HILBERT SPACES AND C^*-ALGEBRAS ARE NOT FINITELY CONCRETE

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ABSTRACT. We show that no faithful functor from the category of Hilbert spaces with linear isometries into the category of sets preserves directed colimits. Thus Hilbert spaces cannot form an abstract elementary class, even up to change of language. We deduce an analogous result for the category of commutative unital C^*-algebras with $*$-homomorphisms. This implies, in particular, that this category is not axiomatizable by a first-order theory, a strengthening of a conjecture of Bankston.

1. Introduction

Consider the category $\mathbf{Hilb}_r$ of Hilbert spaces with linear isometries. The forgetful functor $U : \mathbf{Hilb}_r \to \mathbf{Set}$ gives $\mathbf{Hilb}_r$ the structure of a concrete category. There is however an intuitive sense in which $\mathbf{Hilb}_r$ is "less concrete" than, say, the category of abelian groups with monomorphisms. Indeed, the union of an increasing chain of abelian groups is an abelian group, but this is not the case for Hilbert spaces (one needs to take the completion of the union). In category-theoretic language, directed colimits of Hilbert spaces are not concrete: the functor U does not preserve them. It is natural to ask whether this is a problem intrinsic to the category $\mathbf{Hilb}_r$, or whether the situation can be resolved by a clever choice of an alternative functor U.[1] That is, we consider whether $\mathbf{Hilb}_r$ is *finitely concrete*, in the sense that there exists *some* faithful functor preserving directed colimits from that category into $\mathbf{Set}$, i.e. one that succeeds where the usual forgetful functor fails. This is a subtle question, and some of the ideas we use to show that a category is *not* finitely concrete—as we proceed to do with $\mathbf{Hilb}_r$, Banach spaces, commutative unital C^*-algebras, and so on—have only recently been developed [Hen].

Finite concreteness of a category is an essential first test in determining the extent to which it can be subjected to a model-theoretic analysis. That is, the paradigmatic examples of finitely concrete categories include any category $\mathcal{K}$ which is *elementary*, in the sense that there is a first-order theory T so that $\mathcal{K}$ is equivalent to the category $\mathbf{Mod}(T)$ of T-models and homomorphisms. More generally, any category axiomatizable in the infinitary logic $\mathbb{L}_{\infty,\omega}$ (i.e. (∞,ω)-elementary in the sense of [MP89, p. 58]) is finitely concrete. Still more generally, any abstract elementary category in the sense of [BR12, 5.3], including any abstract elementary class (AEC),

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[1]We note that a change of U amounts to a change in the signature, see e.g. [LR16, 3.5]

cf. [She87], is finitely concrete. Thus we establish, in particular, that not only is $\mathbf{Hilb}_r$ not an AEC with respect to the usual forgetful functor—which is obvious from the failure of the union of chains axiom—this problem is essential: $\mathbf{Hilb}_r$ is *not* equivalent to an AEC, or an elementary category, answering [LRV19, 5.10]. This should also be compared to [Bon17, Ack] which give "near-equivalences" between continuous classes such as $\mathbf{Hilb}_r$ and AECs.

By embedding $\mathbf{Hilb}_r$ into various categories, we can deduce more examples of non-finite concreteness (Example 10). In particular, we show that the category $\mathbf{CC^*Alg}$ of commutative unital C^*-algebras with (unit-preserving) $*$-homomorphisms is not finitely concrete (Theorem 11), hence, in particular, not elementary. This solves a stronger version of a problem of Bankston [Ban82, Ban03] who had conjectured that the dual of the category of compact Hausdorff spaces (equivalently $\mathbf{CC^*Alg}$) was not P-elementary, i.e. not equivalent to a category of the form $\mathbf{Mod}(T)$ closed under products (this weaker conjecture was solved independently by Banaschewski [Ban84] and the second author [Ros89]). The question as to whether $\mathbf{CC^*Alg}$ is elementary was posed in [Ros89], and more recently in [MR17, 1.5].

2. Hilbert spaces

Throughout, we assume basic familiarity with category theory (as exposed for example in [AHS04]) and with accessible categories (see [AR94] or the recent tutorial [Vas]). Let $\mathbf{Set}$ denote the category of sets (with functions as morphisms), and let $\mathbf{Hilb}$ be the category whose objects are (complex) Hilbert spaces and whose morphisms are linear contractions. Note that linear isometries (equivalently orthogonal operators) are exactly the regular monomorphisms in $\mathbf{Hilb}$ (see [AHS04, 7.58(3)]). Thus we let $\mathbf{Hilb}_r$ be the subcategory of $\mathbf{Hilb}$ with the same objects but with linear isometries as morphisms. We observe in passing that $\mathbf{Hilb}_r$ is an $\aleph_1$-accessible category with directed colimits.

From now on we assume:

Hypothesis 1. $U : \mathbf{Hilb}_r \to \mathbf{Set}$ is a fixed faithful functor preserving directed colimits.

Our goal is to show that such a U cannot exist. The following definition follows [Hen, 4.3].

Definition 2. Let A be a Hilbert space, and let $x \in UA$. We say that x is *supported* on a subspace A_0 of A if whenever $f, g : A \to B$ are such that $fi_{A_0,A} = gi_{A_0,A}$ (where $i_{A_0,A}$ denotes the inclusion map $A_0 \to A$), then $f(x) = g(x)$. When A is clear from context, we omit it.

Note that, as is standard, we have abused notation and written $f(x)$ instead of $(Uf)(x)$. The next observation will be used repeatedly:

Remark 3. If A_0 is a subspace of a Hilbert space A and $x_0 \in UA_0$, then $i_{A_0,A}(x_0)$ is supported on A_0.

Lemma 4. Let A be a Hilbert space, and let $x \in UA$. If U preserves directed colimits, then x is supported on some finite dimensional subspace of A.

Proof. A is a directed colimit of its finite-dimensional subspaces. Since U preserves directed colimits, UA is a directed colimits of sets of the form UA_0, for A_0 a finite-dimensional subspace of A. Directed colimits in $\mathbf{Set}$ are unions, so $x = i_{A_0,A}(x_0)$ for some finite-dimensional A_0 and some $x_0 \in UA_0$. Now use Remark 3. $\square$

The next result says that finite-dimensional supports are closed under intersections. This is crucial and not so obvious, since there is no assumption that the concrete functor preserves pullbacks (which are intersections here). The proof follows [Hen, 4.7].

Lemma 5. Let A be an infinite-dimensional Hilbert space, let $x \in UA$ and let A_0, A_1 be finite-dimensional subspaces of A. If x is supported on both A_0 and A_1, then x is supported on $A_0 \cap A_1$.

Proof. Fix a Hilbert space B. For functions $f, g : A \to B$, we write $f \sim^* g$ if either $fi_{A_0,A} = gi_{A_0,A}$ or $fi_{A_1,A} = gi_{A_1,A}$. This is usually not an equivalence relation, so let $\sim$ be its transitive closure. Observe that since x is supported on both A_0 and A_1, we have that $f \sim^* g$ implies $f(x) = g(x)$, hence also $f \sim g$ implies $f(x) = g(x)$.

Let $f, g : A \to B$ be given so that $fi_{A_0 \cap A_1,A} = gi_{A_0 \cap A_1,A}$. We will find $h : A \to B$ so that $f \sim h$ and $g \sim h$, which will imply that $f \sim g$, and hence that $f(x) = g(x)$.

We first fix an infinite-dimensional subspace S of B that is disjoint from the space generated by $f[A_0] \cup f[A_1]$. This is possible as A_0 and A_1 are finite-dimensional, and B is necessarily infinite-dimensional: A is infinite-dimensional, and f is injective. Fix bases $\mathcal{B}_0$ and $\mathcal{B}_1$ for A_0, A_1 respectively so that $\mathcal{B}_0 \cap \mathcal{B}_1$ is a basis for $A_0 \cap A_1$. Fix a basis $\mathcal{C}$ extending $f[\mathcal{B}_0 \cap \mathcal{B}_1]$ for the space S' generated by $S \cup f[\mathcal{B}_0 \cap \mathcal{B}_1]$, and fix injections $\gamma_\ell : \mathcal{B}_\ell \to \mathcal{C}$, $\ell = 0, 1$, so that $\gamma_\ell(v) = f(v)(= g(v))$ whenever $v \in \mathcal{B}_0 \cap \mathcal{B}_1$. Extend γ_ℓ to a morphism $h_\ell : A_\ell \to S'$, and let $h : A \to B$ be a morphism extending both h_0 and h_1. We claim that $f \sim h \sim g$. We prove that $f \sim h$, and a symmetric argument will prove $g \sim h$. First, let $h'_\ell : A \to B$ be an extension of h_ℓ so that $h'_\ell i_{A_{1-\ell},A} = fi_{A_{1-\ell},A}$. This is possible by the assumption on S. Observe that $f \sim h'_\ell$ by definition, but also $h^*_\ell \sim h'_\ell$ for any extension h^*_ℓ of h_ℓ to A. In particular, $h \sim h'_\ell$, hence $f \sim h$, as desired. □

Lemma 6. For every infinite-dimensional Hilbert space A and any $x \in UA$, there is a unique minimal finite-dimensional subspace A_0 of A on which x is supported.

Proof. Combine Lemmas 4 and 5 with the fact that a nontrivial intersection of two finite-dimensional subspaces must have lower dimension. □

Definition 7. For an infinite-dimensional Hilbert space A and $x \in UA$, we call the minimal subspace of A on which x is supported (given by Lemma 6) the *support of x (in A)*. We say that this support is *trivial* if it is the zero space, nontrivial otherwise.

Lemma 8. For any nonzero subspace A_0 of an infinite-dimensional Hilbert space A, there is $x_0 \in UA_0$ such that $i_{A_0,A}(x_0)$ has nontrivial support in A.

Proof. Suppose not. Let $f, g : A \to B$ be any two morphisms such that $fi_{A_0,A} \neq gi_{A_0,A}$ (these are easy to construct: for example, take B to be the direct sum of A with itself, have f send A_0 to its copy in the left component and have g send A_0 to its copy in the right component). We know that f and g agree on the zero space, hence by definition of the support for any $x_0 \in UA_0$, $f(i_{A_0,A}(x_0)) = g(i_{A_0,A}(x_0))$. Thus $U(fi_{A_0,A}) = U(gi_{A_0,A})$ and so $fi_{A_0,A} = gi_{A_0,A}$ by faithfulness of U, a contradiction. □

Theorem 9. No faithful functor from $\mathbf{Hilb}_r$ to $\mathbf{Set}$ preserves directed colimits.

Proof. Suppose for a contradiction that U is a faithful functor from $\mathbf{Hilb}_r$ to $\mathbf{Set}$ preserving directed colimits. By the uniformization theorem [AR94, 2.19] (and see [BR12, 4.3]), there is a cardinal μ_0 such that for all regular cardinals $\mu \geq \mu_0$, U preserves μ-presentable objects. Fix a cardinal $\lambda > \mu_0 + 2^{\aleph_0}$ of countable cofinality. Let A be the Hilbert space of dimension λ, hence of cardinality $\lambda^{\aleph_0} > \lambda$. Note that A is λ^+-presentable, so by definition of μ_0 we also have that UA is λ^+-presentable, hence has cardinality at most λ.

Each nonzero element of A spans a line (i.e. a one-dimensional subspace of A), and each line contains only $|\mathbb{C}| = 2^{\aleph_0}$-many elements. This implies that there are $\lambda^{\aleph_0}$-many distinct lines. Since $|UA| \leq \lambda < \lambda^{\aleph_0}$, there must be a line A_0 that is *not* the support of any $x \in UA$. However, for each $x_0 \in UA_0$, $i_{A_0,A}(x_0)$ is supported on A_0 (Remark 3). By minimality of the support, the support of every element of $i_{A_0,A}[UA_0]$ must be a strict subspace of A_0, i.e. the zero space. In other words, every element of $i_{A_0,A}[UA_0]$ has trivial support. This contradicts Lemma 8. □

3. C^*-algebras and other examples

From now on, let us call a category *finitely concrete* if it admits a faithful functor into $\mathbf{Set}$ preserving directed colimits. We have just shown that $\mathbf{Hilb}_r$ is *not* finitely concrete. Note moreover that if $F : \mathcal{K} \to \mathcal{K}^*$ is faithful and preserves directed colimits and $\mathcal{K}^*$ is finitely concrete, then so is $\mathcal{K}$. Contrapositively, if $\mathcal{K}$ is not finitely concrete and there is a faithful directed-colimit preserving functor into $\mathcal{K}^*$, then $\mathcal{K}^*$ is not finitely concrete either. Using this, we can give more examples of non-finitely concrete categories:

Example 10.

(1) The category $\mathbf{Met}_r$ of complete metric spaces with isometries and the category $\mathbf{Ban}_r$ of Banach spaces with linear isometries, are not finitely concrete. This can be shown via the natural embedding of $\mathbf{Hilb}_r$ into these categories.
(2) The category $\mathbf{Hilb}$ of Hilbert spaces with linear contractions is not finitely concrete. Indeed, the inclusion $\mathbf{Hilb}_r \to \mathbf{Hilb}$ is faithful and preserves directed colimits. This applies more generally, any time we have a non-finitely concrete subcategory $\mathcal{K}$ of a category $\mathcal{K}^*$ that is closed under directed colimits (i.e. where the inclusion preserves directed colimits – naturally, inclusions are always faithful). In particular, we also get that the category $\mathbf{Met}$ of complete metric spaces with contractions and the category $\mathbf{Ban}$ of Banach spaces with linear contractions are not finitely concrete.

Let $\mathbf{CC^*Alg}$ be the category of commutative unital C^*-algebras and unit-preserving $*$-homomorphisms.

Theorem 11. The category $\mathbf{CC^*Alg}$ is not finitely concrete.

Proof. Let $V : \mathbf{CC^*Alg} \to \mathbf{Ban}$ be the forgetful functor (recall that $*$-homomorphisms are, in particular, contractions). It is folklore (see for example [Pes93, §12]) that V has a left adjoint $F : \mathbf{Ban} \to \mathbf{CC^*Alg}$. We note that this also follows from the adjoint functor theorem for locally presentable categories ([AR94, 1.66]) because V preserves limits and $\aleph_1$-directed colimits and both $\mathbf{Ban}$ and $\mathbf{CC^*Alg}$ are locally presentable (see, respectively, [AR94, 1.48], and [AR94, 3.28]—in the second case we need the result of Isbell [Isb82] that $\mathbf{CC^*Alg}$ is a variety of algebras with $\aleph_0$-ary operations).

Moreover, the components of the unit of the adjunction, $\eta_B : B \to VFB$, are linear isometries hence, in particular, monomorphisms. This follows from the fact that any Banach space B can be isometrically embedded into a commutative unital C^*-algebra. Indeed, this algebra can be taken to be the C^*-algebra $C(X)$ of continuous complex-valued functions on the closed unit ball X of the dual space B^* with the weak* topology. Since X is compact (by the Banach-Alaoglu theorem), $C(X)$ is commutative and unital.

Thus F is faithful by a general result about adjoints (see [AHS04, 19.14(1)]). Since F is a left adjoint, it preserves arbitrary colimits. By Example 10(2), then, $\mathbf{CC^*Alg}$ is not finitely concrete. $\square$

As mentioned in the introduction, a category $\mathcal{K}$ is called *elementary* if there is a first-order theory T such that $\mathcal{K}$ is equivalent to the category $\mathbf{Mod}(T)$ of T-models and homomorphisms. Following [Ric71] (see also [AR94, 5.23]), the forgetful functor $\mathbf{Mod}(T) \to \mathbf{Set}$ preserves directed colimits. Thus we obtain as a corollary that $\mathbf{CC^*Alg}$ cannot be elementary.

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