ABSTRACT ELEMENTARY CLASSES STABLE IN $\aleph_0$

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Abstract. We study abstract elementary classes (AECs) that, in $\aleph_0$, have amalgamation, joint embedding, no maximal models and are stable (in terms of the number of orbital types). We prove that such classes exhibit superstable-like behavior at $\aleph_0$. More precisely, there is a superlimit model of cardinality $\aleph_0$ and the class generated by this superlimit has a type-full good $\aleph_0$-frame (a local notion of nonforking independence) and a superlimit model of cardinality $\aleph_1$. This extends the first author's earlier study of $\mathrm{PC}_{\aleph_0}$-representable AECs and also improves results of Hyttinen-Kesälä and Baldwin-Kueker-VanDieren.

1. Introduction

1.1. **Motivation.** In [She87a] (a revised version of which appears as [She09a, Chapter I], from which we cite), the first author introduced *abstract elementary classes* (AECs): a semantic framework generalizing first-order model theory and also encompassing logics such as $\mathbb{L}_{\omega_1,\omega}(Q)$. The first author studied $\mathrm{PC}_{\aleph_0}$-representable AECs (roughly, AECs which are reducts of a class of models of a first-order theory omitting a countable set of types) and generalized and improved some of his earlier results on $\mathbb{L}_{\omega_1,\omega}$ [She83a, She83b] and $\mathbb{L}_{\omega_1,\omega}(Q)$ [She75].

For example, fix a $\mathrm{PC}_{\aleph_0}$-representable AEC $\mathfrak{K}$ and assume for simplicity that it is categorical in $\aleph_0$. Assuming $2^{\aleph_0} < 2^{\aleph_1}$ and $1 \leq \mathbb{I}(\mathfrak{K}, \aleph_1) < 2^{\aleph_1}$, the first author shows [She09a, I.3.8] that $\mathfrak{K}$ has amalgamation in $\aleph_0$. Further, [She09a, I.4, I.5], it has a lot of structure in $\aleph_0$ and assuming more set-theoretic assumptions as well as few models in $\aleph_2$, $\mathfrak{K}$ has a superlimit model in $\aleph_1$ [She09a, I.5.34, I.5.40]. This means roughly (see [She09a, I.3.3]) that there is a saturated model in $\aleph_1$ and that the

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union of an increasing chain of type ω consisting of saturated models of cardinality $\aleph_1$ is saturated.

1.2. **Main result.** The present paper improves this result by removing the need for the extra set-theoretic and structure hypotheses on $\aleph_2$:

Theorem 1.1. Assume $2^{\aleph_0} < 2^{\aleph_1}$. Let $\mathfrak{K}$ be a $\mathrm{PC}_{\aleph_0}$-representable AEC (with $\mathrm{LS}(\mathfrak{K}) = \aleph_0$ and countable vocabulary). If $\mathfrak{K}$ is categorical in $\aleph_0$ and $1 \leq \mathbb{I}(\mathfrak{K}, \aleph_1) < 2^{\aleph_1}$, then $\mathfrak{K}$ has a superlimit model of cardinality $\aleph_1$.

We give the proof of Theorem 1.1 at the end of this introduction. For now, notice that it implies the nontrivial fact that $\mathfrak{K}$ has a model of size $\aleph_2$. However this consequence was known because under the hypotheses of Theorem 1.1, one can change the ordering on $\mathfrak{K}$ to obtain a new class $\mathfrak{K}'$ that has a good $\aleph_0$-frame [She09a, II.3.4] (a local axiomatic notion of nonforking independence. Its existence implies that there is a model of size $\aleph_2$). Note also that the assumption of categoricity in $\aleph_0$ is not really needed (see [She09a, I.3.10]) but then one has to change the class to obtain one that is categorical in $\aleph_0$ and get a superlimit in the new class.

An additional difficulty in [She09a, I.5] is the lack of stability: one can only get that there are $\aleph_1$-many orbital types over countable models. A workaround is to redefine the ordering (but not the class of models) to get a stable class, see [She09a, I.5.29].

1.3. **Outline of the paper.** In this paper, we start with some of the *consequences* of [She09a, Chapter I]: amalgamation (plus joint embedding and no maximal models) in $\aleph_0$ and stability in $\aleph_0$. We show that once we have them we can derive all the rest (e.g. existence of a superlimit in $\aleph_0$ and existence of a good $\aleph_0$-frame) *without* assuming anything else (no need for $2^{\aleph_0} < 2^{\aleph_1}$ or $\mathbb{I}(\mathfrak{K}, \aleph_1) < 2^{\aleph_1}$). In fact, we do *not* need to assume that $\mathfrak{K}$ is $\mathrm{PC}_{\aleph_0}$ (rather, we can *prove* that a certain subclass of $\mathfrak{K}$ is $\mathrm{PC}_{\aleph_0}$, see Theorem 4.2 and Corollary 4.14). Moreover, we do not need to start with full amalgamation but can work in the slightly more general setup of [SV99].

One of the main tool is model-theoretic forcing in the style of Robinson, as used in [She09a, Chapter I]. When assuming amalgamation, the notion is well-behaved. In particular, every formula is decided. We prove (Theorem 4.10) that one can characterize brimmed models (also called limit models in the literature) as those that are homogeneous for orbital types, or equivalently homogeneous for the syntactic types induced by the forcing notion (we call them generic types). This has as immediate consequence that the brimmed model of cardinality $\aleph_0$ is superlimit (Corollary 4.11). This sheds light on an argument of Lessmann [Les05] and answers a question of Fred Drueck (see footnote 3 on [Dru13, p. 25].

We also deduce (Corollary 4.13) that orbital types over countable models are determined by their restrictions to finite sets (this is often called $(< \aleph_0, \aleph_0)$-tameness in the literature, we call it locality). This generalizes a result of Hyttinen and Kesälä, who proved it in the context of *finitary* AECs [HK06, 3.12].

One can then build a good frame (Theorem 4.19) as in the proof of [She09a, II.3.4] but a key new point given by the locality is that this frame will be good$^+$ (a technical condition characterized in Theorem 3.15). Using it, we can obtain the superlimit model in $\aleph_1$.

Another application of the construction of a good frame is that if the class has global amalgamation and all its orbital types are determined by their countable restrictions (this is called $\aleph_0$-tameness in other places in the literature), then $\aleph_0$-stability implies stability in all cardinals. This follows from e.g. the stability transfer in [Vas16b, 5.6] and improves a result of Baldwin-Kueker-VanDieren [BKV06, 3.6] (by removing the hypothesis of ω-locality there; in fact it follows from the rest by the existence of the good frame).

Proof of Theorem 1.1. The global hypotheses of [She09a, I.5] are satisfied, and in particular we have amalgamation in $\aleph_0$. By [She09a, I.5.36], we can assume without loss of generality that $\mathfrak{K}$ is stable in $\aleph_0$. Therefore the hypotheses of Theorem 4.19 hold, hence its conclusion. $\square$

1.4. **Notes.** This paper was written while the second author was working on a Ph.D. thesis under the direction of Rami Grossberg at Carnegie Mellon University and he would like to thank Professor Grossberg for his guidance and assistance in his research in general and in this work specifically.

Note that at the beginning of several sections, we make global hypotheses assumed throughout the section.

2. Preliminaries

We assume familiarity with the basics of AECs, as presented for example in [Gro02, Bal09], or the first three sections of Chapter I together with the first section of Chapter II in [She09a]. We also assume familiarity with good frames (see [She09a, Chapter II]). This section mostly fixes the notation that we will use.

Given a τ-structure M, we write $|M|$ for its universe and $\|M\|$ for its cardinality. We may abuse notation and write e.g $a \in M$ instead of $a \in |M|$. We may even write $\bar{a} \in M$ instead of $\bar{a} \in {}^{<\omega}|M|$.

We write $\mathfrak{K} = (K, \leq_{\mathfrak{K}})$ for an AEC. We may abuse notation and write $M \in \mathfrak{K}$ instead of $M \in K$. For a cardinal λ, we write $\mathfrak{K}_\lambda$ for the AEC restricted to its models of size λ. As shown in [She09a, II.1], any AEC is uniquely determined by its restriction $\mathfrak{K}_{\leq \text{LS}(\mathfrak{K})}$.

When we say that $M \in \mathfrak{K}$ is an *amalgamation base*, we mean (as in [SV99]) that it is an amalgamation base in $\mathfrak{K}_{\|M\|}$, i.e. we do *not* require that larger models can be amalgamated.

Given an AEC $\mathfrak{K}$, we may extend the relation $\leq_{\mathfrak{K}}$ to allow the empty set on the left hand side by requiring that $\emptyset \leq_{\mathfrak{K}} M$ for all $M \in \mathfrak{K}$. This is useful when looking at universal models.

For $M_0 \in \mathfrak{K} \cup \{\emptyset\}$ we say that M is *universal over* M_0 if $M \leq_{\mathfrak{K}} N$ and for any $N \in \mathfrak{K}$ with $M_0 \leq_{\mathfrak{K}} N$, if $\|N\| \leq \|M_0\| + \text{LS}(\mathfrak{K})$, there exists $f : N \xrightarrow[M_0]{} M$. We say that M is (λ, δ)-*brimmed over* M_0 (often also called (λ, δ)-limit e.g. in [SV99, GVV16]) if $\delta < \lambda^+$ is a limit ordinal, $M_0 = \emptyset$ or $M_0 \in \mathfrak{K}_\lambda$, and there exists an increasing continuous chain $\langle N_i : i \leq \delta \rangle$ of members of $\mathfrak{K}_\lambda$ such that N_0 is universal over M_0, $N_\delta = M$, and N_{i+1} is universal over N_i for all $i < \delta$. We say that M is *brimmed over* M_0 if it is $(\|M\|, \delta)$-brimmed over M_0 for some limit $\delta < \|M\|^+$. We say that M is *brimmed* if it is brimmed over $\emptyset$.

The following notion of types already appears in [She87b]. It is called Galois types by many, but we prefer the term *orbital types* here. They are the same types that are defined in [She09a, II.1.9], except we also define them over sets. As pointed out in [Vas16c, Section 2], this causes no additional difficulties.

Definition 2.1. Fix an AEC $\mathfrak{K}$.

(1) We say $(A, N_1, \bar{b}_1) E_{\text{at}} (A, N_2, \bar{b}_2)$ if:
  (a) For $\ell = 1, 2$, $N_\ell \in \mathfrak{K}$, $A \subseteq |N_\ell|$, and $\bar{b}_\ell \in {}^{<\infty}|N_\ell|$.
  (b) There exists $N \in \mathfrak{K}$ and $f_\ell : N_\ell \xrightarrow[A]{} N$, $\ell = 1, 2$, such that $f_1(\bar{b}_1) = \bar{b}_2$.
(2) E_{at} is a reflexive and symmetric relation. Let E be its transitive closure.
(3) Let $\mathbf{ortp}(\bar{b}, A, N)$ be the E-equivalence class of $(\bar{b}, A, N)$.

(4) Define $\mathscr{S}(A, N)$, $\mathscr{S}(M)$, $\mathscr{S}^{<\omega}(M)$, etc. as expected. See for example [Vas16c, Section 2].

Let us say that an AEC $\mathfrak{K}$ is *stable in* λ if for any $M \in \mathfrak{K}_\lambda$, $|\mathscr{S}(M)| \leq \lambda$. This makes sense in any AEC, and is quite well-behaved assuming amalgamation and no maximal models (since then it is known that one can build universal extensions). We will often work in the following axiomatic setup, a slight weakening where full amalgamation is not assumed. This comes from the context derived in [SV99]:

Definition 2.2. Let $\mathfrak{K}$ be an AEC and let λ be a cardinal. We say that $\mathfrak{K}$ is *nicely stable in* λ (or *nicely* λ*-stable*) if:

(1) $\text{LS}(\mathfrak{K}) \leq \lambda$.
(2) $\mathfrak{K}_\lambda \neq \emptyset$.
(3) $\mathfrak{K}$ has joint embedding in λ.
(4) Density of amalgamation bases: For any $M \in \mathfrak{K}_\lambda$, there exists $N \in \mathfrak{K}_\lambda$ such that $M \leq_{\mathfrak{K}} N$ and N is an amalgamation base (in $\mathfrak{K}_\lambda$).
(5) Existence of universal extensions: For any amalgamation base $M \in \mathfrak{K}_\lambda$, there exists an amalgamation base $N \in \mathfrak{K}_\lambda$ such that $M <_{\mathfrak{K}} N$ and N is universal over M.
(6) Any brimmed model in $\mathfrak{K}_\lambda$ is an amalgamation base.

We say that $\mathfrak{K}$ is *very nicely stable in* λ if in addition it has amalgamation in λ.

Remark 2.3. An AEC $\mathfrak{K}$ is very nicely stable in λ if and only if $\text{LS}(\mathfrak{K}) \leq \lambda$, $\mathfrak{K}_\lambda \neq \emptyset$, $\mathfrak{K}$ is stable in λ, and $\mathfrak{K}_\lambda$ has amalgamation, joint embedding, and no maximal models.

We will make use of good frames for types of finite length (not just length one). Their definition is just like for types of length one, see [BVb, 3.8]. We call them *good* $(<\omega, \lambda)$*-frames*. Note that any good λ-frame (i.e. for types of length one) extends to a good $(<\omega, \lambda)$-frame (using independent sequences, see [She09a, III.9.4]) or [BVb, 5.8].

Given a good $(<\omega, \lambda)$-frame $\mathfrak{s}$, we write $\mathscr{S}^{\text{bs}}_{\mathfrak{s}}(M)$ for the basic types over M and $\mathfrak{K}_{\mathfrak{s}}$ for the underlying class of the frames (so for some essentially unique AEC $\mathfrak{K}$, $\mathfrak{K}_{\mathfrak{s}} = \mathfrak{K}_\lambda$). We write $M \leq_{\mathfrak{s}} N$ to mean that $M, N \in \mathfrak{K}_{\mathfrak{s}}$ (so in particular both M and N have cardinality λ) and $M \leq_{\mathfrak{K}} N$.

3. Weak nonforking amalgamation

In this section, we work in a good λ-frame and study a natural weak version of nonforking amalgamation, $\mathrm{LWNF}_{\mathfrak{s}}$. The main results are the existence property (Theorem 3.11) and how the symmetry property of $\mathrm{LWNF}_{\mathfrak{s}}$ is connected to $\mathfrak{s}$ being good$^+$ (Theorem 3.15). All throughout, we assume:

Hypothesis 3.1.

(1) $\mathfrak{s}$ is a good $(< \omega, \lambda)$-frame, except that it may not satisfy the symmetry axiom.
(2) $\mathfrak{K}_{\mathfrak{s}}$ is categorical in λ. Write $\mathfrak{K}$ for the AEC generated by $\mathfrak{K}_{\mathfrak{s}}$.

Remark 3.2. In this section, λ is allowed to be uncountable.

The reason for not assuming symmetry is that we will use some of the results of this section to *prove* that the symmetry axiom holds of a certain nonforking relation in Section 4.

We will use:

Fact 3.3 (II.4.3 in [She09a]). Let $\delta < \lambda^+$ be a limit ordinal divisible by λ. Let $\langle M_i : i \leq \delta \rangle$ be increasing continuous in $\mathfrak{K}_{\mathfrak{s}}$. If for any $i < \delta$ and any $p \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M_i)$, there exists λ-many $j \in [i, \delta)$ such that the nonforking extension of p to M_j is realized in M_{j+1}, then M_δ is brimmed over M_0.

Definition 3.4. Define the following 4-ary relations on $\mathfrak{K}_{\mathfrak{s}}$:

(1) $\mathrm{LWNF}_{\mathfrak{s}}(M_0, M_1, M_2, M_3)$ if and only if $M_0 \leq_{\mathfrak{s}} M_\ell \leq_{\mathfrak{s}} M_3$ for $\ell = 1, 2$ and for any $\bar{b} \in {}^{<\omega}|M_1|$, if $\mathbf{ortp}(\bar{b}, M_2, M_3)$ and $\mathbf{ortp}(\bar{b}, M_0, M_3)$ are basic then $\mathbf{ortp}(\bar{b}, M_2, M_3)$ does not fork over M_0.
(2) $\mathrm{RWNF}_{\mathfrak{s}}(M_0, M_1, M_2, M_3)$ if and only if $\mathrm{LWNF}_{\mathfrak{s}}(M_0, M_2, M_1, M_3)$.
(3) $\mathrm{WNF}_{\mathfrak{s}}(M_0, M_1, M_2, M_3)$ if and only if both $\mathrm{LWNF}_{\mathfrak{s}}(M_0, M_1, M_2, M_3)$ and $\mathrm{RWNF}_{\mathfrak{s}}(M_0, M_1, M_2, M_3)$.

When $\mathfrak{s}$ is clear from context, we write LWNF, RWNF, and WNF.

Remark 3.5. WNF stands for weak nonforking amalgamation, and LWNF, RWNF stand for left (respectively right) weak nonforking amalgamation.

The following result often comes in handy.

Lemma 3.6. Let $\delta < \lambda^+$ be a limit ordinal. Let $\langle M_i : i \leq \delta \rangle$, $\langle N_i : i \leq \delta \rangle$ be increasing continuous in $\mathfrak{K}_{\mathfrak{s}}$. Assume that for each $i \leq j < \delta$, we have that $\mathrm{LWNF}(M_i, N_i, M_j, N_j)$. If for each $i < \delta$, N_i realizes all the basic types over M_i, then N_δ realizes all the basic types over M_δ.

Proof. Let $p \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M_\delta)$. By local character, there exists $i < \delta$ such that p does not fork over M_i. By assumption, there exists $a \in |N_i|$ such that $p \upharpoonright M_i = \mathbf{ortp}(a, M_i, N_i)$. Because for all $j \in [i, \delta)$, $\mathrm{LWNF}(M_i, N_i, M_j, N_j)$, we have by continuity that $\mathbf{ortp}(a, M_\delta, N_\delta)$ does not fork over M_i, hence by uniqueness it must be equal to p. Therefore a realizes p, as needed. $\square$

We will see that there seems to be a clear difference between LWNF and RWNF. The following ordering is defined similarly to $\leq^*_{\lambda^+}$ from [She09a, II.7.2]:

Definition 3.7. For $R \in \{\mathrm{LWNF}, \mathrm{RWNF}, \mathrm{WNF}\}$, define a relation $\leq_R$ on $\mathfrak{K}_{\lambda^+}$ as follows. For $M^0, M^1 \in \mathfrak{K}_{\lambda^+}$, $M^0 \leq_R M^1$ if and only if there exists increasing continuous resolutions $\langle M_i^\ell \in \mathfrak{K}_\lambda : i < \lambda^+ \rangle$ of M^ℓ for $\ell = 0, 1$ such that for all $i < j < \lambda^+$, $R(M_i^0, M_i^1, M_j^0, M_j^1)$.

The following is a straightforward "catching your tail argument", see the proof of [Vas17, 4.6].

Fact 3.8. Let $M, N \in \mathfrak{K}_{\lambda^+}$. If $M \leq_{\mathfrak{K}} N$, then $M \leq_{\mathrm{LWNF}} N$.

Whether $M \leq_{\mathrm{RWNF}} N$ can be concluded as well seems to be a much more complicated question, and in fact is equivalent to $\mathfrak{s}$ being good$^+$ (Theorem 3.15). Observe that an increasing union of a $\leq_{\mathrm{RWNF}}$-increasing chain of saturated models is saturated:

Lemma 3.9. Let $\delta < \lambda^{++}$ be a limit ordinal. If $\langle M_i : i < \delta \rangle$ is a $\leq_{\mathrm{RWNF}}$-increasing sequence of saturated models in $\mathfrak{K}_{\lambda^+}$, then $\bigcup_{i<\delta} M_i$ is saturated.

Proof. Without loss of generality, $\delta = \mathrm{cf}(\delta) < \lambda^+$. Let $M_\delta := \bigcup_{i<\delta} M_i$. We build $\langle M_{i,j} : i \leq \delta, j \leq \lambda^+ \rangle$ such that:

(1) For any $i \leq \delta$, $M_{i,\lambda^+} = M_i$.
(2) For any $i < \delta, j < \lambda^+$, $M_{i,j} \in \mathfrak{K}_{\mathfrak{s}}$.
(3) For any $i \leq \delta$, $\langle M_{i,j} : j < \lambda^+ \rangle$ is increasing and continuous.
(4) For any $j \leq \lambda^+$, $\langle M_{i,j} : i < \delta \rangle$ is increasing and $M_{\delta,j} = \bigcup_{i<\delta} M_{i,j}$.
(5) For any $i_1 < i_2 \leq \delta$, $j_1 < j_2 \leq \lambda^+$, M_{i_2,j_2} realizes all the types in $\mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M_{i_1,j_1})$.

This is easy to do. Now for each $i_1 < i_2 < \delta$, we have by assumption that $M_{i_1} \leq_{\text{RWNF}} M_{i_2}$. Thus the set C_{i_1,i_2} of $j < \lambda^+$ such that for all $j' \in [j, \lambda^+)$, $\text{RWNF}(M_{i_1,j}, M_{i_2,j}, M_{i_1,j'}, M_{i_2,j'})$ is a club. Therefore $C := \bigcap_{i_1 < i_2 < \delta} C_{i_1,i_2}$ is also a club. Hence by renaming without loss of generality for all $i_1 < i_2 < \delta$ and all $j \leq j' < \lambda^+$, $\text{RWNF}(M_{i_1,j}, M_{i_2,j}, M_{i_1,j'}, M_{i_2,j'})$.

Now let $N \leq_{\mathfrak{K}} M_\delta$ be such that $N \in \mathfrak{K}_\lambda$. We want to see that any type over N is realized in M_δ. By Fact 3.3, it is enough to show that any *basic* type over N is realized in M_δ.

Let $j < \lambda^+$ be big-enough such that $N \leq_{\mathfrak{K}} M_{\delta,j}$. It is enough to see that any basic type over $M_{\delta,j}$ is realized in $M_{\delta,j+1}$. To see this, use Lemma 3.6 with $\langle M_i : i \leq \delta \rangle$, $\langle N_i : i \leq \delta \rangle$ there standing for $\langle M_{i,j} : i \leq \delta \rangle$, $\langle M_{i,j+1} : i \leq \delta \rangle$ here. We know that for each $i \leq i' < \delta$, $\text{RWNF}(M_{i,j}, M_{i',j}, M_{i,j+1}, M_{i',j+1})$ and therefore $\text{LWNF}(M_{i,j}, M_{i,j+1}, M_{i',j}, M_{i',j+1})$. Thus the hypotheses of Lemma 3.6 are satisfied. $\square$

The proof of the following fact is a direct limit argument similar to e.g. [GVV16, 5.3]. Note that the symmetry axiom is *not* needed.

Fact 3.10. Let $\alpha < \lambda^+$. Let $\langle M_i : i \leq \alpha \rangle$ be $\leq_{\mathfrak{s}}$-increasing continuous and let $\langle \bar{a}_i : i < \alpha \rangle$ be given such that $\bar{a}_i \in M_{i+1}$ for all $i < \alpha$ and $\mathbf{ortp}(\bar{a}_i, M_i, M_{i+1}) \in \mathscr{S}_{\mathfrak{s}}^{\text{bs}}(M_i)$.

There exists $\langle N_i : i \leq \alpha \rangle$ $\leq_{\mathfrak{s}}$-increasing continuous such that:

(1) $M_i <_{\mathfrak{s}} N_i$ for all $i \leq \alpha$.
(2) $\mathbf{ortp}(\bar{a}_i, N_i, N_{i+1})$ does not fork over M_i.

We are now ready to list some basic properties of weak nonforking amalgamation.

Theorem 3.11. Let $R \in \{\text{LWNF}, \text{RWNF}, \text{WNF}\}$.

(1) Invariance: If $R(M_0, M_1, M_2, M_3)$ and $f : M_3 \cong M_3'$, then $R(f[M_0], f[M_1], f[M_2], M_3')$.
(2) Monotonicity: If $R(M_0, M_1, M_2, M_3)$ and $M_0 \leq_{\mathfrak{s}} M_\ell' \leq_{\mathfrak{s}} M_\ell$ for $\ell = 1, 2$, then $R(M_0, M_1', M_2', M_3)$.
(3) Ambiant monotonicity: If $R(M_0, M_1, M_2, M_3)$ and $M_3 \leq_{\mathfrak{s}} M_3'$, then $R(M_0, M_1, M_2, M_3')$. If $M_3'' \leq_{\mathfrak{s}} M_3$ contains $|M_1| \cup |M_2|$, then $R(M_0, M_1, M_2, M_3'')$.
(4) Continuity: If $\delta < \lambda^+$ is a limit ordinal and $\langle M_i^\ell : i \leq \delta \rangle$ are increasing continuous for $\ell < 4$ with $R(M_i^0, M_i^1, M_i^2, M_i^3)$ for each $i < \delta$, then $R(M_\delta^0, M_\delta^1, M_\delta^2, M_\delta^3)$.

(5) Long transitivity: If $\alpha < \lambda^+$ is an ordinal, $\langle M_i : i \leq \alpha \rangle$, $\langle N_i : i \leq \alpha \rangle$ are increasing continuous and $\text{LWNF}(M_i, N_i, M_{i+1}, N_{i+1})$ for all $i < \alpha$, then $\text{LWNF}(M_0, N_0, M_\alpha, N_\alpha)$.
(6) Existence: If $R \neq \text{WNF}$, $M_0 \leq_{\mathfrak{s}} M_\ell$, $\ell = 1, 2$, then there exists $M_3 \in \mathfrak{K}_\lambda$ and $f_\ell : M_\ell \underset{M_0}{\longrightarrow} M_3$ such that $R(M_0, f_1[M_1], f_2[M_2], M_3)$.

Proof. Invariance and the monotonicity properties are straightforward to prove. Continuity and long transitivity follow directly from the local character, continuity, and transitivity properties of good frames. We prove existence via the following claim:

<u>Claim</u>: There exists $N_0, N_1, N_2, N_3 \in \mathfrak{K}_{\mathfrak{s}}$ such that $\text{LWNF}(N_0, N_1, N_2, N_3)$ and N_ℓ is brimmed over N_0 for $\ell = 1, 2$.

Existence easily follows from the claim: given $M_0 \leq_{\mathfrak{s}} M_\ell$, $\ell = 1, 2$, there is (by categoricity in λ) an isomorphism $f : M_0 \cong N_0$ and (by universality of brimmed models) embeddings $f_\ell : M_\ell \to N_\ell$ extending f for $\ell = 1, 2$. After some renaming, we obtain the desired LWNF-amalgam. To obtain an RWNF-amalgam, reverse the role of M_1 and M_2.

<u>Proof of Claim</u>: Let $\delta := \lambda \cdot \lambda$. We choose $(\bar{M}^\alpha, \bar{a}^\alpha)$ by induction on $\alpha \leq \delta$ such that:

(1) $\bar{M}^\alpha = \langle M_i^\alpha : i \leq \alpha \rangle$ is $\leq_{\mathfrak{s}}$-increasing continuous.
(2) $\bar{a}^\alpha = \langle \bar{a}_i : i < \alpha \rangle$, and $\bar{a}_i \in M_{i+1}^\alpha$ for all $i < \alpha$.
(3) For all $i < \alpha$, $\mathbf{ortp}(\bar{a}_i^\alpha, M_i^\alpha, M_{i+1}^\alpha) \in \mathscr{S}_{\mathfrak{s}}^{\text{bs}}(M_i^\alpha)$.
(4) For each $i \leq \delta$, $\langle M_i^\alpha : \alpha \in [i, \delta] \rangle$ is $<_{\mathfrak{s}}$-increasing continuous.
(5) For each $i < \delta$ and each $\alpha \in (i, \delta]$, $\mathbf{ortp}(\bar{a}_i, M_i^\alpha, M_{i+1}^\alpha)$ does not fork over M_i^i.
(6) If $p \in \mathscr{S}_{\mathfrak{s}}^{\text{bs}}(M_i^\alpha)$ for $i \leq \alpha < \delta$, then for λ-many $\beta \in [\alpha, \delta)$, $\mathbf{ortp}(\bar{a}_\beta, M_\beta^{\beta+1}, M_{\beta+1}^{\beta+1})$ is a nonforking extension of p.
(7) If $i < \alpha < \delta$ and $\mathbf{ortp}(\bar{a}, M_0^\alpha, M_{i+1}^\alpha) \in \mathscr{S}_{\mathfrak{s}}^{\text{bs}}(M_0^\alpha)$, then for some $\beta \in (\alpha, \delta)$ exactly one of the following occurs:
   (a) $\mathbf{ortp}(\bar{a}, M_0^{\beta+1}, M_{i+1}^{\beta+1})$ forks over M_0^α.
   (b) There is no $\langle M_j^* : j \leq i+1 \rangle$ $\leq_{\mathfrak{s}}$-increasing continuous such that:
      (i) $M_j^\beta \leq_{\mathfrak{s}} M_j^*$ for all $j \leq i+1$.
      (ii) $\mathbf{ortp}(\bar{a}_j, M_j^*, M_{j+1}^*)$ does not fork over M_j^β for all $j < i+1$.
      (iii) $\mathbf{ortp}(\bar{a}, M_0^*, M_{i+1}^*)$ forks over M_0^β.

This is possible: Along the construction, we also build an enumeration $\langle(\bar{b}_j^\gamma, k_j^\gamma, i_j^\gamma, \alpha_j^\gamma) : j < \lambda, \gamma < \lambda\rangle$ such that for any $\gamma \in (0, \lambda)$, any $\alpha < \lambda \cdot \gamma$, any $i < \alpha$, any $k \leq i$, and any $\bar{a} \in {}^{<\omega}M_{i+1}^\alpha$, if $\mathbf{ortp}(\bar{a}, M_k^\alpha, M_{i+1}^\alpha) \in \mathscr{S}_{\mathfrak{s}}^{\mathrm{bs}}(M_k^\alpha)$, then there exists $j < \lambda$ so that $\bar{b}_j^\gamma = \bar{a}$, $i_j^\gamma = i$, $k_j^\gamma = k$, and $\alpha_j^\gamma = \alpha$. We require that always $k_j^\gamma \leq i_j^\gamma < \alpha_j^\gamma < \lambda \cdot \gamma$ and the triple $(\bar{b}_j^\gamma, M_{k_j^\gamma}^{\alpha_j^\gamma}, M_{i_j^\gamma+1}^{\alpha_j^\gamma})$ represents a basic type. We make sure that at stage $\lambda \cdot (\gamma + 1)$ of the construction below, $\bar{b}_j^{\gamma'}, k_j^{\gamma'}, i_j^{\gamma'}, \alpha_j^{\gamma'}$ are defined for all $j < \lambda$, $\gamma' \leq \gamma$.

For $\alpha = 0$, take any $M_0^0 \in \mathfrak{K}_{\mathfrak{s}}$. For α limit, let $M_i^\alpha := \bigcup_{\beta \in [i, \alpha)} M_i^\beta$ for $i < \alpha$ and $M_\alpha^\alpha := \bigcup_{i<\alpha} M_i^\alpha$. Now assume that $\bar{M}^\alpha$, $\bar{a}^\alpha$ have been defined for $\alpha < \delta$. We define $\bar{M}^{\alpha+1}$ and $\bar{a}_\alpha$. Fix ρ and $j < \lambda$ such that $\alpha = \lambda \cdot \rho + j$. We consider two cases.

- Case 1: ρ is zero or a limit: Use Fact 3.10 to get $\langle M_i^{\alpha+1} : i \leq \alpha\rangle$ $<_{\mathfrak{s}}$-increasing continuous such that $M_i^\alpha <_{\mathfrak{s}} M_i^{\alpha+1}$ for all $i \leq \alpha$, and for all $i < \alpha$, $\mathbf{ortp}(\bar{a}_i, M_i^{\alpha+1}, M_{i+1}^{\alpha+1})$ does not fork over M_i^α. Pick any $M_{\alpha+1}^{\alpha+1}$ with $M_\alpha^{\alpha+1} <_{\mathfrak{s}} M_{\alpha+1}^{\alpha+1}$ and any $\bar{a}_\alpha \in {}^{<\omega}M_{\alpha+1}^{\alpha+1}$ such that $\mathbf{ortp}(\bar{a}_\alpha, M_\alpha^{\alpha+1}, M_{\alpha+1}^{\alpha+1}) \in \mathscr{S}_{\mathfrak{s}}^{\mathrm{bs}}(M_\alpha^{\alpha+1})$.
- Case 2: ρ is a successor: Say $\rho = \gamma + 1$. Let $\bar{a} := \bar{b}_j^\gamma$, $\alpha_0 := \alpha_j^\gamma$, $k_0 := k_j^\gamma$, $i_0 := i_j^\gamma$. There are two subcases. It is possible that $k_0 \neq 0$ or $k_0 = 0$ and (7b) holds with i, α, β there standing for i_0, α_0, α here. In this case, we proceed as in Case 1 to define $\langle M_i^{\alpha+1} : i \leq \alpha\rangle$. Then we pick $\bar{a}_\alpha$, $M_{\alpha+1}^{\alpha+1}$ such that $\mathbf{ortp}(\bar{a}_\alpha, M_\alpha^\alpha, M_{\alpha+1}^{\alpha+1})$ is the nonforking extension of $\mathbf{ortp}(\bar{a}, M_{i_0}^{\alpha_0}, M_{i_0+1}^{\alpha_0})$.
  On the other hand, it is possible that $k_0 = 0$ and (7b) fails. In this case let $\langle M_j^* : j \leq i_0 + 1\rangle$ witness the failure and set $M_j^{\alpha+1} := M_j^*$ for $j \leq i_0 + 1$. Then continue as in Case 1 and define $\bar{a}_\alpha$, $M_{\alpha+1}^{\alpha+1}$ as before.

This is enough: We choose $\bar{M}^* = \langle M_i^* : i \leq \delta\rangle$ increasing continuous such that M_0^* is brimmed over M_0^δ, $M_i^\delta \leq_{\mathfrak{s}} M_i^*$ for all $i \leq \delta$, and $\mathbf{ortp}(\bar{a}_i, M_i^*, M_{i+1}^*)$ does not fork over M_i^δ. This is possible, see case 1 above. Now:

- M_0^* is brimmed over M_0^δ.
  [Why? By construction].
- If $p \in \mathscr{S}_{\mathfrak{s}}^{\mathrm{bs}}(M_i^\delta)$ for $i < \delta$, then for λ-many $\beta \in [i, \delta)$, $\mathbf{ortp}(\bar{a}_\beta, M_\beta^\delta, M_{\beta+1}^\delta)$ is a nonforking extension of p.

[Why? Pick $i' \in (i, \delta)$ such that p does not fork over $M_i^{i'}$. By (6), we know that for λ-many $\beta \in [i', \delta)$, the nonforking extension of $p \restriction M_i^{i'}$ to $M_\beta^{\beta+1}$ is realized in $M_{\beta+1}^{\beta+1}$ by $\bar{a}_\beta$. But by (5) we also have that $\mathbf{ortp}(\bar{a}_\beta, M_\beta^\delta, M_{\beta+1}^\delta)$ does not fork over M_β^β. In particular by uniqueness $\bar{a}_\beta$ also realizes p.]

- M_δ^δ is brimmed over M_0^δ.
  [Why? We apply Fact 3.3 to the chain $\langle M_i^\delta : i \leq \delta \rangle$, using the previous step.].
- $\mathrm{LWNF}(M_0^\delta, M_\delta^\delta, M_0^*, M_\delta^*)$.
  [Why? Pick $\bar{a} \in {}^{<\omega}M_\delta^\delta$ such that $\mathbf{ortp}(\bar{a}, M_0^\delta, M_\delta^\delta)$ is basic. By local character, there exists $\alpha < \delta$ such that $\mathbf{ortp}(\bar{a}, M_0^\delta, M_\delta^\delta)$ does not fork over M_0^α. Further, we can increase α if necessary and pick $i < \alpha$ such that $\bar{a} \in {}^{<\omega}M_{i+1}^\alpha$. We now apply Clause (7). We know that (7a) fails for all $\beta \in (\alpha, \delta)$ by the choice of α, therefore (7b) must hold for all $\beta \in (\alpha, \delta)$. Now if $\mathbf{ortp}(\bar{a}, M_0^*, M_\delta^*)$ forks over M_0^δ, then it must fork over M_0^β for all high-enough β, but then $\langle M_j^* : j \leq i+1 \rangle$ would contradict Clause (7b). Therefore $\mathbf{ortp}(\bar{a}, M_0^*, M_\delta^*)$ does not fork over M_0^δ, as desired.]

Therefore we can take $(M_0, M_1, M_2, M_3) := (M_0^\delta, M_\delta^\delta, M_0^*, M_\delta^*)$.

$\dagger_{\text{Claim}}$

$\square$

Definition 3.12. Let $R \in \{\mathrm{LWNF}, \mathrm{RWNF}, \mathrm{WNF}\}$.

(1) We say that R has the *symmetry property* if $R(M_0, M_1, M_2, M_3)$ implies $R(M_0, M_2, M_1, M_3)$.
(2) We say that R has the *uniqueness property* if whenever $R(M_0, M_1, M_2, M_3)$ and $R(M_0, M_1, M_2, M_3')$, there exists M_3'' with $M_3' \leq_{\mathfrak{s}} M_3''$ and $f : M_3 \xrightarrow[|M_1| \cup |M_2|]{} M_3''$.

The following are trivial observations about the definitions:

Remark 3.13.

(1) WNF has the symmetry property, and LWNF has the symmetry property if and only if RWNF has the symmetry property if and only if LWNF = RWNF = WNF.
(2) LWNF has the uniqueness property if and only RWNF has it.

Recall from [She09a, III.1.3]:

Definition 3.14. $\mathfrak{s}$ is *good*$^+$ when the following is *impossible*:

There exists an increasing continuous $\langle M_i : i < \lambda^+ \rangle$, $\langle N_i : i < \lambda^+ \rangle$, a basic type $p \in \mathscr{S}^{\text{bs}}_{\mathfrak{s}}(M_0)$, and $\langle \bar{a}_i : i < \lambda^+ \rangle$ such that for any $i < \lambda^+$:

(1) $M_i \leq_{\mathfrak{s}} N_i$.
(2) $\bar{a}_{i+1} \in |M_{i+2}|$ and $\mathbf{ortp}(\bar{a}_{i+1}, M_{i+1}, M_{i+2})$ is a nonforking extension of p, but $\mathbf{ortp}(\bar{a}_{i+1}, N_0, N_{i+2})$ forks over M_0.
(3) $\bigcup_{j<\lambda^+} M_j$ is saturated.

Theorem 3.15. (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) $\Rightarrow$ (4), where:

(1) LWNF has the symmetry property.
(2) $\mathfrak{s}$ is good$^+$.
(3) For $M, N \in \mathfrak{K}_{\lambda^+}$ both saturated, $M \leq_{\mathfrak{K}} N$ implies $M \leq_{\text{WNF}} N$.
(4) There is a superlimit model in $\mathfrak{K}_{\lambda^+}$.

Proof.

- <u>(3) implies (4)</u>: This follows from Lemma 3.9.
- <u>$\neg$(2) implies $\neg$(3)</u>: Fix a witness $\langle M_i : i < \lambda^+ \rangle$, $\langle N_i : i < \lambda^+ \rangle$, $\langle \bar{a}_i : i < \lambda^+ \rangle$, p to the failure of being good$^+$. Write $M_{\lambda^+} := \bigcup_{i<\lambda^+} M_i$, $N_{\lambda^+} := \bigcup_{i<\lambda^+} N_i$. By assumption, M_{λ^+} is saturated. Clearly, increasing the N_i's will not change that we have a witness so without loss of generality N_{λ^+} is also saturated. We claim that $M_{\lambda^+} \not\leq_{\text{RWNF}} N_{\lambda^+}$. We show this by proving that for any $i < \lambda^+$ and any $j \leq i + 1$, $\neg\,\text{RWNF}(M_j, N_j, M_{i+2}, N_{i+2})$. Indeed, $\mathbf{ortp}(\bar{a}_{i+1}, N_j, N_{i+2})$ forks over M_j: if not, then by transitivity $\mathbf{ortp}(\bar{a}_{i+1}, N_j, N_{i+2})$ does not fork over M_0, and hence $\mathbf{ortp}(\bar{a}_{i+1}, N_0, N_{i+2})$ does not fork over M_0, and we know that this is not the case of the witness we selected.
- <u>$\neg$(3) implies $\neg$(2)</u>: Fix M, N saturated in $\mathfrak{K}_{\lambda^+}$ such that $M \leq_{\mathfrak{K}} N$ but $M \not\leq_{\text{RWNF}} N$.

  <u>Claim</u>: For any $A \subseteq |M|$ of size λ, there exists $M_0 \leq_{\mathfrak{s}} M_1 \leq_{\mathfrak{K}} M$ and $N_0 \leq_{\mathfrak{s}} N_1 \leq_{\mathfrak{K}} N$ such that $M_0 \leq_{\mathfrak{s}} N_0$, $M_1 \leq_{\mathfrak{s}} N_1$, $A \subseteq |M_0|$, but $\neg\,\text{RWNF}(M_0, N_0, M_1, N_1)$.

  <u>Proof of Claim</u>: If not, we can use failure of the claim and continuity of RWNF to build increasing continuous resolution $\langle M_i : i \leq \lambda^+ \rangle$, $\langle N_i : i \leq \lambda^+ \rangle$ of M and N respectively such that $\text{RWNF}(M_i, N_i, M_j, N_j)$ for all $i < j < \lambda^+$. Thus $M \leq_{\text{RWNF}} N$, contradicting the assumption. $\dagger_{\text{Claim}}$

  Build $\langle M_i^* : i \leq \lambda^+ \rangle$, $\langle N_i^* : i \leq \lambda^+ \rangle$ increasing continuous resolutions of M, N respectively such that for all $i < \lambda^+$, $M_i^* \leq_{\mathfrak{s}} N_i^*$ and $\neg\,\text{RWNF}(M_{i+1}^*, N_{i+1}^*, M_{i+2}^*, N_{i+2}^*)$. This is possible by the claim. Let $\bar{a}^*_{i+1} \in |M^*_{i+2}|$ witness the RWNF-forking, i.e. $\mathbf{ortp}(\bar{a}^*_{i+1}, N^*_{i+1}, N^*_{i+2})$ forks over M^*_{i+1}. By Fodor's lemma, local character, and stability, there exists a stationary set S, $i_0 < \lambda^+$ and $p \in \mathscr{S}^{\text{bs}}_{\mathfrak{s}}(M^*_{i_0})$ such that for all $i \in S$, $\mathbf{ortp}(\bar{a}^*_{i+1}, M^*_i, M^*_{i+2})$ is the nonforking extension of p. Without loss of generality, i_0 is limit and all elements of S are also limit ordinals.

Now build an increasing continuous sequence of ordinals $\langle j_i : i < \lambda^+ \rangle$ as follows. Let $j_0 := i_0$. For i limit, let $j_i := \sup_{k<i} j_k$. For i successor, pick any $j_i \in S$ with $j_i > j_{i-1}$.

Now for i not the successor of a limit, let $M_i := M^*_{j_i}$, $N_i := N^*_{j_i}$, $\bar{a}_i := \bar{a}^*_{j_i}$. For $i = k + 1$ with k a limit, set $M_i := M^*_{j_k}$, $N_i := N^*_{j_k}$, $\bar{a}_i := \bar{a}^*_{j_i}$. This gives a witness to the failure of being good$^+$.

- (1) implies (3): If LWNF has the symmetry property, then by Remark 3.13, LWNF = RWNF = WNF. By Fact 3.8, it follows that $M \leq_{\mathfrak{K}} N$ implies $M \leq_{\text{WNF}} N$ for any $M, N \in \mathfrak{K}_{\lambda^+}$, so (3) holds.

□

Question 3.16. Are the conditions in Theorem 3.15 all equivalent?

Question 3.17. Is there a good λ-frame $\mathfrak{s}$ such that LWNF$_{\mathfrak{s}}$ does *not* have the symmetry property?

The next result shows that the uniqueness property has strong consequences. The first author has given conditions under which when $\lambda = \aleph_0$, failure of uniqueness implies nonstructure [She09b, VII.4.16].

Theorem 3.18. Assume that $\mathfrak{s}$ is a good $(< \omega, \lambda)$-frame (so it satisfies symmetry). If LWNF has the uniqueness property, then LWNF has the symmetry property and $\mathfrak{s}$ is successful good$^+$ (see [She09a, III.1.1]).

Proof. By [Vas17, 3.11] (used with the pre-$(\leq \lambda, \lambda)$-frame induced by LWNF, recalling Fact 3.8) $\mathfrak{s}$ is weakly successful. This implies that there is a relation NF = NF$_{\mathfrak{s}}$ that is a nonforking relation respecting $\mathfrak{s}$ (see [She09a, II.6.1], in particular it has all the properties listed in Theorem 3.11, as well as uniqueness and symmetry). Now as NF respects $\mathfrak{s}$, we must have that NF(M_0, M_1, M_2, M_3) implies LWNF(M_0, M_1, M_2, M_3). Since LWNF has the uniqueness property and NF has the existence property, it follows from [BGKV16, 4.1] that LWNF = NF. In particular, LWNF has the symmetry property.

To see that $\mathfrak{s}$ is successful good$^+$, it is enough to show that for $M, N \in \mathfrak{K}_{\lambda^+}$, $M \leq_{\mathfrak{K}} N$ implies $M \leq_{\text{NF}} N$ (where $\leq_{\text{NF}}$ is defined as in Definition 3.7). This is immediate from Fact 3.8 and LWNF = NF. $\square$

To prepare for the proof of symmetry in the $\lambda = \aleph_0$ case, we introduce yet another notion of nonforking amalgamation (VWNF stands for "very weak nonforking amalgamation").

Definition 3.19.

(1) For $M \leq_{\mathfrak{s}} N$, $B \subseteq |N|$, $\bar{a} \in {}^{<\omega}N$, we say that $\mathbf{ortp}(\bar{a}, B, N)$ *does not fork over* M if there exists M', N' with $N \leq_{\mathfrak{s}} N'$, $M \leq_{\mathfrak{s}} M' \leq_{\mathfrak{s}} N'$, and $B \subseteq |M'|$ such that $\mathbf{ortp}(\bar{a}, M', N')$ does not fork over M_0.
(2) We define a 4-ary relation $\text{VWNF}_{\mathfrak{s}} = \text{VWNF}$ on $\mathfrak{K}_{\mathfrak{s}}$ by $\text{VWNF}(M_0, M_1, M_2, M_3)$ if and only if $M_0 \leq_{\mathfrak{s}} M_\ell \leq_{\mathfrak{s}} M_3$, $\ell = 1, 2$ and for any $\bar{a} \in {}^{<\omega}M_1$ and any finite $B \subseteq |M_2|$, if $\mathbf{ortp}(\bar{a}, M_0, M_3)$ and $\mathbf{ortp}(\bar{a}, M_2, M_3)$ are both basic, then $\mathbf{ortp}(\bar{a}, B, M_3)$ does not fork over M_0.

Theorem 3.20. Assume that $\mathfrak{s}$ is a type-full good $(< \omega, \lambda)$-frame.

(1) VWNF has the symmetry property: $\text{VWNF}(M_0, M_1, M_2, M_3)$ if and only if $\text{VWNF}(M_0, M_2, M_1, M_3)$.
(2) If for any $M \in \mathfrak{K}_{\mathfrak{s}}$ and any $p \neq q \in \mathscr{S}^{\text{bs}}_{\mathfrak{s}}(M)$ there exists $B \subseteq |M|$ finite such that $p \upharpoonright B \neq q \upharpoonright B$, then VWNF = WNF. In particular, LWNF has the symmetry property.

Proof.

(1) By the symmetry axiom of good frames.
(2) This is observed in [Vas16a, 4.5]. In details, it suffices to show that for $M \leq_{\mathfrak{s}} N$, $p \in \mathscr{S}^{\text{bs}}_{\mathfrak{s}}(N)$ does not fork over M if and only if $p \upharpoonright B$ does not fork over M for all finite $B \subseteq |N|$. Let $q \in \mathscr{S}^{\text{bs}}_{\mathfrak{s}}(N)$ be the nonforking extension of $p \upharpoonright M$. For any finite $B \subseteq |N|$, we have that $q \upharpoonright B = p \upharpoonright B$, by the uniqueness property for (the extended notion of) forking, see [BGKV16, 5.4]. Therefore by the assumption we must have $p = q$, as desired.

$\square$

4. Building a good $\aleph_0$-frame

In this section, we work in $\aleph_0$ and aim to build a good $\aleph_0$-frame from stability and amalgamation.

Hypothesis 4.1. $\mathfrak{K} = (K, \leq_{\mathfrak{K}})$ is an AEC with $\mathrm{LS}(\mathfrak{K}) = \aleph_0$ (and countable vocabulary).

First note that if $\mathfrak{K}$ is stable and has few models, we can say something about its definability:

Theorem 4.2. Assume that $\mathbb{I}(\mathfrak{K}, \aleph_0) \leq \aleph_0$.

(1) The set $\{M \in K_{\aleph_0} : |M| \subseteq \omega\}$ is Borel.
(2) If $\mathfrak{K}$ has amalgamation in $\aleph_0$ and is stable in $\aleph_0$, then the set $\{(M, N) : M \leq_{\mathfrak{K}} N \text{ and } |N| \subseteq \omega\}$ is $\mathbf{\Sigma^1_1}$.

In particular if $\mathfrak{K}$ has amalgamation in $\aleph_0$ and is stable in $\aleph_0$, then $\mathfrak{K}$ is a $\mathrm{PC}_{\aleph_0}$-representable AEC.

Proof.

(1) Fix $M \in \mathfrak{K}_{\aleph_0}$. By Scott's isomorphism theorem, there exists a formula ϕ_M of $\mathbb{L}_{\aleph_1,\aleph_0}(\tau_{\mathfrak{K}})$ such that $N \models \phi_M$ if and only if $M \cong N$. Now observe that the set

$$\{N : N \text{ is a } \tau_{\mathfrak{K}}\text{-structure with } |N| \subseteq \omega \text{ and } N \models \phi_M\}$$

is Borel and use that $\mathbb{I}(\mathfrak{K}, \aleph_0) \leq \aleph_0$.

(2) For $M, N \in \mathfrak{K}_{\aleph_0}$ with $M \leq_{\mathfrak{K}} N$, let us say that N is *almost brimmed over* M if either N is brimmed over M, or N is $\leq_{\mathfrak{K}}$-maximal. Using amalgamation, it is easy to check that if N, N' are both almost brimmed over M, then $N \cong_M N'$. Moreover there always exists an almost brimmed model over any $M \in \mathfrak{K}_{\aleph_0}$.

Fix $\langle M^*_n : n < \omega \rangle$ such that for any $M \in \mathfrak{K}_{\aleph_0}$ there exists $n < \omega$ such that $M \cong M^*_n$ (possible as $\mathbb{I}(\mathfrak{K}, \aleph_0) \leq \aleph_0$). For each $n < \omega$, fix $N^*_n \in \mathfrak{K}_{\aleph_0}$ almost brimmed over M^*_n. We have:

$\circledast_1$ For $M, N \in \mathfrak{K}_{\aleph_0}$:
  (a) There is $n < \omega$ and an isomorphism $f : M^*_n \cong M$.
  (b) If N is almost brimmed over M, then any such f extends to $g : N^*_n \cong N$.

$\circledast_2$ For $M_1, M_2 \in \mathfrak{K}_{\aleph_0}$, $M_1 \leq_{\mathfrak{K}} M_2$ if and only if $M_1 \subseteq M_2$ and for some $n < \omega$, for some (N, f_1, f_2) we have: $M_1 \subseteq M_2 \subseteq N$ and f_ℓ is an isomorphism from (M^*_n, N^*_n) onto (M_ℓ, N).

[Why? The implication "if" holds by the coherence axiom of AECs. The implication "only if" holds as there is $N \in \mathfrak{K}_{\aleph_0}$ which is almost brimmed over M_2 (and so $M_2 \leq_{\mathfrak{K}} N$) hence N is almost brimmed over M_1 and use $\circledast_1$ above.]

The result now follows from $\circledast_2$.

By [BL16, 3.3], it follows that $\mathfrak{K}$ is $\mathrm{PC}_{\aleph_0}$. $\square$

The following appears already in [She09a, I.4.3]:

Definition 4.3. Let $\phi(\bar{x})$ be a formula in $\mathbb{L}_{\infty,\aleph_0}(\tau_{\mathfrak{K}})$ and let $M \in \mathfrak{K}_{\aleph_0}$, $\bar{a} \in {}^{<\omega}M$. We define $M \Vdash_{\mathfrak{K}} \phi[\bar{a}]$ (we will just write $M \Vdash \phi[\bar{a}]$ as $\mathfrak{K}$ is fixed) by induction on ϕ as follows:

- If ϕ is atomic, $M \Vdash \phi[\bar{a}]$ if and only if $M \models \phi[\bar{a}]$.
- If $\phi(\bar{x}) = \wedge_{i<\alpha}\phi_i[\bar{x}]$, then $M \Vdash \phi[\bar{a}]$ if and only if $M \Vdash \phi_i[\bar{a}]$ for all $i < \alpha$.
- If $\phi(\bar{x}) = \exists \bar{y}\psi(\bar{y}, \bar{x})$, then $M \Vdash \phi[\bar{a}]$ if and only if for every $N \in \mathfrak{K}_{\aleph_0}$ with $M \leq_{\mathfrak{K}} N$, there exists $N' \in \mathfrak{K}_{\aleph_0}$ with $N \leq_{\mathfrak{K}} N'$ and $\bar{b} \in {}^{<\omega}N'$ such that $N' \Vdash \psi[\bar{b}, \bar{a}]$.
- If $\phi(\bar{x}) = \neg\psi(\bar{x})$, then $M \Vdash \phi[\bar{a}]$ if and only if for every $N \in \mathfrak{K}_{\aleph_0}$ with $M \leq_{\mathfrak{K}} N$, $N \nVdash \psi[\bar{a}]$.
- If $\phi(\bar{x}) = \forall \bar{y}\psi(\bar{y}, \bar{x})$, then $M \Vdash \phi[\bar{a}]$ if and only if $M \Vdash \neg\exists\bar{y}\neg\psi(\bar{y}, \bar{a})$.
- If $\phi(\bar{x}) = \vee_{i<\alpha}\phi_i(\bar{x})$, then $M \Vdash \phi[\bar{a}]$ if and only if $M \Vdash \neg \wedge_{i<\alpha} \neg\phi_i[\bar{a}]$.

We now state some basic facts about forcing. In particular, forcing is very well-behaved on amalgamation bases.

Lemma 4.4. Let $M, N \in \mathfrak{K}_{\aleph_0}$ with $M \leq_{\mathfrak{K}} N$, $\bar{a} \in {}^{<\omega}M$, and $\phi(\bar{x})$ be an $\mathbb{L}_{\infty,\aleph_0}(\tau_{\mathfrak{K}})$-formula.

Then:

(1) If $M \Vdash \phi[\bar{a}]$, then $N \Vdash \phi[\bar{a}]$.
(2) If $M \Vdash \phi[\bar{a}]$, then $M \nVdash \neg\phi[\bar{a}]$. If $M \nVdash \neg\phi[\bar{a}]$, then there exists $N \in \mathfrak{K}_{\aleph_0}$ with $M \leq_{\mathfrak{K}} N$ such that $N \Vdash \phi[\bar{a}]$.
(3) $M \Vdash \phi[\bar{a}]$ if and only if for every $N \in \mathfrak{K}_{\aleph_0}$ with $M \leq_{\mathfrak{K}} N$ there exists $N' \in \mathfrak{K}_{\aleph_0}$ such that $N \leq_{\mathfrak{K}} N'$ and $N' \Vdash \phi[\bar{a}]$.
(4) If $M \nVdash \phi[\bar{a}]$, then there exists $N \in \mathfrak{K}_{\aleph_0}$ such that $M \leq_{\mathfrak{K}} N$ and $N \Vdash \neg\phi[\bar{a}]$.
(5) If M is an amalgamation base, then either $M \Vdash \phi[\bar{a}]$ or $M \Vdash \neg\phi[\bar{a}]$.
(6) If M is an amalgamation base, then $M \Vdash \phi[\bar{a}]$ if and only if $N \Vdash \phi[\bar{a}]$.
(7) If $M_0 \in \mathfrak{K}_{\aleph_0} \cup \{\emptyset\}$, M is brimmed over M_0, and $N \Vdash \psi[\bar{a}, \bar{b}]$ (with $\bar{b} \in N$), then there exists $\bar{b}' \in M$ such that $M \Vdash \psi[\bar{a}, \bar{b}']$.
(8) If M is a brimmed amalgamation base, then $M \Vdash \phi[\bar{a}]$ if and only if $M \models \phi[\bar{a}]$.

Proof.

(1) Straightforward induction on ϕ.
(2) By definition of $M \Vdash \neg\phi[\bar{a}]$.
(3) Straightforward induction on ϕ.
(4) By the previous part and definition of forcing a negation.
(5) If $M \not\Vdash \phi[\bar{a}]$ and $M \not\Vdash \neg\phi[\bar{a}]$, then by the previous parts there exists extensions $M_1, M_2 \in \mathfrak{K}_{\aleph_0}$ of M which force ϕ and $\neg\phi$ respectively. Use amalgamation to get a contradiction.
(6) We have already shown the left to right direction. For the right to left direction, suppose that $M \not\Vdash \phi[\bar{a}]$. Then by the previous part $M \Vdash \neg\phi[\bar{a}]$ so $N \Vdash \neg\phi[\bar{a}]$ so $N \not\Vdash \phi[\bar{a}]$, as desired.
(7) Since M is brimmed over M_0, there exists $M_1 \in \mathfrak{K}_{\aleph_0}$ such that $\bar{a} \in M_1$, $M_0 \leq_{\mathfrak{K}} M_1 \leq_{\mathfrak{K}} M$, and M is universal over M_1. Let $f : N \underset{M_1}{\longrightarrow} M$. Then $f[N] \Vdash \psi[f(\bar{b}), \bar{a}]$, so $M \Vdash \psi[f(\bar{b}), ba]$, so $\bar{b}' := f(\bar{b})$ is as desired.
(8) Straightforward induction on ϕ, using the previous part for the existential case.

$\square$

Definition 4.5. For $M \in \mathfrak{K}_{\aleph_0}$, $B \subseteq |M|$, and $\bar{a} \in {}^{<\omega}M$, let $\mathrm{gtp}(\bar{a}, B, M)$ (the generic type of $\bar{a}$ over B in M) be the following set:

$$\{\phi(\bar{x}, \bar{b}) \mid \phi(\bar{x}, \bar{y}) \in \mathbb{L}_{\aleph_1,\aleph_0}(\tau_{\mathfrak{K}}), \bar{b} \in {}^{<\omega}B, M \Vdash \phi[\bar{a}, \bar{b}]\}$$

Note that generic types are always rougher than orbital types. See Corollary 4.12 for a converse.

Lemma 4.6. Let $M_1, M_2 \in \mathfrak{K}_{\aleph_0}$ be amalgamation bases, $B \subseteq |M_1| \cap |M_2|$ and $\bar{a}_\ell \in {}^{<\omega}M_\ell$. If $\mathbf{ortp}(\bar{a}_1, B, M_1) = \mathbf{ortp}(\bar{a}_2, B, M_2)$, then $\mathrm{gtp}(\bar{a}_1, B, M_1) = \mathrm{gtp}(\bar{a}_2, B, M_2)$.

Proof. By the definition of orbital types and Lemma 4.4(6). $\square$

Assuming there is a universal extension over M_0, the set of generic types over M_0 will be the set of generic types realized in the universal extension. In particular, it will be countable:

Lemma 4.7. For any $M_0 \in \mathfrak{K}_{\aleph_0} \cup \{\emptyset\}$ and any $M \in \mathfrak{K}_{\aleph_0}$ universal over M_0, we have:

$$\{\mathrm{gtp}(\bar{a}, M_0, M) \mid \bar{a} \in {}^{<\omega}M\} = \{\mathrm{gtp}(\bar{a}, M_0, N) \mid N \in \mathfrak{K}_{\aleph_0}, M_0 \leq_{\mathfrak{K}} N, \bar{a} \in {}^{<\omega}N\}$$

(where by convention we set $\emptyset \leq_{\mathfrak{K}} N$ for every $N \in \mathfrak{K}$)

Proof. Use universality of M and Lemma 4.4(6). $\square$

The following technical lemma shows how to code a generic type inside a single formula.

Lemma 4.8. Assume that $\mathfrak{K}$ is nicely stable in $\aleph_0$ (recall Definition 2.2). Fix an amalgamation base $M_0 \in \mathfrak{K}_{\aleph_0} \cup \{\emptyset\}$. There exists a sequence $\langle \phi_m^{M_0} : m < \omega \rangle$ such that:

(1) For each $m < \omega$, $\phi_m^{M_0}$ is an $\mathbb{L}_{\aleph_1,\aleph_0}(\tau_{\mathfrak{K}})$-formula with parameters from M_0.
(2) For any $M \in \mathfrak{K}_{\aleph_0}$ extending M_0 and any $\bar{a} \in {}^{<\omega}M$, there is a unique $m = m(\bar{a}, M_0, M) < \omega$ such that $M \Vdash \phi_m^{M_0}[\bar{a}]$ and $M \Vdash \neg\phi_{m'}^{M_0}[\bar{a}]$ for all $m' \neq m$.
(3) For any $M \in \mathfrak{K}_{\aleph_0}$ extending M_0 and any $\bar{a}, \bar{b} \in {}^{<\omega}M$, $\operatorname{gtp}(\bar{a}, M_0, M) = \operatorname{gtp}(\bar{b}, M_0, M)$ if and only if $m(\bar{a}, M_0, M) = m(\bar{b}, M_0, M)$.

Proof. Say $\{p_i : i < \omega\} = \{\operatorname{gtp}(\bar{a}, M_0, N) : N \in \mathfrak{K}_{\aleph_0}, M_0 \leq_{\mathfrak{K}} N, \bar{a} \in {}^{<\omega}N\}$ (this set is countable by Lemma 4.7). For each $i \neq j$ in ω, there exists $\psi_{i,j}$ such that $\psi_{i,j} \in p_i$ and $\neg\psi_{i,j} \in p_j$. For $m < \omega$, set $\phi_m^{M_0} := \wedge_{m \neq j} \psi_{m,j}$. It is straightforward to see that this works. $\square$

We have all the tools available to study homogeneous models and show that they coincide with brimmed models.

Definition 4.9. Let D be a set of orbital types and let $M \in \mathfrak{K}$. We say that M is $(D, \aleph_0)$-*homogeneous* if whenever $p \in D$ is the type of an $(n+m)$-elements sequence and $\bar{a} \in {}^{n}M$ realizes p^n (the restriction of p to its first n "variables"), there exists a sequence $\bar{b} \in {}^{m}M$ such that $\bar{a}\bar{b}$ realizes p. When $D = \mathscr{S}^{<\omega}(\emptyset, M)$, we omit it.

Theorem 4.10. Assume that $\mathfrak{K}$ is nicely stable in $\aleph_0$. Let $M_0 \in \mathfrak{K}_{\aleph_0} \cup \{\emptyset\}$ be an amalgamation base, and let $M \in \mathfrak{K}_{\aleph_0}$ be such that $M_0 \leq_{\mathfrak{K}} M$. The following are equivalent:

(1) M is brimmed over M_0
(2) M is $(\mathscr{S}^{<\omega}(M_0), \aleph_0)$-homogeneous.

Proof. First we show:

Claim 1: If M is brimmed over M_0, then M is $\aleph_0$-homogeneous over M_0 in the sense of generic types. That is, if $\bar{a}_1, \bar{b}_1, \bar{a}_2 \in {}^{<\omega}M$ and $\operatorname{gtp}(\bar{a}_1, M_0, M) = \operatorname{gtp}(\bar{a}_2, M_0, M)$, then there exists $\bar{b}_2 \in {}^{<\omega}M$ such that $\operatorname{gtp}(\bar{a}_1\bar{b}_1, M_0, M) = \operatorname{gtp}(\bar{a}_2\bar{b}_2, M_0, M)$.

Proof of Claim 1: Let $m := m(\bar{a}_1, M_0, M)$ and $n := m(\bar{a}_1\bar{b}_1, M_0, M)$ (see Lemma 4.8). Since the generic types are equal, we must have that $m = m(\bar{a}_2, M_0, M)$. Consider the formula

$$\psi(\bar{x}) := \phi_m(\bar{x}) \wedge \exists \bar{y} \phi_n(\bar{x}, \bar{y})$$

where $\ell(\bar{x}) = \ell(\bar{a}_1)$ and $\ell(\bar{y}) = \ell(\bar{b}_1)$. We have that $M \Vdash \psi[\bar{a}_1]$ (the existential part is witnessed by $\bar{b}_1$) so also $M \Vdash \psi[\bar{a}_2]$ by equality of the generic types. By definition of forcing this means that there exists $N \in \mathfrak{K}_{\aleph_0}$ and $\bar{b}_2^* \in {}^{<\omega}N$ such that $M \leq_{\mathfrak{K}} N$ and $N \Vdash \phi_n[\bar{a}_2, \bar{b}_2^*]$. Now by Lemma 4.4(7 (using that M is brimmed over M_0), there exists $\bar{b}_2 \in M$ such that $M \Vdash \phi_n[\bar{a}_2, \bar{b}_2]$, as desired. $\dagger_{\text{Claim 1}}$

Claim 2: If M is brimmed over M_0 and $\bar{a}, \bar{b} \in {}^{<\omega}M$, then $\mathrm{gtp}(\bar{a}, M_0, M) = \mathrm{gtp}(\bar{b}, M_0, M)$ if and only if there is an automorphism of M sending $\bar{a}$ to $\bar{b}$ and fixing M_0 pointwise.

Proof of Claim 2: The right to left direction is clear and the left to right direction is a direct back and forth argument using Claim 1. $\dagger_{\text{Claim 2}}$

From Claim 2, it follows directly that if M is brimmed over M_0 then it is $(\mathscr{S}^{<\omega}(M_0), \aleph_0)$-homogeneous. Conversely, the countable $(\mathscr{S}^{<\omega}(M_0), \aleph_0)$-homogeneous model is unique and so it must also be brimmed over M_0. □

Corollary 4.11. If $\mathfrak{K}$ is nicely stable in $\aleph_0$, then there is a superlimit model of cardinality $\aleph_0$.

Proof. The $\aleph_0$-homogeneous model works (Theorem 4.10 with $M_0 := \emptyset$ implies its existence). □

We deduce the following characterization of types:

Corollary 4.12. Assume that $\mathfrak{K}$ is nicely stable in $\aleph_0$. Let $M_0 \in \mathfrak{K}_{\aleph_0} \cup \{\emptyset\}$ be an amalgamation base. Let $M \in \mathfrak{K}_{\aleph_0}$ be a brimmed model extending M_0 (but *not* necessarily brimmed *over* M_0). Let $\bar{a}_1, \bar{a}_2 \in {}^{<\omega}M$. The following are equivalent:

(1) $\mathbf{ortp}(\bar{a}_1, M_0, M) = \mathbf{ortp}(\bar{a}_2, M_0, M)$.
(2) $\mathrm{gtp}(\bar{a}_1, M_0, M) = \mathrm{gtp}(\bar{a}_2, M_0, M)$.
(3) $\mathrm{tp}_{\mathbb{L}_{\infty,\aleph_0}(\tau_{\mathfrak{K}})}(\bar{a}_1, M_0, M) = \mathrm{tp}_{\mathbb{L}_{\infty,\aleph_0}}(\bar{a}_2, M_0, M)$.
(4) $\mathrm{tp}_{\mathbb{L}_{\aleph_1,\aleph_0}(\tau_{\mathfrak{K}})}(\bar{a}_1, M_0, M) = \mathrm{tp}_{\mathbb{L}_{\aleph_1,\aleph_0}}(\bar{a}_2, M_0, M)$.

Proof. Let N be brimmed over M (hence over M_0). First we prove:

Claim: For any $\bar{a} \in M$, $\operatorname{gtp}(\bar{a}, M_0, M) = \operatorname{gtp}(\bar{a}, M_0, N)$ and $\operatorname{tp}_{\mathbb{L}_{\infty,\aleph_0}(\tau_{\mathfrak{K}})}(\bar{a}, M_0, M) = \operatorname{tp}_{\mathbb{L}_{\infty,\aleph_0}(\tau_{\mathfrak{K}})}(\bar{a}, M_0, N)$

Proof of Claim: This follows from Lemmas 4.4(6),(8). $\dagger_{\text{Claim}}$

Now consider the following statement:

(1)' There is an automorphism of N fixing M_0 sending $\bar{a}_1$ to $\bar{a}_2$.

Using it, we complete the proof of the theorem as follows:

- (1) is equivalent to (1)' (by a back and forth argument).
- (1)' implies (3) (straightforward using the Claim).
- (3) implies (4) (trivial).
- (4) is equivalent to (2) by Lemmas 4.4(6),(8), recalling that generic types are defined using $\mathbb{L}_{\aleph_1,\aleph_0}(\tau)$-formulas.
- (4) implies (1)' by the Claim, the equivalence of (2) with (4), and Claim 2 in the proof of Theorem 4.10.

$\square$

Corollary 4.13 (Locality)**.** Assume that $\mathfrak{K}$ is nicely stable in $\aleph_0$. Let $M \in \mathfrak{K}_{\aleph_0}$ be an amalgamation base. Let $p, q \in \mathscr{S}^{<\omega}(M)$. If $p \neq q$, then there exists $A \subseteq |M|$ finite such that $p \upharpoonright A \neq q \upharpoonright A$.

Proof. Suppose that $p \neq q$. Say $p = \mathbf{ortp}(\bar{a}, M, N)$, $q = \mathbf{ortp}(\bar{b}, M, N)$, with N brimmed over M. By Corollary 4.12, $\operatorname{gtp}(\bar{b}, M, N) \neq \operatorname{gtp}(\bar{b}, M, N)$, so there exists $A \subseteq |M|$ finite such that $\operatorname{gtp}(\bar{a}, A, N) \neq \operatorname{gtp}(\bar{b}, A, N)$. By Lemma 4.6, this implies that $\mathbf{ortp}(\bar{a}, A, N) \neq \mathbf{ortp}(\bar{b}, A, N)$, as desired. $\square$

We have also justified assuming amalgamation in the following sense:

Corollary 4.14. If $\mathfrak{K}$ is nicely stable in $\aleph_0$, then there exists an AEC $\mathfrak{K}' = (K', \leq_{\mathfrak{K}'})$ such that:

(1) $\operatorname{LS}(\mathfrak{K}') = \aleph_0$.
(2) $\mathfrak{K}'_{<\aleph_0} = \emptyset$.
(3) $\tau_{\mathfrak{K}'} = \tau_{\mathfrak{K}}$.
(4) $K' \subseteq K$ and for $M, N \in K'$, $M \leq_{\mathfrak{K}'} N$ if and only if $M \leq_{\mathfrak{K}} N$.
(5) For any $M \in \mathfrak{K}$ there exists $M' \in \mathfrak{K}'$ with $M \leq_{\mathfrak{K}} M'$.
(6) $\mathfrak{K}'$ is categorical in $\aleph_0$.
(7) $\mathfrak{K}'$ is very nicely stable in $\aleph_0$. In particular it has amalgamation in $\aleph_0$.
(8) For $M, N \in \mathfrak{K}'_{\aleph_0}$, $M \leq_{\mathfrak{K}'} N$ implies $M \preceq_{\mathbb{L}_{\infty,\aleph_0}(\tau_{\mathfrak{K}'})} N$.
(9) $\mathfrak{K}'$ is $\mathrm{PC}_{\aleph_0}$.

Proof. Let $M \in \mathfrak{K}_{\aleph_0}$ be superlimit (exists by Corollary 4.11). Let $K'_{\aleph_0} := \{N \in K : N \cong M\}$. Now let $\mathfrak{K}'$ be the AEC generated by $(K'_{\aleph_0}, \leq_{\mathfrak{K}})$. One can easily check that $\mathfrak{K}'$ is nicely stable in $\aleph_0$ and from categoricity in $\aleph_0$ we get amalgamation in $\aleph_0$, hence (7) holds. To see (8), use Corollary 4.12. As for (9), it follows from Theorem 4.2. $\square$

We can now construct the promised good $\aleph_0$-frame. Its nonforking relation will be define terms of splitting. We will work in the class generated by the superlimit so the reader may assume that all the models are brimmed.

Definition 4.15. For $M \in \mathfrak{K}_{\aleph_0}$ brimmed and $A \subseteq |M|$, $p \in \mathscr{S}^{<\omega}(M)$ *splits over* A if there exists an automorphism f of M such that $f(p) \neq p$.

Remark 4.16. Using Corollary 4.12, one can check that (assuming that $\mathfrak{K}$ is nicely stable in $\aleph_0$) this is equivalent to the syntactic definition using $\mathbb{L}_{\aleph_1,\aleph_0}(\tau_{\mathfrak{K}})$-formulas.

The following is proven in [She09a, I.5.6].

Fact 4.17. Assume that $\mathfrak{K}$ is nicely stable in $\aleph_0$ and categorical in $\aleph_0$. If $M \in \mathfrak{K}_{\aleph_0}$ and $p \in \mathscr{S}^{<\omega}(M)$, then there exists $A \subseteq |M|$ finite such that p does not split over A.

Definition 4.18. Assume that $\mathfrak{K}$ is nicely stable in $\aleph_0$. We define a pre-$(<\omega, \aleph_0)$-frame $\mathfrak{s} = (\mathfrak{K}_{\mathfrak{s}}, \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}, \underset{\mathfrak{s}}{\perp})$ by:

(1) $\mathfrak{K}_{\mathfrak{s}} = \mathfrak{K}'_{\aleph_0}$, where $\mathfrak{K}'$ is as given by Corollary 4.14.
(2) $\mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$ is the set of all nonalgebraic types of finite sequences over M.
(3) For $M \leq_{\mathfrak{s}} N$, $p \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(N)$ does not fork over M if and only if there exists a finite $A \subseteq |M|$ so that p does not split over A.

Theorem 4.19. If $\mathfrak{K}$ is nicely stable in $\aleph_0$, then $\mathfrak{s}$ is a categorical type-full good $(<\omega, \aleph_0)$-frame. Moreover $\mathrm{LWNF}_{\mathfrak{s}}$ has the symmetry property (recall Definitions 3.4 and 3.12). In particular, $\mathfrak{s}$ is good$^+$ and $\mathfrak{K}$ has a superlimit of cardinality $\aleph_1$.

Proof. Without loss of generality assume to simplify the notation that the class has already been changed, i.e. $\mathfrak{K} = \mathfrak{K}'$ where $\mathfrak{K}'$ is from Corollary 4.14. Equivalently, $\mathfrak{K}$ is categorical in $\aleph_0$. Once we have shown that $\mathfrak{s}$ is a type-full good frame, the moreover part follows from Corollary 4.13 and Theorem 3.20. The last sentence is by Theorem 3.15 (it is easy to check that if $\mathfrak{K}'$ has a superlimit in $\aleph_1$ then $\mathfrak{K}$ also has one).

Except for symmetry, the axioms of good frames are easy to check (see the proof of [She09a, II.3.4]). For example:

- Local character: Let $\langle M_i : i \leq \delta \rangle$ be increasing continuous in $\mathfrak{K}_{\mathfrak{s}}$. Let $p \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M_\delta)$. By Fact 4.17, there exists a finite $A \subseteq |M_\delta|$ such that p does not split over A. Pick $i < \delta$ such that $A \subseteq |M_i|$. Then p does not fork over M_i.
- Uniqueness: standard, see for example [BVa, 5.5] (and Remark 4.16).
- Extension: follows on general grounds, see [Vas, 3.10].

Symmetry is the hardest to prove, and is done as in [She09a, I.5.30]. We give a full proof for the convenience of the reader.

Suppose that $\mathbf{ortp}(\bar{b}, N_2, N_3)$ does not fork over N_0 and let $\bar{c} \in {}^{<\omega}N_2 \backslash N_1$. We want to find N_1, N_3' such that $N_0 \leq_{\mathfrak{s}} N_1 \leq_{\mathfrak{s}} N_3'$, $N_3 \leq_{\mathfrak{s}} N_3'$, $\bar{b} \in {}^{<\omega}N_1$ and $\mathbf{ortp}(\bar{c}, N_1, N_3')$ does not fork over N_0. Assume for a contradiction that there is no such N_1. Using existence for $\mathrm{LWNF}_{\mathfrak{s}} = \mathrm{LWNF}$ (see Theorem 3.11), as well as the extension property for nonforking, we can increase N_2 and N_3 if necessary and find N_1 such that $\mathrm{LWNF}(N_0, N_1, N_2, N_3)$, N_ℓ is brimmed over N_0, and N_3 is brimmed over N_ℓ for $\ell = 1, 2$. By assumption, $p := \mathbf{ortp}(\bar{c}, N_1, N_3)$ forks over N_0.

<u>Claim 1</u>: Let I be the linear order $[0, \infty) \cap \mathbb{Q}$. There exists an increasing chain $\langle M_s : s \in I \rangle$ such that for any $s < t$ in I, M_s, M_t are in $\mathfrak{K}_{\aleph_0}$ and M_t is brimmed over M_s.

<u>Proof of Claim 1</u>: Fix $\langle M_i^* : i < \omega_1 \rangle$ increasing continuous in $\mathfrak{K}_{\aleph_0}$ such that M_{i+1}^* is brimmed over M_i^* for all $i < \omega_1$. Using undefinability of well-ordering, pick a countable ill-founded model of set theory $\mathfrak{B} = (A, E, \langle M_s : s \in I^* \rangle)$ elementarily equivalent to $(H(\aleph_2), \in, \langle M_i^* : i < \omega_1 \rangle)$. Now I^* contains a copy of the rationals by a general argument on ill-founded models of set theory, see [Fri73, Section 3]). Recalling that $\mathfrak{K}$ is $\mathrm{PC}_{\aleph_0}$ (see Corollary 4.14) and the syntactic characterization of brimmed models (Theorem 4.10), the result follows. $\dagger_{\text{Claim 1}}$

Fix I, $\langle M_s : s \in I \rangle$ as in Claim 1. Fix N_0' such that N_0 is brimmed over N_0' and $p \restriction N_0$ does not fork over N_0'.

For any fixed infinite $J \subseteq I$, write $M_J := \bigcup_{s \in J} M_s$. Assume now that M_I is brimmed over M_J. Let $N_0^J := M_J$, $N_1^J := M_I$. Let N_3^J be brimmed over N_1^J. By categoricity and uniqueness of brimmed models, there exists $f_0 : N_0' \cong M_0$, $f_0^J : N_0 \cong N_0^J$, $f_1^J : N_1 \cong N_1^J$, and $f_3^J : N_3 \cong N_3^J$ such that $f_0 \subseteq f_0^J \subseteq f_1^J \subseteq f_3^J$. Let $f_2^J := f_3^J \restriction N_2$ and let $N_2^J := f_2^J[N_2]$. Note that $\mathrm{LWNF}(N_0^J, N_1^J, N_2^J, N_3^J)$ holds.

Let $p_J := \mathbf{ortp}(f_3^J(\bar{c}), f_3^J[N_1], f_3^J[N_3]) = \mathbf{ortp}(f_3^J(\bar{c}), M_I, N_3^J)$. Since we are assuming that $\mathbf{ortp}(\bar{c}, N_1, N_3)$ forks over N_0, we have that p_J forks over N_0^J. Moreover $p_J \restriction N_0^J$ does not fork over M_0.

Claim 2: If J has no last elements, $I \backslash J$ has no first elements, and $t \in I \backslash J$, then $p_J \restriction M_t$ forks over N_0^J.

Proof of Claim 2: Suppose that $p_J \restriction M_t$ does not fork over N_0^J. Note that M_t is brimmed over M_J. Find N_1' such that $N_0 \leq_{\mathfrak{s}} N_1' \leq_{\mathfrak{s}} N_1$, N_1' is brimmed over N_1, and $f_1^J : N_1' \cong M_t$. Let $\bar{b}' \in {}^{<\omega} N_1'$ be such that $\mathbf{ortp}(\bar{b}', N_0, N_1') = \mathbf{ortp}(\bar{b}, N_0, N_1)$. Since $\text{LWNF}(N_0, N_1, N_2, N_3)$, we know that $\mathbf{ortp}(\bar{b}', N_2, N_3)$ does not fork over N_0, hence by uniqueness $\mathbf{ortp}(\bar{b}, N_2, N_3) = \mathbf{ortp}(\bar{b}', N_2, N_3)$. But we have assumed shown that $\mathbf{ortp}(\bar{c}, N_1', N_3)$ does not fork over N_0 and $\bar{b}' \in {}^{<\omega_1} N_1'$, hence by a simple renaming we obtain a contradiction to our hypothesis that symmetry failed. $\dagger_{\text{Claim 2}}$

Claim 3: If $J_1 \subsetneq J_2$ are both proper initial segments of I with no last elements and $J_2 \backslash J_1$ has no first elements, then $p_{J_1} \neq p_{J_2}$.

Proof of Claim 3: Fix $t \in J_2 \backslash J_1$. By Claim 2, $p_{J_1} \restriction M_t$ forks over $N_0^{J_1}$. We claim that $p_{J_2} \restriction M_t$ does not fork over $N_0^{J_1}$. Indeed recall that $N_0^{J_2} = M_{J_2}$ and by assumption $p_{J_2} \restriction N_0^{J_2}$ does not fork over M_0. Therefore by monotonicity also $p_{J_2} \restriction M_t$ does not fork over $M_{J_1} = N_0^{J_1}$. $\dagger_{\text{Claim 3}}$

To finish, observe that there are $2^{\aleph_0}$ cuts of I as in Claim 3. Therefore stability fails, a contradiction. $\square$

References

[Bal09] John T. Baldwin, *Categoricity*, University Lecture Series, vol. 50, American Mathematical Society, 2009.

[BGKV16] Will Boney, Rami Grossberg, Alexei Kolesnikov, and Sebastien Vasey, *Canonical forking in AECs*, Annals of Pure and Applied Logic **167** (2016), no. 7, 590–613.

[BKV06] John T. Baldwin, David Kueker, and Monica VanDieren, *Upward stability transfer for tame abstract elementary classes*, Notre Dame Journal of Formal Logic **47** (2006), no. 2, 291–298.

[BL16] John T. Baldwin and Paul B. Larson, *Iterated elementary embeddings and the model theory of infinitary logic*, Annals of Pure and Applied Logic **167** (2016), 309–334.

[BVa] Will Boney and Sebastien Vasey, *Chains of saturated models in AECs*, Preprint. URL: `http://arxiv.org/abs/1503.08781v3`.

[BVb] ______, *Tameness and frames revisited*, The Journal of Symbolic Logic, To appear. URL: `http://arxiv.org/abs/1406.5980v6`.

[Dru13] Fred Drueck, *Limit models, superlimit models, and two cardinal problems in abstract elementary classes*, Ph.D. thesis, 2013, Available online. URL: `http://homepages.math.uic.edu/~drueck/thesis.pdf`.

[Fri73] Harvey M. Friedman, *Countable models of set theories*, Cambridge Summer School in Mathematical Logic (A.R.D. Mathias and H. Rogers, eds.), Springer-Verlag, 1973, pp. 539–573.

[Gro02] Rami Grossberg, *Classification theory for abstract elementary classes*, Contemporary Mathematics **302** (2002), 165–204.

[GVV16] Rami Grossberg, Monica VanDieren, and Andrés Villaveces, *Uniqueness of limit models in classes with amalgamation*, Mathematical Logic Quarterly **62** (2016), 367–382.

[HK06] Tapani Hyttinen and Meeri Kesälä, *Independence in finitary abstract elementary classes*, Annals of Pure and Applied Logic **143** (2006), 103–138.

[Les05] Olivier Lessmann, *Upward categoricity from a successor cardinal for tame abstract elementary classes with amalgamation*, The Journal of Symbolic Logic **70** (2005), no. 2, 639–660.

[She75] Saharon Shelah, *Categoricity in $\aleph_1$ of sentences in $L_{\omega_1,\omega}(Q)$*, Israel Journal of Mathematics **20** (1975), no. 2, 127–148.

[She83a] ———, *Classification theory for non-elementary classes I: The number of uncountable models of $\psi \in L_{\omega_1,\omega}$. Part A*, Israel Journal of Mathematics **46** (1983), no. 3, 214–240.

[She83b] ———, *Classification theory for non-elementary classes I: The number of uncountable models of $\psi \in L_{\omega_1,\omega}$. Part B*, Israel Journal of Mathematics **46** (1983), no. 4, 241–273.

[She87a] ———, *Classification of non elementary classes II. Abstract elementary classes*, Classification Theory (Chicago, IL, 1985) (John T. Baldwin, ed.), Lecture Notes in Mathematics, vol. 1292, Springer-Verlag, 1987, pp. 419–497.

[She87b] ———, *Universal classes*, Classification theory (Chicago, IL, 1985) (John T. Baldwin, ed.), Lecture Notes in Mathematics, vol. 1292, Springer-Verlag, 1987, pp. 264–418.

[She09a] ———, *Classification theory for abstract elementary classes*, Studies in Logic: Mathematical logic and foundations, vol. 18, College Publications, 2009.

[She09b] ———, *Classification theory for abstract elementary classes 2*, Studies in Logic: Mathematical logic and foundations, vol. 20, College Publications, 2009.

[SV99] Saharon Shelah and Andrés Villaveces, *Toward categoricity for classes with no maximal models*, Annals of Pure and Applied Logic **97** (1999), 1–25.

[Vas] Sebastien Vasey, *Toward a stability theory of tame abstract elementary classes*, Preprint. URL: `http://arxiv.org/abs/1609.03252v2`.

[Vas16a] ———, *Building independence relations in abstract elementary classes*, Annals of Pure and Applied Logic **167** (2016), no. 11, 1029–1092.

[Vas16b] ———, *Forking and superstability in tame AECs*, The Journal of Symbolic Logic **81** (2016), no. 1, 357–383.

[Vas16c] ______, *Infinitary stability theory*, Archive for Mathematical Logic **55** (2016), 567–592.

[Vas17] ______, *Downward categoricity from a successor inside a good frame*, Annals of Pure and Applied Logic **168** (2017), no. 3, 651–692.

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