FORKING INDEPENDENCE FROM THE CATEGORICAL POINT OF VIEW

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Abstract. We develop the theory of forking-like independence in accessible categories. To do so, we present an axiomatic definition of what we call a stable independence notion and show that this is in fact a purely category-theoretic axiomatization of the properties of model-theoretic forking in a stable first-order theory.

We then show that any coregular locally presentable category with effective unions admits a forking-like independence notion. This includes the cases of Grothendieck toposes and Grothendieck abelian categories.

We also give conditions for existence and canonicity of stable independence notions. Specifically, an accessible category with directed colimits whose morphisms are monomorphisms will have at most one stable independence notion. Moreover, assuming a large cardinal axiom, a cofinal full subcategory will have a stable independence notion if and only if a certain order property fails. This establishes, in particular, a category-theoretic characterization of stability (in the model-theoretic sense) for accessible categories.

Contents

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1. Introduction

1.1. **Background.** Forking is a model-theoretic notion generalizing linear independence in vector spaces and algebraic independence in fields. A central notion of modern model theory, it was developed by Saharon Shelah [She90] for classes of models axiomatized by a stable first-order theory[1]. Recall that a first-order theory T has the *order property* if there exists a model M of T, a formula $\phi(\bar{x};\bar{y})$ and a sequence $\langle \bar{a}_i : i < \omega \rangle$ of finite tuples in M such that for $i, j < \omega$, $M \models \phi[\bar{a}_i; \bar{a}_j]$ if and only if $i < j$. A theory T is *stable* precisely when it does *not* have the order property. For example, both the theory of vector spaces over $\mathbb{Q}$ and the theory of algebraically closed fields of characteristic zero are stable, but the theory of linear orders and that of graphs are not (a formula witnessing the order property inside the random graph, for example, is $\phi(x_1x_2, y_1y_2) := x_1Ey_2 \wedge \neg(x_2Ey_1)$).

Roughly, Shelah defines forking so that a type p over a set B (i.e. a set of formulas over B) *does not fork* over a subset A of B if it is a "generic" extension of $p \upharpoonright A$. In particular, p is essentially determined by $p \upharpoonright A$. Thus nonforking[2] can be seen as a notion of free extension. There is another way to see nonforking: in his survey on stability theory, Makkai [Mak84, A.1] introduces the *anchor symbol* $\mathop{\smile\!\!\!\!\!\!|}$ and defines $B \mathop{\smile\!\!\!\!\!\!|}_A C$ (when working inside a monster model $\mathfrak{C}$) to mean that for any finite tuples $\bar{b}$ of elements from B, $\mathrm{tp}(\bar{b}/AC)$ does not fork over A. This notation $B \mathop{\smile\!\!\!\!\!\!|}_A C$, which can be read as "$B$ is independent from C over A," simplifies the statement of some of the properties of forking. For example, *symmetry* can be written as "$B \mathop{\smile\!\!\!\!\!\!|}_A C$ if and only if $C \mathop{\smile\!\!\!\!\!\!|}_A B$".

Forking is especially well-behaved when the base set A above is a model of the theory[3]. In fact, to understand it category-theoretically, it is useful to consider the case in which *all* the sets under consideration are models, specifically $M_1 \overset{M_3}{\underset{M_0}{\mathop{\smile\!\!\!\!\!\!|}}} M_2$, with $M_0 \preceq M_\ell \preceq M_3$, $\ell = 1, 2$. Here, we write M_3 for an ambient model inside which all the types are computed. One can also view such a quadruple as a commutative diagram of embeddings, also known as an *amalgam*. In this sense, we may identify a nonforking notion over models with a particular choice of such amalgams. Note that consideration of nonforking amalgams plays an important role in model theory, as it leads to definitions such as that of an *independent system of models* (see [Mak84, A.11] or [She83a, She83b]). This is for example a key concept in both the statement and proof of Shelah's main gap theorem [She85], a celebrated achievement of first-order classification theory.

[1]While there are generalizations of the theory of stable forking to certain unstable first-order classes, most notably those axiomatized by NIP and simple theories, the present paper focuses exclusively on the stable case.

[2]It is somewhat unfortunate that the negation of forking, "nonforking" is the positive notion. Nevertheless, this terminology is now well established.

[3]To study forking over sets and not just models, one can consider a certain category of sufficiently algebraically closed sets (Example 3.30(4)) in which the nonforking amalgams determine the behavior of forking over any set.

1.2. **Main questions and earlier work.** The present paper seeks to answer the following questions:

(1) What are the basic category-theoretic properties of nonforking amalgams in the category of models of a stable first-order theory? More precisely, we would like a list of properties that are:
  (a) Invariant under equivalence of categories, e.g. they should not depend on what underlying concrete functor we use to represent the category.
  (b) Canonical: in any reasonable category, there should be at most one notion satisfying the properties of nonforking amalgamation.

(2) In what other categories is there a notion satisfying these properties?

There has already been a substantial amount of work on more purely model-theoretic versions of these questions. Consider, in particular, Harnik and Harrington [HH84], which characterizes forking in a stable first-order theory by a list of four axioms on a relation of inclusion between types. These axioms are, however, not suitably category-theoretic, as they depend on seeing types as sets of formulas. Recently, Boney, Grossberg, Kolesnikov, and the third author [BGKV16] characterized stable forking in the general framework of abstract elementary classes (AECs), encompassing first-order theories but also classes axiomatized by infinitary logics such as $\mathbb{L}_{\infty,\omega}$. The characterization of [BGKV16] is phrased in terms of an anchor relation $\mathrel{\raise0.2ex\hbox{$\smile$}\llap{$\vert$}}$, but still uses the underlying set representations of the objects in the category.

The second question was considered early on by Shelah, with partial results in homogeneous model theory [She70], $\mathbb{L}_{\omega_1,\omega}$ [She83a, She83b], universal classes [She87b], and AECs (e.g. in his two-volume book [She09a, She09b]). In fact, there is a growing body of literature on forking-like notions in AECs. We refer the reader to the survey of Boney and the third author [BV17a], but let us mention in particular the work of Boney and Grossberg [BG17], which generalizes work of Makkai and Shelah [MS90] from $\mathbb{L}_{\kappa,\omega}$ (κ a strongly compact cardinal) to AECs, and builds a global forking-like independence notion on a subclass of sufficiently saturated models of the AEC. Interestingly, the subclass is not itself an AEC, but is still closed under κ-directed unions (as opposed to arbitrary directed unions). This was one motivation for developing κ-AECs [BGL$^+$16]. This is a very broad framework for model theory. In fact, it has a category-theoretic equivalent: per Fact 2.3, a κ-AEC is exactly (up to equivalence of category) an accessible category whose morphisms are monomorphisms (see Makkai-Paré [MP89] or Adamek-Rosický [AR94] on accessible categories and their broader relationship to model theory).

1.3. **Main results.** It therefore seems natural to investigate forking in arbitrary accessible categories (perhaps with all morphisms monomorphisms). The present paper makes the following contributions:

- We define when a category has what we call a *stable independence notion* (Definition 3.18). Roughly, this is a class of distinguished squares that satisfies, in particular, an existence property, a uniqueness property (a weakening of the definition of a pushout), as well as a transitivity property (corresponding to being closed under composition in a double categorical sense). The transitivity property makes the class of independent squares into a category, and we require that this category be *accessible*.

- We show that this is the desired purely category-theoretic axiomatization of forking: in a μ-AEC $\mathbf{K}$, being stable (and, specifically, the accessibility of the category of independent squares) corresponds to having certain local character properties well known to model theorists (Theorem 8.13). Moreover this axiomatization is canonical, assuming that the class has chain bounds (that is, any increasing chain of models has an upper bound, see Definition 7.7). This result, Theorem 9.1, generalizes [BGKV16] to μ-AECs.
- Working purely abstractly, we exhibit a connection between forking and *effective unions* (an exactness property introduced by Barr [Bar88]). Specifically, we show that, if we start with a locally presentable and coregular category $\mathcal{K}$ that has effective unions then $\mathcal{K}_{reg}$, the subcategory of $\mathcal{K}$ containing just the regular monomorphisms of $\mathcal{K}$, has a stable independence notion, see Theorem 5.1. In particular, this covers both Grothendieck toposes and Grothendieck abelian categories. That forking occurs in these contexts seems not to have been recognized before (although it has long been known that forking occurs in classes axiomatized by first-order theories of modules, see [Pre88]).
- Assuming a large cardinal axiom, we characterize precisely when a stable independence notion exists in any μ-AEC with chain bounds (and hence in any accessible category with chain bounds whose morphisms are monomorphisms). This is Corollary 10.3: such a μ-AEC has a cofinal subclass with a stable independence notion if and only if it does not have a certain order property. This implies that the usual "syntactic" definition of stability (note that the usual definition in terms of counting types is too weak in this context, see Example 9.9) is equivalent to a purely category-theoretic statement. Thus model-theoretic stability is invariant under equivalence of category[4]. As a philosophical remark, Shelah [She09a, p. 23] argues that classification-theoretic dividing lines should have both an "internal" and an "external" characterization. If we interpret "external" as "invariant under equivalence of category" and "internal" as "a property satisfied by a fixed model in the class", then we see here this principle in action and obtain evidence that there is a "stability-like" dividing line in the general framework of accessible categories.

1.4. **Notes.** In a sense, this paper falls naturally into two parts: after a brief review of some of the basic concepts that will be used (Section 2), we devote Sections 3 through 4 to the development of an analogue of stable (or nonforking) independence suited to a general category and, working purely abstractly, give conditions on a category under which such a relation is (a) guaranteed to exist, and (b) to take a particular recognizable form, e.g. the independent squares are precisely the pullback squares. In particular, in Section 3 we formulate the axioms of stable independence in a general category, along with examples both abstract (Remark 3.29) and concrete (Example 3.30). In Section 4, we give some background on coregular

[4] For the category of models of a first-order theory, this can also be seen using Shelah's saturation spectrum theorem [She90, VIII.4.7], see also [Ros97]. However no such saturation spectrum theorem is known in arbitrary μ-AECs.

categories and effective unions, which is then put to use in Section 5, where we show that having effective unions implies the existence of a stable independence notion—provided the category is coregular and locally presentable, stable independence corresponds precisely to pullback (Theorem 5.1). This first half is intended to be congenial to a broad mathematical audience, and may be of particular use to those who have previously been reluctant to wade into the details of model-theoretic nonforking.

In the early sections, the reader will notice numerous references to results from the second half of the paper, including the canonicity of the independence notions at play. In this second half, we shift our attention to μ-AECs, which, being concrete, come with a great deal more machinery, and thus allow the development of a richer theory—the results and proofs here are more recognizably model-theoretic, and significantly more technical. We note, however, that μ-AECs are accessible categories with all morphisms monomorphisms, and vice versa—so, in fact, we are working in a vastly more general context than that of, say, Section 5. Section 6 gives some tools to move from an arbitrary category to a μ-AEC by restricting its class of morphisms: this provides a bridge between the paper's two halves. In Section 7, we give some basic model-theoretic tools for use in μ-AECs. In section 8, we consider what the properties of stable independence look like in a μ-AEC and show that they are equivalent to the more model-theoretic local character properties of forking in a stable first-order theory. In Section 9, we prove that stable independence notions are canonical, symmetric, and imply failure of a certain order property. In Section 10, we reverse this and show (assuming a large cardinal axiom) that failure of an order property implies there is a stable independence notion on a subclass of saturated models.

Throughout this paper, we assume the reader is familiar with basic category theory as presented e.g. in [AHS04]. More particularly, we will spend much of our time in accessible, locally presentable, or locally multipresentable categories (see [AR94] for further details). For connections between locally multipresentable categories—and locally polypresentable categories, which also make a brief appearance—and abstract model theory, we point readers to [LRVb]. We also assume some familiarity with μ-AECs and their relationship with accessible categories [BGL$^+$16].

We use the following notational conventions: we write K for a class of τ-structures and $\mathbf{K}$ (boldface) for a pair $(K, \leq_{\mathbf{K}})$, where $\leq_{\mathbf{K}}$ is a partial order. We write $\mathcal{K}$ (script) for a category. We will abuse notation and write $M \in \mathbf{K}$ instead of $M \in K$. For a structure M, we write UM for its universe, and $|UM|$ for the cardinality of its universe. We write $M \subseteq N$ to mean that M is a substructure of N. For α an ordinal, we let ${}^{<\alpha}A$ [resp. ${}^{\alpha}A$] denote the set of sequences of length less than [resp. exactly] α with elements from the set A. We will abuse this notation as well, writing ${}^{<\alpha}M$ in place of ${}^{<\alpha}UM$.

2. Preliminaries

Intuitively, an accessible category is a category with all sufficiently directed colimits and such that every object can be written as a highly directed colimit of "small" objects. Here "small" is interpreted in terms of *presentability*, a notion of size that makes sense in an arbitrary (potentially non-concrete) category. In the category of sets, of course, a set is λ-presentable if and only if its cardinality is less than λ; in an AEC $\mathbf{K}$, the same is true for all $\lambda > \mathrm{LS}(\mathbf{K})$.

Definition 2.1. Let $\mathcal{K}$ be a category and let λ be a regular cardinal.

(1) An object M is *λ-presentable* if its hom-functor $\mathcal{K}(M, -) : \mathcal{K} \to \mathbf{Set}$ preserves λ-directed colimits. Put another way, M is λ-presentable if for any morphism $f : M \to N$ with N a λ-directed colimit $\langle \phi_\alpha : N_\alpha \to N \rangle$, f factors essentially uniquely through one of the N_α, i.e. $f = \phi_\alpha f_\alpha$ for some $f_\alpha : M \to N_\alpha$.
(2) $\mathcal{K}$ is *λ-accessible* if it has λ-directed colimits and $\mathcal{K}$ contains a set S of λ-presentable objects such that every object of $\mathcal{K}$ is a λ-directed colimit of objects in S.
(3) $\mathcal{K}$ is *accessible* if it is λ'-accessible for some regular cardinal λ'.

We will often quote results on accessible categories from [AR94].

Recall from [BGL$^+$16, §2] that a *(μ-ary) abstract class* is a pair $\mathbf{K} = (K, \leq)$ such that K is a class of structures in a fixed μ-ary vocabulary $\tau = \tau(\mathbf{K})$, and $\leq$ is a partial order on K that refines the τ-substructure relation, with K and $\leq$ closed under τ-isomorphism. We say that such a $\mathbf{K}$ is *coherent* if $M_0 \subseteq M_1 \leq_{\mathbf{K}} M_2$ and $M_0 \leq_{\mathbf{K}} M_2$ implies $M_0 \leq_{\mathbf{K}} M_1$.

In any abstract class $\mathbf{K}$, there is a natural notion of morphism: we say that $f : M \to N$ is a *$\mathbf{K}$-embedding* if f is an isomorphism from M onto $f[M]$ and $f[M] \leq_{\mathbf{K}} N$. We can see an abstract class and its $\mathbf{K}$-embeddings as a category. In fact (see [BGL$^+$16, §2]), an abstract class is a replete and iso-full subcategory of the category of τ-structures with injective homomorphisms (insisting, as is customary in model theory, that relation symbols be reflected as well as preserved). Thus we also think of $\mathbf{K}$ as a (concrete) category. Saying that $\mathbf{K}$ has *concrete μ-directed colimits* amounts to saying that for any μ-directed system in $\mathbf{K}$, the union of the system is its colimit. That is, $\mathbf{K}$ satisfies the chain axioms of μ-AECs. In fact, let us now recall the definition of a μ-AEC from [BGL$^+$16, 2.2]:

Definition 2.2. Let μ be a regular cardinal. An abstract class $\mathbf{K}$ is a *μ-abstract elementary class* (or *μ-AEC* for short) if it satisfies the following three axioms:

(1) Coherence: for any $M_0, M_1, M_2 \in \mathbf{K}$, if $M_0 \subseteq M_1 \leq_{\mathbf{K}} M_2$ and $M_0 \leq_{\mathbf{K}} M_2$, then $M_0 \leq_{\mathbf{K}} M_1$.
(2) Chain axioms: if $\langle M_i : i \in I \rangle$ is a μ-directed system in $\mathbf{K}$, then:
   (a) $M := \bigcup_{i \in I} M_i$ is in $\mathbf{K}$.
   (b) $M_i \leq_{\mathbf{K}} M$ for all $i \in I$.
   (c) If $M_i \leq_{\mathbf{K}} N$ for all $i \in I$, then $M \leq_{\mathbf{K}} N$.
(3) Löwenheim-Skolem-Tarski (LST) axiom: there exists a cardinal $\lambda = \lambda^{<\mu} \geq |\tau(\mathbf{K})| + \mu$ such that for any $M \in \mathbf{K}$ and any $A \subseteq UM$, there exists $M_0 \in \mathbf{K}$ with $M_0 \leq_{\mathbf{K}} M$, $A \subseteq UM_0$, and $|UM_0| \leq |A|^{<\mu} + \lambda$. We write $\mathrm{LS}(\mathbf{K})$ for the least such λ.

Note that when $\mu = \aleph_0$, we recover Shelah's definition of an AEC from [She87a]. The connection between μ-AECs and accessible categories is given by [BGL$^+$16, §4]:

Fact 2.3. If $\mathbf{K}$ is a μ-AEC, then it is an $\mathrm{LS}(\mathbf{K})^+$-accessible category with all μ-directed colimits whose morphisms are monomorphisms. Conversely, any μ-accessible category whose morphisms are monomorphisms is equivalent to a μ-AEC.

We end this section by briefly recalling that we can define a notion of Galois (orbital) type in any abstract class. This notion of type was introduced by Shelah for AECs but we use the notation and definitions from [Vas16b, §2]. In particular $\mathrm{gS}^{<\infty}(M)$ denotes the set of Galois types of any length over M. It is defined as:

$$\mathrm{gS}^{<\infty}(M) := \{\mathrm{gtp}(\bar{a}/M; N) \mid M \leq_{\mathbf{K}} N, \bar{a} \in {}^{<\infty}N\}$$

Here, $\mathrm{gtp}(\bar{a}/M; N)$ denotes the Galois types of the sequence $\bar{a}$ over M as computed in N. Loosely speaking, it is defined as the finest notion of type preserving $\mathbf{K}$-embeddings (see [Vas16b, 2.16] for a precise definition). We will also use variations such as $\mathrm{gS}^{\alpha}(M)$ (the length is restricted to be α) or $\mathrm{gS}^{<\infty}(B; N)$ (the base set is B and we only look at types of elements inside N).

3. Stable independence in an arbitrary category

All throughout this section, we assume:

Hypothesis 3.1. We work inside a fixed category $\mathcal{K}$.

The goal of this section is to axiomatize stable amalgams as a particular category of commutative squares in $\mathcal{K}$ (see Definition 3.16). In fact, although we will not stress this perspective, we will identify this notion with a double subcategory of $\mathrm{Sq}(\mathcal{K})$, the usual double category of commutative squares in $\mathcal{K}$.

Definition 3.2. Let $(f_1 : M_0 \to M_1, f_2 : M_0 \to M_2)$ be a span in $\mathcal{K}$.

(1) An *amalgam* of (f_1, f_2) is a cospan $(g_1 : M_1 \to N, g_2 : M_2 \to N)$ such that the following diagram commutes:

$$\begin{tikzcd} M_1 \arrow[r, dotted, "g_1"] & N \\ M_0 \arrow[u, "f_1"] \arrow[r, "f_2"'] & M_2 \arrow[u, dotted, "g_2"'] \end{tikzcd}$$

An *amalgamation diagram* is a quadruple (f_1, f_2, g_1, g_2) such that (f_1, f_2) is a cospan and (g_1, g_2) is an amalgam thereof.

(2) We say that $\mathcal{K}$ has the *amalgamation property* if every span has an amalgam.

(3) Two amalgams $g_1^a : M_1 \to N^a, g_2^a : M_2 \to N^a$, $g_1^b : M_1 \to N^b, g_2^b : M_2 \to N^b$ of (f_1, f_2) are *equivalent* (written $(f_1, f_2, g_1^a, g_2^a) \sim^* (f_1, f_2, g_1^b, g_2^b)$ if there exists N and g^a, g^b making the following diagram commute:

$$\begin{tikzcd}
 & N^b \arrow[r, dotted, "g^b"] & N_* \\
M_1 \arrow[ur, "g_1^b"] \arrow[rr, "g_1^a"] & & N^a \arrow[u, dotted, "g^a"'] \\
M_0 \arrow[u, "f_1"] \arrow[r, "f_2"'] & M_2 \arrow[uu, "g_2^b"] \arrow[ur, "g_2^a"'] &
\end{tikzcd}$$

(4) Let $\sim$ be the transitive closure of $\sim^*$.

Remark 3.3. If $\mathcal{K}$ has the amalgamation property, then $\sim^*$ is already transitive [JS13, 4.3].

The idea of $\sim$ is to identify amalgams whose underlying spans look the same. From this perspective, an independence relation simply consists in the specification of a particular way to amalgamate each span.

Definition 3.4. An *independence relation (on $\mathcal{K}$)* is a set $\mathop{\smile\!\!\!\!\!\!\mid}$ of amalgamation diagrams that is closed under $\sim$. We write $\mathop{\smile\!\!\!\!\!\!\mid}(f_1, f_2, g_1, g_2)$ instead of $(f_1, f_2, g_1, g_2) \in \mathop{\smile\!\!\!\!\!\!\mid}$. We call an amalgamation diagram (f_1, f_2, g_1, g_2) that is in $\mathop{\smile\!\!\!\!\!\!\mid}$ an $\mathop{\smile\!\!\!\!\!\!\mid}$*-independent diagram* (or just an *independent diagram* when $\mathop{\smile\!\!\!\!\!\!\mid}$ is clear from context).

Remark 3.5. We will use the terms *independence relation* and *independence notion* completely interchangeably.

Notation 3.6. We note a certain utility, too, in a diagrammatic representation analogous to that for pullbacks and pushouts; that is, one may wish to represent the assertion $\mathop{\smile\!\!\!\!\!\!\mid}(f_1, f_2, g_1, g_2)$ by the annotated diagram

$$\begin{tikzcd}
M_1 \arrow[r, "g_1"] & N \\
M_0 \arrow[u, "f_1"] \arrow[r, "f_2"'] \arrow[ur, phantom, "\mathop{\smile\!\!\!\!\!\!\mid}"] & M_2 \arrow[u, "g_2"']
\end{tikzcd}$$

This points to the fact that, in axiomatizing abstract independence, we will be delineating the properties of a (double) category of such squares—this is made precise in Definition 3.16 below.

When we are working in a concrete class and the morphisms are simply inclusions of strong structures, we will employ the obvious notational shortcut:

Notation 3.7. For an independence relation $\mathop{\smile\!\!\!\!\!\!\mid}$ on an abstract class $\mathbf{K}$, we write $M_1 \underset{M_0}{\overset{M_3}{\mathop{\smile\!\!\!\!\!\!\mid}}} M_2$ (or $\mathop{\smile\!\!\!\!\!\!\mid}(M_0, M_1, M_2, M_3)$) if $M_0 \leq_{\mathbf{K}} M_\ell \leq_{\mathbf{K}} M_3$ for $\ell = 1, 2$ and $\mathop{\smile\!\!\!\!\!\!\mid}(i_{0,1}, i_{0,2}, i_{1,3}, i_{2,3})$, where $i_{l,k}$ is the $\mathbf{K}$-embedding from M_l to M_k.

Definition 3.8. An independence relation $\mathop{\smile\!\!\!\!\!\!\mid}$ is *invariant* if it is invariant under isomorphisms of amalgamation diagrams (in the expected sense).

One can always switch the left and right hand side of $\mathop{\smile\hspace{-0.9em}|}$ and obtain another independence relation. Thus it is natural to define:

Definition 3.9. For $\mathop{\smile\hspace{-0.9em}|}$ an independence relation on $\mathcal{K}$, we define the *dual* of $\mathop{\smile\hspace{-0.9em}|}$, denoted $\overset{d}{\mathop{\smile\hspace{-0.9em}|}}$ by $\overset{d}{\mathop{\smile\hspace{-0.9em}|}}(f_1, f_2, g_1, g_2)$ if and only if $\mathop{\smile\hspace{-0.9em}|}(f_2, f_1, g_2, g_1)$. We say that $\mathop{\smile\hspace{-0.9em}|}$ is *symmetric* if $\mathop{\smile\hspace{-0.9em}|} = \overset{d}{\mathop{\smile\hspace{-0.9em}|}}$.

The following property is a strengthening of the amalgamation property: it asks that any span have an *independent* amalgam.

Definition 3.10. We say that $\mathop{\smile\hspace{-0.9em}|}$ has the *existence property* (or just *has existence*) if for any cospan (f_1, f_2), there is an amalgam (g_1, g_2) such that $\mathop{\smile\hspace{-0.9em}|}(f_1, f_2, g_1, g_2)$ (so in particular, $\mathcal{K}$ has the amalgamation property).

Remark 3.11. If $\mathop{\smile\hspace{-0.9em}|}$ has the existence property, then $\overset{d}{\mathop{\smile\hspace{-0.9em}|}}$ has the existence property.

The existence property implies that the base is independent over itself. More precisely:

Lemma 3.12. If $\mathop{\smile\hspace{-0.9em}|}$ has the existence property, then any commutative diagram of the form:

$$\begin{tikzcd}
M_0 \arrow[r, "g_1"] & M_3 \\
M_0 \arrow[u, "\mathrm{id}_{M_0}"] \arrow[r, "f_2"'] & M_2 \arrow[u, "g_2"']
\end{tikzcd}$$

is an independent diagram.

Proof. Apply existence to the cospan (id_{M_0}, f_2). We obtain an amalgam $(h_1 : M_0 \to M_3', h_2 : M_2 \to M_3')$ such that $\mathop{\smile\hspace{-0.9em}|}(\mathrm{id}_{M_0}, f_2, h_1, h_2)$. Now (g_1, g_2) is equivalent to (h_1, h_2). Indeed, any amalgam of (g_2, h_2) will witness the equivalence. Since $\mathop{\smile\hspace{-0.9em}|}$ is (by definition of an independence relation) closed under $\sim$, we also have that $\mathop{\smile\hspace{-0.9em}|}(\mathrm{id}_{M_0}, f_2, g_1, g_2)$. $\square$

The statement of the uniqueness property, below, may seem unusual. The idea is that we want every span to have an independent amalgam which is unique, not up to isomorphism (as we always want e.g. to be able to grow the ambient model) but up to equivalence of amalgams:

Definition 3.13. We say that $\mathop{\smile\hspace{-0.9em}|}$ has the *uniqueness property* if whenever $\mathop{\smile\hspace{-0.9em}|}(f_0, f_1, g_1, g_2)$ and $\mathop{\smile\hspace{-0.9em}|}(f_0, f_1, g_1', g_2')$, we have that $(f_0, f_1, g_1, g_2) \sim (f_0, f_1, g_1', g_2')$.

Remark 3.14. If $\mathop{\smile\hspace{-0.9em}|}$ has the uniqueness property, then $\overset{d}{\mathop{\smile\hspace{-0.9em}|}}$ has the uniqueness property.

In order to coherently compose independent squares, the following property is key:

Definition 3.15. Let $\mathop{\smile\mkern-12mu\vert}$ be an independence relation. We say that $\mathop{\smile\mkern-12mu\vert}$ is *right transitive* if whenever we have:

$$\begin{tikzcd}
M_1 \arrow[r, "f_{1,3}"] & M_3 \arrow[r, "f_{3,5}"] & M_5 \\
M_0 \arrow[u, "f_{0,1}"] \arrow[r, "f_{0,2}"'] & M_2 \arrow[u, "f_{2,3}"'] \arrow[r, "f_{2,4}"'] & M_4 \arrow[u, "f_{4,5}"']
\end{tikzcd}$$

with $\mathop{\smile\mkern-12mu\vert}(f_{0,1}, f_{0,2}, f_{1,3}, f_{2,3})$ and $\mathop{\smile\mkern-12mu\vert}(f_{2,3}, f_{2,4}, f_{3,5}, f_{4,5})$, then it is also the case that $\mathop{\smile\mkern-12mu\vert}(f_{0,1}, f_{2,4} \circ f_{0,2}, f_{3,5} \circ f_{1,3}, f_{4,5})$.

We say that $\mathop{\smile\mkern-12mu\vert}$ is *left transitive* if $\overset{d}{\mathop{\smile\mkern-12mu\vert}}$ is right-transitive. We say that $\mathop{\smile\mkern-12mu\vert}$ is *transitive* if it is both left and right transitive.

Diagrammatically, right transitivity means precisely that the situation in the left-most diagram below implies that on the right:

$$\begin{tikzcd}
M_1 \arrow[r, "f_{1,3}"] & M_3 \arrow[r, "f_{3,5}"] & M_5 \\
M_0 \arrow[u, "f_{0,1}"] \arrow[r, "f_{0,2}"'] \arrow[ur, phantom, "\mathop{\smile\mkern-12mu\vert}"] & M_2 \arrow[u, "f_{2,3}"'] \arrow[r, "f_{2,4}"'] \arrow[ur, phantom, "\mathop{\smile\mkern-12mu\vert}"] & M_4 \arrow[u, "f_{4,5}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
M_1 \arrow[r, "f_{3,5}f_{1,3}"] & M_5 \\
M_0 \arrow[u, "f_{0,1}"] \arrow[r, "f_{2,4}f_{0,2}"'] \arrow[ur, phantom, "\mathop{\smile\mkern-12mu\vert}"] & M_4 \arrow[u, "f_{2,3}"']
\end{tikzcd}$$

In other words, right transitivity guarantees that the collection of independent squares is closed under horizontal composition. Dually, left transitivity gives closure under vertical composition. While this (in conjunction with the preceding properties) gives us the structure of double category, we opt instead, for practical reasons, to describe the category of independent squares in more straightforward 1-categorical terms:

Definition 3.16. Given a *right transitive* independence relation $\mathop{\smile\mkern-12mu\vert}$ with existence, define $\mathcal{K}_{\mathrm{NF}}$ (where NF stands for "nonforking") to be the following category:

(1) Its objects are morphisms $f : M_1 \to M_2$ in $\mathcal{K}$.
(2) A morphism in $\mathcal{K}_{NF}$ from $f : M_1 \to M_2$ to $g : N_1 \to N_2$ is a pair $(h_1 : M_1 \to N_1, h_2 : M_2 \to N_2)$ such that the following is an independent diagram:

$$\begin{tikzcd}
M_2 \arrow[r, "h_1"] & N_2 \\
M_1 \arrow[u, "f"] \arrow[r, "h_2"'] & N_1 \arrow[u, "g"']
\end{tikzcd}$$

(3) Composition of morphisms is defined as expected. Note that right transitivity exactly gives that $\mathcal{K}_{\mathrm{NF}}$ is closed under composition and Lemma 3.12 (or really its dual, see Remark 3.11) gives the existence of an identity morphism.

Clearly, $\mathcal{K}_{\mathrm{NF}}$ is a subcategory of the category $\mathcal{K}^2$ of morphisms in $\mathcal{K}$ with the same objects as $\mathcal{K}^2$.

Remark 3.17. Let $\downarrow$ be a right transitive independence relation with the existence property. Then (using Lemma 3.12) $\mathcal{K}$ is isomorphic to a full subcategory of $\mathcal{K}_{\mathrm{NF}}$.

A transitive independence relation with existence and uniqueness is not quite forking-like yet: for this it needs a kind of local character property. We formulate this by requiring that the induced category be accessible:

Definition 3.18. Let $\downarrow$ be an independence relation.

(1) We say that $\downarrow$ is *right accessible* if it is right transitive and $\mathcal{K}_{\mathrm{NF}}$ is accessible (Definition 2.1). We say that $\downarrow$ is *left accessible* if $\overset{d}{\downarrow}$ is right accessible. We say that $\downarrow$ is *accessible* if it is both left and right accessible.
(2) We say that $\downarrow$ is *stable* if it is invariant, symmetric, transitive, accessible, has existence, and has uniqueness.

Remark 3.19. In several examples of interest, $\mathcal{K}_{\mathrm{NF}}$ will actually be an AEC. This is the case for example when $\mathcal{K}$ is the category of models of a stable first-order theory or when $\mathbf{K}$ has an almost fully good independence relation in the sense of [Vas17a, A.2].

While we have made no requirements on the underlying category $\mathcal{K}$, accessibility of $\mathcal{K}_{\mathrm{NF}}$ implies accessibility of $\mathcal{K}$:

Lemma 3.20. If $\downarrow$ is right transitive, right accessible and has existence, then $\mathcal{K}$ is accessible.

Proof. Assume that $\mathcal{K}_{\mathrm{NF}}$ is λ-accessible. At first, we prove that $\mathcal{K}$ has λ-directed colimits. Let $\langle f_{i,j} : M_i \to M_j \mid i,j \in I\rangle$ be a λ-directed diagram in $\mathcal{K}$. We know that $\mathcal{K}_{\mathrm{NF}}$ has λ-directed colimits so (identifying $\mathcal{K}$ with its copy in $\mathcal{K}_{\mathrm{NF}}$, see Remark 3.17), let $\langle (f_i^1, f_i^2) : (\mathrm{id}_{M_i} : M_i \to M_i) \to (g : N_1 \to N_2) \mid i \in I\rangle$ be a colimit in $\mathcal{K}_{\mathrm{NF}}$. We claim that $\langle f_i^2 \mid i \in I\rangle$ is a colimit of the diagram in $\mathcal{K}$. Let $\langle g_i : M_i \to N \mid i \in I\rangle$ be a cocone. Then by Lemma 3.12, $\langle (g_i, g_i) : \mathrm{id}_{M_i} \to \mathrm{id}_N \mid i \in I\rangle$ is a cocone in $\mathcal{K}_{\mathrm{NF}}$. Therefore there must exist a unique morphism (h_1, h_2) such that $h_\ell f_i^\ell = g_i$ for $\ell = 1,2$ and $i \in I$. In particular, $h_2 f_i^2 = g_i$. Moreover, h_2 is unique (in $\mathcal{K}$) with this property: if $hf_i^2 = g_i$ for all $i \in I$, then $hgf_i^1 = g_i$ and, by Lemma 3.12, (hg, h) is a morphism from $g : N_1 \to N_2$ to id_N in $\mathcal{K}_{\mathrm{NF}}$. Hence $h = h_2$.

Let M be an object of $\mathcal{K}$. Then id_M is a λ-directed colimit $(f_i^1, f_1^2) : g_i \to \mathrm{id}_M$ where $g_i : M_i \to N_i$ are λ-presentable in $\mathcal{K}_{\mathrm{NF}}$. In particular, $f_i^2 : N_i \to M$ is a cocone in $\mathcal{K}$. If $h_i : N_i \to M$ is a cocone in $\mathcal{K}$ then $(h_i g_i, g_i) : g_i \to \mathrm{id}_M$ is a cocone in $\mathcal{K}_{\mathrm{NF}}$ and thus there is a unique $t : M \to X$ such that $tf_i^2 = h_i$. Therefore $f_i^2 : N_i \to M$ is a colimit in $\mathcal{K}$. We need to prove that N_i are λ-presentable in $\mathcal{K}$.

Consider a λ-directed colimit $h_j : M_j \to M$ of a diagram $h_{jj'} : M_j \to M_{j'}$ and a morphism $t : L_i \to M$. We get the λ-directed colimit $(h_j, h_j) : \mathrm{id}_{M_j} \to \mathrm{id}_M$ and the morphism $(tg_i, t) : g_i \to \mathrm{id}_M$ in $\mathcal{K}_{\mathrm{NF}}$. Since g_i is λ-presentable in $\mathcal{K}_{\mathrm{NF}}$, (tg_i, t) factorizes through some id_{M_j}. Hence t factorizes through some h_j. It remains to show that this factorization is essentially unique. Let $t = h_{j_1} t_1$ and $t = h_{j_2} t_2$. Then $(tg_i, t) = (h_{j_1}, h_{j_1})(t_1 g_i, t_1)$ and $(tg_i, t), t_2) = (h_{j_2}, h_{j_2})(t_2 g_i, t_2)$.

Since g_i is λ-presentable in $\mathcal{K}_{\mathrm{NF}}$, there is $(h_{j_1 j}, h_{j_2 j})$ such that $(h_{j_1 j}, h_{j_1 j})(t_1 g_i, t_1) = (h_{j_2 j}, h_{j_2 j})(t_2 g_i, t_2)$. Thus $h_{j_1 j} t_1 = h_{j_2 j} t_2$. $\square$

Remark 3.21. The category $\mathcal{K}$ is accessibly embedded (see [AR94, 2.35]) into $\mathcal{K}_{\mathrm{NF}}$. To prove it we expand the proof above and show that $\mathcal{K}$ is closed under λ-directed colimits in $\mathcal{K}_{\mathrm{NF}}$. For this, we have to show that $g : N_1 \to N_2$ is an isomorphism. Since $f_i^1 : M_i \to N_1$ is a cocone of the diagram $f_{ij} : M_i \to M_j$, there is a unique morphism $t : N_2 \to N_1$ such that $t f_i^2 = f_i^1$ for each $i \in I$. Since $(tg, g) : g \to g$ satisfies $(tg, g) \cdot (f_t^1, f_i^2) = (f_i^1, f_i^2)$, we have $tg = \mathrm{id}_{N_1}$. Hence t is the inverse of g.

Note that a nice-enough independence relation will be monotonic in the following sense:

Definition 3.22. Let $\mathop{\smile\!\!\!\!\!\mid}$ be an independence relation. We say that $\mathop{\smile\!\!\!\!\!\mid}$ is *right monotonic* if whenever we have:

$$\begin{tikzcd}
M_1 \arrow[rr, "f_{1,4}"] & & M_4 \\
M_0 \arrow[u, "f_{0,1}"] \arrow[r, "f_{0,2}"'] & M_2 \arrow[r, "f_{2,3}"'] & M_3 \arrow[u, "f_{3,4}"']
\end{tikzcd}$$

if $(f_{0,1}, f_{2,3} \circ f_{0,2}, f_{1,4}, f_{3,4})$ is an independent diagram, then $(f_{0,1}, f_{0,2}, f_{1,4}, f_{3,4} \circ f_{2,3})$ is an independent diagram.

We say that $\mathop{\smile\!\!\!\!\!\mid}$ is *left monotonic* if $\overset{d}{\mathop{\smile\!\!\!\!\!\mid}}$ is right monotonic. We say that $\mathop{\smile\!\!\!\!\!\mid}$ is *monotonic* if it is both left and right monotonic.

Lemma 3.23. Let $\mathop{\smile\!\!\!\!\!\mid}$ be an independence relation with existence and uniqueness. If $\mathop{\smile\!\!\!\!\!\mid}$ is right transitive, then $\mathop{\smile\!\!\!\!\!\mid}$ is right monotonic.

Proof. We start with the diagram from Definition 3.22 and assume that

$$(f_{0,1}, f_{2,3} f_{0,2}, f_{1,4}, f_{3,4})$$

is an independent diagram. Using existence, pick $g_{1,4} : M_1 \to M_4'$, $g_{2,4} : M_2 \to M_4'$ such that $\mathop{\smile\!\!\!\!\!\mid}(f_{0,1}, f_{0,2}, g_{1,4}, g_{2,4})$. Now pick $g_{4,5} : M_4' \to M_5'$, $g_{3,5} : M_3 \to M_5'$ such that $\mathop{\smile\!\!\!\!\!\mid}(g_{2,4}, f_{2,3}, g_{4,5}, g_{3,4})$. By right transitivity, $\mathop{\smile\!\!\!\!\!\mid}(f_{0,1}, f_{2,3} f_{0,2}, g_{4,5} g_{1,4}, g_{3,5})$. By uniqueness, we obtain the following picture:

$$\begin{tikzcd}
 & M_4' \arrow[r, "g_{4,5}"] & M_5' \arrow[r, dotted, "h_5"] & N \\
M_1 \arrow[ur, "g_{1,4}"] \arrow[rr, "f_{1,4}"] & & M_4 \arrow[ur, dotted, "h_4"'] & \\
M_0 \arrow[u, "f_{0,1}"] \arrow[r, "f_{0,2}"'] & M_2 \arrow[uu, "g_{2,4}"] \arrow[r, "f_{2,3}"'] & M_3 \arrow[u, "f_{3,4}"] \arrow[uu, bend right, "g_{3,5}"'] &
\end{tikzcd}$$

This shows that the amalgams $(f_{0,1}, f_{0,2}, g_{1,4}, g_{2,4})$ and $(f_{0,1}, f_{0,2}, f_{1,4}, f_{3,4}f_{2,3})$ are equivalent. Since the first is an independent diagram, the second also is. $\square$

The following variation will also be useful:

Lemma 3.24. Let $\mathop{\smile\hspace{-0.9em}|\hspace{0.5em}}$ be a right monotonic independence relation and consider

$$\begin{tikzcd}
M_1 \arrow[r] & M_3 \arrow[r] & M_5 \\
M_0 \arrow[u] \arrow[r] & M_2 \arrow[u] \arrow[r] & M_4 \arrow[u]
\end{tikzcd}$$

where the outer rectangle is independent. Then the left square is independent.

Proof. Since $\mathop{\smile\hspace{-0.9em}|\hspace{0.5em}}$ is right monotonic, the square

$$\begin{tikzcd}
M_1 \arrow[r] & M_5 \\
M_0 \arrow[u] \arrow[r] & M_2 \arrow[u]
\end{tikzcd}$$

is independent. Since this square is equivalent to the left square, the latter is independent. The equivalence is documented by the diagram

$$\begin{tikzcd}
 & M_3 \arrow[r, dotted] & M_5 \\
M_1 \arrow[ur] \arrow[rr] & & M_5 \arrow[u, dotted] \\
M_0 \arrow[u] \arrow[r] & M_2 \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

$\square$

Remark 3.25. Let $\mathop{\smile\hspace{-0.9em}|\hspace{0.5em}}$ be a right monotonic, right transitive, right accessible independence relation with existence. Then the embedding of $\mathcal{K}_{\mathrm{NF}}$ into $\mathcal{K}^2$ is accessible. In fact, assume that $\mathcal{K}_{\mathrm{NF}}$ is λ-accessible. Consider a λ-directed diagram $D : \mathcal{D} \to \mathcal{K}_{\mathrm{NF}}$. Let $(u_1, u_2) : Dd \to g$ be its colimit in $\mathcal{K}^2$ and $(\bar{u}_1, \bar{u}_2) : Dd \to \bar{g}$ its colimit in $\mathcal{K}_{\mathrm{NF}}$. We get a unique morphism $(t_1, t_2) : g \to \bar{g}$ in $\mathcal{K}^2$. Since the outer rectangle of

$$\begin{tikzcd}
N_i \arrow[r, "u_2"] & N \arrow[r, "t_2"] & \bar{N} \\
M_i \arrow[u, "Dd"] \arrow[r, "u_1"'] & M \arrow[u, "g"'] \arrow[r, "t_1"'] & \bar{M} \arrow[u, "\bar{g}"']
\end{tikzcd}$$

is independent for each $d \in \mathcal{D}$, 3.24 implies that the left squares are independent for each $d \in \mathcal{D}$. Thus g is a colimit of D in $\mathcal{K}_{\mathrm{NF}}$.

We can also define a monotonicity property with respect to the base:

Definition 3.26. Let $\mathop{\smile\!\!\!\!\!\!|}$ be an independence relation. We say that $\mathop{\smile\!\!\!\!\!\!|}$ is *right base-monotonic* if whenever we have:

$$\begin{tikzcd}
M_1 \arrow[rr, "f_{1,4}"] & & M_4 \\
M_0 \arrow[u, "f_{0,1}"] \arrow[r, "f_{0,2}"'] & M_2 \arrow[r, "f_{2,3}"'] & M_3 \arrow[u, "f_{3,4}"']
\end{tikzcd}$$

and the outer square $(f_{0,1}, f_{2,3}f_{0,2}, f_{1,4}, f_{3,4})$ is independent, then there exists $f_{4,4'} : M_4 \to M_4'$ and $f_{1,1'} : M_1 \to M_1'$, $f_{2,1'} : M_2 \to M_1'$, $f_{1',4'} : M_1' \to M_4'$ such that $(f_{2,1'}, f_{2,3}, f_{1',4'}, f_{4,4'} \circ f_{3,4})$ is independent and the diagram below commutes:

$$\begin{tikzcd}
& M_1' \arrow[r, dotted, "f_{1',4'}"'] & M_4' \\
M_1 \arrow[ur, dotted, "f_{1,1'}"] \arrow[rr, "f_{1,4}"] & & M_4 \arrow[u, dotted, "f_{4,4'}"'] \\
M_0 \arrow[u, "f_{0,1}"] \arrow[r, "f_{0,2}"'] & M_2 \arrow[uu, dotted, "f_{2,1'}"' near end] \arrow[r, "f_{2,3}"'] & M_3 \arrow[u, "f_{3,4}"']
\end{tikzcd}$$

As usual, $\mathop{\smile\!\!\!\!\!\!|}$ left base-monotonic means that $\overset{d}{\mathop{\smile\!\!\!\!\!\!|}}$ is right base-monotonic, and base-monotonic means both left and right.

Any stable independence relation is base-monotonic. More precisely:

Lemma 3.27. If $\mathop{\smile\!\!\!\!\!\!|}$ is right transitive and has uniqueness and existence, then $\mathop{\smile\!\!\!\!\!\!|}$ is right base-monotonic.

Proof. This is similar to (E)(b) in [She09a, III.9.6]. In detail, start with the setup of Definition 3.26. By Lemma 3.23, the square $(f_{0,1}, f_{0,2}, f_{1,4}, f_{3,4}f_{2,3})$ is independent. By existence, find $f_{3,3'} : M_3 \to M_3'$ and $f_{4,3'} : M_4 \to M_4'$ such that the square $(f_{2,4}, f_{2,3}, f_{4,3'}, f_{3,3'})$ is independent. By right transitivity, the square $(f_{0,1}, f_{2,3}f_{0,2}, f_{4,3'}f_{1,4}, f_{3,3'})$ is independent. By uniqueness, this square is equivalent to $(f_{0,1}, f_{2,3}f_{0,2}, f_{1,4}, f_{3,4}f_{2,3})$. Amalgamating these two squares, we obtain the desired diagram. □

Let us illustrate what a stable independence relation looks like by considering accessible categories with weak *polycolimits*:

Definition 3.28. A *polyinitial* object is a set $\mathcal{I}$ of objects of a category $\mathcal{K}$ such that for every object M in $\mathcal{K}$:

(1) There is a unique $i \in \mathcal{I}$ having a morphism $i \to M$.

(2) For each $i \in \mathcal{I}$, given $f, g : i \to M$, there is a unique (isomorphism) $h : i \to i$ with $fh = g$.

The *polycolimit* of a diagram D in a category $\mathcal{K}$ is a polyinitial object in the category of cones on D, i.e. a set of cones such that for any cone on D, there will be a unique induced map from exactly one of the members of the set. The *weak polycolimit* of D is defined similarly, except that we waive the uniqueness requirement on the map.

For example, the algebraic closures of the prime fields form a polyinitial object in the category of algebraically closed fields.

We recall, too, the related (and, perhaps, more familiar) concept of a multicolimit, again derived from the notion of a multinitial set of objects: a multiinitial set of objects in a category $\mathcal{K}$ is given by adding replacing (2) above—uniqueness of the morphism $i \to M$ up to isomorphism—by actual uniqueness. That is, for each $M \in \mathcal{K}$, there is a unique $i \in \mathcal{I}$ and a unique $i \to M$.

In the category of fields, for example, the prime fields form a multiinitial object.

We say, incidentally, that a category is locally λ-polypresentable (resp. locally λ-multipresentable) if it is λ-accessible and has all polycolimits (resp. multicolimits). Note that locally λ-presentable implies locally λ-multipresentable implies locally λ-polypresentable implies λ-accessible.

Remark 3.29.

(1) Let $\mathop{\smile\kern-0.8em\mid}$ be a stable independence relation in an accessible category $\mathcal{K}$ with weak polycolimits and all morphisms monomorphisms (for example in a μ-AEC admitting intersections, see [LRVb, 2.4,5.7]). Consider a span $s = (f_1, f_2)$ and its weak polypushout $(p_{i,1}, p_{i,2})$, $i \in I_s$ (note that in general I_s may be empty but here it is not: the existence property of $\mathop{\smile\kern-0.8em\mid}$ implies amalgamation). This means that we have commutative squares

$$\begin{tikzcd} M_1 \arrow[r, "p_{i,2}"] & P_i \\ M_0 \arrow[u, "f_1"] \arrow[r, "f_2"'] & M_2 \arrow[u, "p_{i,1}"'] \end{tikzcd}$$

such that for any commutative square

$$\begin{tikzcd} M_1 \arrow[r, "g_2"] & M \\ M_0 \arrow[u, "f_1"] \arrow[r, "f_2"'] & M_2 \arrow[u, "g_1"'] \end{tikzcd}$$

there exist a unique $i \in I_s$ and (not necessarily unique) $t : P_i \to M$ such that $tp_{i,1} = g_1$ and $tp_{i,2} = g_2$. By closure under $\sim$ and the uniqueness property, there is exactly one $i \in I_s$ such that $M_1 \mathop{\smile\kern-0.8em\mid}_{M_0}^{P_i} M_2$. By monotonicity, these choices are coherent in the sense that given a morphism of spans

$(\mathrm{id}_{M_0}, h_1, h_2) : s \to s'$, we get an induced morphism $P_i \to P_{i'}$ where i is chosen from I_s and i' from $I_{s'}$. For a general object M, the characterization of the relation $M_1 \underset{M_0}{\overset{M}{\downarrow}} M_2$ is by closure under $\sim$: the relation holds if and only if the induced map from the weak polypushout is $P_i \to M$ for the unique P_i such that $M_1 \underset{M_0}{\overset{P_i}{\downarrow}} M_2$.

On the other hand, a relation $\downarrow$ given by a coherent choice of weak polypushouts satisfies all axioms of stable independence with the possible exception of accessibility (and existence, if $\mathcal{K}$ does not have amalgamation). We will see (Theorem 9.1) that there is at most one coherent choice of weak polypushouts giving a stable independence relation.

(2) One cannot expect that an accessible category with weak polycolimits and all morphisms monomorphisms will have pushouts. In this case, we might try to get a stable independence relation by taking all commutative squares

$$\begin{tikzcd} M_1 \arrow[r] & M_3 \\ M_0 \arrow[u] \arrow[r] & M_2 \arrow[u] \end{tikzcd}$$

Without the existence of pushouts, this relation satisfies all axioms of a stable independence relation except the uniqueness axiom.

We note that if $\mathcal{K}$ is an accessible category with multicolimits, pushouts and all morphisms monomorphisms, it is small: consider an object M, the instance O of a multiinitial object with the morphism $O \to M$ and the pushout

$$\begin{tikzcd} M \arrow[r] & M \coprod M \\ O \arrow[u] \arrow[r] & M \arrow[u] \end{tikzcd}$$

Since the codiagonal $M \coprod M \to M$ is a split epimorphism, it is an isomorphism and thus the two coproduct injections $M \to M \coprod M$ are equal. Consequently, for any N in $\mathcal{K}$, M has at most one morphism $M \to N$. Thus $\mathcal{K}$ is thin and, since it is an accessible category, it must therefore be small.

We end this section by giving several examples and non-examples:

Example 3.30.

(1) If $\mathbf{K}$ is the AEC of all sets (ordered with substructure), then $M_1 \underset{M_0}{\overset{M_3}{\downarrow}} M_2$ if and only if $M_0 \subseteq M_\ell \subseteq M_3$, $\ell = 1, 2$, and $M_1 \cap M_2 = M_0$ is a stable independence relation.

(2) For F a field, if $\mathbf{K}$ is the AEC of all F-vector spaces (ordered with subspace), then $M_1 \overset{M_3}{\underset{M_0}{\mathop{\smile\!\!\!\!\!\!|}}} M_2$ if and only if $M_0 \subseteq M_\ell \subseteq M_3$, $\ell = 1, 2$, and $M_1 \cap M_2 = M_0$ is a stable independence relation. We will see that these two examples are instances of Theorem 5.1, which gives a general condition on when pullback induces a stable independence notion.

(3) If $\mathbf{K}$ is the AEC of all algebraically closed fields of characteristic p (for p a fixed prime or 0) ordered by subfield, then $M_1 \overset{M_3}{\underset{M_0}{\mathop{\smile\!\!\!\!\!\!|}}} M_2$ if and only if $M_0 \leq_{\mathbf{K}} M_\ell \leq_{\mathbf{K}} M_3$ and for any finite $A \subseteq |M_1|$, and any $a \in |M_1|$, the transcendence degree of a over AM_2 is the same as the transcendence degree of a over AM_0 defines a stable independence notion. Note however that $M_0 \leq_{\mathbf{K}} M_\ell \leq_{\mathbf{K}} M_3$ and $M_1 \cap M_2 = M_0$ does *not* imply $M_1 \overset{M_3}{\underset{M_0}{\mathop{\smile\!\!\!\!\!\!|}}} M_2$. This is because the pregeometry induced by algebraic closure is not modular. Nevertheless, $\mathbf{K}$ has weak polycolimits, hence we are in the setup described by Remark 3.29(1).

(4) Let T be a stable first-order theory. Write $B \overset{M}{\underset{A}{\mathop{\smile\!\!\!\!\!\!|}}} C$ if and only if $A, B, C \subseteq UM$, $M \models T$, and $\text{tp}(\bar{b}/AC; M)$ does not fork over A (in the original sense of Shelah, see [She90, III.1.4]), where $\bar{b}$ is any enumeration of B and $\text{tp}(\bar{b}/AC; M)$ is the set of first-order formulas with parameters from AC satisfied by $\bar{b}$ in M. There is a certain expansion T^{eq} of T which "eliminates imaginaries" in the sense that for every definable equivalence relation E there is a definable function F_E sending two E-equivalent elements to the same object (i.e. the equivalence classes of E are also elements), see [She90, §III.6]. Expanding to T^{eq} leads to an isomorphic category of models, so without loss of generality $T = T^{eq}$. Then it is known that $B \overset{M}{\underset{A}{\mathop{\smile\!\!\!\!\!\!|}}} C$ if and only if $\text{acl}(AB) \overset{M}{\underset{\text{acl}(A)}{\mathop{\smile\!\!\!\!\!\!|}}} \text{acl}(AC)$, where $\text{acl}(A)$ denotes the set of elements b in M for which there is a formula $\phi(x)$ with parameters from A with only finitely many solutions, one of which is b. Thus it suffices to consider forking over algebraically closed sets. Let $\mathbf{K}$ be the AEC of algebraically closed sets in T, ordered by being a substructure (this is [She09b, V.B.2.1]). Then $\mathop{\smile\!\!\!\!\!\!|}$ induces a stable independence relation on $\mathbf{K}$.

(5) Let $\mathbf{K}$ be the AEC of all graphs (ordered by subgraphs). Define $M_1 \overset{M_3}{\underset{M_0}{\mathop{\smile\!\!\!\!\!\!|}}} M_2$ to hold if and only if $M_0 \subseteq M_\ell \subseteq M_3$, $\ell = 1, 2$, $M_1 \cap M_2 = M_0$, and there are no edges of M_3 going from $M_1 \backslash M_0$ to $M_2 \backslash M_0$. Then $\mathop{\smile\!\!\!\!\!\!|}$ satisfies all the axioms of a stable independence relation except accessibility. One can define a different independence notion (that will similarly satisfy all the axioms except accessibility) similarly by requiring instead that *all* possible cross edges are present between $M_1 \backslash M_0$ and $M_2 \backslash M_0$. This is [She09a, II.6.4], see also [BGKV16, 4.15].

(6) Let $\mathbf{K}$ be the AEC of all graphs, as before. One can modify $\mathop{\smile\!\!\!\!\!\!|}$ so that it satisfies existence, uniqueness, and monotonicity but not transitivity: say $M_1 \underset{M_0}{\overset{M_3}{\mathop{\smile\!\!\!\!\!\!|}}} M_2$ holds if and only if $M_0 \subseteq M_\ell \subseteq M_3$, $\ell = 1, 2$, and:
   (a) If M_0 is a finite graph, then no cross edges are present between $M_1 \backslash M_0$ and $M_2 \backslash M_0$.
   (b) If M_0 is infinite, all possible cross edges are present between $M_1 \backslash M_0$ and $M_2 \backslash M_0$.

   One can similarly construct a non-monotonic independence relation with existence and uniqueness that fails monotonicity (use the isomorphism type of M_2 to decide whether all or no cross edges should be included). It is also easy to construct examples that have existence, but not uniqueness, or uniqueness but not existence, see [BGKV16, 4.16].

4. Coregular categories and effective unions

Recall that a monomorphism [resp. epimorphism] in a category is *regular* if it is an equalizer [resp. coequalizer] of a pair of morphisms (see [AHS04, 7.56]). Let $\mathcal{K}_{reg}$ be the category having the same objects as $\mathcal{K}$ and regular monomorphisms as morphisms.

Remark 4.1. In μ-AECs with disjoint amalgamation (and more generally in any category where spans can be completed to pullback squares), every monomorphism is regular.

A category $\mathcal{K}$ is called *regular* if it has finite limits, coequalizers of kernel pairs and regular epimorphisms are stable under pullbacks (see [Bar71]). Coregularity is the dual of this notion. So, in particular, a locally presentable category $\mathcal{K}$ is coregular if and only if regular monomorphisms are stable under pushouts. In this case, we have the factorization system (epimorphism, regular monomorphisms) in $\mathcal{K}$. This implies that $\mathcal{K}_{reg}$ is quite a well-behaved category:

Lemma 4.2. Let $\mathcal{K}$ be a coregular locally λ-presentable category. Then $\mathcal{K}_{reg}$ is locally λ-multipresentable.

Proof. Following [AR94, 1.62], $\mathcal{K}_{reg}$ is closed under λ-directed colimits in $\mathcal{K}$. Since λ-pure monomorphisms are regular ([AR94, 2.31]), $\mathcal{K}_{reg}$ is accessible by [AR94, 2.34]. $\mathcal{K}_{reg}$ is clearly closed under equalizers in $\mathcal{K}$. Let $p_i : P \to M_i$ be a wide pullback of regular monomorphisms $f_i : M_i \to M$. Since regular monomorphisms form the right part of a factorization system, the composition $f_i p_i$ is a regular monomorphism. For the same reason, the p_i are regular monomorphisms. Thus $\mathcal{K}_{reg}$ has connected colimits and, since it is locally λ-presentable and therefore λ-accessible, it is locally λ-multipresentable. $\square$

We will use the following important fact:

Fact 4.3 ([Rin72])**.** In a coregular locally presentable category, pushouts of regular monomorphisms are pullbacks.

In fact, this is valid in any category where pushouts of regular monomorphisms are regular monomorphisms.

Note, however, that pushouts of regular monomorphisms in $\mathcal{K}$ need not be pushouts in $\mathcal{K}_{reg}$—the induced map from the pushout will not even be a monomorphism, in general, let alone a regular monomorphism. To ask that this be the case is precisely to ask that $\mathcal{K}$ have *effective unions*. The existence of effective unions is an exactness property (introduced by Barr [Bar88]) satisfied by both Grothendieck abelian categories and Grothendieck toposes and which is strong enough to ensure the existence of enough injective objects—see Remark 4.6 below.

Definition 4.4. A locally presentable category $\mathcal{K}$ *has effective unions* if whenever we have a pullback

$$\begin{tikzcd} M_1 \arrow[r] & M_3 \\ M_0 \arrow[u] \arrow[r] & M_2 \arrow[u] \end{tikzcd}$$

in $\mathcal{K}_{reg}$, and the pushout

$$\begin{tikzcd} M_1 \arrow[r] & P \\ M_0 \arrow[u] \arrow[r] & M_2 \arrow[u] \end{tikzcd}$$

in $\mathcal{K}$, the induced morphism $P \to M_3$ is a regular monomorphism.

Remark 4.5. If $\mathcal{K}$ is a coregular locally presentable category then any morphism in $\mathcal{K}_{reg}$ is a regular monomorphism. This follows from the fact that pushouts squares there are pullbacks (Fact 4.3) and Remark 4.1.

Remark 4.6. Effective unions are closely tied to the existence of enough injectives (see e.g. Chapter 4 in [AR94]): let $\mathcal{K}$ be a coregular locally presentable category such that regular monomorphisms are closed under directed colimits (in the sense of [Ros, 2.1(2)]). Then $\mathcal{K}_{reg}$ has directed colimits. Moreover, if $\mathcal{K}$ has effective unions then regular monomorphisms are cofibrantly generated (in the sense given at the beginning of [Ros, p. 7]; the proof is the same as that of [Bek00, 1.12]). In particular, there is a set $\mathcal{S}$ of regular monomorphisms such that an object injective with respect to any $s \in \mathcal{S}$ is injective to all regular monomorphisms. Consequently, these injective objects form an accessible category (see [AR94, 4.7]).

Example 4.7.

(1) Both Grothendieck toposes and Grothendieck abelian categories are locally presentable coregular categories having effective unions and regular monomorphisms closed under directed colimits (see [Bar88]). In particular, they include categories of R-modules and presheaf categories $\mathbf{Set}^{\mathcal{X}}$. The latter include the category of multigraphs.
(2) The category **Gra** of graphs is locally presentable and coregular but it does not have effective unions. It suffices to take the pullback of two vertices of

an edge where the pushout are two vertices without any edge. Thus the embedding of this pushout to the edge is not regular (it would be regular in multigraphs). See also Example 3.30(5).

(3) The category of groups and the category of Boolean algebras are locally presentable and coregular. Regular monomorphisms coincide with monomorphisms and are closed under directed colimits. But groups do not have enough injectives, and the injectives in Boolean algebras are complete Boolean algebras: these do not form an accessible category. Thus neither groups nor Boolean algebras have effective unions (see Remark 4.6).

(4) The category **Ban** of Banach spaces (with linear contractions) is locally presentable and coregular. The regular monomorphisms are isometries and are closed under directed colimits. However **Ban** does not have effective unions. We give two proofs:
   - By Remark 4.6, regular monomorphisms would be cofibrantly generated. This contradicts [Ros, 3.1(2)].
   - If **Ban** had effective unions, the corresponding μ-AEC $\mathbf{K}$ of Banach spaces with isometries would have a stable independence notion by Theorem 5.1. By Corollary 9.8, this means that $\mathbf{K}$ does not have the order property (see Definition 9.6). Take however the Banach space c_0 of complex-valued sequences $\langle a_n : n < \omega \rangle$ going to zero with the supremum norm. Let e_n be the sequence that is one at position n and 0 elsewhere. Let $f_n := \sum_{i \leq n} e_i$. Then $\|e_m + f_n\| = 2$ if and only if $m \leq n$ (see [KM81]). Thus c_0 satisfies an instance of the order property of length ω. By the compactness theorem for continuous first-order logic (see e.g. [BYBHU08]), this implies that $\mathbf{K}$ must have the order property. Contradiction.

(5) The category **Hilb** of Hilbert spaces (with linear isometries) is accessible. Recall that Hilbert spaces are precisely the Banach spaces whose norm satisfies the parallelogram identity. By the note above, **Hilb** is a (full) subcategory of $\mathbf{Ban}_{reg}$. Pullbacks in **Hilb** exist and are calculated as for Banach spaces. If $f : V \to W$ is a morphism in **Hilb** then $W \cong V \oplus V^\perp$ where $V^\perp$ is the orthogonal complement of $f(V)$ in W. Thus a typical span of maps consists of $f : V \to V \oplus W_1$ and $g : V \to V \oplus W_2$, with pushout $V \oplus W_1 \oplus W_2$. Since a general pullback has the form

$$\begin{tikzcd}
V \oplus W_1 \arrow[r] & V \oplus W_1 \oplus W_2 \oplus W_3 \\
V \arrow[u] \arrow[r] & V \oplus W_2 \arrow[u]
\end{tikzcd}$$

and the induced map $V \oplus W_1 \oplus W_2 \to V \oplus W_1 \oplus W_2 \oplus W_3$ is clearly an equalizer, **Hilb** has effective unions.

5. Stable independence and effective unions

Theorem 5.1. Let $\mathcal{K}$ be a coregular locally presentable category. If $\mathcal{K}$ has effective unions, then $\mathcal{K}_{reg}$ has a stable independence relation.

Proof. Put $M_1 \underset{M_0}{\overset{M_3}{\smile\!\!\!\!\!\mid}} M_2$ if and only if the square

$$\begin{tikzcd} M_1 \arrow[r] & M_3 \\ M_0 \arrow[u] \arrow[r] & M_2 \arrow[u] \end{tikzcd}$$

is a pullback. We check that this satisfies all the axioms of stable independence. Invariance under isomorphisms is evident. If $M_1 \underset{M_0}{\overset{M_3}{\smile\!\!\!\!\!\mid}} M_2$ and $M_3 \to M_3'$ then $M_1 \underset{M_0}{\overset{M_3'}{\smile\!\!\!\!\!\mid}} M_2$. Having a commutative square

$$\begin{tikzcd} M_1 \arrow[r] & M_3'' \\ M_0 \arrow[u] \arrow[r] & M_2 \arrow[u] \end{tikzcd}$$

and $M_3'' \to M_3$ such that $M_1 \underset{M_0}{\overset{M_3}{\smile\!\!\!\!\!\mid}} M_2$ then $M_1 \underset{M_0}{\overset{M_3''}{\smile\!\!\!\!\!\mid}} M_2$. This shows that $\smile\!\!\!\!\!\mid$ is closed under equivalence of amalgams.

Given a span $M_1 \leftarrow M_0 \to M_2$, we form the pushout

$$\begin{tikzcd} M_1 \arrow[r] & M_3 \\ M_0 \arrow[u] \arrow[r] & M_2 \arrow[u] \end{tikzcd}$$

Following Fact 4.3, this square is a pullback. Thus $M_1 \underset{M_0}{\overset{M_3}{\smile\!\!\!\!\!\mid}} M_2$, which yields the existence property.

In order to prove the uniqueness property, consider $M_1 \underset{M_0}{\overset{M_3'}{\smile\!\!\!\!\!\mid}} M_2$ and $M_1 \underset{M_0}{\overset{M_3''}{\smile\!\!\!\!\!\mid}} M_2$ with the same span $M_1 \leftarrow M_0 \to M_2$. Form the pushout

$$\begin{tikzcd} M_1 \arrow[r] & P \\ M_0 \arrow[u] \arrow[r] & M_2 \arrow[u] \end{tikzcd}$$

and take the induced regular monomorphisms $P \to M_3'$ and $P \to M_3''$. Then the pushout

$$\begin{tikzcd} M_3' \arrow[r] & M_3 \\ P \arrow[u] \arrow[r] & M_3'' \arrow[u] \end{tikzcd}$$

amalgamates the starting pullbacks.

The transitivity and symmetry properties are also easy to check. We can prove accessibility directly or use the soon to be proven Theorem 8.13: local character holds since the nonforking base is simply given by the pullback, and the witness property is also a straightforward consequence of the definition. $\square$

Remark 5.2.

(1) We note that the assumption of local presentability is largely a matter of convenience, and the argument will go through under a weaker assumption. In particular, the category need only be accessible with pullbacks and pushouts of monomorphisms.

(2) Effective unions were only needed for the uniqueness property. On the other hand, if the independence relation defined by being a pullback square satisfies uniqueness, then the category has effective unions. Consider $M_1 \underset{M_0}{\overset{M_3}{\downarrow}} M_2$ and form the pushout

$$\begin{tikzcd} M_1 \arrow[r] & P \\ M_0 \arrow[u] \arrow[r] & M_2 \arrow[u] \end{tikzcd}$$

Since $M_1 \underset{M_0}{\overset{P}{\downarrow}} M_2$ (by 4.3), we have an amalgam $M_3 \to M \leftarrow P$ whose right leg is the composition $P \to M_3 \to M$. Thus $P \to M_3$ is a regular monomorphism. However we do *not* know whether the converse of Theorem 5.1 holds; that is, whether the existence of a stable independence relation in $\mathcal{K}_{reg}$ implies that $\mathcal{K}$ has effective unions. Put another way, we do not know whether there is an example of a coregular locally presentable category such that $\mathcal{K}_{reg}$ has a stable independence notion which is *not* given by pullbacks. We show in Theorem 10.6 that in any μ-AEC, independent squares over sufficiently saturated models are pullback.

(3) Let $\mathcal{K}$ be a coregular locally presentable category having effective unions. Then in $\mathcal{K}_{reg}$, given a span (f_1, f_2), exactly one instance of a multipushout of f_1 and f_2 is a pullback. This follows from Remark 3.29 and Theorem 5.1.

Example 5.3. Both Grothendieck toposes and Grothendieck abelian categories have stable independence relations. This follows from Theorem 5.1 and Example 4.7(1) and subsumes 3.30(1) and (2). In fact, (2) is valid for R-modules in general.

This was known already for complete first-order theories of modules (see [Pre88]): we have produced an alternate proof of this fact. Similarly, the category of Hilbert spaces (with linear isometries) has a stable independence notion—as **Hilb** is accessible but not locally presentable, we must actually invoke the weakening mentioned in 5.1(1) above. This was also known from the model-theoretic analysis of the corresponding continuous first-order theory, see for example [Iov99].

In the rest of this section, we show that without effective unions, we can still define an independence notion that satisfies all the axioms of stable independence except accessibility.

Definition 5.4. Let $\mathcal{K}$ be a coregular locally presentable category. Define $\overset{*}{\downarrow}$ by taking commutative squares in $\mathcal{K}_{reg}$

$$\begin{tikzcd} M_1 \arrow[r] & M_3 \\ M_0 \arrow[u] \arrow[r] & M_2 \arrow[u] \end{tikzcd}$$

such that the induced morphism from the pushout

$$\begin{tikzcd} M_1 \arrow[r] & P \\ M_0 \arrow[u] \arrow[r] & M_2 \arrow[u] \end{tikzcd}$$

is a regular monomorphism.

Remark 5.5. These are precisely *effective pullback squares*

$$\begin{tikzcd} M_1 \arrow[r] & M_3 \\ M_0 \arrow[u] \arrow[r] & M_2 \arrow[u] \end{tikzcd}$$

in the sense that the unique morphism $P \to M_3$ is a regular monomorphism. In **Gra**, $\overset{*}{\downarrow}$ is the relation $\downarrow$ mentioned in Example 3.30(5).

If $\mathcal{K}$ has effective unions then $\downarrow = \overset{*}{\downarrow}$ where $\downarrow$ is from Theorem 5.1. This uses the fact that pushouts in $\mathcal{K}$ are pullbacks.

Moreover, the next result together with (Theorem 9.1) implies that that whenever $\downarrow$ is *any* stable independence relation on $\mathcal{K}_{reg}$ then $\downarrow = \overset{*}{\downarrow}$.

Theorem 5.6. Let $\mathcal{K}$ be a coregular locally presentable categories and let $\overset{*}{\downarrow}$ be as in Definition 5.4.

Then:

(1) $\overset{*}{\downfree} \subseteq \downfree$ (where $\downfree$ is from Definition 5.1).
(2) $\overset{*}{\downfree}$ is invariant, monotonic, symmetric, transitive, has existence and uniqueness.
(3) If $\mathcal{K}$ is locally λ-presentable then the category $\mathcal{L}$ of $\overset{*}{\downfree}$-independent squares in $\mathcal{K}_{reg}$ (from Definition 3.16) is closed under λ-directed colimits in $\mathcal{K}^2$.
(4) If $\mathcal{K}$ is locally λ-presentable, $\mathcal{K}^{\square}$ is the category of commutative squares in $\mathcal{K}$, and $\mathcal{K}^{\boxdot}$ is the category of effective pullback squares in $\mathcal{K}$, then $\mathcal{K}^{\boxdot}$ is λ-accessible with λ-presentable objects being squares of λ-presentable objects in $\mathcal{K}$.

Proof.

(1) Straightforward.
(2) The properties of $\overset{*}{\downfree}$ are proven as in the proof of Theorem 5.1, except for transitivity. To prove transitivity, consider:

$$\begin{tikzcd} M_1 \arrow[r] & M_3 \arrow[r] & M_5 \\ M_0 \arrow[u] \arrow[r] & M_2 \arrow[u] \arrow[r] & M_4 \arrow[u] \end{tikzcd}$$

where both squares are effective pullbacks. We have to show that the outer rectangle is an effective pullback. Thus we have to show that the induced morphism $p : P \to M_5$ from the pushout

$$\begin{tikzcd} M_1 \arrow[r] & P \\ M_0 \arrow[u] \arrow[r] & M_4 \arrow[u] \end{tikzcd}$$

is a regular monomorphism. This pushout is a composition of pushouts

$$\begin{tikzcd} M_1 \arrow[r] & Q \arrow[r] & P \\ M_0 \arrow[u] \arrow[r] & M_2 \arrow[u] \arrow[r] & M_4 \arrow[u] \end{tikzcd}$$

Consider the pushout

$$\begin{tikzcd} M_3 \arrow[r] & P' \\ Q \arrow[u, "q"] \arrow[r] & P \arrow[u, "\bar{q}"'] \end{tikzcd}$$

where $q : Q \to M_3$ is the induced morphism. Since the left pullback above is effective, q is a regular monomorphism and thus $\bar{q}$ is a regular monomorphism. Since

$$\begin{tikzcd} M_3 \arrow[r] & P' \\ M_2 \arrow[u] \arrow[r] & M_4 \arrow[u] \end{tikzcd}$$

is a pushout and the right pullback above is effective, the induced morphism $p' : P' \to M_5$ is a regular monomorphism. Thus $p = p'\bar{q}$ is a regular monomorphism. Conversely, if the outer rectangle and right square in the proof above are effective pullbacks then the left square is an effective pullback. This follows directly from monotonicity. In fact, the composition of q with $M_3 \to M_5$ is a regular monomorphism as the composition of $Q \to P$ with p. Thus q is a regular monomorphism (see [Ros, 2.1(1)]).

(3) Let $D : \mathcal{D} \to \mathcal{L}$ be a λ-directed diagram where Dd is $f_d : M_d \to N_d$. Let $f : M \to N$ be its colimit in $\mathcal{K}^2$. For each $d \in \mathcal{D}$, the square

$$\begin{tikzcd} M \arrow[r, "f"] & N \\ M_d \arrow[u] \arrow[r, "f_d"'] & N_d \arrow[u] \end{tikzcd}$$

is a pullback because it is a λ-directed colimit of pullbacks

$$\begin{tikzcd} M'_d \arrow[r, "f_{d'}"] & N'_d \\ M_d \arrow[u] \arrow[r, "f_d"'] & N_d \arrow[u] \end{tikzcd}$$

and pullbacks commute with λ-directed colimits in $\mathcal{K}$ (see [AR94] 1.59). Analogously, the pushout

$$\begin{tikzcd} M \arrow[r, "g"] & P \\ M_d \arrow[u] \arrow[r, "f_d"'] & N_d \arrow[u] \end{tikzcd}$$

is a λ-directed colimit of pushouts

$$\begin{tikzcd} M'_d \arrow[r, "g_{d'}"] & P_{d'} \\ M_d \arrow[u] \arrow[r, "f_d"'] & N_d \arrow[u] \end{tikzcd}$$

Thus the induced morphism $p : P \to N$ is a λ-directed colimit of induced morphisms $p_{d'} : P_{d'} \to N_{d'}$. Hence p is a regular monomorphism.

(4) The category $\mathcal{K}^{\boxdot}$ is is accessible and accessibly embedded into the category $\mathcal{K}^{\square}$ because it is given by the following sketch (see [AR94] 2.60)

$$\begin{tikzcd}
A \arrow[rr] \arrow[dd] & & B \arrow[dl] \arrow[dd, "u"] \\
& P \arrow[dr, "p"] & \\
C \arrow[ur] \arrow[rr, "v"'] & & D
\end{tikzcd}$$

where the outer rectangle is a pullback, the inner quadrangle is a pushout and u, v, p are regular monomorphisms. There is a regular cardinal $\mu \triangleright \lambda$ such that $\mathcal{K}^{\boxdot}$ is μ-accessible and the inclusion $G : \mathcal{K}^{\boxdot} \to \mathcal{K}^{\square}$ preserves μ-directed colimits and μ-presentable objects.

$\square$

Remark 5.7. In fact, $\overset{*}{\downfree}$ even has the witness property from Definition 8.7. To see this, use that any locally multipresentable category whose morphisms are monomorphisms is fully tame and short (by the equivalence from [LRVb, 5.9] and the proof of Boney's theorem that universal classes are tame, see [Vas17a, 3.7]), and hence Fact 8.8 applies.

6. From accessible category to μ-AEC

In Lemma 4.2, we examined the relationship between a certain accessible category $\mathcal{K}$ and the category $\mathcal{K}_{reg}$ obtained by restricting to the regular monomorphisms. In this section we prove several more results along these lines, both in connection with $\mathcal{K}_{reg}$ and with $\mathcal{K}_{mono}$, where the morphisms are taken to be all monomorphisms in the original category $\mathcal{K}$. The latter is particularly useful in light of Fact 2.3: accessible categories with all morphisms monomorphisms are μ-AECs, and vice versa. This provides the essential Rosetta stone that allows us to translate the uniqueness and canonicity results derived for μ-AECs below back to the abstract categorical framework. We deduce in particular (using the corresponding result of the third author for universal classes [Vas17a, Vas17b]) that the eventual categoricity conjecture holds for all locally $\aleph_0$-multipresentable categories (Corollary 6.5).

For a category $\mathcal{K}$, let $\mathcal{K}_{mono}$ be the category having the same objects as $\mathcal{K}$ and monomorphisms as morphisms.

Lemma 6.1. Let $\mathcal{K}$ be a λ-accessible category. Then $\mathcal{K}_{mono}$ is accessible and has λ-directed colimits.

Proof. Let $\mathcal{A}$ be a (representative) full subcategory of $\mathcal{K}$ consisting of λ-presentable objects and

$$E : \mathcal{K} \to \mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}$$

be the canonical functor (see [AR94, 1.25]). Then E is a full embedding preserving λ-directed colimits (see [AR94, 1.26]). Clearly, E preserves monomorphisms too. Thus monomorphisms in $\mathcal{K}$ are stable under λ-directed colimits because this is true in $\mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}$ (see [AR94, 1.60]). That is, given λ-directed diagrams $D, D' : \mathcal{D} \to \mathcal{K}$ and a natural monotransformation $\delta : D \to D'$, then $\operatorname{colim} \delta : \operatorname{colim} D \to \operatorname{colim} D'$ is a monomorphism. Consequently, $\mathcal{K}_{mono}$ is closed under λ-directed colimits in $\mathcal{K}$. That is, given a λ-directed colimit of monomorphisms in $\mathcal{K}$, (i) the colimit cocone consists of monomorphisms, and (ii) for every cocone of monomorphisms the factoring morphism is a monomorphism.

Following [AR94, 2.34], there is a regular cardinal $\mu \triangleright \lambda$ such that each object of $\mathcal{K}$ is a μ-directed colimit of μ-presentable λ-pure subobjects of K. Thus $\mathcal{K}_{mono}$ is μ-accessible. $\square$

Remark 6.2. Let μ be a regular cardinal such that $\lambda \trianglelefteq \mu$ and $|\mathcal{A}| > \mu$ where $|\mathcal{A}|$ denotes the cardinality of the set of morphisms of $\mathcal{A}$. Following [AR94, 2.33, 2.34], $\mathcal{K}_{mono}$ is $(\mu^{<\mu})^{+}$-accessible.

When the starting class is locally multipresentable, the index of accessibility is preserved:

Lemma 6.3. Let $\mathcal{K}$ be a locally λ-multipresentable category. Then $\mathcal{K}_{mono}$ is locally λ-multipresentable.

Proof. Following 6.1, we have to prove that each object K of $\mathcal{K}$ is a λ-directed colimit of λ-presentable objects in $\mathcal{K}_{mono}$. Let $a_i : A_i \to K$ be a λ-directed colimit of λ-presentable objects A_i in $\mathcal{K}$. Following [Kel69, 4.5], $\mathcal{K}$ has (strong epimorphism, monomorphism) factorizations; $\mathcal{K}$ is well-powered by [Joh02, A1.4.17]. Let $a_i = a_i'' a_i'$ be a (strong epimorphism, monomorphism) factorization of a_i. Then $a_i'' : B_i \to K$ is a λ-directed colimit of monomorphisms. Following [AR94, 1.69], objects B_i are λ-presentable in $\mathcal{K}_{mono}$. Thus $\mathcal{K}_{mono}$ is λ-accessible.

It remains to show that $\mathcal{K}_{mono}$ has connected limits. Since equalizers are monomorphisms, we only need that wide pullbacks of monomorphisms are monomorphisms. But this is evident. $\square$

Remark 6.4. If K is λ-presentable in $\mathcal{K}$ then it is λ-presentable in $\mathcal{K}_{mono}$. One cannot expect the converse: in a general locally λ-presentable category $\mathcal{K}$, for example, the objects of $\mathcal{K}$ that are λ-presentable in $\mathcal{K}_{mono}$ (the λ-*generated objects* of $\mathcal{K}$) are guaranteed only to be strong quotients of λ-presentables (see [AR94, 1.67, 1.68]).

Corollary 6.5. The eventual categoricity conjecture in the sense of internal sizes (see [LRVb]) holds for locally $\aleph_0$-multipresentable categories.

Proof. By contrast with the remark above, whenever $\mathcal{K}$ is locally $\aleph_0$-multipresentable, the embedding $\mathcal{K}_{mono} \to \mathcal{K}$ preserves μ-presentable objects for all sufficiently large μ (see [BR12, 4.3]). Recalling that any locally $\aleph_0$-multipresentable category is a universal AEC ([LRVb, 5.9]), the eventual categoricity conjecture for locally $\aleph_0$-multipresentable categories follows from the corresponding result for universal AECs in [Vas17a, Vas17b]. $\square$

In order to be able to restrict to *regular* monomorphisms, we will assume existence of pushouts:

Lemma 6.6. Let $\mathcal{K}$ be a λ-accessible category with pushouts. Then $\mathcal{K}_{reg}$ is accessible and has λ-directed colimits.

Proof. We follow the proof of 6.1. We note, first, that regular monomorphisms are stable under λ-directed colimits in $\mathcal{K}$ (see [AHT96] Proposition 2). That is, given λ-directed diagrams $D, D' : \mathcal{D} \to \mathcal{K}$ and a natural transformation $\delta : D \to D'$ such that all δ_d are regular monomorphisms, then $\operatorname{colim} \delta : \operatorname{colim} D \to \operatorname{colim} D'$ is a regular monomorphism.

For the rest, it suffices to know that λ-pure morphisms are regular monomorphisms (see [AHT96, Corollary 1]). $\square$

7. Some model theory of μ-AECs

The following is a simple technical tool to prove the Löwenheim-Skolem-Tarski axiom of μ-AECs. A similar statement appears in [BGL$^+$16, 4.6], but since the proof is short we present it here as well.

Lemma 7.1. Let $\mathbf{K}$ be a coherent abstract class in a μ-ary vocabulary with concrete μ-directed colimits. Let $\theta \geq \mu + |\tau(\mathbf{K})|$ be a cardinal such that:

(1) θ is μ-closed: $\lambda < \theta$ implies $\lambda^{<\mu} < \theta$.
(2) $\operatorname{cf}(\theta) \geq \mu$.
(3) Any $M \in \mathbf{K}$ is θ-closed [LRVa, 4.6]. That is, for any $A \subseteq UM$, if $|A| < \theta$, there exists $M_0 \leq_{\mathbf{K}} M$ with $|UM_0| < \theta$ such that $A \subseteq UM_0$.

Then $\mathbf{K}$ is a μ-AEC and $\operatorname{LS}(\mathbf{K}) \leq \theta^{<\mu} = \theta$.

Proof. It suffices to prove the Löwenheim-Skolem-Tarski axiom. Let $M \in \mathbf{K}$ and let $A \subseteq |M|$. It is enough to find $N \in \mathbf{K}$ such that $N \leq_{\mathbf{K}} M$, $A \subseteq UN$, and $|UN| \leq |A|^{<\mu} + \theta$. Build a μ-directed system $\langle M_s : s \in [A]^{<\mu} \rangle$ such that $s \subseteq UM_s$, $M_s \leq_{\mathbf{K}} M$, and $|UM_s| < \theta$ for all $s \in [A]^{<\mu}$. This is enough: $N := \bigcup_{s \in [A]^{<\mu}} M_s$ is as desired. This is possible: we work by induction on $\alpha := |s|$. Assuming that M_t has been built for $|t| < \alpha$, let $B := s \cup \bigcup_{t \in [s]^{<\alpha}} UM_t$. Note that $|B| < \theta$ by the cardinal arithmetic assumptions. Since M is θ-closed, there is $M_s \leq_{\mathbf{K}} M$ such that $B \subseteq UM_s$ and $|UM_s| < \theta$. By coherence, $M_t \leq_{\mathbf{K}} M_s$ for any $t \in [s]^{<\alpha}$, as desired. $\square$

The following lemma will be useful also in the proof of Theorem 8.13. Note that we do not assume that $\mathbf{K}$ is coherent or that $\leq_{\mathbf{K}}$ extends the substructure relation here.

Lemma 7.2. Let $\mathbf{K} = (K, \leq_{\mathbf{K}})$ be such that:

(1) K is a class of structures in a fixed μ-ary vocabulary $\tau(\mathbf{K})$.
(2) $\leq_{\mathbf{K}}$ is a partial order on K such that $M \leq_{\mathbf{K}} N$ implies $UM \subseteq UN$.
(3) $\mathbf{K}$ is closed under isomorphisms.

Assume that $\mathbf{K}$ is accessible and has concrete μ-directed colimits. Let C be the class of cardinals λ such that for any $M \in \mathbf{K}$, $|UM| < \lambda$ if and only if M is λ_0-presentable for some $\lambda_0 < \lambda$. Then:

(1) C is closed unbounded.
(2) If $\mathbf{K}$ is μ-accessible and $\lambda \in C$ is such that $\lambda = \lambda^{<\mu}$, then for any $M \in \mathbf{K}$ there exists $\langle M_i : i \in \mathcal{I} \rangle$ increasing and λ^+-directed such that:
  (a) $M = \bigcup_{i \in \mathcal{I}} M_i$.
  (b) $|UM_i| \leq \lambda$ and M_i is λ^+-presentable for all $i \in \mathcal{I}$.
  (c) $M_i = \bigcup_{j<i} M_j$ for all $i \in \mathcal{I}$ such that $\{j \in \mathcal{I} \mid j < i\}$ is μ-directed.

Proof.

(1) C is clearly closed. Now given a cardinal λ_0, build $\langle \lambda_i : i \leq \omega \rangle$ increasing continuous such that for all $i < \omega$, for any $M \in \mathbf{K}$, if M is λ_i-presentable, then $|UM| < \lambda_{i+1}$ and if $|UM| \leq \lambda_i$, then M is λ_{i+1}-presentable. This is possible since there is up to isomorphism only a set of objects of cardinality at most λ_i and a set of λ_i-presentable objects. Now λ_ω is in C, as desired.
(2) Let $M \in \mathbf{K}$. We know that M can be written as a μ-directed union of μ-presentable objects $M = \bigcup_{i \in I} N_i$. Now by the cardinal arithmetic assumption, any subset of I of cardinality at most λ can be completed to a μ-directed subset of I of cardinality at most λ. Thus letting $\mathcal{I}$ be the set of μ-directed subsets of I of cardinality at most λ, we have that $M = \bigcup_{J \in \mathcal{I}} N_J$, where $N_J = \bigcup_{j \in J} N_j$. This is a λ^+-directed system and since $\lambda \in C$, $|UN_J| \leq \lambda$. This directly implies that N_J is also λ^+-presentable. Thus, letting for $i \in \mathcal{I}$ $M_i := N_i$, the M_i's are as desired.

□

The next result is similar to [BGL+16, 4.5], but we do *not* require here anything on how presentability ranks relate to cardinalities. Rather, what we need is derived via Lemma 7.2.

Theorem 7.3. Let $\mathbf{K}$ be a coherent abstract class in a μ-ary vocabulary. If $\mathbf{K}$ is accessible and has concrete μ-directed colimits, then $\mathbf{K}$ is a μ-AEC.

Proof. Let C be as given by Lemma 7.2. Let $\mu' \geq \mu$ be such that $\mathbf{K}$ is μ'-accessible and let $\theta \in C$ be such that $\lambda^{<\mu'} + |\tau(\mathbf{K})| < \theta$ for all $\lambda < \theta$ and $\mathrm{cf}(\theta) \geq \mu'$. We check that the third hypothesis of Lemma 7.1 is satisfied. Let $M \in \mathbf{K}$ and let $A \subseteq UM$ be such that $|A| < \theta$. Let $\lambda := \left((|A| + \aleph_0)^{<\mu'}\right)^+$. By [MP89, 2.3.10], $\mathbf{K}$ is λ-accessible. Therefore there is a λ-directed system $\langle M_i : i \in I \rangle$ such that $M = \bigcup_{i \in I} M_i$, with each M_i λ-presentable. Since I is λ-directed and $\lambda > |A|$, there is $i \in I$ such that $A \subseteq UM_i$. By the choice of θ, $|UM_i| < \theta$, so M_i is as desired. □

7.1. **Model-homogeneous models.** The following notions are very close to "κ-closed" and "κ-saturated" from [Ros97]. The only difference is that we use cardinalities instead of presentability ranks. Most of the proofs there carry through with little change, however.

Definition 7.4. Let $\mathbf{K}$ be a μ-AEC. For $\kappa > \mathrm{LS}(\mathbf{K})$, $M \leq_{\mathbf{K}} N$, we say that M is *κ-model-homogeneous in N* if whenever $M_0, N_0 \in \mathbf{K}$ are such that:

(1) $M_0 \leq_{\mathbf{K}} M$.
(2) $M_0 \leq_{\mathbf{K}} N_0 \leq_{\mathbf{K}} N$.
(3) $N_0 \in \mathbf{K}_{<\kappa}$

Then there exists $f : N_0 \to M$ fixing M_0.

We say that M is *locally κ-model-homogeneous* if it is κ-model-homogeneous in N for every $N \in \mathbf{K}$ with $M \leq_{\mathbf{K}} N$. We say that M is *κ-model-homogeneous* if whenever $M_0 \leq_{\mathbf{K}} M$, $M_0 \leq_{\mathbf{K}} N_0$ are such that $N_0 \in \mathbf{K}_{<\kappa}$, there exists $f : N_0 \to M$ fixing M_0.

Remark 7.5. If $\mathbf{K}$ has amalgamation, being locally κ-model-homogeneous is equivalent to being κ-model-homogeneous (see [Ros97, Lemma 3]). Further (still assuming amalgamation), if M is κ-model-homogeneous in N and N is κ-model-homogeneous, then M is κ-model-homogeneous.

Definition 7.6. Let $\mathbf{K}$ be a μ-AEC and let $\kappa > \mathrm{LS}(\mathbf{K})$. We let $\mathbf{K}^{\kappa\text{-lmh}}$ be the abstract class of locally κ-model-homogeneous models in $\mathbf{K}$, ordered with the appropriate restriction of $\leq_{\mathbf{K}}$.

Given an increasing chain $\langle M_i : i < \delta \rangle$ in a μ-AEC $\mathbf{K}$, the union of the chain may not be inside $\mathbf{K}$. The following conditions are a very useful weakening: we require that this chain has an upper bound. This is already used in [Ros97]. Classes of saturated models in AECs, as well as μ-CAECs (e.g. metric classes) are examples satisfying this property, see [BGL+16, 6.7].

Definition 7.7. $\mathbf{K}$ has *directed bounds* if whenever $\langle M_i : i \in I \rangle$ is a directed system, there exists $M \in \mathbf{K}$ such that $M_i \leq_{\mathbf{K}} M$ for all $i \in I$.

$\mathbf{K}$ *has chain bounds* if this holds whenever I is an ordinal.

In practice, we will use chain bounds. It allows us to build locally κ-model-homogeneous models. Moreover a large cardinal axiom (*Vopěnka's Principle*, see Chapter 6 of [AR94]) ensures that the class of all such models is well-behaved. In particular, it has the amalgamation property.

Theorem 7.8. Let $\kappa > \mathrm{LS}(\mathbf{K})$ be regular.

(1) Assume that $\mathbf{K}$ has chain bounds. For any $M \in \mathbf{K}$, there exists $N \in \mathbf{K}$ such that $M \leq_{\mathbf{K}} N$ and N is locally κ-model-homogeneous. Moreover, we can take N so that $|UN| \leq (|UM| + \mathrm{LS}(\mathbf{K}))^{<\kappa}$.
(2) For any $N \in \mathbf{K}$ and any $A \subseteq UN$, there exists $M \in \mathbf{K}$ with $M \leq_{\mathbf{K}} N$ such that M contains A, M is κ-model-homogeneous in N, and $|UM| \leq (|A| + \mathrm{LS}(\mathbf{K}))^{<\kappa}$.
(3) Assume that $\mathbf{K}$ has chain bounds. If either $\mathbf{K}$ has amalgamation or Vopěnka's principle holds, then $\mathbf{K}^{\kappa\text{-lmh}}$ is a κ-AEC with chain bounds.
(4) If κ is strongly compact, then any locally κ-model-homogeneous model M is a global amalgamation base: whenever $M \leq_{\mathbf{K}} M_\ell$, $\ell = 1, 2$, there exists $N \in \mathbf{K}$ and $f_\ell : M_\ell \to N$ fixing M.

Proof.

(1) Similar to [Ros97, Theorem 1].
(2) Similar to the above.
(3) All the axioms are straightforward, except for the Löwenheim-Skolem-Tarski axiom. If $\mathbf{K}$ has amalgamation, this follows from Theorem 7.8 and Remark 7.5. If Vopěnka's principle holds, then by [BGL+16, 4.6], the Löwenheim-Skolem-Tarski axiom holds (though we do not have a bound for it).
(4) As in [Bon14, 7.2], using the fact that one can take ultraproducts in μ-AECs [BGL+16, §5]

□

Now that we have tools to get amalgamation, we investigate to what extent we can work inside a big homogeneous model as in the first-order case (and indeed as in AECs with amalgamation):

Definition 7.9. For $\kappa > \mathrm{LS}(\mathbf{K})$, we say that $M \in \mathbf{K}$ is *κ-universal* if any $M_0 \in \mathbf{K}_{<\kappa}$ embeds into M.

Definition 7.10. We say that $\mathbf{K}$ *has a monster model* if for any $\kappa > \mathrm{LS}(\mathbf{K})$ there exists $M \in \mathbf{K}$ which is both κ-universal and κ-model-homogeneous.

It turns out that building a monster model requires (at least when $\mu = \aleph_1$) chain bounds:

Theorem 7.11. If $\mathbf{K}$ is non-empty, has amalgamation, joint embedding, and chain bounds, then $\mathbf{K}$ has a monster model. Conversely, if $\mathbf{K}$ has a monster model, then $\mathbf{K}$ has amalgamation and joint embedding; if $\mu \leq \aleph_1$, moreover, $\mathbf{K}$ has chain bounds.

Proof. If $\mathbf{K}$ has amalgamation, joint embedding, and has chain bounds, then the construction of a κ-universal κ-model-homogeneous model is standard.

Conversely, if $\mathbf{K}$ has a monster model, then it clearly has amalgamation and joint embedding. To see it has chain bounds when $\mu = \omega_1$, fix $\langle M_i : i < \delta \rangle$ increasing. Without loss of generality, $\delta = \mathrm{cf}(\delta)$. If $\delta \geq \omega_1$, we can use the chain axioms of $\aleph_1$-AECs, so assume without loss of generality that $\delta = \omega$. Let M be κ-universal and κ-model-homogeneous, where $\kappa := \left(\sum_{i<\delta} \|M_i\|^{<\mu} + \mathrm{LS}(\mathbf{K})\right)^+$. We build $\langle f_i : M_i \to M \mid i < \omega \rangle$ increasing as follows:

(1) For $i = 0$, use universality of M.
(2) For i successor, use model-homogeneity of M.

Now let $N \leq_{\mathbf{K}} M$ contain $\bigcup_{i<\omega} f_i[M_i]$ and rename to obtain an upper bound to $\langle M_i : i < \omega \rangle$. □

In conclusion, assuming a large cardinal axiom there is a well-behaved sub-μ-AEC of the original class:

Corollary 7.12. Let $\kappa > \mathrm{LS}(\mathbf{K})$ be strongly compact and assume Vopěnka's principle. If $\mathbf{K}$ has chain bounds, then there exists a subclass $\mathbf{K}^*$ of $\mathbf{K}$ such that:

(1) $\mathbf{K}^*$ is a κ-AEC.

(2) $\mathbf{K}^*$ has amalgamation, joint embedding, and chain bounds. In particular, it has a monster model.
(3) If $\mathbf{K}$ is not empty, $\mathbf{K}^*$ is not empty and can be chosen to have arbitrarily large models if $\mathbf{K}$ has arbitrarily large models.
(4) If $\mathbf{K}$ has joint embedding, then $\mathbf{K}^*$ is cofinal in $\mathbf{K}$, i.e. any $M \in \mathbf{K}$ is contained inside an $N \in \mathbf{K}^*$.

Proof. By Theorem 7.8, $\mathbf{K}^{\kappa\text{-lmh}}$ is a κ-AEC with amalgamation and chain bounds. If $\mathbf{K}$ had joint embedding already, then $\mathbf{K}^{\kappa\text{-lmh}}$ also had joint embedding, and hence one can take $\mathbf{K}^* := \mathbf{K}^{\kappa\text{-lmh}}$. Otherwise, using the equivalence relation induced by "embedding in a common model," one can partition $\mathbf{K}^{\kappa\text{-lmh}}$ into disjoint κ-AECs $\langle \mathbf{K}_i^* : i \in I \rangle$ such that each has amalgamation, joint embedding, and chain bounds. If $\mathbf{K}$ has arbitrarily large models, $\mathbf{K}^{\kappa\text{-lmh}}$ has arbitrarily large models and hence (since I is a set) we can pick $i \in I$ such that $\mathbf{K}_i^*$ has arbitrarily large models. Let $\mathbf{K}^* := \mathbf{K}_i^*$. $\square$

Question 7.13. Can one remove Vopěnka's principle from Corollary 7.12?

One attempt would be to consider a stronger ordering on $\mathbf{K}$, e.g. $M \trianglelefteq N$ if and only if $M \leq_{\mathbf{K}} N$ and $M \preceq_{\mathbb{L}_{\kappa,\kappa}} N$. However in this case we do not know whether having chain bounds is preserved.

8. Stable amalgamation inside a μ-AEC

In this section, we consider independence relation on μ-AECs. The main result is Theorem 8.13, characterizing stable independence in terms of model-theoretic local character properties of forking. All throughout, we assume:

Hypothesis 8.1. We work inside a fixed μ-AEC $\mathbf{K}$. We fix an *invariant* (recall Definition 3.8) independence relation $\mathop{\smile\hskip-0.9em\raise0.4ex\hbox{$\vert$}}$ on $\mathbf{K}$.

We can see $\mathop{\smile\hskip-0.9em\raise0.4ex\hbox{$\vert$}}$ as a relation on Galois types if we introduce some notation. A similar idea is already investigated in [BGKV16, §5.1], but there the left hand side of $\mathop{\smile\hskip-0.9em\raise0.4ex\hbox{$\vert$}}$ is already assumed to be an arbitrary set. Thus the situation here requires slightly more caution.

Definition 8.2. Write $A \overset{N_3}{\underset{N_0}{\overline{\mathop{\smile\hskip-0.9em\raise0.4ex\hbox{$\vert$}}}}} B$ if $N_0 \leq_{\mathbf{K}} N_3$, $A \cup B \subseteq UN_3$, and there exists M_1, M_2, M_3 with $A \subseteq UM_1$, $B \subseteq UM_2$, $N_3 \leq_{\mathbf{K}} M_3$, and $M_1 \overset{M_3}{\underset{N_0}{\mathop{\smile\hskip-0.9em\raise0.4ex\hbox{$\vert$}}}} M_2$.

We say that $\mathrm{gtp}(\bar{a}/B; N_3)$ *does not fork over* N_0 if $\mathrm{ran}(\bar{a}) \overset{N_3}{\underset{N_0}{\overline{\mathop{\smile\hskip-0.9em\raise0.4ex\hbox{$\vert$}}}}} B$ (it is easy to see that this does not depend on the choice of representatives, see also Fact 8.4(2)).

One can think of $\overline{\mathop{\smile\hskip-0.9em\raise0.4ex\hbox{$\vert$}}}$ as the "closure" of $\mathop{\smile\hskip-0.9em\raise0.4ex\hbox{$\vert$}}$. The point is that we allow sets on the left and right hand side.

Remark 8.3. It is tempting to try to define $\overline{\mathop{\smile\hskip-0.9em\raise0.4ex\hbox{$\vert$}}}$ in an arbitrary category as the class of diagrams

$$
\begin{tikzcd}
M_1 \arrow[r] & M_3 \\
M_0 \arrow[ur] & M_2 \arrow[u]
\end{tikzcd}
$$

that can be extended to an independent diagram consisting of M_0, M_1', M_2' and M_3' such that the following commutes:

$$
\begin{tikzcd}
 & M_1' \arrow[r, dotted] & M_3' \\
M_1 \arrow[r] \arrow[ur, dotted] & M_3 \arrow[ur, dotted] & M_2' \arrow[u, dotted] \\
M_0 \arrow[ur] \arrow[uur, dotted] \arrow[urr, dotted] & M_2 \arrow[u] \arrow[ur, dotted] &
\end{tikzcd}
$$

However, when there is already a morphism e.g. from M_0 to M_2, there is no reason to believe that the resulting diagram will also commute with that morphism. This is problematic when one tries to prove, for example, that $\overline{\smile\!\!\!\!\!\!\mid}$ is transitive (see Theorem 8.5). Thus it is unclear to us how to define $\overline{\smile\!\!\!\!\!\!\mid}$ in a non-concrete category.

Properties of $\smile\!\!\!\!\!\!\mid$ generalize to $\overline{\smile\!\!\!\!\!\!\mid}$ as follows:

Fact 8.4. Assume that $\smile\!\!\!\!\!\!\mid$ is monotonic.

(1) Let $M_0 \leq_{\mathbf{K}} M_\ell \leq_{\mathbf{K}} M_3$ for $\ell = 1, 2$. Then $M_1 \mathop{\smile\!\!\!\!\!\!\mid}\limits_{M_0}^{M_3} M_2$ if and only if $M_1 \mathop{\overline{\smile\!\!\!\!\!\!\mid}}\limits_{M_0}^{M_3} M_2$.

(2) (Preservation under **K**-embeddings) Given $M_0 \leq_{\mathbf{K}} M_3$, $A, B \subseteq UM_3$, and $f : M_3 \to N_3$, we have that $A \mathop{\overline{\smile\!\!\!\!\!\!\mid}}\limits_{M_0}^{M_3} B$ if and only if $f[A] \mathop{\overline{\smile\!\!\!\!\!\!\mid}}\limits_{f[M_0]}^{N_3} f[B]$.

(3) (Monotonicity) If $A \mathop{\overline{\smile\!\!\!\!\!\!\mid}}\limits_{M_0}^{M_3} B$ and $A_0 \subseteq A$, $B_0 \subseteq B$, then $A_0 \mathop{\overline{\smile\!\!\!\!\!\!\mid}}\limits_{M_0}^{M_3} B_0$.

(4) (Normality) $A \mathop{\overline{\smile\!\!\!\!\!\!\mid}}\limits_{M_0}^{M_3} B$ if and only if $AM_0 \mathop{\overline{\smile\!\!\!\!\!\!\mid}}\limits_{M_0}^{M_3} BM_0$.

(5) (Base monotonicity) Assume that $\smile\!\!\!\!\!\!\mid$ is right base-monotonic. If $A \mathop{\overline{\smile\!\!\!\!\!\!\mid}}\limits_{M_0}^{M_3} B$, $M_0 \leq_{\mathbf{K}} M_2 \leq_{\mathbf{K}} M_3$, and $UM_2 \subseteq B$, then $A \mathop{\overline{\smile\!\!\!\!\!\!\mid}}\limits_{M_2}^{M_3} B$.

(6) (Existence) Assume that $\smile\!\!\!\!\!\!\mid$ has existence. Whenever $M \leq_{\mathbf{K}} N$ and $p \in \mathrm{gS}^{<\infty}(M)$, there exists $q \in \mathrm{gS}^{<\infty}(N)$ extending p such that q does not fork over M.

(7) (Symmetry) Assume that $\perp$ is symmetric. Then $\overline{\perp}$ is symmetric: $A \underset{M}{\overset{N}{\overline{\perp}}} B$ holds if and only if $B \underset{M}{\overset{N}{\overline{\perp}}} A$ holds.

Proof. This is essentially given by the arguments in [BGKV16, 5.1,5.4]. For the convenience of the reader, we sketch some details:

(1) If $M_1 \underset{M_0}{\overset{M_3}{\perp}} M_2$, then directly from the definition $M_1 \underset{M_0}{\overset{M_3}{\overline{\perp}}} M_2$. For the converse, use monotonicity, invariance, closure under $\sim$, the coherence axiom of μ-AECs, and the definition of $\overline{\perp}$.
(2) Directly from the definitions, invariance, and closure under $\sim$.
(3) Clear from the definition of $\overline{\perp}$.
(4) The right to left direction is by monotonicity. For the left to right direction, suppose that we have M_1, M_2, M_3' witnessing $A \underset{M_0}{\overset{M_3}{\overline{\perp}}} B$, i.e. $M_0 \leq_{\mathbf{K}} M_\ell \leq_{\mathbf{K}} M_3'$ for $\ell = 1, 2$, $M_3 \leq_{\mathbf{K}} M_3'$, $A \subseteq UM_1$, $B \subseteq UM_2$, and $M_1 \underset{M_0}{\overset{M_3'}{\perp}} M_2$. By the first part, $M_1 \underset{M_0}{\overset{M_3'}{\overline{\perp}}} M_2$. By monotonicity, $AM_0 \underset{M_0}{\overset{M_3'}{\overline{\perp}}} BM_0$. Since $\overline{\perp}$ preserves $\mathbf{K}$-embeddings, this implies that $AM_0 \underset{M_0}{\overset{M_3}{\overline{\perp}}} BM_0$, as desired.
(5) Directly from the definition and the coherence axiom of μ-AECs.
(6) Say $p = \mathrm{gtp}(\bar{b}/M; M')$. By existence, find $N' \in \mathbf{K}$ and $f : M' \to N'$ such that $N \leq_{\mathbf{K}} N'$, f fixes M, and $f[M'] \underset{M}{\overset{N'}{\perp}} N$. Let $q := \mathrm{gtp}(f(\bar{b})/N; N')$.
(7) Directly from the definition.

□

Uniqueness and transitivity also generalize. However, the argument is longer than the corresponding one in [BGKV16, 5.4, 5.11] (essentially because we are taking a closure with respect to both the left and right hand side of $\perp$). We sketch a full proof here.

Theorem 8.5. Assume that $\perp$ is transitive and has existence and uniqueness.

(1) (Uniqueness) Given $p, q \in \mathrm{gS}^{<\infty}(B; N)$ with $M \leq_{\mathbf{K}} N$ and $UM \subseteq B \subseteq UN$, if $p \upharpoonright M = q \upharpoonright M$ and p, q do not fork over M, then $p = q$.
(2) (Transitivity) If $M_0 \leq_{\mathbf{K}} M_2 \leq_{\mathbf{K}} M_3$, $A \underset{M_0}{\overset{M_3}{\overline{\perp}}} M_2$ and $A \underset{M_2}{\overset{M_3}{\overline{\perp}}} B$, then $A \underset{M_0}{\overset{M_3}{\overline{\perp}}} B$.

Proof. We proceed via a series of claims.

Claim 1: Assume that $M_1 \underset{M_0}{\overset{M_3}{\downfree}} M_2$ and $M_1' \underset{M_0}{\overset{M_3'}{\downfree}} M_2$. Let $\bar{b}, \bar{b}'$ be enumerations of M_1, M_1' respectively. If $\operatorname{gtp}(\bar{b}/M_0; M_3) = \operatorname{gtp}(\bar{b}'/M_0; M_3')$ (i.e. the enumerations induce an isomorphism between M_1 and M_1' fixing M_0), then $\operatorname{gtp}(\bar{b}/M_2; M_3) = \operatorname{gtp}(\bar{b}'/M_2; M_3')$.

Proof of Claim 1: This follows directly from the uniqueness property as in [Vas16a, 12.6]. $\dagger_{\text{Claim 1}}$.

Claim 2: Assume that $M_2 \underset{M_0}{\overset{M_3}{\downfree}} M_1$ and $M_2 \underset{M_0}{\overset{M_3}{\downfree}} M_1'$. There exists M_1'', M_3', $f : M_1' \to M_1''$ such that f fixes M_0, $M_1 \leq_{\mathbf{K}} M_1'' \leq_{\mathbf{K}} M_3'$, $M_3 \leq_{\mathbf{K}} M_3'$, and $M_2 \underset{M_0}{\overset{M_3'}{\downfree}} M_1''$.

Proof of Claim 2: Let $\bar{a}$ be an enumeration of M_2. Let $p := \operatorname{gtp}(\bar{a}/M_0; M_3)$. By extension, let $q \in \operatorname{gS}^{<\infty}(M_3)$ be such that q extends p and q does not fork over M_0. Say $q = \operatorname{gtp}(\bar{a}'/M_3; M_3'^{*})$, and let M_2' be the model enumerated by $\bar{a}$. We have that $M_2' \underset{M_0}{\overset{M_3^*}{\downfree}} M_3$. By monotonicity, $M_2' \underset{M_0}{\overset{M_3^*}{\downfree}} M_1$. By Claim 1 (where the role of M_1 and M_2 is reversed), $\operatorname{gtp}(\bar{a}/M_1; M_3) = \operatorname{gtp}(\bar{a}'/M_1; M_3^*)$. Let $g : M_3^* \to M_3'$ with $M_3 \leq_{\mathbf{K}} M_3'$ be such that $g(\bar{a}') = \bar{a}$ and g fixes M_1. Let $M_1'' := g[M_3]$ and let $f := g \restriction M_1'$. This is as desired. $\dagger_{\text{Claim 2}}$.

Claim 3: Assume that $M_1 \underset{M_0}{\overset{M_3}{\downfree}} M_2$ and $\bar{b}_1, \bar{b}_2 \in {}^{<\infty} M_1$ are such that $\operatorname{gtp}(\bar{b}_1/M_0; M_3) = \operatorname{gtp}(\bar{b}_2/M_0; M_3)$. Then $\operatorname{gtp}(\bar{b}_1/M_2; M_3) = \operatorname{gtp}(\bar{b}_2/M_2; M_3)$.

Proof of Claim 3: Let $f : M_3 \to M_3'$, $M_3 \leq_{\mathbf{K}} M_3'$ be such that f fixes M_0 and $f(\bar{b}_1) = \bar{b}_2$. Using extension, find $g : M_3' \to M_3''$ that fixes M_1, such that $M_3 \leq_{\mathbf{K}} M_3''$ and $g[M_3'] \underset{M_1}{\overset{M_3''}{\downfree}} M_3$. By transitivity (for $\downfree$), $g[M_3'] \underset{M_0}{\overset{M_3}{\downfree}} M_2$. Let $h := gf$. We have in particular that $h[M_1] \underset{M_0}{\overset{M_3}{\downfree}} M_2$. Letting $\bar{a}$ be an enumeration of M_1 and using Claim 1, this means that $\operatorname{gtp}(\bar{a}/M_2; M_3) = \operatorname{gtp}(h(\bar{a})/M_2; M_3)$. In particular, $\operatorname{gtp}(\bar{b}_1/M_2; M_3) = \operatorname{gtp}(\bar{b}_2/M_2; M_3)$. $\dagger_{\text{Claim 3}}$.

Claim 4 (uniqueness for types over models): Let $M_0 \leq_{\mathbf{K}} M_2$ and let $p_1, p_2 \in \operatorname{gS}^{<\infty}(M_2)$ be given such that both do not fork over M_0 and $p_1 \restriction M_0 = p_2 \restriction M_0$. Then $p_1 = p_2$.

Proof of Claim 4: Say $p_\ell = \operatorname{gtp}(\bar{b}_\ell/M_2; N_\ell)$. Without loss of generality $N := N_1 = N_2$). By definition of nonforking of types, $\bar{b}_\ell \underset{M_0}{\overset{N}{\overline{\downfree}}} M_2$. Expanding the definition of $\overline{\downfree}$ and extending N if necessary, we have that for some M_1^ℓ containing $\bar{b}_\ell$, $\ell = 1, 2$, $M_1^\ell \underset{M_0}{\overset{N}{\downfree}} M_2$. By Claim 2 applied to $\overset{d}{\downfree}$, there exists M_1', N', and $f : M_1^2 \to M_1'$ such that f fixes M_0, $M_1^1 \leq_{\mathbf{K}} M_1'$, $N \leq_{\mathbf{K}} N'$, and $M_1' \underset{M_0}{\overset{N'}{\downfree}} M_2$. Let $\bar{b}_2' := f(\bar{b}_2)$. By Claim 3, $\operatorname{gtp}(\bar{b}_1/M_2; N') = \operatorname{gtp}(\bar{b}_2'/M_2; N')$. Moreover, Claim 1 implies that $\operatorname{gtp}(\bar{b}_2'/M_2; N') = \operatorname{gtp}(\bar{b}_2/M_2; N')$, so we get the desired result. $\dagger_{\text{Claim 4}}$

Claim 5 (transitivity): Let $M_0 \leq_{\mathbf{K}} M_1 \leq_{\mathbf{K}} M_2$ and let $p \in \mathrm{gS}^{<\infty}(M_2)$. If p does not fork over M_1 and $p \upharpoonright M_1$ does not fork over M_0, then p does not fork over M_0 (note that this implies the transitivity statement of the theorem, since by definition of $\overline{\downarrow}$ we can always extend B so that it is a model extending M_2).

Proof of Claim 5: By extension, let $q \in \mathrm{gS}^{<\infty}(M_2)$ extend $p \upharpoonright M_0$ such that q does not fork over M_0. By monotonicity, $q \upharpoonright M_1$ does not fork over M_0. By Claim 4 (applied to $q \upharpoonright M_1$ and $p\ restM_1$), $p \upharpoonright M_1 = q \upharpoonright M_1$. By base monotonicity, q does not fork over M_1. By Claim 4 again, $p = q$. $\dagger_{\text{Claim 5}}$

Claim 6: Let $p \in \mathrm{gS}^{<\infty}(B; N)$ with $M \leq_{\mathbf{K}} N$ and $UM \subseteq B \subseteq UN$. If p does not fork over M, then there exists $q \in \mathrm{gS}^{<\infty}(N)$ such that q extends p and q does not fork over M.

Proof of Claim 6: First note that this is not quite the same as the existence property, since we start of with a type that already does not fork over a smaller base. Write $p = \mathrm{gtp}(\bar{a}/B; N)$. Fix M', N' such that $M \leq_{\mathbf{K}} M' \leq_{\mathbf{K}} N'$, $N \leq_{\mathbf{K}} N'$, $B \subseteq UM'$, and $\bar{a} \overset{N'}{\underset{M}{\overline{\downarrow}}} M'$. Let $p' := \mathrm{gtp}(\bar{a}/M'; N')$. Note that p' extends p and p' does not fork over M. Find $q' \in \mathrm{gS}^{<\infty}(N')$ such that q' extends p' and q' does not fork over M'. By Claim 5 (transitivity), q' does not fork over M. Let $q := q' \upharpoonright N$. By monotonicity, it is as desired. $\dagger_{\text{Claim 7}}$

Claim 7 (uniqueness): Given $p_1, p_2 \in \mathrm{gS}^{<\infty}(B; N)$ with $M \leq_{\mathbf{K}} N$ and $UM \subseteq B \subseteq UN$, if $p_1 \upharpoonright M = p_2 \upharpoonright M$ and p_1, p_2 do not fork over M, then $p_1 = p_2$.

Proof of Claim 7: Using Claim 6, find $q_\ell \in \mathrm{gS}^{<\infty}(N)$ such that q_ℓ does not fork over M and extends p_ℓ for $\ell = 1, 2$. Thus in particular $q_1 \upharpoonright M = p_1 \upharpoonright M = p_2 \upharpoonright M = q_2 \upharpoonright M$. By Claim 4 (uniqueness for types over models), $q_1 = q_2$. In particular, $p_1 = q_1 \upharpoonright B = q_2 \upharpoonright B = p_2$. $\dagger_{\text{Claim 7}}$ □

We can now state the local character property of forking: every type does not fork over a "small" set:

Definition 8.6. We say that $\downarrow$ has *right local character* if for each cardinal α, there exists a cardinal λ (depending on α) such that for any $p \in \mathrm{gS}^{\alpha}(M)$ there exists $M_0 \leq_{\mathbf{K}} M$ with $|UM_0| \leq \lambda$ and p not forking over M_0. Equivalently—avoiding any mention of Galois types, and suitable for an abstract category—given $M \leq_{\mathbf{K}} N$ and $N_1 \leq_{\mathbf{K}} N$, if N_1 has cardinality at most α there exists M_0 of cardinality at most λ and N_1', N', such that $N_1 \leq_{\mathbf{K}} N_1' \leq_{\mathbf{K}} N'$, $N \leq_{\mathbf{K}} N'$, and $N_1' \overset{N}{\underset{M_0}{\downarrow}} M$.

We say that $\downarrow$ has *left local character* if $\overset{d}{\downarrow}$ has right local character. We say that $\downarrow$ has *local character* if it has both left and right local character.

The idea of the definition of local character is as follows: given $M \leq_{\mathbf{K}} N$ and $N_1 \leq_{\mathbf{K}} N$, we may want to write that $N_1 \overset{N}{\underset{N_0}{\downarrow}} M$, where N_0 is the pullback of N_1 and M over N. However pullbacks may not exist (and even when they do, the desired independence may not hold, see Example 3.30(3)). Thus we say instead that we can close the intersection of N_1 and M to a small (meaning of cardinality depending only on α, the size of N_1) model M_0 so that N_1 and M do not interact over M_0. Importantly, we still require that $M_0 \leq_{\mathbf{K}} M$.

We will also study the following locality property, named "model-witness property" in [Vas16a, 3.12(9)]. When $\theta = \aleph_0$ (as in the first-order case), it is often confusingly called the finite character property of forking.

Definition 8.7. Let θ be an infinite cardinal. We say that $\mathop{\smile\!\!\!\!\!\!\vert}$ has the *right* $(<\theta)$-*witness property* if $M_1 \overset{M_3}{\underset{M_0}{\mathop{\smile\!\!\!\!\!\!\vert}}} M_2$ holds whenever $M_0 \leq_{\mathbf{K}} M_\ell \leq_{\mathbf{K}} M_3$, $\ell = 1, 2$, and $M_1 \overset{M_3}{\underset{M_0}{\overline{\mathop{\smile\!\!\!\!\!\!\vert}}}} A$ for all $A \subseteq UM_2$ with $|A| < \theta$.

We say that $\mathop{\smile\!\!\!\!\!\!\vert}$ has the *left* $(<\theta)$-*witness property* if $\overset{d}{\mathop{\smile\!\!\!\!\!\!\vert}}$ has it and we say that $\mathop{\smile\!\!\!\!\!\!\vert}$ has the $(<\theta)$-*witness property* if it has both the left and right one. When θ is omitted, we mean that the witness property holds for some θ.

The witness property is known to follow from appropriate tameness assumptions. Recall from [BG17, 2.8] that $\mathbf{K}$ is *fully* $(<\theta)$-*tame and short* if Galois types are determined by the restrictions of their domain and variables of size less than θ. This holds in particular if $\theta \geq \kappa$, for some $\kappa > \mathrm{LS}(\mathbf{K})$ strongly compact [BGL$^+$16, 5.5] or if $\theta \geq \mu$ and $\mathbf{K}$ is a universal μ-AEC (see the argument inside Remark 5.7). For the convenience of the reader, we replicate the precise statement and proof of the witness property from full tameness and shortness here:

Fact 8.8 (4.5 in [Vas16a])**.** Let θ be an infinite cardinal and assume that $\mathop{\smile\!\!\!\!\!\!\vert}$ is transitive and has existence and uniqueness. If $\mathbf{K}$ is fully $(<\theta)$-tame and short, then $\mathop{\smile\!\!\!\!\!\!\vert}$ has the $(<\theta)$-witness property.

Proof. We prove the right $(<\theta)$-witness property. The left version follows by applying the same argument to $\overset{d}{\mathop{\smile\!\!\!\!\!\!\vert}}$. Assume that $M_0 \leq_{\mathbf{K}} M_\ell \leq_{\mathbf{K}} M_3$, $\ell = 1, 2$, and $M_1 \overset{M_3}{\underset{M_0}{\overline{\mathop{\smile\!\!\!\!\!\!\vert}}}} A$ for all $A \subseteq UM_2$ with $|A| < \theta$. Let $\bar{b}$ be an enumeration of M_1 and let $p := \mathrm{gtp}(\bar{b}/M_2; M_3)$. By extension for $\overline{\mathop{\smile\!\!\!\!\!\!\vert}}$, let $q \in \mathrm{gS}^{<\infty}(M_2)$ extend $p \upharpoonright M_0$ and not fork over M_0. We show that $p = q$, which is enough since $\overline{\mathop{\smile\!\!\!\!\!\!\vert}}$ respects $\mathbf{K}$-embeddings. By full tameness and shortness, it is enough to see that $p \upharpoonright A = q \upharpoonright A$ for any A of size less than θ. Fix such an A. We know that $p \upharpoonright A$ does not fork over M_0 by assumption, and $q \upharpoonright A$ also does not fork over M_0 by monotonicity. Therefore by uniqueness for $\overline{\mathop{\smile\!\!\!\!\!\!\vert}}$ (Theorem 8.5), $p \upharpoonright A = q \upharpoonright A$, as desired. $\square$

Remark 8.9. The argument does not need the full strength of tameness and shortness (in fact it only uses tameness, albeit for types of arbitrary length).

The witness property also makes the local character property more uniform: the cardinal λ from Definition 8.6 becomes a simple function of α:

Lemma 8.10. Assume that $\mathop{\smile\!\!\!\!\!\!\vert}$ is right transitive, has existence, uniqueness, and has the left $(<\theta)$-witness property. If $\mathop{\smile\!\!\!\!\!\!\vert}$ has right local character, then there exists a cardinal λ_0 such that for each $p \in \mathrm{gS}^{\alpha}(M)$, there exists $M_0 \in \mathbf{K}$ with $M_0 \leq_{\mathbf{K}} M$, $|UM_0| \leq \lambda_0 + \alpha^{<(\theta+\mu)}$ and p not forking over M_0.

Proof. Let λ_0 be such that $\lambda_0 = \lambda_0^{<\mu} + \mathrm{LS}(\mathbf{K})$ and satisfy Definition 8.6 with α, λ there standing for θ, λ_0 here. Now given any $p \in \mathrm{gS}^\alpha(M)$, for any $I \subseteq \alpha$ with $|I| < \theta$, pick $M_I \leq_{\mathbf{K}} M$ of cardinality at most λ_0 such that p^I does not fork over M_I. In the end, let $A := \bigcup_{I \subseteq \alpha, |I| < \theta} M_I$. Note that $|A| \leq \lambda_0 + \alpha^{<\theta}$, so one can pick $M_0 \leq_{\mathbf{K}} M$ containing A of size at most $|A|^{<\mu} + \mathrm{LS}(\mathbf{K})$. By base monotonicity and the left $(< \theta)$-witness property, M_0 is as desired. $\square$

We now turn to the main result of this section. If $\mathop{\smile\!\!\!\!\!\!|}\,$ is reasonable, accessibility of $\mathop{\smile\!\!\!\!\!\!|}\,$ is equivalent to the conjunction of the witness and local character properties. We will use the following auxiliary classes:

Definition 8.11. Assume that $\mathop{\smile\!\!\!\!\!\!|}\,$ is monotonic and right transitive. Let $\mathbf{K}_{\mathrm{NF}} = (K_{\mathrm{NF}}, \leq_{\mathbf{K}_{\mathrm{NF}}})$ be the obvious coding of $\mathcal{K}_{\mathrm{NF}}$ into an abstract class: the vocabulary is $\tau(\mathbf{K}_{\mathrm{NF}}) = \tau(\mathbf{K}) \cup \{P\}$, where P is a unary predicate, and we think of the members of K_{NF} as pairs of $\tau(\mathbf{K})$-structures (M, N) satisfying $M \leq_{\mathbf{K}} N$ (so the elements of M are the ones satisfying the predicate). We order $\mathbf{K}_{\mathrm{NF}}$ by $(M_0, M_1) \leq_{\mathbf{K}_{\mathrm{NF}}} (M_2, M_3)$ if and only if $M_1 \overset{M_3}{\underset{M_0}{\mathop{\smile\!\!\!\!\!\!|}}} M_2$. Let $\mathbf{K}^*_{\mathrm{NF}} = (K^*_{\mathrm{NF}}, \leq_{\mathbf{K}^*_{\mathrm{NF}}})$ be the subcategory of $\mathbf{K}_{\mathrm{NF}}$ where the squares are also required to be *pullbacks*. That is, $K^*_{\mathrm{NF}} = K_{\mathrm{NF}}$ and $(M_0, M_1) \leq_{\mathbf{K}^*_{\mathrm{NF}}} (M_2, M_3)$ if and only if $(M_0, M_1) \leq_{\mathbf{K}_{\mathrm{NF}}} (M_2, M_3)$ and $M_1 \cap M_2 = M_0$.

Remark 8.12. Assume that $\mathop{\smile\!\!\!\!\!\!|}\,$ is monotonic and right transitive. Then $\mathbf{K}_{\mathrm{NF}}$ is isomorphic (as a category) to $\mathcal{K}_{\mathrm{NF}}$, so we need not distinguish between the two. We have that $\mathbf{K}_{\mathrm{NF}}$ is closed under isomorphisms, is a partial order, and $\mathbf{K}^*_{\mathrm{NF}}$ satisfies the coherence axiom: if $(M_0, M_1) \subseteq (M_0', M_1') \leq_{\mathbf{K}_{\mathrm{NF}}} (M, N)$ and $(M_0, M_1) \leq_{\mathbf{K}_{\mathrm{NF}}} (M, N)$, then by monotonicity (Lemma 3.24) $(M_0, M_1) \leq_{\mathbf{K}_{\mathrm{NF}}} (M_0', M_1')$. The same argument shows that $\mathbf{K}^*_{\mathrm{NF}}$ also has these properties. However in $\mathbf{K}_{\mathrm{NF}}$, $\leq_{\mathbf{K}_{\mathrm{NF}}}$ may not extend the $\tau(\mathbf{K}_{\mathrm{NF}})$-substructure relation, while $\leq_{\mathbf{K}^*_{\mathrm{NF}}}$ does.

Theorem 8.13 (Characterization of stable independence)**.** Let $\mathbf{K}$ be a μ-AEC and let $\mathop{\smile\!\!\!\!\!\!|}\,$ be an invariant independence relation on $\mathbf{K}$. Assume that $\mathop{\smile\!\!\!\!\!\!|}\,$ is transitive, and has existence and uniqueness. The following are equivalent:

1. $\mathop{\smile\!\!\!\!\!\!|}\,$ is accessible.
2. $\mathbf{K}^*_{\mathrm{NF}}$ is a λ-AEC, for some λ.
3. $\mathop{\smile\!\!\!\!\!\!|}\,$ has the witness and local character properties.

Proof.

- <u>(1) implies (2)</u>: Assume that $\mathop{\smile\!\!\!\!\!\!|}\,$ is accessible. Pick λ_0 such that $\mathbf{K}_{\mathrm{NF}}$ is λ_0-accessible. By Remark 3.25, λ_0-directed colimits in $\mathbf{K}^2$ and $\mathbf{K}_{\mathrm{NF}}$ coincide, so in particular $\mathbf{K}_{\mathrm{NF}}$ has *concrete* λ_0-directed colimits. Together with the definition of $\mathbf{K}^*_{\mathrm{NF}}$, this directly implies that $\mathbf{K}^*_{\mathrm{NF}}$ has concrete λ_0-directed colimits. Now let λ be as given by Lemma 7.2(2) (applied to $\mathbf{K}_{\mathrm{NF}}$). We claim that $\mathbf{K}^*_{\mathrm{NF}}$ is λ^+-accessible. This is enough by Theorem 7.3. To see that $\mathbf{K}^*_{\mathrm{NF}}$ is λ^+-accessible, let $(M, N) \in \mathbf{K}^*_{\mathrm{NF}}$. By how λ was chosen, there is a λ^+-directed (according to the ordering of $\mathbf{K}_{\mathrm{NF}}$) system $\langle (M_i, N_i) : i \in I \rangle$ of objects of cardinality at most λ that is continuous at points of cofinality at least μ and so that $\bigcup_{i \in I} (M_i, N_i) = (M, N)$. By the choice of λ, $\lambda = \lambda^{<\mu}$ so $\mathrm{cf}(\lambda) \geq \mu$. Therefore the usual "catching your tail" argument gives that the set $J := \{i \in I \mid M \cap N_i = M_i\}$ is unbounded in I. Thus $\langle (M_j, N_j) : j \in J \rangle$ witnesses that (M, N) can be written as a λ^+-directed (in $\mathbf{K}^*_{\text{NF}}$) colimit of λ^+-presentable (since they have cardinality at most λ) objects, as desired.

- <u>(2) implies (3)</u>: Fix a regular cardinal λ such that $\mathbf{K}^*_{\text{NF}}$ is a λ-AEC. Fix M_0, M_1, M_2, M_3 such that $M_0 \leq_{\mathbf{K}} M_\ell \leq_{\mathbf{K}} M_3$, $\ell = 1, 2$ and for any $A \subseteq UM_1$ with $|A| \leq \text{LS}(\mathbf{K}^*_{\text{NF}})$, $A \overset{M_3}{\underset{M_0}{\downarrow}} M_2$ holds. We show that $M_1 \overset{M_3}{\underset{M_0}{\downarrow}} M_2$, which will establish the $(< \text{LS}(\mathbf{K}^*_{\text{NF}})^+)$-witness property. Since $\mathbf{K}^*_{\text{NF}}$ is a λ-AEC, we can find a $\text{LS}(\mathbf{K}^*_{\text{NF}})^+$-directed system $\langle (M_0^i, M_1^i) : i \in I \rangle$ such that $(M_0, M_1) = \bigcup_{i \in I} (M_0^i, M_1^i)$ and $|UM_1^i| \leq \text{LS}(\mathbf{K}^*_{\text{NF}})$ for all $i \in I$. This implies in particular that for any $i \in I$, $M_1^i \overset{M_1}{\underset{M_0^i}{\downarrow}} M_0$. We also know that $M_1^i \overset{M_3}{\underset{M_0}{\downarrow}} M_2$. By transitivity for $\overline{\downarrow}$ (Theorem 8.5), $M_1^i \overset{M_3}{\underset{M_0^i}{\downarrow}} M_2$. In other words, $(M_0^i, M_1^i) \leq_{\mathbf{K}^*_{\text{NF}}} (M_2, M_3)$ for all $i \in I$. By smoothness, this implies that $(M_0, M_1) \leq_{\mathbf{K}^*_{\text{NF}}} (M_2, M_3)$, i.e. $M_1 \overset{M_3}{\underset{M_0}{\downarrow}} M_2$, as desired.

  Similarly, local character holds: let $p \in \text{gS}^\alpha(M_1)$. Without loss of generality $\alpha \leq \text{LS}(\mathbf{K})$ (if not use the witness property). Without loss of generality, $p = \text{gtp}(\bar{a}/M_1; M_2)$. Since I is $\text{LS}(\mathbf{K}^*_{\text{NF}})^+$-directed, there is $i \in I$ such that $\bar{a} \in {}^\alpha M_1^i$. Thus in particular p does not fork over M_0^i. Since $|UM_0^i| \leq \text{LS}(\mathbf{K}^*_{\text{NF}})$, we are done.
- <u>(3) implies (1)</u>: Fix a regular cardinal θ such that $\downarrow$ has the $(< \theta)$-witness property and let λ be as given by Lemma 8.10. We may assume without loss of generality that $\lambda = \lambda^{\theta + \mu + \text{LS}(\mathbf{K})}$ and λ is regular. By Remark 8.12, it is enough to prove that $\mathbf{K}_{\text{NF}}$ satisfies the Löwenheim-Skolem-Tarski and chain axioms of λ-AECs.
  - <u>LST axiom</u>: Let $(M, N) \in \mathbf{K}_{\text{NF}}$ and let $A \subseteq UN$. Without loss of generality $|A|$ is infinite. We build $\langle M_i, N_i : i < \lambda \rangle$ increasing in $\mathbf{K}$ such that for all $i < \lambda$:
    1. $A \subseteq UN_0$.
    2. $M_i \leq_{\mathbf{K}} M$, $M_i \leq_{\mathbf{K}} N_i \leq_{\mathbf{K}} N$.
    3. $|UM_i| \leq |UN_i| \leq |A|^{<\lambda}$.
    4. $\bigcup_{j<i} UN_j \overset{N}{\underset{M_i}{\downarrow}} M$.

    This is enough: Let $M_\lambda := \bigcup_{i<\lambda} M_i$, $N_\lambda := \bigcup_{i<\lambda} N_i$. We claim that $N_\lambda \overset{N}{\underset{M_\lambda}{\downarrow}} M$. Indeed, recall that $\text{cf}(\lambda) \geq \theta$ so for any $B \subseteq UN_\lambda$ with $|B| < \theta$, there exists $i < \lambda$ such that $B \subseteq UN_i$. In particular by monotonicity $B \overset{N}{\underset{M_{i+1}}{\downarrow}} M$. By base monotonicity, $B \overset{N}{\underset{M_\lambda}{\downarrow}} M$. Since B was arbitrary, by the $(< \theta)$-witness property we indeed have that $N_\lambda \overset{N}{\underset{M_\lambda}{\downarrow}} M$. However we also have that $M_\lambda \leq_{\mathbf{K}} N_\lambda$, so $N_\lambda \underset{M_\lambda}{\overset{N}{\mathop{\smile}\limits^{\mid}}} M$, and hence $(M_\lambda, N_\lambda) \leq_{\mathbf{K}_{\text{NF}}} (M, N)$, as needed.

This is possible: given $\langle M_j : j < i \rangle$, $\langle N_j : j < i \rangle$, take any $M_i \leq_{\mathbf{K}} M$ such that $|UM_i| \leq |A|^{<\lambda}$, $M_j \leq_{\mathbf{K}} M_i$ for all $j < i$, $A \cap UM \subseteq UM_i$, and $\bigcup_{j<i} N_j \overline{\underset{M_i}{\overset{N}{\mathop{\smile}\limits^{\mid}}}} M$ (use local character, Lemma 8.10, and base monotonicity). Now given $\langle M_j : j \leq i \rangle$ and $\langle N_j : j < i \rangle$, pick any $N_i \leq_{\mathbf{K}} N$ such that $N_j \leq_{\mathbf{K}} N_i$ for all $j < i$, $|UN_i| \leq |A|^{<\lambda}$, $M_i \leq_{\mathbf{K}} N_i$, and $A \subseteq UN_i$.

– <u>Chain axioms</u>: Fix $\langle (M_i, N_i) : i \in I \rangle$ a λ-directed system in $\mathbf{K}_{\text{NF}}$ and let $(M, N) := \bigcup_{i \in I} (M_i, N_i)$. Clearly, $(M, N) \in \mathbf{K}_{\text{NF}}$. We first want to see that $N_i \underset{M_i}{\overset{N}{\mathop{\smile}\limits^{\mid}}} M$. We use the witness property. Fix $A \subseteq UM$ of size less than θ. Since $\theta \leq \lambda$, there exists $j \in I$ such that $i \leq j$ and $A \subseteq UM_j$. Since $N_i \underset{M_i}{\overset{N_j}{\mathop{\smile}\limits^{\mid}}} M_j$ by assumption, we must have by monotonicity that $N_i \overline{\underset{M_i}{\overset{N}{\mathop{\smile}\limits^{\mid}}}} A$. Since A was arbitrary, this implies that $N_i \underset{M_i}{\overset{N}{\mathop{\smile}\limits^{\mid}}} M$. Similarly, smoothness also holds: fix (M', N') such that $(M_i, N_i) \leq_{\mathbf{K}_{\text{NF}}} (M', N')$ for all $i \in I$. We want to show that $N \underset{M}{\overset{N'}{\mathop{\smile}\limits^{\mid}}} M'$. Fix $A \subseteq UN$ of size less than θ and fix $i \in I$ such that $A \subseteq UN_i$. By assumption, $N_i \underset{M_i}{\overset{N'}{\mathop{\smile}\limits^{\mid}}} M'$, so $A \overline{\underset{M_i}{\overset{N'}{\mathop{\smile}\limits^{\mid}}}} M'$, and so by base monotonicity $A \overline{\underset{M}{\overset{N'}{\mathop{\smile}\limits^{\mid}}}} M'$. Since A was arbitrary, the witness property implies that $N \underset{M}{\overset{N'}{\mathop{\smile}\limits^{\mid}}} M'$, as desired.

$\square$

Remark 8.14. The proof goes through if we assume only that $\mathop{\smile}\limits^{\mid}$ is invariant and monotonic, and that $\overline{\mathop{\smile}\limits^{\mid}}$ is transitive.

We deduce that having a stable independence notion implies stability and tameness:

Corollary 8.15. Let $\mathbf{K}$ be a μ-AEC with a stable independence relation. Then:

(1) (Stability for Galois types) For any α, there exists a proper class of cardinals λ such that $M \in \mathbf{K}_\lambda$ implies $|\text{gS}^{<\alpha}(M)| = \lambda$.
(2) (Tameness) For any α, there exists a cardinal λ such that for any $M \in \mathbf{K}$, $p, q \in \text{gS}^{<\alpha}(M)$, $p = q$ if and only if $p \upharpoonright A = q \upharpoonright A$ for all $A \subseteq UM$ with $|A| < \lambda$.

Proof. Use Theorem 8.13 and imitate the argument in [BGKV16, 5.17]: use local character and uniqueness. $\square$

We will see in the next section (Corollary 9.8) that if it has chain bounds then $\mathbf{K}$ also does not have a certain order property.

9. Canonicity, symmetry, and the order property

In this section, we prove (assuming chain bounds) that stable independence is canonical: there is at most one stable independence relation in any class. In fact, the symmetry property is not necessary, and hence can be deduced from the others. We show further that symmetry implies failure of an order property. Combined with Corollary 8.15, this shows that the class has several features of stable first-order theories.

Note that the proof of symmetry and the order property here are different from those in [BGKV16]: since we are working in a more general context than AECs, we do not have Ehrenfeucht-Mostowski models ([BGL$^+$16, 4.12]) and hence cannot directly deduce that the order property implies unstability as in [BGKV16, 5.13]. In fact, this fails in general, see Example 9.9.

Theorem 9.1 (The canonicity theorem)**.** Let $\mathbf{K}$ be a μ-AEC which has chain bounds (see Definition 7.7). Let $\downarrow$ be an invariant, transitive independence notion on $\downarrow$ with existence, uniqueness, and right local character.

Then any other invariant transitive relation on $\mathbf{K}$ satisfying existence, and uniqueness must be $\downarrow$.

Proof sketch. First note that the properties of $\downarrow$ carry over to $\overline{\downarrow}$, by Fact 8.4 and Theorem 8.5. By Theorem 7.11, we can work inside a monster model $\mathfrak{C}$ which is as homogeneous as we need (if $\mathbf{K}$ does not have joint embedding, we can partition it into subclasses that each have joint embedding by looking at the equivalence classes of the relation "embedding into a common model"). We want to imitate the argument of [BGKV16, 4.14]. It falls into two parts. The first part shows that $\downarrow$ must have a property there called (E_+) (a strong existence/extension property, which we recall below). The second part shows that having (E_+) implies canonicity. This latter part is implemented in [BGKV16, 4.8], and does not use the fact that $\mathbf{K}$ is an AEC: the argument works in any μ-AEC (and, in fact, in any coherent abstract class).

It remains to check that $\downarrow$ has (E_+): for any $M \leq_{\mathbf{K}} N_0$ and set A, there is $N \leq_{\mathbf{K}} N_0$ such that for all $N' \equiv_{N_0} N$, there is $N_0' \equiv_M N_0$ with $A \overset{\mathfrak{C}}{\underset{M}{\overline{\downarrow}}} N_0'$ and $N_0' \leq_{\mathbf{K}} N'$ (see [BGKV16, 4.4]). The proof of (E_+) uses independent sequences. These are sequences $\langle A_i : i < \delta \rangle$ (inside a monster model) such that for some $\langle N_i : i < \delta \rangle$ (called the *witnesses*), $\langle N_i : i < \delta \rangle$ is an increasing chain, $\bigcup_{j<i} A_j \subseteq UN_i$ and $A_i \overset{\mathfrak{C}}{\underset{N_0}{\overline{\downarrow}}} N_i$ for all $i < \delta$. We say that $\langle A_i : i < \delta \rangle$ is *independent over M* when there is a witnessing sequence $\langle N_i : i < \delta \rangle$ with $M = N_0$.

The first step [BGKV16, 4.10] is to show that we can always build independent sequences: given A, M, and δ, there exists $\langle A_i : i < \delta \rangle$ that are independent over M and such that the type of each A_i over M is the same as the type of A over M

(for some enumerations of A_i and A). The argument uses the extension property for $\overline{\mathop{\smile\!\!\!\!\!\!|}}$ and goes through in the present setup too.

The second and last ingredient in the proof of (E_+) is to show [BGKV16, 4.11] a certain local character property of independent sequences. The proof uses symmetry, so we will instead use the following variation:

Claim: Let A be a set and let $\lambda \geq \mu$ be a regular cardinal such that any type of a sequence of length $|A|$ does not fork over a model of cardinality strictly less than λ (this exists by right local character). Then whenever $\langle M_i : i < \lambda \rangle$ is a $\stackrel{d}{\mathop{\smile\!\!\!\!\!\!|}}$-independent sequence (i.e. an independent sequence with respect to $\stackrel{d}{\mathop{\smile\!\!\!\!\!\!|}}$) with $M \leq_{\mathbf{K}} M_i$ for all $i < \lambda$, then there is $i < \lambda$ with $A \overset{\mathfrak{c}}{\underset{M}{\mathop{\smile\!\!\!\!\!\!|}}} M_i$.

Proof of Claim: This is the same proof as in [BGKV16, 4.11], but we give it for the convenience of the reader. Let $\langle N_i : i < \lambda \rangle$ witness the independence and let $N_\lambda := \bigcup_{i<\lambda} N_i$. By right local character and base monotonicity, there exists $i < \lambda$ such that $A \overset{\mathfrak{c}}{\underset{N_i}{\mathop{\smile\!\!\!\!\!\!|}}} N_\lambda$. By monotonicity, $A \overset{\mathfrak{c}}{\underset{N_i}{\mathop{\smile\!\!\!\!\!\!|}}} M_i$. Since the M_i's are $\stackrel{d}{\mathop{\smile\!\!\!\!\!\!|}}$-independent, we also have that $N_i \overset{\mathfrak{c}}{\underset{M}{\mathop{\smile\!\!\!\!\!\!|}}} M_i$. Using left transitivity, we get that $A \overset{\mathfrak{c}}{\underset{M}{\mathop{\smile\!\!\!\!\!\!|}}} M_i$, as desired. $\dagger_{\text{Claim}}$

Now that the claim is proven, the argument of [BGKV16, 4.13] goes through to show that $\mathop{\smile\!\!\!\!\!\!|}$ has (E_+), completing the proof. $\square$

Remark 9.2. It is enough to assume that $\mathbf{K}$ has a monster model instead of having chain bounds, see Theorem 7.11.

We deduce the symmetry property:

Corollary 9.3. If $\mathbf{K}$ is a μ-AEC which has chain bounds and $\mathop{\smile\!\!\!\!\!\!|}$ is an invariant, transitive independence notion with existence, uniqueness and right local character, then $\mathop{\smile\!\!\!\!\!\!|}$ is symmetric. Thus if $\mathop{\smile\!\!\!\!\!\!|}$ has in addition the right (or left) witness property, then it is a stable independence relation.

Proof. $\stackrel{d}{\mathop{\smile\!\!\!\!\!\!|}}$ is invariant, transitive, and has existence and uniqueness, so by Theorem 9.1 $\mathop{\smile\!\!\!\!\!\!|} = \stackrel{d}{\mathop{\smile\!\!\!\!\!\!|}}$. The last sentence follows from Theorem 8.13. $\square$

Corollary 9.4. Let $\mathcal{K}$ be a category which has chain bounds and whose morphisms are monomorphisms. Then there is at most one stable independence notion on $\mathcal{K}$.

Proof. By Lemma 3.20, the equivalence between μ-AECs and accessible categories with all morphisms monomorphisms, Theorem 8.13, and Theorem 9.1. $\square$

Question 9.5. Can one prove an even more general canonicity result? What if the morphisms of the category are not all monomorphisms? What if the category does not have chain bounds?

We will use the following definition of the order property, introduced by Shelah for AECs [She99, 4.3]. In the first-order case, it is equivalent to the usual definition.

Definition 9.6. A μ-AEC $\mathbf{K}$ has the *α-order property of length θ* if there exists $M \in \mathbf{K}$ and a sequence $\langle \bar{a}_i : i < \theta \rangle$ such that:

(1) $\bar{a}_i \in {}^{\alpha}|M|$ for all $i < \theta$.
(2) For all $i_0 < j_0 < \theta$ and $i_1 < j_1 < \theta$, $\mathrm{gtp}(\bar{a}_{i_0}\bar{a}_{j_0}/\emptyset; M) \neq \mathrm{gtp}(\bar{a}_{j_1}\bar{a}_{i_1}/\emptyset; M)$.

$\mathbf{K}$ has the *order property* if there exists α such that for all θ, $\mathbf{K}$ has the α-order property of length θ.

Theorem 9.7. Let $\mathbf{K}$ be a μ-AEC and $\mathop{\smile}\limits^{\vert}$ be an invariant, transitive independence notion with existence, uniqueness, and right local character. If $\mathbf{K}$ has chain bounds, or more generally if $\mathop{\smile}\limits^{\vert}$ has symmetry, then $\mathbf{K}$ does *not* have the order property.

Proof. Recall from Corollary 9.3 that chain bounds implies symmetry in the context of the theorem. Assume for a contradiction that $\mathbf{K}$ has the order property. Pick α and θ such that $\mathbf{K}$ has the α-order property of length θ^+, with $\theta = \theta^{<\mu} \geq \mathrm{LS}(\mathbf{K})$ "sufficiently big" (the proof will tell us how big) such that $|\mathrm{gS}^{\alpha}(M)| \leq \theta$ for all $M \in \mathbf{K}_{\leq \theta}$ (exists by the proof of Corollary 8.15). Pick $\langle \bar{a}_i : i < \theta^+ \rangle$ and M witnessing the α-order property of length θ^+. Let $\langle M_i : i < \theta \rangle$ be an increasing sequence of submodels of M such that $\{\bar{a}_j : j < i\} \subseteq UM_i$ for all $i < \theta^+$ and $|UM_i| \leq \theta$. For each $i < \theta^+$ of sufficiently high cofinality, there exists $j < i$ such that $\mathrm{gtp}(\bar{a}_i/M_i)$ does not fork over M_j. By Fodor's lemma, we can therefore assume without loss of generality that $\mathrm{gtp}(\bar{a}_i/M_i; M)$ does not fork over M_0 for all $i < \theta^+$. Since $|\mathrm{gS}^{\alpha}(M_0)| \leq \theta$, we can further assume without loss of generality that $\mathrm{gtp}(\bar{a}_i/M_0; M) = \mathrm{gtp}(\bar{a}_{i'}/M_0; M)$ for all $i < i' < \theta^+$. By uniqueness, this implies that $\mathrm{gtp}(\bar{a}_i/M_i; M) = \mathrm{gtp}(\bar{a}_{i'}/M_i; M)$.

We are now in the following setup: $\bar{a}_0\bar{a}_1$ is a two-element independent sequence over M_0 (in this case, this means that $\bar{a}_0 \mathop{\smile}\limits^{M}_{M_0} \bar{a}_1$), and so by symmetry also $\bar{a}_1 \mathop{\smile}\limits^{M}_{M_0} \bar{a}_0$, so $\bar{a}_1\bar{a}_0$ is a two-element independent sequence over M_0. Now the proof of [BV17b, 4.8] tells us that if we have two two-element independent sequences $\bar{a}\bar{b}$ and $\bar{a}'\bar{b}'$ over a model M_0, so that the types of their individual elements are equal over M_0 (i.e. $\bar{a}$ and $\bar{a}'$ agree over M_0 and $\bar{b}$ and $\bar{b}'$ agree over M_0), then in fact $\bar{a}\bar{b}$ and $\bar{b}\bar{b}'$ have the same type over M_0. Applying this here, we get that $\mathrm{gtp}(\bar{a}_0\bar{a}_1/M_0; M) = \mathrm{gtp}(\bar{a}_1\bar{a}_0/M_0; M)$. This contradicts that the sequence $\langle \bar{a}_i : i < \theta \rangle$ witnessed the order property. $\square$

Corollary 9.8. If $\mathbf{K}$ is a μ-AEC with a stable independence relation, then $\mathbf{K}$ does not have the order property.

Proof. This is a consequence of Theorem 9.7, using the equivalence given by Theorem 8.13. $\square$

Shelah [She, §1] examines several definitions of stability for $\mathbb{L}_{\kappa,\kappa}$, κ a strongly compact cardinal, and shows that, while there are natural implications, there are also several non-equivalence. The following is a simple example:

Example 9.9. Let $\mathbf{K}$ be the $\aleph_1$-AEC of well-orderings, ordered by being a suborder. Then $\mathbf{K}$ has the order property but for every $M \in \mathbf{K}$, $|\mathrm{gS}^{<\omega}(M)| = |UM| + \aleph_0$.

On the other hand it is known that failure of the order property implies stability in terms of counting Galois types. Roughly, this is because the proof of the corresponding first-order fact can be carried out inside a fixed model, see [She09b, §V.A], [She, §1], or the proof of [Vas16b, 4.11].

10. Stable independence on saturated models

Putting together [BG17, BGKV16, Vas16b], one obtains the following converse to Corollary 9.8: Assuming large cardinals, any μ-AEC which has chain bounds and does not have the order property admits a stable independence relation on a subclass of model-homogeneous models. This was essentially observed for AECs with amalgamation by Boney and Grossberg [BG17, 8.2], although categoricity is used there to prove local character. We work from the result of [Vas16b, §5], which focuses on the stable case and avoids any use of categoricity.

Theorem 10.1 (The existence theorem)**.** Assume Vopěnka's principle. Let $\mathbf{K}$ be a μ-AEC which has chain bounds. Let $\kappa > \mathrm{LS}(\mathbf{K})$ be strongly compact. If $\mathbf{K}$ does not have the order property, then the κ-AEC $\mathbf{K}^{\kappa\text{-lmh}}$ of locally κ-model-homogeneous models of $\mathbf{K}$ (see Definition 7.4) has a stable independence relation.

Proof sketch. By Theorem 7.8, $\mathbf{K}^{\kappa\text{-lmh}}$ has amalgamation and is indeed a κ-AEC. First observe that for any $\alpha < \kappa$, $\mathbf{K}$ does not have the α-order property of length κ. If it did, then we would be able to take repeated ultraproducts of the universe to make the sequence longer and get that $\mathbf{K}$ has the α-order property of any length. Moreover, the strongly compact also gives that $\mathbf{K}$ is fully $(< \kappa)$-tame and short (see [BGL+16, 5.5]).

Now define $M_1 \overset{M_3}{\underset{M_0}{\perp}} M_2$ to hold if and only if $M_0 \leq_{\mathbf{K}} M_\ell \leq_{\mathbf{K}} M_3$, $\ell = 1, 2$, all are in $\mathbf{K}^{\kappa\text{-lmh}}$, and for any $\bar{a} \in {}^{<\kappa} M_1$, $\mathrm{gtp}(\bar{a}/M_2; M_3)$ is a κ-coheir over M_0. That is, for any $A \subseteq UM_2$ of cardinality less than κ, $\mathrm{gtp}(\bar{a}/A; M_3)$ is realized inside M_0.

By the proofs of [BG17] or [Vas16b, §5], we get all the required properties. More precisely, it is straightforward to check that $\perp$ is invariant and monotonic. Transitivity, symmetry, and uniqueness are proven as in [BG17, §4] or [Vas16b, 5.15]. The $(< \kappa)$-witness property then follows from the definition. Local character is [Vas16b, 5.15(2c)] and existence is [BG17, 8.2]. Using Theorem 8.13, we get that $\perp$ is stable, as desired. □

Remark 10.2. The assumption of Vopěnka's principle can be removed if $\mathbf{K}$ has amalgamation.

We summarize:

Corollary 10.3. Assume Vopěnka's principle. Let $\mathbf{K}$ be a μ-AEC which has chain bounds. The following are equivalent:

(1) $\mathbf{K}$ does not have the order property.
(2) For some sub-λ-AEC $\mathbf{K}^* \subseteq \mathbf{K}$ which is cofinal in $\mathbf{K}$ and has the same ordering as $\mathbf{K}$, $\mathbf{K}^*$ has a stable independence notion.

Proof. Combine Theorems 9.7 (noting that if $\mathbf{K}$ has the order property, then any of its cofinal subclasses must have it) and 10.1. $\square$

Note that in general, one can always restrict a stable independence relation to a subclass of model-homogeneous models. In fact:

Lemma 10.4. Let $\mathbf{K}$ be a μ-AEC with a stable independence relation $\mathop{\smile}\limits^{\mid}$. Let $\mathbf{K}^*$ be a subclass of $\mathbf{K}$ (ordered by the appropriate restriction of $\leq_{\mathbf{K}}$). If:

(1) $\mathbf{K}^*$ is a λ-AEC, for some λ.
(2) $\mathbf{K}^*$ is cofinal in $\mathbf{K}$ (that is any $M \in \mathbf{K}$ extends to an $N \in \mathbf{K}^*$).

Then the restriction $\overset{*}{\mathop{\smile}\limits^{\mid}}$ of $\mathop{\smile}\limits^{\mid}$ to $\mathbf{K}^*$ is a stable independence relation on $\mathbf{K}^*$.

Proof sketch. We use Theorem 8.13. It is straightforward to check that $\overset{*}{\mathop{\smile}\limits^{\mid}}$ is an invariant, monotonic, symmetric, and transitive independence relation with the witness property. Existence and uniqueness follow from the corresponding properties of $\mathop{\smile}\limits^{\mid}$ and the fact that $\mathbf{K}^*$ is cofinal in $\mathbf{K}$. To see local character, we use local character of $\mathop{\smile}\limits^{\mid}$ together with the fact that $\mathbf{K}^*$ is a λ-AEC, hence satisfies the Löwenheim-Skolem-Tarski axiom. $\square$

Restricting to a subclass of sufficiently homogeneous models, we can also get that any stable square is a pullback square (i.e. is disjoint over the base). The argument is similar to [Vas16a, 12.13] but we give a self-contained version here.

Lemma 10.5. Let $\mathop{\smile}\limits^{\mid}$ be a stable independence relation on a μ-AEC $\mathbf{K}$. Let $\lambda > \mathrm{LS}(\mathbf{K})$ and let $M \leq_{\mathbf{K}} N$ both be in $\mathbf{K}$ with M λ-model-homogeneous. Let $p \in \mathrm{gS}(N)$ and assume that p does not fork over M_0, with $M_0 \leq_{\mathbf{K}} M$ and $|UM_0| < \lambda$. Then p is algebraic (i.e. realized inside its domain) if and only if $p \upharpoonright M$ is algebraic.

Proof. If $p \upharpoonright M$ is algebraic, then p is clearly algebraic. Conversely, assume that p is algebraic. Pick $a \in N$ realizing p and let $N_0 \in \mathbf{K}_{<\lambda}$ be such that $M_0 \leq_{\mathbf{K}} N_0 \leq_{\mathbf{K}} N$ and N_0 contains a. Since M is λ-model-homogeneous, there exists $f : N_0 \to M$ fixing M_0. Now by monotonicity $p \upharpoonright N_0$ does not fork over M_0, so $f(p \upharpoonright N_0)$ does not fork over M_0. Since f fixes M_0, $p \upharpoonright M_0 = f(p \upharpoonright N_0) \upharpoonright M_0$, so by uniqueness $f(p \upharpoonright N_0) = p \upharpoonright f[N_0]$. Since $p \upharpoonright N_0$ is algebraic, this implies that $p \upharpoonright f[N_0]$ also is. Since $f[N_0] \leq_{\mathbf{K}} M$, $p \upharpoonright M$ is algebraic, as desired. $\square$

Theorem 10.6. Let $\mathop{\smile}\limits^{\mid}$ be a stable independence relation on a μ-AEC $\mathbf{K}$. There exists a regular cardinal $\lambda > \mathrm{LS}(\mathbf{K})$ such that if $M_1 \underset{M_0}{\overset{M_3}{\mathop{\smile}\limits^{\mid}}} M_2$ and M_0 is λ-model-homogeneous, then $M_1 \cap M_2 = M_0$.

Proof. By local character, we can pick $\lambda > \mathrm{LS}(\mathbf{K})$ regular so that any type of length one does not fork over a model of cardinality strictly less than λ. Now assume that $M_1 \underset{M_0}{\overset{M_3}{\mathop{\smile}\limits^{\mid}}} M_2$ and M_0 is λ-model-homogeneous. Let $a \in M_1 \cap M_2$. We show that $a \in M_0$. Let $p := \mathrm{gtp}(a/M_2; M_3)$. Note that p is algebraic. By how λ was chosen, there exists $M_0' \leq_{\mathbf{K}} M_0$ such that $M_0' \in \mathbf{K}_{<\lambda}$ and $p \upharpoonright M_0$ does not fork over M_0'.

By transitivity, p does not fork over M_0'. By Lemma 10.5 (where M_0, M, N there stand for M_0', M_0, M_2 here), $p \upharpoonright M_0$ is algebraic, so $a \in M_0$, as desired. $\square$

References

[AHS04] Jiří Adámek, Horst Herrlich, and George E. Strecker, *Abstract and concrete categories*, online edition ed., 2004, Available from `http://katmat.math.uni-bremen.de/acc/`.

[AHT96] J. Adamek, H. Hu, and W. Tholen, *On pure morphisms in accessible categories*, The Journal of Pure and Applied Algebra **107** (1996), 1–8.

[AR94] Jiří Adamek and Jiří Rosický, *Locally presentable and accessible categories*, London Math. Society Lecture Notes, Cambridge University Press, 1994.

[Bar71] Michael Barr, *Exact categories*, Exact Categories and Categories of Sheaves, Lecture Notes in Mathematics, vol. 236, Springer, 1971.

[Bar88] ———, *On categories with effective unions*, Categorical Algebra and its Applications (F. Borceux, ed.), Lecture Notes in Mathematics, vol. 1348, Springer, 1988, pp. 19–35.

[Bek00] Tibor Beke, *Sheafifiable homotopy model category*, Mathematical Proceedings of the Cambridge Philosophical society **129** (2000), 447–475.

[BG17] Will Boney and Rami Grossberg, *Forking in short and tame AECs*, Annals of Pure and Applied Logic **168** (2017), no. 8, 1517–1551.

[BGKV16] Will Boney, Rami Grossberg, Alexei Kolesnikov, and Sebastien Vasey, *Canonical forking in AECs*, Annals of Pure and Applied Logic **167** (2016), no. 7, 590–613.

[BGL+16] Will Boney, Rami Grossberg, Michael J. Lieberman, Jiří Rosický, and Sebastien Vasey, *μ-Abstract elementary classes and other generalizations*, The Journal of Pure and Applied Algebra **220** (2016), no. 9, 3048–3066.

[Bon14] Will Boney, *Tameness from large cardinal axioms*, The Journal of Symbolic Logic **79** (2014), no. 4, 1092–1119.

[BR12] Tibor Beke and Jiří Rosický, *Abstract elementary classes and accessible categories*, Annals of Pure and Applied Logic **163** (2012), 2008–2017.

[BV17a] Will Boney and Sebastien Vasey, *A survey on tame abstract elementary classes*, Beyond first order model theory (José Iovino, ed.), CRC Press, 2017, pp. 353–427.

[BV17b] ———, *Tameness and frames revisited*, The Journal of Symbolic Logic **82** (2017), no. 3, 995–1021.

[BYBHU08] Itay Ben-Yaacov, Alexander Berenstein, C. Ward Henson, and Alexander Usvyatsov, *Model theory for metric structures*, Model theory with applications to algebra and analysis (Zoé Chatzidakis, Dugald Macpherson, Anand Pillay, and Alex Wilkie, eds.), vol. 2, Cambridge University Press, 2008, pp. 315–427.

[HH84] Victor Harnik and Leo Harrington, *Fundamentals of forking*, Annals of Pure and Applied Logic **26** (1984), 245–286.

[Iov99] José Iovino, *Stable Banach spaces and Banach space structures, II: forking and compact topologies*, Models, Algebras, and proofs (X. Caceido and C. Montenegro, eds.), Marcel Dekker, 1999.

[Joh02] P. T. Johnston, *Sketches of an elephant : a topos theory compendium*, Oxford University Press, 2002.

[JS13] Adi Jarden and Saharon Shelah, *Non-forking frames in abstract elementary classes*, Annals of Pure and Applied Logic **164** (2013), 135–191.

[Kel69] G. M. Kelly, *Monomorphisms, epimorphisms, and pull-backs*, Journal of the Australian Mathematical society **9** (1969), 124–142.

[KM81] J.-L. Krivine and B. Maurey, *Espaces de banach stables*, Israel Journal of Mathematics **39** (1981), no. 4, 273–295.

[LRVa] Michael J. Lieberman, Jiří Rosický, and Sebastien Vasey, *Internal sizes in μ-abstract elementary classes*, Preprint. URL: `http://arxiv.org/abs/1708.06782v2`.

[LRVb] ———, *Universal abstract elementary classes and locally multipresentable categories*, Preprint. URL: `http://arxiv.org/abs/1707.09005v2`.

[Mak84] Michael Makkai, *A survey of basic stability theory, with particular emphasis on orthogonality and regular types*, Israel Journal of Mathematics **49** (1984), no. 1, 181–238.

[MP89] Michael Makkai and Robert Paré, *Accessible categories: The foundations of categorical model theory*, Contemporary Mathematics, vol. 104, American Mathematical Society, 1989.

[MS90] Michael Makkai and Saharon Shelah, *Categoricity of theories in $L_{\kappa,\omega}$, with κ a compact cardinal*, Annals of Pure and Applied Logic **47** (1990), 41–97.

[Pre88] Mike Prest, *Model theory and modules*, London Math. Society Lecture Notes, vol. 130, Cambridge University Press, 1988.

[Rin72] Claus Michael Ringel, *The intersection property of amalgamations*, Journal of Pure and Applied Algebra **2** (1972), 341–342.

[Ros] Jiří Rosický, *On the uniqueness of cellular injectives*, Preprint. URL: `http://arxiv.org/abs/1702.08684v1`.

[Ros97] ———, *Accessible categories, saturation and categoricity*, The Journal of Symbolic Logic **62** (1997), no. 3, 891–901.

[She] Saharon Shelah, *Model theory for a compact cardinal*, Preprint. URL: `http://arxiv.org/abs/1303.5247v3`.

[She70] ———, *Finite diagrams stable in power*, Annals of Mathematical Logic **2** (1970), no. 1, 69–118.

[She83a] ———, *Classification theory for non-elementary classes I: The number of uncountable models of $\psi \in L_{\omega_1,\omega}$. Part A*, Israel Journal of Mathematics **46** (1983), no. 3, 214–240.

[She83b] ———, *Classification theory for non-elementary classes I: The number of uncountable models of $\psi \in L_{\omega_1,\omega}$. Part B*, Israel Journal of Mathematics **46** (1983), no. 4, 241–273.

[She85] ———, *Classification of first order theories which have a structure theorem*, Bulletin of the American Mathematical Society **12** (1985), no. 2, 227–232.

[She87a] ———, *Classification of non elementary classes II. Abstract elementary classes*, Classification Theory (Chicago, IL, 1985) (John T. Baldwin, ed.), Lecture Notes in Mathematics, vol. 1292, Springer-Verlag, 1987, pp. 419–497.

[She87b] ———, *Universal classes*, Classification theory (Chicago, IL, 1985) (John T. Baldwin, ed.), Lecture Notes in Mathematics, vol. 1292, Springer-Verlag, 1987, pp. 264–418.

[She90] ———, *Classification theory and the number of non-isomorphic models*, 2nd ed., Studies in logic and the foundations of mathematics, vol. 92, North-Holland, 1990.

[She99] ———, *Categoricity for abstract classes with amalgamation*, Annals of Pure and Applied Logic **98** (1999), no. 1, 261–294.

[She09a] ———, *Classification theory for abstract elementary classes*, Studies in Logic: Mathematical logic and foundations, vol. 18, College Publications, 2009.

[She09b] ———, *Classification theory for abstract elementary classes 2*, Studies in Logic: Mathematical logic and foundations, vol. 20, College Publications, 2009.

[Vas16a] Sebastien Vasey, *Building independence relations in abstract elementary classes*, Annals of Pure and Applied Logic **167** (2016), no. 11, 1029–1092.

[Vas16b] ———, *Infinitary stability theory*, Archive for Mathematical Logic **55** (2016), 567–592.

[Vas17a] ———, *Shelah's eventual categoricity conjecture in universal classes: part I*, Annals of Pure and Applied Logic **168** (2017), no. 9, 1609–1642.

[Vas17b] ———, *Shelah's eventual categoricity conjecture in universal classes: part II*, Selecta Mathematica **23** (2017), no. 2, 1469–1506.

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