

AMALGAMATION FROM CATEGORICITY IN UNIVERSAL CLASSES

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ABSTRACT. We prove that, in universal classes, categoricity in a high-enough cardinal implies amalgamation. The proof stems from ideas of Adi Jarden and Will Boney.

Theorem 0.1. Let K be a universal class. If K is categorical in a high-enough cardinal, then there exists λ such that:

- (1) $K_{\geq \lambda}$ has amalgamation.
- (2) $K_{\geq \lambda}$ is fully good (i.e. it admits a global forking-like notion).

We obtain several categoricity transfers:

Corollary 0.2. Let K be a universal class.

- (1) If K is categorical in a high-enough successor cardinal, then K is categorical on a tail of cardinals.
- (2) Assume that an unpublished claim of Shelah holds, and that $2^\lambda < 2^{\lambda^+}$ for all cardinals λ . If K is categorical in a high-enough cardinal, then K is categorical on a tail of cardinals.

We work in a more general context than universal classes: abstract elementary classes which admit intersections, a notion introduced by Baldwin and Shelah.

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1. INTRODUCTION

Morley’s categoricity theorem [Mor65] states that a first-order countable theory that is categorical in some uncountable cardinal must be categorical in all uncountable cardinals. The result motivated much of the development of first-order classification theory (it was later generalized by Shelah [She74] to uncountable theories).

Toward developing a classification theory for non-elementary classes, one can ask whether there is such a result for infinitary logics, e.g. for an $L_{\omega_1, \omega}$ sentence. In 1971, Keisler proved [Kei71, Section 23] a generalization of Morley’s theorem to this framework assuming in addition that the model in the categoricity cardinal is sequentially homogeneous. Unfortunately Shelah later observed using an example of Marcus [Mar72] that Keisler’s assumption does not follow from categoricity. Still in the later seventies Shelah proposed the following far-reaching conjecture:

Conjecture 1.1 (Open problem D.(3a) in [She90]). If L is a countable language and $\psi \in L_{\omega_1, \omega}$ is categorical in one $\mu \geq \beth_{\omega_1}$, then it is categorical in all $\mu \geq \beth_{\omega_1}$.

This has now become the central test problem in classification theory for non-elementary classes. Shelah alone has more than 2000 pages of approximations (for example [She75, She83a, She83b, MS90, She99, She01, She09a, She09b]). Shelah’s results led him to introduce a semantic framework encompassing many different infinitary logics and algebraic classes [She87a]: abstract elementary classes (AECs). In this framework, we can state an eventual version of the conjecture¹:

Conjecture 1.2 (Shelah’s eventual categoricity conjecture for AECs). An AEC that is categorical in unboundedly many cardinals is categorical on a tail of cardinals.

¹The statement here appears in [She09a, Conjecture N.4.2]. Note that (as an AEC is determined by its models of size at most the Löwenheim-Skolem number) there are only set many AECs with a given Löwenheim-Skolem number, so categoricity in unboundedly many cardinals is equivalent to categoricity in a single high-enough cardinal.

Positive results are known in less general frameworks: For homogeneous model theory by Lessmann [Les00] and more generally for tame² simple finitary AECs by Hyttinen and Kesälä [HK11] (note that these results apply only to countable languages). In uncountable languages, Grossberg and VanDieren proved the conjecture in tame AECs categorical in a successor cardinal [GV06c, GV06a]. Later Will Boney pointed out that tameness follows³ from large cardinals [Bon14b], a result that (as pointed out in [LR]) can also be derived from a 25 year old theorem of Makkai and Paré ([MP89, Theorem 5.5.1]). A combination of this gives that statements much stronger than Shelah’s categoricity conjecture for a successor hold if there exists a proper class of strongly compact cardinals.

The question of whether categoricity in a sufficiently high limit cardinal implies categoricity on a tail remains open (even in tame AECs). Using the additional assumption of shortness, we managed to prove⁴ (see [Vas, Theorem 1.6]):

Fact 1.3. Assume the weak generalized continuum hypothesis⁵ and the result of an unpublished claim of Shelah (see the discussion in [Vas]). Then a fully tame and short AEC with amalgamation that is categorical in a high-enough cardinal is categorical on a tail of cardinals.

Note that Fact 1.3 applies in particular to homogeneous model theory and tame finitary AECs⁶ with uncountable language, a case that could not previously be dealt with.

Now a conjecture of Grossberg made in 1986 (see Grossberg [Gro02, Conjecture 2.3]) is that categoricity in a high-enough cardinal should imply amalgamation. This is especially relevant considering that all the positive results above assume amalgamation. In the presence of large cardinals, Grossberg’s conjecture is known to be true (see [MS90, Proposition 1.13] or the stronger [SK96]). In recent years it has been

²Tameness is a locality property for orbital types introduced by Grossberg and VanDieren in [GV06b].

³Recently Boney and Unger [BU] established that the statement “all AECs are tame” is in fact *equivalent* to a large cardinal axioms (the existence of a proper class of almost strongly compact cardinals).

⁴In [She09a, Theorem IV.7.12], Shelah claims to prove Fact 1.3 without tameness or shortness. However we were unable to verify Shelah’s proof. Also, the statement contains an error (it contradicts Morley’s categoricity theorem).

⁵The weak generalized continuum hypothesis (WGCH) is the statement that for all cardinals λ , $2^\lambda < 2^{\lambda^+}$.

⁶By [HK06, Theorem 4.11], categorical tame finitary AECs are short.

shown that many results that could be proven using large cardinals can be proven using just the model-theoretic assumption of tameness or shortness (see all of the above papers on tameness and for example [Vasb, BV]). Thus one can ask whether tameness suffices to get amalgamation from categoricity. In general, this is not known. The only approximation is a result of Adi Jarden [Jar] discussed more at length in Section 4. Our contribution is a weak version of amalgamation which one can assume alongside tameness to prove Grossberg's conjecture (see Corollary 4.13):

Theorem 1.4. Let K be a tame AEC categorical in unboundedly many cardinals. If K has weak amalgamation (see Definition 4.9), then there exists λ such that $K_{\geq \lambda}$ has amalgamation.

The proof uses a deep result of Shelah showing that a categorical AEC is well-behaved in a specific cardinal, then uses tameness and weak amalgamation to transfer the good behavior up.

We apply our result to *universal classes*. Universal classes were introduced by Shelah in [She87b] as an important framework where he thought finding dividing lines should be possible⁷. For many years, Shelah has claimed a main gap theorem for these classes but the full proof has not appeared in print. The most recent version is Chapter V of [She09b] which contains hundreds of pages of approximations. The methods used are stability theory inside a model (averages) as well as combinatorial tools to build many models.

Here, we show that universal classes are tame⁸ (in fact fully $(< \aleph_0)$ -tame and short) and have weak amalgamation, hence the previous theorem applies, and using Fact 1.3 (or [GV06a] if the categoricity cardinal is a successor) we obtain the corollary in the abstract (see Corollary 4.17 for the details). In particular, the eventual version of Conjecture 1.1 holds when ψ is a universal sentence.

Without assuming weak GCH, we study universal classes using methods that have recently emerged from the study of tame AECs: assuming amalgamation, a central question is the existence of a forking-like notion, the latest development being [Vasa]. We continue this paper and succeed in building a global independence notion (as a simple corollary, we get the disjoint amalgamation property, see Corollary 5.10). In fact, the forking-like notion we obtain works even over arbitrary

⁷We were told by Rami Grossberg that another motivation was to study certain non first-order classes of modules.

⁸This uses an argument of Will Boney.

sets, see Theorem 5.19. As many natural classes of objects appearing in algebra are universal (or just admit intersections, see the next paragraph) we believe that the existence of such an independence notion is likely to have further applications. See also the introduction to [Vas].

All our results are developed in a more general context than universal classes. We study AECs which admit intersections, a notion introduced by Baldwin and Shelah in [BS08, Definition 1.2]. We prove early on (Theorem 3.5) that universal classes are fully tame and short AECs which admit intersections, and our results apply to these more general classes.

This paper was written while working on a Ph.D. thesis under the direction of Rami Grossberg at Carnegie Mellon University and I would like to thank Professor Grossberg for his guidance and assistance in my research in general and in this work specifically. I also thank Will Boney for helpful conversations and pointing me to AECs which admit intersections.

2. AECs WHICH ADMIT INTERSECTIONS

Throughout this paper, we assume familiarity with a basic text on AECs such as [Bal09] or [Gro] and refer the reader to the preliminaries of [Vas] for more details and motivations on the notation and concepts used in this paper.

Recall:

Definition 2.1 ([She87b]). A class of structure K is *universal* if:

- (1) It is a class of L -structures for a fixed language $L = L(K)$, closed under isomorphisms.
- (2) If $\langle M_i : i < \delta \rangle$ is \subseteq -increasing in K , then $\bigcup_{i < \delta} M_i \in K$.
- (3) If $M \in K$ and $M_0 \subseteq M$, then $M_0 \in K$.

Example 2.2.

- (1) The class of models of a universal $L_{\lambda, \omega}$ theory is universal.
- (2) Not all elementary classes are universal but some universal classes are not elementary (locally finite groups are one example).
- (3) Coloring classes [KLH] are universal classes. This shows that the behavior of amalgamation is non-trivial even in universal classes: some coloring classes can have amalgamation up to \beth_α for some $\alpha < \text{LS}(K)^+$ and fail to have it above \beth_α .

Universal classes are abstract elementary classes:

Definition 2.3 (Definition 1.2 in [She87a]). An *abstract elementary class* (AEC for short) is a pair (K, \leq) , where:

- (1) K is a class of L -structured, for some fixed language $L = L(K)$.
- (2) \leq is a partial order (that is, a reflexive and transitive relation) on K .
- (3) (K, \leq) respects isomorphisms: If $M \leq N$ are in K and $f : N \cong N'$, then $f[M] \leq N'$.
- (4) If $M \leq N$, then $M \subseteq N$.
- (5) Coherence: If $M_0, M_1, M_2 \in K$ satisfy $M_0 \leq M_2$, $M_1 \leq M_2$, and $M_0 \subseteq M_1$, then $M_0 \leq M_1$;
- (6) Tarski-Vaught axioms: Suppose δ is a limit ordinal and $\langle M_i \in K : i < \delta \rangle$ is an increasing chain. Then:
 - (a) $M_\delta := \bigcup_{i < \delta} M_i \in K$ and $M_0 \leq M_\delta$.
 - (b) If there is some $N \in K$ so that for all $i < \delta$ we have $M_i \leq N$, then we also have $M_\delta \leq N$.
- (7) Löwenheim-Skolem-Tarski axiom: There exists a cardinal $\lambda \geq |L(K)| + \aleph_0$ such that for any $M \in K$ and $A \subseteq |M|$, there is some $M_0 \leq M$ such that $A \subseteq |M_0|$ and $\|M_0\| \leq |A| + \lambda$. We write $\text{LS}(K)$ for the minimal such cardinal

We often will not distinguish between the class K and the pair (K, \leq) .

Remark 2.4. If K is a universal class, then (K, \subseteq) is an AEC with $\text{LS}(K) = |L(K)| + \aleph_0$. We will use this fact freely.

We now recall the definition of AECs that admit intersections, a notion introduced by Baldwin and Shelah. It is interesting to note that Baldwin and Shelah thought of admitting intersections as a weak version of amalgamation (see the conclusion of [BS08]). However their paper did not study it systematically.

Definition 2.5 (Definition 1.2 in [BS08]). Let K be an AEC.

- (1) Let $N \in K$ and let $A \subseteq |N|$. N *admits intersections over A* if there is $M_0 \leq N$ such that $|M_0| = \bigcap \{M \leq N \mid A \subseteq |M|\}$. N *admits intersections* if it admits intersections over all $A \subseteq |N|$.
- (2) K *admits intersections* if N admits intersections for all $N \in K$.

Example 2.6.

- (1) If K is a universal class, then K admits intersections.
- (2) If K is a class of models of a first-order theory, then when (K, \subseteq) admits intersections has been characterized by Rabin [Rab62].

- (3) The examples in [BS08] admit intersections. Since they are not tame, they cannot be universal classes (see Theorem 3.5).
- (4) Many classes appearing in algebra admit intersections. For example⁹, let K be the class of algebraically closed normed fields (we code the value group with an additional predicate), ordered by $F_1 \leq F_2$ if and only if F_1 is a subfield of F_2 , the value groups are the same, and the valuations coincide on F_1 . Then K admits intersections. Again, K is not universal (as it is not closed under substructure).
- (5) Zilber's quasiminimal pregeometry classes ([Zil05], see [Kir10] for an exposition) admit intersections (by the characterization of Theorem 2.11).
- (6) If \mathfrak{C} is a monster model for a first-order theory T , we can let K be the class of (isomorphic copies of) algebraically closed subsets of \mathfrak{C} , ordered by the substructure relation. Then K admits intersections (but is not necessarily closed under substructure, so not necessarily universal).

In the rest of this section, we give several equivalent definitions of admitting intersections and deduce some properties of these classes. All throughout this paper, we assume:

Hypothesis 2.7. K is an AEC.

Definition 2.8. Let $M \in K$ and let $A \subseteq |M|$ be a set. M is *minimal over A* if $A \subseteq |M|$, and if $M' \leq M$ contains A , then $M' = M$. M is *minimal over A in N* if in addition $M \leq N$.

Definition 2.9. Let $N \in K$. We say \mathcal{F} is a *set of Skolem functions for N* if:

- (1) \mathcal{F} is a set, and each element f of \mathcal{F} is a function from N^n to N , for some $n < \omega$.
- (2) For all $A \subseteq |N|$, $M := \mathcal{F}[A] := \bigcup \{f[A] \mid f \in \mathcal{F}\}$ is such that $M \leq N$ and contains A .

Remark 2.10. The proof of Shelah's presentation theorem [She87a, Lemma 1.8] gives that for each $N \in K$, there is \mathcal{F} a set of Skolem functions for N with $|\mathcal{F}| \leq \text{LS}(K)$.

Theorem 2.11. Let K be an AEC and let $N \in K$. The following are equivalent:

- (1) N admits intersections.

⁹This is Example 1.15 in [Gro02].

- (2) There is an operator $\text{cl} := \text{cl}^N : \mathcal{P}(|N|) \rightarrow \mathcal{P}(|N|)$ such that for all $A, B \subseteq |N|$ and all $M \leq N$:
- (a) $\text{cl}(A) \leq N$.
 - (b) $A \subseteq \text{cl}(A)$.
 - (c) $A \subseteq B$ implies $\text{cl}(A) \subseteq \text{cl}(B)$.
 - (d) $\text{cl}(M) = M$.
- (3) For each $A \subseteq |N|$, there is a unique minimal model over A in N .
- (4) There is a set \mathcal{F} of Skolem functions for N such that:
- (a) $|\mathcal{F}| \leq \text{LS}(K)$.
 - (b) For all $M \leq N$, we have $\mathcal{F}[M] = M$.

Moreover the operator $\text{cl}^N : \mathcal{P}(|N|) \rightarrow \mathcal{P}(|N|)$ with the properties in (2) is unique and if it exists then it has the following characterizations:

- $\text{cl}^N(A) = \bigcap \{M \leq N \mid A \subseteq |M|\}$.
- $\text{cl}^N(A) = \mathcal{F}[A]$, for any set of Skolem functions \mathcal{F} for N such that $\mathcal{F}[M] = M$ for all $M \leq N$.
- $\text{cl}^N(A)$ is the unique minimal model over A in N .

Proof.

- (1) implies (2): Let $\text{cl}^N(A) := \bigcap \{M \leq N \mid A \subseteq |M|\}$. Even without hypotheses on N , (2b), (2c), and (2d) are satisfied. Since N admits intersections, (2a) is also satisfied.
- (2) implies (3): Let $A \subseteq |N|$. Let cl be as given by (2). Let $M := \text{cl}(A)$. By (2a), $M \leq N$. By (2b), $A \subseteq |M|$. Moreover if $M' \leq N$ contains A , then by (2c), $|M| \subseteq |\text{cl}(M')|$ but by (2d), $\text{cl}(M') = M'$. Thus by coherence and (2a) $M \leq M'$. This shows both that M is minimal over A and that it is unique.
- (3) implies (4): We slightly change the proof of Shelah's presentation theorem as follows: Let $\chi := \text{LS}(K)$. For each $\bar{a} \in {}^{<\omega}|N|$, let $\langle b_i^{\bar{a}} : i < \chi \rangle$ be an enumeration (possibly with repetitions) of the unique minimal model over $\text{ran}(\bar{a})$ in N . For each $n < \omega$ and $i < \chi$, we let $f_i^n : N^n \rightarrow N$ be $f_i^n(\bar{a}) := b_i^{\bar{a}}$. Let $\mathcal{F} := \{f_i^n \mid i < \chi, n < \omega\}$. Then $|\mathcal{F}| \leq \text{LS}(K)$ and if $A \subseteq |N|$, we claim that $\mathcal{F}[A]$ is minimal over A in N . This shows in particular that \mathcal{F} is as required.

Let $M := \mathcal{F}[A]$. By definition, $M = \bigcup_{\bar{a} \in {}^{<\omega}|A|} \mathcal{F}[\text{ran}(\bar{a})]$. Now if $\bar{a} \in {}^{<\omega}A$, $M_{\bar{a}} := \mathcal{F}[\text{ran}(\bar{a})] = \{b_i^{\bar{a}} : i < \chi\}$ is the unique minimal model over $\text{ran}(\bar{a})$ in N . Thus if $\text{ran}(\bar{a}) \subseteq \text{ran}(\bar{b})$, we must have (by coherence) $M_{\bar{a}} \leq M_{\bar{b}}$. It follows that $M \in K$ and

by the axioms of AECs also $M \leq N$. Of course, M contains A . Now if $M' \leq M$ contains A , then for all $\bar{a} \in {}^{<\omega}A$, $\bar{a} \in {}^{<\omega}|M'|$, so as $M_{\bar{a}}$ is minimal over $\text{ran}(\bar{a})$, $M_{\bar{a}} \leq M'$. It follows that $M \leq M'$ so $M = M'$.

- (4) implies (1): Let \mathcal{F} be as given by (4). Let $A \subseteq |N|$. Let $M := \mathcal{F}[A]$. By definition of Skolem functions, M contains A and $M \leq N$. We claim that $M = \bigcap \{M' \leq N \mid A \subseteq |M'|\}$. Indeed, if $M' \leq N$ contains A , then by the hypothesis on \mathcal{F} , $M = \mathcal{F}[A] \subseteq \mathcal{F}[M'] = M'$.

The moreover part follows from the arguments above. \square

Definition 2.12. For $N \in K$ let $\text{cl}^N : \mathcal{P}(|N|) \rightarrow \mathcal{P}(|N|)$ be defined by $\text{cl}^N(A) := \bigcap \{M \leq N \mid A \subseteq |M|\}$.

Theorem 2.11 allows us to deduce several properties of the operator cl^N .

Proposition 2.13.

- (1) Let $M \leq N \in K$ and let $A, B \subseteq |N|$.
 - (a) Invariance: If $f : N \cong N'$, then $f[\text{cl}^N(A)] = \text{cl}^{f[N]}(f[A])$.
 - (b) Monotonicity 1: $A \subseteq \text{cl}^N(A)$.
 - (c) Monotonicity 2: $A \subseteq B$ implies $\text{cl}^N(A) \subseteq \text{cl}^N(B)$.
 - (d) Monotonicity 3: If $A \subseteq |M|$, then $\text{cl}^N(A) \subseteq \text{cl}^M(A)$. Moreover if N admits intersections over A , then M admits intersections over A and $\text{cl}^N(A) = \text{cl}^M(A)$.
 - (e) Idempotence: $\text{cl}^N(M) = M$.
 - (f) Finite character: If N admits intersections, then if $B \subseteq \text{cl}^N(A)$ is finite, there exists $A_0 \subseteq A$ finite such that $B \subseteq \text{cl}^N(A_0)$.
- (2) Equality of Galois types is witnessed by an isomorphism: Assume K admits intersections. Then $\text{gtp}(\bar{a}_1; N_1) = \text{gtp}(\bar{a}_2; N_2)$ ¹⁰ if and only if there exists $f : \text{cl}^{N_1}(\bar{a}_1) \cong \text{cl}^{N_2}(\bar{a}_2)$ such that $f(\bar{a}_1) = \bar{a}_2$.

Proof.

- (1) Trivial given Theorem 2.11: For finite character, use the characterization in terms of Skolem functions. For monotonicity 3, let $M_0 := \text{cl}^N(A)$. We have $M_0 \leq N$ since N admits intersections over A . Since $M \leq N$ contains A , we must have $|M_0| \subseteq |M|$. By coherence, $M_0 \leq M$, and by minimality, $M_0 = \text{cl}^M(A)$.

¹⁰We write $\text{gtp}(\bar{a}; N)$ instead of $\text{gtp}(\bar{a}/\emptyset; N)$.

- (2) Let $M_1 := \text{cl}^{N_1}(\bar{a}_1)$, $M_2 := \text{cl}^{N_2}(\bar{a}_2)$. Since N_ℓ admits intersections, we have $M_\ell \leq N_\ell$, $\ell = 1, 2$ so the right to left direction follows. Now assume $\text{gtp}(\bar{a}_1; N_1) = \text{gtp}(\bar{a}_2; N_2)$. It suffices to prove the result when the equality is atomic (then we can compose the isomorphisms in the general case). So let $N \in K$ and $f_\ell : N_\ell \rightarrow N$ witness atomic equality, i.e. $f_1(\bar{a}_1) = f_2(\bar{a}_2)$. By invariance and monotonicity 3, $f_\ell[M_\ell] = \text{cl}^{f_\ell[N_\ell]}(f_\ell(\bar{a}_\ell)) = \text{cl}^N(f_\ell(\bar{a}_\ell))$. Since $f_1(\bar{a}_1) = f_2(\bar{a}_2)$, we must have that $f_1[M_1] = f_2[M_2]$. Thus $f := (f_2 \upharpoonright M_2)^{-1} \circ (f_1 \upharpoonright M_1)$ is as desired.

□

Remark 2.14. That equality of Galois types is witnessed by an isomorphism was already observed (without proof) in [BS08, Lemma 1.3]. Baldwin and Shelah also claim that equality of Galois types is atomic equality (see Definition 4.5), but this does not seem to follow.

Note in particular the following:

Corollary 2.15.

- (1) Assume that for every $M \in K$ and every $A \subseteq |M|$, there is $N \geq M$ such that N admits intersections over A . Then K admits intersections.
- (2) $N \in K$ admits intersections if and only if it admits intersections over every finite $A \subseteq |N|$.

Proof.

- (1) By monotonicity 3.
- (2) By the proof of Theorem 2.11.

□

Remark 2.16. The second result is implicit in the discussion after Remark 4.3 in [BS08].

Before ending this section, we point out a technical disadvantage of the definition of admitting intersection: it is not closed under the tail of the AEC: if K admits intersections and λ is a cardinal, then it is not clear that $K_{\geq \lambda}$ admits intersections. Thus we will work with a slightly weaker definition:

Definition 2.17. For K an AEC and $M \in K$, let K_M be the AEC defined by adding constant symbols for the elements of M and requiring

that M embeds inside every model of K_M . That is, $L(K_M) = L(K) \cup \{c_a \mid a \in |M|\}$, where the c_a s are new constant symbols, and

$$K_M := \{(N, c_a^N)_{a \in |M|} \mid N \in K \text{ and there exists } f : M \rightarrow N \text{ and } c_a^N = f(a)\}$$

We order K_M by $(N_1, c_a^{N_1})_{a \in |M|} \leq (N_2, c_a^{N_2})_{a \in |M|}$ if and only if $N_1 \leq N_2$ and $c_a^{N_1} = c_a^{N_2}$ for all $a \in |M|$.

Definition 2.18. For P a property of AECs and $M \in K$, K has P above M if K_M has P . K locally has P if it has P above every $M \in K$.

Remark 2.19. K locally admits intersections if and only if for every $M \leq N$ in K and every $A \subseteq |N|$ containing M , $\text{cl}^N(A) \leq N$.

Remark 2.20. If K locally has P , then for every cardinal λ , $K_{\geq \lambda}$ locally has P .

3. UNIVERSAL CLASSES ARE FULLY TAME AND SHORT

In this section, we show that universal classes are fully ($< \aleph_0$)-tame and short. The basic argument for Theorem 3.5 is due to Will Boney and will also appear in [Bon].

Note that it is impossible to extend this result to AECs which admits intersections: [BS08] gives several counterexamples. One could hope that showing that categoricity in a high-enough cardinal implies tameness (a conjecture of Grossberg and VanDieren, see [GV06a, Conjecture 1.5]) is easier in AECs which admits intersections, but we have been unable to make progress in that direction and leave it to further work.

The key of the argument for tameness of universal classes is that the isomorphism characterizing the equality of Galois type is unique. We abstract this feature into a definition:

Definition 3.1. K is *pseudo-universal* if it admits intersections and for any $N_1, N_2, \bar{a}_1, \bar{a}_2$, if $\text{gtp}(\bar{a}_1; N_1) = \text{gtp}(\bar{a}_2; N_2)$ and $f, g : \text{cl}^{N_1}(\bar{a}_1) \cong \text{cl}^{N_2}(\bar{a}_2)$ are such that $f(\bar{a}_1) = g(\bar{a}_1) = \bar{a}_2$, then $f = g$.

Example 3.2.

- (1) In universal classes, $\text{cl}^N(A)$ is just the substructure of N generated by A . Thus universal classes are pseudo-universal.
- (2) If \mathfrak{C} is a monster model for a first-order theory T , we can let K be the class of (isomorphic copies) of definably closed¹¹ sets

¹¹That is, an element which is definable from finitely many parameters in the set must be in the set.

of \mathfrak{C} , ordered by elementary substructure. Then K is a pseudo-universal AEC, but it need not be universal.

- (3) We show below that pseudo-universal classes are $(< \aleph_0)$ -tame, hence the AECs in [BS08] admit intersections but are not pseudo-universal.

We quickly recall the definitions of tameness and shortness: Tameness as a property of AECs was introduced by Grossberg and VanDieren in [GV06b] (it was isolated from a proof in [She99]). It says that Galois types are determined by small restrictions of their domain. Shortness was introduced by Will Boney in [Bon14b, Definition 3.3]. It says that Galois types are determined by restrictions to small length. We will use the notation of [Vasb, Definition 2.19].

Definition 3.3. Let Γ be a class (possibly proper) of Galois types in K . Let κ be an infinite cardinal.

- (1) K is $(< \kappa)$ -tame for Γ if for any $p \neq q$ in Γ , if $A := \text{dom}(p) = \text{dom}(q)$, then there exists $A_0 \subseteq A$ such that $|A_0| < \kappa$ and $p \upharpoonright A_0 \neq q \upharpoonright A_0$.
- (2) K is $(< \kappa)$ -short for Γ if for any $p \neq q$ in Γ , if $\alpha := \ell(p) = \ell(q)$, then there exists $I \subseteq \alpha$ such that $|I| < \kappa$ and $p^I \neq q^I$.
- (3) κ -tame means $(< \kappa^+)$ -tame, similarly for short.
- (4) We say K is $(< \kappa)$ -tame if it is $(< \kappa)$ -tame for types of length one over models (i.e. for $\bigcup_{M \in K} \text{gS}(M)$).
- (5) We say K is *fully* $(< \kappa)$ -tame if it is $(< \kappa)$ -tame for all types over models (i.e. for $\bigcup_{M \in K} \text{gS}^{<\infty}(M)$), similarly for short.

Definition 3.4. Let $\bar{a}_\ell \in {}^\alpha N_\ell$ and let κ be an infinite cardinal. We write $(\bar{a}_1, N_1) \equiv_{<\kappa} (\bar{a}_2, N_2)$ if for every $I \subseteq \alpha$ of size less than κ , $\text{gtp}(\bar{a}_1 \upharpoonright I; N_1) = \text{gtp}(\bar{a}_2 \upharpoonright I; N_2)$.

Theorem 3.5. If K is pseudo-universal, then K is $(< \aleph_0)$ -tame and short for any collection of types.

Proof. Let $\bar{a}_\ell \in {}^\alpha N_\ell$, $\ell = 1, 2$. Let $M_\ell := \text{cl}^{N_\ell}(\text{ran}(\bar{a}_\ell))$. Assume that $(\bar{a}_1, N_1) \equiv_{<\aleph_0} (\bar{a}_2, N_2)$. We want to show that $\text{gtp}(\bar{a}_1; N_1) = \text{gtp}(\bar{a}_2; N_1)$. The result then easily follows (as $\text{gtp}(\bar{b}/A; N_1) = \text{gtp}(\bar{c}/A; N_2)$ if and only if $\text{gtp}(\bar{b}\bar{a}; N_1) = \text{gtp}(\bar{c}\bar{a}; N_2)$ for some enumeration \bar{a} of A).

For each finite $I \subseteq \alpha$, let $M_{\ell,I} := \text{cl}^{N_\ell}(\text{ran}(\bar{a}_\ell \upharpoonright I))$. By definition of $\equiv_{<\aleph_0}$, for each finite $I \subseteq \alpha$, $\text{gtp}(\bar{a}_1 \upharpoonright I; N_1) = \text{gtp}(\bar{a}_2 \upharpoonright I; N_2)$. Therefore (because K admits intersections) there exists $f_I : M_{1,I} \cong M_{2,I}$ such that $f_I(\bar{a}_1 \upharpoonright I) = \bar{a}_2 \upharpoonright I$. Moreover by definition of pseudo-universal, f_I is unique with that property. This means in particular that if $I \subseteq J \subseteq \alpha$

are both finite, we must have $f_I \subseteq f_J$. By finite character of the closure operator, $M_\ell = \bigcup_{I \in [\alpha]^{<\aleph_0}} M_{\ell,I}$ and so letting $f := \bigcup_{I \in [\alpha]^{<\aleph_0}} f_I$, we have that $f : M_1 \cong M_2$ and $f(\bar{a}_1) = \bar{a}_2$. This witnesses that $\text{gtp}(\bar{a}_1; M_1) = \text{gtp}(\bar{a}_2; M_2)$ and so (since $M_\ell \leq N_\ell$), $\text{gtp}(\bar{a}_1; N_1) = \text{gtp}(\bar{a}_2; N_2)$. \square

Remark 3.6. The proof shows that if K is locally pseudo-universal (see Definition 2.18), then K is fully ($< \aleph_0$)-tame and short (i.e. it is ($< \aleph_0$)-tame and short for types over models).

4. AMALGAMATION FROM CATEGORICITY

We investigate how to get amalgamation from categoricity in tame AECs admitting intersections. Recall:

Definition 4.1. An AEC K has amalgamation if for any $M_0 \leq M_\ell$, $\ell = 1, 2$, there exists $N \in K$ and embeddings $f_\ell : M_\ell \xrightarrow{M_0} N$. We say that K has λ -amalgamation if this holds for the models in K_λ . We define similarly *disjoint amalgamation*, where we require in addition that $f_1[M_1] \cap f_2[M_2] = M_0$.

We will use the following deep result of Shelah. Recall that a good λ -frame is a notion of forking for types of length one over models of size λ , see [She09a, Definition II.2.1].

Fact 4.2. If K is categorical in unboundedly many cardinals, then there exists a categoricity cardinal $\lambda \geq \text{LS}(K)$ such that K has a good λ -frame (i.e. there exists a good λ -frame \mathfrak{s} such that $K_{\mathfrak{s}} = K_\lambda$). In particular, K has λ -amalgamation.

Proof. We will refer to results in Chapter IV of [She09a]. As in Definition IV.3.2.2, we say (μ, θ) is *in the model-completeness spectrum of* K if $\theta \leq \mu$ and for any $M \leq N$ with $M \in K_\theta$, $N \in K_\mu$, we have that $M \preceq_{L(K)_{\infty, \theta+}} N$. Now by Conclusion IV.2.14, for each θ , there exists $\mu(\theta)$ such that for all $\mu \geq \mu(\theta)$, if K is categorical in μ then (θ, μ) is in the model-completeness spectrum of K . Now pick $\langle \lambda_i : i \leq \omega \cdot \omega \rangle$ increasing continuous such that for all $i \leq \omega \cdot \omega$:

- (1) $\lambda_0 > \text{LS}(K)$
- (2) K is categorical in λ_i
- (3) $\lambda_{i+1} > \beth_{(2^{\lambda_i})^+} + \mu(\lambda_i)$.

This is possible: for successors, there is no problem. For i limit, let $\lambda_i := \sup_{j < i} \lambda_j$. Then $\beth_{\lambda_i} = \lambda_i$ and $\text{cf}(\lambda_i) = \aleph_0$. Therefore by the proof of Observation IV.3.5, K is categorical in λ_i . This is enough: let

$\lambda := \lambda_{\omega \cdot \omega}$. Note that λ is a limit of fixed points of the beth function of cofinality \aleph_0 . Moreover K is categorical in some $\mu \geq \mu(\lambda)$, so by Theorem IV.4.10, K has a good λ -frame. \square

Remark 4.3. The λ given by Fact 4.2 is potentially very big. In fact, Shelah's argument for the existence of $\mu(\theta)$ in the proof above is nonconstructive so no upper bound on it can be deduced.

Thus it is reasonable to assume that we have a good λ -frame, and we want to transfer amalgamation above it. Our inspiration is a recent result of Adi Jarden, presented at a talk in South Korea in the Summer of 2014.

Fact 4.4 (Corollary 4.10 in [Jar]). Assume K has a good λ -frame where the class of uniqueness triples satisfies the existence property and K is strongly λ -tame, then K has λ^+ -amalgamation.

We will not give the definition of the class of uniqueness triples here (but see Definition 5.5 and Fact 5.7). It suffices to say that they are a version of domination for good frames. As for strong tameness, it is a variation of tameness relevant when amalgamation fails to hold. Recall that λ -tameness asks for two types that are equal on all their restrictions of size λ to be equal. The strong version asks them to be *atomically equal*, i.e. there is a map witnessing it that amalgamates the two models in which the types are computed, see Definition 4.5. Jarden's result is interesting, since it shows that tameness, a locality property that we see as quite mild compared to assuming amalgamation, can be of some use to proving amalgamation. The downside is that we have to ask for a strengthened version.

While Jarden proved much more than λ^+ -amalgamation, it has been pointed out by Will Boney (in a private communication) that if one only wants amalgamation, the hypothesis that uniqueness triples satisfy the existence property is not necessary. The reason is that the methods of [Bon14a] can be used to transfer enough of the good frame to λ^+ so that the extension property holds, and the extension property implies amalgamation.

We make the argument precise here and also show that less than strong tameness is needed (in particular, it suffices to assume tameness and that the AEC admits intersections). We first fix some notation.

Definition 4.5. Let $\lambda \geq \text{LS}(K)$.

- (1) K^3 is the set of triples (a, M, N) such that $M \leq N$ and $a \in N$.
 K_λ^3 is the set of such triples where the models are in K_λ .

- (2) For $(a_\ell, M_0, M_\ell) \in K^3$, $\ell = 1, 2$, we say (a_1, M_0, M_1) and (a_2, M_0, M_2) are *atomically equivalent* and write $(a_1, M_0, M_1)E_{\text{at}}(a_2, M_0, M_2)$ if there exists $N \in K$ and $f_\ell : M_\ell \xrightarrow{M_0} N$ such that $f_1(a_1) = f_2(a_2)$. E_{at} is a symmetric and reflexive relation on K^3 and we let E be its transitive closure.
- (3) We say (a_1, M_1, N_1) *atomically extends* (a_0, M_0, N_0) if $M_1 \geq M_0$ and $(a_1, M_0, N_1)E_{\text{at}}(a_0, M_0, N_0)$.
- (4) We say $M \in K_\lambda$ has the *type extension property* if for any $N \geq M$ in K_λ and any $p \in \text{gS}(M)$, there exists $q \in \text{gS}(N)$ extending p .
- (5) We say M has the *strong type extension property* if for any $N \geq M$, whenever $(a, M, M') \in K_\lambda^3$, there exists $(b, N, N') \in K_\lambda^3$ atomically extending (a, M, M') .

We say K_λ has the *[strong] type extension property* (or K has the *[strong] type extension property in λ*) if every $M \in K_\lambda$ has it.

Remark 4.6. For $(a, M, N) \in K^3$, we have that by definition $\text{gtp}(a/M; N) = [(a, M, N)]_E$.

Remark 4.7. Let $\lambda \geq \text{LS}(K)$. It is well known that if K has λ -amalgamation, then $E \upharpoonright K_\lambda^3 = E_{\text{at}} \upharpoonright K_\lambda^3$. Moreover, K has amalgamation if and only if K_λ has the strong type extension property.

We think of the type extension property as saying that amalgamation cannot fail because there are “fundamentally incompatible” elements in the two models we want to amalgamate. Rather, the reason amalgamation fails is because we simply “do not have enough models” to witness that two types are equal in one step. It would be useful to formalize this intuition but so far we have failed to do so.

We are interested in conditions implying that the type extension property (not the strong one) is enough to get amalgamation. For this, it turns out that it is enough to require that the AEC admits intersections. However we can even require a weaker condition:

Definition 4.8. Let $(a_\ell, M, N_\ell) \in K_\lambda^3$, $\ell = 1, 2$. We say $(a_1, M, N_1)E_{\text{at}}^-(a_2, M, N_2)$ if for $\ell = 1, 2$, there exists $N'_\ell \leq N_\ell$ containing a_ℓ and M such that $(a_\ell, M, N'_\ell)E_{\text{at}}(a_{3-\ell}, M, N_{3-\ell})$.

Definition 4.9. K has *weak amalgamation* if $E = E_{\text{at}}^-$. Similarly define what it means for K to have weak λ -amalgamation.

Remark 4.10. If K locally admits intersections, $(a_\ell, M, N_\ell) \in K_\lambda^3$, $\ell = 1, 2$ and $(a_1, M, N_1)E(a_2, M, N_2)$, then Proposition 2.13 $N'_\ell :=$

$\text{cl}^{N_\ell}(|M| \cup \{a_\ell\})$ witness that $(a_1, M, N_1)E_{\text{at}}^-(a_2, M, N_2)$. Thus in that case, $E \restriction K_\lambda^3 = E_{\text{at}}^- \restriction K_\lambda^3$, so K has weak amalgamation.

The key result is:

Theorem 4.11. Let K be an AEC and $\lambda \geq \text{LS}(K)$. Assume K_λ has the type extension property. The following are equivalent:

- (1) K has λ -amalgamation.
- (2) $E \restriction K_\lambda^3 = E_{\text{at}}^- \restriction K_\lambda^3$.
- (3) K has weak λ -amalgamation.

In particular, if K admits intersections and has the type extension property, then it has amalgamation.

Proof. (1) implies (2) implies (3) is easy. We prove (3) implies (1).

Assume $E \restriction K_\lambda^3 = E_{\text{at}}^- \restriction K_\lambda^3$.

Claim 1. For every triple (M_0, M_1, M_2) of models in K_λ so that $M_0 < M_1$ and $M_0 \leq M_2$, there exists $M'_1 \leq M_1$ and $M'_2 \geq M_2$ in K_λ such that $M_0 < M'_1$, and there exists $g : M'_1 \xrightarrow{M_0} M'_2$.

$$\begin{array}{ccc}
 & M_1 & \\
 & \uparrow & \\
 & M'_1 & \cdots \xrightarrow{g} M'_2 \\
 & \uparrow & \uparrow \\
 M_0 & \longrightarrow & M_2
 \end{array}$$

Proof of claim 1. Let $M_0 < M_\ell$ be models in K_λ , $\ell = 1, 2$. Pick any $a_1 \in |M_1| \setminus |M_0|$. Let $p := \text{gtp}(a_1/M_0; M_1)$. By the type extension property, there exists $q \in \text{gS}(M_2)$ extending p . Pick $M_2^* \geq M_2$ and $a_2 \in |M_2^*|$ such that $q = \text{gtp}(a_2/M_2; M_2^*)$. Since E is E_{at}^- over the domain of interest, we have $(a_1, M_0, M_1)E_{\text{at}}^-(a_2, M_0, M_2^*)$. Let $M'_1 \leq M_1$ contain a_1 and M_0 such that $(a_1, M_0, M'_1)E_{\text{at}}(a_2, M_0, M_2^*)$. By definition, we have that there exists $M'_2 \geq M_2^*$ such that M'_1 embeds into M'_2 over M_0 , as needed.

Now we obtain amalgamation by repeatedly applying Claim 1. Since the result is key to subsequence arguments, we give full details below.

Claim 2. For every triple (M_0, M_1, M_2) of models in K_λ so that $M_0 < M_1$ and $f : M_0 \rightarrow M_2$, there exists $M'_1 \leq M_1$, $M'_2 \geq M_2$ in K_λ and $g : M'_1 \rightarrow M'_2$ such that $M_0 < M'_1$ and $f \subseteq g$.

$$\begin{array}{ccc}
 & M_1 & \\
 & \uparrow & \\
 M'_1 & \cdots \xrightarrow{g} & M'_2 \\
 & \uparrow & \uparrow \\
 M_0 & \xrightarrow{f} & M_2
 \end{array}$$

Proof of claim 2. Let M_0, M_1, M_2 and f be as given by the hypothesis. Let \widehat{M}_2 and \widehat{f} be such that $f \subseteq \widehat{f}$, $M_0 \leq \widehat{M}_2$ and $\widehat{f} : \widehat{M}_2 \cong M_2$. Now apply Claim 1 to $(M_0, M_1, \widehat{M}_2)$ to obtain $M'_1 \leq M_1$ with $M_0 < M'_1$, $\widehat{M}_2' \geq \widehat{M}_2$ and $\widehat{g} : M'_1 \xrightarrow{M_0} \widehat{M}_2'$. Now let \widehat{f}', M'_2 be such that $M'_2 \geq M_2$ and $\widehat{f}' : \widehat{M}_2' \cong M'_2$ extends \widehat{f} . Let $g := \widehat{f}' \circ \widehat{g}$. Since \widehat{g} fixes M_0 and \widehat{f}' extends f , g extends f , as desired.

Now let $M_0 \leq M$ and $M_0 \leq N$ be in K_λ . We want to amalgamate M and N over M_0 . We try to build $\langle M_i : i < \lambda^+ \rangle$, $\langle N_i : i < \lambda^+ \rangle$ increasing continuous in K_λ and $\langle f_i : i < \lambda^+ \rangle$ an increasing continuous sequence of embeddings such that for all $i < \lambda^+$:

- (1) $M_i \leq M$.
- (2) $f_i : M_i \xrightarrow{M_0} N_i$.
- (3) $N_0 = N$.
- (4) $M_i < M_{i+1}$.

This is impossible since then $\bigcup_{i < \lambda^+} M_i$ has cardinality λ^+ but is a K -substructure of M which has cardinality λ . Now for $i = 0$, we can take $N_0 = N$ and $f_0 = \text{id}_{M_0}$ and for i limit we can take unions. Therefore there must be some $\alpha < \lambda^+$ such that $f_\alpha, M_\alpha, N_\alpha$ are defined but we cannot define $f_{\alpha+1}, M_{\alpha+1}, N_{\alpha+1}$. If $M_\alpha < M$, we can use Claim 2 (with M_0, M_1, M_2, f there standing for $M_\alpha, M, N_\alpha, f_\alpha$ here) to get $M_{\alpha+1} \leq M$ with $M_\alpha < M_{\alpha+1}$ and $N_{\alpha+1} \geq N_\alpha$ with $f_{\alpha+1} : M_{\alpha+1} \rightarrow N_{\alpha+1}$ extending f_α (so M'_1, M'_2, g in Claim 2 stand for $M_{\alpha+1}, N_{\alpha+1}, f_{\alpha+1}$ here). Thus we can continue the induction, which we assumed was impossible. Therefore $M_\alpha = M$, so $f_\alpha : M \xrightarrow{M_0} N_\alpha$ amalgamates M and N over M_0 , as desired. \square

We are now ready to formally state the amalgamation transfer:

Theorem 4.12. Let K be an AEC. Let $\lambda \geq \text{LS}(K)$ and assume \mathfrak{s} is a good λ -frame with underlying class K_λ . If:

- (1) K is λ -tame.
- (2) $K_{\geq \lambda}$ has weak amalgamation.

Then $K_{\geq \lambda}$ has amalgamation.

Proof. We extend \mathfrak{s} to models of size greater than λ by defining $\geq \mathfrak{s}$ as in [She09a, Section II.2] (or see [Bon14a, Definition 2.7]). Even without any hypotheses, Shelah has shown that $\geq \mathfrak{s}$ has local character, density of basic types, and transitivity. Moreover, tameness implies that it has uniqueness. Now work by induction on $\mu \geq \lambda$ to show that K has μ -amalgamation. When $\mu = \lambda$ this follows from the definition of a good frame so assume $\mu > \lambda$. As in [Bon14a, Theorem 5.13], we can prove that $\geq \mathfrak{s}$ has the extension property for models of size μ . In particular, K_μ has the type extension property for basic types. The proof of Theorem 4.11 shows that this suffices to get μ -amalgamation. \square

Corollary 4.13. Let K be a tame AEC categorical in unboundedly many cardinals. If K has weak amalgamation, then there exists λ such that $K_{\geq \lambda}$ has amalgamation.

Proof. By Fact 4.2, we can find $\lambda \geq \text{LS}(K)$ such that K_λ has a good frame and K is λ -tame. By Theorem 4.12, $K_{\geq \lambda}$ has amalgamation. \square

Corollary 4.14. Let K be an AEC categorical in unboundedly many cardinals. If K is tame and locally admits intersections, then there exists λ such that $K_{\geq \lambda}$ has amalgamation.

Proof. By Remark 4.10, K has weak amalgamation. Now apply Corollary 4.13. \square

Corollary 4.15. Let K be a locally pseudo-universal AEC. If K is categorical in unboundedly many cardinals, then there exists λ such that $K_{\geq \lambda}$ has amalgamation.

Proof. By Theorem 3.5 (and the remark following it), K is tame. Now apply Corollary 4.14. \square

We can apply these results to Shelah's categoricity conjecture and improve [Vas, Theorem 1.6].

Corollary 4.16. Let K be a tame AEC with weak amalgamation. Assume K is categorical in unboundedly many cardinals.

- (1) If K is categorical in a high-enough successor cardinal, then K is categorical on a tail of cardinals.

- (2) Assume weak GCH and an unpublished claim of Shelah (Claim 1.5 in [Vasa]). If K is fully tame and short, then K is categorical on a tail of cardinals.

Proof. By Corollary 4.13, we can actually assume without loss of generality that K has amalgamation. Now:

- (1) Apply [GV06a] (and [She99] can also give a downward transfer).
- (2) Apply Theorem 1.6 in [Vasa].

□

In particular, we obtain the corollary in the abstract.

Corollary 4.17. Let K be a locally pseudo-universal AEC which is categorical in unboundedly many cardinals.

- (1) If K is categorical in a high-enough successor cardinal, then K is categorical on a tail of cardinals.
- (2) Assume weak GCH and an unpublished claim of Shelah (Claim 1.5 in [Vasa]). Then K is categorical on a tail of cardinals.

Proof. By Theorem 3.5, K is fully tame and short. Moreover, universal classes admit intersections and therefore have weak amalgamation (Remark 4.10). Now apply Corollary 4.16. □

5. INDEPENDENCE

We investigate the properties of independence in AECs admitting intersections. Recall that [Vasa, Theorem 15.6] showed that a fully tame and short AEC with amalgamation categorical in unboundedly many cardinals eventually admits a nice independence notion. We want to specialize this result to AECs admitting intersections and prove more properties of forking there. In particular we prove that the independence relation satisfies the axioms of [BGKV] (partially answering Question 7.1 there). We start by recalling some definitions.

Definition 5.1 (Definition 8.1 in [Vasa]). $\mathbf{i} = (K, \perp)$ is a *fully good independence relation* if:

- (1) K is an AEC with $K_{<LS(K)} = \emptyset$ and $K \neq \emptyset$.
- (2) K has amalgamation, joint embedding, and no maximal models.
- (3) K is stable in all cardinals.

- (4) \mathbf{i} is a $(< \infty, \geq \text{LS}(K))$ -independence relation (see [Vasa, Definition 3.6]). That is, \perp is a relation on quadruples (M, A, B, N) with $M \leq N$ and $A, B \subseteq |N|$ satisfying invariance, monotonicity, and normality. We write $A \underset{M}{\perp}^N B$ instead of $\perp(M, A, B, N)$, and we also say $\text{gtp}(\bar{a}/B; N)$ does not fork over M for $\text{ran}(\bar{a}) \underset{M}{\perp}^N B$.
- (5) \mathbf{i} has base monotonicity, disjointness ($A \underset{M}{\perp}^N B$ implies $A \cap B \subseteq |M|$), symmetry, uniqueness, extension, and the local character properties:
- (a) If $p \in \text{gS}^\alpha(M)$, there exists $M_0 \leq M$ with $\|M_0\| \leq |\alpha| + \text{LS}(K)$ such that p does not fork over M_0 .
 - (b) If $\langle M_i : i \leq \delta \rangle$ is increasing continuous, $p \in \text{gS}^\alpha(M_\delta)$ and $\text{cf}(\delta) > \alpha$, then there exists $i < \delta$ such that p does not fork over M_i .
- (6) \mathbf{i} has the $\text{LS}(K)$ -witness property: $A \underset{M}{\perp}^N B$ if and only if for all $A_0 \subseteq A$ and $B_0 \subseteq B$ with $|A_0| + |B_0| \leq \text{LS}(K)$, we have that $A_0 \underset{M}{\perp}^N B_0$.
- (7) \mathbf{i} has full model continuity: if for $\ell < 4$, $\langle M_i^\ell : i \leq \delta \rangle$ are increasing continuous such that for all $i < \delta$, $M_i^0 \leq M_i^\ell \leq M_i^3$ for $\ell = 1, 2$ and $M_i^1 \underset{M_i^0}{\perp}^{M_i^3} M_i^2$, then $M_\delta^1 \underset{M_\delta^0}{\perp}^{M_\delta^3} M_\delta^2$.

We say that K is *fully good* if there exists \perp such that (K, \perp) is fully good.

In this terminology, we have:

Fact 5.2 (Theorem 15.1.3 in [Vasa]). Let K be a fully $(< \kappa)$ -tame and short AEC with amalgamation.

If $\kappa = \beth_\kappa > \text{LS}(K)$, and K is categorical in a $\mu > \lambda_0 := (2^\kappa)^{+5}$, then $K_{\geq \lambda}$ is fully good except that it only has extension over models of size λ (or types of length at most λ). Here we have set $\lambda := \min(\mu, \beth_{(2^{\lambda_0})^+})$.

Thus it is reasonable to assume:

Hypothesis 5.3.

- (1) K locally admits intersections.

- (2) $\mathbf{i} = (K, \perp)$ is a fully good independence relation, except that extension only holds for models of size $\text{LS}(K)$ (or types of length at most $\text{LS}(K)$).

Our goal is to prove that \mathbf{i} is actually fully good, i.e. extension holds. For this, we want to study how the closure operator interacts with independence. The key lemma is:

Lemma 5.4. If $A \underset{M_0}{\overset{N}{\perp}} B$, then $\text{cl}^N(A) \underset{M_0}{\overset{N}{\perp}} \text{cl}^N(B)$.

Proof. By normality, without loss of generality $|M_0| \subseteq A, B$. Using symmetry, it is enough to show that $A \underset{M_0}{\overset{N}{\perp}} \text{cl}^N(B)$. By the witness property and finite character of the closure operator, we can assume without loss of generality that $|A| \leq \text{LS}(K)$. Therefore by extension there exists $N' \geq N$ and $M \geq M_0$ such that $M \leq N'$, M contains B , and $A \underset{M_0}{\overset{N'}{\perp}} M$.

By definition, $\text{cl}^N(B) = \text{cl}^{N'}(B)$ is contained in M , so $A \underset{M_0}{\overset{N'}{\perp}} \text{cl}^N(B)$, so $A \underset{M_0}{\overset{N}{\perp}} \text{cl}^N(B)$. \square

An abstract way of stating Lemma 5.4 is via domination triples.

Definition 5.5. (a, M, N) is a *domination triple* if $M \leq N$, $a \in |N| \setminus |M|$, and for any $N' \geq N$ and any $B \subseteq |N'|$, if $a \underset{M}{\overset{N'}{\perp}} B$, then $N \underset{M}{\overset{N'}{\perp}} B$.

Lemma 5.6. Let $M \leq N$ and let $a \in |N| \setminus |M|$. Then $(a, M, \text{cl}^N(\{a\} \cup |M|))$ is a domination triple.

Proof. Directly from Lemma 5.4. \square

In our framework, domination triples are the same as the uniqueness triples of [She09a, Definition II.5.3]:

Fact 5.7 (Lemma 11.7 in [Vas]). Let $\mu \geq \text{LS}(K)$ and let $\mathfrak{s} := \text{pre}(\mathbf{i}^{\leq 1} \upharpoonright K_\mu)$ (That is, we restrict \mathbf{i} to types of length one and models of size μ , see [Vas, Definition 3.8]). Note that \mathfrak{s} is a good μ -frame.

For $M, N \in K_\mu$, (a, M, N) is a domination triple if and only if it is a uniqueness triple in \mathfrak{s} .

We get:

Theorem 5.8. \mathbf{i} has extension. Hence it is a fully good independence relation.

Proof. Let $\mu \geq \text{LS}(K)$ and let $\mathfrak{s} := \text{pre}(\mathbf{i}^{\leq 1} \upharpoonright K_\mu)$. By Lemma 5.6 and Fact 5.7 \mathfrak{s} has the so-called existence property for uniqueness triples (see [She09a, Definition II.5.3]). By section II.5 of [She09a] (and the results of section 12 in [Vasa]) \mathfrak{s} induces an independence relation \mathbf{i}' for types of length at most μ over models of size μ that is well-behaved (i.e. it has all of the properties of a fully good independence relation except full model continuity and disjointness). By the canonicity of such relations (see the proofs of Corollary 5.19 and Theorem 6.13 in [BGKV]), \mathbf{i}' must be the same as $\mathbf{i}^{\leq \mu} \upharpoonright K_\mu$, the restriction of \mathbf{i} to size μ . Thus for all $\mu \geq \text{LS}(K)$, \mathbf{i} has extension for types of length at most μ over models of size μ . By the proof of [Vasa, Lemma 14.13], this suffices to conclude that \mathbf{i} has extension. \square

Remark 5.9. The proof show that instead of the AEC admitting intersections, it is enough to assume that for each μ , the restriction of \mathbf{i} to a good frame in μ has the existence property for uniqueness triples. Unfortunately the proof in [Vasa, Section 11] only works when the frame is restricted to the saturated models of size μ .

Corollary 5.10. K has disjoint amalgamation.

Proof. Because \mathbf{i} has existence, extension and disjointness. \square

Another consequence of having a closure operator is:

Theorem 5.11 (Finite character of independence). $A \downarrow_{M_0}^N B$ if and only if for all finite $A_0 \subseteq A$ and $B_0 \subseteq B$, $A_0 \downarrow_{M_0}^N B_0$. That is, \mathbf{i} has the $(< \aleph_0)$ -witness property.

Proof. By symmetry it is enough to show that if $A_0 \downarrow_{M_0}^N B$ for all finite $A_0 \subseteq A$, then $A \downarrow_{M_0}^N B$. For each finite $A_0 \subseteq A$, let $M_{A_0} := \text{cl}^N(|M_0| \cup A_0)$. Let $M := \text{cl}^N(|M_0| \cup B)$. By Lemma 5.4, $M_{A_0} \downarrow_{M_0}^N M$ for each finite $A_0 \subseteq A$. Let $M_A := \text{cl}^N(|M_0| \cup A)$. It is easy to see that

$\langle M_{A_0} \mid A_0 \in [A]^{<\aleph_0} \rangle$ is a directed system with union M_A . Therefore by full model continuity, $M_A \underset{M_0}{\overset{N}{\downarrow}} M$, and so $A \underset{M_0}{\overset{N}{\downarrow}} B$. \square

Remark 5.12. Thus we have that the axioms in [She09b, Chapter V.B] are satisfied by $(K, \downarrow, \text{cl})$.

For the next two results, we drop our hypotheses.

Theorem 5.13. Let K be a fully $(< \kappa)$ -tame and short AEC with amalgamation. Assume further that K locally admits intersections.

If $\kappa = \beth_\kappa > \text{LS}(K)$, and K is categorical in a $\mu > \lambda_0 := (2^\kappa)^{+5}$, then $K_{\geq \lambda}$ is fully good, where $\lambda := \min(\mu, \beth_{(2^{\lambda_0})^+})$. Moreover the independence relation has the $(< \aleph_0)$ -witness property.

Proof. Combine Fact 5.2, Theorem 5.8, and Theorem 5.11. \square

Remark 5.14. If K is not categorical but only superstable (see [Vasa, Definition 10.1]), then we can generalize the result (using [Vasa, Theorem 15.1]) provided that for all λ , $K^{\lambda\text{-sat}}$ (the class of λ -saturated models in K) locally admits intersections.

Corollary 5.15. Let K be a fully tame and short AEC which locally admits intersections. If K is categorical in unboundedly many cardinals, then there exists λ such that $K_{\geq \lambda}$ is fully good (and the independence relation has the $(< \aleph_0)$ -witness property).

Proof. By Corollary 4.14, K eventually has amalgamation. Now apply Theorem 5.13. \square

5.1. Set bases. We end by showing that it is possible to extend the independence relation to define forking not only over models but also over sets. In the terminology of [HL02], K is simple (note that the paper gives an example due to Shelah of a class that has a fully good independence relation, yet is not simple).

For our methods to work, we have to assume that K admits intersections, i.e. not just locally. To see that this is not a big loss, recall that if K is categorical in unboundedly many cardinals and has amalgamation, then the models in the categoricity cardinals are saturated, so for $M \in \text{LS}(K)$, K_M will also be categorical in unboundedly many cardinals.

Hypothesis 5.16. K admits intersections.

Definition 5.17. Let $N \in K$ and $A, B, C \subseteq |N|$. Define $B \overset{N}{\underset{A}{\downarrow}} C$ to hold if and only if $\text{cl}^N(AB) \overset{N}{\underset{\text{cl}^N(A)}{\downarrow}} \text{cl}^N(AC)$.

We define properties such as invariance, monotonicity, etc. just as for the model-based version of independence.

Remark 5.18. When $A \leq N$, this agrees with the previous definition of independence.

Theorem 5.19.

- (1) \downarrow has invariance, left and right monotonicity, base monotonicity, and normality.
- (2) \downarrow has symmetry, finite character (i.e. the $(< \aleph_0)$ -witness property), existence and transitivity.
- (3) \downarrow has extension.
- (4) Let $N \in K$ and let $\langle B_i : i < \delta \rangle$ be an increasing chain of sets. Let $B_\delta := \bigcup_{i < \delta} B_i$ and assume $B_\delta \subseteq |N|$. Let $p \in \text{gS}^\alpha(B; N)$. If $\text{cf}(\delta) > \alpha$, then there exists $i < \delta$ such that p does not fork over B_i .
- (5) If $p \in \text{gS}^\alpha(B; N)$, there exists $A \subseteq B$ such that p does not fork over A and $|A| < |\alpha|^+ + \aleph_0$.

Proof.

- (1) Easy.
- (2) Easy.
- (3) By transitivity and extension of i.
- (4) By local character for i.
- (5) By finite character, it is enough to show it when $\alpha < \omega$. Work by induction on $\lambda := |B|$. If $\lambda < \aleph_0$, take $A = B$ and use the existence property. If $\lambda \geq \aleph_0$, write $B = \bigcup_{i < \lambda} B_i$, where $|B_i| < \lambda$ for all $i < \lambda$. By the previous result, there exists $i < \lambda$ such that p does not fork over B_i . Now apply the induction hypothesis and transitivity.

□

Remark 5.20. Thus in this framework types of finite length really do not fork over a finite set. This removes the need for a special chain version of local character (i.e. if $\langle M_i : i \leq \delta \rangle$ is increasing continuous, $p^{<\omega} \in \text{gS}(M_\delta)$, there exists $i < \delta$ such that p does not fork over M_i).

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