

FORKING AND SUPERSTABILITY IN TAME AECS

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ABSTRACT. We prove that any tame abstract elementary class categorical in a suitable cardinal has an eventually global good frame: a forking-like notion defined on all types of single elements. This gives the first known general construction of a good frame in ZFC. We show that we already obtain a well-behaved independence relation assuming only a superstability-like hypothesis instead of categoricity. These methods are applied to obtain an upward stability transfer theorem from categoricity and tameness, as well as new conditions for uniqueness of limit models.

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1. INTRODUCTION

In 2009, Shelah published a two volume book [She09a, She09b] on classification theory for abstract elementary classes. The central new structural notion is that of a good λ -frame (for a given abstract elementary class (AEC) K): a generalization of first-order forking to types over models of size λ in K (see Section 2.4 below for

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the precise definition). The existence of a good frame shows that K is very well-behaved at λ and the aim was to use this frame to deduce more on the structure of K above λ . Part of this program has already been accomplished through several hundreds of pages of hard work (see for example [She01], [She09a, Chapter 2 and 3], [JS12, JS13, JS, Jar]). Among many other results, Shelah shows that good frames exist under strong categoricity assumptions and additional set-theoretic hypotheses:

Fact 1.1 (Theorem II.3.7 in [She09a]). Assume $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$ and the weak diamond ideal in λ^+ is not λ^{++} -saturated.

Let K be an abstract elementary class with $\text{LS}(K) \leq \lambda$. Assume:

- (1) K is categorical in λ and λ^+ .
- (2) $0 < I(\lambda^{++}, K) < \mu_{\text{unif}}(\lambda^{++}, 2^{\lambda^+})$

Then K has a good λ -frame.

It is a major open problem whether the set-theoretic hypotheses in Fact 1.1 are necessary. In this paper, we show that if the class already has some *global* structure, then good frames are much easier to build. For example we prove (in ZFC):

Theorem 1.2. Let K be an abstract elementary class with amalgamation, joint embedding, and no maximal models. Assume K is categorical in a high-enough successor λ^+ . Then K has a type-full good λ -frame.

By the main theorem of [She99], the hypotheses of Theorem 1.2 imply K is categorical in λ . On the other hand, we do not need any set-theoretic hypothesis and we do not need to know anything about the number of models in λ^{++} . Moreover, the frame Shelah constructs typically defines a notion of nonforking only for a restricted class of basic types (the minimal types). With a lot of effort, he then manages to show [She09a, Section III.9] that under some set-theoretic hypotheses one can always extend a frame to be type-full. In our frame, nonforking is directly defined for every type. This is technically very convenient and closer to the first-order intuition. Of course, we pay for this luxury by assuming amalgamation and no maximal models.

Our proof relies on two key properties of AECs. The first one is tameness (a locality property of Galois types, see Definition 2.4), and assuming it lets us relax the successor assumption in Theorem 1.2:

Theorem 1.3. Let K be an abstract elementary class with amalgamation, joint embedding, and no maximal models. Assume K is μ -tame and categorical in some cardinal λ such that $\text{cf}(\lambda) > \mu$. Then K has a type-full good $\geq \lambda$ -frame.

That is, not only do we obtain a good λ -frame, but we can also extend this frame to any model of size $\geq \lambda$ (this last step essentially follows from earlier work

of Boney [Bonb]). Hence we obtain a global forking notion above λ , although only defined for 1-types. A forking notion for types of all lengths is obtained in [BG] (using stronger tameness hypotheses than ours) but the authors assume the extension property for coheir, and it is unclear when this holds, even assuming categoricity everywhere. Thus our result partially answers [BGKV, Question 7.1]. We also obtain new theorems whose statements do not mention frames:

Corollary 1.4. Let K be an abstract elementary class with amalgamation, joint embedding, and no maximal models. Assume K is μ -tame and categorical in some cardinal λ such that $\text{cf}(\lambda) > \mu$. Then K is stable everywhere.

Remark 1.5. Shelah already established in [She99] that categoricity in $\lambda > \text{LS}(K)$ implies stability below λ (assuming amalgamation, joint embedding, and no maximal models). The first upward stability transfer for tame AECs appeared in [GV06b]. Later, [BKV06] gave some variations, showing for example \aleph_0 -stability and a strong form of tameness implies stability everywhere. Our upward stability transfer improves on [BKV06, Corollary 4.7] which showed that categoricity in a successor λ implies stability in λ .

Corollary 1.6. Let K be an abstract elementary class with amalgamation, joint embedding, and no maximal models. Assume K is μ -tame and categorical in some cardinal λ such that $\text{cf}(\lambda) > \mu$. Then K has a unique limit model in every $\lambda' \geq \lambda$.

Remark 1.7. This is also new and complements the conditions for uniqueness of limit models given in [She99], [Van06], and [GVV].

The second key property in our proof is a technical condition we call local character of μ -nonsplitting for \triangleleft -chains (see Definition 3.10). This follows from categoricity in a cardinal of cofinality larger than μ and we believe it is a good candidate for a definition of superstability, at least in the tame context. Under this hypothesis, we already obtain a forking notion that is well-behaved for μ^+ -saturated base models and can prove the upward stability transfer given by Corollary 1.4. Local character of nonsplitting already played a key role in other papers such as [SV99], [Van06], and [GVV].

Even if this notion of superstability fails to hold, we can still look at the length of the chains for which μ -nonsplitting has local character (analogous to the cardinal $\kappa(T)$ in the first-order context). Using GCH, we can generalize one direction of the first-order characterization of the stability spectrum (Theorem 7.6).

The paper is structured as follows: In Section 2, we review background in the theory of AECs and give the definition of good frames. In Section 3, we fix a cardinal μ and build a μ -frame-like object named a *skeletal frame*. This is done using the weak extension and uniqueness properties of nonsplitting isolated by VanDieren [Van02], together with the assumption of local character of nonsplitting. In Section 4, we show that some of the properties of our skeletal frame in μ lift to

cardinals above μ (and in fact become better than they were in μ). This is done using the same methods as in [She09a, Section II.2].

In Section 5, we show assuming tameness that the other properties of the skeletal frame lift as well and similarly become better, so that we obtain (if we restrict ourselves to μ^+ -saturated models and so, assuming categoricity in the right cardinal, to all models) all the properties of a good frame except perhaps symmetry. This uses the ideas from [Bonb]. Next in Section 6 we show how to get symmetry by using more tameness together with the order property (this is where we really use that we have structure properties holding globally and not only at a few cardinals). Finally, we put everything together in Section 7 (Theorem 1.2 is proven there as Theorem 7.4 and Theorem 1.3 as Theorem 7.3). In Section 8, we conclude.

At the beginning of Sections 3, 4, 5, and 6, we give hypotheses that are assumed to hold everywhere in those sections. We made an effort to show clearly how much of the structural properties (amalgamation, tameness, superstability, etc.) are used at each step, but our construction is new even for the case of a totally categorical AEC K with amalgamation, joint embedding, no maximal models, and $\text{LS}(K)$ -tameness. It might help the reader to keep this case in mind throughout.

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2. PRELIMINARIES

2.1. Abstract elementary classes. We assume the reader is familiar with the definition of an abstract elementary class (AEC) and the basic related concepts. See [Gro02] for an introduction.

For the rest of this section, fix an AEC K . We denote the partial ordering on K by \leq , and write $M < N$ if $M \leq N$ and $M \neq N$. For R a binary relation on K and δ an ordinal, an *R -increasing chain* $(M_i)_{i < \delta}$ is a sequence of models in K such that for all $i < \delta$, if $i + 1 < \delta$ then $R(M_i, M_{i+1})$. The chain is *continuous* if it is \leq -increasing and for any limit $i < \delta$, $M_i = \bigcup_{j < i} M_j$. When we talk of an increasing chain, we mean a \leq -increasing chain. Strictly increasing means $<$ -increasing.

For K an abstract elementary class and \mathcal{F} an interval¹ of cardinals of the form $[\lambda, \theta)$, where $\theta > \lambda \geq \text{LS}(K)$ is either a cardinal or ∞ , let $K_{\mathcal{F}} := \{M \in K \mid \|M\| \in \mathcal{F}\}$. We write K_{λ} instead of $K_{\{\lambda\}}$, $K_{\geq \lambda}$ instead of $K_{[\lambda, \infty)}$ and $K_{\leq \lambda}$ instead of $K_{[\text{LS}(K), \lambda]}$.

¹The definitions that follow make sense for an arbitrary set of cardinals \mathcal{F} , but the proofs of most of the facts below require that \mathcal{F} is an interval.

The following properties of AECs are classical:

Definition 2.1 (Amalgamation, joint embedding, no maximal models). Let \mathcal{F} be an interval of cardinals as above.

- (1) $K_{\mathcal{F}}$ has *amalgamation* if for any $M_0 \leq M_\ell \in K_{\mathcal{F}}$, $\ell = 1, 2$ there exists $N \in K_{\mathcal{F}}$ and $f_\ell : M_\ell \xrightarrow{M_0} N$, $\ell = 1, 2$.
- (2) $K_{\mathcal{F}}$ has *joint embedding* if for any $M_\ell \in K_{\mathcal{F}}$, $\ell = 1, 2$ there exists $N \in K_{\mathcal{F}}$ and $f_\ell : M_\ell \rightarrow N$, $\ell = 1, 2$.
- (3) $K_{\mathcal{F}}$ has *no maximal models* if for any $M \in K_{\mathcal{F}}$ there exists $N > M$ in $K_{\mathcal{F}}$.

Fact 2.2. Let \mathcal{F} be an interval of cardinals as above.

- (1) If K_μ has no maximal models for all $\mu \in \mathcal{F}$, then $K_{\mathcal{F}}$ has no maximal models.
- (2) If K_μ has amalgamation for all $\mu \in \mathcal{F}$, then $K_{\mathcal{F}}$ has amalgamation.

Proof. No maximal models is straightforward and amalgamation is [She09a, Conclusion I.2.12]. \square

Finally, we will also use:

Lemma 2.3. Let $\mathcal{F} = [\lambda, \theta]$ be an interval of cardinals as above. If $K_{\mathcal{F}}$ has amalgamation and K_λ has joint embedding, then $K_{\mathcal{F}}$ has joint embedding.

Proof sketch. Let $M_\ell \in K_{\mathcal{F}}$, $\ell = 1, 2$. Pick $M'_\ell \leq M_\ell$ of size λ . Use joint embedding on M'_1, M'_2 , then use amalgamation. \square

2.2. Galois types, stability, and tameness. We assume familiarity with Galois types (see [Gro02, Section 6]). For $M \in K$, we write $S^\alpha(M)$ for the set of Galois types of sequences of length α over M . We will at one point also consider types over the empty set, which are defined analogously (see e.g. [She, Definition 1.4]). We write $S(M)$ for $S^1(M)$. We write $S^{\text{na}}(M)$ for the set of *nonalgebraic* 1-types over M , that is:

$$S^{\text{na}}(M) := \{\text{gtp}(a/M; N) \mid a \in N \setminus M, M \leq N \in K\}$$

From now on, we will write $\text{tp}(a/M; N)$ for $\text{gtp}(a/M; N)$.

We briefly review the notion of tameness. Although it appears implicitly (for saturated models) in [She99], tameness as a property of AECs was first introduced in [GV06b] and used to prove a stability spectrum theorem. It was later used in [GV06a] to prove an upward categoricity transfer. Our definition follows [Bonc, Definition 3.1].

Definition 2.4 (Tameness). Let $\lambda > \kappa > \text{LS}(K)$. Let α be a cardinal. We say that K is (κ, λ) -tame for α -length types if for any $M \in K_{\leq \lambda}$ and any $p, q \in S^\alpha(M)$, if $p \neq q$, then there exists $M_0 \leq M$ of size $\leq \kappa$ such that $p \restriction M_0 \neq q \restriction M_0$. We define similarly $(\kappa, < \lambda)$ -tame, $(< \kappa, \lambda)$ -tame, etc. When $\lambda = \infty$, we omit it. When $\alpha = 1$, we omit it. We say that K is *fully κ -tame* if it is κ -tame for all lengths.

We also recall that we can define a notion of stability:

Definition 2.5 (Stability). Let $\lambda \geq \text{LS}(K)$ and α be cardinals. We say that K is α -stable in λ if for any $M \in K_\lambda$, $|S^\alpha(M)| \leq \lambda$.

We say that K is *stable* in λ if it is 1-stable in λ .

We say that K is α -stable if it is α -stable in λ for some $\lambda \geq \text{LS}(K)$. We say that K is *stable* if it is 1-stable in λ for some $\lambda \geq \text{LS}(K)$. We write “unstable” instead of “not stable”.

We define similarly stability for $K_{\mathcal{F}}$, e.g. $K_{\mathcal{F}}$ is stable if and only if K is stable in λ for some $\lambda \in \mathcal{F}$.

Remark 2.6. If $\alpha < \beta$, and K is β -stable in λ , then K is α -stable in λ .

The following follows from [Bona, Theorem 1.1].

Fact 2.7. Let $\lambda \geq \text{LS}(K)$. Let α be a cardinal. Assume K is stable in λ and $\lambda^\alpha = \lambda$. Then K is α -stable in λ .

2.3. Universal and limit extensions.

Definition 2.8 (Universal and limit extensions). For $M, N \in K$, we say that N is *universal over M* (written $M <_{\text{univ}} N$) if and only if $M < N$ and for any $M' \geq M$ with $\|M'\| = \|M\|$, M' can be embedded inside N over M . We also write $N >_{\text{univ}} M$ for $M <_{\text{univ}} N$.

For $\mu \geq \text{LS}(K)$ and $0 < \delta < \mu^+$ an ordinal, we say that N is (μ, δ) -limit over M (written $M <_{\mu, \delta} N$) if and only if $M \leq N \in K_\mu$ and there is a $<_{\text{univ}}$ -increasing chain $(M_i)_{i \leq \delta}$ with $M_0 = M$, $M_\delta = N$ and $M_\delta = \bigcup_{i < \delta} M_i$ if δ is limit. We also write $N >_{\mu, \delta} M$ for $M <_{\mu, \delta} N$.

We say that N is μ -limit over M if it is μ -limit over M for some limit δ . We say that N is δ -limit if it is $(\|N\|, \delta)$ -limit over M for some M . We say that it is *limit* if it is δ -limit for some limit δ .

Definition 2.9. A model $N \in K$ is μ -model-homogeneous if for any $M \leq N$ with $\|M\| < \mu$, we have $M <_{\text{univ}} N$. N is *model-homogeneous* if it is $\|N\|$ -model-homogeneous.

Fact 2.10. Let $\mu \geq \text{LS}(K)$. Assume K_μ has amalgamation, no maximal models, and is stable. For any $M \in K_\mu$, there exists $N >_{\text{univ}} M$ in K_μ . Therefore there is a model-homogeneous $N > M$ in K_{μ^+} .

Proof. The first part is by [She09a, Claim II.1.16.1(a)]. The second part follows from iterating the first part μ^+ many times. \square

Remark 2.11. By [She01, Lemma 0.26], for $\mu > \text{LS}(K)$, N is μ -model-homogeneous if and only if it is μ -saturated.

Proposition 2.12. Let $M_0, M_1, M_2 \in K_\mu$, $\mu \geq \text{LS}(K)$ and $0 < \delta < \mu^+$. Then:

- (1) $M_0 <_{\mu, \delta} M_1$ implies $M_0 <_{\text{univ}} M_1$.
- (2) $M_0 <_{\text{univ}} M_1 \leq M_2$ implies $M_0 <_{\text{univ}} M_2$.
- (3) Assume K_μ has amalgamation. Then $M_0 \leq M_1 <_{\mu, \delta} M_2$ implies $M_0 <_{\mu, \delta} M_2$.
- (4) Assume K_μ has amalgamation, no maximal models, and is stable. Then there exists M'_0 such that $M_0 <_{\mu, \delta} M'_0$.
- (5) Conversely, if for every $M_0 \in K_\mu$ there exists $M'_0 \in K_\mu$ such that $M_0 <_{\text{univ}} M'_0$, then K_μ has amalgamation, no maximal models, and is stable.

Proof.

- (1) Fix $(N_i)_{i \leq \delta}$ witnessing that $M_0 <_{\mu, \delta} M_1$. Let $M'_0 \geq M_0$ have size μ . Since $\delta > 0$, N_1 is well defined, and is universal over $N_0 = M_0$, hence M'_0 can be embedded inside N_1 over M_0 , and hence since $N_1 \leq M_1$ can be embedded inside M_1 over M_0 .
- (2) Let $M'_0 \geq M_0$ have size μ . Since M'_0 embed inside M_1 over M_0 , it also embeds inside M_2 over M_0 .
- (3) Let $(N_i)_{i \leq \delta}$ witness $M_0 <_{\mu, \delta} M_1$. We show that $M_0 <_{\text{univ}} N_1$. This is enough since then $M_0 \frown (N_i)_{0 < i \leq \delta}$ will witness that $M_0 <_{\mu, \delta}$. Let $M'_0 \geq M_0$ have size μ . By amalgamation, find $M'_1 \geq M_1$ and $h : M'_0 \xrightarrow{M_0} M'_1$. Now use universality of M_2 over M_1 to find $g : M'_1 \xrightarrow{M_1} M_2$. Let $f := g \circ h$. Then $f : M'_0 \xrightarrow{M_0} M_2$, as desired.
- (4) Iterate Fact 2.10 δ many times.
- (5) Let $M_0 \in K_\mu$ and let $M'_0 >_{\text{univ}} M_0$ be in K_μ . M'_0 witnesses that M_0 is not maximal in K_μ . Moreover, M_0 is an amalgamation base, since any two models of size μ extending M_0 can be amalgamated over M_0 inside M'_0 . Finally, all types over M_0 are realized in M'_0 which has size μ , there can be at most μ many of them, so stability follows.

\square

We give orderings satisfying the conclusion of Proposition 2.12 a name:

Definition 2.13 (Abstract universal ordering). An *abstract universal ordering* on K_μ is a binary relation \triangleleft on K_μ satisfying the following properties. For any $M_0, M_1, M_2 \in K_\mu$:

- (1) $M_0 \triangleleft M_1$ implies $M_0 <_{\text{univ}} M_1$.
- (2) There exists $N_0 \triangleright M_0$ in K_μ .
- (3) $M_0 \leq M_1 \triangleleft M_2$ implies $M_0 \triangleleft M_2$.

Note that this implies that \triangleleft is a strict partial ordering on K_μ extending $<$.

For $0 < \delta < \mu^+$, a model $M \in K_\mu$ is (δ, \triangleleft) -*limit* if there exists a \triangleleft -increasing chain $(M_i)_{i < \delta}$ in K_μ such that $M = \bigcup_{i < \delta} M_i$. M is \triangleleft -*limit* if there exists a limit δ such that M is (δ, \triangleleft) -limit.

Remark 2.14. Assume K_μ has amalgamation, no maximal models, and is stable. Then by Proposition 2.12, for any $0 < \delta < \mu^+$, $<_{\mu, \delta}$ is an abstract universal ordering on K_μ . Moreover, the existence of *any* abstract universal ordering on K_μ implies that $<_{\text{univ}}$ is an abstract universal ordering, and hence that K_μ has amalgamation, no maximal models, and is stable.

Let $\text{LS}(K) \leq \mu < \lambda$. Even assuming stability everywhere, it is unclear whether there should be any μ^+ -model-homogeneous model in λ (think for example of the case $\text{cf}(\lambda) = \omega$). The following tells us we can at least get an approximation to one:

Lemma 2.15. Let $\text{LS}(K) \leq \mu^+ < \lambda$. Assume $K_{[\mu, \lambda]}$ has amalgamation, no maximal models, and is stable in μ' for unboundedly many $\mu < \mu' < \lambda$ (that is, for any $\mu < \mu' < \lambda$, there exists $\mu' \leq \mu'' < \lambda$ such that $K_{\mu''}$ is stable).

For any $N_0 \in K_{[\mu, \lambda]}$, there exists $(N_i)_{i < \lambda} <_{\text{univ}}$ -increasing continuous in $K_{[\mu, \lambda]}$ with each N_{i+1} is μ^+ -model-homogeneous. Moreover any $M \geq N_0$ in $K_{[\mu, \lambda]}$ can be embedded inside $N := \bigcup_{i < \lambda} N_i$ over N_0 .

Proof. We build $(N_i)_{i < \lambda}$ by induction. N_0 is already given and without loss of generality $\|N_0\| \geq \mu^+$. Take unions at limits and for a given N_i , first take $N'_i \geq N_i$ such that $K_{\|N'_i\|}$ is stable, and iterate Fact 2.10 μ^+ -many times to pick $N_{i+1} >_{\text{univ}} N'_i$ (and so by Lemma 2.12 also $N_{i+1} >_{\text{univ}} N_i$) in $K_{\|N'_i\|}$ that is also μ^+ -model homogeneous.

Now given $M \geq N_0$ in $K_{[\mu, \lambda]}$, let $(M_i)_{i \leq \lambda}$ be an increasing continuous resolution of M with $\|M_i\| < \lambda$ for all $i < \lambda$ and $M_0 = N_0$. Inductively build $(f_i)_{i \leq \lambda}$ an increasing continuous chain of K -embeddings such that for each $i \leq \lambda$, $f_i : M_i \xrightarrow{M_0} N_i$. This is easy since $N_{i+1} >_{\text{univ}} N_i$ for all $i < \lambda$. Then f_λ embeds M into N . \square

2.4. Good frames. Good frames were first defined in [She09a, Chapter II]. The idea is to provide a localized (i.e. only for base models of a given size λ) axiomatization of a forking-like notion for (a “nice enough” set of) 1-types. Jarden

and Shelah (in [JS13]) later gave a slightly more general definition, not assuming the existence of a superlimit model and dropping some of the redundant clauses. We will use a slight variation here: we assume the models come from $K_{\mathcal{F}}$, for \mathcal{F} an interval, instead of just K_{λ} . We first adapt the definition of a pre- λ -frame from [She09a, Definition III.0.2.1] to such an interval:

Definition 2.16 (Pre-Frame). Let \mathcal{F} be an interval of the form $[\lambda, \theta)$, where λ is a cardinal, and $\theta > \lambda$ is either a cardinal or ∞ .

A *pre- \mathcal{F} -frame* is a triple $\mathfrak{s} = (K, \perp, S^{\text{bs}})$, where:

- (1) K is an abstract elementary class with $\lambda \geq \text{LS}(K)$, $K_{\lambda} \neq \emptyset$.
- (2) $S^{\text{bs}} \subseteq \bigcup_{M \in K_{\mathcal{F}}} S^{\text{na}}(M)$. For $M \in K_{\mathcal{F}}$, we write $S^{\text{bs}}(M)$ for $S^{\text{bs}} \cap S^{\text{na}}(M)$.
- (3) \perp is a relation on quadruples of the form (M_0, M_1, a, N) , where $M_0 \leq M_1 \leq N$, $a \in N$, and M_0, M_1, N are all in $K_{\mathcal{F}}$. We write $\perp(M_0, M_1, a, N)$ or $a \underset{M_0}{\overset{N}{\perp}} M_1$ instead of $(M_0, M_1, a, N) \in \perp$.
- (4) The following properties hold:
 - (a) Invariance: If $f : N \cong N'$ and $a \underset{M_0}{\overset{N}{\perp}} M_1$, then $f(a) \underset{f[M_0]}{\overset{N'}{\perp}} f[M_1]$. If $\text{tp}(a/M_1; N) \in S^{\text{bs}}(M_1)$, then $\text{tp}(f(a)/f[M_1]; N') \in S^{\text{bs}}(f[M_1])$.
 - (b) Monotonicity: If $a \underset{M_0}{\overset{N}{\perp}} M_1$, $M_0 \leq M'_0 \leq M'_1 \leq M_1 \leq N' \leq N \leq N''$ with $a \in N'$ and $N'' \in K_{\mathcal{F}}$, then $a \underset{M'_0}{\overset{N'}{\perp}} M'_1$ and $a \underset{M'_0}{\overset{N''}{\perp}} M'_1$.
 - (c) Nonforking types are basic: If $a \underset{M}{\overset{N}{\perp}} M$, then $\text{tp}(a/M; N) \in S^{\text{bs}}(M)$.

We write λ -frame instead of $\{\lambda\}$ -frame, $(\geq \lambda)$ -frame instead of $[\lambda, \infty)$ -frame. We sometimes drop the \mathcal{F} when it is clear from context.

A pre-frame is *type-full* if $S^{\text{bs}}(M) = S^{\text{na}}(M)$ for all $M \in K_{\mathcal{F}}$.

For $\mathcal{F}' \subseteq \mathcal{F}$ an interval, we let $\mathfrak{s} \upharpoonright \mathcal{F}'$ denote the pre- \mathcal{F}' -frame defined in the obvious way by restricting the basic types and \perp to models in $K_{\mathcal{F}'}$. For $\lambda' \in \mathcal{F}$, we write $\mathfrak{s} \upharpoonright \lambda'$ instead of $\mathfrak{s} \upharpoonright \{\lambda'\}$.

By the invariance and monotonicity properties, \perp is really a relation on types. This justifies the next definition.

Definition 2.17. If $\mathfrak{s} = (K, \perp, S^{\text{bs}})$ is a pre- \mathcal{F} -frame, $p \in S(M_1)$ is a type, we say p *does not fork over* M_0 if $a \underset{M_0}{\overset{N}{\perp}} M_1$ for some (equivalently any) a and N such that $p = \text{tp}(a/M_1; N)$.

Remark 2.18. We could have started from (K, \perp) and defined the basic types as those that do not fork over their own domain. The existence property of good frames (see below) would then hold for free. Since we are sometimes interested in studying frames that only satisfy existence over a certain class of models (like the saturated models), we will not adopt this approach.

Definition 2.19 (Good frame). Let \mathcal{F} be as above.

A *good \mathcal{F} -frame* is a pre- \mathcal{F} -frame $(K, \perp, S^{\text{bs}})$ satisfying in addition:

- (1) $K_{\mathcal{F}}$ has amalgamation, joint embedding, and no maximal model.
- (2) bs-Stability: $|S^{\text{bs}}(M)| \leq \|M\|$ for all $M \in K_{\mathcal{F}}$.
- (3) Density of basic types: If $M < N$ and $M, N \in K_{\mathcal{F}}$, then there is $a \in N$ such that $\text{tp}(a/M; N) \in S^{\text{bs}}(M)$.
- (4) Existence: If $M \in K_{\mathcal{F}}$ and $p \in S^{\text{bs}}(M)$, then p does not fork over M .
- (5) Extension: If $p \in S(N)$ does not fork over M , and $N' \geq N$ is in $K_{\mathcal{F}}$, then there is $q \in S(N')$ extending p that does not fork over M .
- (6) Uniqueness: If $p, q \in S(N)$ do not fork over M and $p \upharpoonright M = q \upharpoonright M$, then $p = q$.
- (7) Symmetry: If $a_1 \perp_{M_0}^N M_2$, $a_2 \in M_2$, and $\text{tp}(a_2/M_0; N) \in S^{\text{bs}}(M_0)$, then there is M_1 containing a_1 and $N' \geq N$ such that $a_2 \perp_{M_0}^{N'} M_1$.
- (8) Local character: If δ is a limit ordinal, $(M_i)_{i \leq \delta}$ is an increasing chain in $K_{\mathcal{F}}$ with $M_\delta = \bigcup_{i < \delta} M_i$, and $p \in S^{\text{bs}}(M_\delta)$, then there exists $i < \delta$ such that p does not fork over M_i .
- (9) Continuity: If δ is a limit ordinal, $(M_i)_{i \leq \delta}$ is an increasing chain in $K_{\mathcal{F}}$ with $M_\delta = \bigcup_{i < \delta} M_i$, $p \in S(M_\delta)$ is so that $p \upharpoonright M_i$ does not fork over M_0 for all $i < \delta$, then p does not fork over M_0 .
- (10) Transitivity²: If $M_0 \leq M_1 \leq M_2$, $p \in S(M_2)$ does not fork over M_1 and $p \upharpoonright M_1$ does not fork over M_0 , then p does not fork over M_0 .

For \mathbb{L} a list of properties, a $\text{good}^{-\mathbb{L}}$ \mathcal{F} -frame is a pre- \mathcal{F} -frame that satisfies all the properties of good frames except possibly the ones in \mathbb{L} . In this paper, \mathbb{L} will only contain symmetry and/or bs-stability. We abbreviate symmetry by S , bs-stability by St , and write good^- for $\text{good}^{-(S, St)}$.

We say that K has a good \mathcal{F} -frame if there is a good \mathcal{F} -frame where K is the underlying AEC (and similarly for good^-).

Remark 2.20. Using \mathcal{F} instead of a single cardinal λ is only a convenience: just like an abstract elementary class K is determined by $K_{\text{LS}(K)}$, a good^- \mathcal{F} -frame \mathfrak{s} is determined by $\mathfrak{s} \upharpoonright \lambda$, where $\lambda := \min(\mathcal{F})$. More precisely, if \mathfrak{t} is a good^- \mathcal{F} -frame

²This actually follows from uniqueness and extension, see [She09a, Claim II.2.18].

such that $\mathfrak{t} \restriction \lambda = \mathfrak{s} \restriction \lambda$, then the arguments from [She09a, Section II.2] show that $\mathfrak{t} = \mathfrak{s}$.

Note that local character implies nonforking is always witnessed by a model of small size:

Proposition 2.21. Assume \mathcal{F} is an interval of cardinals with minimum λ . Assume $\mathfrak{s} = (K, \perp, S^{\text{bs}})$ is a pre- \mathcal{F} -frame satisfying local character and transitivity. If $M \in K_{\mathcal{F}}$ and $p \in S^{\text{bs}}(M)$, then there exists $M' \in K_{\lambda}$ such that p does not fork over M' .

Proof. By induction on $\lambda' := \|M\|$. If $\lambda' = \lambda$, then since local character implies existence, we can take $M' := M$. Otherwise, $\lambda' > \lambda$ so we can take a resolution $(M_i)_{i < \lambda'}$ of M such that $\lambda \leq \|M_i\| < \lambda'$ for all $i < \lambda'$. By local character, there exists $i < \lambda'$ such that p does not fork over M_i . By monotonicity, $p \restriction M_i$ does not fork over M_i , so must be basic. By the induction hypothesis, there exists $M' \in K_{\lambda}$ such that $p \restriction M_i$ does not fork over M' . By transitivity, p does not fork over M' . \square

3. A SKELETAL FRAME FROM NONSPLITTING

Hypothesis 3.1.

- (1) K is an abstract elementary class. $\mu \geq \text{LS}(K)$ is a cardinal. $K_{\mu} \neq \emptyset$.
- (2) K_{μ} has amalgamation.

In this section, we start our quest for a good frame. Recall the following generalization of first-order splitting from [She99, Definition 3.2]:

Definition 3.2. For $p \in S(N)$, we say that p μ -splits over M if $M \leq N$ and there exists $N_1, N_2 \in K_{\mu}$ so that $M \leq N_{\ell} \leq N$ for $\ell = 1, 2$, and $h : N_1 \cong_M N_2$ such that $h(p \restriction N_1) \neq p \restriction N_2$.

When μ is clear from context, we drop it.

Remark 3.3 (Monotonicity). If $p \in S(N)$ does not μ -split over M and $M \leq M' \leq N' \leq N$ are in K_{μ} , then $p \restriction N'$ does not μ -split over M' .

Remark 3.4. If $\mathfrak{s} = (K, \perp, S^{\text{bs}})$ is a good⁻ μ -frame, and $p \in S(N)$ does not fork over M , then p does not μ -split over M (this is an easy exercise in the use of the uniqueness property, see e.g. [BGKV, Lemma 4.2]). Thus splitting can be seen as a first approximation to a forking notion.

Our starting point will be the following extension and uniqueness properties of splitting, first isolated by VanDieren [Van02, Theorem II.7.9, Theorem II.7.11]. Intuitively, they tell us that the usual uniqueness and extension property of a

nonforking notion hold provided we have enough room (concretely, the base model has to be “shifted” by a universal extension).

Fact 3.5. Let $M_0 <_{\text{univ}} M \leq N$ with $M_0, M, N \in K_\mu$. Then:

- (1) Weak uniqueness: If $p_\ell \in S(N)$ does not split over M_0 , $\ell = 1, 2$, $p_1 \upharpoonright M = p_2 \upharpoonright M$, then $p_1 = p_2$.
- (2) Weak extension: If $p \in S(M)$ does not split over M_0 , then there exists $q \in S(N)$ extending p that does not split over M_0 . Moreover, q is nonalgebraic if p is.

Proof. See [Van06, Theorem I.4.12] for weak uniqueness. For weak extension, use universality to get $h : N \xrightarrow{M_0} M$. Further extend h to an isomorphism $\hat{h} : \hat{N} \cong_{M_0} \hat{M}$. So that \hat{M} contains a realization a of p . Let $a' := \hat{h}^{-1}(a)$, and let $q := \text{tp}(a/N; \hat{N})$. The proof of [Van06, Theorem I.4.10] shows q is indeed an extension of p that does not split over M_0 . In addition if q is algebraic, $a' \in N$ so $a = h(a') \in M$, so p is algebraic. \square

We will mostly use those two properties instead of the exact definition of splitting. However, they characterize splitting in the following sense:

Proposition 3.6. Assume K_μ has amalgamation, no maximal models, and is stable. Let \mathfrak{s} be a type-full pre- μ -frame with underlying AEC K . The following are equivalent.

- (1) For all types $p \in S(N)$, $N \in K_\mu$ and $M \leq N$, if p does not fork over M , then for any $M <_{\text{univ}} M' \leq N$, p does not split over M' .
- (2) Nonforking satisfies weak uniqueness and weak extension (i.e. the conclusion of Fact 3.5 holds with “split” replaced by “fork”).

Proof. Left as an exercise (not used). \square

We also obtain a weak transitivity property:

Proposition 3.7 (Weak transitivity of nonsplitting). Let $M_0 \leq M_1 <_{\text{univ}} M'_1 \leq M_2$ in K_μ . Let $p \in S(M_2)$. If $p \upharpoonright M'_1$ does not split over M_0 and p does not split over M_1 , then p does not split over M_0 .

Proof. By weak extension, find $q \in S(M_2)$ extending $p \upharpoonright M'_1$ and not splitting over M_0 . By monotonicity, q does not split over M_1 . By weak uniqueness, $p = q$, as needed. \square

We now turn to building a forking notion that will satisfy a version of uniqueness and extension (see Definition 2.19) in K_μ . The idea is simple enough: we want to say that a type does not fork over M if there is a “small” substructure M_0 of M

over which the type does not *split*. Fact 3.5 suggests that “small” should mean “such that M is a universal extension of M_0 ”, and this is exactly how we define it:

Definition 3.8 (Nonforking). For $M_0 \leq M \leq N$ in K_μ , we say $p \in S(N)$ *explicitly does not fork over* (M_0, M) if:

- (1) $M_0 <_{\text{univ}} M \leq N$.
- (2) p does not split over M_0 .

We say p *does not fork over* M if there exists M_0 so that p explicitly does not fork over (M_0, M) .

When we say that a type p explicitly does not fork over (M_0, M) , we think of M as the base, and M_0 as the *explicit* witness to the nonforking. It would be nice if we could get rid of the witness entirely and get that nonforking satisfies extension and uniqueness, but uniqueness seems to depend on the particular witness.

Transitivity is also problematic: although we manage to get a weak version depending on the particular witnesses, we still do not know how to prove the witness-free version. This was stated as [Bal09, Exercise 12.9] but Baldwin later admitted [Bal] there was a mistake in his proof.

If instead we define “ p does not fork* over M ” to mean “for all $M_0 <_{\text{univ}} M$ there exists M'_0 with $M_0 \leq M'_0 <_{\text{univ}} M$ and p explicitly does not fork over (M'_0, M) ” then extension and uniqueness (and thus transitivity) hold, but local character (assuming local character of nonsplitting) is problematic. Thus it seems we have to carry along the witness in our definition of nonforking, and this makes the resulting independence notion quite weak (hence the name “skeletal”). However, we will see in the next sections that (assuming some tameness and homogeneity) our skeletal μ -frame transfers to a much better-behaved frame *above* μ . In particular, full uniqueness and transitivity will hold there.

Lemma 3.9 (Basic properties of nonforking). Below, all models are in K_μ .

- (1) Monotonicity: If $p \in S(N)$ explicitly does not fork over (M_0, M) , $M_0 \leq M'_0 \leq M \leq M' \leq N' \leq N$ and $M'_0 <_{\text{univ}} M'$, then $p \upharpoonright N'$ explicitly does not fork over (M'_0, M') . In particular, if $p \in S(N)$ does not fork over M and $M \leq M' \leq N' \leq N$, then $p \upharpoonright N'$ does not fork over M' .
- (2) Extension: If $p \in S(N)$ explicitly does not fork over (M_0, M) and $N' \geq N$, then there is $q \in S(N')$ extending p that explicitly does not fork over (M_0, M) . If p is nonalgebraic, then q is nonalgebraic.
- (3) Uniqueness: If $p_\ell \in S(N)$ explicitly does not fork over (M_0, M) , $\ell = 1, 2$, and $p_1 \upharpoonright M = p_2 \upharpoonright M$, then $p_1 = p_2$.
- (4) Transitivity: Let $M_1 \leq M_2 \leq M_3$ and let $p \in S(M_3)$. If $p \upharpoonright M_2$ explicitly does not fork over (M_0, M_1) and p explicitly does not fork over (M'_0, M_2) for $M_0 \leq M'_0$, then p explicitly does not fork over (M_0, M_1) .

- (5) Nonalgebraicity: If $p \in S(N)$ does not fork over M and $p \restriction M$ is not algebraic, then p is not algebraic.

Proof. Monotonicity follows directly from the definition (and Proposition 2.12.(2)), extension and uniqueness are just restatements of Fact 3.5, and transitivity is a restatement of Proposition 3.7. For nonalgebraicity, assume $p \restriction M$ is nonalgebraic. Then it has a nonalgebraic nonforking extension to N by extension, and this extension must be p by uniqueness, so the result follows. \square

Assuming some local character for nonsplitting, we obtain weak versions of the local character and continuity properties:

Definition 3.10. Let R be a binary relation on K_μ , and let κ be a regular cardinal. We say that μ -nonsplitting has κ -local character for R -increasing chains if for any R -increasing $(M_i)_{i < \delta}$ with $\text{cf}(\delta) \geq \kappa$, $M_\delta = \bigcup_{i < \delta} M_i$, and any $p \in S(M_\delta)$, there is $i < \delta$ so that p does not split over M_i .

Remark 3.11. If K_μ is stable, then by [GV06b, Fact 4.6] μ -nonsplitting has μ^+ -local character for \leq -increasing chains.

Lemma 3.12. Let \triangleleft be an abstract universal ordering on K_μ , and let κ be a regular cardinal. Assume nonsplitting has κ -local character for \triangleleft -increasing chains. Then:

- (1) κ -local character for \triangleleft -increasing chains: If $(M_i)_{i < \delta}$ is a \triangleleft -increasing chain in K_μ with $\text{cf}(\delta) \geq \kappa$, $M_\delta = \bigcup_{i < \delta} M_i$ and $p \in S(M_\delta)$, then there exists $i < \delta$ so that p explicitly does not fork over (M_i, M_{i+1}) .
- (2) κ -continuity for \triangleleft -increasing chains: If $(M_i)_{i < \delta}$ is a \triangleleft -increasing chain in K_μ with $\text{cf}(\delta) \geq \kappa$, $M_\delta = \bigcup_{i < \delta} M_i$ and $p \in S(M_\delta)$ such that $p \restriction M_i$ does not fork over M_0 for all $i < \delta$, then p does not fork over M_0 . Moreover, if in addition $p \restriction M_i$ explicitly does not fork over (M'_0, M_0) for all $i < \delta$ (i.e. the witness is always the same), then p explicitly does not fork over (M'_0, M_0) .
- (3) Existence over $(\geq \kappa, \triangleleft)$ -limits: If $M \in K_\mu$ is (δ, \triangleleft) -limit for some δ with $\text{cf}(\delta) \geq \kappa$, then any $p \in S(M)$ does not fork over M . In fact, if $p_0, \dots, p_{n-1} \in S(M)$, $n < \omega$, then there exists $M_0 <_{\text{univ}} M$ such that p_i explicitly does not fork over (M_0, M) for all $i < n$.

Proof.

- (1) Follows from κ -local character of nonsplitting for \triangleleft -increasing chains.
- (2) By κ -local character, there exists $i < \delta$ so that p explicitly does not fork over (M_i, M_{i+1}) . By assumption, there exists $M'_0 <_{\text{univ}} M_0$ so that $p \restriction M_{i+1}$ explicitly does not fork over (M'_0, M_0) . Since $M'_0 \leq M_i$, we can apply transitivity to obtain that p explicitly does not fork over (M'_0, M_0) . The proof of the moreover part is similar.

(3) By local character and monotonicity.

□

Thus if nonsplitting has \aleph_0 -local character for \triangleleft -increasing chains for some abstract universal ordering \triangleleft and if all models in K_μ are \triangleleft -limit (e.g. if K_μ is categorical), then it seems we are very close to having a good^{-S} μ -frame, but the witnesses must be carried along, which as observed above is rather annoying. Also, local character and continuity only hold for \triangleleft -chains.

In the next sections, we show that these problems disappear when we transfer our skeletal frame above μ . Note that Shelah's construction of a good frame already takes advantage of that phenomenon. A similar idea is also exploited in the definition of a rooted minimal type in Grossberg and VanDieren's categoricity transfer from tameness [GV06a, Definition 2.6].

4. GOING UP WITHOUT TAMENESS

Hypothesis 4.1.

- (1) K is an abstract elementary class. $\mu \geq \text{LS}(K)$ is a cardinal. $K_\mu \neq \emptyset$.
- (2) \triangleleft is an abstract universal ordering on K_μ . In particular (by Remark 2.14), K_μ has amalgamation, no maximal models, and is stable.

In [She09a, Section II.2], Shelah showed how to extend a good μ -frame to all models in $K_{\geq \mu}$. The resulting object will in general not be a good frame, but several of the properties are nevertheless preserved. In this section, we apply the same procedure on our skeletal frame and show Shelah's arguments still go through, assuming the base models are μ^+ -homogeneous. In the next section, we will assume tameness to prove more properties of nonforking.

We first define nonforking in $K_{>\mu}$ in exactly the same way as Shelah:

Definition 4.2. Assume $N \in K_{>\mu}$, $M \in K_{\geq \mu}$ and $p \in S^{\text{na}}(N)$. We say that p *does not fork over* M if $M \leq N$ and there exists $M' \leq M$ in K_μ such that for all $N' \in K_\mu$ with $M' \leq N' \leq N$, $p \upharpoonright N'$ does not fork over M' .

We define the relation \perp by $a \perp_M^{\hat{N}} N$ if and only if $M, N, \hat{N} \in K_{\geq \mu}$, $a \in \hat{N}$, and $\text{tp}(a/N; \hat{N})$ does not fork over M .

For technical reasons, we also need to define explicit nonforking over a model of size μ :

Definition 4.3 (Explicit nonforking in $K_{>\mu}$). Assume $N \in K_{>\mu}$, $M_0 \leq M \in K_\mu$, and $p \in S^{\text{na}}(N)$. We say that p *explicitly does not fork over* (M_0, M) if for all

$N' \in K_\mu$ with $M \leq N' \leq N$, we have $p \restriction N'$ explicitly does not fork over (M_0, M) (see Definition 3.8). Equivalently, p does not μ -split over M_0 and $M_0 <_{\text{univ}} M \leq N$.

Proposition 4.4. $\mathfrak{s}_0 := (K, \perp, S^{\text{na}})$ is a type-full pre- $[\mu, \infty)$ -frame.

Proof. The properties to check follow directly from the definition of nonforking. \mathfrak{s}_0 is type-full since we defined the basic types to be all the nonalgebraic types. \square

In K_μ we had by definition that a type which does not fork over M also explicitly does not fork over (M_0, M) for some witness M_0 . This is not necessarily the case in $K_{\geq \mu}$: take for example $N \in K_{> \mu}$ and $M \in K_\mu$ and assume $p \in S(N)$ does not fork over M . Then for all $N' \in K_\mu$ with $M \leq N' \leq N$, $p \restriction N'$ does not fork over M , i.e. there is a witness M'_0 such that $p \restriction N'$ explicitly does not fork over (M'_0, M) , but there could be different witnesses for different N' s.

The next lemma shows that this can be avoided if we have enough homogeneity. This is crucial to our proofs of transitivity, uniqueness, and extension.

Lemma 4.5. Assume $M \leq N$ in $K_{\geq \mu^+}$ and M is μ^+ -model-homogeneous. Assume $p \in S(N)$ does not fork over M . Then there exists $M'_0, M' \in K_\mu$ with $M'_0 \leq M' \leq M$ such that p explicitly does not fork over (M'_0, M') (i.e. nonforking over M' is witnessed by the same M'_0 uniformly).

Proof. By definition, there is $M'_0 \leq M$ in K_μ such that p does not fork over M'_0 . Since M is μ^+ -model-homogeneous, one can pick $M' >_{\text{univ}} M'_0$ in K_μ with $M' \leq M$. By monotonicity (Lemma 3.9.(1)), p does not fork over (M'_0, M') . \square

Lemma 4.6 (Existence). Let $M \in K_{\geq \mu^+}$ be μ^+ -model-homogeneous. Then $p \in S^{\text{na}}(M)$ if and only if p does not fork over M .

Proof. If p does not fork over M , then p is nonalgebraic by definition. Now assume p is nonalgebraic. By [GV06b, Fact 4.6], there is $M'_0 \leq M$ in K_μ such that p does not μ -split over M'_0 . Pick $M' >_{\text{univ}} M'_0$ in K_μ so that $M' \leq M$. This is possible by μ^+ -model-homogeneity. We have that p explicitly does not fork over (M'_0, M') , so does not fork over M' , as needed. \square

Lemma 4.7 (Transitivity). If $M_0 \leq M_1 \leq M_2$ are in $K_{\geq \mu}$, M_1 is μ^+ -model-homogeneous, $p \in S^{\text{na}}(M_2)$ is such that $p \restriction M_1$ does not fork over M_0 and p does not fork over M_1 , then p does not fork over M_0 .

Proof. Find $M'_0 \leq M_0$ in K_μ such that $p \restriction M_1$ does not fork over M'_0 . Using monotonicity and Lemma 4.5, we can also find $M'_0 \leq M'_1 <_{\text{univ}} M''_1 \leq M_1$ in K_μ such that p explicitly does not fork over (M'_1, M''_1) . By transitivity in K_μ (Lemma 3.9.(4)), p does not fork over M'_0 , and hence over M_0 . \square

Lemma 4.8 (Local character). Assume nonsplitting has κ -local character for \triangleleft -increasing chains. If $\text{cf}(\delta) \geq \kappa$, $(M_i)_{i \leq \delta}$ is an increasing chain in $K_{\geq \mu^+}$ with $M_\delta = \bigcup_{i < \delta} M_i$, M_i is μ^+ -model-homogeneous for $i < \delta$, and $p \in S^{\text{na}}(M_\delta)$, then there is $i < \delta$ such that p does not fork over M_i .

Proof. Without loss of generality, δ is regular. If $\delta \geq \mu^+$, then M_δ is also μ^+ -model-homogeneous so one can pick $N^* \leq M_\delta$ in K_μ witnessing existence (use Lemma 4.6) and find $i < \delta$ with $N^* \leq M_i$, so p does not fork over M_i as needed. Now assume $\delta < \mu^+$. We imitate the proof of [She09a, Claim II.2.11.5]. Assume the conclusion fails. Build $(N_i)_{i \leq \delta}$ \triangleleft -increasing continuous in K_μ , $(N'_i)_{i \leq \delta}$ \leq -increasing continuous in K_μ such that for all $i < \delta$:

- (1) $N_i \leq M_i$.
- (2) $N_i \leq N'_i \leq M_\delta$.
- (3) $p \upharpoonright N'_{i+1}$ explicitly forks over (N_i, N_{i+1}) .
- (4) $\bigcup_{j \leq i} (N'_j \cap M_{i+1}) \subseteq |N_{i+1}|$.

This is possible. For $i = 0$, let $N_0 \leq M_0$ be any model in K_μ , and $M_\delta \geq N'_0 \geq N_0$ be a model in K_μ . For i limit, take unions. For the successor case, assume $i = j + 1$. Choose $N_i \leq M_i$ satisfying (4) with $N_i \supset N_j$ (possible since M_i is μ^+ -model-homogeneous). By assumption, p forks over M_i , hence explicitly forks over (N_j, N_i) , and so by definition of forking and monotonicity there exists $M_\delta \geq N'_i \geq N_i$, $N'_i \geq N'_j$ with $p \upharpoonright N'_i$ explicitly forking over (N_j, N_i) . It is as required.

This is enough. By local character in K_μ , there is $i < \delta$ such that $p \upharpoonright N_\delta$ explicitly does not fork over (N_i, N_{i+1}) . By (2) and (4), $N'_\delta \leq N_\delta$. Thus $p \upharpoonright N'_{i+1}$ explicitly does not fork over (N_i, N_{i+1}) , contradicting (3). □

Lemma 4.9 (Continuity). Assume nonsplitting has κ -local character for \triangleleft -increasing chains. If $\text{cf}(\delta) \geq \kappa$, $(M_i)_{i \leq \delta}$ is an increasing chain in $K_{\geq \mu^+}$ with $M_\delta = \bigcup_{i < \delta} M_i$, M_i μ^+ -model-homogeneous for $i < \delta$, and $p \in S(M_\delta)$ is so that $p \upharpoonright M_i$ does not fork over M_0 for all $i < \delta$, then p_δ does not fork over M_0 .

Proof. Follows directly from κ -local character and transitivity, see [She09a, Claim II.2.17.3]. □

Remark 4.10. In the statements of local character and continuity, we assumed that M_i was μ^+ -model-homogeneous for all $i < \delta$, but *not* that their union M_δ was μ^+ -model-homogeneous.

5. A TAME GOOD^{-S} FRAME

Hypothesis 5.1.

- (1) K is an abstract elementary class. $\mu \geq \text{LS}(K)$ is a cardinal. $K_\mu \neq \emptyset$.
- (2) \triangleleft is an abstract universal ordering on K_μ . In particular (by Remark 2.14), K_μ has amalgamation, no maximal models, and is stable.
- (3) κ is the least regular cardinal such that nonsplitting has κ -local character for \triangleleft -increasing chains in K_μ .
- (4) $\lambda > \mu$ is such that:
 - (a) K is (μ, λ) -tame.
 - (b) $K_{[\mu, \lambda]}$ has amalgamation.
 - (c) $K_{[\mu, \lambda]}$ has no maximal models.

Remark 5.2. κ plays a similar role as the cardinal $\kappa(T)$ in the first-order setup. By Remark 3.11 and Hypothesis 5.1.(2), $\kappa \leq \mu^+$. In the end, we will be able to obtain a good frame only when $\kappa = \aleph_0$, but studying the general case leads to results on the stability spectrum.

Boney showed in [Bonb] that given a good μ -frame, tameness implies that Shelah's extension of the frame to $\geq \mu$ is actually a good frame. In this section, we apply the ideas of his proof (assuming the base models are μ^+ -model-homogeneous) to our skeletal frame, i.e. we prove stability, extension and uniqueness using tameness. Note that uniqueness is actually *equivalent* to (λ, μ) -tameness by [Bonb, Theorem 3.2]. The easiest case is when $\lambda = \mu^+$. Then we know a model-homogeneous model exists in K_λ , and this simplifies some of the proofs.

Lemma 5.3 (Uniqueness). Let $M \leq N$ be models in $K_{[\mu, \lambda]}$. Let $p, q \in S(N)$. Assume $p \upharpoonright M = q \upharpoonright M$.

- (1) If $M \in K_\mu$ and p, q explicitly do not fork over (M_0, M) for some $M_0 <_{\text{univ}} M$, then $p = q$.
- (2) If $M \in K_{[\mu^+, \lambda]}$ is μ^+ -model-homogeneous and p, q do not fork over M , then $p = q$.

Proof. (1) follows from uniqueness in K_μ (Lemma 3.9.(3)) and tameness. To see (2), use monotonicity and Lemma 4.5, to find $M'_0 <_{\text{univ}} M' \leq M$ in K_μ such that both p and q explicitly do not fork over (M'_0, M') . Now apply (1). \square

Interestingly, we already have enough machinery to obtain a stability transfer theorem. First recall:

Fact 5.4. K_{μ^+} is stable.

Proof. This could be done using the method of proof of Theorem 5.6 (exercise), but this is also [BKV06, Theorem 1]. \square

Lemma 5.5. Assume that $\lambda > \mu^+$, $\text{cf}(\lambda) \geq \kappa$, and there are unboundedly (in the same sense as in the statement of Lemma 2.15) many $\mu \leq \lambda' < \lambda$ such that $K_{\lambda'}$ is stable. Then K_λ is stable.

Proof. Let $M \in K_\lambda$. By Lemma 2.15, M can be embedded inside some $\widehat{M} \in K_\lambda$ which can be written as $\bigcup_{i < \lambda} M_i$, with $(M_i)_{i < \lambda}$ an increasing chain³ of μ^+ -model-homogeneous models in $K_{[\mu^+, \lambda)}$. From amalgamation, we know that Galois types can be extended, so $|S(M)| \leq |S(\widehat{M})|$, and so we can assume without loss of generality that $M = \widehat{M}$. Let $(p_j)_{j < \lambda^+}$ be types in $S(M)$. By κ -local character, for each $j < \lambda^+$ there is $i_j < \lambda$ such that p_j does not fork over M_{i_j} . By the pigeonhole principle, we may assume $i_j = i_0$ for all $j < \lambda^+$. Taking i_0 bigger if necessary, we may assume that $K_{\|M_{i_0}\|}$ is stable. Thus $|S(M_{i_0})| \leq \|M_{i_0}\| \leq \lambda$, so by the pigeonhole principle again, we can assume that there is $q \in S(M_{i_0})$ such that $p_j \upharpoonright M_{i_0} = q$ for all $j < \lambda^+$. By uniqueness, $p_j = p_{j'}$ for each $j, j' < \lambda^+$, so the result follows. \square

We can now prove stability transfers up. Recall that λ is the cardinal above μ fixed in Hypothesis 5.1.(4). Recall also that we already have stability in μ by Hypothesis 5.1.(2).

Theorem 5.6 (The superstability theorem). If $\kappa = \aleph_0$, then K_λ is stable.

Proof. We work by induction on λ . If $\lambda = \mu^+$, this is Fact 5.4 and if $\lambda > \mu^+$ this is given by Lemma 5.5 and the induction hypothesis. \square

Assuming the generalized continuum hypothesis (GCH), we can also say something for arbitrary κ :

Theorem 5.7. Assume GCH. If $\lambda^{<\kappa} = \lambda$, then K_λ is stable.

Proof. By induction on λ . If $\lambda = \mu^+$, this is Fact 5.4, so assume $\lambda > \mu^+$. By König's theorem, $\text{cf}(\lambda) \geq \kappa$. If λ is successor, then $\lambda^\mu = \lambda$ by GCH, so by [GV06b, Corollary 6.4], K is stable in λ . If λ is limit there exists a sequence of successor cardinals $(\lambda_i)_{i < \text{cf}(\lambda)}$ increasing cofinal in λ with $\lambda_0 \geq \mu^+$. Since without loss of generality $\kappa \leq \mu^+$ (Remark 3.11), GCH implies that $\lambda_i^{<\kappa} = \lambda_i$, so by the induction hypothesis, K is stable in λ_i for all $i < \text{cf}(\lambda)$. Apply Lemma 5.5 to conclude. \square

We now prove extension. This follows from compactness in the first-order case, but we make crucial use of the superstability hypothesis $\kappa = \aleph_0$ in the general case.

Lemma 5.8. Assume $\kappa = \aleph_0$. Let $\delta < \lambda^+$ be a limit ordinal. Assume $(M_i)_{i \leq \delta}$ is an increasing continuous sequence in $K_{[\mu, \lambda)}$ with $M_0 \in K_\mu$. Let $(p_i)_{i < \delta}$ be an increasing continuous sequence of types with $p_i \in S(M_i)$ for all $i < \delta$, and p_i explicitly does not fork over (M'_0, M_0) . Assume that one of the following holds:

³Explicitly, we take $(N_i)_{i < \lambda}$ as given by Lemma 2.15 for some $N_0 \leq M$ in K_{μ^+} , and let $M_i := N_{i+1}$. Note that the chain $(M_i)_{i < \lambda}$ will *not* be continuous.

- (1) $(M_i)_{i < \delta}$ is \triangleleft -increasing in K_μ .
- (2) For all $i < \delta$, M_{i+1} is μ^+ -model-homogeneous.

Then there exists a unique $p_\delta \in S(M_\delta)$ extending each p_i and explicitly not forking over (M'_0, M_0) .

Proof. This is similar to the argument in [GV06a, Corollary 2.22], but we give some details. We focus on (1) (the proof of the other case is completely similar). Build by induction $(f_{i,j})_{i < j < \delta}$, $(a_i)_{i < \delta}$, and increasing continuous $(N_i)_{i < \delta}$ such that for all $i < j < \delta$:

- (1) $M_i \leq N_i$, $a_i \in N_i$.
- (2) $f_{i,j} : N_i \rightarrow N_j$.
- (3) For $j < k < \delta$, $f_{j,k} \circ f_{i,j} = f_{i,k}$.
- (4) $f_{i,j}$ fixes M_i .
- (5) $f_{i,j}(a_i) = a_j$.
- (6) $p_i = \text{tp}(a_i/M_i; N_i)$.

This is enough. Let $(N_\delta, (f_{i,\delta})_{i < \delta})$ be the direct limit of the system $(N_i, f_{i,j})_{i < j < \delta}$, and let $a_\delta := f_{0,\delta}(a_0)$, $p_\delta := \text{tp}(a_\delta/M_\delta; N_\delta)$. One easily checks that p_δ extends each p_i , $i < \delta$, and so using continuity for \triangleleft -increasing chains (Lemma 3.12.(2)), explicitly does not fork over (M'_0, M_0) . Finally, p_δ is unique by Lemma 5.3.

This is possible. For $i = 0$, we take a_0 and N_0 so that $\text{tp}(a_0/M_0; N_0) = p_0$. For i limit, we let $(N_i, f_{i_0,i})_{i_0 < i}$ be the direct limit of the system $(N_{i_0}, f_{i_0,j_0})_{i_0 < j_0 < i}$, and let $a_i := f_{0,i}(a_0)$. By continuity for \triangleleft -increasing chains, $\text{tp}(a_i/M_i; N_i)$ explicitly does not fork over (M'_0, M_0) , and so by uniqueness, it must equal p_i . For $i = i_0 + 1$ successor, find a_i and $N'_i \geq M_i$ such that $p_i = \text{tp}(a_i/M_i; N'_i)$. Since $p_i \upharpoonright M_{i_0} = p_{i_0}$, we can use the definition of types to amalgamate N_{i_0} and N'_i over M_{i_0} : there exists $N_i \geq N'_i$ and $f_{i_0,i} : N_{i_0} \xrightarrow{M_{i_0}} N_i$ so that $f_{i_0,i}(a_{i_0}) = a_i$. Define $f_{i'_0,i} := f_{i_0,i} \circ f_{i'_0,i_0}$ for all $i'_0 < i_0$. \square

Lemma 5.9 (Extension). Assume $\kappa = \aleph_0$. Let $M \leq N$ be in $K_{[\mu^+, \lambda]}$ with M and N μ^+ -model-homogeneous, and let $p \in S^{\text{ma}}(M)$. Then there is $q \in S(N)$ extending p that does not fork over M .

Proof. We imitate the proof of [Bonb, Theorem 5.3]. By existence and Lemma 4.5, there exists $M'_0 <_{\text{univ}} M_0 \leq M$ in K_μ with p explicitly nonforking over (M'_0, M_0) . Work by induction on λ . If $N \in K_{<\lambda}$, use the induction hypothesis, so assume $N \in K_\lambda$. There are two cases: either $\lambda = \mu^+$ or $\lambda > \mu^+$.

Assume first $\lambda > \mu^+$. By transitivity and Lemma 2.15, we can assume without loss of generality that $N = \bigcup_{i < \lambda} N_i$, where $(N_i)_{i \leq \lambda}$ is a $<_{\text{univ}}$ -increasing continuous chain in $K_{[\mu^+, \lambda]}$, each N_{i+1} is μ^+ -model-homogeneous, and N_0 extends M_0 . Now inductively build a \leq -increasing continuous $(M_i)_{i \leq \lambda}$ with $M_\lambda = M$ so that $M_0 \leq$

$M_i \leq N_i$ for all $i < \lambda$ (we allow repetitions). Set $p_i := p \upharpoonright M_i$ and note that by monotonicity, p_i explicitly does not fork over (M'_0, M_0) .

We inductively build an increasing $(q_i)_{i \leq \lambda}$ with $q_i \in S(N_i)$, $p_i \leq q_i$, and q_i explicitly does not fork over (M'_0, M_0) . For $i = 0$, use extension in $K_{<\lambda}$ to find q_0 as needed. For $i = j + 1$, use extension to find a nonforking extension $q_i \in S(N_i)$ of q_j that explicitly does not fork over (M'_0, M_0) . By uniqueness, $q_i \geq p_i$. At limits, use Lemma 5.8 and uniqueness. $q := q_\lambda$ is as desired.

If $\lambda = \mu^+$, the construction is exactly the same except we use extension in K_λ at successor steps and the first case of Lemma 5.8 at limit steps. Note that since N is μ^+ -model-homogeneous, $N = \bigcup_{i < \mu^+} N_i$, where $(N_i)_{i < \mu^+}$ is a \triangleleft -increasing continuous chain in K_μ . \square

Definition 5.10. Let $\mathfrak{s} := \mathfrak{s}_0 \upharpoonright \lambda$, where \mathfrak{s}_0 is the pre-frame from Proposition 4.4.

Corollary 5.11. Assume:

- (1) $\kappa = \aleph_0$.
- (2) K_μ has joint embedding.
- (3) K_λ has no maximal models.
- (4) All the models in K_λ are μ^+ -model-homogeneous.

Then \mathfrak{s} is a type-full good $^{-S}$ λ -frame.

Proof. It is easy to see \mathfrak{s} is a type-full pre- λ -frame. K_λ has amalgamation and no maximal models by hypothesis. It has joint embedding since K_μ has joint embedding and $K_{[\mu, \lambda]}$ has amalgamation (see Lemma 2.3). Stability holds by Theorem 5.6. Density of basic types is always true in a type-full frame. For the other properties, see Lemmas 4.6, 4.7, 4.8, 4.9, 5.3, and 5.9 (note that the original statement of extension in Definition 2.19 follows from Lemma 5.9 and transitivity). \square

Lemma 5.12. Assume K is categorical in λ and $\kappa = \aleph_0$. Then:

- (1) $K_{[\mu, \lambda]}$ has joint embedding and K_λ (and hence $K_{[\mu, \lambda]}$) has no maximal models.
- (2) If in addition $\text{cf}(\lambda) > \mu$, then all the models in K_λ are μ^+ -model-homogeneous.

Proof. To see (2), use stability to build $(M_i)_{i \leq \lambda}$ $<_{\text{univ}}$ -increasing continuous with $M_i \in K_{[\mu, \lambda]}$ for all $i < \lambda$. Then M_λ is μ^+ -model-homogeneous.

For (1), K_λ has joint embedding by categoricity so use that $K_{[\mu, \lambda]}$ has no maximal models. To see K_λ has no maximal model, let $N \in K_\lambda$ be given. First assume $\lambda = \mu^+$. Build a \triangleleft -increasing continuous chain $(M_i)_{i \leq \mu^+}$, and $a \in N$ such that for all $i < \mu^+$:

- (1) $M_i \in K_\mu$, $M_i \leq N$.

- (2) $a \notin M_0$.
- (3) $\text{tp}(a/M_i; N)$ does not fork over M_0 .

This is enough. $M_{\mu^+} \in K_{\lambda^+}$. Moreover by Lemma 3.9.(5), $a \notin M_i$ for all $i < \mu^+$, so $a \notin M_{\mu^+}$. Thus $M_{\mu^+} < N$. By categoricity, the result follows.

This is possible. Pick a \triangleleft -limit $M_0 \leq N_1$ in K_μ (this is possible by model-homogeneity of N_1), and pick any $a \in N \setminus M_0$. At limits, take unions and use continuity (Lemma 3.12.(2)) to see the requirements are maintained. For a successor $i = j + 1$, use extension and some renaming. In details, pick an arbitrary $M'_i \triangleright M_j$ with $M'_i \leq N$ (possible by model-homogeneity). By extension (Lemma 3.9.(2)), there is $q \in S(M'_i)$ that does not fork over M_0 and extends $p_j := \text{tp}(a/M_j; N)$. Since N is saturated, there is $a' \in N$ realizing q . Pick $N \geq N_i \geq M'_i$ containing a' and a . By assumption, $\text{tp}(a'/M_j; N_i) = p_j = \text{tp}(a/M_j; N_i)$. Thus there is $N'_i \geq N_i$ and $f : N_i \xrightarrow{M_j} N'_i$ such that $f(a') = a$ and without loss of generality $N'_i \leq N$. Let $M_i := f[M'_i]$ and use invariance to see it is as desired.

If $\lambda > \mu^+$, the proof is completely similar: if there is $N_1 > N$, we are done, so assume not. Then amalgamation implies N must be model-homogeneous. Build a $<_{\text{univ}}$ -increasing continuous $(M_i)_{i \leq \lambda}$ and $a \in N$ such that for all $i < \lambda$:

- (1) $M_i \in K_{[\mu^+, \lambda)}$, $M_i \leq N$.
- (2) M_{i+1} is μ^+ -model-homogeneous.
- (3) $\text{tp}(a/M_i; N)$ does not fork over M_0 .

As before, this is possible and the result follows. \square

Corollary 5.13. Assume K is categorical in λ , $\kappa = \aleph_0$, and $\text{cf}(\lambda) > \mu$, then \mathfrak{s} is a type-full good $^{-S}$ λ -frame.

Proof. Lemma 5.12 tells us all the hypotheses of Corollary 5.11 are satisfied. \square

Note that categoricity in λ is not the only hypothesis giving that all models in K_λ are μ^+ -model-homogeneous. For example:

Fact 5.14 (Theorem 5.4 in [BG]). Assume K has amalgamation, is categorical in a cardinal θ so that K_θ has a μ^+ -model-homogeneous model (this holds if e.g. $\theta^\mu = \theta$). Then every member of $K_{\geq \chi}$ is μ^+ -model-homogeneous, where $\chi := \min(\theta, \sup_{\gamma < \mu} \beth_{(2^\gamma)^+})$.

6. GETTING SYMMETRY

Hypothesis 6.1. $\mathfrak{s} = (K, \perp, S^{\text{bs}})$ is a good $^-$ \mathcal{F} -frame, where \mathcal{F} is an interval of cardinals of the form $[\lambda, \theta)$ for λ a cardinal and $\theta > \lambda$ either a cardinal or ∞ .

Note that we do *not* assume \mathfrak{s} satisfies bs-stability. It will hold in the setup of the previous sections, but the arguments of this section work just as well without it. Note in passing that bs-stability and stability are equivalent:

Fact 6.2 ([She09a], Claim II.4.2.1). For any $\lambda' \in \mathcal{F}$, $\mathfrak{s} \upharpoonright \lambda'$ satisfies bs-stability if and only if K is stable in λ' .

Moreover, eventual stability will hold:

Proposition 6.3.

- (1) If $2^\lambda \in \mathcal{F}$, then K is stable in 2^λ .
- (2) Assume $\chi_0 \in \mathcal{F}$ and K is stable in χ_0 . Then K is stable in every $\chi \geq \chi_0$ with $\chi \in \mathcal{F}$.

In particular, if χ is a cardinal with $2^\lambda \leq \chi < \theta$, then K is stable in χ .

Proof.

- (1) Let $\chi := 2^\lambda$. By Fact 6.2, it is enough to show that $\mathfrak{s} \upharpoonright \chi$ satisfies bs-stability. Let $M \in K_\chi$, and let $(p_i)_{i < \chi^+}$ be elements of $S^{\text{bs}}(M)$. Let $(M_i)_{i < \chi}$ be a resolution of M . For each $i < \chi^+$, local character implies there exists $j_i < \chi$ such that p_i does not fork over M_{j_i} . By the pigeonhole principle, we can assume without loss of generality that $j_i = j_0$ for all $i < \chi^+$. By Proposition 2.21 and transitivity, there exists $M' \leq M_{j_0}$ such that p_i does not fork over M' for all $i < \chi^+$. We know that $|S(M')| \leq 2^\lambda = \chi$, so by the pigeonhole principle again, we can assume that there is $q \in S(M')$ such that $p_i \upharpoonright M' = q$ for all $i < \chi^+$. By uniqueness, $p_i = p_{i'}$ for all $i, i' < \chi^+$, and the result follows.
- (2) By the proof of [Bonb, Theorem 4.1].

□

We would like to give conditions under which \mathfrak{s} has symmetry. A useful fact⁴ is that it is enough to look at $\mathfrak{s} \upharpoonright \lambda$:

Fact 6.4 (Theorem 7.3 in [BV]). \mathfrak{s} has symmetry if and only if $\mathfrak{s} \upharpoonright \lambda$ has symmetry.

Since we are not assuming anything about how \mathfrak{s} is defined, we will work by contradiction: We will show that if θ is big enough and symmetry fails, then we get the order property, a nonstructure property which implies unstability. This is how the symmetry property of forking was originally proven in the first-order context, see [She90, Theorem III.4.13]. The same approach was later used in a non-elementary setup in [She75, Theorem 5.1], and generalized in [BGKV, Theorem 5.14]. We will rely on the proof of the latter.

⁴This is not crucial to our argument, but enables us to obtain an explicit upper bound on the amount of tameness needed.

The definitions and fact below do not need Hypothesis 6.1.

Definition 6.5. Let α , χ and γ be cardinals. A model N has the (α, χ) -order property of length γ if there exists $M \leq N$ of size $\leq \chi$ (we also allow M to be empty) and $(\bar{a}_i)_{i < \gamma}$, $\bar{a}_i \in {}^\alpha N$ so that for any $i_0 < i_1 < \gamma$ and $j_0 < j_1 < \gamma$, $\text{tp}(\bar{a}_{i_0} \bar{a}_{i_1} / A; N) \neq \text{tp}(\bar{a}_{j_0} \bar{a}_{j_1} / A; N)$. If $\chi = 0$, we omit it.

K has the (α, χ) -order property of length γ if some $N \in K$ has it. K has the (α, χ) -order property if it has the (α, χ) -order property for all lengths (we sometimes also say K has the (α, χ) -order property of length ∞). K has the order property if it has the α -order property for some α .

Remark 6.6. If N has the (α, χ) -order property of length γ , then it has the $(\alpha + \chi)$ -order property of length γ .

Definition 6.7. Given a cardinal χ , define $h(\chi) := \beth_{(2^\chi)^+}$.

Fact 6.8.

- (1) If K has the (α, χ) -order property of length $h(\alpha + \chi + \text{LS}(K))$, then K has the (α, χ) -order property.
- (2) If K has the (α, χ) -order property, then it is α -unstable in χ' for all $\chi' \geq \chi$.

Proof. The statements essentially appear in [She99, Claim 4.5.3, Claim 4.7.2]. The proof of (1) is an application of Morley's method together with Shelah's presentation theorem, and a proof of a statement similar to (2) is sketched in [BGKV, Fact 5.13]. \square

Fact 6.9. If \mathfrak{s} does not have symmetry, then K has the $(2, \lambda)$ -order property of length θ .

Proof. By Fact 6.4, $\mathfrak{s} \upharpoonright \lambda$ does not have symmetry. The result now follows by exactly the same proof as [BGKV, Corollary 5.15]. \square

Corollary 6.10. If $\theta \geq h(\lambda)$, then \mathfrak{s} has symmetry.

Proof. If \mathfrak{s} does not have symmetry, then by Fact 6.9 and Fact 6.8.(1), K has the $(2, \lambda)$ -order property and hence by Fact 6.8.(2) is 2-unstable in 2^λ . By Theorem 2.7, K is unstable in 2^λ , contradicting Proposition 6.3 (note that $2^\lambda < h(\lambda) \leq \theta$). \square

Thus it seems quite a big gap between λ and θ is needed. On the other hand, [Bonb] tells us that with enough tameness we can make \mathcal{F} bigger:

Fact 6.11. Let $\theta' \geq \theta$ and let $\mathcal{F}' := [\lambda, \theta']$. Assume $K_{\mathcal{F}'}$ has amalgamation and no maximal models, and K is (λ, θ') -tame. Then \mathfrak{s} can be extended to a good $^-$ $[\lambda, \theta')$ -frame. If \mathfrak{s} has bs-stability, the extended frame will also have bs-stability. If \mathfrak{s} is type-full, then the extended frame will also be type-full.

Proof. By Remark 2.20, \mathfrak{s} is determined by $\mathfrak{s} \upharpoonright \lambda$, so we can apply [Bonb, Theorem 1.1]: its proof only uses the tameness for 2-types hypothesis to obtain symmetry. Note that no maximal models follows from symmetry there, so we need to assume it as an additional hypothesis. Note also that joint embedding follows from Lemma 2.3. The proof of Lemma 4.8 gives us that the extended frame is type-full if \mathfrak{s} is. \square

Remark 6.12. We could replace (λ, θ') -tameness by (λ', θ') -tameness in the above, where $\lambda' \in \mathcal{F}$. This turns out to be equivalent (at least if we consider tameness for basic types) since the uniqueness property of \mathfrak{s} gives us (λ, λ') -tameness for basic types.

Corollary 6.13. Let $\mathcal{F}' := [\lambda, h(\lambda))$. Assume $K_{\mathcal{F}'}$ has amalgamation and no maximal models, and K is $(\lambda, < h(\lambda))$ -tame. Then \mathfrak{s} has symmetry.

Proof. Using Fact 6.11, we can assume without loss of generality that $\mu \geq h(\lambda)$. Now use Corollary 6.10. \square

7. THE MAIN THEOREMS

We finally have our promised good frame:

Theorem 7.1. Assume:

- (1) K is an abstract elementary class. $\mu \geq \text{LS}(K)$ is a cardinal.
- (2) $K_\mu \neq \emptyset$ has joint embedding.
- (3) \triangleleft is an abstract universal ordering on K_μ . In particular (by Remark 2.14), K_μ has amalgamation, no maximal models, and is stable.
- (4) Nonsplitting has \aleph_0 -local character for \triangleleft -increasing chains in K_μ .
- (5) $\lambda > \mu$ is such that:
 - (a) K is $(\mu, < h(\lambda))$ -tame.
 - (b) $K_{[\mu, h(\lambda))}$ has amalgamation and no maximal models.
 - (c) All the models in K_λ are μ^+ -model-homogeneous.

Then K has a type-full good $[\lambda, h(\lambda))$ -frame.

Proof. Corollary 5.11, gives us a good^{-S} λ -frame \mathfrak{s} . \mathfrak{s} can be extended (by Fact 6.11) to a good $[\lambda, h(\lambda))$ frame which will also have symmetry by Corollary 6.10. \square

We can use categoricity to derive some of the hypotheses above. We will use:

Fact 7.2. Assume K has amalgamation, joint embedding and no maximal models. Assume K is categorical in λ . Then:

- (1) K is stable in all $\text{LS}(K) \leq \mu < \lambda$.

- (2) For any $\text{LS}(K) \leq \mu < \text{cf}(\lambda)$ and any limit $\delta < \mu^+$, μ -nonsplitting has \aleph_0 -local character for \triangleleft -chains, where $\triangleleft := <_{\mu, \delta}$.
- (3) Let $h_2 := h(h(\text{LS}(K)))$. Assume λ is a successor cardinal and $\lambda > \lambda_0 \geq h_2$. Then K is (h_2, λ_0) -tame and categorical in λ_0 . In addition, the model of size λ_0 is saturated.

Proof. (1) is [She99, Claim 1.7]. (2) is [She99, Lemma 6.3], and (3) were originally stated (with a lower Hanf number) in [She99, Main Claim II.2.3] and [She99, Theorem II.2.7]. A full proof (with discussion on whether it is possible to lower the h_2 bound) can be found in [Bal09, Section 14]. \square

Theorem 7.3. Let K be an abstract elementary class and let λ be a cardinal such that $\text{cf}(\lambda) > \mu \geq \text{LS}(K)$. Let $\mathcal{F} := [\lambda, h(\lambda))$, $\mathcal{F}' := [\mu, h(\lambda))$. Assume:

- (1) $K_{\mathcal{F}'}$ has amalgamation and no maximal models.
- (2) K_λ is categorical.
- (3) K is $(\mu, < h(\lambda))$ -tame.

Then K has a type-full good \mathcal{F} -frame.

Proof. First, $K_\lambda \neq \emptyset$ by categoricity. By Lemma 5.12, $K_{\mathcal{F}'}$ has joint embedding and all models in K_λ are μ^+ -model-homogeneous. By Fact 7.2, μ -nonsplitting has \aleph_0 -local character for \triangleleft -chains, where $\triangleleft := <_{\lambda, \omega}$. This shows all the hypotheses of Theorem 7.3 are satisfied. \square

Assuming categoricity in a big-enough successor, we obtain the tameness assumption:

Theorem 7.4. Let K be an abstract elementary class. Let $\mu := h_2 := h(h(\text{LS}(K)))$. Let $\lambda := \mu^+$. Assume K has amalgamation, joint embedding, and is categorical in some successor $\theta \geq h(\lambda)$.

Let $\mathcal{F} := [\lambda, \theta)$. Then there is a type-full good \mathcal{F} -frame with underlying AEC K .

Proof. Since $\theta \geq h(\text{LS}(K))$, K has arbitrarily large models and so using joint embedding K has no maximal models. By Fact 7.2, K is categorical in λ and K is $(\mu, < h(\lambda))$ -tame. Apply Theorem 7.3. \square

Notice that one also obtains that categoricity (at a cardinal of high-enough cofinality) and tameness implies stability everywhere. This improves on [BKV06, Corollary 4.7]:

Theorem 7.5. Let K be an abstract elementary class with amalgamation, joint embedding, and no maximal models. Assume K is categorical in some λ such that $\text{cf}(\lambda) > \mu \geq \text{LS}(K)$ and K is (μ, μ') -tame. Then K is stable in all $\theta \in [\text{LS}(K), \mu']$. In particular, if $\mu' = \infty$, then K is stable everywhere.

Proof. By Fact 7.2, μ -nonsplitting has \aleph_0 -local character for \triangleleft -chains, where $\triangleleft := \prec_{\mu, \omega}$ and K is stable everywhere below and at μ . Apply Theorem 5.6 to see K is stable everywhere in $(\mu, \mu']$. \square

This result is much more local than the other results of this section. For example, we do not need to assume that $\mu' \geq h(\mu)$. Moreover, as Theorem 5.6 shows, the categoricity hypothesis can be replaced by μ -nonsplitting having \aleph_0 -local character for \triangleleft -chains, for some abstract universal ordering \triangleleft on K_μ . In this case, the no maximal model hypothesis is not necessary.

Assuming the generalized continuum hypothesis (GCH), we obtain a more general stability spectrum theorem:

Theorem 7.6. Assume GCH. Let K be an abstract elementary class with amalgamation, joint embedding, and no maximal models. Assume K is μ -tame for $\mu \geq \text{LS}(K)$, \triangleleft is an abstract universal ordering on K_μ , and μ -nonsplitting has κ -local character for \triangleleft -increasing chains. Then K is stable in all $\lambda \geq \mu$ with $\lambda = \lambda^{<\kappa}$.

Proof. K is stable in μ since we have an abstract universal ordering on K_μ . If $\lambda > \mu$, the result follows from Theorem 5.7. \square

Remark 7.7. If K is the class of models of a complete first-order theory, the conditions for stability given by Corollary 7.6 are very close⁵ to optimal (see [She90, Corollary III.3.8]).

Remark 7.8. Let K be an abstract elementary class with amalgamation, joint embedding, and no maximal models. Assume K is χ -tame and stable in some $\mu \geq h(\chi)$. Then [GV06b, Theorem 4.13] shows that for some $\kappa < h(\chi)$, μ -nonsplitting has κ -local character. Thus we have:

Corollary 7.9. Assume GCH. Let K be an abstract elementary class with amalgamation, joint embedding, and no maximal models. Assume K is χ -tame and stable in some $\mu \geq \text{LS}(K)$. Then there is $\kappa < h(\chi)$ such that K is stable in all $\lambda \geq \mu$ with $\lambda^{<\kappa} = \lambda$.

Proof. If $\mu < h(\chi)$, then by [GV06b, Corollary 6.4] one can take $\kappa := \mu^+$, so assume $\mu \geq h(\chi)$. By the previous remark, there is $\kappa < h(\chi)$ such that μ -nonsplitting has κ -local character. The result now follows from Corollary 7.6. \square

Remark 7.10. In the upcoming [Vas], we will use different methods to prove Corollary 7.9 in ZFC. We do not know whether Corollary 7.6 also holds in ZFC (although it is clear from the proof that much less than GCH is needed).

⁵The least regular cardinal κ such that nonsplitting has κ -local character will be at most the successor of $\kappa(T)$.

We can also apply our good frame to the question of uniqueness of limit models:

Theorem 7.11 (Uniqueness of limit models). Assume the hypotheses of Theorem 7.3 hold. Then K has a unique limit model in any $\mu' \in \mathcal{F}$. In fact, if $M_0 \in K_{\mu'}$ and M_ℓ is (μ', δ_ℓ) -limit over M_0 for $\ell = 1, 2$ and δ_ℓ a limit ordinal, then $M_1 \cong_{M_0} M_2$.

In particular, if K has amalgamation, joint embedding, no maximal models, is categorical in λ and is μ -tame for some $\mu < \text{cf}(\lambda)$, then K has a unique limit model in any $\mu' \geq \lambda$.

Proof. By Theorem 7.3, K has a good \mathcal{F} -frame \mathfrak{s} . In particular, K is stable in μ' , so one can iterate Fact 2.10 to build a (μ', δ) -limit model for any desired $\delta < (\mu')^+$. To see uniqueness, apply [She09a, Lemma II.4.8] (see [Bonb, Theorem 9.2] for a detailed proof of that result). \square

We see this theorem as an encouraging approximation to generalizing the upward categoricity transfer result of [GV06a] (which assumes categoricity in a successor cardinal) to categoricity in a limit cardinal.

Remark 7.12. Uniqueness of limit models of cardinality μ was asserted to follow from categoricity in some $\lambda^+ > \mu$ already in [SV99]. However, the authors later admitted that an error found by VanDieren in 1999 was fatal. VanDieren [Van06, Van13] proves uniqueness with the additional assumption that unions of amalgamation bases are amalgamation bases (but does not use tameness). It is still open whether uniqueness of limit models follows from categoricity only.

Remark 7.13. A variation on Theorem 7.11 is [BG, Corollary 6.10], which uses stronger locality assumptions but manages to obtain uniqueness of limit models below the categoricity cardinal without any cofinality restriction.

8. CONCLUSION AND FURTHER WORK

Assuming amalgamation, joint embedding, no maximal models, and tameness, we have given superstability-like conditions under which an abstract elementary class has a type-full good frame \mathfrak{s} , i.e. a forking-like notion for 1-types. These arguments would work just as well to get a notion of independence for all n -types, with $n < \omega$. The proof of extension breaks down, however, for types of infinite length (difficulties in obtaining the extension property in the absence of compactness is one of the reasons⁶ it was assumed as an axiom in [BG]).

Shelah's approach around this in [She09a, Chapter II] is to show that if the frame is *weakly successful* (a uniqueness condition for certain kinds of amalgamations), then

⁶Another reason was Shelah's example (see [HL02, Section 4]) of an \aleph_0 -stable non-simple diagram, but we have shown that we do not get into trouble as long as we restrict the base of our types to be sufficiently saturated models.

it has a notion of nonforking for types of models. In [She09a, Chapter III], Shelah has several hundreds of pages of approximations on when weak successfulness can be transferred across cardinals, but even assuming $\mathfrak{s} \upharpoonright \lambda$ is weakly successful for every λ , it is not clear how we can get a good nonforking notion for models of different sizes. This is one direction further work could focus on.

Another (non-orthogonal) direction would be to find applications for such a forking notion. As mentioned in the previous section, we believe it could be useful in proving categoricity transfer theorems. Moreover, the frame built in Section 5 is only well-behaved for μ^+ -saturated models, and it would be interesting to know at what cardinals such models exist. This calls for tools to deal with unions of saturated models and we plan to explore this further in future work.

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