INFINITARY STABILITY THEORY

SEBASTIEN VASEY

Abstract. We introduce a new device in the study of abstract elementary classes (AECs): Galois Morleyization, which consists in expanding the models of the class with a relation for every Galois type of length less than a fixed cardinal κ. We show:

Theorem 0.1 (The semantic-syntactic correspondence)**.** An AEC K is fully $(< \kappa)$-tame and type short if and only if Galois types are syntactic in the Galois Morleyization.

This exhibits a correspondence between AECs and the syntactic framework of stability theory inside a model. We use the correspondence to make progress on the stability theory of tame and type short AECs. The main theorems are:

Theorem 0.2. Let K be a $(< \kappa)$-tame AEC with amalgamation, $\kappa \geq \text{LS}(K)$. The following are equivalent:
(1) K is stable in some $\lambda \geq \kappa$.
(2) K does not have the order property.
(3) There exists $\mu \leq \lambda_0 < \beth_{(2^\kappa)^+}$ such that K is stable in any $\lambda \geq \lambda_0$ with $\lambda = \lambda^{<\mu}$.

Theorem 0.3. Let K be a fully $(< \kappa)$-tame and type short AEC with amalgamation, $\kappa = \beth_\kappa > \text{LS}(K)$. If K is stable, then the class of κ-saturated models of K admits an independence notion ($(< \kappa)$-coheir) which, except perhaps for extension, has the properties of forking in a first-order stable theory.

Contents

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1. Introduction

Abstract elementary classes (AECs) are sometimes described as a purely semantic framework for model theory. It has been shown, however, that AECs are closely connected with more syntactic objects. See for example Shelah's presentation theorem [She87a, Lemma 1.8], or Kueker's [Kue08, Theorem 7.2] showing that an AEC with Löwenheim-Skolem number λ is closed under L_{∞,λ^+}-elementary equivalence.

Another framework for non-elementary model theory is stability theory inside a model (introduced in Rami Grossberg's 1981 master thesis and studied for example in [Gro91a, Gro91b] or [She87b, Chapter I], see [She09b, Chapter V.A] for a more recent version). There the methods are very syntactic but it is believed (see for example the remark on p. 116 of [Gro91a]) that they can help the resolution of more semantic questions, such as Shelah's categoricity conjecture for $L_{\omega_1,\omega}$.

In this paper, we establish a correspondence between these two frameworks. We show that results from stability theory inside a model directly translate to results about *tame* abstract elementary classes. Recall that an AEC is $(< \kappa)$-tame if its Galois (i.e. orbital) types are determined by their restrictions to domains of size less than κ. Tameness as a property of AEC was first isolated (from an argument in [She99]) by Grossberg and VanDieren [GV06b] and used to prove an upward categoricity transfer [GV06a, GV06c], which Boney [Bon14] used to prove Shelah's categoricity conjecture for successors from class-many strongly compact cardinals.

The basic idea of the translation is the observation (appearing for example in [Bon14, p. 15] or [Lie11, p. 206]) that in a $(< \kappa)$-tame abstract elementary class, Galois types over domains of size less than κ play a role analogous to first-order formulas. We make this observation precise by expanding the language of such an AEC with a relation symbol for every Galois type of size less than κ, and looking at $L_{\kappa,\kappa}$-formulas in the expanded language. We call this expansion the *Galois Morleyization*[1] of the AEC. Thinking of a type as the set of its small restrictions, we can then prove the *semantic-syntactic correspondence* (Theorem 3.16):

[1]We thank Rami Grossberg for suggesting the name.

Galois types in the AEC correspond to quantifier-free syntactic types in its Galois Morleyization.

The correspondence gives us a new method to prove results in tame abstract elementary classes:

(1) Prove a syntactic result in the Galois Morleyization of the AEC (e.g. using tools from stability theory inside a model).
(2) Translate to a semantic result in the AEC using the semantic-syntactic correspondence.
(3) Push the semantic result further using known (semantic) facts about AECs, maybe combined with more hypotheses on the AEC (e.g. amalgamation).

As an application, we prove (Theorem 4.13):

Theorem 1.1. Let K be a $(< \kappa)$-tame AEC with amalgamation, $\kappa \geq \mathrm{LS}(K)$. The following are equivalent:

(1) K is stable in *some* cardinal greater than or equal to κ.
(2) K does not have the order property.
(3) There exists $\mu \leq \lambda_0 < \beth_{(2^\kappa)^+}$ such that K is stable in any $\lambda \geq \lambda_0$ with $\lambda^{<\mu} = \lambda$.

This gives the equivalence between no order property and stability in tame AECs and generalizes one direction of the stability spectrum theorem of homogeneous model theory ([She70, Theorem 4.4], see also [GL02, Corollary 3.11]). The syntactic part of the proof is not new (it is a straightforward generalization of Shelah's first-order proof [She90, Theorem 2.10]) and we are told by Rami Grossberg that proving such results was one of the reason tameness was introduced (in fact theorems with the same spirit appear in [GV06b]). However we believe it is challenging to give a transparent proof of the result using Galois types only. The reason is that the classical proof uses local types and it is not clear how to naturally define them semantically.

Our method has other applications: Theorem 5.13 (formalizing Theorem 0.3 from the abstract) shows that in stable fully tame and short AECs, the coheir independence relation has some of the properties of a well-behaved independence notion. This is used in [Vas] to build a global independence notion from superstability. In [BV], we also use syntactic methods to investigate chains of Galois-saturated models.

Precursors to this work include Makkai and Shelah's study of classes of models of an $L_{\kappa,\omega}$ sentence for κ a strongly compact cardinal [MS90]: there they prove [MS90, Proposition 2.10] that Galois and syntactic

$\Sigma_1(L_{\kappa,\kappa})$-types are really the same (so in particular those classes are $(<\kappa)$-tame). One can see our result as a generalization to tame AECs. Also, the construction of the Galois Morleyization when $\kappa = \aleph_0$ (so the language remains finitary) appears in [Kan, Section 2.4]. Moreover it has been pointed out to us[2] that a device similar to Galois Morleyization is used in [Ros81, Section 3] to present any concrete category as a class of models of an infinitary theory. However the use of Galois Morleyization to translate results of stability theory inside a model to AECs is new.

This paper is organized as follows. In section 2, we review some preliminaries. In section 3, we introduce *abstract Morleyizations* of AECs and the main example: Galois Morleyizations. We then prove the semantic-syntactic correspondence. In section 4, we investigate various order properties and prove Theorem 1.1. In section 5, we study the coheir independence relation.

We end with a note on how AECs compare to some other non first-order framework like homogeneous model theory (see [She70]). There is an example (due to Marcus, see [Mar72]) of an $L_{\omega_1,\omega}$-axiomatizable class which is categorical in all uncountable cardinals but does not have an $\aleph_1$-sequentially-homogeneous model. For $n < \omega$, an example due to Hart and Shelah (see [HS90, BK09]) has amalgamation, no maximal models, and is categorical in all $\aleph_k$ with $k \leq n$, but no higher. By [GV06c], the example cannot be $\aleph_k$-tame for $k < n$. However if κ is a strongly compact cardinal, the example will be fully $(<\kappa)$-tame and type short by the main result of [Bon14]. The discussion on p. 74 of [Bal09] gives more non-homogeneous examples.

In general, classes from homogeneous model theory or quasiminimal pregeometry classes (see [Kir10]) are special cases of AECs that are always fully $(<\aleph_0)$-tame and type short, while finitary AECs (see [HK06]) are in our opinion also very close to being fully $(<\aleph_0)$-tame and type short (see for example [HK06, Theorem 4.11]). In this paper we work with the much more general assumption of $(<\kappa)$-tameness and type shortness for a possibly uncountable κ.

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[2]By Jonathan Kirby.

thinking of Galois types as formulas, and Jonathan Kirby for valuable comments.

2. Preliminaries

We review some of the basics of abstract elementary classes and fix some notation. The reader is advised to skim through this section quickly and go back to it as needed.

2.1. Set theoretic terminology.

We will often use the following function:

Definition 2.1 (Hanf function)**.** For λ an infinite cardinal, define $h(\lambda) := \beth_{(2^\lambda)^+}$.

Note that for λ infinite, $\lambda = \beth_\lambda$ if and only if for all $\mu < \lambda$, $h(\mu) < \lambda$.

Definition 2.2. For κ an infinite cardinal, let κ_r be the least regular cardinal $\geq \kappa$. That is, κ_r is κ^+ if κ is singular and κ otherwise.

2.2. Syntax.

Our notation is standard, but since we will work with infinitary objects and need to be quite precise, we review the basics. We will often work with the logic $L_{\kappa,\kappa}$, see [Dic75] for the definition and basic results.

Definition 2.3. An *infinitary language* is a language where we also allow relation and function symbols of infinite arity. For simplicity, we require the arity to be an ordinal. An infinitary language is $(<\kappa)$*-ary* if all its symbols have arity strictly less than κ. A *finitary language* is a $(<\aleph_0)$-ary language.

For L an infinitary language, ϕ an $L_{\kappa,\kappa}$-formula and $\bar{x}$ a sequence of variables, we write $\phi = \phi(\bar{x})$ to emphasize that the free variables of ϕ appear among $\bar{x}$ (recall that a $L_{\kappa,\kappa}$-formula must have fewer than κ-many free variables, but we allow $\ell(\bar{x}) \geq \kappa$). We use a similar notation for sets of formulas. When $\bar{a}$ is an element in some L-structure and $\phi(\bar{x}, \bar{y})$ is a formula, we often abuse notation and say that $\psi(\bar{x}) = \phi(\bar{x}, \bar{a})$ is a formula (again, we allow $\ell(\bar{a}) \geq \kappa$). We say $\phi(\bar{x}, \bar{a})$ is a *formula over A* if $\bar{a} \in {}^{<\infty}A$.

Definition 2.4. For ϕ a formula over a set, let $\mathrm{FV}(\phi)$ denote the set of free variables of ϕ (that is, the smallest sequence $\bar{x}$ such that $\phi = \phi(\bar{x})$), $\ell(\phi) := \ell(\mathrm{FV}(\phi))$, and $\mathrm{dom}(\phi)$ be the smallest set A such that ϕ is over A. Define similarly the meaning of $\mathrm{FV}(p)$, $\ell(p)$, and $\mathrm{dom}(p)$ on a set p of formulas.

Definition 2.5. For L an infinitary language, M an L-structure, $A \subseteq |M|$, $\bar{b} \in {}^{<\infty}|M|$, and Δ a set of L-formulas (in some logic), let[3]

$$\mathrm{tp}_{\Delta}(\bar{b}/A; M) := \{\phi(\bar{x}; \bar{a}) \mid \phi(\bar{x}, \bar{y}) \in \Delta \text{ and } M \models \phi[\bar{b}, \bar{a}]\}$$

We will most often work with $\Delta = qL_{\kappa,\kappa}$, the set of *quantifier-free* $L_{\kappa,\kappa}$-formulas. In fact, we will more often than not work with quantifier-free formulas and so we may forget to say a formula is quantifier-free if it is clear from context. When κ and L are also clear from context, we write $\mathrm{tp}(\bar{b}/A; M)$ for $\mathrm{tp}_{qL_{\kappa,\kappa}}(\bar{b}/A; M)$.

Definition 2.6. For M an L-structure, Δ a set of L-formulas, $A \subseteq |M|$, α an ordinal or ∞, let

$$\mathrm{S}_{\Delta}^{<\alpha}(A; M) := \{\mathrm{tp}_{\Delta}(\bar{b}/A; M) \mid \bar{b} \in {}^{<\alpha}|M|\}$$

Define similarly the variations for $\leq \alpha$, α, etc. When we just write S, we mean $\mathrm{S}^1_{qL_{\kappa,\kappa}}$, and similarly if we write only one of the two parameters.

2.3. **Abstract classes.** We review the definition of an abstract elementary class. Abstract elementary classes (AECs) were introduced by Shelah in [She87a]. The reader unfamiliar with AECs can consult [Gro02] for an introduction.

We first define more general objects that we will sometimes use. Abstract classes are already defined in [Gro], while μ-abstract elementary classes should be introduced in [BGV]. We will mostly use them to deal with Morleyizations and classes of saturated models of an AEC.

Definition 2.7. An *abstract class* (AC for short) is a pair $(K, \leq)$, where:

(1) K is a class of L-structure, for some fixed infinitary language L (that we will denote by $L(K)$). We say $(K, \leq)$ is $(< \mu)$-*ary* if L is $(< \mu)$-ary.
(2) $\leq$ is a partial order (that is, a reflexive and transitive relation) on K.
(3) If $M \leq N$ are in K and $f : N \cong N'$, then $f[M] \leq N'$ and both are in K.
(4) If $M \leq N$, then $M \subseteq N$.

[3] Of course, we have in mind a canonical sequence of variables $\bar{x}$ of order types $\ell(\bar{b})$ that should really be part of the notation but (as is customary) we always omit this detail.

Remark 2.8. We do not always strictly distinguish between K and $(K, \leq)$.

Notation 2.9. For K an abstract class, $M, N \in K$, we write $M < N$ when $M \leq N$ and $M \neq N$.

Definition 2.10. Let K be an abstract class. A sequence $\langle M_i : i < \delta \rangle$ of elements of K is *increasing* if for all $i < j < \delta$, $M_i \leq M_j$. *Strictly increasing* means $M_i < M_j$ for $i < j$. $\langle M_i : i < \delta \rangle$ is *continuous* if for all limit $i < \delta$, $M_i = \bigcup_{j<i} M_j$.

Notation 2.11. For K an abstract class, we use notations such as K_λ, $K_{\geq \lambda}$, $K_{<\lambda}$ for the models in K of size λ, $\geq \lambda$, $< \lambda$, respectively.

Definition 2.12. Let $(I, \leq)$ be a partially-ordered set.

(1) We say that I is μ-*directed* provided for every $J \subseteq I$ if $|J| < \mu$ then there exists $r \in I$ such that $r \geq s$ for all $s \in J$ (thus $\aleph_0$-directed is the usual notion of directed set)
(2) Let $(K, \leq)$ be an abstract class. An indexed system $\langle M_i : i \in I \rangle$ of models in K is μ-*directed* if I is a μ-directed set and $i < j$ implies $M_i \leq M_j$.

Definition 2.13. Let μ be an infinite cardinal and let $(K, \leq)$ be a $(< \mu)$-ary abstract class. We say that $(K, \leq)$ is a μ-*abstract elementary class* (μ-AEC for short) if:

(1) Coherence: If $M_0, M_1, M_2 \in K$ satisfy $M_0 \leq M_2$, $M_1 \leq M_2$, and $M_0 \subseteq M_1$, then $M_0 \leq M_1$;
(2) Tarski-Vaught axioms: Suppose $\langle M_i \in K : i \in I \rangle$ is a μ-directed system. Then:
  (a) $\bigcup_{i \in I} M_i \in K$ and, for all $j \in I$, we have $M_j \leq \bigcup_{i \in I} M_i$.
  (b) If there is some $N \in K$ so that for all $i \in I$ we have $M_i \leq N$, then we also have $\bigcup_{i \in I} M_i \leq N$.
(3) Löwenheim-Skolem-Tarski axiom: There exists a cardinal $\lambda = \lambda^{<\mu} \geq |L(K)| + \mu$ such that for any $M \in K$ and $A \subseteq |M|$, there is some $M_0 \leq M$ such that $A \subseteq |M_0|$ and $\|M_0\| \leq |A|^{<\mu} + \lambda$. We write $\mathrm{LS}(K)$ for the minimal such cardinal[4].

When $\mu = \aleph_0$, we omit it and simply call K an *abstract elementary class* (AEC for short).

In any abstract class, we can define a notion of embedding:

[4]Pedantically, $\mathrm{LS}(K)$ really depends on μ but μ will always be clear from context.

Definition 2.14. Let K be an AC. We say a function $f : M \to N$ is a *K-embedding* if $M, N \in K$ and $f : M \cong f[M] \leq N$. For $A \subseteq |M|$, we write $f : M \underset{A}{\to} N$ to mean that f fixes A pointwise. Unless otherwise stated, when we write $f : M \to N$ we mean that f is an embedding.

Here are three of the most important property an AC can have:

Definition 2.15. Let K be an AC.

(1) K has *amalgamation* if for any $M_0 \leq M_\ell$ in K, $\ell = 1, 2$, there exists $N \in K$ and $f_\ell : M_\ell \underset{M_0}{\longrightarrow} N$.
(2) K has *joint embedding* if for any M_ℓ in K, $\ell = 1, 2$, there exists $N \in K$ and $f_\ell : M_\ell \to N$.
(3) K has *no maximal models* if for any $M \in K$ there exists $N \in K$ with $M < N$.

2.4. **Galois types.** Let K be an abstract class. We define here a semantic notion of types for K. This was first introduced in [She87b, Definition II.1.9]. While Galois types are usually only defined over models, here we define them over sets. This is not harder and is often notationally convenient[5]. Note however that Galois types over sets are in general not too well-behaved. For example, they can sometimes fail to have an extension if their domain is not an amalgamation base.

Definition 2.16.

(1) Let K^3 be the set of triples of the form $(\bar{b}, A, N)$, where $N \in K$, $A \subseteq |N|$, and $\bar{b}$ is a sequence of elements from N.
(2) For $(\bar{b}_1, A_1, N_1), (\bar{b}_2, A_2, N_2) \in K^3$, we say $(\bar{b}_1, A_1, N_1) E_{\text{at}} (\bar{b}_2, A_2, N_2)$ if $A := A_1 = A_2$, and there exists $f_\ell : N_\ell \underset{A}{\to} N$ such that $f_1(\bar{b}_1) = f_2(\bar{b}_2)$.
(3) Note that E_{at} is a symmetric and reflexive relation on K^3. We let E be the transitive closure of E_{at}.
(4) For $(\bar{b}, A, N) \in K^3$, let $\text{gtp}(\bar{b}/A; N) := [(\bar{b}, A, N)]_E$. We call such an equivalence class a *Galois type*.
(5) For $p = \text{gtp}(\bar{b}/A; N)$ a Galois type, define[6] $\ell(p) := \ell(\bar{b})$ and $\text{dom}(p) := A$.

[5]For example, types over the empty sets are used here in the definition of the Galois Morleyization. They appear implicitly in the definition of the order property in [She99, Definition 4.3] and explicitly in [GV06b, Notation 1.9].

[6]It is easy to check that this does not depend on the choice of representatives.

(6) We say a Galois types $p = \mathrm{gtp}(\bar{b}/A; N)$ is *algebraic* if $\bar{b} \in {}^{\ell(\bar{b})}A$ (it is easy to check this does not depend on the choice of representatives). We mostly use this when $\ell(p) = 1$.

We can go on to define the restriction of a type (if $A_0 \subseteq \mathrm{dom}(p)$, $I \subseteq \ell(p)$, we will write $p^I \restriction A_0$ when the realizing sequence is restricted to I and the domain is restricted to A_0), the image of a type under an isomorphism, or what it means for a type to be realized. Just as in [She09a, Observation II.1.11.4], we have:

Fact 2.17. If K has amalgamation, then $E = E_{\mathrm{at}}$.

Note that the proof goes through, even though we only have amalgamation over models, not over all sets.

Definition 2.18.

(1) Let $N \in K$, $A \subseteq |N|$, and α be an ordinal. Define:

$$\mathrm{gS}^\alpha(A; N) := \{\mathrm{gtp}(\bar{b}/A; N) \mid \bar{b} \in {}^\alpha|N|\}$$

(2) For $M \in K$ and α an ordinal, let:

$$\mathrm{gS}^\alpha(M) := \bigcup_{N \geq M} \mathrm{gS}^\alpha(M; N)$$

(3) For α an ordinal, let:

$$\mathrm{gS}^\alpha(\emptyset) := \bigcup_{N \in K} \mathrm{gS}^\alpha(\emptyset; N)$$

When $\alpha = 1$, we omit it. Similarly define $\mathrm{gS}^{<\alpha}$, where α is allowed to be ∞.

Next, we recall the definition of tameness, a locality property of types. Tameness was introduced by Grossberg and VanDieren in [GV06b] and used to get an upward stability transfer (and an upward categoricity transfer in [GV06c]). Later on, Boney showed in [Bon14] that it followed from large cardinals and also introduced a dual property he called *type shortness*.

Definition 2.19 (Definitions 3.1 and 3.3 in [Bon14])**.** Let K be an abstract class and let Γ be a class (possibly proper) of Galois types in K. Let κ be an infinite cardinal.

(1) K is $(<\kappa)$-*tame for* Γ if for any $p \neq q$ in Γ, if $A := \mathrm{dom}(p) = \mathrm{dom}(q)$, then there exists $A_0 \subseteq A$ such that $|A_0| < \kappa$ and $p \restriction A_0 \neq q \restriction A_0$.

(2) K is $(<\kappa)$-*type short for* Γ if for any $p \neq q$ in Γ, if $\alpha := \ell(p) = \ell(q)$, then there exists $I \subseteq \alpha$ such that $|I| < \kappa$ and $p^I \neq q^I$.
(3) κ-tame means $(<\kappa^+)$-tame, similarly for type short.
(4) We usually just say "short" instead of "type short".
(5) Usually, Γ will be a class of types over models only, and we often specify it in words. For example, $(<\kappa)$-*short for types of length* α means $(<\kappa)$-short for $\bigcup_{M\in K} \mathrm{gS}^\alpha(M)$.
(6) We say K is $(<\kappa)$-tame if it is $(<\kappa)$-tame for types of length one.
(7) We say K is *fully* $(<\kappa)$-tame if it is $(<\kappa)$-tame for $\bigcup_{M\in K} \mathrm{gS}^{<\infty}(M)$, similarly for short.

We discuss the natural notion of stability in this context. Our definition is slightly unusual: we define what it means for a *model* to be stable in a given cardinal, and get a local notion of stability that is equivalent (in AECs) to the usual notion if amalgamation holds, but behaves better if amalgamation fails. Note that we count the number of types over an arbitrary set, not (as is common in AECs) only over models. In case the abstract class has a Löwenheim-Skolem number and we work above it this is equivalent, as any type in $\mathrm{gS}^{<\alpha}(A; N)$ can be extended[7] to $\mathrm{gS}^{<\alpha}(B; N)$ when $A \subseteq B$, so $|\mathrm{gS}^{<\alpha}(A; N)| \leq |\mathrm{gS}^{<\alpha}(B; N)|$.

Definition 2.20 (Stability)**.** Let K be an abstract class. Let α be a cardinal, μ be a cardinal. A model $N \in K$ is $(<\alpha)$-*stable in* μ if for all $A \subseteq |N|$ of size $\leq \mu$, $|\mathrm{gS}^{<\alpha}(A; N)| \leq \mu$. Here and below, α-stable means $(<(\alpha^+))$-stable. We say "stable" instead of "1-stable".

K is $(<\alpha)$-*stable in* μ if every $N \in K$ is $(<\alpha)$-stable in μ. K is $(<\alpha)$-*stable* if it is $(<\alpha)$-stable in unboundedly many cardinals.

Define similarly *syntactically stable* for syntactic types (in this paper, the quantifier-free $L_{\kappa,\kappa}$-types where κ is clear from context).

The next fact spells out the connection between stability for different lengths and tameness.

Fact 2.21. Let K be an AEC and let $\mu \geq \mathrm{LS}(K)$.

(1) [Bon, Theorem 3.1]: If K is stable in μ, K_μ has amalgamation, and $\mu^\alpha = \mu$, then K is α-stable in μ.

[7]Note that this does not use any amalgamation because we work inside the same model N.

(2) [GV06b, Corollary 6.4][8]: If K has amalgamation, is μ-tame, and stable in μ, then K is stable in all λ such that $\lambda^\mu = \lambda$.
(3) If K has amalgamation, is μ-tame, and is stable in μ, then K is α-stable (in unboundedly many cardinals), for all cardinals α.

Proof of (3). Given cardinals $\lambda_0 \geq \mathrm{LS}(K)$ and α, let $\lambda := (\lambda_0)^{\alpha+\mu}$. Combining the first two statements gives us that K is α-stable in λ. $\square$

Finally, we define a notion of saturation using Galois types. Note that we again define the local notions (but our definitions are equivalent to the usual ones assuming amalgamation).

Definition 2.22. Let K be an abstract class, $M \in K$ and μ be an infinite cardinal.

(1) For $N \geq M$, M is *μ-saturated*[9] *in N* if for any $A \subseteq |M|$ of size less than μ, any $p \in \mathrm{gS}^{<\mu}(A; N)$ is realized in M.
(2) M is *μ-saturated* if it is μ-saturated in N for all $N \geq M$. When $\mu = \|M\|$, we omit it.
(3) We write $K^{\mu\text{-sat}}$ for the class of μ-saturated models of $K_{\geq\mu}$ (ordered by the ordering of K).

Remark 2.23.

(1) We defined saturation also when $\mu \leq \mathrm{LS}(K)$. This is why we look at types over sets and not only over models. In an AEC, when $\mu > \mathrm{LS}(K)$, this is equivalent to the usual definition (see also our remark in the definition of stability).
(2) We could similarly define what it means for a *set* to be saturated in a model (this is useful in [BV]).
(3) It is easy to check that if K is an AEC with amalgamation and $\mu > \mathrm{LS}(K)$, then $K^{\mu\text{-sat}}$ is a μ-AEC with $\mathrm{LS}(K^{\mu\text{-sat}}) = \mathrm{LS}(K)^{<\mu}$.

3. The semantic-syntactic correspondence

3.1. Abstract and Galois Morleyization.

Definition 3.1. Let K be an AC with language $L := L(K)$, and let $\widehat{L}$ be an infinitary language extending L. An *abstract $\widehat{L}$-Morleyization* of K is a class $\widehat{K}$ of $\widehat{L}$-structures satisfying the following properties:

[8] The result we want can easily be seen to follow from the proof there: see [Bal09, Theorem 12.10].

[9] Pedantically, we should really say "Galois-saturated" to differentiate this from being syntactically saturated. In this paper, we will only discuss Galois saturation.

(1) The map $\widehat{M} \mapsto \widehat{M} \upharpoonright L$ is a bijection from $\widehat{K}$ onto K. For $M \in K$, we will write $\widehat{M}$ for the unique element of $\widehat{K}$ whose reduct is M. When we write "$\widehat{M} \in \widehat{K}$", it is understood that $M = \widehat{M} \upharpoonright L(K)$.
(2) Invariance: For $M, N \in K$, if $f : M \cong N$, then $f : \widehat{M} \cong \widehat{N}$.
(3) Monotonicity: If $M \leq N$ are in K, then $\widehat{M} \subseteq \widehat{N}$.

We say an abstract Morleyization $\widehat{K}$ is $(< \kappa)$-*ary* if $\widehat{L}$ is $(< \kappa)$-ary. Usually, we drop the "abstract" and just talk about a Morleyization.

Example 3.2.

(1) For K an abstract class, K is an L-Morleyization of K itself. This is because $\leq$ must extend $\subseteq$.
(2) Let T be a complete first-order theory in a language L. Let $K := (\mathrm{Mod}(T), \preceq)$. It is common to expand L to $\widehat{L}$ by adding a relation symbol for every first-order L-formula. We then expand T (to $\widehat{T}$) and every model M of T in the expected way (to some $\widehat{M}$) and obtain a new theory in which every formula is equivalent to an atomic one (this is commonly called the *Morleyization* of the theory and is the reason for our choice of terminology). Then $\widehat{K} := \mathrm{Mod}(\widehat{T})$ is a $\widehat{L}$-Morleyization of K.
(3) Let K be an abstract class with $L := L(K)$ and let κ be an infinite cardinal. Add a κ-ary predicate P to L, forming a language $\widehat{L}$. Expand each $M \in K$ to a $\widehat{L}$-structure by defining $P^{\widehat{M}}(\bar{a})$ to hold if and only if $\bar{a}$ is the universe of a $\leq$-submodel of M (this is more or less what Shelah does in [She09a, Definition IV.1.9.1]). Then $\widehat{K}$ is an $\widehat{L}$-Morleyization of K.

Our main example of an abstract Morleyization is the *Galois Morleyization*:

Definition 3.3. Let K be an abstract class and let κ be an infinite cardinal. Define an expansion $\widehat{L}$ of $L(K)$ by adding a relation symbol R_p of arity $\ell(p)$ for each $p \in \mathrm{gS}^{<\kappa}(\emptyset)$. Expand each $N \in K$ to a $\widehat{L}$-structure $\widehat{N}$ by specifying that for each $\bar{a} \in \widehat{N}$, $R_p^{\widehat{N}}(\bar{a})$ holds exactly when $\mathrm{gtp}(\bar{a}/\emptyset; N) = p$. It is straightforward to check that $\widehat{K}$ is a $(< \kappa)$-ary $\widehat{L}$-Morleyization of K. We write $\widehat{K}^{<\kappa}$ and $\widehat{L}^{<\kappa}$ for $\widehat{K}$ and $\widehat{L}$ respectively. We call $\widehat{K}^{<\kappa}$ the $(< \kappa)$-*Galois Morleyization* of K.

Remark 3.4. Let K be an AEC and κ be an infinite cardinal. Then $|L(\widehat{K}^{<\kappa})| \leq |\mathrm{gS}^{<\kappa}(\emptyset)| + |L| \leq 2^{<(\kappa + \mathrm{LS}(K)^+)}$.

Note that a Morleyization can naturally be made into an abstract class:

Definition 3.5. Let $(K, \leq)$ be an abstract class and let $\widehat{K}$ be a Morleyization of K. Define an ordering $\widehat{\leq}$ on $\widehat{K}$ by $\widehat{M} \widehat{\leq} \widehat{N}$ if and only if $M \leq N$.

Remark 3.6. For simplicity, we will abuse notation and write $(\widehat{K}, \leq)$ rather than $(\widehat{K}, \widehat{\leq})$. As usual, when the ordering is clear from context we omit it.

The next propositions are easy but conceptually quite interesting[10].

Proposition 3.7. Let $(K, \leq)$ be an abstract class with $L := L(K)$. Let $\widehat{K}$ be an $\widehat{L}$-Morleyization of K.

(1) $(\widehat{K}, \leq)$ is an abstract class.
(2) If every chain in K has an upper bound, then every chain in $\widehat{K}$ has an upper bound.
(3) Galois types are the same in K and $\widehat{K}$: $\operatorname{gtp}(\bar{a}_1/A; N_1) = \operatorname{gtp}(\bar{a}_2/A; N_2)$ if and only if $\operatorname{gtp}(\bar{a}_1/A; \widehat{N}_1) = \operatorname{gtp}(\bar{a}_2/A; \widehat{N}_2)$.
(4) Assume K is a μ-AEC and $\widehat{K}$ is a $(< \mu)$-ary Morleyization of K. Then $(\widehat{K}, \leq)$ is a μ-AEC with $\operatorname{LS}_{<\mu}(\widehat{K}) = \operatorname{LS}_{<\mu}(K) + |\widehat{L}|^{<\mu}$.
(5) Let $L \subseteq \widehat{L}' \subseteq \widehat{L}$. Then $\widehat{K} \upharpoonright \widehat{L}' := \{\widehat{M} \upharpoonright \widehat{L}' \mid \widehat{M} \in \widehat{K}\}$ is an $\widehat{L}'$-Morleyization of K.
(6) If $\widehat{\widehat{K}}$ is a Morleyization[11] of $\widehat{K}$, then $\widehat{\widehat{K}}$ is a Morleyization of K.

Proof. All are straightforward. As an example, we show that if K is a μ-AEC, $\widehat{K}$ is a $(< \mu)$-ary Morleyization, and $\langle \widehat{M}_i : i \in I \rangle$ is a μ-directed system in $\widehat{K}$, then letting $M := \bigcup_{i \in I} M_i$, we have that $\bigcup_{i \in I} \widehat{M}_i = \widehat{M}$ (so in particular $\bigcup_{i \in I} \widehat{M}_i \in \widehat{K}$). Let R be a relation symbol in $\widehat{L}$ of arity α. Let $\bar{a} \in {}^{\alpha}|\widehat{M}|$. Assume $\widehat{M} \models R[\bar{a}]$. We show $\bigcup_{i \in I} \widehat{M}_i \models R[\bar{a}]$. The converse is done by replacing R by $\neg R$, and the proof with function symbols is similar. Since $\widehat{L}$ is $(< \mu)$-ary, $\alpha < \mu$. Since I is μ-directed, $\bar{a} \in {}^{\alpha}|M_j|$ for some $j \in I$. Since $M_j \leq M$, the monotonicity axiom implies $\widehat{M}_j \subseteq \widehat{M}$. Thus $\widehat{M}_j \models R[\bar{a}]$, and this holds for all $j' \geq j$. Thus by definition of the union, $\bigcup_{i \in I} \widehat{M}_i \models R[\bar{a}]$. □

[10] We believe it would be worthwhile to isolate the "category-theoretic essence" of Morleyizations (of course this is quite vague as stated) so that we can have a better understanding of the big picture.

[11] Where of course we think of $\widehat{K}$ as an abstract class with the ordering induced from K.

Remark 3.8. A word of warning: if K is an AEC and $\widehat{K}$ is an $\widehat{L}$-abstract Morleyization of K, then K and $\widehat{K}$ are isomorphic as categories. In particular, any directed system in $\widehat{K}$ has a colimit. However, if $L(\widehat{K})$ is not finitary the colimit of a directed system in $\widehat{K}$ may *not* be the union: relations may need to contain more elements.

3.2. **Formulas and syntactic types.** From now on until the end of the section, we assume:

Hypothesis 3.9. K is an abstract class with $L := L(K)$, κ is an infinite cardinal, $\widehat{K}$ is a $(< \kappa)$-ary $\widehat{L}$-Morleyization of K.

We will adopt the conventions of Section 2.2. In particular when we talk about a formula we mean a (usually quantifier-free) $\widehat{L}_{\kappa,\kappa}$-formula. For $N \in K$, we write $\mathrm{tp}(\bar{a}/A; \widehat{N})$ for $\mathrm{tp}_{q\widehat{L}_{\kappa,\kappa}}(\bar{a}/A; \widehat{N})$, and $\mathrm{S}^\alpha(A; \widehat{N})$ for $\mathrm{S}^\alpha_{q\widehat{L}_{\kappa,\kappa}}(A; \widehat{N})$ (and similarly for other variations). We may write $N \models \phi[\bar{a}]$ for $\widehat{N} \models \phi[\bar{a}]$.

Remark 3.10. When κ is clear from context, we sometimes say that a set is *small* if it has cardinality strictly less than κ, or that a type is *small* if its domain and length are small.

Proposition 3.11. Let $\phi(\bar{x})$ be a quantifier-free formula, $M \in K$, and $\bar{a} \in M$. If $f : M \to N$, then $\widehat{M} \models \phi[\bar{a}]$ if and only if $\widehat{N} \models \phi[f(\bar{a})]$.

Proof. Directly from the invariance and monotonicity properties of Morleyizations. $\square$

In general, Galois and syntactic types (even in the Morleyization) are different. However, Galois types are always finer than syntactic types in the Morleyization:

Lemma 3.12. Let $N_1, N_2 \in K$, $A \subseteq |N_\ell|$ for $\ell = 1, 2$. Let $\bar{b}_\ell \in N_\ell$. If $\mathrm{gtp}(\bar{b}_1/A; N_1) = \mathrm{gtp}(\bar{b}_2/A; N_2)$, then $\mathrm{tp}(\bar{b}_1/A; \widehat{N}_1) = \mathrm{tp}(\bar{b}_2/A; \widehat{N}_2)$.

Proof. By transitivity of equality, it is enough to show that if $(\bar{b}_1, A, N_1) E_{\mathrm{at}} (\bar{b}_2, A, N_2)$, then $\mathrm{tp}(\bar{b}_1/A; \widehat{N}_1) = \mathrm{tp}(\bar{b}_2/A; \widehat{N}_2)$. So assume $(\bar{b}_1, A, N_1) E_{\mathrm{at}} (\bar{b}_2, A, N_2)$. Then there exists $N \in K$ and $f_\ell : N_\ell \xrightarrow[A]{} N$ such that $f_1(\bar{b}_1) = f_2(\bar{b}_2)$. Let $\phi(\bar{x})$ be a formula over A. Assume $\widehat{N}_1 \models \phi[\bar{b}_1]$. By Proposition 3.11, $\widehat{N} \models \phi[f_1(\bar{b}_1)]$, so $\widehat{N} \models \phi[f_2(\bar{b}_2)]$, so by Proposition 3.11 again, $\widehat{N}_2 \models \phi[\bar{b}_2]$. Replacing ϕ by $\neg\phi$, we get the converse, so $\mathrm{tp}(\bar{b}_1/A; \widehat{N}_1) = \mathrm{tp}(\bar{b}_2/A; \widehat{N}_2)$. $\square$

Note that this used that syntactic types were quantifier-free. We have justified the following definition:

Definition 3.13. For a Galois type p, let let p^s be the corresponding syntactic type *in the Morleyization*. That is, if $p = \mathrm{gtp}(\bar{b}/A; N)$, then $p^s := \mathrm{tp}(\bar{b}/A; \widehat{N})$.

Proposition 3.14. Let $N \in K$, $A \subseteq |N|$. Let α be an ordinal. The map $p \mapsto p^s$ from $\mathrm{gS}^\alpha(A; N)$ to $\mathrm{S}^\alpha(A; \widehat{N})$ is a surjection.

Proof. If $\mathrm{tp}(\bar{b}/A; \widehat{N}) = q \in \mathrm{S}^\alpha(A; \widehat{N})$, then by definition $\left(\mathrm{gtp}(\bar{b}/A; N)\right)^s = q$. $\square$

Remark 3.15. To investigate formulas with quantifiers, we could define a different version of Galois types using isomorphisms rather than embeddings, and remove the monotonicity axiom from the definition of a Morleyization. As we have no use for it, we avoid this approach.

3.3. **On when Galois types are syntactic.** We have seen in Proposition 3.14 that $p \mapsto p^s$ is a surjection, so Galois types are always finer than syntactic type in the Morleyization. It is natural to ask when they are the same, i.e. when $p \mapsto p^s$ is a *bijection*. For $\widehat{K}$ a Morleyization of $\widehat{K}^{<\kappa}$ (see Definition 3.3), note that this will mostly be used when $\widehat{K} = \widehat{K}^{<\kappa}$), we characterize when this is the case using shortness and tameness (Definition 2.19). Note that we make no hypothesis on K. In particular, amalgamation is not needed.

Theorem 3.16 (The semantic-syntactic correspondence)**.** Assume $\widehat{K}$ is a ($(< \kappa)$-ary, recall Hypothesis 3.9) Morleyization of $\widehat{K}^{<\kappa}$.

Let Γ be a family of Galois types. The following are equivalent:

(1) K is $(< \kappa)$-tame and short for Γ.
(2) The map $p \mapsto p^s$ is a bijection from Γ onto $\Gamma^s := \{p^s \mid p \in \Gamma\}$.

Proof.

- (1) implies (2): By Lemma 3.12, the map $p \mapsto p^s$ with domain Γ is well-defined and it is clearly a surjection onto Γ^s. It remains to see it is injective. Let $p, q \in \Gamma$ be distinct. If they do not have the same domain or the same length, then $p^s \neq q^s$, so assume that $A := \mathrm{dom}(p) = \mathrm{dom}(q)$ and $\alpha := \ell(p) = \ell(q)$. Say $p = \mathrm{gtp}(\bar{b}/A; N)$, $q = \mathrm{gtp}(\bar{b}'/A; N')$. By the tameness and shortness hypotheses, there exists $A_0 \subseteq A$ and $I \subseteq \alpha$ of size less than κ such that $p_0 := p^I \upharpoonright A_0 \neq q^I \upharpoonright A_0 =: q_0$. Let $\bar{a}_0$

be an enumeration of A_0, and let $\bar{b}_0 := \bar{b} \restriction I$, $\bar{b}'_0 := \bar{b}' \restriction I$. Let $p'_0 := \text{gtp}(\bar{b}_0\bar{a}_0/\emptyset; N)$, and let $\phi := R_{p'_0}(\bar{x}_0, \bar{a}_0)$, where $\bar{x}_0$ is a sequence of variables of type I. Since $\bar{b}_0$ realizes p_0 in N, $\widehat{N} \models \phi[\bar{b}_0]$, and since $\bar{b}'_0$ realizes q_0 in N' and $q_0 \neq p_0$, $\widehat{N'} \models \neg\phi[\bar{b}'_0]$. Thus $\phi(\bar{x}_0) \in p^s$, $\neg\phi(\bar{x}_0) \in q^s$. By definition, $\phi(\bar{x}_0) \notin q$ so $p^s \neq q^s$.

- <u>(2) implies (1)</u>: Let $p, q \in \Gamma$ be distinct with domain A and length α. Say $p = \text{gtp}(\bar{b}/A; N)$, $q = \text{gtp}(\bar{b}'/A; N')$. By hypothesis, $p^s \neq q^s$ so there exists $\phi(\bar{x})$ over A such that (without loss of generality) $\phi(\bar{x}) \in p$ but $\neg\phi(\bar{x}) \in q$. Let $A_0 := \text{dom}(\phi)$, $\bar{x}_0 := \text{FV}(\phi)$ (note that A_0 and $\bar{x}_0$ have size strictly less than κ). Let $\bar{b}_0$, $\bar{b}'_0$ be the corresponding subsequences of $\bar{b}$ and $\bar{b}'$ respectively. Let $p_0 := \text{gtp}(\bar{b}_0/A_0; N)$, $q_0 := \text{gtp}(\bar{b}'_0/A_0; N')$. Then it is straightforward to check that $\phi \in p_0^s$, $\neg\phi \in q_0^s$, so $p_0^s \neq q_0^s$ and hence (by Lemma 3.12) $p_0 \neq q_0$. Thus A_0 and I witness tameness and shortness respectively.

$\square$

Remark 3.17. The proof shows that (2) implies (1) is valid when $\widehat{K}$ is any Morleyization of K.

Remark 3.18. Since Galois types in K and $\widehat{K}^{<\kappa}$ are the same (Proposition 3.7), nothing interesting happens if we take the Galois Morleyization a second time, i.e. $\widehat{\widehat{K}^{<\kappa}}^{<\kappa}$ is essentially the same as $\widehat{K}^{<\kappa}$.

Corollary 3.19. Assume $\widehat{K}$ is a Morleyization of $\widehat{K}^{<\kappa}$.

(1) K is fully $(<\kappa)$-tame and short if and only if for any $M \in K$ the map $p \mapsto p^s$ from $\text{gS}^{<\infty}(M)$ to $\text{S}^{<\infty}(M)$ is a bijection[12].
(2) K is $(<\kappa)$-tame if and only if for any $M \in K$ the map $p \mapsto p^s$ from $\text{gS}(M)$ to $\text{S}(M)$ is a bijection.

Proof. By Theorem 3.16 applied to $\Gamma := \bigcup_{M \in K} \text{gS}^{<\infty}(M)$ and $\Gamma := \bigcup_{M \in K} \text{gS}(M)$ respectively. $\square$

Remark 3.20. For $M \in K$, $p, q \in \text{gS}(M)$, say $pE_{<\kappa}q$ if and only if $p \restriction A_0 = q \restriction A_0$ for all $A_0 \subseteq |M|$ with $|A_0| < \kappa$. Of course, if K is $(<\kappa)$-tame, then $E_{<\kappa}$ is just equality. More generally, the proof of Theorem 3.16 shows that if $\widehat{K}$ is a Morleyization of $\widehat{K}^{<\kappa}$, then $pE_{<\kappa}q$ if and only if $p^s = q^s$. Thus in that case syntactic types in the Morleyization can be seen as $E_{<\kappa}$-equivalence classes of Galois types. Note that $E_{<\kappa}$ appears in the work of Shelah, see for example [She99, Definition 1.8].

[12] We have set $\text{S}^{<\infty}(M) := \bigcup_{N \geq M} \text{S}^{<\infty}(M; \widehat{N})$. Similarly define $\text{S}(M)$.

4. Order properties and stability spectrum

In this section, we start applying the semantic-syntactic correspondence (Theorem 3.16) to prove new structural results about AECs. Recall from the introduction that the general method we use to prove a result about AECs using syntactic methods goes as follows:

(1) Prove a syntactic result in the Galois Morleyization of the AEC (e.g. using tools from stability theory inside a model).
(2) Translate to a semantic result in the AEC using the semantic-syntactic correspondence.
(3) Push the semantic result further using known (semantic) facts about AECs, maybe combined with more hypotheses on the AEC (e.g. amalgamation).

In our proof of Theorem 4.13, Fact 4.11 gives the first step, while Facts 4.5 (AECs have a Hanf number for the order property) and 2.21 (In tame AECs with amalgamation, stability behaves reasonably well) are keys for the third step.

Throughout this section, we make the following hypotheses:

Hypothesis 4.1.

(1) K is an abstract elementary class.
(2) κ is an infinite cardinal, $\widehat{L}$ is a $(< \kappa)$-ary language.
(3) $\widehat{K}$ is a class of $\widehat{L}$-structures.

Since we want to state general definitions, we do not assume that $\widehat{K} = \widehat{K}^{<\kappa}$, or even that $\widehat{K}$ is an $\widehat{L}$-Morleyization of K. Eventually, this will of course be assumed.

4.1. Several order properties.

The next definition is a natural syntactic extension of the first-order order property. A related definitions appears already in [She72] and has been well studied (see for example [GS86, GS]).

Definition 4.2 (Syntactic order property)**.** Let β and μ be cardinals. A model $\widehat{M} \in \widehat{K}$ has the *syntactic β-order property of length μ* if there exists $\langle \bar{a}_i : i < \mu \rangle$ inside $\widehat{M}$ with $\ell(\bar{a}_i) = \beta$ for all $i < \mu$ and a quantifier-free formula $\phi(\bar{x}, \bar{y})$ such that for all $i, j < \mu$, $\widehat{M} \models \phi[\bar{a}_i, \bar{a}_j]$ if and only if $i < j$.

Let α be a cardinal. $\widehat{M}$ has the *syntactic $(< \alpha)$-order property of length μ* if it has the syntactic β-order property of length μ for some $\beta < \alpha$.

$\widehat{M}$ has the *syntactic order property of length* μ if it has the syntactic $(< \kappa)$-order property of length μ.

$\widehat{K}$ *has the syntactic* β*-order of length* μ if some $\widehat{M} \in \widehat{K}$ has it. $\widehat{K}$ *has the syntactic order property* if it has the syntactic order property for every length.

Arguably the most natural semantic definition of the order property in AECs appears in [She99, Definition 4.3]. For simplicity, we have removed one parameter from the definition.

Definition 4.3. Let α and μ be cardinals. A model $M \in K$ has the *Galois* α*-order property of length* μ if there exists $\langle \bar{a}_i : i < \mu \rangle$ inside M with $\ell(\bar{a}_i) = \alpha$ for all $i < \mu$, such that for any $i_0 < j_0 < \mu$ and $i_1 < j_1 < \mu$, $\text{gtp}(\bar{a}_{i_0}\bar{a}_{j_0}/\emptyset; N) \neq \text{gtp}(\bar{a}_{j_1}\bar{a}_{i_1}/\emptyset; N)$.

We usually drop the "Galois" and define variations such as "K has the α-order property" as in Definition 4.2.

Notice this is more general than the syntactic order property, since α is not required to be less than κ. However, when $\widehat{K} = \widehat{K}^{<\kappa}$ and α is small, the two properties are equivalent. Notice that this does not use any tameness.

Proposition 4.4. Let α and μ be cardinals with $\alpha < \kappa$. Assume $\widehat{K}$ is an $\widehat{L}$-Morleyization of K and work inside a model N. The syntactic α-order property of length μ implies the α-order property of length μ.

Conversely, if $\widehat{K}$ is a Morleyization of $\widehat{K}^{<\kappa}$, letting $\chi := |\text{gS}^{\alpha+\alpha}(\emptyset)|$, if $\mu \geq \left(2^{\lambda+\chi}\right)^+$, then the α-order property of length μ implies the syntactic α-order property of length λ.

In particular, if $\widehat{K}$ is a Morleyization of $\widehat{K}^{<\kappa}$, the α-order property (in K) and the syntactic α-order property (in $\widehat{K}$) are equivalent.

Proof. That the syntactic order property implies the Galois one is a consequence of invariance[13]. For the converse, let $\langle \bar{a}_i : i < \mu \rangle$ witness the Galois order property. By the Erdős-Rado theorem used on the coloring $(i < j) \mapsto \text{gtp}(\bar{a}_i\bar{a}_j/\emptyset; N)$, we get that (without loss of generality), $\langle \bar{a}_i : i < \lambda \rangle$ is such that whenever $i < j$, $\text{gtp}(\bar{a}_i\bar{a}_j/\emptyset; N) = p \in \text{gS}^{\alpha+\alpha}(\emptyset)$. But then (since by assumption $\text{gtp}(\bar{a}_i\bar{a}_j/\emptyset; N) \neq \text{gtp}(\bar{a}_j\bar{a}_i/\emptyset; N)$), $\phi(\bar{x}, \bar{y}) := R_p(\bar{x}, \bar{y})$ witnesses the syntactic order property. □

[13]Note: we are using that everything in sight is quantifier-free.

We will see later that assuming some tameness (and $\widehat{K}$ is a Morleyization of $\widehat{K}^{<\kappa}$), the order property is actually equivalent to the syntactic order property (even for non-small αs).

In the next part, we heavily use the assumption of no syntactic order property *of length* κ. We now look at how it compares to the (long) order property. Note that Proposition 4.4 already tells us that (if $\widehat{K}$ is a Morleyization of $\widehat{K}^{<\kappa}$) the $(<\kappa)$-order property implies the syntactic order property of length κ. To get an equivalence, we will assume κ is a fixed point of the Beth function. Recall:

Fact 4.5. Let α be a cardinal. If K has the α-order property of length μ for all $\mu < h(\alpha + \mathrm{LS}(K))$, then K has the α-order property.

Proof. By the same proof as [She99, Claim 4.5.3]. □

Corollary 4.6. Let $\widehat{K}$ be an $\widehat{L}$-Morleyization of K. Assume $\beth_\kappa = \kappa > \mathrm{LS}(K)$.

(1) If $\widehat{K}$ has the syntactic order property of length κ, then K has the $(<\kappa)$-order property.
(2) If $\widehat{K}$ is a Morleyization of $\widehat{K}^{<\kappa}$, then $\widehat{K}$ has the syntactic order property of length κ if and only if K has the $(<\kappa)$-order property.

Proof.

(1) For some $\alpha < \kappa$, $\widehat{K}$ has the syntactic α-order property of length κ, and thus by Proposition 4.4 the α-order property of length κ. Since $\kappa = \beth_\kappa$, $h(|\alpha| + \mathrm{LS}(K)) < \kappa$, so by Fact 4.5, K has the α-order property.
(2) By the first part and Proposition 4.4.

□

For completeness, we also discuss the following semantic variation of the syntactic order property of length κ that appears in [BG, Definition 4.2] (but is adapted from a previous definition of Shelah, see there for more background):

Definition 4.7. For $\kappa > \mathrm{LS}(K)$, K has the *weak κ-order property* if there are $\alpha, \beta < \kappa$, $M \in K_{<\kappa}$, $N \geq M$, types $p \neq q \in \mathrm{gS}^{\alpha+\beta}(M)$, and sequences $\langle \bar{a}_i : i < \kappa \rangle$, $\langle \bar{b}_i : i < \kappa \rangle$ from N so that for all $i, j < \kappa$:

(1) $i \leq j$ implies $\mathrm{gtp}(\bar{a}_i \bar{b}_j / M; N) = p$.

(2) $i > j$ implies $\mathrm{gtp}(\bar{a}_i \bar{b}_j / M; N) = q$.

Lemma 4.8. Let $\kappa > \mathrm{LS}(K)$.

(1) If K has the $(< \kappa)$-order property, then K has the weak κ-order property.
(2) If K has the weak κ-order property, and $\widehat{K}$ is a Morleyization of $\widehat{K}^{<\kappa}$, then $\widehat{K}$ has the syntactic order property of length κ.

In particular, if $\kappa = \beth_\kappa$, then the weak κ-order property, the $(< \kappa)$-order property of length κ, and the $(< \kappa)$-order property are equivalent.

Proof.

(1) Assume K has the $(< \kappa)$-order property. To see the weak order property, let $\alpha < \kappa$ be such that K has the α-order property. Fix an $N \in K$ such that N has a long-enough α-order property. Pick any $M \in K_{<\kappa}$ with $M \leq N$. By using the Erdős-Rado theorem twice, we can assume we are given $\langle \bar{c}_i : i < \kappa \rangle$ such that whenever $i < j < \kappa$, $\mathrm{gtp}(\bar{c}_i \bar{c}_j / M; N) = p$, and $\mathrm{gtp}(\bar{c}_j \bar{c}_i / M; N) = q$, for some $p \neq q \in \mathrm{gS}(M)$.
    For $l < \kappa$, let $j_l := 2l$, and $k_l := 2l + 1$. Then $j_l, k_l < \kappa$, and $l \leq l'$ implies $j_l < k_{l'}$, whereas $l > l'$ implies $j_l > k_{l'}$. Thus the sequences defined by $\bar{a}_l := \bar{c}_{j_l}$, $\bar{b}_l := \bar{c}_{k_l}$ are as required.
(2) Assume K has the weak κ-order property and let $M, N, p, q, \langle \bar{a}_i : i < \kappa \rangle$, $\langle \bar{b}_i : i < \kappa \rangle$ witness it. For $i < \kappa$, Let $\bar{c}_i := \bar{a}_i \bar{b}_i$ and $\phi(\bar{x}_1 \bar{x}_2; \bar{y}_1 \bar{y}_2) := R_p(\bar{y}_1, \bar{x}_2)$. This witnesses the syntactic order property of length κ in $\widehat{K}^{<\kappa}$.

$\square$

4.2. **Order property and stability.** We now want to relate stability in terms of the number of types (see Definition 2.20) and the order property and use this to find many stability cardinals.

Note that if $\widehat{K}$ is a Morleyization of $\widehat{K}^{<\kappa}$, stability will coincide with syntactic stability given enough tameness and shortness (see Theorem 3.16). In general, they could be different, but by invariance, stability always implies syntactic stability (and so syntactic unstability implies unstability). This contrasts with the situation with the order properties, where the syntactic and regular order property are equivalent without tameness (see Proposition 4.4).

The basic relationship between the order property and stability is given by:

Fact 4.9. If K has the α-order property and $\mu \geq |\alpha| + \text{LS}(K)$, then K is not α-stable in μ. If in addition $\alpha < \kappa$ and $\widehat{K}$ is a Morleyization of $\widehat{K}^{<\kappa}$, then $\widehat{K}$ is not even syntactically α-stable in μ.

Proof. [She99, Claim 4.8.2] is the first sentence. The proof (see [BGKV, Fact 5.13]) generalizes (using the syntactic order property) to get the second sentence. ☐

This shows that the order property implies unstability and we now work towards a syntactic converse. The key is:

Fact 4.10 (Theorem V.A.1.19 in [She09b])**.** Let $\widehat{N} \in \widehat{K}$. Let $\alpha < \kappa$. Let $\chi \geq (|\widehat{L}| + 2)^{<\kappa}$ be a cardinal. If $\widehat{N}$ does not have the syntactic $(< \kappa)$-order property of length χ^+, then whenever $\lambda = \lambda^\chi + \beth_2(\chi)$, $\widehat{N}$ is (syntactically) $(< \kappa)$-stable in λ.

Note that since everything is done inside a model, the next corollary does not need any amalgamation.

Corollary 4.11. Assume $\widehat{K} = \widehat{K}^{<\kappa}$. The following are equivalent:

(1) $\widehat{K}$ is syntactically $(< \kappa)$-stable in *some* cardinal greater than or equal to $\text{LS}(K) + \kappa$.
(2) K does not have the $(< \kappa)$-order property.
(3) There exists[14] $\mu \leq \lambda_0 < h(\kappa + \text{LS}(K))$ such that $\widehat{K}$ is syntactically $(< \kappa)$-stable in any $\lambda \geq \lambda_0$ with $\lambda^{<\mu} = \lambda$. In particular, $\widehat{K}$ is syntactically $(< \kappa)$-stable.

Proof. (3) clearly implies (1). (1) implies (2): If K has the $(< \kappa)$-order property, then by Fact 4.9 it cannot be syntactically $(< \kappa)$-stable in any cardinal above $\text{LS}(K) + \kappa$ (note that α-unstability for $\alpha < \kappa$ implies $(< \kappa)$-unstability).

Finally (2) implies (3). Assume K does not have the $(< \kappa)$-order property. By the contrapositive of Fact 4.5, for each $\alpha < \kappa$, there exists $\mu_\alpha < h(|\alpha| + \text{LS}(K)) \leq h(\kappa + \text{LS}(K))$ such that K does not have the α-order property of length μ_α. Since $2^{<(\kappa + \text{LS}(K)^+)} < h(\kappa + \text{LS}(K))$, we can without loss of generality assume that $2^{<(\kappa + \text{LS}(K)^+)} \leq \mu_\alpha$ for all $\alpha < \kappa$. Let $\chi := \sup_{\alpha < \kappa} \mu_\alpha$. Then K does not have the $(< \kappa)$-order property of length χ. Since $\text{cf}(h(\kappa + \text{LS}(K))) = (2^{\kappa + \text{LS}(K)})^+ > \kappa$, $\chi < h(\kappa + \text{LS}(K))$. Let $\mu := \chi^+$ and $\lambda_0 := \beth_2(\chi)$. It is easy to

[14] The cardinal μ is closely related to the local character cardinal $\bar{\kappa}$ for nonsplitting. See for example [GV06b, Theorem 4.13].

check that $\mu \leq \lambda_0 < h(\kappa + \mathrm{LS}(K))$. Finally, note that by Remark 3.4, $|\widehat{L}| \leq 2^{<(\kappa+\mathrm{LS}(K)^+)}$, so $\chi \geq (|\widehat{L}|+2)^{<\kappa}$. Now apply Fact 4.10 (note that by definition of λ_0, if $\lambda = \lambda^\chi \geq \lambda_0$, then $\lambda = \lambda^\chi + \beth_2(\chi)$.) $\square$

Remark 4.12. Shelah [She, Theorem 3.3] claims (without proof) a version of (1) implies (3).

Assuming $(< \kappa)$-tameness for types of length $< \kappa$, we can of course convert the above result to a statement about Galois types. To replace "$(< \kappa)$-stable" by just "stable" (and get away with only tameness for types of length one) we will use amalgamation together with Fact 2.21.

Theorem 4.13. Assume K has amalgamation and is $(< \kappa)$-tame. The following are equivalent:

(1) K is stable in *some* cardinal greater than or equal to $\mathrm{LS}(K)+\kappa$.
(2) K does not have the order property.
(3) K does not have the $(< \kappa)$-order property.
(4) There exists $\mu \leq \lambda_0 < h(\kappa + \mathrm{LS}(K))$ such that K is stable in any $\lambda \geq \lambda_0$ with $\lambda^{<\mu} = \lambda$.

In particular, K is stable if and only if K does not have the order property.

Proof. Clearly, (4) implies (1) and (2) implies (3). (1) implies (2): If K has the α-order property, then by Fact 4.9 it cannot be α-stable in any cardinal above $\mathrm{LS}(K) + \kappa + |\alpha|$. By Fact 2.21.(3), K is not stable in any cardinal greater than or equal to $\kappa + \mathrm{LS}(K)$, so (1) fails. Finally, (3) implies (4) by combining Corollary 4.11 and Corollary 3.19. $\square$

5. Coheir

We look at the natural generalization of coheir (introduced in [LP79] for first-order logic) to our context. A definition of coheir for classes of models in $L_{\kappa,\omega}$ was first introduced in [MS90] and later adapted to general AECs in [BG]. We give a slightly more conceptual definition here and show that coheir has many of the properties of forking in a stable first-order theory. This improves on [BG] which assumed that coheir had the extension property.

Hypothesis 5.1.

(1) κ is an infinite cardinal.
(2) K is a κ-AEC, $\widehat{K}$ is a $(< \kappa)$-ary $\widehat{L}$-Morleyization of K.

(3) If $M \leq N$ are in K, then $\widehat{M} \preceq_{\Sigma_1(\widehat{L}_{\kappa,\kappa})} \widehat{N}$ (that is, for any quantifier-free $\widehat{L}_{\kappa,\kappa}$ formula $\phi(\bar{x}, \bar{y})$ with $\ell(\bar{x}) + \ell(\bar{y}) < \kappa$ and any $\bar{a} \in {}^{<\kappa}\widehat{M}$, we have $\widehat{M} \models \exists \bar{x}\phi[\bar{x}, \bar{a}] \Leftrightarrow \widehat{N} \models \exists \bar{x}\phi[\bar{x}, \bar{a}]$).

Remark 5.2. In [She09b, Definition V.A.0.9], Shelah gives weaker relations than $\preceq_{\Sigma_1(\widehat{L}_{\kappa,\kappa})}$ in (3) that also suffice for our purpose, but we do not see the need to introduce them here.

The reader can see $\widehat{K}$ as the class in which coheir is computed syntactically, while K is the class in which it is used semantically. In applications, we will start with an AEC with amalgamation K^0, let K be the κ-AEC of κ-saturated models of K^0 (for $\kappa > \mathrm{LS}(K^0)$) and let $\widehat{K} := \widehat{K}^{<\kappa}$. Note that in this case, when $\widehat{K}$ is a $(<\kappa)$-ary Morleyization of $\widehat{K}^{<\kappa}$, condition (3) is actually equivalent to asking for all models in K to be κ-saturated.

Definition 5.3. Let $\widehat{N} \in \widehat{K}$, $A \subseteq |\widehat{N}|$, and p be a set of formulas over $\widehat{N}$.

(1) p is a *heir over* A if for any formula $\phi(\bar{x}; \bar{b}) \in p$ over A, there exists $\bar{a} \in A$ such that $\phi(\bar{x}; \bar{a}) \in p \restriction A$.
(2) p is a *coheir over* A *in* $\widehat{N}$ if for any $\phi(\bar{x}) \in p$ there exists $\bar{a} \in A$ such that $\widehat{N} \models \phi[\bar{a}]$. When $\widehat{N}$ is clear from context, we drop it.

Remark 5.4. Working in $\widehat{N} \in \widehat{K}$, let $\bar{c}$ be a permutation of $\bar{c}'$, and A, B be sets. Then $\mathrm{tp}(\bar{c}/B; \widehat{N})$ is a coheir over A if and only if $\mathrm{tp}(\bar{c}'/B; \widehat{N})$ is a coheir over A. Similarly for heir. Thus we can talk about $\mathrm{tp}(C/B; \widehat{N})$ being a heir/coheir over A without worrying about the enumeration of C.

Remark 5.5. We may talk about $(<\kappa)$-coheir or $(<\kappa)$-heir if κ is not clear from context.

We will mostly look at coheir, but the next proposition tells us how to express one in term of the other.

Proposition 5.6. $\mathrm{tp}(\bar{a}/A\bar{b}; \widehat{N})$ is a heir over A if and only if $\mathrm{tp}(\bar{b}/A\bar{a}; \widehat{N})$ is a coheir over A.

Proof. Straightforward. $\square$

We now see coheir as an independence relation.

Definition 5.7. Write $A \underset{M}{\overset{N}{\perp}} B$ if $M \leq N$ are in K and $\text{tp}(A/|M| \cup B; \widehat{N})$ is a coheir over $|M|$. We also say[15] $\text{gtp}(A/B; N)$ *is a coheir over* M.

Remark 5.8. This depends on κ and $\widehat{K}$ but we hide this detail.

Note that in the case we are interested in, there is a more semantic definition:

Proposition 5.9. Assume $\widehat{K}$ is a $(<\kappa)$-ary Morleyization of $\widehat{K}^{<\kappa}$.

Then $p \in \text{gS}^{<\infty}(B; N)$ is a coheir over $M \leq N$ if and only if for any $I \subseteq \ell(p)$ and any $B_0 \subseteq B$, if $|I_0| + |B_0| < \kappa$, $p^I \restriction B_0$ is realized in M.

Proof. Straightforward $\square$

This shows that in the relevant cases, our definition is equivalent to that of [BG, Definition 3.2] (and to [BGKV, Definition 3.8], if we require that the right hand side contains the base). For completeness, we show that our definition of heir also agrees with the semantic definition given in [BG, Definition 6.1].

Proposition 5.10. Let K^0 be an AEC with amalgamation and joint embedding. Let $\kappa > \text{LS}(K^0)$ and let K be the κ-AEC of κ-saturated models of K^0. Assume $\widehat{K}$ is a $(<\kappa)$-ary $\widehat{L}$-Morleyization of $\widehat{K}^{<\kappa}$. Let $M_0 \leq M \leq N$ be in K, $\bar{a} \in {}^{<\infty}|\widehat{N}|$.

Then $\text{tp}(\bar{a}/M; \widehat{N})$ is a heir over M_0 if and only if for all $(<\kappa)$-sized $I \subseteq \ell(\bar{a})$ and $(<\kappa)$-sized $M_0^- \leq M$, $M_0^- \leq M^- \leq M$ (where we also allow M_0^- to be empty), there is $f : M^- \xrightarrow[M_0^-]{} M_0$ such that $\text{gtp}(\bar{a}/M; N)$ extends $f(\text{gtp}((\bar{a} \restriction I)/M^-; N))$.

Proof. Assume first $\text{tp}(\bar{a}/M; \widehat{N})$ is a heir over M_0 and let $I \subseteq \ell(\bar{a})$, $M_0^- \leq M^- \leq M$ be $(<\kappa)$-sized, with M_0^- possibly empty. Let $p := \text{gtp}((\bar{a} \restriction I)/M^-; N)$. Let $\bar{b}_0$ be an enumeration of M_0^- and let $\bar{b}$ be an enumeration of $|M^-| \backslash |M_0^-|$. Let $q := \text{gtp}((\bar{a} \restriction I)\bar{b}_0\bar{b}/\emptyset; N)$. Consider the formula $\phi(\bar{x}; \bar{b}; \bar{b}_0) := R_q(\bar{x}; \bar{b}; \bar{b}_0)$, where $\bar{x}$ are the free variables in $\text{tp}(\bar{a}/M; \widehat{N})$ and we assume for simplicity the I-indiced variables are picked out by $R_q(\bar{x}, \bar{b}, \bar{b}_0)$. Then ϕ is in $\text{tp}(\bar{a}/M; \widehat{N})$. By the syntactic definition of heir, there is $\bar{c} \in {}^{<\kappa}|M_0|$ such that $\phi(\bar{x}; \bar{c}; \bar{b}_0)$ is in $\text{tp}(\bar{a}/M_0; \widehat{N})$. By definition of $\widehat{K}^{<\kappa}$ this means that $\text{gtp}((\bar{a} \restriction I)\bar{b}\bar{b}_0/\emptyset; N) = \text{gtp}((\bar{a} \restriction I)\bar{c}\bar{b}_0/\emptyset)$. By definition of Galois

[15]It is easy to check this does not depend on the choice of representatives.

types, saturation of M_0, joint embedding, and amalgamation, there is $f : M^- \xrightarrow[M_0^-]{} M_0$ witnessing the equality, which is as desired.

The converse is similar. $\square$

Remark 5.11. The notational difficulties encountered in the above proof and the complexity of the semantic definition of heir show the convenience of using a syntactic notation rather than working purely semantically.

We now investigate the properties of coheir. For the convenience of the reader, we explicitly prove the uniqueness property (we have to slightly adapt the proof of (U) from [MS90, Proposition 4.8]). For the others, they are either straightforward or we can just quote.

Lemma 5.12. Let $M \leq N, N'$. Assume $\widehat{M}$ does *not* have the syntactic $(< \kappa)$-order property of length κ. Let $\bar{a} \in {}^{<\infty}N$, $\bar{a}' \in {}^{<\infty}N'$, $\bar{b} \in {}^{<\infty}M$ be given such that:

1. $\mathrm{tp}(\bar{a}/M; \widehat{N}) = \mathrm{tp}(\bar{a}'/M; \widehat{N'})$
2. $\mathrm{tp}(\bar{a}/M\bar{b}; \widehat{N})$ is a coheir over M.
3. $\mathrm{tp}(\bar{b}/M\bar{a}'; \widehat{N'})$ is a coheir over M.

Then $\mathrm{tp}(\bar{a}/M\bar{b}; \widehat{N}) = \mathrm{tp}(\bar{a}'/M\bar{b}; \widehat{N'})$.

Proof. We suppose not and prove that $\widehat{M}$ has the syntactic $(< \kappa)$-order property of length κ. Assume that $\mathrm{tp}(\bar{a}/M\bar{b}; \widehat{N}) \neq \mathrm{tp}(\bar{a}'/M\bar{b}; \widehat{N'})$ and pick $\phi(\bar{x}, \bar{y})$ a formula over M witnessing it:

$$\widehat{N} \models \phi[\bar{a}; \bar{b}] \text{ but } \widehat{N'} \models \neg\phi[\bar{a}'; \bar{b}] \tag{1}$$

(note that we can assume without loss of generality that $\ell(\bar{a}) + \ell(\bar{b}) < \kappa$).

Define by induction on $i < \kappa$ $\bar{a}_i$, $\bar{b}_i$ *in* M such that for all $i, j < \kappa$:

1. $\widehat{M} \models \phi[\bar{a}_i, \bar{b}]$.
2. $\widehat{M} \models \phi[\bar{a}_i, \bar{b}_j]$ if and only if $i \leq j$.
3. $\widehat{N} \models \neg\phi[\bar{a}, \bar{b}_i]$.

Note that since $\bar{b}_i \in {}^{<\kappa}M$, (3) is equivalent to $\widehat{N'} \models \neg\phi[\bar{a}', \bar{b}_i]$.

<u>This is enough:</u> Then $\chi(\bar{x}_1, \bar{y}_1, \bar{x}_2, \bar{y}_2) := \phi(\bar{x}_1, \bar{y}_2) \wedge \bar{x}_1\bar{y}_1 \neq \bar{x}_2\bar{y}_2$ together with the sequence $\langle \bar{a}_i\bar{b}_i : i < \kappa \rangle$ witness the syntactic $(< \kappa)$-order property of length κ.

<u>This is possible:</u> Suppose that $\bar{a}_j, \bar{b}_j$ have been defined for all $j < i$. Note that by the induction hypothesis and (1) we have:

$$\widehat{N} \models \bigwedge_{j<i} \phi[\bar{a}_j, \bar{b}] \wedge \bigwedge_{j<i} \neg\phi[\bar{a}, \bar{b}_j] \wedge \phi[\bar{a}, \bar{b}]$$

Since $\text{tp}(\bar{a}/A\bar{b}; \widehat{N})$ is a coheir over M, there is $\bar{a}'' \in {}^{<\kappa}M$ such that:

$$\widehat{N} \models \bigwedge_{j<i} \phi[\bar{a}_j, \bar{b}] \wedge \bigwedge_{j<i} \neg\phi[\bar{a}'', \bar{b}_j] \wedge \phi[\bar{a}'', \bar{b}]$$

Note that all the data in the equation above is in M, so as $M \leq N$, the monotonicity axiom of abstract Morleyizations implies $\widehat{M} \subseteq \widehat{N}$, so $\widehat{M}$ also models the above. By monotonicity again, $\widehat{N'}$ models the above. We also know that $\widehat{N'} \models \neg\phi[\bar{a}', \bar{b}]$. Thus we have:

$$\widehat{N'} \models \bigwedge_{j<i} \phi[\bar{a}_j, \bar{b}] \wedge \bigwedge_{j<i} \neg\phi[\bar{a}'', \bar{b}_j] \wedge \phi[\bar{a}'', \bar{b}] \wedge \neg\phi[\bar{a}', \bar{b}]$$

Since $\text{tp}(\bar{b}/M\bar{a}'; \widehat{N})$ is a coheir over M, there is $\bar{b}'' \in {}^{<\kappa}M$ such that:

$$\widehat{N'} \models \bigwedge_{j<i} \phi[\bar{a}_j, \bar{b}''] \wedge \bigwedge_{j<i} \neg\phi[\bar{a}'', \bar{b}_j] \wedge \phi[\bar{a}'', \bar{b}''] \wedge \neg\phi[\bar{a}', \bar{b}'']$$

Let $\bar{a}_i := \bar{a}''$, $\bar{b}_i := \bar{b}''$. It is easy to check that this works. $\square$

Theorem 5.13 (Properties of coheir)**.**

(1) (a) Invariance: If $f : N \cong N'$ and $A \overset{N}{\underset{M}{\downarrow}} B$, then $f[A] \overset{N'}{\underset{f[M]}{\downarrow}} f[B]$.

(b) Monotonicity: If $A \overset{N}{\underset{M}{\downarrow}} B$ and $M \leq M' \leq N_0 \leq N$, $A_0 \subseteq A$, $B_0 \subseteq B$, $|M'| \subseteq B$, $A_0 \cup B_0 \subseteq |N_0|$, then $A_0 \overset{N_0}{\underset{M'}{\downarrow}} B_0$.

(c) Normality: If $A \overset{N}{\underset{M}{\downarrow}} B$, then $A \cup |M| \overset{N}{\underset{M}{\downarrow}} B \cup |M|$.

(d) Disjointness: If $A \overset{N}{\underset{M}{\downarrow}} B$, then $A \cap B \subseteq |M|$.

(e) Left and right existence: $A \overset{N}{\underset{M}{\downarrow}} M$ and $M \overset{N}{\underset{M}{\downarrow}} B$.

(f) Left and right $(<\kappa)$-set-witness: $A \underset{M}{\overset{N}{\mathop{\smile}\limits^{\mid}}} B$ if and only if for all $A_0 \subseteq A$ and $B_0 \subseteq B$ of size less than κ, $A_0 \underset{M}{\overset{N}{\mathop{\smile}\limits^{\mid}}} B_0$.

(g) Strong left transitivity: If $M_1 \underset{M_0}{\overset{N}{\mathop{\smile}\limits^{\mid}}} B$ and $A \underset{M_1}{\overset{N}{\mathop{\smile}\limits^{\mid}}} B$, then $A \underset{M_0}{\overset{N}{\mathop{\smile}\limits^{\mid}}} B$.

(2) If $\widehat{K}$ does not have the syntactic $(<\kappa)$-order property of length κ, then[16]:

(a) Symmetry: $A \underset{M}{\overset{N}{\mathop{\smile}\limits^{\mid}}} B$ if and only if $B \underset{M}{\overset{N}{\mathop{\smile}\limits^{\mid}}} A$.

(b) Strong right transitivity: If $A \underset{M_0}{\overset{N}{\mathop{\smile}\limits^{\mid}}} M_1$ and $A \underset{M_1}{\overset{N}{\mathop{\smile}\limits^{\mid}}} B$, then $A \underset{M_0}{\overset{N}{\mathop{\smile}\limits^{\mid}}} B$.

(c) Set local character: For all cardinals α, all $p \in \mathrm{gS}^\alpha(M)$, there exists $M_0 \leq M$ with $\|M_0\| \leq \mu_\alpha := \left(\alpha + \mathrm{LS}(K) + |\widehat{L}|\right)^{<\kappa_r}$ such that p is a coheir over M_0. In particular, if K^0 is an AEC with amalgamation, $\kappa > \mathrm{LS}(K^0)$, K is the κ-AEC of κ-saturated models of K^0, and $\widehat{K} = \widehat{K}^{<\kappa}$, then $\mu_\alpha = (\alpha + 2)^{<\kappa_r}$.

(d) Syntactic uniqueness: If $M_0 \leq M \leq N_\ell$ for $\ell = 1, 2$, $|M_0| \subseteq B \subseteq |M|$. $q_\ell \in \mathrm{S}^{<\infty}(B; \widehat{N_\ell})$, $q_1 \upharpoonright M_0 = q_2 \upharpoonright M_0$ and q_ℓ is a coheir over M_0 in $\widehat{N_\ell}$ for $\ell = 1, 2$, then $q_1 = q_2$.

(e) Stability: For α a cardinal, $\widehat{K}$ is (syntactically) α-stable in all $\lambda \geq \mathrm{LS}(K)$ such that $\lambda^{\mu_\alpha} = \lambda$.

(f) If $\widehat{K}$ is a $(<\kappa)$-ary Morleyization of $\widehat{K}^{<\kappa}$ and K is $(<\kappa)$-tame and short for types of length less than α, then coheir has uniqueness for types of length less than α: If $p, q \in \mathrm{gS}^{<\alpha}(M)$ are coheir over $M_0 \leq M$ and $p \upharpoonright M_0 = q \upharpoonright M_0$, then $p = q$.

Proof. Observe that (except for the last uniqueness property), one can work in $\widehat{K}$ and prove the properties there using purely syntactic methods (so amalgamation is never needed for example). More specifically, (1) is straightforward. As for (2), symmetry is exactly as in[17] [Pil82, Proposition 3.1], strong right transitivity follows from strong left transitivity and symmetry, syntactic uniqueness is by symmetry and Lemma

[16] Note that (by Theorem 4.13 and Lemma 4.8) this holds in particular if K is the class of κ-saturated models of a stable AEC K^0 with amalgamation, $\widehat{K}$ is a $(<\kappa)$-ary Morleyization of $\widehat{K}^{<\kappa}$ and $\kappa = \beth_\kappa > \mathrm{LS}(K^0)$.

[17] Note that a proof of symmetry of nonforking from no order property already appears in [She78], but Pillay's proof for coheir is the one we use here.

5.12, and set local character is as in the proof of $(B)_\mu$ in [MS90, Proposition 4.8]. For the particular case of set local character, note that by Remark 3.4 $|\widehat{L}^{<\kappa}| \leq |\text{gS}^{<\kappa}(\emptyset)| + |L|$. Since (by amalgamation) any model of K^0 has a κ-saturated extension, the Galois types of K^0 and K are in bijection, so $|\text{gS}^{<\kappa}(\emptyset)| \leq 2^{<(\kappa + \text{LS}(K^0)^+)} = 2^{<\kappa}$ (we have used that $\kappa > \text{LS}(K^0)$). By Remark 2.23, $\text{LS}(K) = \text{LS}(K^0)^{<\kappa} \leq 2^{<\kappa_r}$ so the result follows.

The proof of stability is as in the first-order case. To get the last uniqueness property, use the translation between Galois and syntactic types (Theorem 3.16). $\square$

Remark 5.14. We can give localized version of some of the above results. For example in the statement of the symmetry property it is enough to assume M does not have the syntactic $(<\kappa)$-order property of length κ. We could also have been more precise and state the uniqueness property in terms of being $(<\kappa)$-tame and short for $\{q_1, q_2\}$, where q_1, q_2 are two Galois types we are comparing.

Remark 5.15. We can use Theorem 5.13.(2e) to get another proof of the equivalence between (syntactic) stability and no order property in AECs.

Remark 5.16. The extension property (given $p \in \text{gS}^{<\infty}(M)$, $N \geq M$, p has an extension to N which is a coheir over M) seems to be more problematic. In [BG], Boney and Grossberg simply assumed it (they also showed that it followed from κ being strongly compact [BG, Theorem 8.2.1]). Here we do not need to assume it but are unable to prove it. In [Vas], we prove it assuming more superstability-like hypotheses[18].

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E-mail address: `sebv@cmu.edu`

URL: `http://math.cmu.edu/~svasey/`

Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, Pennsylvania, USA