SHELAH'S EVENTUAL CATEGORICITY CONJECTURE IN UNIVERSAL CLASSES

SEBASTIEN VASEY

Abstract. We show that, in universal classes, categoricity in two suitable cardinals implies categoricity on a tail of cardinals:

Theorem 0.1. Let K be a universal class. For an infinite cardinal θ, write $h(\theta) := \beth_{(2^\theta)^+}$. If K is categorical in two cardinals λ and μ with $\mu \geq h(|L(K)| + \aleph_0)$ and $\lambda > h(h(\mu))$, then K is categorical in all $\lambda' \geq h(h(\mu))$.

The proof stems from ideas of Adi Jarden and Will Boney, and also relies on results of Shelah. We deduce that Shelah's eventual categoricity conjecture holds in universal classes. As opposed to previous works, the proof is in ZFC and does not use the assumption of categoricity in a successor cardinal. The argument generalizes to abstract elementary classes (AECs) with amalgamation that satisfy a locality property and where certain prime models exist:

Theorem 0.2. Let K be an AEC with amalgamation and no maximal models. Assume that K is fully $\mathrm{LS}(K)$-tame and short and for all M, N in K with $M \leq N$ and any element $a \in N$ there is a prime model $M_a \leq N$ over $M \cup \{a\}$. If K is categorical in a $\lambda > h(h(\mathrm{LS}(K)))$, then K is categorical in all $\lambda' \geq h(h(\mathrm{LS}(K)))$.

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1. Introduction

Morley's categoricity theorem [Mor65] states that a first-order countable theory that is categorical in some uncountable cardinal must be categorical in all uncountable cardinals. The result motivated much of the development of first-order classification theory (it was later generalized by Shelah [She74] to uncountable theories).

Toward developing a classification theory for non-elementary classes, one can ask whether there is such a result for infinitary logics, e.g. for an $L_{\omega_1,\omega}$ sentence. In 1971, Keisler proved [Kei71, Section 23] a generalization of Morley's theorem to this framework assuming in addition that the model in the categoricity cardinal is sequentially homogeneous. Unfortunately Shelah later observed using an example of Marcus [Mar72] that Keisler's assumption does not follow from categoricity. Still in the later seventies Shelah proposed the following far-reaching conjecture:

Conjecture 1.1 (Open problem D.(3a) in [She90])**.** If L is a countable language and $\psi \in L_{\omega_1,\omega}$ is categorical in one $\mu \geq \beth_{\omega_1}$, then it is categorical in all $\mu \geq \beth_{\omega_1}$.

This has now become the central test problem in classification theory for non-elementary classes. Shelah alone has more than 2000 pages of approximations (for example [She75, She83a, She83b, MS90, She99, She01, She09a, She09b]). Shelah's results led him to introduce a semantic framework encompassing many different infinitary logics and algebraic classes [She87a]: abstract elementary classes (AECs). In this framework, we can state an eventual version of the conjecture[1]:

[1] The statement here appears in [She09a, Conjecture N.4.2].

Conjecture 1.2 (Shelah's eventual categoricity conjecture for AECs)**.** An AEC that is categorical in a high-enough cardinal is categorical on a tail of cardinals.

Remark 1.3. A more precise statement is that there should be a function $\mu \mapsto \lambda_\mu$ such that every AEC K categorical in some $\lambda \geq \lambda_{\text{LS}(K)}$ is categorical in all $\lambda' \geq \lambda_{\text{LS}(K)}$. By a similar argument as for the existence of Hanf numbers [Han60] (see [Bal09, Conclusion 15.13]), Shelah's eventual categoricity conjecture for AECs is equivalent to the statement that an AEC categorical in *unboundedly many* cardinals is categorical on a tail of cardinals. We will use this equivalence freely.

Positive results are known in less general frameworks: For homogeneous model theory by Lessmann [Les00] and more generally for tame[2] simple finitary AECs by Hyttinen and Kesälä [HK11] (note that these results apply only to countable languages). In uncountable languages, Grossberg and VanDieren proved the conjecture in tame AECs categorical in a successor cardinal [GV06c, GV06a]. Later Will Boney pointed out that tameness follows[3] from large cardinals [Bon14b], a result that (as pointed out in [LR]) can also be derived from a 25 year old theorem of Makkai and Paré ([MP89, Theorem 5.5.1]). A combination of this gives that statements much stronger than Shelah's categoricity conjecture for a successor hold if there exists a proper class of strongly compact cardinals.

The question of whether categoricity in a sufficiently high *limit* cardinal implies categoricity on a tail remains open (even in tame AECs). The central tool there are *good λ-frames*, a local notion of forking which is the main concept in [She09a]. After developing the theory of good λ-frames over several hundreds of pages, Shelah claims to be able to prove the following (see [She09a, Discussion III.12.40], a proof should appear in [She]):

Claim 1.4. Assume the weak generalized continuum hypothesis[4]. Let K be an AEC such that there is an ω-successful[5] good λ-frame with

[2]Tameness is a locality property for orbital types introduced by Grossberg and VanDieren in [GV06b].

[3]Recently Boney and Unger [BU] established that the statement "all AECs are tame" is in fact *equivalent* to a large cardinal axioms (the existence of a proper class of almost strongly compact cardinals).

[4]The weak generalized continuum hypothesis (WGCH) is the statement that for all cardinals λ, $2^\lambda < 2^{\lambda^+}$.

[5]See Appendix A for a definition of good frames and the related technical terms.

underlying class K_λ. If K is categorical in λ and in some $\mu > \lambda^{+\omega}$, then K is categorical in *all* $\mu > \lambda^{+\omega}$.

Using the additional locality assumption of shortness and Shelah's unpublished claim, we managed to prove[6] (see [Vasb, Theorem 1.6]):

Fact 1.5. Assume the weak generalized continuum hypothesis and Claim 1.4. Then a fully tame and short AEC with amalgamation that is categorical in a high-enough cardinal is categorical on a tail of cardinals.

Note that Fact 1.5 applies in particular to homogeneous model theory and fully tame and short finitary AECs with uncountable language, a case that could not previously be dealt with.

Now a conjecture of Grossberg made in 1986 (see Grossberg [Gro02, Conjecture 2.3]) is that categoricity of an AEC in a high-enough cardinal should imply amalgamation (above a certain Hanf number). This is especially relevant considering that all the positive results above assume amalgamation. In the presence of large cardinals, Grossberg's conjecture is known to be true (This was first pointed out by Will Boney for general AECs, see [Bon14b, Theorem 4.3] and the discussion around Theorem 7.6 there. The key is that the proofs in [MS90, Proposition 1.13] or the stronger [SK96] which are for classes of models of an $L_{\kappa,\omega}$ sentence, κ a large cardinal, carry over to AECs K with $\mathrm{LS}(K) < \kappa$). In recent years it has been shown that many results that could be proven using large cardinals can be proven using just the model-theoretic assumption of tameness or shortness (see all of the above papers on tameness and for example [Vasc, BVb]). Thus one can ask whether tameness suffices to get amalgamation from categoricity. In general, this is not known. The only approximation is a result of Adi Jarden [Jarb] discussed more at length in Section 4. Our contribution is a weak version of amalgamation which one can assume alongside tameness to prove Grossberg's conjecture:

Corollary 4.15. Let K be tame AEC categorical in a high-enough cardinal. If K is eventually syntactically characterizable[7] and has weak amalgamation (see Definition 4.10), then there exists λ such that $K_{\geq\lambda}$ has amalgamation.

[6] In [She09a, Theorem IV.7.12], Shelah claims to prove Fact 1.5 without tameness or shortness. However we were unable to verify Shelah's proof. Also, the statement contains an error (it contradicts Morley's categoricity theorem).

[7] A technical condition discussed more at length in Section 4.

The proof uses a deep result of Shelah showing that a categorical AEC is well-behaved in a specific cardinal, then uses tameness and weak amalgamation to transfer the good behavior up.

We apply our result to *universal classes*. Universal classes were introduced by Shelah in [She87b] as an important framework where he thought finding dividing lines should be possible[8]. For many years, Shelah has claimed a main gap theorem for these classes but the full proof has not appeared in print. The most recent version is Chapter V of [She09b] which contains hundreds of pages of approximations. The methods used are stability theory inside a model (averages) as well as combinatorial tools to build many models. Combining Shelah's tools with arguments in [She09a, Chapter IV], we can actually bypass[9] Corollary 4.15 to show:

Theorem 5.6. Let K be a universal class. If K is categorical in a $\lambda \geq \beth_{\left(2^{|L(K)|+\aleph_0}\right)^+}$, then $K_{\geq \lambda}$ has amalgamation and no maximal models.

We also show that universal classes are tame[10] (in fact fully $(< \aleph_0)$-tame and short) and have weak amalgamation. Thus using Fact 1.5 we already get Shelah's categoricity conjecture for universal classes assuming WGCH and Claim 1.4. If the universal class is categorical in a successor, we can use [GV06a] instead to get a categoricity transfer in ZFC.

By relying on Shelah's analysis of frames in [She09a, Chapter III] as well as the frame transfer theorems in [Bon14a, BVc], we can also prove that Claim 1.4 holds in ZFC for universal classes (this makes use of the methods of proof of Corollary 4.15). After some Hanf number computations, we obtain Theorem 0.1 in the abstract (see Corollary 6.21). In particular, the eventual version of Conjecture 1.1 is true when ψ is a universal sentence (see Corollary 6.22). Note that the result also holds in uncountable languages.

Our results apply to a more general context than universal classes: fully tame and short AECs with amalgamation and no maximal models which have a prime model over every set of the form $M \cup \{a\}$ for M

[8]We were told by Rami Grossberg that another motivation was to study certain non first-order classes of modules.

[9]The problem is with the thorny technical condition of eventual syntactic characterizability, see the discussion in Section 4.

[10]This uses an argument of Will Boney.

a model (this is Theorem 0.2 in the abstract, see Theorem 6.18 for a proof).

Note that existence of prime models over sets of the form $M \cup \{a\}$ already played a crucial role in the proof of the categoricity transfer theorem for excellent classes of models of an $L_{\omega_1,\omega}$ sentences [She83b, Theorem 5.9] (in fact, our proof works also in this setting). Theorem 0.2 shows that, at least assuming amalgamation, tameness and shortness, this is the *only* obstacle. Note that all these follow by large cardinals by the results of [Bon14b] so we obtain (writting $h(\theta) := \beth_{(2^\theta)^+}$ as in the abstract):

Theorem 1.6. Let K be an AEC and let $\kappa > \mathrm{LS}(K)$ be strongly compact. Assume that in $K_{\geq \kappa}$, there are prime models over sets of the form $M \cup \{a\}$. If K is categorical in a $\lambda > h(h(\kappa))$, then K is categorical in all $\lambda' \geq h(h(\kappa))$.

The paper is organized as follows. In Section 2, we recall the definition of universal classes and more generally of AECs which admit intersections (a notion introduced by Baldwin and Shelah in [BS08]), give examples, and prove some basic properties. In Section 3, we prove that universal classes are fully $(< \aleph_0)$-tame and short. In Section 4 we give conditions under which amalgamation follows from categoricity (in more general classes than universal classes). In Section 5, we prove that amalgamation follows from categoricity in universal classes. In Section 6, we prove a categoricity transfer in universal classes that have amalgamation.

To avoid cluttering the paper, we have written the technical definitions and results on independence needed for the paper (but not crucial to a conceptual understanding) in Appendix A. In Appendix B, we prove Fact 6.10, a result of Shelah which is crucial to our argument but whose proof is only implicit in Shelah's book. In Appendix C, we give some properties of independence in AECs which admit intersections.

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2. AECs which admit intersections

Throughout this paper, we assume familiarity with a basic text on AECs such as [Bal09] or [Gro] and refer the reader to the preliminaries of [Vasc] for more details and motivations on the notation and concepts used here.

Recall:

Definition 2.1 ([She87b])**.** A class of structure K is *universal* if:

(1) It is a class of L-structures for a fixed language $L = L(K)$, closed under isomorphisms.
(2) If $\langle M_i : i < \delta \rangle$ is $\subseteq$-increasing in K, then $\bigcup_{i<\delta} M_i \in K$.
(3) If $M \in K$ and $M_0 \subseteq M$, then $M_0 \in K$.

Example 2.2.

(1) The class of models of a universal $L_{\lambda,\omega}$ theory is universal.
(2) Not all elementary classes are universal but some universal classes are not elementary (locally finite groups are one example).
(3) Coloring classes [KLH] are universal classes. This shows that the behavior of amalgamation is non-trivial even in universal classes: some coloring classes can have amalgamation up to $\beth_\alpha$ for some $\alpha < \text{LS}(K)^+$ and fail to have it above $\beth_\alpha$. Other universal classes with non-trivial amalgamation spectrum appear in [BKL].

Universal classes are abstract elementary classes:

Definition 2.3 (Definition 1.2 in [She87a])**.** An *abstract elementary class* (AEC for short) is a pair $(K, \leq)$, where:

(1) K is a class of L-structured, for some fixed language $L = L(K)$.
(2) $\leq$ is a partial order (that is, a reflexive and transitive relation) on K.
(3) $(K, \leq)$ respects isomorphisms: If $M \leq N$ are in K and $f : N \cong N'$, then $f[M] \leq N'$.
(4) If $M \leq N$, then $M \subseteq N$.
(5) Coherence: If $M_0, M_1, M_2 \in K$ satisfy $M_0 \leq M_2$, $M_1 \leq M_2$, and $M_0 \subseteq M_1$, then $M_0 \leq M_1$;
(6) Tarski-Vaught axioms: Suppose δ is a limit ordinal and $\langle M_i \in K : i < \delta \rangle$ is an increasing chain. Then:
   (a) $M_\delta := \bigcup_{i<\delta} M_i \in K$ and $M_0 \leq M_\delta$.

(b) If there is some $N \in K$ so that for all $i < \delta$ we have $M_i \leq N$, then we also have $M_\delta \leq N$.

(7) Löwenheim-Skolem-Tarski axiom: There exists a cardinal $\lambda \geq |L(K)| + \aleph_0$ such that for any $M \in K$ and $A \subseteq |M|$, there is some $M_0 \leq M$ such that $A \subseteq |M_0|$ and $\|M_0\| \leq |A| + \lambda$. We write $\mathrm{LS}(K)$ for the minimal such cardinal.

We often will not distinguish between the class K and the pair $(K, \leq)$.

Remark 2.4. If K is a universal class, then $(K, \subseteq)$ is an AEC with $\mathrm{LS}(K) = |L(K)| + \aleph_0$. We will use this fact freely. Note that K may have finite models, and it is the case in many examples, see [BKL].

We now recall the definition of AECs that admit intersections, a notion introduced by Baldwin and Shelah. It is interesting to note that Baldwin and Shelah thought of admitting intersections as a weak version of amalgamation (see the conclusion of [BS08]).

Definition 2.5 (Definition 1.2 in [BS08])**.** Let K be an AEC.

(1) Let $N \in K$ and let $A \subseteq |N|$. N *admits intersections over* A if there is $M_0 \leq N$ such that $|M_0| = \bigcap \{M \leq N \mid A \subseteq |M|\}$. N *admits intersections* if it admits intersections over all $A \subseteq |N|$.
(2) K *admits intersections* if N admits intersections for all $N \in K$.

Example 2.6.

(1) If K is a universal class, then K admits intersections.
(2) If K is a class of models of a first-order theory, then when $(K, \subseteq)$ admits intersections has been characterized by Rabin [Rab62].
(3) The examples in [BS08] admit intersections. Since they are not tame, they cannot be universal classes (see Theorem 3.7).
(4) Many classes appearing in algebra admit intersections. For example[11], let K be the class of algebraically closed valued fields (we code the value group with an additional predicate), ordered by $F_1 \leq F_2$ if and only if F_1 is a subfield of F_2, the value groups are the same, and the valuations coincide on F_1. Then K admits intersections. Again, K is not universal (as it is not closed under substructure).
(5) If $\mathfrak{C}$ is a monster model for a first-order theory T, we can let K be the class of (isomorphic copies of) algebraically closed subsets of $\mathfrak{C}$, ordered by the substructure relation. Then K admits intersections (but is not necessarily closed under substructure, so not necessarily universal).

[11]This is Example 1.15 in [Gro02].

In the rest of this section, we give several equivalent definitions of admitting intersections and deduce some properties of these classes. All throughout this paper, we assume:

Hypothesis 2.7. K is an AEC.

Definition 2.8. Let $M \in K$ and let $A \subseteq |M|$ be a set. M is *minimal over A* if whenever $M' \leq M$ contains A, then $M' = M$. M is *minimal over A in N* if in addition $M \leq N$.

Definition 2.9. Let $N \in K$. We say $\mathcal{F}$ is a *set of Skolem functions for N* if:

(1) $\mathcal{F}$ is a non-empty set, and each element f of $\mathcal{F}$ is a function from N^n to N, for some $n < \omega$.
(2) For all $A \subseteq |N|$, $M := \mathcal{F}[A] := \bigcup\{f[A] \mid f \in \mathcal{F}\}$ is such that $M \leq N$ and contains A.

Remark 2.10. The proof of Shelah's presentation theorem (see [She09a, Lemma I.1.7]) gives that for each $N \in K$, there is $\mathcal{F}$ a set of Skolem functions for N with $|\mathcal{F}| \leq \mathrm{LS}(K)$.

Theorem 2.11. Let K be an AEC and let $N \in K$. The following are equivalent:

(1) N admits intersections.
(2) There is an operator $\mathrm{cl} := \mathrm{cl}^N : \mathcal{P}(|N|) \to \mathcal{P}(|N|)$ such that for all $A, B \subseteq |N|$ and all $M \leq N$:
   (a) $\mathrm{cl}(A) \leq N$.
   (b) $A \subseteq \mathrm{cl}(A)$.
   (c) $A \subseteq B$ implies $\mathrm{cl}(A) \subseteq \mathrm{cl}(B)$.
   (d) $\mathrm{cl}(M) = M$.
(3) For each $A \subseteq |N|$, there is a unique minimal model over A in N.
(4) There is a set $\mathcal{F}$ of Skolem functions for N such that:
   (a) $|\mathcal{F}| \leq \mathrm{LS}(K)$.
   (b) For all $M \leq N$, we have $\mathcal{F}[M] = M$.

Moreover the operator $\mathrm{cl}^N : \mathcal{P}(|N|) \to \mathcal{P}(|N|)$ with the properties in (2) is unique and if it exists then it has the following characterizations:

- $\mathrm{cl}^N(A) = \bigcap\{M \leq N \mid A \subseteq |M|\}$.
- $\mathrm{cl}^N(A) = \mathcal{F}[A]$, for any set of Skolem functions $\mathcal{F}$ for N such that $\mathcal{F}[M] = M$ for all $M \leq N$.
- $\mathrm{cl}^N(A)$ is the unique minimal model over A in N.

Proof.

- <u>(1) implies (2)</u>: Let $\mathrm{cl}^N(A) := \bigcap\{M \leq N \mid A \subseteq |M|\}$. Even without hypotheses on N, (2b), (2c), and (2d) are satisfied. Since N admits intersections, (2a) is also satisfied.
- <u>(2) implies (3)</u>: Let $A \subseteq |N|$. Let cl be as given by (2). Let $M := \mathrm{cl}(A)$. By (2a), $M \leq N$. By (2b), $A \subseteq |M|$. Moreover if $M' \leq N$ contains A, then by (2c), $|M| \subseteq |\mathrm{cl}(M')|$ but by (2d), $\mathrm{cl}(M') = M'$. Thus by coherence and (2a) $M \leq M'$. This shows both that M is minimal over A and that it is unique.
- <u>(3) implies (4)</u>: We slightly change the proof of [She09a, Lemma I.1.7] as follows: Let $\chi := \mathrm{LS}(K)$. For each $\bar{a} \in {}^{<\omega}|N|$, let $\langle b_i^{\bar{a}} : i < \chi \rangle$ be an enumeration (possibly with repetitions) of the unique minimal model over $\mathrm{ran}(\bar{a})$ in N. For each $n < \omega$ and $i < \chi$, we let $f_i^n : N^n \to N$ be $f_i^n(\bar{a}) := b_i^{\bar{a}}$. Let $\mathcal{F} := \{f_i^n \mid i < \chi, n < \omega\}$. Then $|\mathcal{F}| \leq \mathrm{LS}(K)$ and if $A \subseteq |N|$, we claim that $\mathcal{F}[A]$ is minimal over A in N. This shows in particular that $\mathcal{F}$ is as required.

  Let $M := \mathcal{F}[A]$. By definition, $M = \bigcup_{\bar{a} \in {}^{<\omega}|A|} \mathcal{F}[\mathrm{ran}(\bar{a})]$. Now if $\bar{a} \in {}^{<\omega}A$, $M_{\bar{a}} := \mathcal{F}[\mathrm{ran}(\bar{a})] = \{b_i^{\bar{a}} : i < \chi\}$ is the unique minimal model over $\mathrm{ran}(\bar{a})$ in N. Thus if $\mathrm{ran}(\bar{a}) \subseteq \mathrm{ran}(\bar{b})$, we must have (by coherence) $M_{\bar{a}} \leq M_{\bar{b}}$. It follows that $M \in K$ and by the axioms of AECs also $M \leq N$. Of course, M contains A. Now if $M' \leq M$ contains A, then for all $\bar{a} \in {}^{<\omega}A$, $\bar{a} \in {}^{<\omega}|M'|$, so as $M_{\bar{a}}$ is minimal over $\mathrm{ran}(\bar{a})$, $M_{\bar{a}} \leq M'$. It follows that $M \leq M'$ so $M = M'$.
- <u>(4) implies (1)</u>: Let $\mathcal{F}$ be as given by (4). Let $A \subseteq |N|$. Let $M := \mathcal{F}[A]$. By definition of Skolem functions, M contains A and $M \leq N$. We claim that $M = \bigcap\{M' \leq N \mid A \subseteq |M'|\}$. Indeed, if $M' \leq N$ contains A, then by the hypothesis on $\mathcal{F}$, $M = \mathcal{F}[A] \subseteq \mathcal{F}[M'] = M'$.

The moreover part follows from the arguments above. $\square$

Definition 2.12. For $N \in K$ let $\mathrm{cl}^N : \mathcal{P}(|N|) \to \mathcal{P}(|N|)$ be defined by $\mathrm{cl}^N(A) := \bigcap\{M \leq N \mid A \subseteq |M|\}$.

Theorem 2.11 allows us to deduce several properties of the operator cl^N.

Proposition 2.13. Let $M \leq N \in K$ and let $A, B \subseteq |N|$.

(1) Invariance: If $f : N \cong N'$, then $f[\mathrm{cl}^N(A)] = \mathrm{cl}^{N'}(f[A])$.
(2) Monotonicity 1: $A \subseteq \mathrm{cl}^N(A)$.

(3) Monotonicity 2: $A \subseteq B$ implies $\mathrm{cl}^N(A) \subseteq \mathrm{cl}^N(B)$.
(4) Monotonicity 3: If $A \subseteq |M|$, then $\mathrm{cl}^N(A) \subseteq \mathrm{cl}^M(A)$. Moreover if N admits intersections over A, then M admits intersections over A and $\mathrm{cl}^N(A) = \mathrm{cl}^M(A)$.
(5) Idempotence: $\mathrm{cl}^N(M) = M$.
(6) Finite character: If N admits intersections, then if $B \subseteq \mathrm{cl}^N(A)$ is finite, there exists a finite $A_0 \subseteq A$ such that $B \subseteq \mathrm{cl}^N(A_0)$.

Proof. Straightforward given Theorem 2.11: For finite character, use the characterization in terms of Skolem functions. For monotonicity 3, let $M_0 := \mathrm{cl}^N(A)$. We have $M_0 \leq N$ since N admits intersections over A. Since $M \leq N$ contains A, we must have $|M_0| \subseteq |M|$. By coherence, $M_0 \leq M$, and by minimality, $M_0 = \mathrm{cl}^M(A)$. □

Note in particular the following:

Corollary 2.14.

(1) Assume that for every $M \in K$ and every $A \subseteq |M|$, there is $N \geq M$ such that N admits intersections over A. Then K admits intersections.
(2) $N \in K$ admits intersections if and only if it admits intersections over every finite $A \subseteq |N|$.

Proof.

(1) By Monotonicity 3.
(2) By the proof of Theorem 2.11.

□

Remark 2.15. The second result is implicit in the discussion after Remark 4.3 in [BS08].

Before stating the next proposition, we recall that any AECs admits a semantic notion of types. This was first introduced in [She87b, Definition II.1.9]. We use the notation of [Vasc, Definition 2.16].

Definition 2.16 (Galois types)**.**

(1) Let K^3 be the set of triples of the form $(\bar{b}, A, N)$, where $N \in K$, $A \subseteq |N|$, and $\bar{b}$ is a sequence of elements from N.
(2) For $(\bar{b}_1, A_1, N_1), (\bar{b}_2, A_2, N_2) \in K^3$, we write $(\bar{b}_1, A_1, N_1) E_{\mathrm{at}} (\bar{b}_2, A_2, N_2)$ if $A := A_1 = A_2$, and there exists $f_\ell : N_\ell \underset{A}{\to} N$ such that $f_1(\bar{b}_1) = f_2(\bar{b}_2)$. We call E_{at} *atomic equivalence of triples* and say that two triples are *atomically equivalent*.

(3) Note that E_{at} is a symmetric and reflexive relation on K^3. We let E be the transitive closure of E_{at}.
(4) For $(\bar{b}, A, N) \in K^3$, let $\mathrm{gtp}(\bar{b}/A; N) := [(\bar{b}, A, N)]_E$. We call such an equivalence class a *Galois type*.
(5) For $p = \mathrm{gtp}(\bar{b}/A; N)$ a Galois type, define[12] $\ell(p) := \ell(\bar{b})$ and $\mathrm{dom}(p) := A$.
(6) We say a Galois types $p = \mathrm{gtp}(\bar{b}/A; N)$ is *algebraic* if $\bar{b} \in {}^{\ell(\bar{b})}A$ (it is easy to check this does not depend on the choice of representatives). We mostly use this when $\ell(p) = 1$.
(7) For $N \in K$, $A \subseteq |N|$, and α an ordinal, we let $\mathrm{gS}^\alpha(A; N) := \{\mathrm{gtp}(\bar{b}/A; N) \mid \bar{b} \in {}^\alpha|N|\}$. When $\alpha = 1$, we omit it. For $M \in K$, we write $\mathrm{gS}^\alpha(M)$ for $\bigcup_{M' \geq M} \mathrm{gS}^\alpha(M; M')$. We similarly define $\mathrm{gS}^{<\infty}(M)$, etc.

We can go on to define the restriction of a type (if $A_0 \subseteq \mathrm{dom}(p)$, $I \subseteq \ell(p)$, we will write $p^I \upharpoonright A_0$ when the realizing sequence is restricted to I and the domain is restricted to A_0), the image of a type under an isomorphism, or what it means for a type to be realized.

The next result says that in AECs admitting intersections, equality of Galois types is witnessed by an isomorphism. This can be seen as a weak version of amalgamation (see Section 4).

Proposition 2.17. Assume K admits intersections. Then $\mathrm{gtp}(\bar{a}_1/\emptyset; N_1) = \mathrm{gtp}(\bar{a}_2/\emptyset; N_2)$ if and only if there exists $f : \mathrm{cl}^{N_1}(\bar{a}_1) \cong \mathrm{cl}^{N_2}(\bar{a}_2)$ such that $f(\bar{a}_1) = \bar{a}_2$.

Proof. Let $M_1 := \mathrm{cl}^{N_1}(\bar{a}_1)$, $M_2 := \mathrm{cl}^{N_2}(\bar{a}_2)$. Since N_ℓ admits intersections, we have $M_\ell \leq N_\ell$, $\ell = 1, 2$ so the right to left direction follows. Now assume $\mathrm{gtp}(\bar{a}_1/\emptyset; N_1) = \mathrm{gtp}(\bar{a}_2/\emptyset; N_2)$. It suffices to prove the result when the equality is atomic (then we can compose the isomorphisms in the general case). So let $N \in K$ and $f_\ell : N_\ell \to N$ witness atomic equality, i.e. $f_1(\bar{a}_1) = f_2(\bar{a}_2)$. By invariance and monotonicity 3, $f_\ell[M_\ell] = \mathrm{cl}^{f[N_\ell]}(f_\ell(\bar{a}_\ell)) = \mathrm{cl}^N(f_\ell(\bar{a}_\ell))$. Since $f_1(\bar{a}_1) = f_2(\bar{a}_2)$, we must have that $f_1[M_1] = f_2[M_2]$. Thus $f := (f_2 \upharpoonright M_2)^{-1} \circ (f_1 \upharpoonright M_1)$ is as desired. $\square$

Remark 2.18. Proposition 2.17 was already observed (without proof) in [BS08, Lemma 1.3]. Baldwin and Shelah also claim that $E = E_{\mathrm{at}}$ (see Definition 2.16), but this does not seem to follow.

Before ending this section, we point out a technical disadvantage of the definition of admitting intersection. The notion is not closed under the

[12]It is easy to check that this does not depend on the choice of representatives.

tail of the AEC: If K admits intersections and λ is a cardinal, then it is not clear that $K_{\geq \lambda}$ admits intersections. Thus we will work with a slightly weaker definition:

Definition 2.19. For K an AEC and $M \in K$, let K_M be the AEC defined by adding constant symbols for the elements of M and requiring that M embeds inside every model of K_M. That is, $L(K_M) = L(K) \cup \{c_a \mid a \in |M|\}$, where the c_a's are new constant symbols, and

$$K_M := \{(N, c_a^N)_{a \in |M|} \mid N \in K \text{ and } a \mapsto c_a^N \text{ is a } K\text{-embedding from } M \text{ into } N\}$$

We order K_M by $(N_1, c_a^N)_{a \in |M|} \leq (N_2, c_a^{N_2})$ if and only if $N_1 \leq N_2$ and $c_a^{N_1} = c_a^{N_2}$ for all $a \in |M|$.

Definition 2.20. For P a property of AECs and $M \in K$, K *has* P *above* M if K_M has P. K *locally has* P if it has P above every $M \in K$.

Remark 2.21. K locally admits intersections if and only if for every $M \leq N$ in K and every $A \subseteq |N|$ *which contains* M, $\text{cl}^N(A) \leq N$.

Remark 2.22. If K locally has P, then for every cardinal λ, $K_{\geq \lambda}$ locally has P.

3. Universal classes are fully tame and short

In this section, we show that universal classes are fully $(< \aleph_0)$-tame and short. The basic argument for Theorem 3.7 is due to Will Boney and will also appear in [Bonb].

Note that it is impossible to extend this result to AECs which admits intersections: [BS08] gives several counterexamples. One could hope that showing that categoricity in a high-enough cardinal implies tameness (a conjecture of Grossberg and VanDieren, see [GV06a, Conjecture 1.5]) is easier in AECs which admits intersections, but we have been unable to make progress in that direction and leave it to further work.

The key of the argument for tameness of universal classes is that the isomorphism characterizing the equality of Galois type is unique. We abstract this feature into a definition:

Definition 3.1. K is *pseudo-universal* if it admits intersections and for any N_1, N_2, $\bar{a}_1$, $\bar{a}_2$, if $\text{gtp}(\bar{a}_1/\emptyset; N_1) = \text{gtp}(\bar{a}_2/\emptyset; N_2)$ and $f, g : \text{cl}^{N_1}(\bar{a}_1) \cong \text{cl}^{N_2}(\bar{a}_2)$ are such that $f(\bar{a}_1) = g(\bar{a}_1) = \bar{a}_2$, then $f = g$.

Example 3.2.

(1) In universal classes, $\mathrm{cl}^N(A)$ is just the substructure of N generated by A. Thus universal classes are pseudo-universal.
(2) If $\mathfrak{C}$ is a monster model for a first-order theory T, we can let K be the class of (isomorphic copies) of definably closed[13] sets of $\mathfrak{C}$, ordered by elementary substructure. Then K is a pseudo-universal AEC, but it need not be universal.
(3) We show below that pseudo-universal classes are $(< \aleph_0)$-tame, hence the AECs in [BS08] admit intersections but are not pseudo-universal.

We quickly recall the definitions of tameness and shortness: Tameness as a property of AECs was introduced by Grossberg and VanDieren in [GV06b] (it was isolated from a proof in [She99]). It says that Galois types are determined by small restrictions of their domain. Shortness[14] was introduced by Will Boney in [Bon14b, Definition 3.3]. It says that Galois types are determined by restrictions to small length. We will use the notation of [Vasc, Definition 2.19].

Definition 3.3. Let κ be an infinite cardinal.

(1) K is *$(< \kappa)$-tame* if for any $M \in K$ and any $p \neq q$ in $\mathrm{gS}(M)$, there exists $A \subseteq |M|$ with $|A| < \kappa$ such that $p \upharpoonright A \neq q \upharpoonright A$.
(2) K is *fully $(< \kappa)$-tame and short* if for any $M \in K$, any ordinal α, and any $p \neq q$ in $\mathrm{gS}^\alpha(M)$, there exists $I \subseteq \alpha$ and $A \subseteq |M|$ such that $|I| + |A| < \kappa$ and $p^I \upharpoonright A \neq q^I \upharpoonright A$.
(3) κ-tame means $(< \kappa^+)$-tame, similarly for fully κ-tame and short.

Definition 3.4. Let $\bar{a}_\ell \in {}^\alpha|N_\ell|$ and let κ be an infinite cardinal. We write $(\bar{a}_1, N_1) \equiv_{<\kappa} (\bar{a}_2, N_2)$ if for every $I \subseteq \alpha$ of size less than κ, $\mathrm{gtp}(\bar{a}_1 \upharpoonright I/\emptyset; N_1) = \mathrm{gtp}(\bar{a}_2 \upharpoonright I/\emptyset; N_2)$.

The next proposition says roughly that it is enough to show shortness for types over the empty set. This appears already as [Bon14b, Theorem 3.5]. We repeat the argument here for convenience.

Proposition 3.5. Let κ be an infinite cardinal. Assume that for every α, $N_\ell \in K$, $\bar{a}_\ell \in {}^\alpha|N_\ell|$, $\ell = 1, 2$, we have that $(\bar{a}_1, N_1) \equiv_{<\kappa} (\bar{a}_2, N_2)$ implies $\mathrm{gtp}(\bar{a}_1/\emptyset; N_1) = \mathrm{gtp}(\bar{a}_2/\emptyset; N_2)$. Then K is fully $(< \kappa)$-tame and short.

[13]That is, an element which is definable from finitely many parameters in the set must be in the set.

[14]What we call "Shortness" is called "Type shortness" by Boney, but in this paper we never write the "Type".

Proof. Let β be an ordinal, $M \in K$, $p, q \in \text{gS}^\beta(M)$. Assume that $p^I \restriction A = q^I \restriction A$ for all $I \subseteq \beta$ and $A \subseteq |M|$ of size less than κ. Say $p = \text{gtp}(\bar{a}_1/M; N_1)$, $q = \text{gtp}(\bar{a}_2/M; N_2)$. Let $\bar{b}$ be an enumeration of $|M|$ and let $p' := \text{gtp}(\bar{a}_1\bar{b}/\emptyset; N_1)$, $q' := \text{gtp}(\bar{a}_2\bar{b}/\emptyset; N_2)$. By assumption, $(p')^{I'} = (q')^{I'}$ for all I' of size less than κ. In other words, $(\bar{a}_1\bar{b}, N_1) \equiv_{<\kappa} (\bar{a}_2\bar{b}, N_2)$. Therefore by our locality assumption $p' = q'$, and from the definition of Galois types this implies that $p = q$. $\square$

Remark 3.6. By a similar argument, we can show that pseudo-universal classes are locally pseudo-universal (recall Definition 2.20).

Theorem 3.7. If K is pseudo-universal, then K is fully $(< \aleph_0)$-tame and short.

Proof. Let $N_\ell \in K$, $\bar{a}_\ell \in {}^\alpha|N_\ell|$, $\ell = 1, 2$. Assume that $(\bar{a}_1, N_1) \equiv_{<\aleph_0} (\bar{a}_2, N_2)$. We show that $\text{gtp}(\bar{a}_1/\emptyset; N_1) = \text{gtp}(\bar{a}_2/\emptyset; N_1)$, which is enough by Proposition 3.5. Let $M_\ell := \text{cl}^{N_\ell}(\text{ran}(\bar{a}_\ell))$.

For each finite $I \subseteq \alpha$, let $M_{\ell,I} := \text{cl}^{N_\ell}(\text{ran}(\bar{a}_\ell \restriction I))$. By definition of $\equiv_{<\aleph_0}$, for each finite $I \subseteq \alpha$, $\text{gtp}(\bar{a}_1 \restriction I/\emptyset; N_1) = \text{gtp}(\bar{a}_2 \restriction I/\emptyset; N_2)$. Therefore (because K admits intersections) there exists $f_I : M_{1,I} \cong M_{2,I}$ such that $f_I(\bar{a}_1 \restriction I) = \bar{a}_2 \restriction I$. Moreover by definition of pseudo-universal, f_I is unique with that property. This means in particular that if $I \subseteq J \subseteq \alpha$ are both finite, we must have $f_I \subseteq f_J$. By finite character of the closure operator, $M_\ell = \bigcup_{I \in [\alpha]^{<\aleph_0}} M_{\ell,I}$ and so letting $f := \bigcup_{I \in [\alpha]^{<\aleph_0}} f_I$, we have that $f : M_1 \cong M_2$ and $f(\bar{a}_1) = \bar{a}_2$. This witnesses that $\text{gtp}(\bar{a}_1/\emptyset; M_1) = \text{gtp}(\bar{a}_2/\emptyset; M_2)$ and so (since $M_\ell \leq N_\ell$), $\text{gtp}(\bar{a}_1/\emptyset; N_1) = \text{gtp}(\bar{a}_2/\emptyset; N_2)$. $\square$

We can localize Theorem 3.7 to obtain more generally (Remark 3.6):

Corollary 3.8. If K is locally pseudo-universal, then K is fully $\text{LS}(K)$-tame and short.

Proof. Let $M \in K$ and let $p, q \in \text{gS}^\alpha(M)$. Assume that $p^I \restriction A = q^I \restriction A$ for all $A \subseteq |M|$ of at most size $\text{LS}(K)$ and all $I \subseteq \alpha$ of size at most $\text{LS}(K)$. We want to see that $p = q$. Without loss of generality $\|M\| \geq \text{LS}(K)$. Let $M_0 \leq M$ have size $\text{LS}(K)$. We know that $p^I \restriction M_0 = q^I \restriction M_0$ for all $I \subseteq \alpha$ of size at most $\text{LS}(K)$. Since K is locally pseudo-universal, K_{M_0} (see Definition 2.19) is pseudo-universal. By Theorem 3.7, K_{M_0} is fully $(< \aleph_0)$-tame and short. Translating to K, this means that for any $N \geq M_0$, any $p', q' \in \text{gS}^\beta(N)$, if $(p')^I \restriction (|M_0| \cup A) = (q')^I \restriction (|M_0| \cup A)$ for all finite I and A, then $p' = q'$. Setting N, p', q' to stand for M, p, q, we get that $p = q$, as desired. $\square$

4. Amalgamation from categoricity

We investigate how to get amalgamation from categoricity in tame AECs admitting intersections. In what follows, we will often use Remark 1.3 without comments. Recall:

Definition 4.1. An AEC K has amalgamation if for any $M_0 \leq M_\ell$, $\ell = 1, 2$, there exists $N \in K$ and embeddings $f_\ell : M_\ell \xrightarrow[M_0]{} N$. We say that K has λ-amalgamation if this holds for the models in K_λ. We define similarly *disjoint amalgamation*, where we require in addition that $f_1[M_1] \cap f_2[M_2] = M_0$.

We will use the concept of a good λ-frame, a notion of forking for types of length one over models of size λ, see [She09a, Definition II.2.1] or Appendix A. The following claim is a deep result of Shelah which says that good λ-frames exist in categorical classes.

Claim 4.2. If K is categorical in unboundedly many cardinals, then there exists a categoricity cardinal $\lambda \geq \text{LS}(K)$ such that K has a good λ-frame (i.e. there exists a good λ-frame $\mathfrak{s}$ such that $K_{\mathfrak{s}} = K_\lambda$). In particular, K has λ-amalgamation.

The statement is implicit in Chapter IV of [She09a], but in June 2015 Will Boney and the author identified a gap in a key part of Shelah's proof. A fix was later announced but an error was found. See the discussion in [BVa]. The key condition is:

Definition 4.3 (Definition 2.1 in [BVa])**.** An AEC K is $L_{\infty,\theta}$*-syntactically characterizable* if whenever $M, N \in K$, if $M \leq N$ then $M \preceq_{L_{\infty,\theta}} N$. We say that K is *eventually syntactically characterizable* if for every infinite cardinal θ, there exists λ such that $K_{\geq\lambda}$ is $L_{\infty,\theta}$-syntactically characterizable.

The problematic part of Shelah's proof is a claim that an AEC categorical in unboundedly many cardinals is eventually syntactically characterizable (see [She09a, Conclusion IV.2.14]). However the following weakenings are true:

Fact 4.4.

(1) [BVa, Proposition 1.3] If K has amalgamation and no maximal models and is categorical in unboundedly many cardinals, then K is eventually syntactically characterizable.

(2) [She09a, Claim IV.1.12.(1)] Let θ be an infinite[15] cardinal. If K is categorical in a $\lambda = \lambda^{<\theta} \geq \text{LS}(K)$, then $K_{\geq \lambda}$ is $L_{\infty,\theta}$-syntactically characterizable.
(3) [She09a, Conclusion IV.2.12.(1)] If K is categorical in cardinals of arbitrarily large cofinality (that is, for every θ there exists λ such that K is categorical in λ and $\text{cf}(\lambda) \geq \theta$), then K is eventually syntactically characterizable.

From an eventually syntactically characterizable AEC that is categorical in unboundedly many cardinals, Shelah's proof of Claim 4.2 goes through:

Fact 4.5 (Theorem 2.12 in [BVa]). If K is eventually syntactically characterizable and categorical in unboundedly many cardinals, then there exists a categoricity cardinal $\lambda \geq \text{LS}(K)$ such that K has a good λ-frame.

Thus it is reasonable to assume that we have a good λ-frame, and we want to transfer amalgamation above it. Our inspiration is a recent result of Adi Jarden, presented at a talk in South Korea in the Summer of 2014.

Fact 4.6 (Corollary 7.16 in [Jarb]). Assume K has a good λ-frame where the class of uniqueness triples satisfies the existence property and K is strongly λ-tame, then K has λ^+-amalgamation.

We will not give the definition of the class of uniqueness triples here (but see Definition A.19 and Fact A.20). It suffices to say that they are a version of domination for good frames. As for strong tameness, it is a variation of tameness relevant when amalgamation fails to hold. Recall that λ-tameness asks for two types that are equal on all their restrictions of size λ to be equal. The strong version asks them to be *atomically equal*, i.e. there is a map witnessing it that amalgamates the two models in which the types are computed, see Definition 2.16. Jarden's result is interesting, since it shows that tameness, a locality property that we see as quite mild compared to assuming amalgamation, can be of some use to proving amalgamation. The downside is that we have to ask for a strengthened version.

While Jarden proved much more than λ^+-amalgamation, it has been pointed out by Will Boney (in a private communication) that if one only wants amalgamation, the hypothesis that uniqueness triples satisfy the

[15]Shelah assumes in addition that $\theta > \text{LS}(K)$, but the proof shows that it is not necessary.

existence property is not necessary. The reason is that the methods of [Bon14a] can be used to transfer enough of the good frame to λ^+ so that the extension property holds, and the extension property implies amalgamation.

We make the argument precise here and also show that less than strong tameness is needed (in particular, it suffices to assume tameness and that the AEC admits intersections). We first fix some notation.

Definition 4.7. Let $\lambda \geq \text{LS}(K)$.

(1) $K^{3,1}$ is the set of triples (a, M, N) such that $M \leq N$ and $a \in N$. $K_\lambda^{3,1}$ is the set of such triples where the models are in K_λ (the difference with Definition 2.16 is that we require the base to be a model and the sequence $\bar{b}$ to have length one).
(2) We say $(a_1, M_1, N_1) \in K^{3,1}$ *atomically extends* $(a_0, M_0, N_0) \in K^{3,1}$ if $M_1 \geq M_0$ and $(a_1, M_0, N_1) E_{\text{at}} (a_0, M_0, N_0)$ (recall Definition 2.16)
(3) We say $M \in K_\lambda$ has the *type extension property* if for any $N \geq M$ in K_λ and any $p \in \text{gS}(M)$, there exists $q \in \text{gS}(N)$ extending p.
(4) We say M has the *strong type extension property* if for any $N \geq M$, whenever $(a, M, M') \in K_\lambda^{3,1}$, there exists $(b, N, N') \in K_\lambda^{3,1}$ atomically extending (a, M, M').

We say K_λ *has the [strong] type extension property* (or K *has the [strong] type extension property in* λ) if every $M \in K_\lambda$ has it.

Remark 4.8. It is well-known (see for example [Gro]) that if K has amalgamation, then $E = E_{\text{at}}$. Similarly if $\lambda \geq \text{LS}(K)$ and K has λ-amalgamation, then $E \restriction K_\lambda^{3,1} = E_{\text{at}} \restriction K_\lambda^{3,1}$. Moreover, K has λ-amalgamation if and only if K_λ has the strong type extension property.

We think of the type extension property as saying that amalgamation cannot fail because there are "fundamentally incompatible" elements in the two models we want to amalgamate. Rather, the reason amalgamation fails is because we simply "do not have enough models" to witness that two types are equal in one step. It would be useful to formalize this intuition but so far we have failed to do so.

We are interested in conditions implying that the type extension property (not the strong one) is enough to get amalgamation. For this, it turns out that it is enough to require that the AEC admits intersections. However we can even require a weaker condition:

Definition 4.9 (Weak atomic equivalence)**.** Let $(a_\ell, M, N_\ell) \in K_\lambda^{3,1}$, $\ell = 1, 2$. We say $(a_1, M, N_1) E_{\text{at}}^- (a_2, M, N_2)$ (in words, (a_1, M, N_1) and (a_2, M, N_2) are *weakly atomically equivalent*) if for $\ell = 1, 2$, there exists $N_\ell' \leq N_\ell$ containing a_ℓ and M such that $(a_\ell, M, N_\ell') E_{\text{at}} (a_{3-\ell}, M, N_{3-\ell})$.

Definition 4.10. K has *weak amalgamation* if $E \restriction K^{3,1} = E_{\text{at}}^- \restriction K^{3,1}$, i.e. equivalence of triples is the same as *weak* atomic equivalence of triples. Similarly define what it means for K to have weak λ-amalgamation.

Remark 4.11. K has weak amalgamation if and only if whenever $\text{gtp}(a_1/M; N_1) = \text{gtp}(a_2/M; N_2)$, there exists $N_1' \leq N_1$ containing a_1 and M and there exists $N \geq N_2$ and $f : N_1' \underset{M}{\longrightarrow} N$ so that $f(a_1) = a_2$.

Remark 4.12. If K locally admits intersections, $(a_\ell, M, N_\ell) \in K_\lambda^{3,1}$, $\ell = 1, 2$ and $(a_1, M, N_1) E (a_2, M, N_2)$, then by Proposition 2.13, $N_\ell' := \text{cl}^{N_\ell}(|M| \cup \{a_\ell\})$ witnesses that $(a_1, M, N_1) E_{\text{at}}^- (a_2, M, N_2)$. Thus in that case, $E \restriction K_\lambda^{3,1} = E_{\text{at}}^- \restriction K_\lambda^{3,1}$, so K has weak amalgamation.

Intuitively, weak amalgamation requires only that points that have the same Galois types can be amalgamated. The key result is:

Theorem 4.13. Let K be an AEC and $\lambda \geq \text{LS}(K)$. Assume K_λ has the type extension property. The following are equivalent:

(1) K has λ-amalgamation.
(2) $E \restriction K_\lambda^{3,1} = E_{\text{at}} \restriction K_\lambda^{3,1}$ (i.e. equivalence of triples is the same as atomic equivalence of triples).
(3) K has weak λ-amalgamation (i.e. equivalence of triples is the same as *weak* atomic equivalence of triples).

In particular, if K admits intersections and has the type extension property, then it has amalgamation.

Proof. (1) implies (2) implies (3) is easy. We prove (3) implies (1).

Assume $E \restriction K_\lambda^{3,1} = E_{\text{at}}^- \restriction K_\lambda^{3,1}$. The idea of the proof is as follows: we want to amalgamate a triple (M_0, M, N), $M_0 \leq M$, $M_0 \leq N$. We use weak amalgamation first to amalgamate some smaller triple (M_0, M', N') with $M_0 < M' \leq M$, $M_0 < N' \leq N$, then proceed inductively to amalgamate the entire triple. Claim 1 below shows that there exists a smaller triple which can be amalgamated and Claim 2 is a renaming of Claim 1. We then use Claim 2 repeatedly to amalgamate the full triple.

<u>Claim 1</u>. For every triple (M_0, M_1, M_2) of models in K_λ so that $M_0 < M_1$ and $M_0 \leq M_2$, there exists $M_1' \leq M_1$ and $M_2' \geq M_2$ in K_λ such that $M_0 < M_1'$, and there exists $g : M_1' \underset{M_0}{\longrightarrow} M_2'$.

$$\begin{tikzcd}
M_1 & \\
M_1' \arrow[u] \arrow[r, dotted, "g"'] & M_2' \\
M_0 \arrow[u] \arrow[r] & M_2 \arrow[u, dotted]
\end{tikzcd}$$

<u>Proof of claim 1</u>. Let $M_0 < M_\ell$ be models in K_λ, $\ell = 1, 2$. Pick any $a_1 \in |M_1| \backslash |M_0|$. Let $p := \text{gtp}(a_1/M_0; M_1)$. By the type extension property, there exists $q \in \text{gS}(M_2)$ extending p. Pick $M_2^* \geq M_2$ and $a_2 \in |M_2^*|$ such that $q = \text{gtp}(a_2/M_2; M_2^*)$. Since E is E_{at}^- over the domain of interest, we have $(a_1, M_0, M_1) E_{\text{at}}^- (a_2, M_0, M_2^*)$. Let $M_1' \leq M_1$ contain a_1 and M_0 such that $(a_1, M_0, M_1') E_{\text{at}} (a_2, M_0, M_2^*)$. By definition, we have that there exists $M_2' \geq M_2^*$ such that M_1' embeds into M_2' over M_0, as needed. $\dagger_{\text{Claim 1}}$

Now we obtain amalgamation by repeatedly applying Claim 1. Since the result is key to subsequent arguments, we give full details below.

<u>Claim 2</u>. For every triple (M_0, M_1, M_2) of models in K_λ so that $M_0 < M_1$ and $f : M_0 \to M_2$, there exists $M_1' \leq M_1$, $M_2' \geq M_2$ in K_λ and $g : M_1' \to M_2'$ such that $M_0 < M_1'$ and $f \subseteq g$.

$$\begin{tikzcd}
M_1 & \\
M_1' \arrow[u] \arrow[r, dotted, "g"'] & M_2' \\
M_0 \arrow[u] \arrow[r, "f"'] & M_2 \arrow[u, dotted]
\end{tikzcd}$$

<u>Proof of claim 2</u>. Let M_0, M_1, M_2 and f be as given by the hypothesis. Let $\widehat{M_2}$ and $\widehat{f}$ be such that $f \subseteq \widehat{f}$, $M_0 \leq \widehat{M_2}$ and $\widehat{f} : \widehat{M_2} \cong M_2$. Now apply Claim 1 to $(M_0, M_1, \widehat{M_2})$ to obtain $M_1' \leq M_1$ with $M_0 < M_1'$, $\widehat{M_2}' \geq \widehat{M_2}$ and $\widehat{g} : M_1' \underset{M_0}{\longrightarrow} \widehat{M_2}'$. Now let $\widehat{f}'$, M_2' be such that $M_2' \geq M_2$

and $\widehat{f}' : \widehat{M_2}' \cong M_2'$ extends $\widehat{f}$. Let $g := \widehat{f}' \circ \widehat{g}$. Since $\widehat{g}$ fixes M_0 and $\widehat{f}'$ extends f, g extends f, as desired. $\dagger_{\text{Claim 2}}$

Now let $M_0 \leq M$ and $M_0 \leq N$ be in K_λ. We want to amalgamate M and N over M_0. We try to build $\langle M_i : i < \lambda^+ \rangle$, $\langle N_i : i < \lambda^+ \rangle$ increasing continuous in K_λ and $\langle f_i : i < \lambda^+ \rangle$ an increasing continuous sequence of embeddings such that for all $i < \lambda^+$:

(1) $M_i \leq M$.
(2) $f_i : M_i \xrightarrow[M_0]{} N_i$.
(3) $N_0 = N$.
(4) $M_i < M_{i+1}$.

This is impossible since then $\bigcup_{i<\lambda^+} M_i$ has cardinality λ^+ but is a K-substructure of M which has cardinality λ. Now for $i = 0$, we can take $N_0 = N$ and $f_0 = \text{id}_{M_0}$ and for i limit we can take unions. Therefore there must be some $\alpha < \lambda^+$ such that f_α, M_α, N_α are defined but we cannot define $f_{\alpha+1}$, $M_{\alpha+1}$, $N_{\alpha+1}$. If $M_\alpha < M$, we can use Claim 2 (with M_0, M_1, M_2, f there standing for M_α, M, N_α, f_α here) to get $M_{\alpha+1} \leq M$ with $M_\alpha < M_{\alpha+1}$ and $N_{\alpha+1} \geq N_\alpha$ with $f_{\alpha+1} : M_{\alpha+1} \to N_{\alpha+1}$ extending f_α (so M_1', M_2', g in Claim 2 stand for $M_{\alpha+1}$, $N_{\alpha+1}$, $f_{\alpha+1}$ here). Thus we can continue the induction, which we assumed was impossible. Therefore $M_\alpha = M$, so $f_\alpha : M \xrightarrow[M_0]{} N_\alpha$ amalgamates M and N over M_0, as desired. $\square$

We are now ready to formally state the amalgamation transfer:

Theorem 4.14. Let K be an AEC. Let $\lambda \geq \text{LS}(K)$ and assume $\mathfrak{s}$ is a good λ-frame with underlying class K_λ. If:

(1) K is λ-tame.
(2) $K_{\geq\lambda}$ has weak amalgamation.

Then $K_{\geq\lambda}$ has amalgamation.

Proof. We extend $\mathfrak{s}$ to models of size greater than λ by defining $\geq \mathfrak{s}$ as in [She09a, Section II.2] (or see [Bon14a, Definition 2.7]). Even without any hypotheses, Shelah has shown that $\geq \mathfrak{s}$ has local character, density of basic types, and transitivity. Moreover, tameness implies that it has uniqueness. Now work by induction on $\mu \geq \lambda$ to show that K has μ-amalgamation. When $\mu = \lambda$ this follows from the definition of a good frame so assume $\mu > \lambda$. As in [Bon14a, Theorem 5.13], we can prove that $\geq \mathfrak{s}$ has the extension property for models of size μ (the key is that the directed system argument only uses amalgamation below μ).

In particular, K_μ has the type extension property for basic types. The proof of Theorem 4.13 together with the density of basic types shows that this suffices to get μ-amalgamation. $\square$

Corollary 4.15. Let K be a tame AEC that is eventually syntactically characterizable and categorical in unboundedly many cardinals. If K has weak amalgamation, then there exists λ such that $K_{\geq \lambda}$ has amalgamation.

Proof. By Fact 4.5, we can find $\lambda \geq \mathrm{LS}(K)$ such that K_λ has a good frame and K is λ-tame. By Theorem 4.14, $K_{\geq \lambda}$ has amalgamation. $\square$

Corollary 4.16. Let K be an eventually syntactically charaterizable AEC categorical in unboundedly many cardinals. If K is tame and locally admits intersections, then there exists λ such that $K_{\geq \lambda}$ has amalgamation.

Proof. By Remark 4.12, K has weak amalgamation. Now apply Corollary 4.15. $\square$

Corollary 4.17. Let K be locally pseudo-universal AEC. If K is eventually syntactically characterizable and categorical in unboundedly many cardinals, then there exists λ such that $K_{\geq \lambda}$ has amalgamation.

Proof. By Corollary 3.8, K is tame. Now apply Corollary 4.16. $\square$

We can apply these results to Shelah's categoricity conjecture and improve Fact 1.5. When K has primes, this will be further improved in Theorem 6.18.

Corollary 4.18. Let K be a tame AEC with weak amalgamation. Assume K is categorical in unboundedly many cardinals.

(1) If K is categorical in a high-enough successor cardinal, then K is categorical on a tail of cardinals.
(2) Assume weak GCH and an unpublished claim of Shelah (Claim 1.4). If K is eventually syntactically characterizable and fully tame and short, then K is categorical on a tail of cardinals.

Proof. By Corollary 4.15 (using Fact 4.4 to see that K is eventually syntactically characterizable in (1)), we can assume without loss of generality that K has amalgamation. Now:

(1) Apply [GV06a] (and [She99] can also give a downward transfer).
(2) Apply Fact 1.5.

□

In the next section, we show that we can remove the hypothesis that K is eventually syntactically characterizable in Corollary 4.17, provided that K is a universal class (i.e. not "locally" or "pseudo"). However even if K is a universal class we will use Theorem 4.14 later to transfer categoricity (see Theorem 6.16).

5. Amalgamation in categorical universal classes

In this section we assume:

Hypothesis 5.1. K is a universal class.

As in the abstract, we will use the following notation:

Notation 5.2. For θ an infinite cardinal, write $h(\theta) := \beth_{(2^\theta)^+}$.

Our goal is to prove that if K is categorical in a cardinal above $h(\text{LS}(K))$, then K has amalgamation above that cardinal. We will cite extensively from Chapter V of [She09b], an updated version of [She87b]. In [She09b, Section V.B.2], Shelah shows that a universal class that does not have the order property (see below) can be ordered with a certain relation $\leq^*$ so that it has amalgamation (but $(K, \leq^*)$ may not be as nice as $(K, \subseteq)$, for example while it will be an AEC, it may not admit intersections). We will argue that assuming categoricity in a high-enough λ, K does not have the order property and $\leq^*$ restricted to $K_{\geq \lambda}$ is just substructure. Thus $K_{\geq \lambda}$ has amalgamation. We first by specializing the order property from [She09b, Definition V.A.1.1] to the quantifier-free version for universal classes:

Definition 5.3. K has the *order property* if there exists a quantifier-free formula $\phi(\bar{x}, \bar{y}, \bar{z})$ such that for every χ, there exists $M \in K$, $\bar{c} \in {}^{\ell(\bar{z})}|M|$, and sequences $\langle \bar{a}_i : i < \chi \rangle$, $\langle \bar{b}_i : i < \chi \rangle$ from M (with $\ell(\bar{a}_i) = \ell(\bar{x})$, $\ell(\bar{b}_i) = \ell(\bar{y})$ for all $i < \chi$) so that for all $i, j < \chi$, $M \models \phi[\bar{a}_i; \bar{b}_j; \bar{c}]$ if and only if $i < j$.

Fact 5.4. Assume that K is categorical in a $\lambda \geq h(\text{LS}(K))$. Then K does not have the order property.

Proof. In [She09b, Claim V.B.2.6], Shelah argues that if K has the order property, then it would have 2^λ-many models of size λ, contradicting categoricity. Since the Shelah's construction of many models is very technical, we sketch an easier proof. Since K has a model of size $h(\text{LS}(K))$, K has arbitrarily large models, hence we can use

Ehrenfeucht-Mostowski models. The standard argument (due to Morley) shows if $M \in K_\lambda$, $\mu \in [\mathrm{LS}(K), \lambda)$, and $A \subseteq |M|$ is such that $|A| \leq \mu$, then M realizes at most μ-many syntactic quantifier-free types over A. However if K had the order property, we would be able to build a set $A \subseteq |M|$ with $|A| \leq \mathrm{LS}(K)$ but at least $\mathrm{LS}(K)^+$ types are realized in M (see e.g. the proof of [BGKV, Fact 5.13]). This is a contradiction. $\square$

From no order property, Shelah shows that K satisfies the axioms of a certain framework axiomatizing first-order forking:

Fact 5.5 (V.B.2.8 and V.B.2.9 in [She09b])**.** If K does not have the order property, then there exists a partial order $\leq^*$ (Shelah calls it $\leq_{\mathfrak{s}}$) on K such that:

(1) For $M, N \in K$, $M \leq^* N$ implies $M \subseteq N$ and $M \preceq_{L_{\infty,\omega}} N$ implies $M \leq^* N$ (this last part follows from [She09b, Lemma V.A.4.4]).
(2) $(K, \leq^*)$ has amalgamation (in fact much more).

We can now prove the main theorem of this section:

Theorem 5.6. If K is categorical in a $\lambda \geq h(\mathrm{LS}(K))$, then $K_{\geq \lambda}$ has amalgamation, joint embedding, and no maximal models.

Proof. By Fact 5.4, K does not have the order property. Let $\leq^*$ be as given by Fact 5.5. By Fact 4.4.(2) (with $\theta := \aleph_0$), for any $M, N \in K_{\geq \lambda}$, $M \subseteq N$ implies $M \preceq_{L_{\infty,\omega}} N$. By the properties of $\leq^*$, it follows that $\leq^*$ is just substructure. Thus $K_{\geq \lambda}$ has amalgamation. $K_{\geq \lambda}$ also has joint embedding (use categoricity in λ and amalgamation). Since K has a model of size $h(\mathrm{LS}(K))$, K has arbitrarily large models, so by joint embedding $K_{\geq \lambda}$ has no maximal models. $\square$

6. Categoricity transfer in AECs with primes

In this section, we prove a categoricity transfer for AECs that have primes. Prime triples were introduced in [She09a, Section III.3], see also [Jara].

Definition 6.1.

(1) Let $M \in K$ and let $A \subseteq |M|$. M is *prime over A* if for any enumeration $\bar{a}$ of A and any $N \in K$, whenever $\mathrm{gtp}(\bar{a}/\emptyset; M) = \mathrm{gtp}(\bar{b}/\emptyset; N)$, there exists $f : M \to N$ such that $f(\bar{a}) = \bar{b}$.

(2) (a, M, N) is a *prime triple* if $M \leq N$, $a \in |N|$, and N is prime over $|M| \cup \{a\}$.
(3) K *has primes* if for any $p \in \mathrm{gS}(M)$ there exists a prime triple (a, M, N) such that $p = \mathrm{gtp}(a/M; N)$.
(4) K *weakly has primes* if whenever $\mathrm{gtp}(a_1/M; N_1) = \mathrm{gtp}(a_2/M; N_2)$, there exists $M_1 \leq M$ containing a_1 and N_1 and $f : M_1 \underset{M}{\longrightarrow} N_2$ such that $f(a_1) = a_2$. Similarly define what it means for K_λ to have or weakly have primes.

Remark 6.2. For $M \leq N$ and $a \in |N|$, (a, M, N) is a prime triple if and only if whenever $\mathrm{gtp}(b/M; N') = \mathrm{gtp}(a/M; N)$, there exists $f : N \underset{M}{\longrightarrow} N'$ such that $f(a) = b$. Thus if K has primes, then K weakly has primes.

Remark 6.3. If K admits intersections, $M \leq N$, and $a \in |N|$, $(a, M, \mathrm{cl}^N(|M| \cup \{a\}))$ is a prime triple. Thus K has primes.

Assume K is an AEC categorical in $\lambda := \mathrm{LS}(K)$ (this is a reasonable assumption as we can always restrict ourselves to the class of λ-saturated models of K). Our goal is to prove (with more hypotheses) that if K is categorical in a $\theta > \lambda$ then it is categorical in all $\theta' \geq \lambda$. To accomplish this, we will show that K_λ is *uni-dimensional*. In [She09a, Section III.2], Shelah gives several possible generalization of the first-order definition in [She90, Definition V.2.2]. We have picked what seems to be the most convenient to work with:

Definition 6.4 (Definition III.2.2.6 in [She09a])**.** Let $\lambda \geq \mathrm{LS}(K)$. K_λ is *weakly uni-dimensional* if for every $M < M_\ell$, $\ell = 1, 2$ all in K_λ, there is $c \in |M_2| \backslash |M|$ such that $\mathrm{gtp}(c/M; M_2)$ has more than one extension in $\mathrm{gS}(M_1)$.

To understand this definition, it might be helpful to look at the negation: there exists $M < M_\ell$, $\ell = 1, 2$ all in K_λ such that for all $c \in |M_2| \backslash |M|$, $\mathrm{gtp}(c/M; M_2)$ has exactly one extension in $\mathrm{gS}(M_1)$. Working in a good frame, this one extension must be the nonforking extension (so in particular $\mathrm{gtp}(c/M; M_2)$ is omitted in M_1). It turns out that for any $c \in |M_2| \backslash |M|$ and $d \in |M_1| \backslash |M|$, $\mathrm{gtp}(c/M; M_2)$ and $\mathrm{gtp}(d/M; M_1)$ are orthogonal (in a suitable sense, see Appendix B), so they will generate two different dimensions.

Fact 6.5 (Claim III.2.3.(4) in [She09a])**.** Let $\lambda \geq \mathrm{LS}(K)$. If K_λ is weakly uni-dimensional, is categorical in λ, is stable in λ, and has λ-amalgamation, then[16] K is categorical in λ^+.

If K is λ-tame and has amalgamation, then categoricity in λ^+ is enough by the categoricity transfer of Grossberg and VanDieren:

Fact 6.6 (Theorem 6.3 in [GV06a])**.** Assume K is an $\mathrm{LS}(K)$-tame AEC with amalgamation and no maximal models. If K is categorical in $\mathrm{LS}(K)$ and $\mathrm{LS}(K)^+$, then K is categorical in all $\mu \geq \mathrm{LS}(K)$.

Thus the hard part is showing that $K_{\mathrm{LS}(K)}$ is weakly uni-dimensional. We proceed by contradiction.

Definition 6.7 (III.12.39.(d) in [She09a])**.** Let $M \in K$ and let $p \in \mathrm{gS}(M)$. We define[17] $K_{\neg^* p}$ to be the class of $N \in K_M$ (recall Definition 2.19) such that $f(p)$ has a unique extension to $\mathrm{gS}(N \upharpoonright L(K))$. Here $f : M \to N$ is given by $f(a) := c_a^N$. We order $K_{\neg^* p}$ with the strong substructure relation induced from K_M.

Remark 6.8. Let $p \in \mathrm{gS}(M)$ be nonalgebraic and let $M \leq N$. If we are working in a good frame and p has a unique extension to $\mathrm{gS}(N)$, then it must be the nonforking extension. Thus p is omitted in N. However even if p is omitted in N, p could have two nonalgebraic extensions to $\mathrm{gS}(N)$, so $K_{\neg^* p}$ need not be the same as the class of models omitting p.

In general, we do *not* claim that $K_{\neg^* p}$ is an AEC. Nevertheless it is an abstract class in the sense introduced by Grossberg in [Gro], see [Vasc, Definition 2.7]. Thus we can define notions such as amalgamation, Galois types, and tameness there just as in AECs. The following gives an easy criteria for when $K_{\neg^* p}$ *is* an AEC:

Proposition 6.9. Let $\mathfrak{s} = (K, \perp)$ be a type-full good $(\geq \lambda)$-frame (so $\lambda = \mathrm{LS}(K)$ and $K_{<\lambda} = \emptyset$). Let $M \in K$ and let $p \in \mathrm{gS}(M)$. Then $K_{\neg^* p}$ is an AEC.

Proof. All the axioms are easy except closure under chains. So let δ be a limit ordinal and let $\langle N_i : i < \delta \rangle$ be increasing continuous in $K_{\neg^* p}$. Identify models in K with their expansions in K_M, assuming without

[16]In [She09a, Claim III.2.3.(4)], Shelah assumes more generally the existence of a good λ-frame, but the proof shows that the hypotheses mentionned here suffice. In any case, we will only use Fact 6.5 inside a good frame.

[17]Shelah calls the class K^*.

loss of generality that $M \leq N_0$, i.e. the map $a \mapsto c_a^{N_0}$ for $a \in M$ is the identity. Let $N_\delta := \bigcup_{i<\delta} M_i$. We have that $N_\delta \upharpoonright L(K) \in K$. Now if $p_1, p_2 \in \mathrm{gS}(N_\delta \upharpoonright L(K))$ are two extensions of p, by local character there exists $i < \delta$ such that p_1 and p_2 do not fork over N_i. Since p has a unique extension to N_i, $p_1 \upharpoonright N_i = p_2 \upharpoonright N_i$. By uniqueness, $p_1 \upharpoonright N_\delta = p_2 \upharpoonright N_\delta$. $\square$

In fact, Shelah gave a criteria for when $K_{\neg^* p}$ has a good λ-frame:

Fact 6.10 (Claim III.12.39 in [She09a])**.** Let $\mathfrak{s}$ be a good λ-frame with underlying class K_λ. Assume $\mathfrak{s}$ is type-full, good$^+$, successful (see appendix A for the definitions of these terms), and K_λ has primes. Assume further that K is categorical in λ.

If K_λ is not weakly uni-dimensional, then there exists $M \in K_\lambda$ and $p \in \mathrm{gS}(M)$ such that $\mathfrak{s} \upharpoonright K_{\neg^* p}$ (the restriction of $\mathfrak{s}$ to models in $K_{\neg^* p}$) is a type-full good λ-frame.

Since this result is crucial to our argument and Shelah's proof is only implicit, we have included a proof in Appendix B.

Note that the hypotheses of Fact 6.10 are reasonable. In fact, it is known that they follow from categoricity in fully tame and short AECs with amalgamation:

Fact 6.11 (Theorem 15.6 in [Vasb])**.** Let K be a fully $(<\kappa)$-tame and short AEC with amalgamation. Let λ, μ be cardinals such that:

$$\mathrm{LS}(K) < \kappa = \beth_\kappa < \lambda = \beth_\lambda \leq \mu$$

Assume further that $\mathrm{cf}(\lambda) \geq \kappa$. If K is categorical in μ, then K is categorical in λ and there exists a type-full successful good λ-frame $\mathfrak{s}$ with underlying class K_λ.

From Proposition A.22, it will follow that the frame given by Fact 6.11 is also good$^+$. If in addition the AEC has primes (e.g. if it is universal), then the hypotheses are satisfied. Of course, the Hanf numbers in Fact 6.11 are not optimal. We give the following improvement in Appendix A:

Theorem 6.12. Let K be a fully $(< \mathrm{LS}(K))$-tame and short AEC with amalgamation. If K is categorical in a $\mu \geq h(\mathrm{LS}(K))$, then there exists $\lambda_0 < h(\mathrm{LS}(K))$ such that for all $\lambda \geq \lambda_0$ where K is categorical in λ, there exists a type-full successful good$^+$ λ-frame with underlying class K_λ.

Proof. Combine Corollaries A.18 and A.23. □

Now we reach a crucial point. For the purpose of a categoricity transfer, it would be enough to show that $K_{\neg^* p}$ above has arbitrarily large models, since this means that there are non-saturated models in every cardinal above λ. Unfortunately, even if K is fully tame and short and has amalgamation, it is not easy to get a handle on $K_{\neg^* p}$. For example, it is not clear if it has amalgamation or even if it is tame. In [She09a, Discussion III.12.40] Shelah claims to be able to show using enough instances of weak GCH that $\mathfrak{s} \upharpoonright K_{\neg^* p}$ above has arbitrarily large models (this is essentially how Claim 1.4 is proven) and this is the key to the proof of Fact 1.5.

We make the situation where $K_{\neg^* p}$ is well-behaved into a definition:

Definition 6.13. K is *nice* if:

(1) K has weak amalgamation.
(2) For any $M \in K$ and any $p \in \mathrm{gS}(M)$, $K_{\neg^* p}$ has weak amalgamation and if K is $\|M\|$-tame, then so is $K_{\neg^* p}$.

Note that if K is a universal class, then $K_{\neg^* p}$ also is universal (using that K is fully $(< \aleph_0)$-tame and short, we can prove as in Proposition 6.9 that it is an AEC), hence K is nice! More generally:

Proposition 6.14. If K weakly has primes, then K is nice.

Proof. Weak amalgamation follows from the definition of weakly having primes. Now let $M \in K$ and $p \in \mathrm{gS}(M)$. Observe that $K_{\neg^* p}$ weakly has primes, because if $N \in K_{\neg^* p}$, $N_0 \leq N \upharpoonright L(K)$ is in K, and $M \leq N_0$, then the natural expansion of N_0 is in $K_{\neg^* p}$. Therefore $K_{\neg^* p}$ also has weak amalgamation. If in addition K is $\|M\|$-tame, then so is $K_{\neg^* p}$: indeed if $N \in K_{\neg^* p}$, $q_1, q_2 \in \mathrm{gS}(N)$, and the two types are equal in K, then since $K_{\neg^* p}$ weakly has primes there is a map witnessing equality of the types in $K_{\neg^* p}$ also. □

The following fact is the key to our argument. It was first proven under slightly stronger hypotheses by Will Boney [Bon14a]. The interesting consequence to us is that it gives a local criteria for a tame AEC to have arbitrarily large models.

Fact 6.15 (Theorem 1.1 in [BVc])**.** If $\mathfrak{s}$ is a good λ-frame on K_λ, K is λ-tame and has amalgamation, then $\mathfrak{s}$ extends to a good $(\geq \lambda)$-frame on $K_{\geq \lambda}$. In particular, $K_{\geq \lambda}$ has no maximal models and is stable in every cardinals above λ.

Theorem 6.16. Let $\mathfrak{s}$ be a good λ-frame with underlying AEC K. Assume $\mathfrak{s}$ is type-full, good$^+$, successful, and K_λ has primes. Assume also that K is categorical in λ, λ-tame, and nice. The following are equivalent.

(1) K_λ is weakly uni-dimensional.
(2) K is categorical in all $\mu \geq \lambda$.
(3) K is categorical in some $\theta > \lambda$.

Proof. Replacing K with $K_{\geq \lambda}$, assume without loss of generality that $\lambda = \mathrm{LS}(K)$ and $K_{<\mathrm{LS}(K)} = \emptyset$. First note that K has amalgamation by Theorem 4.14. By Fact 6.15, $\mathfrak{s}$ extends to a good $(\geq \lambda)$-frame on K. In particular, K has no maximal models and is stable in every cardinal. Moreover by Proposition 6.9, $K_{\neg^* p}$ is an AEC for all $p \in \mathrm{gS}(M)$ and $M \in K$.

If K_λ is weakly uni-dimensional, then by Fact 6.5, K is categorical in λ^+. By Fact 6.6, K is categorical in all $\mu \geq \lambda$. So (1) implies (2). Of course, (2) implies (3). It remains to show (3) implies (1). We show the contrapositive.

Assume that K is *not* weakly uni-dimensional. Let $M \in K_\lambda$ and $p \in \mathrm{gS}(M)$ be as given by Fact 6.10. Let $\mathfrak{s}_{\neg^* p} := \mathfrak{s} \upharpoonright K_{\neg^* p}$, the restriction of $\mathfrak{s}$ to models in $K_{\neg^* p}$. Since K is nice, $K_{\neg^* p}$ has weak amalgamation and since K is also λ-tame, $K_{\neg^* p}$ is λ-tame. Since $\mathfrak{s}_{\neg^* p}$ is a good λ-frame, Theorem 4.14 gives that $K_{\neg^* p}$ has amalgamation. By Fact 6.15, $K_{\neg^* p}$ has no maximal models and is stable in every cardinals. Now let $\theta > \lambda$. By stability, K has a saturated model of size θ. Moreover since $K_{\neg^* p}$ has arbitrarily large models there must exist $\widehat{N} \in K_{\neg^* p}$ of size θ. By construction, $\widehat{N} \upharpoonright L(K)$ is not saturated of size θ. Therefore K is not categorical in θ. $\square$

We are now ready to prove a categoricity transfer in fully tame and short AECs with amalgamation. We need one more fact:

Fact 6.17. If K is a $\mathrm{LS}(K)$-tame AEC with amalgamation and no maximal models which is categorical in a $\lambda \geq h(h(\mathrm{LS}(K)))$ and the model of size λ is saturated, then K is categorical in $h(h(\mathrm{LS}(K)))$.

Proof. By the proof of [She99, Theorem II.1.6] (or see [Bal09, Theorem 14.8]). $\square$

Theorem 6.18. Let K be a fully $\mathrm{LS}(K)$-tame and short AEC with amalgamation and no maximal models[18] such that $K_{\geq h(h(\mathrm{LS}(K)))}$ has primes. If K is categorical in a $\theta > h(h(\mathrm{LS}(K)))$, then K is categorical in all $\theta' \geq h(h(\mathrm{LS}(K)))$.

Proof. First, K is categorical in $\lambda := h(h(\mathrm{LS}(K)))$ by Fact 6.17. Now apply Theorem 6.12 (to $K_{\geq \mathrm{LS}(K)^+}$) and Theorem 6.16. $\square$

The only place where shortness is used above is to get the existence property for uniqueness triples (i.e. that the good frame is successful). The proof shows that it is enough to assume that for some λ, $K_{\geq \lambda}$ is almost fully good, i.e. it has a nice-enough global independence relation (see A.1 for a more precise definition). One can ask:

Question 6.19. Can the full tameness and shortness hypothesis be weakened to just being $\mathrm{LS}(K)$-tame?

Corollary 6.20. Let K be a locally pseudo-universal AEC with amalgamation and no maximal models. If K is categorical in a $\theta > h(h(\mathrm{LS}(K)))$, then K is categorical in all $\theta' \geq h(h(\mathrm{LS}(K)))$.

Proof. By Corollary 3.8, K is fully $\mathrm{LS}(K)$-tame and short. By Remark 6.3, K has primes. Now apply Theorem 6.18. $\square$

We can now prove Theorem 0.1 from the abstract:

Corollary 6.21. Let K be a universal class. If K is categorical in two cardinals λ and μ with $\mu \geq h(|L(K)| + \aleph_0)$ and $\lambda > h(h(\mu))$, then K is categorical in all $\lambda' \geq h(h(\mu))$.

Proof. Note that $\mathrm{LS}(K) = |L(K)| + \aleph_0$. By Theorem 5.6, $K_{\geq \mu}$ has amalgamation and no maximal models. By Remark 2.22, $K_{\geq \mu}$ is locally universal, hence (see Example 3.2.(1)) locally pseudo-universal. Now apply Corollary 6.20 to $K_{\geq \mu}$. $\square$

Corollary 6.22. A universal class which is categorical in a high-enough cardinal is categorical on a tail of cardinals.

Proof. By Remark 1.3, a universal class categorical in a high-enough cardinal is categorical in unboundedly many cardinals, so we can apply Corollary 6.21. $\square$

[18]Note that there is never any need to mention joint embedding, since it follows from amalgamation, no maximal models, and categoricity.

We believe that solving the following question will be central to the resolution of Shelah's categoriciy conjecture.

Question 6.23. If K is fully tame and short, has amalgamation, and is categorical in unboundedly many cardinals, does there exists λ such that $K_{\geq \lambda}$ has primes?

Note that by [Bon14b], a positive answer would imply that Shelah's categoricity conjecture follows from the existence of a proper class of strongly compact cardinals. Moreover, we conjecture that a converse is true, namely:

Conjecture 6.24. Let K be a fully tame and short AEC. Assume K has amalgamation and is categorical on a tail of cardinals. Then there exists λ such that $K_{\geq \lambda}$ has primes.

Remark 6.25. Assume K is fully tame and short with amalgamation is categorical on a tail of cardinals. Using Fact A.3 or Corollary A.18, we can get a fully good independence relation and by the methods of [She09a, Claim III.4.9], we get that K_λ has primes when λ is a high-enough *successor* cardinal. The proof also works when λ is limit and regular but the case of λ singular seems more complicated.

Appendix A. A good frame below the Hanf number

In this appendix, we give all the results needed for the proof of Theorem 6.12. We also define all the technical terms related to good frames used there. Good frames were introduced by Shelah in [She09a, Chapter II] but we use the notation and definitions in [Vasb] (we also extensively use its results). The reader is invited to consult this paper for more motivation and background on the concepts used here.

The first definition is that of a *global* forking-like notion:

Definition A.1 (Definition 8.1 in [Vasb])**.** $\mathfrak{i} = (K, \mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$\scriptstyle|$}\hskip0.5em})$ is a *fully good independence relation* if:

(1) K is an AEC with $K_{<\text{LS}(K)} = \emptyset$ and $K \neq \emptyset$.
(2) K has amalgamation, joint embedding, and no maximal models.
(3) K is stable in all cardinals.
(4) $\mathfrak{i}$ is a $(<\infty, \geq \text{LS}(K))$-independence relation (see [Vasb, Definition 3.6]). That is, $\mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$\scriptstyle|$}\hskip0.5em}$ is a relation on quadruples (M, A, B, N)

with $M \leq N$ and $A, B \subseteq |N|$ satisfying invariance, monotonicity, and normality. We write $A \underset{M}{\overset{N}{\perp}} B$ instead of $\perp(M, A, B, N)$, and we also say $\mathrm{gtp}(\bar{a}/B; N)$ does not fork over M for $\mathrm{ran}(\bar{a}) \underset{M}{\overset{N}{\perp}} B$.

(5) $\mathfrak{i}$ has base monotonicity, disjointness ($A \underset{M}{\overset{N}{\perp}} B$ implies $A \cap B \subseteq |M|$), symmetry, uniqueness, extension, and the local character properties:
   (a) If $p \in \mathrm{gS}^\alpha(M)$, there exists $M_0 \leq M$ with $\|M_0\| \leq |\alpha| + \mathrm{LS}(K)$ such that p does not fork over M_0.
   (b) If $\langle M_i : i \leq \delta \rangle$ is increasing continuous, $p \in \mathrm{gS}^\alpha(M_\delta)$ and $\mathrm{cf}(\delta) > \alpha$, then there exists $i < \delta$ such that p does not fork over M_i.

(6) $\mathfrak{i}$ has the left and right ($\leq \mathrm{LS}(K)$)-witness properties: $A \underset{M}{\overset{N}{\perp}} B$ if and only if for all $A_0 \subseteq A$ and $B_0 \subseteq B$ with $|A_0|+|B_0| \leq \mathrm{LS}(K)$, we have that $A_0 \underset{M}{\overset{N}{\perp}} B_0$.

(7) $\mathfrak{i}$ has full model continuity: if for $\ell < 4$, $\langle M_i^\ell : i \leq \delta \rangle$ are increasing continuous such that for all $i < \delta$, $M_i^0 \leq M_i^\ell \leq M_i^3$ for $\ell = 1, 2$ and $M_i^1 \underset{M_i^0}{\overset{M_i^3}{\perp}} M_i^2$, then $M_\delta^1 \underset{M_\delta^0}{\overset{M_\delta^3}{\perp}} M_\delta^2$.

We say that $\mathfrak{i}$ is *good* if it has all the properties above except full model continuity. We say that K is *[fully] good* if there exists $\perp$ such that $(K, \perp)$ is [fully] good.

We will use the following variation:

Definition A.2. $\mathfrak{i} = (K, \perp)$ is *almost fully good* if it satisfies Definition A.1 except that only the following types are required to have a nonforking extension:

(1) Types that do not fork over *saturated* models.
(2) Type that do not fork over models of size $\mathrm{LS}(K)$.
(3) Types of length at most $\mathrm{LS}(K)$.

As before, we say that K is *almost fully good* if there exists $\perp$ such that $(K, \perp)$ is almost fully good. If we drop "fully" we mean that full model continuity need not hold.

In this terminology, we have:

Fact A.3 (Theorem 15.1.(3) in [Vasb]). Let K be a fully $(<\kappa)$-tame and short AEC with amalgamation.

If $\kappa = \beth_\kappa > \mathrm{LS}(K)$, and K is categorical in a $\mu > \lambda_0 := \left(2^\kappa\right)^{+5}$, then $K_{\geq \lambda}$ is almost fully good, where we have set $\lambda := \min(\mu, h(\lambda_0))$.

A localization of fully good independence relation are Shelah's good λ-frames. Roughly speaking, we simply require the types to have length one and the models to have a fixed size λ. We only give the definition of a *type-full* good λ-frame here, since this is the one that we can build here. In [She09a, Section II.2], Shelah has a more general definition where he only requires a dense class of basic types to satisfy the properties of forking: this is also what we call a good λ-frame (without the "type-full") in this paper, e.g. in Theorem 4.14. We use the definition in [Vasb, Definition 8.1.2] and refer to Remark 3.5 there for why this is equivalent (in the type-full case) to Shelah's definition in [She09a, Section II.2].

Definition A.4. $\mathfrak{s} = (K_{\mathfrak{s}}, \mathop{\smile\!\!\!\!\!\!\!\;|}\;)$ is a *type-full good λ-frame* if:

(1) There exists an AEC K with $\lambda = \mathrm{LS}(K)$, $K_\lambda = K_{\mathfrak{s}}$. Below, we require that all the models be in $K_{\mathfrak{s}}$.
(2) $K_{\mathfrak{s}} \neq \emptyset$.
(3) $K_{\mathfrak{s}}$ has amalgamation, joint embedding, and no maximal models.
(4) $K_{\mathfrak{s}}$ is stable in λ.
(5) $\mathop{\smile\!\!\!\!\!\!\!\;|}\;$ is a relation on quadruples (M_0, a, M, N) with $M_0 \leq M \leq N$ and $a \in |N|$ satisfying invariance, monotonicity, and normality. As before, we write $a \overset{N}{\underset{M_0}{\mathop{\smile\!\!\!\!\!\!\!\;|}\;}} M$ instead of $\mathop{\smile\!\!\!\!\!\!\!\;|}\;(M_0, a, M, N)$, and we also say $\mathrm{gtp}(a/M; N)$ does not fork over M_0 for $a \overset{N}{\underset{M_0}{\mathop{\smile\!\!\!\!\!\!\!\;|}\;}} M$.
(6) $\mathfrak{s}$ has base monotonicity, disjointness, full symmetry (if $a \overset{N}{\underset{M_0}{\mathop{\smile\!\!\!\!\!\!\!\;|}\;}} M$, $b \in |M|$, then there exists $N' \geq N$ and $M_0' \geq M_0$ with $M_0' \leq N'$, $a \in |M_0'|$, and $b \overset{N'}{\underset{M_0}{\mathop{\smile\!\!\!\!\!\!\!\;|}\;}} M_0'$), uniqueness, extension, and the local character property: If $\langle M_i : i \leq \delta \rangle$ is increasing continuous, $p \in \mathrm{gS}(M_\delta)$, then there exists $i < \delta$ such that p does not fork over M_i.

We define similarly "type-full good $(\geq \lambda)$-frame", where we allow the models in $K_{\mathfrak{s}}$ to have sizes in $K_{\geq \lambda}$ (but still work with types of length one).

Notation A.5. When $\mathfrak{i} = (K, \downfree)$ is an almost good independence relation and $\lambda \geq \text{LS}(K)$, we write $\text{pre}(\mathfrak{i}^{\leq 1}) \upharpoonright K_\lambda$ for the type-full good λ-frame obtained by restricting $\downfree$ to types of length one and models in K_λ. Similarly for $\text{pre}(\mathfrak{i}^{\leq 1}) \upharpoonright K_{\geq \lambda}$.

Assuming tameness and amalgamation, good frames can be built from a superstability-like condition (the superstability condition already appears implicitly in [SV99] and is developed further in [Van06, Van13, GVV, Vasa, Vasb, BVb]). The construction of a good frame appears implicitly already in [Vasa]:

Definition A.6 (Superstability, see Definition 10.1 in [Vasb])**.**

(1) For $M, N \in K$, say $M <_{\text{univ}} N$ (N is *universal over* M) if and only if $M < N$ and whenever we have $M' \geq M$ such that $\|M'\| \leq \|N\|$, then there exists $f : M' \xrightarrow[M]{} N$. Say $M \leq_{\text{univ}} N$ if and only if $M = N$ or $M <_{\text{univ}} N$.
(2) $p \in \text{gS}(N)$ *μ-splits over* M if $M \leq N$, $M \in K_\mu$, and there exists $N_1, N_2 \in K_\mu$ with $M \leq N_\ell \leq N$, $\ell = 1, 2$, and an isomorphism $f : N_1 \cong_M N_2$, such that $f(p \upharpoonright N_1) \neq p \upharpoonright N_2$.
(3) K is *μ-superstable* if:
   (a) $\text{LS}(K) \leq \mu$.
   (b) There exists $M \in K_\mu$ such that for any $M' \in K_\mu$ there is $f : M' \to M$ with $f[M'] <_{\text{univ}} M$.
   (c) If $\langle M_i : i < \delta \rangle$ is increasing in K_μ such that $i < \delta$ implies $M_i <_{\text{univ}} M_{i+1}$ and $p \in \text{gS}(\bigcup_{i<\delta} M_i)$, then there exists $i < \delta$ such that p does not μ-split over M_i.

Definition A.7. For λ a cardinal, let $K^{\lambda\text{-sat}}$ be the class of λ-saturated models in $K_{\geq \lambda}$.

Fact A.8 (Theorem 10.8 in [Vasb])**.** Assume K is μ-superstable, μ-tame, and has amalgamation. Then:

(1) K is μ'-superstable for all $\mu' \geq \mu$. In particular, $K_{\geq \mu}$ has joint embedding, no maximal models, and is stable in all cardinals.
(2) If $\lambda > \mu$ is such that $K^{\lambda\text{-sat}}$ is an AEC and $\text{LS}(K^{\lambda\text{-sat}}) = \lambda$, then there exists a type-full good $(\geq \lambda)$-frame with underlying AEC $K^{\lambda\text{-sat}}$.

The analysis of chains of saturated models in [BVb] shows that when λ is high-enough, $K^{\lambda\text{-sat}}$ is an AEC:

Fact A.9 (Theorem 6.11 and Remark 6.12 in [BVb])**.** Assume K is a $(< \text{LS}(K))$-tame μ-superstable AEC with amalgamation. Then there exists $\lambda_0 < h(\mathrm{LS}(K)) + \mu^{++}$ such that for all $\lambda \geq \lambda_0$ $K^{\lambda\text{-sat}}$ is an AEC with $\mathrm{LS}(K^{\lambda\text{-sat}}) = \lambda$.

Definition A.10. Let us say that K *admits saturated unions above* λ_0 if the conclusion of the Fact holds for λ_0. That is, K admits saturated unions above λ_0 if for all $\lambda \geq \lambda_0$, $K^{\lambda\text{-sat}}$ is an AEC with $\mathrm{LS}(K^{\lambda\text{-sat}}) = \lambda$.

Finally, from the analysis of Shelah and Villaveces in [SV99, Theorem 2.2.1], we obtain that superstability follows from categoricity (this was previously observed to follow from categoricity in a cardinal of cofinality above the tameness cardinal in [Vasa] and from categoricity above a fixed point of the beth function in [Vasb, Theorem 10.16]):

Fact A.11 ([GV])**.** Assume K has amalgamation. If K is categorical in a $\theta \geq h(\mathrm{LS}(K))$, then there exists $\mu < h(\mathrm{LS}(K))$ such that K is μ-superstable.

Corollary A.12. Assume K is $(< \mathrm{LS}(K))$-tame and has amalgamation. If K is categorical in a $\theta \geq h(\mathrm{LS}(K))$, then there exists $\lambda < h(\mathrm{LS}(K))$ such that:

(1) K admits saturated unions above λ.
(2) There exists a type-full good $(\geq \lambda)$-frame with underlying class $K^{\lambda\text{-sat}}$.

Proof. Combine Facts A.11, A.9, and A.8. □

It remains to see how to build a fully good (i.e. global) independence relation from just a local good frame. This is done using shortness, together with a property Shelah calls *successfulness* (we do not give the exact definition of uniqueness triple, the relation $\leq^{\mathrm{NF}}_{\lambda^+}$, or the successor frame, as we have no use for it).

Definition A.13 (Definition III.1.1 in [She09a])**.** Let $\mathfrak{s}$ be a type-full good λ-frame.

(1) $\mathfrak{s}$ is *weakly successful* if for any $M \in K_\lambda$ and any nonalgebraic $p \in \mathrm{gS}(M)$, there exists $N \geq M$ and $a \in |N|$ such that $p = \mathrm{gtp}(a/M; N)$ and (a, M, N) is a uniqueness triple (see [She09a, Definition II.5.3]).
(2) $\mathfrak{s}$ is *successful* if in addition the class $(K^{\lambda^+\text{-sat}}_{\lambda^+}, \leq^{\mathrm{NF}}_{\lambda^+})$ (see [JS13, Definition 10.1.1]) is an AEC.
(3) [She09a, Definition III.1.12] $\mathfrak{s}$ is ω*-successful* if for all $n < \omega$, the nth successor frame $\mathfrak{s}^{+n}$ (see [She09a, Definition III.1.12]) is a type-full successful good λ-frame.

We can obtain an ω-successful frame using existence of a sufficiently well-behaved global independence relation:

Fact A.14 (Theorem 11.21 in [Vasb]). Assume $\mathfrak{i}$ is a $(<\infty, \geq \text{LS}(K))$-independence relation on K such that:

(1) $\mathfrak{s} := \text{pre}(\mathfrak{i}^{\leq 1})$ is a type-full good $(\geq \text{LS}(K))$-frame.
(2) K admits saturated unions above $\text{LS}(K)$.
(3) $\mathfrak{i}$ has base monotonicity, uniqueness for types over models, and the left and right $(\leq \text{LS}(K))$-witness properties.
(4) $\mathfrak{i}$ has the following local character property: for every $n < \omega$, if $\lambda := \text{LS}(K)^{+(n+1)}$, then for every increasing continuous $\langle M_i : i \leq \lambda \rangle$ and every $p \in \text{gS}^{<\lambda}(M_\lambda)$, there exists $i < \lambda$ such that p does not fork (in the sense of $\mathfrak{i}$) over M_i.

Then $\mathfrak{s}$ is ω-successful.

In [Vasb], we used $(<\kappa)$-satisfiability as $\mathfrak{i}$ above. The inconvenient is that we use that $\kappa = \beth_\kappa > \text{LS}(K)$. Now we show we can use an independence relation induced by μ-nonsplitting instead of $(<\kappa)$-satisfiability. We need one more fact:

Fact A.15. Assume K has amalgamation and is stable in $\mu \geq \text{LS}(K)$. Let $M \in K_{\geq \mu}$ and let $p \in \text{gS}^{<\kappa}(M)$. If $\mu = \mu^{<\kappa}$, then there exists $M_0 \in K_\mu$ with $M_0 \leq M$ such that p does not μ-split over M_0.

Proof. By [She99, Claim 3.3] (or [GV06b, Fact 4.6]), it is enough to show that $|\text{gS}^{<\kappa}(N)| = \mu$ for every $N \in K_\mu$. This holds by stability in μ and [Bona, Theorem 3.1]. $\square$

Lemma A.16. Assume:

(1) K has amalgamation.
(2) K is μ-superstable.
(3) K is fully $(<\text{LS}(K))$-tame and short.
(4) $\mu = \mu^{<\text{LS}(K)}$.

Then there exists $\lambda < h(\text{LS}(K)) + \mu^{++}$ such that:

(1) K admits saturated unions above λ.
(2) There is an ω-successful type-full good λ-frame with underlying class K_λ.

Proof sketch. Let $\lambda < h(\text{LS}(K)) + \mu^{++}$ be such that K admits saturated unions above λ (exists by Fact A.9). Define a $(<\infty, \geq \lambda)$ independence relation $\mathfrak{i} = (K_{\mathfrak{i}}, \perp)$ as follows:

- $K_{\mathfrak{i}} = K^{\lambda\text{-sat}}$.
- $p \in \mathrm{gS}^\alpha(M)$ does not fork (in the sense of $\mathfrak{i}$) over $M_0 \leq M$ if and only if:
  - $M_0, M \in K^{\lambda\text{-sat}}$.
  - For every $I \subseteq \alpha$ with $|I| < \mathrm{LS}(K)$, there exists $M_0' \leq M_0$ in K_μ such that p^I does not μ-split over M_0'.

We claim that $\mathfrak{i}$ satisfies the hypotheses of Fact A.14 (where K there is $K_{\geq\lambda}$ here). By Fact A.15 and superstability, we have that $\mathfrak{i}$ induces a $\mathrm{LS}(K)$-generator for a weakly good $(< \mathrm{LS}(K))$-independence relation (in the sense of [Vasb, Definition 7.3]), as well as a $\mathrm{LS}(K)$-generator for a good (≤ 1)-independence relation (see [Vasb, Definition 8.5]). It follows from [Vasb, Theorems 7.5, 8.9] that $\mathfrak{s} := \mathrm{pre}(\mathfrak{i}^{\leq 1})$ is a type-full good $(\geq \lambda)$-frame and $\mathfrak{i}^{<\mathrm{LS}(K)}$ (the restriction of $\mathfrak{i}$ to types of length less than $\mathrm{LS}(K)$) has base monotonicity, uniqueness for types over models, transitivity, and so that any type does not fork over a model of size $\mathrm{LS}(K)$.

Now, it is easy to see using shortness that $\mathfrak{i}$ also has uniqueness for types over models. By definition, it also has base monotonicity, transitivity, and the left $(< \mathrm{LS}(K))$-witness property. Now from transitivity and the local character property mentioned in the previous paragraph, we get ([Vasb, Proposition 4.3.6]) that $\mathfrak{i}$ has the right $(\leq \mathrm{LS}(K))$-witness property. Thus all the hypotheses of Fact A.14 are satisfied, so $\mathfrak{s}$ is ω-successful. $\square$

From an ω-successful good λ-frame, we obtain the desired global independence relation:

Fact A.17. Let $\mathfrak{s} = (K_\lambda, \mathop{\smile\!\!\!\!\!\!|}\,)$ be an ω-successful good λ-frame which is categorical in λ. If K is fully $< \mathrm{cf}(\lambda)$-tame and short and has amalgamation, then $K^{\lambda^{+3}\text{-sat}}$ is almost fully good.

Proof. By [Vasb, Theorems 12.16, 13.6, 14.15] $\square$

Corollary A.18. Assume K has amalgamation and is fully $(< \mathrm{LS}(K))$-tame and short. If K is categorical in a $\theta \geq h(\mathrm{LS}(K))$, then there exists $\lambda < h(\mathrm{LS}(K))$ such that K admits saturated unions above λ and $K^{\lambda\text{-sat}}$ is almost fully good.

Proof. By Fact A.11, there exists $\mu < h(\mathrm{LS}(K))$ such that K is μ-superstable. By Fact A.8, we can make μ bigger if necessary to assume that $\mu = \mu^{<\mathrm{LS}(K)}$. By Lemma A.16, there exists $\lambda_0 < h(\mathrm{LS}(K))$ such that K admits saturated unions above λ_0 and there is an ω-successful good λ_0-frame $\mathfrak{s}$ with underlying class $K^{\lambda_0\text{-sat}}$. Now it is easy to check that the successor frame $\mathfrak{s}^+$ is an ω-successful good λ_0^+-frame with underlying AEC $K^{\lambda_0^+\text{-sat}}$, so without loss of generality λ_0 is already a regular cardinal. Now apply Fact A.17 (note that by uniqueness of saturated models, $K^{\lambda_0\text{-sat}}$ is categorical in λ_0). $\square$

Note for future reference that in almost good AECs, uniqueness triples have an easier definition.

Definition A.19. Let $\mathfrak{i} = (K, \mathop{\smile}\limits^{\mid})$ be an almost good independence relation. (a, M, N) is a *domination triple* if $M \leq N$, $a \in |N| \backslash |M|$, and for any $N' \geq N$ and any $B \subseteq |N'|$, if $a \overset{N'}{\underset{M}{\mathop{\smile}\limits^{\mid}}} B$, then $N \overset{N'}{\underset{M}{\mathop{\smile}\limits^{\mid}}} B$.

Fact A.20 (Lemma 11.7 in [Vasb])**.** Let $\mathfrak{i} = (K, \mathop{\smile}\limits^{\mid})$ be an almost good independence relation. Let $\mu \geq \text{LS}(K)$ and let $\mathfrak{s} := \text{pre}(\mathfrak{i}^{\leq 1}) \upharpoonright K_\mu$.

For $M, N \in K_\mu$, (a, M, N) is a domination triple if and only it is a uniqueness triple in $\mathfrak{s}$.

We continue the proof of Theorem 6.12 by showing that the frame induced by an almost good independence relation is good$^+$, a technical property of frames:

Definition A.21 (Definition III.1.3 in [She09a])**.** Let $\mathfrak{s} = (K_\lambda, \mathop{\smile}\limits^{\mid})$ be a type-full good λ-frame. $\mathfrak{s}$ is good$^+$ if the following is *impossible*: There exists increasing continuous chains $\langle M_i : i \leq \lambda^+ \rangle$, $\langle N_i : i \leq \lambda^+ \rangle$, a type $p^* \in \text{gS}(M_0)$, and a sequence $\langle a_i : i < \lambda^+ \rangle$ such that for all $i < \lambda^+$:

(1) M_{λ^+} is λ^+-saturated.
(2) $M_i \leq N_i$ and they are both in K_λ.
(3) $a_{i+1} \in |M_{i+2}|$.
(4) $\text{gtp}(a_{i+1}/M_{i+1}; M_{i+2})$ is a nonforking extension of p^*.
(5) $\text{gtp}(a_{i+1}/N_0; N_{i+2})$ forks over M_0.

Proposition A.22. If $\mathfrak{i} = (K, \mathop{\smile}\limits^{\mid})$ is an almost good independence relation, then $\text{pre}(\mathfrak{i}^{\leq 1}) \upharpoonright K_{\text{LS}(K)}$ is good$^+$.

Proof. Suppose $\langle M_i : i \leq \lambda^+ \rangle$, $\langle N_i : i \leq \lambda^+ \rangle$, $\langle a_i : i < \lambda^+ \rangle$, and p^* witness the failure of being good$^+$. By local character, there exists $i < \lambda^+$ such that $N_0 \overset{N_{\lambda^+}}{\underset{M_i}{\mathop{\smile}\limits^{\mid}}} M_{\lambda^+}$. By symmetry and monotonicity, we must have that $a_{i+1} \overset{N_{\lambda^+}}{\underset{M_i}{\mathop{\smile}\limits^{\mid}}} N_0$, i.e. $\text{gtp}(a_{i+1}/N_0; N_{i+2})$ does not fork over

M_i. By transitivity and base monotonicity, $\mathrm{gtp}(a_{i+1}/N_0; N_{i+2})$ does not fork over M_0, contradiction. $\square$

Corollary A.23. Assume $\mathfrak{i} = (K, \mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}})$ is an almost good independence relation and K admits saturated unions above $\mathrm{LS}(K)$. Let $\lambda > \mathrm{LS}(K)$ and let $\mathfrak{s} := \mathrm{pre}(\mathfrak{i}^{\leq 1}) \upharpoonright K^{\lambda\text{-sat}}$. Then $\mathfrak{s}$ is ω-successful and good$^+$.

Proof. By Fact A.14 and Proposition A.22 (applied to the restriction of $\mathfrak{i}$ to λ-saturated models). $\square$

Appendix B. Frames that are not weakly uni-dimensional

In this appendix, we give a proof of Fact 6.10. We work with the following hypotheses:

Hypothesis B.1.

(1) $\mathfrak{s} = (K_\lambda, \mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}})$ is a type-full successful good$^+$ λ-frame.
(2) K_λ has primes.
(3) K is categorical in λ.

We will use the orthogonality calculus developed in [She09a, Chapter III].

Definition B.2 (Definition III.6.2 in [She09a])**.**

(1) Let $M \in K_\lambda$ and let $p, q \in \mathrm{gS}(M)$ be nonalgebraic. We say that p and q are *weakly orthogonal* if whenever (a, M, N) is a uniqueness triple with $\mathrm{gtp}(a/M; N) = q$, then p has a unique extension to $\mathrm{gS}(N)$. We say that p and q are *orthogonal*, written $p \perp q$ if for every $N \geq M$, the nonforking extensions to N p', q' of p and q respectively are weakly orthogonal.
(2) Let $M_\ell \in K_\lambda$ and $p_\ell \in \mathrm{gS}(M_\ell)$ be nonalgebraic, $\ell = 1, 2$. We say that p_1 and p_2 are *orthogonal* if there exists $N \geq M_\ell$ such that the nonforking extensions to N p_1', p_2' of p_1 and p_2 respectively are orthogonal.

Fact B.3 (Claims III.6.7, III.6.8 in [She09a])**.** Let $M \in K_\lambda$ and $p, q \in \mathrm{gS}(M)$ be nonalgebraic.

(1) [She09a, Claim III.6.3] p is weakly orthogonal to q if and only if there exists a uniqueness triple (a, M, N) such that $\mathrm{gtp}(a/M; N) = q$ and p has a unique extension to $\mathrm{gS}(N)$.
(2) [She09a, Claim III.6.7.2] $p \perp q$ if and only if $q \perp p$.
(3) [She09a, Claim III.6.8.5] p and q are orthogonal if and only if they are weakly orthogonal.

We will also use the following without comments:

Fact B.4 (Claim III.3.7 in [She09a])**.** If (a, M, N) is a prime triple, then it is a uniqueness triple.

Some orthogonality calculus gives us a useful description of the types in $K_{\neg^* p}$ (recall Definition 6.7).

Lemma B.5. Let $M \in K_\lambda$ and let $p \in \mathrm{gS}(M)$ be nonalgebraic. Let $N \in K_{\neg^* p}$ be of size λ such that the map $a \mapsto c_a^N$ is the identity (so $M \leq N \restriction L(K)$). For any $N_0 \leq N \restriction L(K)$ with $M \leq N_0$ and any $q \in \mathrm{gS}(N_0; N)$, $p \perp q$.

Proof. Let p' be the nonforking extension of p to N_0. By Fact B.3, it is enough to show that p' is weakly orthogonal to q. Let (a, N_0, N') be a prime triple such that $\mathrm{gtp}(a/N_0; N') = q$ and $N' \leq N$ (exists since we are assuming that K_λ has primes). Then since p has a unique extension to N it has a unique extension to N', which must be the nonforking extension so p' also has a unique extension to N'. By Fact B.4, (a, N_0, N') is a uniqueness triple and by Fact B.3 again, this suffices to conclude that p' and q are weakly orthogonal. □

The next lemma justifies the "uni-dimensional" terminology: if the class is *not* uni-dimensional, then there are two orthogonal types.

Lemma B.6. If K_λ is not weakly uni-dimensional, there exists $M \in K_\lambda$ and types $p, q \in \mathrm{gS}(M)$ such that $p \perp q$.

Proof. Assume K_λ is not weakly uni-dimensional. This means that there exists $M < M_\ell$, $\ell = 1, 2$, all in K_λ such that for any $c \in |M_2| \backslash |M|$, $\mathrm{gtp}(c/M; M_2)$ has a unique extension to $\mathrm{gS}(M_1)$. Pick any $c \in |M_2| \backslash |M|$ and let $p := \mathrm{gtp}(c/M; M_2)$. Then there is a natural expansion of M_1 to $K_{\neg^* p}$. So pick any $d \in |M_1| \backslash |M|$ and let $q := \mathrm{gtp}(d/M; M_1)$. By Lemma B.5, $p \perp q$, as desired. □

We can now prove Fact 6.10. We restate it here for convenience:

Fact B.7. If K_λ is not weakly uni-dimensional, then there exists $M \in K_\lambda$ and $p \in \mathrm{gS}(M)$ such that $\mathfrak{s} \restriction K_{\neg^* p}$ (the restriction of $\mathfrak{s}$ to the models in $K_{\neg^* p}$) is a type-full good λ-frame.

Proof. Assume K_λ is not weakly uni-dimensional. By Lemma B.6, there exists $M \in K_\lambda$ and types $p, q \in \mathrm{gS}(M)$ such that $p \perp q$.

Let $\mathfrak{s}_{\neg^* p} := \mathfrak{s} \restriction K_{\neg^* p}$. We check that it is a type-full good λ-frame. For ease of notation, we identify a model $N \in K_{\neg^* p}$ and its reduct to K.

For $N \geq M$, we write p_N for the nonforking extension of p to $\text{gS}(N)$, and similarly for q_N.

- $K_{\neg^* p}$ is not empty, since (the natural expansion of) M is in it.
- $(K_{\neg^* p})_\lambda$ is an AEC in λ (that is, its models of size λ behave like an AEC, see [She09a, Definition II.1.18]) by the proof of Proposition 6.9.
- Nonforking has many of the usual properties: monotonicity, invariance, disjointness, local character, continuity, and transitivity all trivially follow from the definition of $K_{\neg^* p}$.
- Nonforking has the uniqueness property: Let $N \in K_{\neg^* p}$ have size λ. Without loss of generality $M \leq N$. Let $N' \geq N$ be in $K_{\neg p}$ of size λ and let $r_1, r_2 \in \text{gS}(N')$ be nonforking over N and such that $r_1 \upharpoonright N = r_2 \upharpoonright N$. Say $r_\ell = \text{gtp}(a_\ell / N'; N_\ell)$. Now in K, $r_1 = r_2$, and since K_λ has primes, the equality is witnessed by an embedding $f : M_1 \xrightarrow[N]{} N_2$, with $M_1 \leq N_1$. Since $N_1 \in K_{\neg^* p}$, $M_1 \in K_{\neg^* p}$, and so $r_1 = r_2$ also in $K_{\neg^* p}$ (this is similar to the proof of Proposition 6.14).
- Nonforking has the extension property. Let $N \in K_{\neg^* p}$ have size λ. Without loss of generality, $M \leq N$. Let $r \in \text{gS}(N)$ be nonalgebraic and let $N' \geq N$ be in $K_{\neg^* p}$ of size λ. Let $r' \in \text{gS}(N')$ be the nonforking extension of r to N' (in K). Let (a, N', N'') be a prime triple such that $\text{gtp}(a/N'; N'') = r'$. By Lemma B.5, $r \perp p$. Thus r' is weakly orthogonal to $p_{N'}$ and hence $p_{N''}$ is the unique extension of $p_{N'}$ to N''. Now if p' is an extension of p to $\text{gS}(N'')$, then $p' \upharpoonright N' = p_{N'}$ as $N' \in K_{\neg^* p}$, so $p' = p_{N''}$ by the previous sentence. This shows that $N'' \in K_{\neg^* p}$, so as $r' \in \text{gS}(N'; N'')$, r' is a Galois type in $K_{\neg^* p}$, as desired.
- $K_{\neg^* p}$ has λ-amalgamation: because $(K_{\neg^* p})_\lambda$ has the type extension property and weak λ-amalgamation (as K_λ, and hence $(K_{\neg^* p})_\lambda$, has primes, see the proof of Proposition 6.14), thus one can apply Theorem 4.13.
- $K_{\neg^* p}$ has λ-joint embedding: since any model contains a copy of M, this is a consequence of λ-amalgamation over M.
- $K_{\neg^* p}$ is stable in λ: because $K_{\neg^* p}$ has "fewer" Galois types than K, and K is stable in λ.
- $(K_{\neg^* p})_\lambda$ has no maximal models: This is where we use the negation of weakly uni-dimensional. Let $N \in K_{\neg^* p}$ be of size λ and without loss of generality assume $M \leq N$. Recall from above that there is a nonalgebraic type $q \in \text{gS}(M)$ such that $p \perp q$. Let q_N be the nonforking extension of q to N and let (a, N, N') be a prime triple such that $q = \text{gtp}(a/N; N')$. As in the proof of

the extension property, $N' \in K_{\neg^* p}$. Moreover as $a \in |N'| \backslash |N|$, $N < N'$, as needed.

- $\mathfrak{s}_{\neg^* p}$ is type-full: because $\mathfrak{s}$ is.
- $\mathfrak{s}_{\neg^* p}$ has full symmetry: Assume $a \overset{N}{\underset{N_0}{\perp}} N_1$, for $N_0, N_1, N \in K_{\neg^* p}$, $M \leq N_0 \leq N_1 \leq N$, and $a \in |N|$. Let $b \in |N_1|$. Without loss of generality, $a \notin |N_1|$ (if $a \in |N_1|$, then $a \in |N_0|$ by disjointness and as $b \overset{N}{\underset{N_0}{\perp}} N_0$, N_0 and N witness the full symmetry). By full symmetry in $\mathfrak{s}$, there exists $N_0', N' \in K$ such that $N \leq N'$, $N_0 \leq N_0' \leq N'$, and $b \overset{N'}{\underset{N_0}{\perp}} N_0'$ (note that the first use of $\perp$ was in $\mathfrak{s}_{\neg^* p}$ and the second in $\mathfrak{s}$, but since the first is just the restriction of the first to models in $K_{\neg^* p}$, we do not make the difference). Now let N_0'' be such that $N_0 \leq N_0'' \leq N_0'$ and (a, N_0, N_0'') is a prime triple. Since $r = \text{gtp}(a/N_0; N_0'') = \text{gtp}(a/N_0; N)$ is orthogonal to p (by Lemma B.5), we have that $N_0'' \in K_{\neg^* p}$. By monotonicity, $b \overset{N'}{\underset{N_0}{\perp}} N_0''$. Now let (b, N_0'', N'') be a prime triple with $N'' \leq N'$. By the same argument as before, $N'' \in K_{\neg^* p}$ and by monotonicity, $b \overset{N''}{\underset{N_0}{\perp}} N_0''$. Since all the models are in $K_{\neg^* p}$, this shows that the nonforking happens in $\mathfrak{s}_{\neg^* p}$, as needed.

We have checked all the properties and therefore $\mathfrak{s}_{\neg^* p}$ is a type-full good λ-frame. □

Appendix C. Independence in universal classes

We investigate the properties of independence in universal classes (more generally in AECs admitting intersections). Recall that [Vasb, Theorem 15.6] showed that a fully tame and short AEC with amalgamation categorical in unboundedly many cardinals eventually admits a well-behaved independence notion. We want to specialize this result to AECs admitting intersections and prove more properties of forking there. Here, we prove that the independence relation satisfies the axioms of [BGKV] (partially answering Question 7.1 there). Moreover it has a finite character property (Theorem C.7) and can be extended to an independence relation over sets (Theorem C.14). A simple corollary is the disjoint amalgamation property (Corollary C.6).

While none of the results are used in this paper, we believe they shed further light on how the existence a closure operator helps in the structural analysis of an AEC. Since many classes of interests to algebraists admit intersections, we believe the existence of a well-behaved independence notion there is likely to have further applications.

By Fact A.3 or Corollary A.18, it is reasonable to assume:

Hypothesis C.1.

(1) K locally admits intersections.
(2) $\mathfrak{i} = (K, \mathop{\smile\!\!\!\!\!|}\,)$ is an almost fully good independence relation (see Definition A.1).

Our goal is to prove that $\mathfrak{i}$ is actually fully good, i.e. extension holds. Note that if we knew that K was categorical above the Löwenheim-Skolem number, we could use the categoricity transfer of Section 6. However here we do not make any categoricity assumption and our approach is easier: we study how the closure operator interacts with independence. The key lemma is:

Lemma C.2. If $A \overset{N}{\underset{M_0}{\mathop{\smile\!\!\!\!\!|}}} B$, then $\operatorname{cl}^N(A) \overset{N}{\underset{M_0}{\mathop{\smile\!\!\!\!\!|}}} \operatorname{cl}^N(B)$.

Proof. By normality, without loss of generality $|M_0| \subseteq A, B$. Using symmetry, it is enough to show that $A \overset{N}{\underset{M_0}{\mathop{\smile\!\!\!\!\!|}}} \operatorname{cl}^N(B)$. By the witness property and finite character of the closure operator, we can assume without loss of generality that $|A| \leq \operatorname{LS}(K)$. Therefore by extension there exists $N' \geq N$ and $M \geq M_0$ such that $M \leq N'$, M contains B, and $A \overset{N'}{\underset{M_0}{\mathop{\smile\!\!\!\!\!|}}} M$. By definition, $\operatorname{cl}^N(B) = \operatorname{cl}^{N'}(B)$ is contained in M, so $A \overset{N'}{\underset{M_0}{\mathop{\smile\!\!\!\!\!|}}} \operatorname{cl}^N(B)$, so $A \overset{N}{\underset{M_0}{\mathop{\smile\!\!\!\!\!|}}} \operatorname{cl}^N(B)$. □

An abstract way of stating Lemma C.2 is via domination triples (recall Definition A.19).

Lemma C.3. Let $M \leq N$ and let $a \in |N| \backslash |M|$. Then $(a, M, \operatorname{cl}^N(\{a\} \cup |M|))$ is a domination triple.

Proof. Directly from Lemma C.2. □

In our framework, domination triples are the same as the uniqueness triples of [She09a, Definition II.5.3] by Fact A.20, thus we get:

Theorem C.4. $\mathfrak{i}$ has extension. Hence it is a fully good independence relation.

Proof. Let $\mu \geq \text{LS}(K)$ and let $\mathfrak{s} := \text{pre}(\mathfrak{i}^{\leq 1} \restriction K_\mu)$. By Lemma C.3 and Fact A.20 $\mathfrak{s}$ has the so-called existence property for uniqueness triples (see [She09a, Definition II.5.3]). By Section II.6 of [She09a] (and the results of section 12 in [Vasb]) $\mathfrak{s}$ induces an independence relation $\mathfrak{i}'$ for types of length at most μ over models of size μ that is well-behaved (i.e. it has all of the properties of a fully good independence relation except full model continuity and disjointness). By the canonicity of such relations (see the proofs of Corollary 5.19 and Theorem 6.13 in [BGKV]), $\mathfrak{i}'$ must be the same as $\mathfrak{i}^{\leq \mu} \restriction K_\mu$, the restriction of $\mathfrak{i}$ to size μ. Thus for all $\mu \geq \text{LS}(K)$, $\mathfrak{i}$ has extension for types of length at most μ over models of size μ. By the proof of [Vasb, Lemma 14.13], this suffices to conclude that $\mathfrak{i}$ has extension. $\square$

Remark C.5. The proof shows that instead of the AEC admitting intersections, it is enough to assume that for each μ, the restriction of $\mathfrak{i}$ to a good frame in μ has the existence property for uniqueness triples. Unfortunately the proof in [Vasb, Section 11] only works when the frame is restricted to the saturated models of size μ.

Corollary C.6. K has disjoint amalgamation.

Proof. Because $\mathfrak{i}$ has existence, extension and disjointness. $\square$

Another consequence of having a closure operator is:

Theorem C.7 (Finite character of independence)**.** $A \underset{M_0}{\overset{N}{\downarrow}} B$ if and only if for all finite $A_0 \subseteq A$ and $B_0 \subseteq B$, $A_0 \underset{M_0}{\overset{N}{\downarrow}} B_0$. That is, $\mathfrak{i}$ has the $(< \aleph_0)$-witness property.

Proof. By symmetry it is enough to show that if $A_0 \underset{M_0}{\overset{N}{\downarrow}} B$ for all finite $A_0 \subseteq A$, then $A \underset{M_0}{\overset{N}{\downarrow}} B$. For each finite $A_0 \subseteq A$, let $M_{A_0} := \text{cl}^N(|M_0| \cup A_0)$. Let $M := \text{cl}^N(|M_0| \cup B)$. By Lemma C.2, $M_{A_0} \underset{M_0}{\overset{N}{\downarrow}} M$ for each finite $A_0 \subseteq A$. Let $M_A := \text{cl}^N(|M_0| \cup A)$. It is easy to see that

$\langle M_{A_0} \mid A_0 \in [A]^{<\aleph_0} \rangle$ is a directed system with union M_A. Therefore by full model continuity, $M_A \underset{M_0}{\overset{N}{\mathop{\smile}\limits^{\mid}}} M$, and so $A \underset{M_0}{\overset{N}{\mathop{\smile}\limits^{\mid}}} B$. $\square$

Remark C.8. Thus we have that the axioms in [She09b, Chapter V.B] are satisfied by $(K, \mathop{\smile}\limits^{\mid}, \text{cl})$.

For the next two results, we drop our hypotheses.

Theorem C.9. Let K be a fully $(< \text{LS}(K))$-tame and short AEC with amalgamation. Assume further that K locally admits intersections.

If K is categorical in a $\mu \geq h(\text{LS}(K))$, then there exists $\lambda < h(\text{LS}(K))$ such that $K_{\geq \lambda}$ is fully good. Moreover the independence relation has the $(< \aleph_0)$-witness property.

Proof. Combine Corollary A.18, Theorem C.4, and Theorem C.7. $\square$

Remark C.10. If K is not categorical but only superstable (see Definition A.6), then we can generalize the result (using [Vasb, Theorem 15.1]) provided that for all λ, $K^{\lambda\text{-sat}}$ (the class of λ-saturated models in K) locally admits intersections.

C.1. **Set bases.** We end by showing that it is possible to extend the independence relation to define forking not only over models but also over sets. In the terminology of [HL02], K is simple (note that the paper gives an example due to Shelah of a class that has a fully good independence relation, yet is not simple).

For our methods to work, we have to assume that K admits intersections, i.e. not just locally. To see that this is not a big loss, recall that if K is categorical in unboundedly many cardinals and has amalgamation, then the models in the categoricity cardinals are saturated, so for $M \in \text{LS}(K)$, K_M will also be categorical in unboundedly many cardinals.

Hypothesis C.11. K admits intersections.

Definition C.12. Let $N \in K$ and $A, B, C \subseteq |N|$. Define $B \underset{A}{\overset{N}{\mathop{\smile}\limits^{\mid}}} C$ to hold if and only if $\text{cl}^N(AB) \underset{\text{cl}^N(A)}{\overset{N}{\mathop{\smile}\limits^{\mid}}} \text{cl}^N(AC)$.

We define properties such as invariance, monotonicity, etc. just as for the model-based version of independence.

Remark C.13. When $A \leq N$, this agrees with the previous definition of independence.

Theorem C.14.

(1) $\mathop{\smile}\limits^{\mid}$ has invariance, left and right monotonicity, base monotonicity, and normality.
(2) $\mathop{\smile}\limits^{\mid}$ has symmetry, finite character (i.e. the $(< \aleph_0)$-witness property), existence and transitivity.
(3) $\mathop{\smile}\limits^{\mid}$ has extension.
(4) Let $N \in K$ and let $\langle B_i : i < \delta \rangle$ be an increasing chain of sets. Let $B_\delta := \bigcup_{i<\delta} B_i$ and assume $B_\delta \subseteq |N|$. Let $p \in \text{gS}^\alpha(B; N)$. If $\text{cf}(\delta) > \alpha$, then there exists $i < \delta$ such that p does not fork over B_i.
(5) If $p \in \text{gS}^\alpha(B; N)$, there exists $A \subseteq B$ such that p does not fork over A and $|A| < |\alpha|^+ + \aleph_0$.

Proof.

(1) Easy.
(2) Easy.
(3) By transitivity and extension of $\mathfrak{i}$.
(4) By local character for $\mathfrak{i}$.
(5) By finite character, it is enough to show it when $\alpha < \omega$. Work by induction on $\lambda := |B|$. If $\lambda < \aleph_0$, take $A = B$ and use the existence property. If $\lambda \geq \aleph_0$, write $B = \bigcup_{i<\lambda} B_i$, where $|B_i| < \lambda$ for all $i < \lambda$. By the previous result, there exists $i < \lambda$ such that p does not fork over B_i. Now apply the induction hypothesis and transitivity.

$\square$

Remark C.15. Thus in this framework types of finite length really do not fork over a finite set. This removes the need for a special chain version of local character (i.e. if $\langle M_i : i \leq \delta \rangle$ is increasing continuous, $p^{<\omega} \in \text{gS}(M_\delta)$, there exists $i < \delta$ such that p does not fork over M_i).

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E-mail address: `sebv@cmu.edu`

URL: `http://math.cmu.edu/~svasey/`

Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, Pennsylvania, USA