, we say that p *does not fork over* (M_0, M) if for all $M \leq N_0 \leq N$ with $N_0 \in K_\mu$ we have $p \upharpoonright N_0$ does not fork over (M_0, M) (see Definition 3.8). Equivalently, p does not μ-split over M_0 and $M_0 <_{\mathrm{univ}} M \leq N$.

We say that p *does not fork over* M if for all $N_0 \leq N$ in K_μ, $p \upharpoonright N_0$ does not fork over M.

For $M \in K_{\geq\mu}$, we say that p *does not fork over* M if there exists $M_0 \leq M$ in K_μ such that p does not fork over M_0.

We define the relation $\underset{M}{\overset{\widehat{N}}{\downarrow}}$ by $N \underset{M}{\overset{\widehat{N}}{\downarrow}} a$ if and only if $M, N, \widehat{N} \in K_{\geq\mu}$, $a \in \widehat{N}$, and $\mathrm{tp}(a/N; \widehat{N})$ does not fork over M.

The following observation is very helpful and will be used in the proof of extension and uniqueness:

Lemma 4.3. Let $M \leq N$ in $K_{\geq\mu^+}$ and M is μ^+-model-homogeneous. Assume $p \in S(N)$ does not fork over M. Then there exists $M_0' \leq M_0 \leq M$ in K_μ such that p does not fork over (M_0', M_0) (i.e. nonforking over M_0 is witnessed by the same M_0' uniformly).

Proof. By definition, there is $M_0' \leq M$ in K_μ such that p does not fork over M_0'. Since M is μ^+-model-homogeneous, one can pick $M_0 >_{\mathrm{univ}} M_0'$ in K_μ with $M_0 \leq M$. By monotonicity (Lemma 3.9.(1)), p does not fork over (M_0', M_0). $\square$

Proposition 4.4. $\mathfrak{s}_0 := (K, \downarrow, S^{\mathrm{na}})$ is a type-full pre-$[\mu, \infty)$-frame.

Proof. The properties to check follow directly from the definition of nonforking. $\mathfrak{s}_0$ is type-full since we defined the basic types to be all the nonalgebraic types. $\square$

Lemma 4.5 (Existence)**.** Let $M \in K_{\geq\mu^+}$ be μ^+-model-homogeneous. Then $p \in S^{\mathrm{na}}(M)$ if and only if p does not fork over M.

Proof. If p does not fork over M, then p is nonalgebraic by definition. Now assume p is nonalgebraic. By [GV06b, Fact 4.6], there is $M_0 \leq M$ in K_μ such that p does not μ-split over M_0. Pick $M_0' >_{\mathrm{univ}} M_0$ in K_μ so that $M_0' \leq M$. This is possible by μ^+-model-homogeneity. Then p does not fork over M_0', as needed. $\square$

Lemma 4.6 (Transitivity). *If $M_0 \leq M_1 \leq M_2$ are in $K_{\geq \mu}$, M_1 is μ^+-model-homogeneous, $p \in S^{\mathrm{na}}(M_2)$ is such that $p \restriction M_1$ does not fork over M_0 and p does not fork over M_1, then p does not fork over M_0.*

Proof. Find $M_0' \leq M_0$ in K_μ such that $p \restriction M_1$ does not fork over M_0'. Using monotonicity, we can also find $M_0' \leq M_1' \leq M_1$ in K_μ such that p does not fork over M_1'. By model-homogeneity, we can pick $M_1' \lhd M_1'' \leq M_1$ in K_μ. We then have that p does not fork over (M_1', M_1''). By transitivity in K_μ (Lemma 3.9.(4)), p does not fork over M_0', and hence over M_0. $\square$

Lemma 4.7 (Local character). *Assume nonsplitting has κ-local character for $\lhd$-increasing chains. If $\mathrm{cf}(\delta) \geq \kappa$, $(M_i)_{i \leq \delta}$ is an increasing chain in $K_{\geq \mu^+}$ with $M_\delta = \bigcup_{i < \delta} M_i$, M_i is μ^+-model-homogeneous for $i < \delta$, and $p \in S^{\mathrm{na}}(M_\delta)$, then there is $i < \delta$ such that p does not fork over M_i.*

Proof. Without loss of generality, δ is regular. If $\delta \geq \mu^+$, then M_δ is also μ^+-model-homogeneous so one can pick $N^* \leq M_\delta$ in K_μ witnessing existence (use Lemma 4.5) and find $i < \delta$ with $N^* \leq M_i$, so p does not fork over M_i as needed. Now assume $\delta < \mu^+$. We imitate the proof of [She09b, Claim 2.11.5]. Assume the conclusion fails. Build $(N_i)_{i \leq \delta}$ $\lhd$-increasing continuous in K_μ, $(N_i')_{i \leq \delta}$ $\leq$-increasing continuous in K_μ such that for all $i < \delta$:

(1) $N_i \leq M_i$.
(2) $N_i \leq N_i' \leq M_\delta$.
(3) $p \restriction N_{i+1}'$ forks over (N_i, N_{i+1}).
(4) $\bigcup_{j \leq i}(N_j' \cap M_{i+1}) \subseteq |N_{i+1}|$.

This is possible. For $i = 0$, let $N_0 \leq M_0$ be any model in K_μ, and $M_\delta \geq N_0' \geq N_0$ be a model in K_μ. For i limit, take unions. For the successor case, assume $i = j + 1$. Choose $N_i \leq M_i$ satisfying (4) with $N_i \rhd N_j$ (possible since M_i is μ^+-model-homogeneous). By assumption, p forks over M_i, hence over (N_j, N_i), and so by definition of forking and monotonicity there exists $M_\delta \geq N_i' \geq N_i$, $N_i' \geq N_j'$ with $p \restriction N_i'$ forks over (N_j, N_i). It is as required.

This is enough. By local character in K_μ, there is $i < \delta$ such that $p \restriction N_\delta$ does not fork over (N_i, N_{i+1}). By (2) and (4), $N_\delta' \leq N_\delta$. Thus $p \restriction N_{i+1}'$ does not fork over (N_i, N_{i+1}), contradicting (3).

$\square$

Lemma 4.8 (Continuity). *Assume nonsplitting has κ-local character for $\lhd$-increasing chains. If $\mathrm{cf}(\delta) \geq \kappa$, $(M_i)_{i \leq \delta}$ is an increasing chain in $K_{\geq \mu^+}$ with $M_\delta = \bigcup_{i < \delta} M_i$, M_i μ^+-model-homogeneous for $i < \delta$, and $p \in S(M_\delta)$ is so that $p \restriction M_i$ does not fork over M_0 for all $i < \delta$, then p_δ does not fork over M_0.*

Proof. Follows directly from κ-local character and transitivity, see [She09b, Claim 2.17.3]. $\square$

Remark 4.9. In the statements of local character and continuity, we did *not* assume that M_δ was μ^+-model-homogeneous.

5. A tame good frame without symmetry

Hypothesis 5.1.

(1) K is an abstract elementary class. $\mu \geq \mathrm{LS}(K)$ is a cardinal. $K_\mu \neq \emptyset$.

(2) $\lhd$ is a good ordering on K_μ. In particular (by Remark 2.14), K_μ has amalgamation, no maximal models, and is stable.

(3) κ is the least regular cardinal such that nonsplitting has κ-local character for $\lhd$-increasing chains in K_μ.

(4) $\lambda > \mu$ is such that:
   (a) K is (μ, λ)-tame.
   (b) $K_{[\mu,\lambda]}$ has amalgamation.
   (c) $K_{[\mu,\lambda)}$ has no maximal models.

Remark 5.2. κ plays a similar role as the cardinal $\kappa(T)$ in the first-order setup. By Remark 3.11 and Hypothesis 5.1.(2), $\kappa \leq \mu^+$. In the end, we will be able to obtain a good frame only when $\kappa = \aleph_0$, but the general case is useful to investigate the stability spectrum.

Boney showed in [Bonb] that given a good μ-frame, tameness implies that Shelah's extension of the frame to $\geq \mu$ is actually a good frame. In this section, we apply the ideas of his proof (assuming the base models are μ^+-model-homogeneous) to our poor man's frame, i.e. we prove stability, extension and uniqueness using tameness. Note that uniqueness is actually *equivalent* to (λ, μ)-tameness by [Bonb, Theorem 3.2]. The easiest case is when $\lambda = \mu^+$. Then we know a model-homogeneous model exists in K_λ, and this simplifies some of the proofs.

Lemma 5.3 (Uniqueness)**.** Let $M \leq N$ be in $K_{[\mu,\lambda]}$. Let $p, q \in S(N)$. Assume $p \restriction M = q \restriction M$.

(1) If $M \in K_\mu$ and p, q do not fork over (M_0, M) for some $M_0 <_{\mathrm{univ}} M$, then $p = q$.

(2) If $M \in K_{[\mu^+,\lambda]}$ is μ^+-model-homogeneous and p, q do not fork over M, then $p = q$.

Proof. (1) follows from uniqueness in K_μ (Lemma 3.9.(3)) and tameness. To see (2), use monotonicity and Lemma 4.3, to find $M_0' <_{\mathrm{univ}} M_0 \leq M$ in K_μ such that both p and q do not fork over (M_0', M_0). Now apply (1). $\square$

Interestingly, we already have enough machinery to obtain a stability transfer theorem. First recall:

Fact 5.4. K_{μ^+} is stable.

Proof. This could be done using the method of proof of Theorem 5.6 (exercise), but this is also [BKV06, Theorem 1]. $\square$

Lemma 5.5. Assume that $\lambda > \mu^+$, $\mathrm{cf}(\lambda) \geq \kappa$, and there are unboundedly many $\mu \leq \lambda' < \lambda$ such that $K_{\lambda'}$ is stable (in the same sense as in the statement of Lemma 2.15). Then K_λ is stable.

Proof. Let $M \in K_\lambda$. By Lemma 2.15, M can be embedded inside some $\widehat{M} \in K_\lambda$ which can be written as $\bigcup_{i<\lambda} M_i$, with $(M_i)_{i<\lambda}$ an increasing chain[3] of μ^+-model-homogeneous models in $K_{[\mu^+,\lambda)}$. From amalgamation, we know that Galois types can be extended, so $|S(M)| \leq |S(\widehat{M})|$, and so we can assume without loss of generality that $M = \widehat{M}$. Let $(p_j)_{j<\lambda^+}$ be types in $S(M)$. By κ-local character, for each $j < \lambda^+$ there is $i_j < \lambda$ such that p_j does not fork over M_{i_j}. By the pigeonhole principle, we may assume $i_j = i_0$ for all $j < \lambda^+$. Taking i_0 bigger if necessary, we may assume that $K_{\|M_{i_0}\|}$ is stable. Thus $|S(M_{i_0})| \leq \|M_{i_0}\| \leq \lambda$, so by the pigeonhole principle again, we can assume that there is $q \in S(M_{i_0})$ such that $p_j \restriction M_{i_0} = q$ for all $j < \lambda^+$. By uniqueness, $p_j = p'_j$ for each $j \neq j'$, so the result follows. $\square$

We can now prove stability transfers up. Recall that λ is the cardinal above μ fixed in Hypothesis 5.1.(4). Recall also that we already have stability in μ by Hypothesis 5.1.(2).

Theorem 5.6 (The superstability theorem). If $\kappa = \aleph_0$, then K_λ is stable.

Proof. We work by induction on λ. If $\lambda = \mu^+$, this is Fact 5.4 and if $\lambda > \mu^+$ this is given by Lemma 5.5 and the induction hypothesis. $\square$

Assuming the generalized continuum hypothesis (GCH), we can also say something for arbitrary κ:

Theorem 5.7. Assume GCH. If $\lambda^{<\kappa} = \lambda$, then K_λ is stable.

Proof. By induction on λ. If $\lambda = \mu^+$, this is Fact 5.4, so assume $\lambda > \mu^+$. By König's theorem, $\mathrm{cf}(\lambda) \geq \kappa$. If λ is successor, then $\lambda^\mu = \lambda$ by GCH, so by [GV06b, Corollary 6.4], K is stable in λ. If λ is limit there exists a sequence of successor cardinals $(\lambda_i)_{i<\mathrm{cf}(\lambda)}$ increasing cofinal in λ with $\lambda_0 \geq \mu^+$. Since without loss of

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[3]Explicitly, we take $(N_i)_{i<\lambda}$ as given by Lemma 2.15 for some $N_0 \leq M$ in K_{μ^+}, and let $M_i := N_{i+1}$. Note that the chain $(M_i)_{i<\lambda}$ will *not* be continuous.

generality $\kappa \leq \mu^+$ (Remark 3.11), GCH implies that $\lambda_i^{<\kappa} = \lambda_i$, so by the induction hypothesis, K is stable in λ_i for all $i < \mathrm{cf}(\lambda)$. Apply Lemma 5.5 to conclude.   □

We now prove extension. This follows from compactness in the first-order case, but we make crucial use of the superstability hypothesis $\kappa = \aleph_0$ in the general case.

Lemma 5.8. Assume $\kappa = \aleph_0$. Let $\delta < \lambda^+$ be a limit ordinal. Assume $(M_i)_{i \leq \delta}$ is an increasing continuous sequence in $K_{[\mu, \lambda)}$ with $M_0 \in K_\mu$. Let $(p_i)_{i < \delta}$ be an increasing continuous sequence of types with $p_i \in S(M_i)$ for all $i < \delta$, and p_i does not fork over (M_0', M_0). Assume that one of the following holds:

  (1) $(M_i)_{i < \delta}$ is $\lhd$-increasing in K_μ.
  (2) For all $i < \delta$, M_{i+1} is μ^+-model-homogeneous.

Then there exists a unique $p_\delta \in S(M_\delta)$ extending each p_i and not forking over (M_0', M_0).

Proof. This is similar to the argument in [GV06a, Corollary 2.22], but we give some details. We focus on (1) (the proof of the other case is completely similar). Build by induction $(f_{i,j})_{i < j < \delta}$, $(a_i)_{i < \delta}$, and increasing continuous $(N_i)_{i < \delta}$ such that for all $i < j < \delta$:

  (1) $M_i \leq N_i$, $a_i \in N_i$.
  (2) $f_{i,j} : N_i \to N_j$.
  (3) For $j < k < \delta$, $f_{j,k} \circ f_{i,j} = f_{i,k}$.
  (4) $f_{i,j}$ fixes M_i.
  (5) $f_{i,j}(a_i) = a_j$.
  (6) $p_i = \mathrm{tp}(a_i/M_i; N_i)$.

This is enough. Let $(N_\delta, (f_{i,\delta})_{i < \delta})$ be the direct limit of the system $(N_i, f_{i,j})_{i < j < \delta}$, and let $a_\delta := f_{0,\delta}(a_0)$, $p_\delta := \mathrm{tp}(a_\delta/M_\delta; N_\delta)$. One easily checks that p_δ extends each p_i, $i < \delta$, and so using continuity for $\lhd$-increasing chains (Lemma 3.12.(2)), does not fork over (M_0', M_0). Finally, p_δ is unique by Lemma 5.3.

This is possible. For $i = 0$, we take a_0 and N_0 so that $\mathrm{tp}(a_0/M_0; N_0) = p_0$. For i limit, we let $(N_i, f_{i_0,i})_{i_0 < i}$ be the direct limit of the system $(N_{i_0}, f_{i_0,j_0})_{i_0 < j_0 < i}$, and let $a_i := f_{0,i}(a_0)$. By continuity for $\lhd$-increasing chains, $\mathrm{tp}(a_i/M_i; N_i)$ does not fork over (M_0', M_0), and so by uniqueness, it must equal p_i. For $i = i_0 + 1$ successor, find a_i and $N_i' \geq M_i$ such that $p_i = \mathrm{tp}(a_i/M_i; N_i')$. Since $p_i \upharpoonright M_{i_0} = p_{i_0}$, we can use the definition of types to amalgamate N_{i_0} and N_i' over M_{i_0}: there exists $N_i \geq N_i'$ and $f_{i_0,i} : N_{i_0} \xrightarrow[M_{i_0}]{} N_i$ so that $f_{i_0,i}(a_{i_0}) = a_i$. Define $f_{i_0',i} := f_{i_0,i} \circ f_{i_0',i_0}$ for all $i_0' < i_0$.   □

Lemma 5.9 (Extension). Assume $\kappa = \aleph_0$. Let $M \leq N$ be in $K_{[\mu^+,\lambda]}$ with M and N μ^+-model-homogeneous, and let $p \in S^{\mathrm{na}}(M)$. Then there is $q \in S(N)$ extending p that does not fork over M.

Proof. We imitate the proof of [Bonb, Theorem 5.3]. By existence and Lemma 4.3, there exists $M_0' <_{\mathrm{univ}} M_0 \leq M$ in K_μ with p nonforking over (M_0', M_0). Work by induction on λ. If $N \in K_{<\lambda}$, use the induction hypothesis, so assume $N \in K_\lambda$. There are two cases: either $\lambda = \mu^+$ or $\lambda > \mu^+$.

Assume first $\lambda > \mu^+$. By transitivity and Lemma 2.15, we can assume without loss of generality that $N = \bigcup_{i<\lambda} N_i$, where $(N_i)_{i\leq\lambda}$ is a $<_{\mathrm{univ}}$-increasing continuous chain in $K_{[\mu^+,\lambda)}$, each N_{i+1} is μ^+-model-homogeneous, and N_0 extends M_0. Now inductively build a $\leq$-increasing continuous $(M_i)_{i\leq\lambda}$ with $M_\lambda = M$ so that $M_0 \leq M_i \leq N_i$ for all $i < \lambda$ (we allow repetitions). Set $p_i := p \restriction M_i$ and note that by monotonicity, p_i does not fork over (M_0', M_0).

We inductively build an increasing $(q_i)_{i\leq\lambda}$ with $q_i \in S(N_i)$, $p_i \leq q_i$, and q_i does not fork over (M_0', M_0). For $i = 0$, use extension in $K_{<\lambda}$ to find q_0 as needed. For $i = j + 1$, use extension to find a nonforking extension $q_i \in S(N_i)$ of q_j that does not fork over (M_0', M_0). By uniqueness, $q_i \geq p_i$. At limits, use Lemma 5.8 and uniqueness. $q := q_\lambda$ is as desired.

If $\lambda = \mu^+$, the construction is exactly the same except we use extension in K_λ at successor steps and the first case of Lemma 5.8 at limit steps. Note that since N is μ^+-model-homogeneous, $N = \bigcup_{i<\mu^+} N_i$, where $(N_i)_{i<\mu^+}$ is a $\lhd$-increasing continuous chain in K_μ. $\qquad\square$

Definition 5.10. Let $\mathfrak{s} := \mathfrak{s}_0 \restriction \lambda$, where $\mathfrak{s}_0$ is the pre-frame from Proposition 4.4.

Corollary 5.11. Assume:

(1) $\kappa = \aleph_0$.
(2) K_μ has joint embedding.
(3) K_λ has no maximal models.
(4) All the models in K_λ are μ^+-model-homogeneous.

Then $\mathfrak{s}$ is a type-full good λ-frame without symmetry.

Proof. It is easy to see $\mathfrak{s}$ is a type-full pre-λ-frame. K_λ has amalgamation and no maximal models by hypothesis. It has joint embedding since K_μ has joint embedding and $K_{[\mu,\lambda]}$ has amalgamation (see Lemma 2.3). Stability holds by Theorem 5.6. Density of basic types is always true in a type-full frame. For the other properties, see Lemmas 4.5, 4.6, 4.7, 4.8, 5.3, and 5.9 (note that the original statement of extension in Definition 2.18 follows from Lemma 5.9 and transitivity). $\qquad\square$

Lemma 5.12. Assume K is categorical in λ and $\kappa = \aleph_0$. Then:

(1) $K_{[\mu,\lambda]}$ has joint embedding and K_λ (and hence $K_{[\mu,\lambda]}$) has no maximal models.
(2) If in addition $\text{cf}(\lambda) > \mu$, then all the models in K_λ are μ^+-model-homogeneous.

Proof. To see (2), use stability to build $(M_i)_{i \leq \lambda} <_{\text{univ}}$-increasing continuous with $M_i \in K_{[\mu,\lambda)}$ for all $i < \lambda$. Then M_λ is μ^+-model-homogeneous.

For (1), K_λ has joint embedding by categoricity so use that $K_{[\mu,\lambda]}$ has no maximal models. To see K_λ has no maximal model, let $N \in K_\lambda$ be given. First assume $\lambda = \mu^+$. Build a $\lhd$-increasing continuous chain $(M_i)_{i \leq \mu^+}$, and $a \in N$ such that for all $i < \mu^+$:

(1) $M_i \in K_\mu$, $M_i \leq N$.
(2) $a \notin M_0$.
(3) $\text{tp}(a/M_i; N)$ does not fork over M_0.

This is enough. $M_{\mu^+} \in K_{\lambda^+}$. Moreover by Lemma 3.9.(5), $a \notin M_i$ for all $i < \mu^+$, so $a \notin M_{\mu^+}$. Thus $M_{\mu^+} < N$. By categoricity, the result follows.

This is possible. Pick a $\lhd$-limit $M_0 \leq N_1$ in K_μ (this is possible by model-homogeneity of N_1), and pick any $a \in N \backslash M_0$. At limits, take unions and use continuity (Lemma 3.12.(2)) to see the requirements are maintained. For a successor $i = j + 1$, use extension and some renaming. In details, pick an arbitrary $M_i' \rhd M_j$ with $M_i' \leq N$ (possible by model-homogeneity). By extension (Lemma 3.9.(2)), there is $q \in S(M_i')$ that does not fork over M_0 and extends $p_j := \text{tp}(a/M_j; N)$. Since N is saturated, there is $a' \in N$ realizing q. Pick $N \geq N_i \geq M_i'$ containing a' and a. By assumption, $\text{tp}(a'/M_j; N_i) = p_j = \text{tp}(a/M_j; N_i)$. Thus there is $N_i' \geq N_i$ and $f : N_i \xrightarrow[M_j]{} N_i'$ such that $f(a') = a$ and without loss of generality $N_i' \leq N$. Let $M_i := f[M_i']$ and use invariance to see it is as desired.

If $\lambda > \mu^+$, the proof is completely similar: if there is $N_1 > N$, we are done, so assume not. Then amalgamation implies N must be model-homogeneous. Build a $<_{\text{univ}}$-increasing continuous $(M_i)_{i \leq \lambda}$ and $a \in N$ such that for all $i < \lambda$:

(1) $M_i \in K_{[\mu^+,\lambda)}$, $M_i \leq N$.
(2) M_{i+1} is μ^+-model-homogeneous.
(3) $\text{tp}(a/M_i; N)$ does not fork over M_0.

As before, this is possible and the result follows. $\square$

Corollary 5.13. Assume K is categorical in λ, $\kappa = \aleph_0$, and $\text{cf}(\lambda) > \mu$, then $\mathfrak{s}$ is a type-full good λ-frame without symmetry.

Proof. Lemma 5.12 tells us all the hypotheses of Corollary 5.11 are satisfied. $\square$

Note that categoricity in λ is not the only hypothesis giving that all models in K_λ μ^+-model-homogeneous. For example:

Fact 5.14 (Theorem 5.4 in [BG])**.** Assume K has amalgamation, is categorical in a cardinal θ so that K_θ has a μ^+-model-homogeneous model (this holds if e.g. $\theta^\mu = \theta$). Then every member of $K_{\geq \chi}$ is μ^+-model-homogeneous, where $\chi :=$ $\min(\theta, \sup_{\gamma < \mu} \beth_{(2^\gamma)^+})$.

6. Getting symmetry

Hypothesis 6.1. $\mathfrak{s} = (K, \underset{\smile}{\downarrow}, S^{\text{bs}})$ is a good $\mathcal{F}$-frame without symmetry, where $\mathcal{F}$ is an interval of the form $[\lambda, \theta)$, λ is a cardinal, and $\theta > \lambda$ is either a cardinal or ∞.

The definition of a good frame only tells us that we have stability for basic types. However, Shelah showed:

Fact 6.2 ([She09b], Claim 4.2.1)**.** For any $\lambda' \in \mathcal{F}$, K is stable in λ'.

We would like to give conditions under which $\mathfrak{s}$ has symmetry. The first observation is that it is enough to consider models in K_λ:

Fact 6.3 ([BVa])**.** $\mathfrak{s}$ has symmetry if and only if $\mathfrak{s} \upharpoonright \lambda$ has symmetry.

Since we are not assuming anything about how $\mathfrak{s}$ is defined, we will work by contradiction: We will show that if θ is big enough and symmetry fails, then we get the order property, a nonstructure property which contradicts Fact 6.2. This is how the symmetry property of forking was originally proven in the first-order context, see [She90, Theorem III.4.13]. The same approach was later used in a non-elementary context in [She75, Theorem 5.1], and generalized in [BGKV, Theorem 5.14]. We will rely on the proof of the latter.

The definitions and fact below do not need Hypothesis 6.1.

Definition 6.4. Let α, χ and γ be cardinals. A model N has the (α, χ)-*order property of length* γ if there exists $M \leq N$ of size $\leq \chi$ (we also allow M to be empty) and $(\bar{a}_i)_{i < \gamma}$, $\bar{a}_i \in {}^\alpha N$ so that for any $i_0 < i_1 < \gamma$ and $j_0 < j_1 < \gamma$, $\text{tp}(\bar{a}_{i_0} \bar{a}_{i_1} / A; N) \neq \text{tp}(\bar{a}_{j_1} \bar{a}_{j_0} / A; N)$. If $\chi = 0$, we omit it.

K has the (α, χ)-*order property of length* γ if some $N \in K$ has it. K has the (α, χ)-*order property* if it has the (α, χ)-order property for all lengths (we sometimes also say K has the (α, χ)-order property of length ∞). K has the *order property* if it has the α-order property for some α.

Remark 6.5. If N has the (α, χ)-order property of length γ, then it has the $(\alpha + \chi)$-order property of length γ.

Definition 6.6. Given a cardinal χ, define $h(\chi) := \beth_{(2^\chi)^+}$.

Fact 6.7.

(1) If K has the (α, χ)-order property of length $h(\alpha + \chi + \mathrm{LS}(K))$, then K has the (α, χ)-order property.

(2) If K has the (α, χ)-order property, then it is α-unstable in χ' for all $\chi' \geq \chi$.

Proof. The statements essentially appear in [She99, Claim 4.5.3, Claim 4.7.2]. The proof of (1) is an application of Morley's method together with Shelah's presentation theorem, and a proof of a statement similar to (2) is sketched in [BGKV, Fact 5.13].   □

Fact 6.8. If $\mathfrak{s}$ does not have symmetry, then K has the $(2, \lambda)$-order property of length θ.

Proof. By Fact 6.3, $\mathfrak{s} \restriction \lambda$ does not have symmetry. The result now follows by exactly the same proof as [BGKV, Corollary 5.15].   □

Corollary 6.9. If $\theta \geq h(\lambda)$, then $\mathfrak{s}$ has symmetry.

Proof. If $\mathfrak{s}$ does not have symmetry, then by Theorem 6.8 and Fact 6.7.(1), K has the $(2, \lambda)$-order property and hence by Fact 6.7.(2) is 2-unstable in λ. By Fact 2.7, K is unstable in λ, contradicting Fact 6.2.   □

Thus it seems quite a big gap between λ and θ is needed. On the other hand, [Bonb] tells us that with enough tameness we can make $\mathcal{F}$ bigger:

Fact 6.10. Let $\theta' \geq \theta$ and let $\mathcal{F}' := [\lambda, \theta')$. Assume $K_{\mathcal{F}'}$ has amalgamation and no maximal models, and K is (λ, θ')-tame. Then $\mathfrak{s}$ can be extended to a good $[\lambda, \theta')$-frame without symmetry. If $\mathfrak{s}$ is type-full, then the extended frame will also be type-full.

Proof. By Remark 2.19, $\mathfrak{s}$ is determined by $\mathfrak{s} \restriction \lambda$, so we can apply [Bonb, Theorem 1.1]: its proof only uses the tameness for 2-types hypothesis to obtain symmetry. Note that no maximal models follows from symmetry there, so we need to assume it as an additional hypothesis. Note also that joint embedding follows from Lemma 2.3. The proof of Lemma 4.7 gives us that the extended frame is type-full if $\mathfrak{s}$ is.   □

Remark 6.11. We could replace (λ, θ')-tameness by (λ', θ')-tameness in the above, where $\lambda' \in \mathcal{F}$. This turns out to be equivalent (at least if we speak of tameness for basic types) since the uniqueness property of $\mathfrak{s}$ gives us (λ, λ')-tameness for basic types.

Corollary 6.12. Let $\mathcal{F}' := [\lambda, h(\lambda))$. Assume $K_{\mathcal{F}'}$ has amalgamation and no maximal models, and K is $(\lambda, < h(\lambda))$-tame. Then $\mathfrak{s}$ has symmetry.

Proof. Using Fact 6.10, we can assume without loss of generality that $\mu \geq h(\lambda)$. Now use Corollary 6.9.   □

7. THE MAIN THEOREMS

We finally have our promised good frame:

Theorem 7.1. Assume:

(1) K is an abstract elementary class. $\mu \geq \mathrm{LS}(K)$ is a cardinal.
(2) $K_\mu \neq \emptyset$ has joint embedding.
(3) $\lhd$ is a good ordering on K_μ. In particular (by Remark 2.14), K_μ has amalgamation, no maximal models, and is stable.
(4) Nonsplitting has $\aleph_0$-local character for $\lhd$-increasing chains in K_μ.
(5) $\lambda > \mu$ is such that:
   (a) K is $(\mu, < h(\lambda))$-tame.
   (b) $K_{[\mu, h(\lambda))}$ has amalgamation and no maximal models.
   (c) All the models in K_λ are μ^+-model-homogeneous.

Then K has a type-full good $[\lambda, h(\lambda))$-frame.

Proof. Corollary 5.11, gives us a good λ-frame $\mathfrak{s}$ without symmetry. $\mathfrak{s}$ can be extended (by Fact 6.10) to a good $[\lambda, h(\lambda))$ frame which will also have symmetry by Corollary 6.9. $\qquad\square$

We can use categoricity to get rid of some of the hypotheses above. We will use:

Fact 7.2. Assume K has amalgamation, joint embedding and no maximal models. Assume K is categorical in λ. Then:

(1) K is stable in all $\mathrm{LS}(K) \leq \mu < \lambda$.
(2) For any $\mathrm{LS}(K) \leq \mu < \mathrm{cf}(\lambda)$ and any limit $\delta < \mu^+$, μ-nonsplitting has $\aleph_0$-local character for $\lhd$-chains, where $\lhd := <_{\mu, \delta}$.
(3) Let $h_2 := h(h(\mathrm{LS}(K)))$. Assume λ is a successor cardinal and $\lambda > \lambda_0 \geq h_2$. Then K is (h_2, λ_0)-tame and categorical in λ_0. In addition, the model of size λ_0 is saturated.

Proof. (1) is [She99, Claim 1.7]. (2) is [She99, Lemma 6.3], and (3) is [She99, Main Claim II.2.3] and [She99, Theorem II.2.7]. $\qquad\square$

Theorem 7.3. Let K be an abstract elementary class and let λ be a cardinal such that $\mathrm{cf}(\lambda) > \mu \geq \mathrm{LS}(K)$. Let $\mathcal{F} := [\lambda, h(\lambda))$, $\mathcal{F}' := [\mu, h(\lambda))$. Assume:

(1) $K_{\mathcal{F}'}$ has amalgamation and no maximal models.
(2) K_λ is categorical.
(3) K is $(\mu, < h(\lambda))$-tame.

Then K has a type-full good $\mathcal{F}$-frame.

Proof. First, $K_\lambda \neq \emptyset$ by categoricity. By Lemma 5.12, $K_{\mathcal{F}'}$ has joint embedding and all models in K_λ are μ^+-model-homogeneous. By Fact 7.2, μ-nonsplitting has $\aleph_0$-local character for $\lhd$-chains, where $\lhd := <_{\lambda,\omega}$. This shows all the hypotheses of Theorem 7.3 are satisfied. $\qquad\square$

Assuming categoricity in a big-enough successor, we get rid of the tameness assumption:

Theorem 7.4. Let K be an abstract elementary class. Let $\mu := h_2 := h(h(\mathrm{LS}(K)))$. Let $\lambda := \mu^+$. Assume K has amalgamation, joint embedding, and is categorical in some successor $\theta \geq h(\lambda)$.

Let $\mathcal{F} := [\lambda, \theta)$. Then there is a type-full good $\mathcal{F}$-frame with underlying AEC K.

Proof. Since $\theta \geq h(\mathrm{LS}(K))$, K has arbitrarily large models and so using joint embedding K has no maximal models. By Fact 7.2, K is categorical in λ and K is $(\mu, < h(\lambda))$-tame. Apply Theorem 7.3. $\qquad\square$

Notice that one also obtains that categoricity (at a cardinal of high-enough cofinality) and tameness implies stability everywhere. This improves on [BKV06, Corollary 4.7].

Theorem 7.5. Let K be an abstract elementary class with amalgamation, joint embedding, and no maximal models. Assume K is categorical in some λ such that $\mathrm{cf}(\lambda) > \mu \geq \mathrm{LS}(K)$ and K is (μ, μ')-tame. Then K is stable in all $\theta \in [\mathrm{LS}(K), \mu']$. In particular, if $\mu' = \infty$, then K is stable everywhere.

Proof. By Fact 7.2, μ-nonsplitting has $\aleph_0$-local character for $\lhd$-chains, where $\lhd := <_{\mu,\omega}$ and K is stable everywhere below and at μ. Apply Theorem 5.6 to see K is stable everywhere in $(\mu, \mu']$. $\qquad\square$

This result is much more local than the other results of this section. For example, we do not need to assume that $\mu' \geq h(\mu)$. Moreover, as Theorem 5.6 shows, the categoricity hypothesis can be replaced by μ-nonsplitting having $\aleph_0$-local character for $\lhd$-chains, for some good ordering $\lhd$ on K_μ. In this case, the no maximal model hypothesis is not necessary.

Assuming the generalized continuum hypothesis (GCH), we obtain a more general stability spectrum theorem:

Theorem 7.6. Assume GCH. Let K be an abstract elementary class with amalgamation, joint embedding, and no maximal models. Assume K is μ-tame for $\mu \geq \mathrm{LS}(K)$, $\lhd$ is a good ordering on K_μ, and μ-nonsplitting has κ-local character for $\lhd$-increasing chains. Then K is stable in all $\lambda \geq \mu$ with $\lambda = \lambda^{<\kappa}$.

Proof. K is stable in μ since we have a good ordering on K_μ. If $\lambda > \mu$, the result follows from Theorem 5.7. $\qquad\square$

Remark 7.7. If K is the class of models of a complete first-order theory, the conditions for stability given by Corollary 7.6 are very close[4] to optimal (see [She90, Corollary III.3.8]).

Remark 7.8. Let K be an abstract elementary class with amalgamation, joint embedding, and no maximal models. Assume K is χ-tame and stable in some $\mu \geq h(\chi)$. Then [GV06b, Theorem 4.13] shows that for some $\kappa < h(\chi)$, μ-nonsplitting has κ-local character. Thus we have:

Corollary 7.9. Assume GCH. Let K be an abstract elementary class with amalgamation, joint embedding, and no maximal models. Assume K is χ-tame and stable in some $\mu \geq \mathrm{LS}(K)$. Then there is $\kappa < h(\chi)$ such that K is stable in all $\lambda \geq \mu$ with $\lambda^{<\kappa} = \lambda$.

Proof. If $\mu < h(\chi)$, then by [GV06b, Corollary 6.4] one can take $\kappa := \mu^+$, so assume $\mu \geq h(\chi)$. By the previous remark, there is $\kappa < h(\chi)$ such that μ-nonsplitting has κ-local character. The result now follows from Corollary 7.6. $\square$

Remark 7.10. In the upcoming [Vas], we will use different methods to prove Corollary 7.9 in ZFC. We do not know whether Corollary 7.6 also holds in ZFC (although it is clear from the proof that much less than GCH is needed).

We can also apply our good frame to the question of uniqueness of limit models:

Theorem 7.11 (Uniqueness of limit models). Assume the hypotheses of Theorem 7.3 hold. Then K has a unique limit model in any $\mu' \in \mathcal{F}$. In fact, if $M_0 \in K_{\mu'}$ and M_ℓ is (μ', δ_l)-limit over M_0 for $\ell = 1, 2$ and δ_l a limit ordinal, then $M_1 \cong_{M_0} M_2$.

In particular, if K has amalgamation, joint embedding, no maximal models, is categorical in λ and is μ-tame for some $\mu < \mathrm{cf}(\lambda)$, then K has a unique limit model in any $\mu' \geq \lambda$.

Proof. By Theorem 7.3, K has a good $\mathcal{F}$-frame $\mathfrak{s}$. In particular, K is stable in μ', so one can iterate Fact 2.10 to build a (μ', δ)-limit model for any desired $\delta < (\mu')^+$. To see uniqueness, apply [She09b, Lemma 4.8] (see [Bonb, Theorem 9.2] for a detailed proof of that result). $\square$

We see this theorem as an encouraging approximation to generalizing the upward categoricity transfer result of [GV06a] (which assumes categoricity in a successor cardinal) to categoricity in a limit cardinal.

Remark 7.12. Uniqueness of limit models of cardinality μ was asserted to follow from categoricity in some $\lambda^+ > \mu$ already in [SV99]. However, the authors later admitted that an error found by VanDieren in 1999 was fatal. VanDieren

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[4]The least regular cardinal κ such that nonsplitting has κ-local character will be at most the *successor* of $\kappa(T)$.

[Van06, Van13] proves uniqueness with the additional assumption that unions of amalgamation bases are amalgamation bases (but does not use tameness). It is still open whether uniqueness of limit models follows from categoricity only.

Remark 7.13. A variation on Theorem 7.11 is [BG, Corollary 6.10], which uses stronger locality assumptions but manages to obtain uniqueness of limit models below the categoricity cardinal without any cofinality restriction.

8. Conclusion and further work

Assuming amalgamation, joint embedding, no maximal models, and tameness, we have given superstability-like conditions under which an abstract elementary class has a global type-full good frame $\mathfrak{s}$, i.e. a forking-like notion for 1-types. These arguments would work just as well to get a notion of independence for all n-types, with $n < \omega$. The proof of extension breaks down, however, for types of infinite length (difficulties in obtaining the extension property in the absence of compactness is one of the reasons[5] it was assumed as an axiom in [BG]).

Shelah's approach around this in [She09b] is to show that if the frame is *weakly successful* (a uniqueness condition for certain kinds of amalgamations), then it has a notion of nonforking for types of models. In [She09e], Shelah has several hundreds of pages of approximations on when weak successfulness can be transferred across cardinals, but even assuming $\mathfrak{s} \restriction \lambda$ is weakly successful for every λ, it is not clear how we can get a good nonforking notion for models of different sizes. This is one direction further work could focus on.

Another (non-orthogonal) direction would be to find applications for such a global good frame. As mentioned in the previous section, we believe it could be useful in proving categoricity transfer theorems. Moreover, the frame built in Section 5 is only well-behaved for μ^+-saturated models, and it would be interesting to know at what cardinals such models exist. This calls for tools to deal with unions of saturated models and we plan to explore this further in future work.

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