A DOWNWARD CATEGORICITY TRANSFER FOR TAME ABSTRACT ELEMENTARY CLASSES

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ABSTRACT. We prove a downward transfer from categoricity in a successor in tame abstract elementary classes (AECs). This complements the upward transfer of Grossberg and VanDieren and improves the Hanf number in Shelah's downward transfer (provided the class is tame).

Theorem 0.1. Let $\mathcal{K}$ be an AEC with amalgamation. If $\mathcal{K}$ is $\mathrm{LS}(\mathcal{K})$-weakly tame and categorical in a successor $\lambda \geq \beth_{\left(2^{\mathrm{LS}(\mathcal{K})}\right)^+}$, then $\mathcal{K}$ is categorical in all $\lambda' \geq \beth_{\left(2^{\mathrm{LS}(\mathcal{K})}\right)^+}$.

The argument uses orthogonality calculus and gives alternate proofs to both the Shelah and the Grossberg-VanDieren transfers.

We deduce Shelah's categoricity conjecture in universal classes with amalgamation:

Theorem 0.2. Let $\mathcal{K}$ be a universal class with amalgamation and arbitrarily large models. If $\mathcal{K}$ is categorical in some $\lambda > |L(K)| + \aleph_0$, then $\mathcal{K}$ is categorical in all $\lambda' \geq \min(\lambda, \beth_{\left(2^{|L(K)|+\aleph_0}\right)^+})$.

Heavily using results of Shelah and assuming the weak generalized continuum hypothesis, we can also deal with categoricity in a limit cardinal and prove the categoricity conjecture in tame AECs with amalgamation:

Theorem 0.3. Assume $2^\theta < 2^{\theta^+}$ for every cardinal θ, as well as an unpublished claim of Shelah. Let $\mathcal{K}$ be a $\mathrm{LS}(\mathcal{K})$-tame AEC with amalgamation and arbitrarily large models. If $\mathcal{K}$ is categorical in some $\lambda > \mathrm{LS}(\mathcal{K})$, then $\mathcal{K}$ is categorical in all $\lambda' \geq \min(\lambda, \beth_{\left(2^{\mathrm{LS}(\mathcal{K})}\right)^+})$.

Contents

1. Introduction

In his milestone paper on AECs with amalgamation [She99], Shelah proves:

Fact 1.1. Let $\mathcal{K}$ be an AEC with amalgamation[1]. If $\mathcal{K}$ is categorical in a successor $\lambda \geq h(h(\mathrm{LS}(\mathcal{K})))$, then $\mathcal{K}$ is categorical in all $\lambda' \in [h(h(\mathrm{LS}(\mathcal{K}))), \lambda]$.

where we have set:

Notation 1.2. For λ an infinite cardinal, $h(\lambda) := \beth_{(2^\lambda)^+}$.

It has been asked by Baldwin [Bal09, Problem D.1.(5)] whether the bound in Fact 1.1 can be lowered to $h(\mathrm{LS}(\mathcal{K}))$. Indeed an earlier result of Makkai and Shelah [MS90] is that if $\mathcal{K}$ is a class of models of an $L_{\kappa,\omega}$ sentence, for κ a strongly compact cardinal, then one can reduce the Hanf number above to $h(\mathrm{LS}(\mathcal{K}) + \kappa)$. In fact, an upward transfer also holds:

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[1]See Remark 7.2 for why joint embedding or no maximal models need not be assumed.

Fact 1.3. If $\mathcal{K}$ is the class of models of an $L_{\kappa,\omega}$-sentence for κ a strongly compact cardinal and $\mathcal{K}$ is categorical in a successor $\lambda \geq h(|L| + \kappa)$, then $\mathcal{K}$ is categorical in every $\lambda' \geq h(|L| + \kappa)$.

The upward part was later generalized by Grossberg and VanDieren [GV06c, GV06a] to:

Fact 1.4. Let $\mathcal{K}$ be an AEC with amalgamation and no maximal models. Let $\lambda > \mathrm{LS}(\mathcal{K})$. If $\mathcal{K}$ is λ-tame and categorical in λ^+, then $\mathcal{K}$ is categorical in all $\lambda' \geq \lambda^+$.

Recall that tameness is a locality property for Galois types, also introduced by Grossberg and VanDieren [GV06b]: we say that an AEC $\mathcal{K}$ is $(< \kappa)$-tame if every Galois types is determined by its restrictions to models of size less than κ. λ-*tame* means $(< \lambda^+)$-tame. See [Lie11] for an equivalent definition in terms of a natural topology on Galois types being Hausdorff.

Why is Fact 1.4 a generalization of the upward part of Makkai and Shelah? Because Boney [Bon14b] proved that if $\mathcal{K}$ is an AEC with amalgamation and $\kappa > \mathrm{LS}(\mathcal{K})$ is strongly compact, then $\mathcal{K}$ is $(< \kappa)$-tame[2]. A common theme in the study of tame AECs is that many results that hold above a strongly compact already hold assuming tameness. In this vein, many results proven in [MS90] have been shown to hold in tame AECs. See for example [BKV06, BG, Vasa, BVa, Vasc, Vasb]. While it is known that the statement "all AECs are tame" is equivalent to the existence of a proper class of almost strongly compact cardinals [BU], many examples of tame AECs are known (see the introduction to [GV06b] or the upcoming [BVb]).

Here we generalize the downward part of Fact 1.3 to any tame AEC:

Main Theorem 7.7. Let $\mathcal{K}$ be a $\mathrm{LS}(\mathcal{K})$-tame AEC with amalgamation. If $\mathcal{K}$ is categorical in a successor $\lambda \geq h(\mathrm{LS}(\mathcal{K}))$, then $\mathcal{K}$ is categorical in all $\lambda' \geq h(\mathrm{LS}(\mathcal{K}))$.

This answers the aforementioned question of Baldwin in case the AEC is $\mathrm{LS}(\mathcal{K})$-tame. In fact, Theorem 7.7 can be generalized to *weakly tame* AECs (that is, we only ask that Galois types over *saturated* models be determined by their small restrictions, see Definition 2.10). Since in [She99, Main Claim II.2.3], Shelah proves that an AEC (with amalgamation and no maximal models) categorical in a successor λ is

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[2]Boney also proves the same conclusion if $\mathcal{K}$ is the class of models of an $L_{\kappa,\omega}$-sentence and $\kappa \geq \mathrm{LS}(\mathcal{K})$.

$h(\mathrm{LS}(\mathcal{K}))$-weakly tame below λ, we obtain an alternate[3] proof of Fact 1.1. The details are in Corollary 7.12. We similarly get an alternate proof of a special case of Fact 1.4 (see Corollary C.6), where we require that the categoricity happens above $h(\mathrm{LS}(\mathcal{K}))^4$.

To prove Theorem 7.7, we first use the methods of [Vasa] to build a global good frame (a forking-like notion for types of elements over models). The local notion for models of size λ, good λ-frames, are the main notion in Shelah's book on AEC [She09]. We then develop orthogonality calculus in this setup (versions of some of our results have been independently derived by Villaveces and Zambrano [VZ14]), heavily inspired from Shelah's development of orthogonality calculus in successful good λ-frames [She09, Section III.6], and use it to define a notion of unidimensionality similar to what is defined in [She09, Section III.2]. We show unidimensionality in λ is equivalent to categoricity in λ^+ and use orthogonality calculus to transfer unidimensionality across cardinals. While we work in a more global setup than Shelah's, we do not assume that the good frames we work with are successful [She09, Definition III.1.1], so we do *not* assume that the forking relation is defined for types of models (it is only defined for types of elements). To get around this difficulty, we use the theory of *independent sequences* introduced by Shelah for good λ-frames in [She09, Section III.5] and developed in [BVc] for global good frames.

An interesting methodological point compared to the proof of the categoricity transfer of [She99] is that we do not use models of set theory to prove the transfer of "no Vaughtian pair" (see $(*)_9$ in the proof of [She99, Theorem II.2.7], or [Bal09, Theorem 14.12]). As outlined above, the method of proof of Theorem 7.7 is much more local, allowing us for example to also show:

Corollary 7.4. Let $\mathcal{K}$ be a $\mathrm{LS}(\mathcal{K})$-weakly tame AEC with amalgamation and arbitrarily large models. Let $\mathrm{LS}(\mathcal{K}) < \lambda_0 < \lambda_1$. If λ_1 is a successor cardinal and $\mathcal{K}$ is categorical in λ_0 and λ_1, then $\mathcal{K}$ is categorical in all $\lambda \in [\lambda_0, \lambda_1]$

While we believe that the methods of [She99] are not sufficient to prove Corollary 7.4, we noticed after posting a first draft of this paper that they *are* enough to prove the downward part of Theorem 7.7. We give

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[3]Note that our alternate proof is not self-contained and still uses much material from [She99] and later work of Shelah and others.

[4]That the result holds under only *weak* tameness is implicit in Grossberg and VanDieren's proof and stated explicitly in [Bal09, Theorem 15.11.(2)].

the details in Section 3 but stress once again that the methods of this paper have further applications.

For example, the reader may ask how this work differs from the categoricity transfers in [Vase, Vasd]. There we do *not* assume that the categoricity cardinal is a successor but assume the existence of prime models over sets of the form $M \cup \{a\}$. Moreover the only purpose of orthogonality calculus there is to prove a technical lemma saying that a certain class of models omitting[5] a type has a good frame. Here we use orthogonality globally and develop more of its properties, without assuming existence of prime models. At the end of Section 7, we use our methods to give improvements on several results in [Vase, Vasd], including Theorem 0.2 from the abstract (see Theorem 7.15). We also prove a strong form of Shelah's categoricity conjecture for AECs with primes:

Theorem 7.14. Let $\mathcal{K}$ be a $\mathrm{LS}(\mathcal{K})$-tame AEC with amalgamation and arbitrarily large models. Assume that $\mathcal{K}$ has primes over models of the form $M \cup \{a\}$. If $\mathcal{K}$ is categorical in *some* $\lambda > \mathrm{LS}(\mathcal{K})$, then $\mathcal{K}$ is categorical in *all* $\lambda' \geq \min(\lambda, h(\mathrm{LS}(\mathcal{K})))$.

Remark 1.5. The case $\lambda = \mathrm{LS}(\mathcal{K})^+$ is allowed here. Thus this missing case in the upward transfer of Grossberg and VanDieren [GV06a] can be taken care of assuming the existence of primes. We also show how to deal with it assuming the weak generalized continuum hypothesis, see Theorem 0.3.

Finally, the reader may also ask how to deal with categoricity in a limit cardinal *without* assuming the existence of prime models. In [She09, Theorem IV.7.12], Shelah claims assuming the weak generalized continuum hypothesis[6] that if $\mathcal{K}$ is an AEC with amalgamation[7], then categoricity in some $\lambda \geq h(\aleph_{\mathrm{LS}(\mathcal{K})^+})$ implies categoricity in all $\lambda' \geq h(\aleph_{\mathrm{LS}(\mathcal{K})^+})$[8]. Shelah's proof relies on an unpublished claim (whose proof should appear in [Sheb]), as well as PCF theory and long constructions of linear orders from [She09, Sections IV.5,IV.6]. We have not fully checked it. In [Vasb], we gave a way to work around the use of PCF theory and the construction of linear orders (though still using Shelah's unpublished claim) by using the locality assumption of full tameness

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[5]Or really, models over which a certain type has a unique extension.

[6]That is, $2^\theta < 2^{\theta^+}$ for all cardinals θ.

[7]Shelah only assumes some instances of amalgamation and no maximal models at specific cardinals, see the discussion in Section 8.

[8]Shelah gives a stronger, erroneous statement (it contradicts Morley's categoricity theorem) but this is what his proof gives.

and shortness (a stronger assumption than tameness introduced by Will Boney in his Ph.D. thesis, see [Bon14b, Definition 3.3]). A consequence was the consistency of Shelah's eventual categoricity conjecture from the existence of a proper class of strongly compact cardinals.

In Section 8, we give an exposition of Shelah's proof that does not use PCF or the construction of linear orders and also prove Theorem 0.3 from the abstract (see Theorem 8.9). This uses a recent result of VanDieren and the author [VVa], showing that a model at a high-enough categoricity cardinal must have some degree of saturation (regardless of the cofinality of the cardinal). We deduce (still using the aforementioned unpublished claim of Shelah) that Shelah's eventual categoricity conjecture is consistent assuming the existence of a proper class of *measurable* cardinals[9]. Furthermore we give explicit upper bounds (see Theorem 8.20). Moreover, we give two ZFC consequences of Shelah's methods[10]: an improvement on the Hanf number for constructing good frames (Theorem 8.14) and a nontrivial restriction on the categoricity spectrum *below* the Hanf number of an AEC with amalgamation and no maximal models (Theorem 8.15).

For clarity, we emphasize once again that Theorem 0.3 is due to Shelah when the Hanf number is $h(\aleph_{\mathrm{LS}(\mathcal{K})^+})$ (and then tameness is not needed). The main contribution of Section 8 is a clear outline of Shelah's proof that avoids several of his harder arguments, as well as several applications of his methods, including an application of Theorem 7.7 to reduce the Hanf number to H_1 when the AEC is $\mathrm{LS}(\mathcal{K})$-tame.

This paper was written while working on a Ph.D. thesis under the direction of Rami Grossberg at Carnegie Mellon University and I would like to thank Professor Grossberg for his guidance and assistance in my research in general and in this work specifically. I also thank John Baldwin and Monica VanDieren for helpful feedback on an earlier draft of this paper.

2. Background

We give some background on superstability and independence that will be used in the next sections. We assume familiarity with the basics of AECs as laid out in e.g. [Bal09] or the forthcoming [Gro]. We will use

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[9]This is implicit in [She09, Chapter IV].

[10]Really, of a simple lemma in his proof that obtains weak tameness from categoricity in certain cardinals below the Hanf number.

the notation from the preliminaries of [Vasc]. Everywhere below, $\mathcal{K}$ is an AEC.

The definition of superstability below is already implicit in [SV99] and has since then been studied in several papers, e.g. [Van06, GVV, Vasb, BVa, GV, VVb]. We will use the definition from [Vasb, Definition 10.1].

Definition 2.1. For $M_0 \leq M$ in $\mathcal{K}$, M is *universal over* M_0 if for any $N \in \mathcal{K}_{\|M_0\|}$ with $M_0 \leq N$, there exists $f : N \xrightarrow[M_0]{} M$.

Definition 2.2. $\mathcal{K}$ is *μ-superstable* (or *superstable in μ*) if:

(1) $\mu \geq \mathrm{LS}(\mathcal{K})$.
(2) $\mathcal{K}_\mu$ is nonempty, has amalgamation, joint embedding, and no maximal models.
(3) $\mathcal{K}$ is stable in μ[11], and:
(4) μ-splitting in $\mathcal{K}$ satisfies the following locality property: for all limit ordinal $\delta < \mu^+$ and every increasing continuous sequence $\langle M_i : i \leq \delta \rangle$ in $\mathcal{K}_\mu$ with M_{i+1} universal over M_i for all $i < \delta$, if $p \in \mathrm{gS}(M_\delta)$, then there exists $i < \delta$ so that p does not μ-split over M_i.

In our setup, superstability follows from categoricity. If (as will be the case in most of this paper) the AEC is categorical in a successor, this is due to Shelah and appears as [She99, Lemma 6.3]. The heart of the proof in the general case appears as [SV99, Theorem 2.2.1] and the result is stated for classes with amalgamation in [GV, Theorem 6.3].

Fact 2.3 (The Shelah-Villaveces theorem). Let $\mu \geq \mathrm{LS}(\mathcal{K})$. If $\mathcal{K}$ is has amalgamation, no maximal models, and is categorical in a $\lambda > \mu$, then $\mathcal{K}$ is μ-superstable.

Note also that tameness implies that superstability transfers up:

Fact 2.4 (Proposition 10.10 in [Vasb]). If $\mathcal{K}$ has amalgamation, is $\mathrm{LS}(\mathcal{K})$-superstable, and $\mathrm{LS}(\mathcal{K})$-tame, then $\mathcal{K}$ is superstable in every $\mu \geq \mathrm{LS}(\mathcal{K})$.

Recall the definition of a limit model (see [GVV] for history and motivation). Here we give a global definition, where we permit the limit model and the base to have different sizes.

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[11]That is, $|\mathrm{gS}(M)| \leq \mu$ for all $M \in \mathcal{K}_\mu$. Some authors call this "Galois-stable". Here we omit the "Galois" prefix, also in the naming of other concepts such as saturated models.

Definition 2.5. Let $M_0 \leq M$ be models in $\mathcal{K}_{\geq \mathrm{LS}(\mathcal{K})}$. M is *limit over* M_0 if there exists a limit ordinal δ and a strictly increasing continuous sequence $\langle N_i : i \leq \delta \rangle$ such that:

(1) $N_0 = M_0$.
(2) $N_\delta = M$.
(3) For all $i < \delta$, N_{i+1} is universal over N_i.

We say that M is *limit* if it is limit over some $M' \leq M$.

Definition 2.6. Assume that $\mathcal{K}$ has amalgamation.

(1) For $\lambda > \mathrm{LS}(\mathcal{K})$, $\mathcal{K}^{\lambda\text{-sat}}$ is the class of λ-saturated models in $\mathcal{K}_{\geq \lambda}$. We order it with the strong substructure relation inherited from $\mathcal{K}$.
(2) We also define $\mathcal{K}^{\mathrm{LS}(\mathcal{K})\text{-sat}}$ to be the class of models $M \in \mathcal{K}_{\geq \mathrm{LS}(\mathcal{K})}$ such that for all $A \subseteq |M|$ with $|A| \leq \mathrm{LS}(\mathcal{K})$, there exists a limit model $M_0 \leq M$ with $M_0 \in \mathcal{K}_{\mathrm{LS}(\mathcal{K})}$ and $A \subseteq |M_0|$. We order $\mathcal{K}^{\mathrm{LS}(\mathcal{K})\text{-sat}}$ with the strong substructure relation inherited from $\mathcal{K}$.

Remark 2.7. If $\mathcal{K}$ has amalgamation and is stable in $\mathrm{LS}(\mathcal{K})$, then $\mathcal{K}^{\mathrm{LS}(\mathcal{K})\text{-sat}}_{\mathrm{LS}(\mathcal{K})}$ is the class of limit models in $\mathcal{K}_{\mathrm{LS}(\mathcal{K})}$.

Together with superstability, a powerful tool is the symmetry property for splitting, first isolated by VanDieren [Vana]:

Definition 2.8. Let $\mu \geq \mathrm{LS}(\mathcal{K})$ and assume that $\mathcal{K}$ has amalgamation in μ. $\mathcal{K}$ exhibits *symmetry for μ-splitting* (or *μ-symmetry* for short) if whenever models $M, M_0, N \in \mathcal{K}_\mu$ and elements a and b satisfy the conditions 1-4 below, then there exists M^b a limit model over M_0, containing b, so that $\mathrm{gtp}(a/M^b)$ does not μ-split over N.

(1) M is universal over M_0 and M_0 is a limit model over N.
(2) $a \in M \backslash M_0$.
(3) $\mathrm{gtp}(a/M_0)$ is non-algebraic and does not μ-split over N.
(4) $\mathrm{gtp}(b/M)$ is non-algebraic and does not μ-split over M_0.

We will use that symmetry follows from categoricity in a successor cardinal. The same results holds when $\mathcal{K}$ is categorical in a high-enough cardinal, see Fact 8.16.

Fact 2.9 (Corollary 5.2 in [VVb]). Let $\mu \geq \mathrm{LS}(\mathcal{K})$. Assume that $\mathcal{K}$ has amalgamation, no maximal models, and is categorical in a cardinal λ with $\mathrm{cf}(\lambda) > \mu$ (or just that the model of size λ is μ^+-saturated). Then $\mathcal{K}$ is μ-superstable and has μ-symmetry.

Let us also recall the definition of weak tameness. We use the notation from [Bal09, Definition 11.6]

Definition 2.10. Let χ, μ be cardinals with $\mathrm{LS}(\mathcal{K}) \leq \chi \leq \mu$. Assume that $\mathcal{K}_{[\chi,\mu]}$ has amalgamation. $\mathcal{K}$ is (χ, μ)-*weakly tame* if for any saturated $M \in \mathcal{K}_\mu$, any $p, q \in \mathrm{gS}(M)$, if $p \neq q$, there exists $M_0 \in \mathcal{K}_\chi$ with $M_0 \leq M$ and $p \restriction M_0 \neq q \restriction M_0$. For $\theta \geq \mu$, $\mathcal{K}$ is $(\chi, < \theta)$-*weakly tame* if it is (χ, μ)-weakly tame for every $\mu \in [\chi, \theta)$. $(\chi, \leq \theta)$-weakly tame means $(\chi, < \theta^+)$-weakly tame. Finally, $\mathcal{K}$ is χ-*weakly tame* if it is (χ, μ)-weakly tame for every $\mu \geq \chi$.

2.1. **Good frames.** In [She09, Definition II.2.1][12], Shelah introduces *good frames*, a local notion of independence for AECs. This is the central concept of his book and has seen many other applications, such as a proof of Shelah's categoricity conjecture for universal classes [Vase]. A *good λ-frame* is a triple $\mathfrak{s} = (\mathcal{K}_\lambda, \underset{\smile}{\perp}, \mathrm{gS}^{\mathrm{bs}})$ where:

(1) $\mathcal{K}$ is a nonempty AEC which has amalgamation in λ, joint embedding in λ, no maximal models in λ, and is stable in λ.
(2) For each $M \in \mathcal{K}_\lambda$, $\mathrm{gS}^{\mathrm{bs}}(M)$ (called the set of *basic types* over M) is a set of nonalgebraic Galois types over M satisfying (among others) the *density property*: if $M < N$ are in $\mathcal{K}_\lambda$, there exists $a \in |N| \backslash |M|$ such that $\mathrm{gtp}(a/M; N) \in \mathrm{gS}^{\mathrm{bs}}(M)$.
(3) $\underset{\smile}{\perp}$ is an (abstract) independence relation on types of length one over models in $\mathcal{K}_\lambda$ satisfying the basic properties of first-order forking in a superstable theory: invariance, monotonicity, extension, uniqueness, transitivity, local character, and symmetry (see [She09, Definition II.2.1]).

As in [She09, Definition II.6.35], we say that a good λ-frame $\mathfrak{s}$ is *type-full* if for each $M \in \mathcal{K}_\lambda$, $\mathrm{gS}^{\mathrm{bs}}(M)$ consists of *all* the nonalgebraic types over M. We focus on type-full good frames in this paper and hence just write $\mathfrak{s} = (\mathcal{K}_\lambda, \underset{\smile}{\perp})$. For notational simplicity, we extend forking to algebraic types by specifying that algebraic types do not fork over their domain. Given a type-full good μ-frame $\mathfrak{s} = (\mathcal{K}_\lambda, \underset{\smile}{\perp})$ and $M_0 \leq M$ both in $\mathcal{K}_\lambda$, we say that a nonalgebraic type $p \in \mathrm{gS}(M)$ *does not $\mathfrak{s}$-fork over M_0* if it does not fork over M_0 according to the abstract independence relation $\underset{\smile}{\perp}$ of $\mathfrak{s}$. When $\mathfrak{s}$ is clear from context, we omit it and just say that *p does not fork over M_0*. We say that a good λ-frame

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[12]The definition here is simpler and more general than the original: We will *not* use Shelah's axiom (B) requiring the existence of a superlimit model of size λ. Several papers (e.g. [JS13]) define good frames without this assumption.

$\mathfrak{s}$ is *on* $\mathcal{K}_\lambda$ if its underlying class is $\mathcal{K}_\lambda$. We might also just say that $\mathfrak{s}$ is *on* $\mathcal{K}$.

We will use the following without comments. See [VVb, Fact 3.6] for a proof.

Fact 2.11. If $\mathfrak{s}$ is a type-full good λ-frame on $\mathcal{K}_\lambda$, then $\mathcal{K}$ is λ-superstable.

It was pointed out in [Vasa] (and further improvements in [Vasb, Section 10] or [VVb, Theorem 6.12]) that tameness can be combined with superstability to build a good frame. This can also be done using only weak tameness:

Fact 2.12 (Theorem 5.1 in [VVa]). Let $\lambda > \mu \geq \mathrm{LS}(\mathcal{K})$. Assume that $\mathcal{K}$ is superstable in every $\chi \in [\mu, \lambda]$ and has λ-symmetry.

If $\mathcal{K}$ is (μ, λ)-weakly tame, then there exists a type-full good λ-frame with underlying class $\mathcal{K}_\lambda^{\lambda\text{-sat}}$ (so in particular, $\mathcal{K}_\lambda^{\lambda\text{-sat}}$ is the initial segment of an AEC).

Once we have a good λ-frame, we can enlarge it so that the forking relation works over larger models. For $\mathcal{F} = [\lambda, \theta)$ an interval with $\theta \geq \lambda$ a cardinal or ∞, we define a *type-full good $\mathcal{F}$-frame* similarly to a type-full good λ-frame but require forking to be defined over models in $\mathcal{K}_\mathcal{F}$ (similarly, the good properties hold of the class $\mathcal{K}_\mathcal{F}$, e.g. $\mathcal{K}$ is stable in every $\mu \in \mathcal{F}$). See [Vasa, Definition 2.21] for a precise definition. For a type-full good $\mathcal{F}$-frame $\mathfrak{s} = (\mathcal{K}_\mathcal{F}, \smile)$ and $\mathcal{K}'$ a subclass of $\mathcal{K}_\mathcal{F}$, we define the restriction $\mathfrak{s} \upharpoonright \mathcal{K}'$ of $\mathfrak{s}$ to $\mathcal{K}'$ in the natural way (see [Vasb, Definition 3.15.(2)]).

Fact 2.13 (Corollary 6.9 in [BVc]). Let $\theta > \lambda \geq \mathrm{LS}(\mathcal{K})$. Let $\mathcal{F} := [\lambda, \theta)$. Assume that $\mathcal{K}_\mathcal{F}$ has amalgamation. Let $\mathfrak{s}$ be a type-full good λ-frame on $\mathcal{K}_\lambda$. If $\mathcal{K}$ is $(\lambda, < \theta)$-tame[13], then there exists a type-full good $\mathcal{F}$-frame $\mathfrak{s}'$ extending $\mathfrak{s}$: $\mathfrak{s}' \upharpoonright \mathcal{K}_\lambda = \mathfrak{s}$.

We obtain:

Proposition 2.14. Let $\mathcal{K}$ be an AEC with amalgamation and no maximal models. Assume that $\mathcal{K}$ is categorical in a successor $\lambda > \mathrm{LS}(\mathcal{K})^+$.

(1) If $\mu \in (\mathrm{LS}(\mathcal{K}), \lambda)$ is such that $\mathcal{K}$ is $(\mathrm{LS}(\mathcal{K}), \mu)$-weakly tame, then there exists a type-full good μ-frame with underlying class $\mathcal{K}_\mu^{\mu\text{-sat}}$.

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[13]This is defined as in Definition 2.10, i.e. for every $M \in \mathcal{K}_{[\lambda, \theta)}$ and every $p, q \in \mathrm{gS}(M)$, if $p \neq q$, there exists $M_0 \leq M$ so that $M_0 \in \mathcal{K}_\lambda$ and $p \upharpoonright M_0 \neq q \upharpoonright M_0$.

(2) Let $\theta > \mathrm{LS}(\mathcal{K})^+$. If $\mathcal{K}$ is $(\mathrm{LS}(\mathcal{K}), < \theta)$-tame, then there exists a type-full good $[\mathrm{LS}(\mathcal{K})^+, \theta)$-frame with underlying class $\mathcal{K}^{\mathrm{LS}(\mathcal{K})^+\text{-sat}}_{[\mathrm{LS}(\mathcal{K})^+, \theta)}$.

Proof. By Fact 2.9, $\mathcal{K}$ is μ-superstable and has μ-symmetry for every $\mu \in [\mathrm{LS}(\mathcal{K}), \lambda)$. Now:

(1) By Fact 2.12.
(2) Use the previous part with $\mu = \mathrm{LS}(\mathcal{K})^+$, then apply Fact 2.13. Note that by (the proof of) [VVb, Theorem 6.8], $\mathcal{K}^{\mathrm{LS}(\mathcal{K})^+\text{-sat}}_{[\mathrm{LS}(\mathcal{K})^+, \theta)}$ is the initial segment of an AEC.

$\square$

Assuming only weak tameness, we can show that if $\mathfrak{s}$ is a (type-full) good μ-frame and $\mathfrak{s}'$ is a good λ-frame with $\mu < \lambda$ and the underlying class of $\mathfrak{s}'$ is the saturated models in the underlying class of $\mathfrak{s}$, then forking in $\mathfrak{s}'$ can be described in terms of forking in $\mathfrak{s}$. This is proven as Theorem A.1 in the appendix and is used to replace tameness by weak tameness in the main theorem.

3. Shelah's omitting type theorem

In this section, we give a nonlocal proof of Theorem 7.7 using the methods of [She99]. We will present a more powerful local proof in the next sections. All throughout, we assume:

Hypothesis 3.1. $\mathcal{K}$ is an AEC with amalgamation.

The key of the proof is what we call Shelah's omitting type theorem, a generalization of Morley's omitting type theorem for AECs. The result initially appears as [She99, Lemma II.1.6]. A full proof can be found in [Shea, Lemma 8.7] or [Bal09, Theorem 14.8]. We state a simplified version for one type only.

Fact 3.2 (Shelah's omitting type theorem). Let $M_0 \leq M$ both be in $\mathcal{K}_{\geq \mathrm{LS}(\mathcal{K})}$ and let $p \in \mathrm{gS}(M_0)$. Assume that M omits $p/E_{\mathrm{LS}(\mathcal{K})}$. That is, for every $a \in |M|$, there is $M_0' \leq M_0$ with $\|M_0'\| = \mathrm{LS}(\mathcal{K})$ such that $\mathrm{gtp}(a/M_0'; M) \neq p \restriction M_0'$.

If $\beth_{(2^{\mathrm{LS}(\mathcal{K})})^+}(\|M_0\|) \leq \|M\|$, then for every $\mu \geq \mathrm{LS}(\mathcal{K})$, there exists $M_0' \leq M_0$ with $\|M_0'\| \leq \mathrm{LS}(\mathcal{K})$ and $N \in \mathcal{K}_\mu$ such that $M_0' \leq N$ and N omits $p \restriction M_0'$. In particular, there is a non $\mathrm{LS}(\mathcal{K})^+$-saturated model in every cardinal.

Note that when $\mathcal{K}$ is $\mathrm{LS}(\mathcal{K})$-tame, M above omits $p/E_{\mathrm{LS}(\mathcal{K})}$ if and only if M omits p. Thus we obtain the following generalization of the second main result of [MS90]:

Theorem 3.3. If $\mathcal{K}$ is $\mathrm{LS}(\mathcal{K})$-tame, has arbitrarily large models, and is categorical in a $\lambda > \mathrm{LS}(\mathcal{K})$, then $\mathcal{K}$ is categorical in all cardinals of the form $\beth_\delta$, where $\left(2^{\mathrm{LS}(\mathcal{K})}\right)^+$ divides δ.

Proof. Let δ be a limit ordinal such that $\left(2^{\mathrm{LS}(\mathcal{K})}\right)^+$ divides δ. We prove that every model of size $\beth_\delta$ is saturated. Suppose for a contradiction that $M \in \mathcal{K}_{\beth_\delta}$ is not saturated. Let $M_0 \leq M$ and $p \in \mathrm{gS}(M_0)$ be such that $\|M_0\| < \|M\|$ and p is omitted in M. By tameness, $p/E_{\mathrm{LS}(\mathcal{K})}$ is omitted in M. Since $\left(2^{\mathrm{LS}(\mathcal{K})}\right)^+$ divides δ, we have that $\beth_{\left(2^{\mathrm{LS}(\mathcal{K})}\right)^+}(\|M_0\|) \leq \|M\|$. Therefore by Fact 3.2 there is a non $\mathrm{LS}(\mathcal{K})^+$-saturated model in every cardinal, in particular in λ. This is a contradiction because the model of size λ is saturated. Why? By Fact 2.3, $\mathcal{K}$ is $\mathrm{LS}(\mathcal{K})$-superstable. Using tameness, by [Vasa, Theorem 5.6], $\mathcal{K}$ is stable in λ. Therefore it is easy to build a μ^+-saturated model in λ for every $\mu < \lambda$, and the result follows using categoricity. $\square$

Remark 3.4. Generalizations of this results hold also in *weakly* tame AECs. Since this is peripheral in this section, we do not state them.

We obtain a proof of Theorem 7.7.

Theorem 3.5. If $\mathcal{K}$ is $\mathrm{LS}(\mathcal{K})$-tame and is categorical in *some successor* $\lambda \geq h(\mathrm{LS}(\mathcal{K}))$, then $\mathcal{K}$ is categorical in *all* $\lambda' \geq h(\mathrm{LS}(\mathcal{K}))$.

Proof sketch. By Theorem 3.3, $\mathcal{K}$ is categorical in $h(\mathrm{LS}(\mathcal{K}))$. Now proceed as in the proof of the main result of [She99], see e.g. [Bal09, Theorem 14.14]. $\square$

Note that this method does not allow us to go lower than the Hanf number, even if we know for example that $\mathcal{K}$ is categorical below it (Shelah's argument for transferring Vaughtian pairs is not local enough). The goal of the next sections is to fix this.

4. Local orthogonality

In this section, we assume:

Hypothesis 4.1. $\mathfrak{s} = (\mathcal{K}_\lambda, \bot)$ is a type-full good λ-frame.

Our aim is to develop some orthogonality calculus as in [She09, Section III.6]. There Shelah works in a good λ-frame that is *successful* (see [She09, Definition III.1.1]). Note that by [She09, Claim III.9.6] such a good frame can be extended to a type-full one. Thus the framework we work in is more general. In fact, it is strictly more general: Hart and Shelah [HS90] have shown that for an arbitrary $n < \omega$ there exists a sentence $\phi \in L_{\omega_1,\omega}$ which is categorical in every $\aleph_m$, $m \leq n$ but not in $\aleph_{n+1}$. By [Bon14a, Theorem 10.2], when $n \geq 1$, there exists a type-full good $\aleph_{n-1}$-frame $\mathfrak{t}$ on $\mathcal{K}_{\aleph_{n-1}}$ (where $\mathcal{K}$ is the class of models of ψ ordered by an appropriate fragment), but the frame cannot be extended to a good $\aleph_n$-frame. Therefore by [She09, II.9.1] $\mathfrak{t}$ is not successful.

One of the main component of the definition of successful is the existence property for uniqueness triples (see [She09, Definition III.1.1] or [JS13, Definition 4.1.(5)]). We showed in [Vasb, Lemma 11.7] that this property is equivalent to a version of domination assuming the existence of a global independence relation. We start by developing a replacement for uniqueness triples in our setup.

Crucial in this section is the uniqueness of limit models, first proven by Shelah in [She09, Claim II.4.8] (see also [Bon14a, Theorem 9.2]).

Fact 4.2. Let $M_0, M_1, M_2 \in \mathcal{K}_\lambda$.

(1) If M_1 and M_2 are limit models, then $M_1 \cong M_2$.
(2) If in addition M_1 and M_2 are both limit over M_0, then $M_1 \cong_{M_0} M_2$.

The following appears in [She09, Claim 1.21]. It is stated for M, N superlimit but the proof goes through if M and N are limit models.

Fact 4.3 (The conjugation property)**.** If $M \leq N$ are limit models in $\mathcal{K}_\lambda$ and $p \in \mathrm{gS}(N)$ does not fork over M, then there exists $f : N \cong M$ so that $f(p) = p \restriction M$.

The next definition is a replacement for a global definition of forking. It already plays a role in [MS90] (see Lemma 4.17 there) and [BVa] (see Definition 3.10 there). A similar notion is called "smooth independence" in [VZ14].

Definition 4.4. For $M \in \mathcal{K}_{\mathcal{F}}$ and $p \in \mathrm{gS}^{<\infty}(M)$, we say that p *does not 1-$\mathfrak{s}$-fork* over M_0 if $M_0 \leq M$ and for $I \subseteq \ell(p)$ with $|I| = 1$, we have that p^I does not $\mathfrak{s}$-fork over M_0.

Notation 4.5. We write $[A]^1 \underset{M_0}{\overset{N}{\downarrow}} M$ if for some (any) enumeration $\bar{a}$ of A, $\mathrm{gtp}(\bar{a}/M; N)$ does not 1-$\mathfrak{s}$-fork over M_0. That is, $a \underset{M_0}{\overset{N}{\downarrow}} M$ for all $a \in A$.

Remark 4.6 (Disjointness). Because nonforking extensions of nonalgebraic types are nonalgebraic, if $[A]^1 \underset{M_0}{\overset{N}{\downarrow}} M$, then $|M| \cap A \subseteq |M_0|$.

The next definition is modeled on [Vasb, Definition 11.5] but uses 1-forking instead of a global independence relation.

Definition 4.7. (a, M, N) is a *weak domination triple in* $\mathfrak{s}$ if $M, N \in K_\lambda$, $M \leq N$, $a \in |N| \backslash |M|$ and for any $N' \geq N$ and $M' \leq N$ with $M \leq M'$ and $M', N' \in K_\lambda$, if $a \underset{M}{\overset{N'}{\downarrow}} M'$, then $[N]^1 \underset{M}{\overset{N'}{\downarrow}} M'$.

We now want to show the existence property for weak domination triples: For any type $p \in \mathrm{gS}(M)$, there exists a weak domination triple (a, M, N) with $p = \mathrm{gtp}(a/M; N)$. We manage to do it for limit models. The proof is a more local version of [Vasb, Lemma 11.12] (which adapted [MS90, Proposition 4.22]). First, we prove that some local character holds for 1-forking:

Lemma 4.8. Let $\langle M_i : i \leq \lambda^+ \rangle$ be increasing continuous with $M_i \in \mathcal{K}_\lambda$ for all $i < \lambda^+$. Let $N \geq M_{\lambda^+}$ and let $A \subseteq |N|$ be such that $|A| \leq \lambda$. Let $N_i \leq N$ be in $\mathcal{K}_\lambda$ and contain $A \cup |M_i|$. Then there exists $i < \lambda^+$ such that $[A]^1 \underset{M_i}{\overset{N_j}{\downarrow}} M_j$ for all $j \geq i$.

Proof. We first show:

<u>Claim.</u> For each $a \in A$ there exists $i_a < \lambda^+$ so that $a \underset{M_{i_a}}{\overset{N_j}{\downarrow}} M_j$ for all $j \geq i_a$.

This is enough because then we can take $i := \sup_{a \in A} i_a$.

<u>Proof of claim.</u> By local character, for each limit $j < \lambda^+$, there exists $i_j < j$ so that $a \underset{M_{i_j}}{\overset{N_j}{\downarrow}} M_j$. By Fodor's lemma, there exists $i_a < \lambda^+$ such that for unboundedly many $j < \lambda^+$, $i_j = i_a$. By monotonicity of forking, i_a is as desired. $\square$

Lemma 4.9. Assume $\langle M_i : i < \lambda^+ \rangle$, $\langle N_i : i < \lambda^+ \rangle$ are increasing continuous in $\mathcal{K}_\lambda$ such that $M_i \le N_i$ for all $i < \lambda^+$. Then there exists $i < \lambda^+$ such that $[N_i]^1 \underset{M_i}{\overset{N_j}{\downdownarrows}} M_j$ for all $j \ge i$.

Proof. By Lemma 4.8, for each $i < \lambda^+$, there exists $j_i < \lambda^+$ such that $[N_i]^1 \underset{M_{j_i}}{\overset{N_j}{\downdownarrows}} M_j$ for all $j \ge j_i$. Let $i^* < \lambda^+$ be such that $j_i < i^*$ for all $i < i^*$. Then it is easy to check that $[N_{i^*}]^1 \underset{M_{i^*}}{\overset{N_j}{\downdownarrows}} M_j$ for all $j \ge i^*$, which is as needed. $\square$

Theorem 4.10. Let $M \in K_\lambda$ be a limit model. For each nonalgebraic $p \in \mathrm{gS}(M)$, there exists a weak domination triple (a, M, N) such that $p = \mathrm{gtp}(a/M; N)$.

Proof. Assume not. We know that M is a limit model. Therefore by local character there exists $M^* \in K_\lambda$ such that M is limit over M^* and p does not fork over M^*.

Claim. For any limit $M' \ge M$ with $M' \in \mathcal{K}_\lambda$, if $q \in \mathrm{gS}(M')$ is the nonforking extension of p, then there is no weak domination triple (b, M', N') such that $q = \mathrm{gtp}(b/M'; N')$.

Proof of claim. By the conjugation property (Fact 4.3), there exists $f : M' \cong M$ such that $f(q) = p$. Now use that weak domination triples are invariant under isomorphisms. $\dagger_{\mathrm{Claim}}$

We construct $\langle M_i : i < \lambda^+ \rangle$, $\langle N_i : i < \lambda^+ \rangle$ increasing continuous such that for all $i < \lambda^+$:

(1) $M_0 = M$.
(2) $M_i \le N_i$ are both in $\mathcal{K}_\lambda$.
(3) M_{i+1} is limit over M_i and N_{i+1} is limit over N_i.
(4) $\mathrm{gtp}(a/M_i; N_i)$ is the nonforking extension of p. In particular, $a \underset{M_0}{\overset{N_i}{\downdownarrows}} M_i$.
(5) $[N_i]^1 \underset{M_i}{\overset{N_{i+1}}{\not\downdownarrows}} M_{i+1}$.

This is enough, since then we get a contradiction to Lemma 4.9. This is possible: If $i = 0$, let $N_0 \in \mathcal{K}_\lambda$ be such that $p = \mathrm{gtp}(a/M; N_0)$. At limits, take unions. Now assume everything up to i has been constructed. By the claim, (a, M_i, N_i) cannot be a weak domination triple. This

means there exists $M_i' \geq M_i$ and $N_i' \geq N_i$ all in $\mathcal{K}_\lambda$ such that $a \underset{M_i}{\overset{N_i'}{\smile}} M_i'$
but $[N_i]^1 \underset{M_i}{\overset{N_i'}{\not\smile}} M_i'$. By the extension property of forking, pick $M_{i+1} \in \mathcal{K}_\lambda$
limit over M_i containing M_i' and $N_{i+1} \geq N_i'$ such that N_{i+1} is limit over
N_i and $a \underset{M_i}{\overset{N_{i+1}}{\smile}} M_{i+1}$.   □

We now give a definition of orthogonality in terms of independent sequences.

Definition 4.11 (Independent sequence, III.5.2 in [She09]). Let α be an ordinal.

(1) $\langle a_i : i < \alpha \rangle \frown \langle M_i : i \leq \alpha \rangle$ is said to be *independent in* (M, M', N) when:
   (a) $(M_i)_{i \leq \alpha}$ is increasing continuous in $\mathcal{K}_\lambda$.
   (b) $M \leq M' \leq M_0$ and $M, M' \in \mathcal{K}_\lambda$.
   (c) $M_\alpha \leq N$ and $N \in \mathcal{K}_\lambda$.
   (d) For every $i < \alpha$, $a_i \underset{M}{\overset{M_{i+1}}{\smile}} M_i$.

   $\langle a_i : i < \alpha \rangle \frown \langle M_i : i \leq \alpha \rangle$ is said to be *independent over M* when it is independent in (M, M_0, M_α).
(2) $\bar{a} := \langle a_i : i < \alpha \rangle$ is said to be *independent in* (M, M_0, N) when $M \leq M_0 \leq N$, $\bar{a} \in {}^\alpha|N|$, and for some $\langle M_i : i \leq \alpha \rangle$ and a model N^+ such that $M_\alpha \leq N^+$, $N \leq N^+$, and $\langle a_i : i < \alpha \rangle \frown \langle M_i : i \leq \alpha \rangle$ is independent over M. When $M = M_0$, we omit it and just say that $\bar{a}$ is independent in (M, N).

Remark 4.12. We will use the definition above when $\alpha = 2$. In this case, we have that $\langle ab \rangle$ is independent in (M, N) if and only if $a \underset{M}{\overset{N}{\smile}} b$ (technically, the right hand side of the $\smile$ relation must be a model but we can remedy this by extending the nonforking relation in the natural way, as in the definition of the minimal closure in [BGKV, Definition 3.4]).

Definition 4.13. Let $M \in \mathcal{K}_\lambda$ and let $p, q \in \mathrm{gS}(M)$ be nonalgebraic. We say that p is *weakly orthogonal to* q and write $p \underset{\mathrm{wk}}{\overset{\mathfrak{s}}{\perp}} q$ (or just $p \underset{\mathrm{wk}}{\perp} q$ if $\mathfrak{s}$ is clear from context) if for all $N \geq M$ and all $a, b \in |N|$ such that $\mathrm{gtp}(a/M; N) = p$ and $\mathrm{gtp}(b/M; N) = q$, $\langle ab \rangle$ is independent in (M, N).

We say that p *is orthogonal to* q (written $p \overset{\mathfrak{s}}{\perp} q$, or just $p \perp q$ if $\mathfrak{s}$ is clear from context) if for every $N \in \mathcal{K}_\lambda$ with $N \geq M$, $p' \underset{\text{wk}}{\perp} q'$, where p', q' are the nonforking extensions to N of p and q respectively.

Remark 4.14. Definition 4.13 is equivalent to Shelah's ([She09, Definition III.6.2]), see [She09, Claim III.6.4.(2)] assuming that $\mathfrak{s}$ is *successful*. By a similar proof (and assuming that $\mathfrak{s}$ has primes), it is also equivalent to the definition in terms of primes in [Vasd, Definition 2.2].

We will use the following consequence of symmetry:

Fact 4.15 (Theorem 4.2 in [JS12]). For any $M_0 \leq M \leq N$ all in $\mathcal{K}_\lambda$, if $a, b \in |N| \backslash |M_0|$, then $\langle ab \rangle$ is independent in (M_0, M, N) if and only if $\langle ba \rangle$ is independent in (M_0, M, N).

Lemma 4.16. Let $M \in \mathcal{K}_\lambda$. Let $p, q \in \mathrm{gS}(M)$ be nonalgebraic.

(1) If M is limit, then $p \perp q$ if and only if $p \underset{\text{wk}}{\perp} q$.

(2) $p \underset{\text{wk}}{\perp} q$ if and only if $q \underset{\text{wk}}{\perp} p$.

(3) If $p \underset{\text{wk}}{\perp} q$, then whenever (a, M, N) is a weak domination triple representing q, p is omitted in N. In particular, if M is limit, there exists $N \in \mathcal{K}_\lambda$ with $M < N$ so that p is omitted in N.

Proof.

(1) By the conjugation property (Fact 4.3). See the proof of [Vasd, Lemma 2.6].

(2) By Fact 4.15.

(3) Let $N' \geq N$ be in $\mathcal{K}_\lambda$ and let $b \in |N'|$ realize p. We have that $\langle ab \rangle$ is independent in (M, N'). Therefore there exists $N'' \geq N$ in $\mathcal{K}_\lambda$, $M' \in \mathcal{K}_\lambda$ so that $M \leq M' \leq N''$, $b \in |M'|$, and $a \overset{N''}{\underset{M}{\smile}} M'$. By domination, $[N]^1 \overset{N''}{\underset{M}{\smile}} M'$, so by disjointness (Remark 4.6), $b \notin |N|$. The last sentence follows from the existence property for weak domination triple (Theorem 4.10).

$\square$

5. Unidimensionality

Hypothesis 5.1. $\mathfrak{s} = (\mathcal{K}_\lambda, \perp)$ is a type-full good λ-frame and $\mathcal{K}$ is categorical in λ.

In this section we give a definition of unidimensionality similar to the ones in [She90, Definition V.2.2] or [She09, Section III.2]. We show that $\mathfrak{s}$ is unidimensional if and only if $\mathcal{K}$ is categorical in λ^+ (this uses categoricity in λ). In the next section, we will show how to transfer unidimensionality across cardinals, hence getting the promised categoricity transfer. In [She09, Section III.2], Shelah gives several different definitions of unidimensionality and also shows (see [She09, III.2.3, III.2.9]) that the so-called "weak-unidimensionality" is equivalent to categoricity in λ^+ (hence our definition is equivalent to Shelah's weak unidimensionality) but it is unclear how to transfer it across cardinals without assuming that the frame is successful.

Note that the hypothesis of categoricity in λ implies that the model of size λ is limit, hence weak orthogonality and orthogonality coincide, see Lemma 4.16.

Rather than defining what it means to be unidimensional, we find it clearer to define what it means to *not* be unidimensional:

Definition 5.2. $\mathfrak{s}$ is *unidimensional* if the following is *false*: for every $M \in \mathcal{K}_\lambda$ and every nonalgebraic $p \in \mathrm{gS}(M)$, there exists $M' \in \mathcal{K}_\lambda$ with $M' \geq M$ and nonalgebraic $p', q \in \mathrm{gS}(M')$ so that p' extends p and $p' \perp q$.

We first give an equivalent definition using minimal types:

Definition 5.3. For $M \in \mathcal{K}_\lambda$, a type $p \in \mathrm{gS}(M)$ is *minimal* if for every $M' \geq M$ with $M' \in \mathcal{K}_\lambda$, p has a unique nonalgebraic extension to $\mathrm{gS}(M')$.

Remark 5.4. If $M \leq N$ are in $\mathcal{K}_\lambda$ and $p \in \mathrm{gS}(N)$ is nonalgebraic such that $p \upharpoonright M$ is minimal, then p does not fork over M (because the nonforking extension of $p \upharpoonright M$ has to be p).

By the proof of $(*)_5$ in [She99, Theorem II.2.7]:

Fact 5.5 (Density of minimal types)**.** For any $M \in \mathcal{K}_\lambda$ and nonalgebraic $p \in \mathrm{gS}(M)$, there exists $M' \in \mathcal{K}_\lambda$ and $p' \in \mathcal{K}_\lambda$ such that $M \leq M'$, p' extends p, and p' is minimal.

Lemma 5.6. The following are equivalent:

(1) $\mathfrak{s}$ is not unidimensional.
(2) For every $M \in \mathcal{K}_\lambda$ and every minimal $p \in \mathrm{gS}(M)$, there exists $M' \geq M$ with $M' \in \mathcal{K}_\lambda$ and $p', q \in \mathrm{gS}(M')$ nonalgebraic so that $p' \perp q$.

(3) For every $M \in \mathcal{K}_\lambda$ and every minimal $p \in \mathrm{gS}(M)$, there exists a nonalgebraic $q \in \mathrm{gS}(M)$ with $p \perp q$.

Proof. (1) implies (2) because (2) is a special case of (1). Conversely, (2) implies (1): given $M \in \mathcal{K}_\lambda$ and $p \in \mathrm{gS}(M)$, first use density of minimal types to extend p to a minimal $p' \in \mathrm{gS}(M')$ (so $M' \in \mathcal{K}_\lambda$ $M \leq M'$). Then apply (2).

Also, if (3) holds, then (2) holds with $M = M'$. Conversely, assume that (2) holds. Let $p \in \mathrm{gS}(M)$ be minimal and let p', q, M' witness (2), i.e. $p', q \in \mathrm{gS}(M')$, p' extends p and $p' \perp q$. By Remark 5.4, p' does not fork over M. By the conjugation property (Fact 4.3), there exists $f : M' \cong M$ so that $f(p') = p$. Thus $p \perp f(q)$, hence (2) holds. $\square$

We use the characterization to show that unidimensionality implies categoricity in λ^+. This is similar to [MS90, Proposition 4.25] but the proof is slightly more involved since our definition of unidimensionality is weaker. We start with a version of density of minimal types inside a fixed model. We will use the following fact, whose proof is a straightforward direct limit argument:

Fact 5.7 (Theorem 11.1 in [Bal09]). Let $\langle M_i : i \leq \omega \rangle$ be an increasing continuous chain in $\mathcal{K}_\lambda$ and for each $i < \omega$, let $p_i \in \mathrm{gS}(M_i)$ be such that $j < i$ implies $p_i \restriction M_j = p_j$. Then there exists $p \in \mathrm{gS}(M_\omega)$ so that $p \restriction M_i = p_i$ for all $i < \omega$.

Lemma 5.8. Let $M_0 \leq M$ with $M_0 \in \mathcal{K}_\lambda$ and $M \in \mathcal{K}_{>\lambda}$. Let $p \in \mathrm{gS}(M_0)$. Then there exists $M_1 \in \mathcal{K}_\lambda$ with $M_0 \leq M_1 \leq M$ and $q \in \mathrm{gS}(M_1)$ so that q extends p and for all $M' \leq M$ with $M' \in \mathcal{K}_\lambda$, $M_1 \leq M'$, any extension of q to $\mathrm{gS}(M')$ does not fork over M_1.

Proof. Suppose not. Build $\langle N_i : i < \omega \rangle$ increasing in $\mathcal{K}_\lambda$ and $\langle q_i : i < \omega \rangle$ such that for all $i < \omega$:

(1) $N_0 = M_0$, $q_0 = p$.
(2) $N_i \leq M$.
(3) $q_i \in \mathrm{gS}(N_i)$ and q_{i+1} extends q_i.
(4) q_{i+1} forks over N_i.

This is possible since we assumed that the lemma failed. This is enough: let $N_\omega := \bigcup_{i<\omega} N_i$. Let $q \in \mathrm{gS}(N_\omega)$ extend each q_i (exists by Fact 5.7). By local character, there exists $i < \omega$ such that q does not fork over N_i, so $q \restriction N_{i+1} = q_{i+1}$ does not fork over N_i, contradiction. $\square$

Lemma 5.9. If $\mathfrak{s}$ is unidimensional, then $\mathcal{K}$ is categorical in λ^+.

Proof. Assume that $\mathcal{K}$ is *not* categorical in λ^+. We show that (2) of Lemma 5.6 holds so $\mathfrak{s}$ is not unidimensional. Let $M_0 \in \mathcal{K}_\lambda$ and let $p \in \mathrm{gS}(M_0)$ be minimal. We consider two cases:

<u>Case 1</u>. There exists $M \in \mathcal{K}_{\lambda^+}$, $M_1 \in \mathcal{K}_\lambda$ with $M_0 \leq M_1 \leq M$ and an extension $p' \in \mathrm{gS}(M_1)$ of p so that p' is omitted in M.

Let $c \in |M| \backslash |M_1|$. Fix $M' \leq M$ in $\mathcal{K}_\lambda$ containing c so that $M_1 \leq M'$ and let $q := \mathrm{gtp}(c/M_1; M')$. We claim that $q \underset{\mathrm{wk}}{\perp} p'$ (and so by Lemma 4.16, $p' \perp q$, as needed). Let $N \in \mathcal{K}_\lambda$ be such that $N \geq M_1$ and let $a, b \in |N|$ be such that $p' = \mathrm{gtp}(b/M_1; N)$, $q = \mathrm{gtp}(a/M_1; N)$. We want to see that $\langle ba \rangle$ is independent in (M_1, N). We have that $\mathrm{gtp}(a/M_1; N) = \mathrm{gtp}(c/M_1; M')$, so let $N' \in \mathcal{K}_\lambda$ with $M' \leq N'$ and $f : N \underset{M_1}{\longrightarrow} N'$ witness it, i.e. $f(a) = c$. Let $b' := f(b)$. We have that $\mathrm{gtp}(b'/M'; N')$ extends p', and $b' \notin |M'|$ since p' is omitted in M, hence by minimality $\mathrm{gtp}(b'/M'; N')$ does not fork over M_1. In particular, $\langle cb' \rangle$ is independent in (M_1, N'). By invariance and monotonicity, $\langle ba \rangle$ is independent in (M_1, N).

<u>Case 2</u>. Not Case 1: For every $M \in \mathcal{K}_{\lambda^+}$, every $M_1 \in \mathcal{K}_\lambda$ with $M_0 \leq M_1 \leq M$, every extension $p' \in \mathrm{gS}(M_1)$ of p is realized in M.

By categoricity in λ and non-categoricity in λ^+, we can find $M \in \mathcal{K}_{\lambda^+}$ with $M_0 \leq M$ and $q_0 \in \mathrm{gS}(M_0)$ omitted in M. Let $M_1 \in \mathcal{K}_\lambda$, $M_0 \leq M_1 \leq M$ and $q \in \mathrm{gS}(M_1)$ extend q_0 so that any extension of q to a model $M' \leq M$ in $\mathcal{K}_\lambda$ does not fork over M_1 (this exists by Lemma 5.8). Let $p' \in \mathrm{gS}(M_1)$ be a nonalgebraic extension of p. By assumption, p is realized by some $c \in |M|$. Now by the same argument as above (reversing the roles of p' and q), $p' \underset{\mathrm{wk}}{\perp} q$, hence $p' \perp q$, as desired.   □

For the converse of Lemma 5.9, we will use:

Fact 5.10 (Theorem 6.1 in [GV06a]). Assume that $\mathcal{K}$ is categorical in λ^+. Then there exists $M \in \mathcal{K}_\lambda$ and a minimal type $p \in \mathrm{gS}(M)$ which is realized in every $N \in \mathcal{K}_\lambda$ with $M < N$.

Remark 5.11. The proof of Fact 5.10 uses categoricity in λ in a strong way (it uses that the union of an increasing chain of limit models is limit).

Lemma 5.12. If $\mathcal{K}$ is categorical λ^+, then $\mathfrak{s}$ is unidimensional.

Proof. By Fact 5.10, there exists $M \in \mathcal{K}_\lambda$ and a minimal $p \in \mathrm{gS}(M)$ so that p is realized in every $N > M$. Now assume for a contradiction that $\mathcal{K}$ is *not* unidimensional. Then by Lemma 5.6, there exists a

nonalgebraic $q \in \mathrm{gS}(M)$ such that $p \perp q$. By Lemma 4.16.(3) (note that M is limit by categoricity in λ), there exists $N \in \mathcal{K}_\lambda$ with $N > M$ so that p is omitted in N, a contradiction to the choice of p. $\qquad \square$

Theorem 5.13. $\mathfrak{s}$ is unidimensional if and only if $\mathcal{K}$ is categorical in λ^+.

Proof. By Lemmas 5.9 and 5.12. $\qquad \square$

6. Global orthogonality

Hypothesis 6.1.

  (1) $\mathcal{K}$ is an AEC.
  (2) $\theta > \mathrm{LS}(\mathcal{K})$ is a cardinal or ∞. We set $\mathcal{F} := [\mathrm{LS}(\mathcal{K}), \theta)$.
  (3) $\mathfrak{s} = (\mathcal{K}_\mathcal{F}, \underset{\smile}{\perp})$ is a type-full good $\mathcal{F}$-frame.

We start developing the theory of orthogonality and unidimensionality in a more global context (with no real loss, the reader can think of $\theta = \infty$ as being the main case). The main problem is to show that for M sufficiently saturated, if $p, q \in \mathrm{gS}(M)$ do not fork over M_0, then $p \perp q$ if and only if $p \upharpoonright M_0 \perp q \upharpoonright M_0$. This can be done with the conjugation property in case $\|M_0\| = \|M\|$ but in general one needs to use more tools from the study of independent sequences. We start by recalling a few facts that we will use without further mention:

Fact 6.2. For any $\lambda \in \mathcal{F}$ with $\lambda > \mathrm{LS}(\mathcal{K})$, $\mathcal{K}^{\lambda\text{-sat}}_{[\lambda,\theta)}$ is the initial segment of an AEC with Löwenheim-Skolem number λ.

Proof. By uniqueness of limit models 4.2 and [Vanb, Corollary 3], see also the proof of [VVb, Theorem 6.6]. $\qquad \square$

Fact 6.3. For $\lambda \in \mathcal{F}$, $M \in \mathcal{K}^{\lambda\text{-sat}}_\lambda$ if and only if M is limit.

Proof. This is trivial if $\lambda = \mathrm{LS}(\mathcal{K})$, so assume that $\lambda > \mathrm{LS}(\mathcal{K})$. If M is limit, then by uniqueness of limit models, M is saturated. Conversely if M is saturated, then it must be unique, hence isomorphic to a limit model. $\qquad \square$

We will also use a few more facts about independent sequences:

Fact 6.4 (Corollary 5.6 in [BVc]). Independent sequences of length two satisfy the axioms of a good $\mathcal{F}$-frame. For example:

(1) Monotonicity: If $\langle ab \rangle$ is independent in (M_0, M, N) and $M_0 \leq M_0' \leq M' \leq M \leq N \leq N'$, then $\langle ab \rangle$ is independent in (M_0', M', N').

(2) Continuity: If $\langle M_i : i \leq \delta \rangle$ is increasing continuous, $M_\delta \leq N$, and $\langle ab \rangle$ is independent in (M_0, M_i, N) for all $i < \delta$, then $\langle ab \rangle$ is independent in (M_0, M_δ, N).

Remark 6.5. The reader may ask inside which frame we work when we say that $\langle ab \rangle$ is independent (the global frame $\mathfrak{s}$ or its restriction?). By monotonicity, the answer does not matter, i.e. the independent sequences are the same either way. Similarly, if $\lambda > \mathrm{LS}(\mathcal{K})$ and $M_0, M, N \in \mathcal{K}_{[\lambda,\theta)}^{\lambda\text{-sat}}$, then $\langle ab \rangle$ is independent in (M_0, M, N) with respect to $\mathfrak{s}$ if and only if it is independent in (M_0, N) with respect to $\mathfrak{s} \upharpoonright \mathcal{K}_{[\lambda,\theta)}^{\lambda\text{-sat}}$ (i.e. we can require the models witnessing the independence to be saturated). This is a simple consequence of the extension property.

We now define global orthogonality.

Definition 6.6. Let $M \in \mathcal{K}_\mathcal{F}$. For $p, q \in \mathrm{gS}(M)$ nonalgebraic, we write $p \perp q$ for $p \overset{\mathfrak{s} \upharpoonright \mathcal{K}_{\|M\|}}{\perp} q$, and $p \underset{\mathrm{wk}}{\perp} q$ for $p \overset{\mathfrak{s} \upharpoonright \mathcal{K}_{\|M\|}}{\underset{\mathrm{wk}}{\perp}} q$ (recall Definition 4.13).

Note that a priori we need not have that if $p \perp q$ and p', q' are nonforking extensions of p and q to big models, then $p' \perp q'$. This will be proven first.

Lemma 6.7. Let δ be a limit ordinal. Let $\langle M_i : i \leq \delta \rangle$ be increasing continuous in $\mathcal{K}_\mathcal{F}$. Let $p, q \in \mathrm{gS}(M_\delta)$ be nonalgebraic and assume that $p \upharpoonright M_i \underset{\mathrm{wk}}{\perp} q \upharpoonright M_i$ for all $i < \delta$. Then $p \underset{\mathrm{wk}}{\perp} q$.

Proof. By the continuity property of independent sequences (Fact 6.4). □

Lemma 6.8. Let δ be a limit ordinal. Let $\langle M_i : i \leq \delta \rangle$ be increasing continuous in $\mathcal{K}_\mathcal{F}$. Let $p, q \in \mathrm{gS}(M_\delta)$ be nonalgebraic and assume that $p \upharpoonright M_i \perp q \upharpoonright M_i$ for all $i < \delta$. Then $p \perp q$.

Proof. By local character, there exists $i < \delta$ so that both p and q do not fork over M_i. Without loss of generality, $i = 0$. Let $\lambda := \|M_\delta\|$. If there exists $i < \delta$ so that $\lambda = \|M_i\|$, then the result follows from the definition of orthogonality. So assume that $\|M_i\| < \lambda$ for all $i < \delta$. Let $M' \geq M_\delta$ be in $\mathcal{K}_\lambda$ and let p', q' be the nonforking extensions to M'

of p, q respectively. We want to see that $p' \underset{\text{wk}}{\perp} q'$. Let $\langle M_i' : i \leq \delta \rangle$ be an increasing continuous resolution of M' such that $M_i \leq M_i'$ and $\|M_i'\| = \|M_i\|$ for all $i < \delta$. We know that $p' \upharpoonright M_i'$ does not fork over M_0, hence over M_i and similarly $q' \upharpoonright M_i'$ does not fork over M_i. Therefore by definition of orthogonality, $p' \upharpoonright M_i' \underset{\text{wk}}{\perp} q' \upharpoonright M_i'$. By Lemma 6.7, $p' \underset{\text{wk}}{\perp} q'$. $\square$

Lemma 6.9. Let $M_0 \leq M$ be both in $\mathcal{K}_{\mathcal{F}}$. Let $p, q \in \mathrm{gS}(M)$ be nonalgebraic so that both do not fork over M_0. If $p \upharpoonright M_0 \perp q \upharpoonright M_0$, then $p \perp q$.

Proof. Let $\delta := \mathrm{cf}(\|M\|)$. Build $\langle N_i : i \leq \delta \rangle$ increasing continuous such that $N_0 = M_0$, $N_\delta = M$, and $p \upharpoonright N_i \perp q \upharpoonright N_i$ for all $i \leq \delta$. This is easy: at successor steps, we require $\|N_i\| = \|N_{i+1}\|$ and use the definition of orthogonality. At limit steps, we use Lemma 6.8. Then $p \upharpoonright N_\delta \perp q \upharpoonright N_\delta$, but $N_\delta = M$ so $p \perp q$. $\square$

Question 6.10. Is the converse true? That is if $M_0 \leq M$ are in $\mathcal{K}_{\mathcal{F}}$, $p, q \in \mathrm{gS}(M)$ do not fork over M_0 and $p \perp q$, do we have that $p \upharpoonright M_0 \perp q \upharpoonright M_0$?

An answer to this question would be useful in order to transfer unidimensionality up in a more conceptual way than below. With a very mild additional hypothesis, we give a positive answer in Theorem C.4 of the appendix.

We now go back to studying unidimensionality.

Definition 6.11. For $\lambda \in \mathcal{F}$, we say that $\mathfrak{s}$ is λ-*unidimensional* if the following is *false*: for every limit $M \in \mathcal{K}_\lambda$ and every nonalgebraic $p \in \mathrm{gS}(M)$, there exists a limit $M' \geq M$ in $\mathcal{K}_\lambda$ and $p', q \in \mathrm{gS}(M')$ so that p' extends p and $p' \perp q$.

Remark 6.12. When $\lambda > \mathrm{LS}(\mathcal{K})$, $\mathfrak{s}$ is λ-unidimensional if and only if $\mathfrak{s} \upharpoonright \mathcal{K}_\lambda^{\lambda\text{-sat}}$ is unidimensional (see Definition 5.2). If $\mathcal{K}$ is categorical in $\mathrm{LS}(\mathcal{K})$, this also holds when $\lambda = \mathrm{LS}(\mathcal{K})$ (if $\mathcal{K}$ is not categorical in $\mathrm{LS}(\mathcal{K})$, we do not know that $\mathcal{K}^{\mathrm{LS}(\mathcal{K})\text{-sat}}$ is an AEC).

Our next goal is to prove (assuming categoricity in $\mathrm{LS}(\mathcal{K})$) that λ-unidimensionality is equivalent to μ-unidimensionality for every $\lambda, \mu \in \mathcal{F}$. We will use another characterization of λ-unidimensionality when $\lambda > \mathrm{LS}(\mathcal{K})$:

Lemma 6.13. Let $\lambda > \mathrm{LS}(\mathcal{K})$ be in $\mathcal{F}$. The following are equivalent:

(1) $\mathfrak{s}$ is *not* λ-unidimensional.
(2) There exists a saturated $M \in \mathcal{K}_\lambda$ and nonalgebraic types $p, q \in \mathrm{gS}(M)$ such that p is minimal and $p \perp q$.

Proof.

- (1) implies (2): Assume that $\mathfrak{s}$ is not λ-unidimensional. Let $M \in \mathcal{K}_\lambda^{\lambda\text{-sat}}$ and let $p \in \mathrm{gS}(M)$ be minimal (exists by density of minimal types and uniqueness of saturated models). By Lemma 5.6, there exists $q \in \mathrm{gS}(M)$ so that $p \perp q$.
- (2) implies (1): Let $M \in \mathcal{K}_\lambda^{\lambda\text{-sat}}$ and let $p, q \in \mathrm{gS}(M)$ be non-algebraic so that p is minimal and $p \perp q$. We show that $\mathcal{K}^{\lambda\text{-sat}}$ is *not* categorical in λ^+, which is enough by Theorem 5.13. Fix $N \leq M$ in $\mathcal{K}_{\mathrm{LS}(\mathcal{K})}$ so that p does not fork over N. Build a strictly increasing continuous chain $\langle M_i : i \leq \lambda^+ \rangle$ such that for all $i < \lambda^+$:
  (1) $M_i \in \mathcal{K}_\lambda^{\lambda\text{-sat}}$.
  (2) $M_0 = M$.
  (3) p is omitted in M_i.

  This is enough, since then p is omitted in M_{λ^+} so $M_{\lambda^+} \in \mathcal{K}_{\lambda^+}$ cannot be saturated. This is possible: at limits we take unions and for $i = 0$ we set $M_0 := M$. Now let $i = j + 1$ be given. Let $p' \in \mathrm{gS}(M_j)$ be the nonforking extension of p. By uniqueness of saturated models, there exists $f : M_j \cong_N M_0$. By uniqueness of nonforking extension, $f(p') = p$. By Lemma 4.16.(3), there exists $M' \geq M_0$ in $\mathcal{K}_\lambda^{\lambda\text{-sat}}$ so that p is omitted in M'. Let $M_{j+1} := f^{-1}[M']$. Then p' is omitted in M_{j+1}. Since p is minimal, p is omitted in $|M_{j+1}| \setminus |M_j|$, and hence by induction in M_{j+1}.

$\square$

An issue in transferring unidimensionality up is that we do not have a converse to Lemma 6.9 (see Question 6.10), so we will "cheat" and use the following transfer which follows from the proof of [GV06a, Theorem 6.3]:

Fact 6.14. If $\mathcal{K}$ is categorical in $\mathrm{LS}(\mathcal{K})$ and $\mathrm{LS}(\mathcal{K})^+$, then $\mathcal{K}$ is categorical in all $\mu \in [\mathrm{LS}(\mathcal{K}), \theta]$.

Theorem 6.15. Assume that $\mathcal{K}$ is categorical in $\mathrm{LS}(\mathcal{K})$. Let λ and μ both be in $\mathcal{F}$. Then $\mathfrak{s}$ is λ-unidimensional if and only if $\mathfrak{s}$ is μ-unidimensional.

Proof. Without loss of generality, $\mu < \lambda$. We first show that if $\mathfrak{s}$ is *not* μ-unidimensional, then $\mathfrak{s}$ is *not* λ-unidimensional. Assume that $\mathfrak{s}$ is not μ-unidimensional. Let $M_0 \in \mathcal{K}_\mu^{\mu\text{-sat}}$ and let $p \in \mathrm{gS}(M_0)$ be minimal (exists by density of minimal types). By definition (and the proof of Lemma 5.6), there exists $q \in \mathrm{gS}(M_0)$ so that $p \perp q$. Now let $M \in \mathcal{K}_\lambda^{\lambda\text{-sat}}$ be such that $M_0 \leq M$. Let p', q' be the nonforking extensions to M of p and q respectively. By Lemma 6.9, $p' \perp q'$. By Lemma 6.13, $\mathfrak{s}$ is not λ-unidimensional.

Conversely, assume that $\mathfrak{s}$ is μ-unidimensional. By the first part, $\mathfrak{s}$ is $\mathrm{LS}(\mathcal{K})$-unidimensional. By Theorem 5.13, $\mathcal{K}$ is categorical in $\mathrm{LS}(\mathcal{K})^+$. By Fact 6.14, $\mathcal{K}$, and hence $\mathcal{K}^{\lambda\text{-sat}}$, is categorical in λ^+. By Theorem 5.13 again, $\mathfrak{s}$ is λ-unidimensional. $\square$

We obtain the promised categoricity transfer. Note that it suffices to assume that $\mathcal{K}^{\lambda\text{-sat}}$ (not $\mathcal{K}$) is categorical in λ^+.

Corollary 6.16. Assume that $\mathcal{K}$ is categorical in $\mathrm{LS}(\mathcal{K})$ and let $\lambda \in \mathcal{F}$. If $\mathcal{K}^{\lambda\text{-sat}}$ is categorical in λ^+, then $\mathcal{K}$ is categorical in every $\mu \in [\mathrm{LS}(\mathcal{K}), \theta]$.

Proof. Assume that $\mathcal{K}^{\lambda\text{-sat}}$ is categorical in λ^+. We prove by induction on $\mu \in [\mathrm{LS}(\mathcal{K}), \theta]$ that $\mathcal{K}$ is categorical in μ. By assumption, $\mathcal{K}$ is categorical in $\mathrm{LS}(\mathcal{K})$. Now let $\mu \in (\mathrm{LS}(\mathcal{K}), \theta]$ and assume that $\mathcal{K}$ is categorical in every $\mu_0 \in [\mathrm{LS}(\mathcal{K}), \mu)$. If μ is limit, then it is easy to see that every model of size μ must be saturated, hence $\mathcal{K}$ is categorical in μ. Now assume that μ is a successor, say $\mu = \mu_0^+$ for $\mu_0 \in \mathcal{F}$. By assumption, $\mathcal{K}^{\lambda\text{-sat}}$ is categorical in λ^+. By Theorem 5.13, $\mathfrak{s}$ is λ-unidimensional. By Theorem 6.15, $\mathfrak{s}$ is μ_0-unidimensional. By Theorem 5.13, $\mathcal{K}^{\mu_0\text{-sat}}$ is categorical in μ_0^+. By the induction hypothesis, $\mathcal{K}$ is categorical in μ_0, hence $\mathcal{K}^{\mu_0\text{-sat}} = \mathcal{K}_{\geq \mu_0}$, so $\mathcal{K}$ is categorical in $\mu_0^+ = \mu$, as desired. $\square$

We finish this section by combining our results with the categoricity transfer in tame AECs with primes of [Vasd]. As there, we say that $\mathcal{K}$ *has primes* if it has primes over sets of the form $M \cup \{a\}$. That is (see [She09, Section II.3]), if $M \leq N$ are in $\mathcal{K}$ and $a \in |N| \backslash |M|$, there exists $N_0 \leq N$ so that $M \leq N_0$, $a \in |N_0|$, and whenever $N' \in \mathcal{K}$, $b \in |N'|$ are such that $\mathrm{gtp}(b/M; N') = \mathrm{gtp}(a/M; N)$, there exists $f : N_0 \xrightarrow[M]{} N'$ with $f(a) = b$. We define localizations such as "$\mathcal{K}_\mathcal{F}$ has primes" in the natural way.

The value of AECs with primes is that a categoricity transfer from categoricity in a *limit* cardinal holds. The next fact is a local version of [Vasd, Theorem 2.16] (which follows directly from its proof):

Fact 6.17. Assume that $\mathcal{K}$ is categorical in $\mathrm{LS}(\mathcal{K})$ and $\mathcal{K}_{\mathcal{F}}$ has primes. If $\mathcal{K}$ is categorical in *some* $\lambda \in [\mathrm{LS}(\mathcal{K})^+, \theta]$, then $\mathcal{K}$ is categorical in *every* $\mu \in [\mathrm{LS}(\mathcal{K})^+, \theta]$.

Combining Fact 6.17 with Corollary 6.16, we obtain:

Theorem 6.18. Assume that $\mathcal{K}$ is categorical in $\mathrm{LS}(\mathcal{K})$ and let $\lambda_0 \in \mathcal{F}$. If there exists $\lambda \in [\lambda_0^+, \theta]$ such that:

(1) $\mathcal{K}_{[\lambda_0, \lambda)}^{\lambda_0\text{-sat}}$ has primes.
(2) $\mathcal{K}^{\lambda_0\text{-sat}}$ is categorical in λ.

Then $\mathcal{K}$ is categorical in *every* $\mu \in [\mathrm{LS}(\mathcal{K}), \theta]$.

Proof. By Fact 6.17 (where $\mathcal{K}, \mathrm{LS}(\mathcal{K}), \theta$ there stand for $\mathcal{K}^{\lambda_0\text{-sat}}, \lambda_0, \lambda$ here), $\mathcal{K}^{\lambda_0\text{-sat}}$ is categorical in λ_0^+. By Corollary 6.16, $\mathcal{K}$ is categorical in every $\mu \in [\mathrm{LS}(\mathcal{K}), \theta]$. $\square$

Remark 6.19. This shows that it is enough to assume existence of primes *below* the categoricity cardinal to get categoricity everywhere.

7. Categoricity at a successor or with primes

We prove the main results of this paper. All throughout, we assume:

Hypothesis 7.1. $\mathcal{K}$ is an AEC with amalgamation and no maximal models.

Remark 7.2. Let $\mathcal{K}$ be an AEC with amalgamation. Then we can write $\mathcal{K}$ as $\bigcup_{i \in I} K^i$, where the $\mathcal{K}^i$s are disjoint AECs with $\mathrm{LS}(\mathcal{K}^i) = \mathrm{LS}(\mathcal{K})$, and each K^i has amalgamation and joint embedding (see the notion of a diagram in [She09, Definition I.2.2] or [Bal09, Lemma 16.14]). Assume in addition that $\lambda \geq \mathrm{LS}(\mathcal{K})$ is such that $\mathcal{K}_\lambda$ has joint embedding (e.g. $\mathcal{K}$ is categorical in λ). Then using amalgamation $\mathcal{K}_{\geq \lambda}$ has joint embedding. So there is a unique $i^* \in I$ so that $(K^{i^*})_{\geq \lambda} = \mathcal{K}$, so when $i \neq i^*$, $(K^i)_{\geq \lambda} = \emptyset$. Moreover if $\mathcal{K}$ has arbitrarily large models (e.g. it has a model of size $h(\mathrm{LS}(\mathcal{K}))$), then $\mathcal{K}^{i^*}$ has no maximal models and there exists $\chi < h(\mathrm{LS}(\mathcal{K}))$ so that for $i \neq i^*$, $(K^i)_{\geq \chi} = \emptyset$. Therefore for the purpose of proving a conclusion about $\mathcal{K}_{\geq \min(\chi, \lambda)}$ (such as "$\mathcal{K}$ is categorical in all $\lambda' \geq \min(\lambda, H_1)$"), we can restrict ourselves to K^{i^*}, i.e. assume without loss of generality that $\mathcal{K}$ has joint embedding

and no maximal models. In particular, "no maximal models" can be replaced by "arbitrarily large models" (which in turns follows if $\mathcal{K}$ is categorical in a $\lambda \geq H_1$) in Corollary 7.4, in the last sentence of Theorem 7.7, in Corollary 7.12, in Theorem 7.14, and in Corollary 7.15.

We will use the notation from Chapter 14 of [Bal09] by writing $H_1 := h(\mathrm{LS}(\mathcal{K}))$ and $H_2 := h(H_1)$ (recall Notation 1.2). The main lemma collects both the case of categoricity in a successor and the case of categoricity in a limit (but then we assume that the AEC has primes).

Lemma 7.3 (Main lemma). Let $\theta \geq \lambda > \mathrm{LS}(\mathcal{K})^+$ be such that $\mathcal{K}$ is $(\mathrm{LS}(\mathcal{K}), < \theta)$-weakly tame. Assume that $\mathcal{K}$ is categorical in λ. Then $\mathcal{K}^{\mathrm{LS}(\mathcal{K})^+\text{-sat}}$ is categorical in all $\mu \in [\mathrm{LS}(\mathcal{K})^+, \theta]$ provided that one of the following holds:

   (1) λ is a successor.
   (2) $\mathcal{K}$ is $(\mathrm{LS}(\mathcal{K})^+, < \lambda)$-tame, the model of size λ is saturated, and there exists $\lambda_0 \in [\mathrm{LS}(\mathcal{K})^+, \lambda)$ so that $\mathcal{K}^{\lambda_0\text{-sat}}_{[\lambda_0,\lambda)}$ has primes.

Proof. We prove the conclusion assuming the first condition. The proof assuming the second condition is similar (use Theorem 6.18). We first prove the result assuming that $\mathcal{K}$ is $(\mathrm{LS}(\mathcal{K}), < \theta)$-tame (not just weakly). We will then explain how to modify the proof so that it works also for weak tameness.

By Proposition 2.14, there exists a type-full good $[\mathrm{LS}(\mathcal{K})^+, \theta]$-frame $\mathfrak{s}$ with underlying class $\mathcal{K}_{\mathfrak{s}} := \mathcal{K}^{\mathrm{LS}(\mathcal{K})^+\text{-sat}}_{[\mathrm{LS}(\mathcal{K})^+,\theta)}$. Moreover, $\mathcal{K}_{\mathfrak{s}}$ is categorical in $\mathrm{LS}(\mathcal{K})^+$. Thus we can apply Corollary 6.16 where $\mathcal{K}, \mathrm{LS}(\mathcal{K})$ there stand for $\mathcal{K}^{\mathrm{LS}(\mathcal{K})^+\text{-sat}}, \mathrm{LS}(\mathcal{K})^+$ here.

Now let us argue that $(\mathrm{LS}(\mathcal{K}), < \theta)$-weak tameness suffices. So assume that $\mathcal{K}$ is $(\mathrm{LS}(\mathcal{K}), < \theta)$-weakly tame. For the upward part, by Fact 1.4, $\mathcal{K}^{\mathrm{LS}(\mathcal{K})^+\text{-sat}}$ is categorical in all $\mu \in [\lambda^+, \theta]$ (that the tameness assumption can be weakened to only $(\mathrm{LS}(\mathcal{K}), < \theta)$-weak tameness is implicit in Grossberg and VanDieren's paper and stated explicitly in Chapter 13 of [Bal09]).

It remains to show the downward part, i.e. that $\mathcal{K}^{\mathrm{LS}(\mathcal{K})^+\text{-sat}}$ is categorical in all $\mu \in [\mathrm{LS}(\mathcal{K})^+, \lambda^+)$. By Proposition 2.14, for each $\mu \in [\mathrm{LS}(\mathcal{K})^+, \lambda^+)$, there exists a type-full good μ-frame with underlying class $\mathcal{K}^{\mu\text{-sat}}_\mu$. By Theorem A.1, the frames agree with each other, i.e. for $\mu < \mu'$ both in $[\mathrm{LS}(\mathcal{K})^+, \lambda^+)$, forking in the good μ'-frame can be described naturally in terms of forking in the good μ-frame. This is enough to make the proof of Corollary 6.16 go through $(\mathcal{K}, \mathrm{LS}(\mathcal{K}), \theta)$

there now stand for $\mathcal{K}^{\mathrm{LS}(\mathcal{K})^+\text{-sat}}, \mathrm{LS}(\mathcal{K})^+, \lambda^+$ here), we just have to make sure that anytime a resolution is taken, all the components are saturated. □

Corollary 7.4. Let $\mathrm{LS}(\mathcal{K}) < \lambda_0 < \lambda_1$. Assume that $\mathcal{K}$ is $(\mathrm{LS}(\mathcal{K}), <\lambda_1)$-weakly tame. If λ_1 is a successor and $\mathcal{K}$ is categorical in λ_0 and λ_1, then $\mathcal{K}$ is categorical in all $\lambda \in [\lambda_0, \lambda_1]$.

Proof. By Lemma 7.3 (with θ, λ there standing for λ_1, λ_1 here), $\mathcal{K}^{\mathrm{LS}(\mathcal{K})^+\text{-sat}}$ is categorical in all $\mu \in [\mathrm{LS}(\mathcal{K})^+, \lambda_1]$. Moreover by the proof of Lemma 7.3, $\mathcal{K}$ is stable in every $\mu \in [\mathrm{LS}(\mathcal{K})^+, \lambda_1)$, hence the model of size λ_0 is saturated. Therefore $\mathcal{K}^{\mathrm{LS}(\mathcal{K})^+\text{-sat}}_{\geq \lambda_0} = \mathcal{K}_{\geq \lambda_0}$, and the result follows. □

Remark 7.5. We can allow $\lambda_0 = \mathrm{LS}(\mathcal{K})$ but then the proof is more complicated: we do not know how to build a good $\mathrm{LS}(\mathcal{K})$-frame so have to work with $\mathrm{LS}(\mathcal{K})$-splitting.

To deduce the main theorem of this paper, we will use Morley's omitting type theorem for AECs [She99, II.1.10]. To state an optimal result, we will actually use a slightly stronger conclusion (replacing H_1 by some $\chi < H_1$) that is implicit e.g. in [She99] but to the best of our knowledge, a proof of this stronger result has not appeared in print before. We include a proof (similar to the proof of [BG, Theorem 5.4], though there is an additional step involved) for the convenience of the reader.

Fact 7.6 (The AEC omitting type theorem). Let $\lambda > \mathrm{LS}(\mathcal{K})$. If every model in $\mathcal{K}_\lambda$ is $\mathrm{LS}(\mathcal{K})^+$-saturated, then there exists $\chi < H_1$ such that every model in $\mathcal{K}_{\geq \chi}$ is $\mathrm{LS}(\mathcal{K})^+$-saturated.

Proof sketch. Suppose not. Then for every $\chi \in [\mathrm{LS}(\mathcal{K}), H_1)$, there exists $M_\chi \in \mathcal{K}_\chi$ which is not $\mathrm{LS}(\mathcal{K})^+$-saturated. Pick witnesses $M_{0,\chi} \leq M_\chi$ and $p_\chi \in \mathrm{gS}(M_{0,\chi})$ such that $\|M_{0,\chi}\| = \mathrm{LS}(\mathcal{K})$ and M_χ omits p_χ. Now there are only $2^{\mathrm{LS}(K)}$ isomorphism types of Galois types over models of size $\mathrm{LS}(\mathcal{K})$, and $\mathrm{cf}(H_1) = \left(2^{\mathrm{LS}(\mathcal{K})}\right)^+ > 2^{\mathrm{LS}(\mathcal{K})}$, so there exists $N \in \mathcal{K}_{\mathrm{LS}(\mathcal{K})}, p \in \mathrm{gS}(M)$, and an unbounded $S \subseteq [\mathrm{LS}(\mathcal{K}), H_1)$ such that for all $\chi \in S$, p_χ is isomorphic to p (in the natural sense). Look at the AEC $\mathcal{K}_{\neg p}$ of all the models of $\mathcal{K}$ omitting p, with constants added for N (see e.g. the definition of $\mathcal{K}^+$ in the proof of [BG, Theorem 5.4]). For each $\chi \in S$, an appropriate expansion of a copy of M_χ is in $K_{\neg p}$. $\mathcal{K}_{\neg p}$ has Löwenheim-Skolem number $\mathrm{LS}(\mathcal{K})$, so by Shelah's presentation theorem and Morley's omitting type theorem (for first-order theories), $\mathcal{K}_{\neg p}$ has arbitrarily large models, contradicting the assumptions on λ. □

Theorem 7.7. Let $\theta \geq \lambda > \mathrm{LS}(\mathcal{K})^+$ be such that $\mathcal{K}$ is $(\mathrm{LS}(\mathcal{K}), < \theta)$-weakly tame and λ is a successor. If $\mathcal{K}$ is categorical in λ, then there exists $\chi < H_1$ such that $\mathcal{K}$ is categorical in all $\mu \in [\min(\chi, \lambda), \theta]$.

In particular, if $\mathcal{K}$ is $\mathrm{LS}(\mathcal{K})$-weakly tame and categorical in a successor $\lambda \geq H_1$, then there exists $\chi < H_1$ such that $\mathcal{K}$ is categorical in all $\mu \geq \chi$.

Proof. By Lemma 7.3, $\mathcal{K}^{\mathrm{LS}(\mathcal{K})^+\text{-sat}}$ is categorical in all $\mu \in [\mathrm{LS}(\mathcal{K})^+, \theta]$. Since λ is regular, the model of size λ is saturated hence $\mathrm{LS}(\mathcal{K})^+$-saturated, so every model in $\mathcal{K}_{\geq \lambda}$ is $\mathrm{LS}(\mathcal{K})^+$-saturated. By Fact 7.6, there exists $\chi < H_1$ such that every model in $\mathcal{K}_{\geq \chi}$ is $\mathrm{LS}(\mathcal{K})^+$-saturated. Thus every model in $\mathcal{K}_{\geq \min(\chi, \lambda)}$ is $\mathrm{LS}(\mathcal{K})^+$-saturated, that is:

$$\mathcal{K}^{\mathrm{LS}(\mathcal{K})^+\text{-sat}}_{\geq \min(\chi, \lambda)} = \mathcal{K}_{\geq \min(\chi, \lambda)}$$

The result follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Next, we deduce Shelah's downward categoricity transfer [She99]. We will use the following fact which appears as [She99, Main Claim II.2.3] (a simplified and improved argument is in [Bal09, Theorem 11.15]):

Fact 7.8. Let $\lambda > \mu \geq H_1$. Assume that $\mathcal{K}$ is categorical in λ, and the model of cardinality λ is μ^+-saturated. Then there exists $\chi < H_1$ such that $\mathcal{K}$ is (χ, μ)-weakly tame.

To obtain the best statement, we will use that we can find a uniform bound on χ above:

Proposition 7.9. Let $\lambda > \mu \geq H_1$. Assume that $\mathcal{K}$ is categorical in λ, and the model of cardinality λ is H_1^+-saturated. Then there exists $\chi < H_1$ such that if the model of size λ is μ^+-saturated, then $\mathcal{K}$ is (χ, μ)-weakly tame.

Proof. By Fact 7.8 (applied with $\mu := H_1^+$), there exists $\chi < H_1$ so that $\mathcal{K}$ is (χ, H_1)-weakly tame. Now assume that the model of size λ is μ^+-saturated. By Fact 7.8, there exists $\chi' < H_1$ such that $\mathcal{K}$ is (χ', μ)-weakly tame. In particular, $\mathcal{K}$ is (H_1, μ)-weakly tame. Now by Fact 2.9, $\mathcal{K}$ is μ'-superstable and has μ'-symmetry for every $\mu' \in [\mathrm{LS}(\mathcal{K}), \mu']$. By [Vanb, Corollary 3] or more explicitly [VVb, 6.6, 6.7], $\mathcal{K}^{H_1\text{-sat}}$ is an AEC with Löwenheim-Skolem number H_1. Thus we can combine (χ, H_1)-weak and (H_1, μ)-weak tameness to get (χ, μ)-weak tameness. $\qquad\square$

Remark 7.10. While this will not be used, H_1^+ can be replaced by H_1, see Corollary 8.13.

Remark 7.11. This proposition and the next corollary sheds some light on [Bal09, Remark 14.15], where the question of whether H_2 can be replaced by $h(\chi)$ for a suitable $\chi < H_2$ (as Shelah claims in [She99]) is discussed. The methods of this paper yield a positive answer.

Corollary 7.12. If $\mathcal{K}$ is categorical in a successor $\lambda > H_2$, then there exists $\chi < H_1$ such that $\mathcal{K}$ is categorical in all $\mu \in [h(\chi), \lambda]$.

Proof. Again, we can use Remark 7.2 to assume without loss of generality that $\mathcal{K}$ has no maximal models. Since λ is a successor, the model of size λ is saturated. By Proposition 7.9, there exists $\chi < H_1$ such that $\mathcal{K}$ is $(\chi, < \lambda)$-weakly tame. By Theorem 7.7 (where $\mathcal{K}, \mathrm{LS}(\mathcal{K}), \lambda, \theta$ there stand for $\mathcal{K}_{\geq \chi}, \chi, \lambda, \lambda$ here), $\mathcal{K}$ is categorical in all $\mu \in [h(\chi), \lambda]$. $\square$

An alternate proof of a special case of the upward transfer of Grossberg and VanDieren [GV06c, GV06a] can also be obtained, see Corollary C.6 in the appendix. We can similarly deduce several consequences on tame AECs with primes. One of the main result of [Vasd] was (the point compared to Shelah's downward categoricity transfer [She99] is that λ need *not* be a successor):

Fact 7.13 (Theorem 3.8 in [Vasd]). Assume that $\mathcal{K}$ is H_2-tame and $\mathcal{K}_{\geq H_2}$ has primes. If $\mathcal{K}$ is categorical in some $\lambda > H_2$, then it is categorical in all $\lambda' \geq H_2$.

Using the methods of this paper, we can obtain categoricity in more cardinals provided that $\mathcal{K}$ has more tameness:

Theorem 7.14. Assume that $\mathcal{K}$ is $\mathrm{LS}(\mathcal{K})$-tame and $\mathcal{K}$ has primes (or just $\mathcal{K}_{\geq \mu}$ has prime, for some μ). If $\mathcal{K}$ is categorical in some $\lambda > \mathrm{LS}(\mathcal{K})$, then $\mathcal{K}$ is categorical in all $\lambda' \geq \min(\lambda, H_1)$.

Proof. By Theorem 3.3, $\mathcal{K}$ is categorical in a proper class of cardinals. By Fact 7.13 (applied to $\mathcal{K}_{\geq \mu}$, where μ is such that $\mathcal{K}_{\geq \mu}$ has primes), $\mathcal{K}$ is categorical in a successor cardinal. By Theorem 7.7, $\mathcal{K}$ is categorical in all $\lambda' \geq H_1$. By Corollary 7.4 (with λ_0, λ_1 there standing for λ, H_1^+ here), $\mathcal{K}$ is categorical also in all $\lambda' \in [\lambda, H_1^+]$. $\square$

Specializing to universal classes, recall [Vase, Theorem 6.8] that the eventual categoricity conjecture holds there with a Hanf number of $\beth_{H_1}$. We can use our main theorem to obtain the full categoricity conjecture (i.e. the Hanf number is H_1) *assuming amalgamation.*

Corollary 7.15. Let $\mathcal{K}$ be a universal class with amalgamation and arbitrarily large models[14]. If $\mathcal{K}$ is categorical in a $\lambda > \mathrm{LS}(\mathcal{K})$, then $\mathcal{K}$ is categorical in all $\lambda' \geq \min(\lambda, H_1)$.

Proof. By a result of Boney [Bon][15], $\mathcal{K}$ is $\mathrm{LS}(\mathcal{K})$-tame. Also [Vase, Remark 5.3], $\mathcal{K}$ has primes. Thus we can apply Theorem 7.14. □

8. Categoricity in a limit without primes

In this section, we prove Theorem 0.3 from the abstract and more generally give an exposition of Shelah's proof of the eventual categoricity conjecture in AECs with amalgamation [She09, Theorem IV.7.12]. Note that while the eventual version of Theorem 0.3 is due to Shelah, here we improve the Hanf number to H_1, and moreover give alternate proofs for several of the hard steps in Shelah's argument.

Most of the results of this section will use the weak generalized continuum hypothesis. We adopt the following notation:

Notation 8.1. For a cardinal λ, $\mathrm{WGCH}(\lambda)$ is the statement "$2^\lambda < 2^{\lambda^+}$". More generally, for S a class of cardinals, $\mathrm{WGCH}(S)$ is the statement "$\mathrm{WGCH}(\lambda)$ for all $\lambda \in S$". WGCH will stand for $\mathrm{WGCH}(\mathrm{Card})$, where Card is the class of all cardinals.

We assume familiarity with Ehrenfeucht-Mostowski models (we use the definitions and notation outlined at the beginning of [She09, Chapter IV] or in [Bal09, Section 6.2]), as well as with the definitions of a *weakly successful*, *successful*, and *ω-successful* good λ-frame (see [She09, Definition III.1.1]). We use the notation from [JS13]. We will say a good λ-frame is *successful*[+] if it is successful and $\leq^{\mathrm{NF}}_{\lambda^+}$ is just $\leq_{\lambda^+}$ on the saturated models in $\mathcal{K}_{\lambda^+}$, see [JS13, Definition 6.1.4]. We say a good λ-frame is *ω-successful*[+] if it is ω-successful and successful[+]. Everywhere below, $\mathcal{K}$ is an AEC.

We first state the unpublished Claim of Shelah mentioned in the introduction. This stems from [She09, Discussion III.12.40]. A proof should appear in [Sheb].

Claim 8.2. Let $\mathfrak{s}$ be an ω-successful[+] good λ-frame on $\mathcal{K}$. Assume $\mathrm{WGCH}([\lambda, \lambda^{+\omega}))$. If $\mathcal{K}^{\lambda^{+\omega}\text{-sat}}$ is categorical in some $\mu > \lambda^{+\omega}$, then $\mathcal{K}^{\lambda^{+\omega}\text{-sat}}$ is categorical in all $\mu' > \lambda^{+\omega}$.

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[14]See Hypothesis 7.1 and the remark following it.
[15]A full proof appears as [Vase, Corollary 3.8].

Next, we discuss how to obtain an ω-successful$^+$ good frame from a good frame. The proof of the following fact is contained in the proof of [She09, Theorem IV.7.12] (see $\odot_4$ there). Note that as opposed to the results in [She09, Section II.5], we do *not* assume that $\mathcal{K}$ has few models in λ^{+2}.

Fact 8.3. Assume WGCH(λ). If $\mathfrak{s}$ is a good λ-frame on $\mathcal{K}$, $\mathcal{K}$ is categorical in λ, has amalgamation in λ^+ and is stable in λ^+, then $\mathfrak{s}$ is weakly successful.

We can obtain the stability hypothesis and successfulness using weak tameness:

Fact 8.4 (Theorem 4.5 in [BKV06])**.** Let $\lambda \geq \mathrm{LS}(\mathcal{K})$, be such that $\mathcal{K}$ has amalgamation in λ and λ^+, and $\mathcal{K}$ is stable in λ. If $\mathcal{K}$ is (λ, λ^+)-weakly tame, then $\mathcal{K}$ is stable in λ^+.

Fact 8.5 (Corollary 7.19 in [Jar])**.** If $\mathfrak{s}$ is a weakly successful good λ-frame on $\mathcal{K}$, $\mathcal{K}$ is categorical[16] in λ, $\mathcal{K}$ has amalgamation in λ^+, and $\mathcal{K}$ is (λ, λ^+)-weakly tame, then $\mathfrak{s}$ is successful$^+$.

Remark 8.6. Although we will not need it, the converse (i.e. obtaining weak tameness from being successful$^+$) is also true, see [JS13, Theorem 7.1.13.(b)].

Corollary 8.7. Assume WGCH(λ). If $\mathfrak{s}$ is a good λ-frame on $\mathcal{K}$, $\mathcal{K}$ is categorical in λ, has amalgamation in λ^+, and is (λ, λ^+)-weakly tame, then $\mathfrak{s}$ is successful$^+$.

Proof. By Fact 8.4, $\mathcal{K}$ is stable in λ^+. By Fact 8.3, $\mathfrak{s}$ is weakly successful. By Fact 8.5, $\mathfrak{s}$ is successful$^+$. $\square$

Using Fact 2.12 to build the good frame, we obtain:

Lemma 8.8. Assume that $\mathcal{K}$ has amalgamation and no maximal models. Assume WGCH($[\mathrm{LS}(\mathcal{K}), \mathrm{LS}(\mathcal{K})^{+\omega})$) and Claim 8.2. If $\mathcal{K}$ is categorical in a $\lambda > \mathrm{LS}(\mathcal{K})^{+\omega}$ and:

  (1) The model of size λ is $\mathrm{LS}(\mathcal{K})^{++}$-saturated.
  (2) $\mathcal{K}$ is $(\mathrm{LS}(\mathcal{K}), < \mathrm{LS}(\mathcal{K})^{+\omega})$-weakly tame.

Then $\mathcal{K}^{\mathrm{LS}(\mathcal{K})^{+\omega}\text{-sat}}$ is categorical in all $\lambda' > \mathrm{LS}(\mathcal{K})^{+\omega}$. In particular, $\mathcal{K}$ is categorical in all $\lambda' \geq \min(\lambda, \sup_{n<\omega} h(\mathrm{LS}(\mathcal{K})^{+n}))$.

---

[16]Note that by [She09, Claim III.1.21], this implies the conjugation property, so Hypothesis 6.5 in [Jar] is satisfied.

Proof. By Fact 2.9, $\mathcal{K}$ has symmetry in $\mathrm{LS}(\mathcal{K})$ and $\mathrm{LS}(\mathcal{K})^+$. By Fact 2.3, $\mathcal{K}$ is superstable in every $\chi \in [\mathrm{LS}(\mathcal{K}), \lambda)$. By Fact 2.12, there is a good $\mathrm{LS}(\mathcal{K})^+$-frame $\mathfrak{s}$ on $\mathcal{K}^{\mathrm{LS}(\mathcal{K})^+\text{-sat}}$. By repeated applications of Corollary 8.7, $\mathfrak{s}$ is ω-successful$^+$. By Claim 8.2, $\mathcal{K}^{\mathrm{LS}(\mathcal{K})^{+\omega}\text{-sat}}$ is categorical in all $\lambda' > \mathrm{LS}(\mathcal{K})^{+\omega}$, and hence by Fact 7.6, $\mathcal{K}$ is categorical in every $\lambda' \geq \sup_{n<\omega} h(\mathrm{LS}(\mathcal{K})^{+\omega})$. In particular by Fact 2.3, $\mathcal{K}$ is stable in λ, so the model of size λ is saturated (hence $\mathrm{LS}(\mathcal{K})^{+\omega}$-saturated), and so $\mathcal{K}$ must be categorical in all $\lambda' \geq \min(\lambda, \sup_{n<\omega} h(\mathrm{LS}(\mathcal{K})^{+\omega}))$. $\qquad\square$

As a corollary, we obtain Theorem 0.3 from the abstract. We restate it here in a slightly stronger form:

Theorem 8.9. Assume WGCH and Claim 8.2. Assume that $\mathcal{K}$ has amalgamation, arbitrarily large models, and is $\mathrm{LS}(\mathcal{K})$-tame. If $\mathcal{K}$ is categorical in a $\lambda > \mathrm{LS}(\mathcal{K})$, then there exists $\chi < H_1$ such that $\mathcal{K}$ is categorical in all $\lambda' \geq \min(\lambda, \chi)$.

Proof. By Remark 7.2, without loss of generality $\mathcal{K}$ has no maximal models. By Theorem 3.3 we can assume without loss of generality that $\lambda \geq H_1$ (then we can use Corollary 7.4 to transfer categoricity downward). By Fact 2.3, $\mathcal{K}$ is $\mathrm{LS}(\mathcal{K})$-superstable. By Fact 2.4[17], $\mathcal{K}$ is stable in λ, hence the model of size λ is saturated. By Lemma 8.8, $\mathcal{K}$ is categorical in all $\lambda' \geq h(\mathrm{LS}(\mathcal{K})^{+\omega})$. In particular, $\mathcal{K}$ is categorical in $(h(\mathrm{LS}(\mathcal{K})^{+\omega}))^+$. By Theorem 7.7, there exists $\chi < H_1$ such that $\mathcal{K}$ is categorical in all $\lambda' \geq \chi$. $\qquad\square$

Remark 8.10. One can ask if, as in Theorem 7.7, tameness can be replaced by just weak tameness. We believe that this is possible but the proof is harder, so we delay it to a future paper.

Without tameness, we can obtain the hypotheses of Lemma 8.8 from categoricity in a high-enough cardinal. The weak tameness will be obtained using the following variation on Fact 7.8:

Fact 8.11 (Claim IV.7.2 in [She09])**.** Let $\chi > \mathrm{LS}(\mathcal{K})$. If:

(1) $\mathcal{K}_{<\chi}$ has amalgamation.
(2) $\mathrm{cf}(\chi) > \mathrm{LS}(\mathcal{K})$.
(3) Φ is a proper for linear orders, and if $\theta \in (\mathrm{LS}(\mathcal{K}), \chi)$, I is a θ-wide[18] linear order, then $\mathrm{EM}_{L(\mathcal{K})}(I, \Phi)$ is θ-saturated.

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[17]In our case, we only use stability in λ, which follows by [Vasa, Theorem 5.6].

[18]A linear order is *θ-wide* if for every $\theta_0 < \theta$, I contains an increasing sequence of length θ_0^+, see [She09, Definition IV.0.14.(1)].

Then there exists $\chi_* \in (\mathrm{LS}(\mathcal{K}), \chi)$ such that $\mathcal{K}$ is $(\chi_*, < \chi)$-weakly tame.

Condition (3) in Fact 8.11 can be derived from categoricity if the model in the categoricity cardinal is sufficiently saturated. This is implicit in [She99] and appears as [Bal09, Lemma 10.11].

Fact 8.12. If $\mathcal{K}$ has amalgamation and no maximal models, $\chi > \mathrm{LS}(\mathcal{K})$, $\mathcal{K}$ is categorical in a $\lambda \geq \chi$, so that the model of size λ is χ-saturated, then for every Φ proper for linear orders, if $\theta \in (\mathrm{LS}(\mathcal{K}), \chi)$ and I is a θ-wide linear order, we have that $\mathrm{EM}_{L(\mathcal{K})}(I, \Phi)$ is θ-saturated.

Corollary 8.13. Assume that $\mathcal{K}$ has amalgamation, no maximal models, and is categorical in a $\lambda > \mathrm{LS}(\mathcal{K})$.

(1) Let χ be a limit cardinal such that $\mathrm{cf}(\chi) > \mathrm{LS}(\mathcal{K})$. If the model of size λ is χ-saturated, then there exists $\chi_0 < \chi$ such that $\mathcal{K}$ is $(\chi_0, < \chi)$-weakly tame.
(2) If[19] the model of size λ is H_1-saturated, then there exists $\chi_0 < H_1$ such that whenever $\chi \geq H_1$ is so that the model of size λ is χ-saturated, we have that $\mathcal{K}$ is $(\chi_0, < \chi)$-weakly tame.

Proof.

(1) By Fact 8.11 (using Fact 8.12 to see that (3) is satisfied).
(2) By the first part (with χ there standing for H_1 here), there exists $\chi_0 < \chi$ such that $\mathcal{K}$ is $(\chi_0, < H_1)$-weakly tame. Now continue as in the proof of Proposition 7.9.

$\square$

We pause briefly to give two applications of (the first part of) Corollary 8.13. First, we obtain an improvement on the Hanf number for the construction of a good frame in [VVa, Corollary 5.4] (χ below can be less than H_1, e.g. $\chi = \aleph_{\mathrm{LS}(\mathcal{K})^+}$).

Theorem 8.14. Assume that $\mathcal{K}$ has amalgamation and no maximal models. Let χ be a limit cardinal such that $\mathrm{cf}(\chi) > \mathrm{LS}(\mathcal{K})$ and assume that $\mathcal{K}$ is categorical in a $\lambda \geq \chi$. If the model of size λ is χ-saturated, then there exists $\chi_0 < \chi$ such that for all $\mu \in [\chi_0, \chi)$, there is a good μ-frame on $\mathcal{K}^{\mu\text{-sat}}$.

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[19]This second part is a small improvement on Proposition 7.9 (as H_1^+ there is replaced by H_1 here). It will not be used.

Proof. By Fact 2.9, $\mathcal{K}$ has χ_0-symmetry for every $\chi_0 < \chi$. By Fact 2.3, $\mathcal{K}$ is also superstable in every $\mu \in [\mathrm{LS}(K), \lambda)$. Now by Corollary 8.13, there exists $\chi_0 < \chi$ so that $\mathcal{K}$ is $(\chi_0, < \chi)$-weakly tame. We finish by applying Fact 2.12. $\square$

Second, we can study the categoricity spectrum *below* the Hanf number of an AEC with amalgamation and no maximal models. While the categoricity spectrum in such AECs must be a closed set, we show (in ZFC) that there are other restrictions:

Theorem 8.15. Assume that $\mathcal{K}$ has amalgamation and no maximal models. Let χ be a limit cardinal such that $\mathrm{cf}(\chi) > \mathrm{LS}(\mathcal{K})$. If $\mathcal{K}$ is categorical in unboundedly many successor cardinals below χ, then there exists $\chi_0 < \chi$ such that $\mathcal{K}$ is categorical in every $\lambda \in [\chi_0, \chi]$.

In particular (setting $\chi := \aleph_{\mathrm{LS}(\mathcal{K})^+}$), if $\mathcal{K}$ is categorical in $\aleph_{\alpha+1}$ for unboundedly many $\alpha < \mathrm{LS}(\mathcal{K})^+$, then there exists $\alpha_0 < \mathrm{LS}(\mathcal{K})^+$ such that $\mathcal{K}$ is categorical in $\aleph_\beta$ for every $\beta \in [\alpha_0, \mathrm{LS}(\mathcal{K})^+]$.

Proof. By amalgamation, every model of size χ is saturated. In particular $\mathcal{K}$ is categorical in χ. By Corollary 8.13 (with λ, χ there standing for χ, χ here), there exists $\chi_0' < \chi$ such that $\mathcal{K}$ is $(\chi_0', < \chi)$-weakly tame. By making χ_0' bigger if necessary, we can assume without loss of generality that $\chi_0' > \mathrm{LS}(\mathcal{K})$ and $\mathcal{K}$ is categorical in $\chi_0 := (\chi_0')^+$. By the upward categoricity transfer of Grossberg and Vandieren (Fact 1.4) or Theorem 7.7, $\mathcal{K}$ is categorical in every $\lambda \in [\chi_0, \chi]$. $\square$

To obtain the first condition in Lemma 8.8 (i.e. that the model in the categoricity cardinal is sufficiently saturated), we will use[20]:

Fact 8.16 (Corollary 4.7 in [VVa]). Assume that $\mathcal{K}$ has amalgamation and no maximal models. Let $\chi \geq \mathrm{LS}(\mathcal{K})$ and assume that $\mathcal{K}$ is categorical in a $\lambda \geq h(\chi)$, then $\mathcal{K}$ has χ-symmetry and (if $\chi > \mathrm{LS}(\mathcal{K})$) the model of size λ is χ-saturated.

We can now give a proof of [She09, Theorem IV.7.12]. Note that, while we give a slightly different proof to attempt to convince doubters, the result is due to Shelah. In fact, Shelah assumes amalgamation more locally but we haven't fully checked his general proof. As explained in the introduction, we avoid relying on PCF theory or on Shelah's construction of certain linear orders in [She09, Sections IV.5, IV.6].

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[20]Shelah claims [She09, Claim IV.7.8] a slightly stronger result using PCF theory and the existence of certain linear orders, but we have not checked his proof.

Fact 8.17. Assume Claim 8.2. Assume that $\mathcal{K}$ has amalgamation. Let λ and χ be cardinals such that:

(1) $\mathcal{K}$ is categorical in λ.
(2) χ is a limit cardinal with $\mathrm{cf}(\chi) > \mathrm{LS}(\mathcal{K})$.
(3) For all $\chi_0 < \chi$, $h(\chi_0) < \lambda$.
(4) For unboundedly many $\chi_0 < \chi$, $\mathrm{WGCH}([\chi_0, \chi_0^{+\omega}])$.

Then there exists $\chi_* < \chi$ such that $\mathcal{K}$ is categorical in all $\lambda' \geq h(\chi_*)$.

Proof. By Remark 7.2, without loss of generality $\mathcal{K}$ has no maximal models. By Fact 8.16 (used with χ there standing for χ_0^+ here, for each $\chi_0 < \chi$), the model of size λ is χ-saturated. By Corollary 8.13, there exists $\chi_0 < \chi$ so that $\mathcal{K}$ is $(\chi_0, < \chi)$-weakly tame. Increasing χ_0 if necessary, assume without loss of generality that $\mathrm{WGCH}((\chi_0, \chi_0^{+\omega}])$. Now apply Lemma 8.8 with $\mathcal{K}$ there standing for $\mathcal{K}_{\geq \chi_0}$ here. We get that $\mathcal{K}$ is categorical in all $\lambda' \geq h(\chi_0^{+\omega})$, so we obtain the desired conclusion with $\chi_* := \chi_0^{+\omega}$. $\square$

Remark 8.18. The proof of Fact 8.17 given above goes through assuming only that $\mathcal{K}$ has amalgamation below the categoricity cardinal λ.

Corollary 8.19. Assume Claim 8.2 and WGCH. If $\mathcal{K}$ has amalgamation and is categorical in a $\lambda \geq h(\aleph_{\mathrm{LS}(\mathcal{K})^+})$, then $\mathcal{K}$ is categorical in all $\lambda' \geq h(\aleph_{\mathrm{LS}(\mathcal{K})^+})$.

Proof. Set $\chi := \aleph_{\mathrm{LS}(\mathcal{K})^+}$ in Fact 8.17. $\square$

We can also state a version using large cardinals. This is implicit in Shelah's work (see the remark after [She09, Theorem IV.7.12]), but to the best of our knowledge, the details have not appeared in print before.

Theorem 8.20. Assume Claim 8.2 and WGCH. Let $\kappa > \mathrm{LS}(\mathcal{K})$ be a measurable cardinal[21]. If $\mathcal{K}$ is categorical in some $\lambda \geq h(\kappa)$, then $\mathcal{K}$ is categorical in all $\lambda' \geq h(\kappa)$[22]

Proof. We quote freely from [SK96, She01]. Note that while the results there are stated when $\mathcal{K}$ is the class of models of an $L_{\kappa,\omega}$-theory, Boney

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[21]Instead of $\kappa > \mathrm{LS}(\mathcal{K})$, it is enough to assume that $\mathrm{LS}(\mathcal{K})$ is essentially below κ, see [Bon14b, Definition 2.10].

[22]The proof gives that there exists $\chi < h(\kappa)$ such that $\mathcal{K}$ is categorical in all $\lambda' \geq \chi$.

observed that the proofs go through just as well in an AEC $\mathcal{K}$ with $\kappa > \mathrm{LS}(\mathcal{K})$, see the discussion around [Bon14b, Theorem 7.6].

By the main theorem of [SK96], $\mathcal{K}_{[\kappa,\lambda)}$ has amalgamation (and no maximal models, by taking ultrapowers). Note that by Remark 8.18 we do not need amalgamation in $\mathcal{K}_{\geq\lambda}$. By [She01, Claim 1.16], the model of size λ is saturated. Let $\chi := \aleph_{\kappa^+}$. By Corollary 8.13, there exists $\chi_0 < \chi$ so that $\mathcal{K}$ is $(\chi_0, < \chi)$-weakly tame. By Lemma 8.8 (with $\mathcal{K}$ there standing for $\mathcal{K}_{\geq\chi_0}$ here), $\mathcal{K}$ is categorical in all $\lambda' \geq h(\chi)$. In particular, $\mathcal{K}$ is categorical in $(h(\chi))^+$. By [She01, Theorem 3.16] (or by Theorem 7.7, since by [She01, Corollary 3.7], $\mathcal{K}$ has enough tameness), $\mathcal{K}$ is also categorical in all $\lambda' \in [h(\kappa), h(\chi))$. $\qquad\square$

APPENDIX A. MORE ON CANONICITY

Here, we prove:

Theorem A.1. Let $\mathcal{K}$ be an abstract elementary class and let $\theta > \mathrm{LS}(\mathcal{K})$ be such that $\mathcal{K}_{\mathrm{LS}(\mathcal{K}),\theta}$ has amalgamation and $\mathcal{K}$ is $(\mathrm{LS}(\mathcal{K}), \theta)$-weakly tame. If $\mathfrak{s}$ is a type-full good $\mathrm{LS}(\mathcal{K})$-frame with underlying class $\mathcal{K}_{\mathrm{LS}(\mathcal{K})}$ and $\mathfrak{s}'$ is a type-full good θ-frame with underlying class $\mathcal{K}_\theta^{\theta\text{-sat}}$, then for every $M \leq N$ in $\mathcal{K}_\theta^{\theta\text{-sat}}$ and $p \in \mathrm{gS}(N)$, p does not $\mathfrak{s}'$-fork over M if and only if there exists $M_0 \leq M$ with $M_0 \in \mathcal{K}_{\mathrm{LS}(\mathcal{K})}$ so that $p \upharpoonright N_0$ does not $\mathfrak{s}$-fork over M_0 for every $N_0 \in \mathcal{K}_{\mathrm{LS}(\mathcal{K})}$ with $M_0 \leq N_0 \leq N$.

This is used to prove Theorem 7.7 when only $\mathrm{LS}(\mathcal{K})$-weak tameness holds, see the proof there. Intuitively, this says that forking in $\mathfrak{s}'$ can be described by forking in $\mathfrak{s}$ in a canonical way (i.e. using Shelah's description of the extended frame, see [She09, Section II.2]). We will use the following result, which gives an explicit description of forking in any categorical good frame:

Fact A.2 (The canonicity theorem, 9.6 in [Vasb]). Let $\mathfrak{s}$ be a type-full good λ-frame with underlying class $\mathcal{K}_\lambda$. If $M \leq N$ are limit models in $\mathcal{K}_\lambda$, then for any $p \in \mathrm{gS}(N)$, p does not $\mathfrak{s}$-fork over M if and only if there exists $M' \in \mathcal{K}_\lambda$ such that M is limit over M' and p does not λ-split over M'.

Proof of Theorem A.1. Note that by uniqueness of limit models, every model in $\mathcal{K}_\theta^{\theta\text{-sat}}$ is limit.

For $M, N \in \mathcal{K}_\theta^{\theta\text{-sat}}$ with $M \leq N$, let us say that $p \in \mathrm{gS}(N)$ *does not* $(\geq \mathfrak{s})$-*fork over* M if it satisfies the condition in the statement of the theorem, namely there exists $M_0 \leq M$ with $M_0 \in \mathcal{K}_{\mathrm{LS}(\mathcal{K})}$ so that $p \upharpoonright N_0$

does not $\mathfrak{s}$-fork over M_0 for every $N_0 \in \mathcal{K}_{\mathrm{LS}(\mathcal{K})}$ with $M_0 \leq N_0 \leq N$. Let us say that p *does not θ-fork over* M if it satisfies the description of the canonicity theorem, namely there exists $M' \in \mathcal{K}_\theta^{\theta\text{-sat}}$ such that M is limit over M' and p does not θ-split over M'. Notice that by the canonicity theorem (Fact A.2), p does not $\mathfrak{s}'$-fork over M if and only if p does not θ-fork over M. Thus it is enough to show that p does not $(\geq \mathfrak{s})$-fork over M if and only if p does not θ-fork over M. We first show one direction:

<u>Claim</u>. Let $M \leq N$ both be in $\mathcal{K}_\theta^{\theta\text{-sat}}$ and let $p \in \mathrm{gS}(N)$. If p does not $(\geq \mathfrak{s})$-fork over M, then p does not θ-fork over M.

<u>Proof of Claim</u>. We know that M is limit, so let $\langle M_i : i < \delta \rangle$ witness it, i.e. δ is limit, for all $i < \delta$, $M_i \in \mathcal{K}_\theta^{\theta\text{-sat}}$, $M_i <_{\mathrm{univ}} M_{i+1}$, and $\bigcup_{i<\delta} M_i = M$. By [She09, Claim II.2.11.(5)], there exists $i < \delta$ such that $p \upharpoonright M$ does not $(\geq \mathfrak{s})$-fork over M_i. By [She09, Claim II.2.11.(4)], p does not $(\geq \mathfrak{s})$-fork over M_i. By weak tameness and the uniqueness property of $\mathfrak{s}$, $(\geq \mathfrak{s})$-forking has the uniqueness property (see the proof of [Bon14a, Theorem 3.2]). By [BGKV, Lemma 4.2], $(\geq \mathfrak{s})$-nonforking must be extended by θ-nonsplitting, so p does not θ-split over M_i. Therefore p does not θ-fork over M, as desired. $\dagger_{\mathrm{Claim}}$.

Now as observed above, $(\geq \mathfrak{s})$-forking has the uniqueness property. Also, θ-forking has the extension property (as $\mathfrak{s}'$-forking has it). The claim tells us that θ-nonforking extends $(\geq \mathfrak{s})$-forking and hence by [BGKV, Lemma 4.1], they are the same. $\square$

Appendix B. Superstability for long types

We generalize Definition 2.2 and use it to prove the extension property for 1-forking (recall Definition 4.4). This is used to give a converse to Lemma 6.9 in the next appendix (but is not needed for the main body of this paper). Everywhere below, $\mathcal{K}$ is an AEC.

Definition B.1. Let $\alpha \leq \omega$ be a cardinal. $\mathcal{K}$ is $(< \alpha, \mu)$-*superstable* (or $(< \alpha)$-*superstable in* μ) if it satisfies Definition B.1 except that in addition in condition (4) there we allow $p \in \mathrm{gS}^{<\alpha}(M_\delta)$ rather than just $p \in \mathrm{gS}(M_\delta)$ (that is, p need not have length one). $(\leq \alpha, \mu)$-*superstable* means $(< \alpha^+, \mu)$-superstable. When $\alpha = 2$, we omit it (that is, μ-superstable means $(< 2, \mu)$-superstable which is the same as $(\leq 1, \mu)$-superstable).

While not formally equivalent to Definition 2.2, Definition B.1 is very close. For example, the proof of Fact 2.3 also gives:

Fact B.2. Let $\mu \geq \mathrm{LS}(\mathcal{K})$. If $\mathcal{K}$ is has amalgamation, no maximal models, and is categorical in a $\lambda > \mu$, then $\mathcal{K}$ is $(<\omega, \mu)$-superstable.

Even without categoricity, we can obtain eventual $(<\omega)$-superstability from just (≤ 1)-superstability and tameness. This uses another equivalent definition of superstability: solvability:

Theorem B.3. Assume $\mathcal{K}$ has amalgamation, no maximal models, and is $(<\mathrm{LS}(\mathcal{K}))$-tame. If $\mathcal{K}$ is $\mathrm{LS}(\mathcal{K})$-superstable, then there exists $\mu_0 < H_1$ such that $\mathcal{K}$ is $(<\omega)$-superstable in every $\mu \geq \mu_0$.

Proof sketch. By [GV, Theorem 5.43], there exists $\mu_0 < H_1$ such that $\mathcal{K}$ is (μ_0, μ)-solvable for every $\mu \geq \mu_0$. This means ([She09, Definition IV.1.4.(1)]) that for every $\mu \geq \mu_0$, there exists an EM Blueprint Φ so that $\mathrm{EM}(I, \Phi)$ is a superlimit in $\mathcal{K}$ for every linear order I of size μ. Intuitively, it gives a weak version of categoricity in μ. As observed in [GV, Section 6], this weak version is enough for the proof of the Shelah-Villaveces theorem to go through, hence by Fact B.2, $\mathcal{K}$ is $(<\omega)$-superstable in μ for every $\mu \geq \mu_0$. $\qquad\square$

Remark B.4. If $\mathcal{K}$ has amalgamation, is $\mathrm{LS}(\mathcal{K})$-tame for types of length less than ω, and is $(<\omega, \mathrm{LS}(\mathcal{K}))$-superstable, then (by the proof of [Vasb, Proposition 10.10]) $\mathcal{K}$ is $(<\omega)$-superstable in every $\mu \geq \mathrm{LS}(\mathcal{K})$. However here we want to stick to using regular tameness (i.e. tameness for types of length one).

To prove the extension property for 1-forking, we will use:

Fact B.5 (Extension property for splitting, 2.12.(3) in [VVa]). Let $\mathcal{K}$ be an AEC, $\theta > \mathrm{LS}(\mathcal{K})$. Let $\alpha \leq \omega$ be a cardinal and assume that $\mathcal{K}$ is $(<\alpha)$-superstable in every $\mu \in [\mathrm{LS}(\mathcal{K}), \theta)$. Let $M_0 \leq M \leq N$ be in $\mathcal{K}_{[\mathrm{LS}(\mathcal{K}), \theta)}$, with M limit over M_0. Let $p \in \mathrm{gS}^{<\alpha}(M)$ be such that p does not $\mathrm{LS}(\mathcal{K})$-split over M_0. Then there exists an extension $q \in \mathrm{gS}^{<\alpha}(N)$ of p which does not $\mathrm{LS}(\mathcal{K})$-split over M_0.

Theorem B.6. Let $\theta > \mathrm{LS}(\mathcal{K})$. Write $\mathcal{F} := [\mathrm{LS}(\mathcal{K}), \theta)$. Let $\mathfrak{s}$ be a type-full good $\mathcal{F}$-frame with underlying class $\mathcal{K}_\mathcal{F}$. Let $\alpha \leq \omega$ be a cardinal and assume that $\mathcal{K}$ is $(<\alpha, \mu)$-superstable for every $\mu \in \mathcal{F}$. Let $M \leq N$ be in $\mathcal{K}_\mathcal{F}$ with M a limit model. Let $p \in \mathrm{gS}^{<\alpha}(M)$. Then there exists $q \in \mathrm{gS}^{<\alpha}(N)$ that extends p so that q does not 1-$\mathfrak{s}$-fork over M (recall Definition 4.4).

Proof. Without loss of generality, N is a limit model. Let $\mu := \|M\|$. By $(<\alpha)$-superstability, there exists $M_0 \in \mathcal{K}_\mu$ such that M is limit over M_0 and p does not μ-split over M_0. By Fact B.5, there exists $q \in \mathrm{gS}(N)$

extending p so that q does not μ-split over M_0. We claim that q does not 1-$\mathfrak{s}$-fork over M. Let $I \subseteq \ell(p)$ have size one. By monotonicity of splitting, q^I does not μ-split over M_0. By local character, let $N_0 \leq N$ be such that $M \leq N_0$, $N_0 \in \mathcal{K}_\mu$, N_0 is limit, and q^I does not $\mathfrak{s}$-fork over N_0. By monotonicity of splitting again, $q^I \upharpoonright N_0$ does not μ-split over M_0. By the canonicity theorem (Fact A.2) applied to the frame $\mathfrak{s} \upharpoonright \mathcal{K}_\mu$, $q^I \upharpoonright N_0$ does not $\mathfrak{s}$-fork over M. By transitivity, q^I does not $\mathfrak{s}$-fork over M, as desired. $\qquad\square$

Appendix C. More on global orthogonality

Assuming superstability for types of length two, we prove a converse to Lemma 6.9, partially answering Question 6.10. We then prove a few more facts about global orthogonality and derive an alternative proof of a special case of the upward categoricity transfer of Grossberg and VanDieren [GV06a]. This material is not needed for the main body of this paper.

Hypothesis C.1.

(1) $\mathcal{K}$ is an AEC.
(2) $\theta > \mathrm{LS}(\mathcal{K})$ is a cardinal or ∞. We set $\mathcal{F} := [\mathrm{LS}(\mathcal{K}), \theta)$.
(3) $\mathfrak{s} = (\mathcal{K}_\mathcal{F}, \downarrow)$ is a type-full good $\mathcal{F}$-frame.
(4) $\mathcal{K}$ is (≤ 2)-superstable in every $\mu \in \mathcal{F}$.

Remark C.2. Compared to Hypothesis 6.1, we have added (≤ 2)-superstability. Note that this would follow automatically if $\mathfrak{s}$ was a type-full good frame for types of length two, hence it is quite a minor addition. It also holds if $\mathcal{K}$ is categorical above $\mathcal{F}$ (Fact B.2) or even if it is just tame (Theorem B.3).

Lemma C.3. Let $M_0 \leq M$ be both in $\mathcal{K}_\mathcal{F}$ with $M_0 \in \mathcal{K}_{\mathrm{LS}(\mathcal{K})}$ limit. Let $p, q \in \mathrm{gS}(M)$ be nonalgebraic so that both do not fork over M_0. If $p \underset{\mathrm{wk}}{\perp} q$, then $p \upharpoonright M_0 \underset{\mathrm{wk}}{\perp} q \upharpoonright M_0$.

Proof. Assume that $p \upharpoonright M_0 \underset{\mathrm{wk}}{\not\perp} q \upharpoonright M_0$. We show that $p \underset{\mathrm{wk}}{\not\perp} q$. Let $N \geq M_0$, $a, b \in |N|$ realize in N $p \upharpoonright M_0$ and $q \upharpoonright M_0$ respectively and such that $\langle ab \rangle$ is not independent in (M_0, N). Let $r := \mathrm{gtp}(ab/M_0; N)$. By Theorem B.6, there exists $r' \in \mathrm{gS}^2(M)$ that extends r and so that r' does not 1-$\mathfrak{s}$-fork over M_0 (recall Definition 4.4). Let $N' \geq M$ and let $\langle a'b' \rangle$ realize r' in N'. Then $\mathrm{gtp}(a'/M; N')$ does not fork over M_0 and extends $p \upharpoonright M_0$, hence a' must realize p in N'. Similarly, b' realizes q. We claim that $\langle a'b' \rangle$ is *not* independent in (M, N'),

hence $p \not\perp_{\mathrm{wk}} q$. If $\langle a'b' \rangle$ were independent in (M, N'), there would exist $N'' \geq N'$ and $M' \leq N''$ so that $M \leq M'$, $b \in |M'|$, and $\mathrm{gtp}(a'/M'; N'')$ does not fork over M. By transitivity, $\mathrm{gtp}(a'/M'; N'')$ does not fork over M_0. This shows that $\langle a'b' \rangle$ is independent in (M_0, N''), so since $\mathrm{gtp}(a'b'/M_0; N'') = \mathrm{gtp}(ab/M_0; N)$, we must have that $\langle ab \rangle$ is independent in (M_0, N), a contradiction. $\square$

We obtain:

Theorem C.4. Let $M \in \mathcal{K}_{\mathcal{F}}$ and $p, q \in \mathrm{gS}(M)$. Then:

(1) If $M \in \mathcal{K}_{\mathcal{F}}^{\mathrm{LS}(\mathcal{K})\text{-sat}}$, then $p \perp q$ if and only if $p \perp_{\mathrm{wk}} q$.

(2) If $M \in \mathcal{K}_{\mathcal{F}}^{\mathrm{LS}(\mathcal{K})\text{-sat}}$, then $p \perp q$ if and only if $q \perp p$.

(3) If $M_0 \in \mathcal{K}_{\mathcal{F}}^{\mathrm{LS}(\mathcal{K})\text{-sat}}$ is such that $M_0 \leq M$ and both p and q do not fork over M_0, then $p \perp q$ if and only if $p \restriction M_0 \perp q \restriction M_0$.

Proof.

(1) If $p \perp q$, then $p \perp_{\mathrm{wk}} q$ by definition. Conversely, assume that $p \perp_{\mathrm{wk}} q$. Fix a limit $M_0 \in \mathcal{K}_{\mathrm{LS}(\mathcal{K})}$ such that $M_0 \leq M$ and both p and q do not fork over M_0. By Lemma C.3, $p \restriction M_0 \perp_{\mathrm{wk}} q \restriction M_0$. By Lemma 4.16.(1), $p \restriction M_0 \perp q \restriction M_0$. By Lemma 6.9, $p \perp q$.

(2) A similar proof, using (2) instead of (1) in Lemma 4.16.

(3) By local character and transitivity, we can fix a limit $M_0' \in \mathcal{K}_{\mathrm{LS}(\mathcal{K})}$ such that $M_0' \leq M_0$ and both p and q do not fork over M_0'. Now by what has been proven above and Lemmas 6.9 and C.3, $p \perp q$ if and only if $p \restriction M_0' \perp q \restriction M_0'$ if and only if $p \restriction M_0 \perp q \restriction M_0$.

$\square$

We can now give another proof of the upward transfer of unidimensionality (the second part of the proof of Theorem 6.15). This does not use Fact 6.14.

Lemma C.5. Let $\mu < \lambda$ be in $\mathcal{F}$. If $\mathfrak{s}$ is μ-unidimensional, then $\mathfrak{s}$ is λ-unidimensional.

Proof. Assume that $\mathfrak{s}$ is *not* λ-unidimensional. Let $M_0 \in \mathcal{K}_\mu$ be limit and let $p_0 \in \mathrm{gS}(M_0)$ be minimal. We show that there exists a limit $M_0' \in \mathcal{K}_\mu$, $p_0', q_0' \in \mathrm{gS}(M_0')$ such that p_0' extends p_0 and $p_0' \perp q_0'$. This will show that $\mathcal{K}$ is not μ-unidimensional by Lemma 5.6. Let $M \in \mathcal{K}_\lambda$

be saturated such that $M_0 \leq M$ and let $p \in \mathrm{gS}(M)$ be the nonforking extension of p_0. By non-λ-unidimensionality (and Lemma 5.6), there exists $q \in \mathrm{gS}(M)$ so that $p \perp q$. Let $M_0' \in \mathcal{K}_\mu$ be limit such that $M_0 \leq M_0' \leq M$ and q does not fork over M_0'. Let $p_0' := p \restriction M_0'$, $q_0' := q \restriction M_0'$. By Theorem C.4, $p_0' \perp q_0'$, as desired. $\qquad\square$

We obtain the promised alternate proof to Grossberg-VanDieren. Note however that the hypotheses we have are stronger than in [GV06a]: we ask for more tameness and categoricity above the Hanf number. Intuitively, this is because we are using a lot of room to setup the abstract machinery of good frames and obtain $(< \omega)$-superstability.

For this corollary, we drop Hypothesis C.1.

Corollary C.6. Let $\mathcal{K}$ be an AEC with amalgamation. If $\mathcal{K}$ is $(< \mathrm{LS}(\mathcal{K}))$-tame and categorical in a successor $\lambda \geq H_1$, then $\mathcal{K}$ is categorical in all $\mu \geq \lambda$.

Proof. By Remark 7.2, we can assume without loss of generality that $\mathcal{K}$ has no maximal models. By Fact 2.3, $\mathcal{K}$ is $\mathrm{LS}(\mathcal{K})$-superstable. By Theorem B.3, $\mathcal{K}$ is $(< \omega)$-superstable in every $\mu \geq H_1$. Now say $\lambda = \lambda_0^+$. By Proposition 2.14, there exists a type-full good $(\geq \lambda_0)$-frame $\mathfrak{s}$ with underlying class $\mathcal{K}_{\geq H_1}^{\mathrm{LS}(\mathcal{K})^+\text{-sat}}$. By Fact 6.2, we can restrict the frame further to have underlying class $\mathcal{K}_{\geq \lambda_0}^{\lambda_0\text{-sat}}$. By Corollary 6.16 (using Lemma C.5 to transfer unidimensionality up), $\mathcal{K}^{\lambda_0\text{-sat}}$ is categorical in every $\mu \geq \lambda_0$. Now $\mathcal{K}_{\geq \lambda}^{\lambda_0\text{-sat}} = \mathcal{K}_{\geq \lambda}$ (by categoricity in λ), so the result follows. $\qquad\square$

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