SHELAH-VILLAVECES REVISITED

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Abstract. We study uniqueness of limit models in abstract elementary classes (AECs) with no maximal models. We prove (assuming instances of diamonds) that categoricity in a cardinal of the form $\mu^{+(n+1)}$ implies the uniqueness of limit models of cardinality $\mu^+, \mu^{++}, \dots, \mu^{+n}$. This sheds light on a paper of Shelah and Villaveces, who were the first to consider uniqueness of limit models in this context. We also prove that (again assuming instances of diamonds) in an AEC with no maximal models, tameness (a locality property for types) together with categoricity in a proper class of cardinals imply categoricity on a tail of cardinals. This is the first categoricity transfer theorem in that setup and answers a question of Baldwin.

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1. Introduction

The main test question for developing a classification theory in abstract elementary classes (AECs) is Shelah's eventual categoricity conjecture: An AEC categorical in *some* high-enough cardinal should be categorical in *all* high-enough cardinals. A version of this conjecture for classes axiomatized in $\mathbb{L}_{\omega_1,\omega}$ already appears as an open problem in [She78]. Forty years later, the conjecture is still open. Numerous partial approximations exist (we do not attempt to be exhaustive here, see the introduction of [Vasd] for a history). In a milestone paper Shelah [She99] proved that the conjecture holds in AECs with the amalgamation property when the categoricity cardinal is a successor. In [She09, Section IV.7], Shelah asserts that he can remove the successor hypothesis assuming the weak generalized continuum hypothesis ($2^\mu < 2^{\mu^+}$ for all cardinals μ) and an unpublished claim (to appear in [She]). An overview of Shelah's proof is given in [Vasb, Section 11].

In another direction, [SV99], Shelah and Villaveces started the study of AECs that only have no maximal models (but may not have full amalgamation). Assuming instances of diamonds, they prove density of amalgamation bases, existence of universal extensions, as well as a superstability property below the categoricity cardinal. They also assert that a property called *uniqueness of limit models* holds in every cardinal below the categoricity cardinal (see [GVV16] for an introduction to limit models; we assume basic familiarity with them for the rest of this paper). Several problems with Shelah and Villaveces' proof were isolated in the second author's [Van06, Van13], and later also in [Van16]. Progress on the question of uniqueness of limit models has been made in the full amalgamation setup [GVV16, Van16, VV, Vasc], resulting in a full proof that in that particular case it indeed follows from categoricity [Vasc, Corollary 5.7.(2)]. In the original setup of Shelah and Villaveces, the question of uniqueness of limit models is still open.

In [Van], uniqueness of limit models is studied (assuming no maximal models and instances of diamonds) in the more restricted setup of categoricity in a cardinal of the form μ^{+n}. It is established that uniqueness of limit models in μ^+ is equivalent to several other statements, including that the union of a chain of length less than μ^+ of saturated models (in $\mathbf{K}_{\mu^+}$) is saturated.

We continue this work here and show that all these equivalent statements unconditionally hold: limit models in μ^+ (and more generally in $\mu^{++}, \dots, \mu^{+(n-1)}$) *are* unique (see Corollary 4.12). The proof uses the theory of Ehrenfeucht-Mostowski (EM) models, more precisely a close study of when they are amalgamation bases. We still do not know whether uniqueness of limit models holds below μ (or even at μ), nor what happens when the categoricity cardinal is a limit.

In Section 5 (which, aside from some terminology, is independent from the rest of the paper), we prove a categoricity transfer in the Shelah-Villaveces setup. Specifically, Baldwin [Bal06, Question 23] has asked whether the categoricity transfer of Grossberg and the second author in tame AECs with amalgamation [GV06c, GV06a] can be generalized to tame AECs with only no maximal models. We answer this positively assuming the previously mentioned unproven claim of Shelah, but can only obtain an eventual transfer: assuming instances of diamonds, tameness, no maximal models, and categoricity in a proper class of cardinals, then categoricity on a tail of cardinals holds (see Corollary 5.18). Note that a Hanf number argument gives us that there exists a map $\mu \mapsto \lambda_\mu$ such that for any AEC $\mathbf{K}$, categoricity in $\lambda \geq \lambda_{\mathrm{LS}(\mathbf{K})}$ implies categoricity in a proper class of cardinals. Therefore Shelah's eventual categoricity conjecture consistently holds in tame AECs with no maximal models, but we are unable to give a bound on the "high-enough" threshold. More precisely, we use two categoricity cardinals that are above what we call a nice fixed point: a cardinal $\mu > \mathrm{LS}(\mathbf{K})$ such that $\mathrm{cf}(\mu) = \aleph_0$ and $\chi = \beth_\chi$ for unboundedly-many $\chi < \mu$.

The proof of the categoricity transfer heavily uses the theory of good frames developed by Shelah in [She09, Chapters II, III, IV]. Although we have tried to include a very short crash course on good frames at the beginning of Section 5, familiarity with good frames is assumed there. In the other sections, familiarity with a basic text on AECs such as [Bal09] should be enough.

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2. Background

We will use the following set-theoretic notation:

Notation 2.1.

(1) For $\chi < \mu$ both regular cardinals,

$$S^\mu_\chi := \{\alpha < \mu : \mathrm{cf}(\alpha) = \chi\}$$

(2) For μ an infinite cardinal, let CHWD_μ stand for the Continuum Hypothesis and Weak Diamond statement "$2^\mu = \mu^+$ and $\Phi_{\mu^+}(S^{\mu^+}_{\mathrm{cf}(\mu)})$ holds" ($\Phi_{\mu^+}(S)$ denotes the weak diamond at S: see [DS78] for the definition when $\mu = \aleph_0$; for general μ, see e.g. [Van06, Definition I.3.2]).
(3) For Θ a class of cardinals, CHWD_Θ means that CHWD_μ holds for every $\mu \in \Theta$.
(4) Let GCHWD stand for the global version: $\mathrm{CHWD}_{[\aleph_0,\infty)}$.

Note that the actual definition of the weak diamond principle is not important as we only use it to quote results, e.g. Fact 2.10.

Everywhere in this paper, $\mathbf{K}$ denotes a fixed AEC (not necessarily satisfying amalgamation or no maximal models). We assume that the reader is familiar with the definitions of amalgamation, no maximal models, Galois types, (Galois) stability, and (Galois) saturation. We will use the notation from the preliminaries of [Vas16].

In particular, $\mathrm{gtp}(\bar{b}/M; N)$ denotes the Galois types of $\bar{b}$ over M as computed inside N (where $M \leq_{\mathbf{K}} N$; note that the general definition of Galois types does not assume amalgamation). We let $\mathrm{gS}^\alpha(M; N)$ denote the set of all Galois types of sequences of length α over M computed in N, and let $\mathrm{gS}^\alpha(M)$ denote the set of all Galois types of sequences of length α over M (computed in any extension N of M). When $\alpha = 1$, we omit it.

When working with EM models, we will use the notation from [She09, Chapter IV]:

Definition 2.2. [She09, Definition IV.0.8]

(1) Relative to a class $\mathbf{K}$, we say that Φ *is proper for linear orders* iff $\Phi = \{p_n : n < \omega\}$, where p_n is an n-variable quantifier-free type in a fixed vocabulary $\tau(\Phi) \supset \tau(\mathbf{K})$ and the types in Φ can be used to generate a $\tau(\Phi)$-structure $\mathrm{EM}(I, \Phi)$ for each linear order I; that is, $\mathrm{EM}(I, \Phi)$ is the closure under the functions of $\tau(\Phi)$ of the universe of I and for any $i_0 < \ldots < i_{n-1}$ in I, $i_0 \ldots i_{n-1}$ realizes p_n.
(2) For $\mu \geq \mathrm{LS}(\mathbf{K})$, let $\Upsilon_\mu[\mathbf{K}]$ be the set of Φ proper for linear orders with:
    (a) $|\tau(\Phi)| \leq \mu$.

(b) If I is a linear order of cardinality λ, $\mathrm{EM}_{\tau(\mathbf{K})}(I, \Phi) \in \mathbf{K}_{\lambda+|\tau(\Phi)|+\mathrm{LS}(\mathbf{K})}$, where $\tau(\mathbf{K})$ is the vocabulary of $\mathbf{K}$ and $\mathrm{EM}_{\tau(\mathbf{K})}(I, \Phi)$ denotes the reduct of $\mathrm{EM}(I, \Phi)$ to $\tau(\mathbf{K})$. Here we are implicitly also assuming that $\tau(\mathbf{K}) \subseteq \tau(\Phi)$.

(c) For $I \subseteq J$ linear orders, $\mathrm{EM}_{\tau(\mathbf{K})}(I, \Phi) \leq_{\mathbf{K}} \mathrm{EM}_{\tau(\mathbf{K})}(J, \Phi)$.

We call Φ as above an *EM blueprint*.

The following is a consequence of Shelah's presentation theorem. We will use it without explicit mention.

Fact 2.3. Let $\mu \geq \mathrm{LS}(\mathbf{K})$. $\mathbf{K}$ has arbitrarily large models if and only if $\Upsilon_\mu[\mathbf{K}] \neq \emptyset$.

Since we will work in a setup that does not have full amalgamation, we give the precise definition of all the variations on saturation that we will use:

Definition 2.4. Let $\mathbf{K}$ be an AEC and let $M, N \in \mathbf{K}$.

(1) We say that M is an *amalgamation base* if any M_1 and M_2 *of the same size as* M can be amalgamated over M.
(2) For $\mu > \mathrm{LS}(\mathbf{K})$, we say that N is *μ-universal over M* if $M \leq_{\mathbf{K}} N$ and for any $M' \in \mathbf{K}_{<\mu}$ with $M \leq_{\mathbf{K}} M'$, there exists $f : M' \underset{M}{\longrightarrow} N$. When $\mu = \|M\|^+$, we omit it.
(3) We say that N is *model-homogeneous over M* if N is $\|N\|^+$-universal over M.
(4) We say that N is *saturated over M* if any $p \in \mathrm{gS}(M)$ is realized in N.
(5) We say that N is *(μ, δ)-limit over M* if:
    (a) $\delta \leq \mu^+$ is a limit ordinal.
    (b) There exists an increasing continuous chain $\langle M_i : i < \delta \rangle$ in $\mathbf{K}_\mu$ such that $M_0 = M$, $N = \bigcup_{i<\delta} M_i$, and M_{i+1} is universal over M_i for all $i < \delta$.

    For a regular cardinal χ, we say that N is *$(\mu, \geq \chi)$-limit over M* if N is (μ, δ)-limit with $\mathrm{cf}(\delta) \geq \chi$. We say that N is *limit over M* if N is $(\|M\|, \delta)$-limit over M for some $\delta \leq \|M\|^+$. We say that N is *limit* if it is limit over M for some $M \in \mathbf{K}_{\|N\|}$.

Remark 2.5.

(1) If N is universal over M, then M is an amalgamation base.
(2) If N is limit over M, then N is universal over M.
(3) Let $M_0 \leq_{\mathbf{K}} M_1 \leq_{\mathbf{K}} M_2$ all be in $\mathbf{K}_\mu$.
    (a) If M_1 is universal over M_0, then M_2 is universal over M_0.

(b) If M_2 is universal [(μ, δ)-limit] over M_1 and M_0 is an amalgamation base, then M_2 is universal [(μ, δ)-limit] over M_0.

Definition 2.6. Let $\mathbf{K}$ be an AEC. Let $\mu \geq \mathrm{LS}(\mathbf{K})$ and let $M \in \mathbf{K}$.

We say that M is *universal [model-homogeneous] [saturated] over μ-amalgamation bases* if for any amalgamation base $M_0 \in \mathbf{K}_\mu$ with $M_0 \leq_{\mathbf{K}} M$, M is universal [model-homogeneous] [saturated] over M_0.

The following axiomatic setup comes from the context derived in [SV99]:

Definition 2.7. Let μ be a cardinal. We say that $\mathbf{K}$ is *nicely stable in μ* (or *nicely μ-stable*) if:

(1) $\mathrm{LS}(\mathbf{K}) \leq \mu$.
(2) $\mathbf{K}_\mu \neq \emptyset$.
(3) $\mathbf{K}$ has joint embedding in μ.
(4) Existence of universal extensions in μ: For any $M \in \mathbf{K}_\mu$, there exists $N \in \mathbf{K}_\mu$ such that $M <_{\mathbf{K}} N$ and N is universal over a model containing M.
(5) Any limit model in $\mathbf{K}_\mu$ is an amalgamation base.

Remark 2.8. Using Remark 2.5, we have the following consequences of the definition:

(1) For any $M \in \mathbf{K}_\mu$, there exists an amalgamation base $N \in \mathbf{K}_\mu$ with $M \leq_{\mathbf{K}} N$
(2) For any amalgamation base $M \in \mathbf{K}_\mu$, there exists an amalgamation base $N \in \mathbf{K}_\mu$ such that N is universal over M.

Note that full amalgamation and stability imply nice stability: existence of universal extensions is due to Shelah, see e.g. [Bal09, Chapter 9] for proofs.

Fact 2.9. Let $\mu \geq \mathrm{LS}(\mathbf{K})$ be such that $\mathbf{K}_\mu \neq \emptyset$ and $\mathbf{K}_\mu$ has joint embedding, amalgamation, and no maximal models. If $\mathbf{K}$ is stable in μ (i.e. $|\mathrm{gS}(M)| \leq \mu$ for every $M \in \mathbf{K}_\mu$), then $\mathbf{K}$ is nicely stable in μ.

It follows (see e.g. [She99, Claim I.1.7]) that nice stability holds below the categoricity cardinal of an AEC with amalgamation. If the AEC only has no maximal models, Shelah and Villaveces [SV99] have shown that one can also obtain it assuming instances of diamonds. See also [Van06, Theorem I.3.13] for the version assuming weak diamond.

Fact 2.10. Assume that $\mathbf{K}$ has arbitrarily large models. Let $\lambda > \mathrm{LS}(\mathbf{K})$. Assume that $\mathbf{K}$ is categorical in λ and $\mathbf{K}_{<\lambda}$ has no maximal models. Let $\mu \in [\mathrm{LS}(\mathbf{K}), \lambda)$.

(1) If $\mathbf{K}$ has amalgamation in μ, then $\mathbf{K}$ is nicely stable in μ.
(2) If CHWD_μ holds (recall Notation 2.1), then $\mathbf{K}$ is nicely stable in μ.

We will use without mention the two basic facts about limit models (appearing already e.g. in [SV99, Fact 1.3.6]): they exist and they are unique when their lengths have the same cofinality. Of course the question of uniqueness when they have different lengths is much harder and occupies us for the rest of this paper.

Fact 2.11. Assume that $\mathbf{K}$ is nicely stable in μ. Let $M \in \mathbf{K}_\mu$ be an amalgamation base.

(1) For any limit ordinal $\delta \leq \mu^+$, there exists a (μ, δ)-limit over M.
(2) If N_1 is (μ, δ_1)-limit over M, N_2 is (μ, δ_2)-limit over M, and $\mathrm{cf}(\delta_1) = \mathrm{cf}(\delta_2)$, then $N_1 \cong_M N_2$.

3. Nice stability in successive cardinals

In this section, we study AECs nicely stable in several successive cardinals and show that saturated models behave quite well, e.g. (μ^{+n}, μ^+)-limits are μ-saturated. The following notion is key and was isolated by the second author in [Van, Definition 8].

Definition 3.1. For $\mathbf{K}$ an AEC, we say that $M \in \mathbf{K}$ is *dense with μ-amalgamation bases* if for any $A \subseteq |M|$ with $|A| \leq \mu$, there exists an amalgamation base $M_0 \in \mathbf{K}_\mu$ with $A \subseteq |M_0|$ and $M_0 \leq_{\mathbf{K}} M$.

We will use the following three basic facts.

Fact 3.2. Assume that $\mathbf{K}$ is nicely stable in μ. The following are equivalent for $M \in \mathbf{K}_{\mu^+}$:

(1) M is (μ, μ^+)-limit.
(2) M is dense with μ-amalgamation bases and saturated over μ-amalgamation bases.
(3) M is dense with μ-amalgamation bases and model-homogeneous over μ-amalgamation bases.

Proof. (1) implies (2) and (3) implies (1) are both straightforward. That (2) implies (3) is [Van, Lemma 1]. $\square$

Fact 3.3. Assume that $\mathbf{K}$ is nicely stable in μ. Let $\delta < \mu^{++}$ be a limit ordinal. If $\langle M_i : i < \delta \rangle$ is an increasing chain of (μ, μ^+)-limit models, then $\bigcup_{i<\delta} M_i$ is dense with μ-amalgamation bases.

In particular, if $\mathbf{K}$ is also nicely stable in μ^+ then any limit model in $\mathbf{K}_{\mu^+}$ is dense with μ-amalgamation bases.

Proof. By the proof of [Van, Lemma 2]. □

Fact 3.4 (Proposition 1 in [Van])**.** Assume that $\mathbf{K}$ is nicely stable in μ and μ^+. Let $M \in \mathbf{K}_\mu$ and $N \in \mathbf{K}_{\mu^+}$. Then N is (μ, μ^+)-limit over M if and only if N is (μ^+, μ^+)-limit over a model containing M.

We generalize these facts by replacing μ^+ there by μ^{+n}:

Theorem 3.5. Let $n < \omega$. Assume that $\mathbf{K}$ is nicely stable in μ^{+m} for every $m \leq n$.

(1) If M is $(\mu^{+n}, \geq \mu^+)$-limit, then M is dense with μ-amalgamation bases and model-homogeneous over μ-amalgamation bases.
(2) If $\delta < \mu^{+(n+2)}$ is a limit ordinal and $\langle M_i : i < \delta \rangle$ is an increasing sequence of $(\mu^{+n}, \geq \mu^+)$-limits, then $\bigcup_{i<\delta} M_i$ is dense with μ-amalgamation bases.
(3) If $n \geq 1$, any limit model in $\mathbf{K}_{\mu^{+n}}$ is dense with μ-amalgamation bases.
(4) If $M \in \mathbf{K}_{\mu^{+(n+1)}}$ is dense with μ-amalgamation bases, then M is saturated over μ-amalgamation bases if and only if M is model-homogeneous over μ-amalgamation bases.

Note that there are no restrictions on the cofinality of δ in (2).

Proof.

(1) For $n = 0$, this is Facts 3.2 and 3.4. For $n = m + 1$, suppose that M is (μ^{+n}, δ)-limit with $\mathrm{cf}(\delta) \geq \mu^+$. WLOG we can write $M = \cup_{i<\delta} M_i$ with M_{i+1} being (μ^{+n}, μ^{+n})-limit over M_i. By Fact 3.4, M_{i+1} is also (μ^{+m}, μ^{+n})-limit and, by hypothesis, is dense with μ-amalgamation bases and model homogeneous over μ-amalgamation bases. Since $\mathrm{cf}(\delta) \geq \mu^+$, any $A \subset |M|$ of size $\leq \mu$ shows up in some M_{i+1}. Thus, M is dense with μ-amalgamation bases and model homogeneous over μ-amalgamation bases.
(2) Let $M_\delta := \bigcup_{i<\delta} M_i$. If $\mathrm{cf}(\delta) \geq \mu^+$, then any subset of $|M_\delta|$ of size μ is contained in some M_i, so the result follows directly from the first part. Now assume that $\mathrm{cf}(\delta) \leq \mu$. Given $A \subseteq |M_\delta|$ with $|A| \leq \mu$, we build $\langle N_i : i \leq \delta \rangle$ increasing continuous such that for all $i < \delta$, $N_i \in \mathbf{K}_\mu$, $N_i \leq_{\mathbf{K}} M_{i+1}$, $A_i \cap |M_i| \subseteq |N_{i+1}|$, and N_{i+1} is universal over N_i. This is possible by the first part. In the end, $A \subseteq |N_\delta|$ and N_δ is (μ, δ)-limit, so N_δ is the desired amalgamation base by Fact 3.3.

(3) Follows from the previous part (n there stands for $n-1$ here) as in the "in particular" part of Fact 3.3.
(4) The right to left direction is trivial and the left to right direction is as in the proof of Fact 3.2, using the first part to get enough homogeneity.

$\square$

We do not know whether Theorem 3.5 generalizes to an infinite interval of cardinals. This is the source of all our later difficulties in generalizing results about categoricity in μ^{+n} to categoricity in an arbitrary λ.

Question 3.6. Let $\mathrm{LS}(\mathbf{K}) \leq \mu < \lambda$ and assume that $\mathbf{K}$ is nicely stable in every $\theta \in [\mu, \lambda)$. If $M \in \mathbf{K}_{[\mu,\lambda)}$ is $(\|M\|, \geq \mu^+)$-limit, is M dense with μ-amalgamation bases and model-homogeneous over μ-amalgamation bases?

4. When are EM models amalgamation bases?

In this section, we will work with EM blueprints (so implicitly we assume that $\mathbf{K}$ has arbitrarily large models). We write $\tau := \tau(\mathbf{K})$. We want to give conditions under which $\mathrm{EM}_\tau(I, \Phi)$ is an amalgamation base, at least for a big family of I's (really we want to see when this follows from categoricity). Further, we also want that for suitable I and J, $\mathrm{EM}_\tau(I, \Phi)$ is universal over $\mathrm{EM}_\tau(J, \Phi)$. To this end, we will study the following property:

Definition 4.1. We say that $\mathbf{K}$ is Φ-nicely *superstable in* μ (or Φ-*nicely μ-superstable*) if:

(1) $\mu \geq \mathrm{LS}(\mathbf{K})$.
(2) Φ is an EM blueprint for $\mathbf{K}$ with $|\tau(\Phi)| \leq \mu$.
(3) $\mathbf{K}$ has joint embedding in μ.
(4) For any ordinal $\alpha \in [\mu, \mu^+)$, there exists an ordinal $\beta \in [\alpha, \mu^+)$ such that $\mathrm{EM}_\tau(\beta, \Phi)$ is universal over a model containing $\mathrm{EM}_\tau(\alpha, \Phi)$.
(5) Any limit model in $\mathbf{K}_\mu$ is an amalgamation base.

At first sight, this looks more like stability than superstability. However we will see (Corollary 4.8) that it is really a variation of solvability, a version of superstability introduced by Shelah in [She09, Definition IV.1.3]. In particular, what is called μ-superstability in [Van, Definition 4] follows from Φ-nice μ-superstability (Corollary 4.9).

The following consequences of the definition will be used without comments.

Lemma 4.2. Assume that $\mathbf{K}$ is Φ-nicely superstable in μ.

(1) For any ordinal $\alpha \in [\mu, \mu^+)$, there exists $\beta \in [\alpha, \mu^+)$ such that:
 (a) $\mathrm{EM}_\tau(\beta, \Phi)$ is an amalgamation base.
 (b) If $\mathrm{EM}_\tau(\alpha, \Phi)$ is an amalgamation base, then $\mathrm{EM}_\tau(\beta, \Phi)$ is universal over $\mathrm{EM}_\tau(\alpha, \Phi)$.
(2) For any $M \in \mathbf{K}_\mu$, there exists $\beta < \mu^+$ and $f : M \to \mathrm{EM}_\tau(\beta, \Phi)$.
(3) $\mathbf{K}$ is nicely stable in μ.

Proof.

(1) We prove (1a), and (1b) then follows from the definition of Φ-nice μ-superstability and Remark 2.5. We build $\langle \alpha_i : i < \omega \rangle$ strictly increasing such that for all $i < \omega$:
 (a) $\alpha_0 = \alpha$.
 (b) $\alpha_i < \mu^+$.
 (c) $\mathrm{EM}_\tau(\alpha_{i+1}, \Phi)$ is universal over a model containing $\mathrm{EM}_\tau(\alpha_i, \Phi)$.

 This is possible by the definition of Φ-nice μ-superstability. Let $\beta := \sup_{i<\omega} \alpha_i$. Say $\mathrm{EM}_\tau(\alpha_{i+1}, \Phi)$ is universal over M_i, with $\mathrm{EM}_\tau(\alpha_i, \Phi) \leq_{\mathbf{K}} M_i \leq_{\mathbf{K}} \mathrm{EM}_\tau(\alpha_{i+1}, \Phi)$. We then have that M_i is an amalgamation base and moreover M_{i+1} is universal over M_i (Remark 2.5). Further, letting $M := \bigcup_{i<\omega} M_i$, we have that $M = \mathrm{EM}_\tau(\beta, \Phi)$ and M is by construction limit over M_0, hence an amalgamation base.
(2) Let $M \in \mathbf{K}_\mu$. Let $\alpha \in [\mu, \mu^+)$ be such that $\mathrm{EM}_\tau(\alpha, \Phi)$ is an amalgamation base, and let $\beta \in [\alpha, \mu^+)$ be such that $\mathrm{EM}_\tau(\beta, \Phi)$ is universal over $\mathrm{EM}_\tau(\alpha, \Phi)$. Note that α and β exist by the first part. By joint embedding, there exists $N \in \mathbf{K}_\mu$ with $\mathrm{EM}_\tau(\alpha, \Phi) \leq_{\mathbf{K}} N$ and $g : M \to N$. By universality of $\mathrm{EM}_\tau(\beta, \Phi)$, there exists $h : N \xrightarrow[\mathrm{EM}_\tau(\alpha,\Phi)]{} \mathrm{EM}_\tau(\beta, \Phi)$. Now let $f := h \circ g$.
(3) This follows directly from the first and second parts.

$\square$

The following easy lemma gives a sufficient condition for Φ-nice μ-superstability:

Lemma 4.3. Assume that $\mathbf{K}$ is nicely stable in μ and let Φ be an EM blueprint for $\mathbf{K}$ with $|\tau(\Phi)| \leq \mu$. Assume that for any $\alpha \in [\mu, \mu^+)$, there exists a set u of ordinals such that $|u| = \mu$, $\alpha \subseteq u$, and $\mathrm{EM}_\tau(u, \Phi)$ is universal over a model containing $\mathrm{EM}_\tau(\alpha, \Phi)$. Then $\mathbf{K}$ is Φ-nicely μ-superstable.

Proof. Let $\alpha \in [\mu, \mu^+)$. Let u be as given by the assumption of the lemma. We have that $\mathrm{EM}_\tau(u, \Phi)$ is isomorphic to $\mathrm{EM}_\tau(\mathrm{otp}(u), \Phi)$ via a canonical map that fixes $\mathrm{EM}_\tau(\alpha, \Phi)$. Since $\beta := \mathrm{otp}(u)$ is in $[\alpha, \mu^+)$, the result follows. □

As motivation, note that assuming full amalgamation Φ-nice μ-superstability follows from categoricity (this will not be used):

Fact 4.4. Assume that $\mathbf{K}$ has arbitrarily large models and let $\lambda > \mathrm{LS}(\mathbf{K})$. Assume that $\mathbf{K}$ is categorical in λ and $\mathbf{K}_{<\lambda}$ has amalgamation and no maximal models. For any $\mu \in [\mathrm{LS}(\mathbf{K}), \lambda)$ and any EM blueprint Φ for $\mathbf{K}$ with $|\tau(\Phi)| \leq \mu$, $\mathbf{K}$ is Φ-nicely superstable in μ.

Proof. It is implicit in [She99] that $\mathbf{K}$ is Φ-nicely superstable in μ when the model of cardinality λ is saturated. By [Vasc], the model of cardinality λ *is* saturated. For a more precise argument, follow the proof of [Vasc, Corollary 5.1]. □

The question we will consider is:

Question 4.5. Assume that $\mathbf{K}$ has arbitrarily large models and let $\lambda > \mathrm{LS}(\mathbf{K})$. Assume that $\mathbf{K}$ is categorical in λ and $\mathbf{K}_{<\lambda}$ has no maximal model. Assume also that $\mathrm{CHWD}_{[\mathrm{LS}(\mathbf{K}),\lambda)}$ holds (recall Notation 2.1). Given $\mu \in [\mathrm{LS}(\mathbf{K}), \lambda)$, does there exist an EM blueprint Φ such that $\mathbf{K}$ is Φ-nicely superstable in μ?

That is, can we replace amalgamation with no maximal models and some set-theoretic assumptions in Fact 4.4. As with Question 3.6, we are stuck when λ is not of the form $\mu^{+(n+1)}$. The rest of this section gives a positive answer when $\lambda = \mu^{+(n+1)}$ and draws some consequences on uniqueness of limit models. The key lemma is:

Lemma 4.6. Let $n < \omega$ and assume that $\mathbf{K}$ is nicely stable in μ^{+m} for every $m \leq n$. Let Φ be an EM blueprint with $|\tau(\Phi)| \leq \mu$. If there exists an ordinal γ such that $\mathrm{EM}_\tau(\gamma, \Phi)$ is $(\mu^{+n}, \geq \mu^+)$-limit, then $\mathbf{K}$ is Φ-nicely superstable in μ.

Proof. Let $M := \mathrm{EM}_\tau(\gamma, \Phi)$. By Theorem 3.5, M is dense with μ-amalgamation bases and model-homogeneous over μ-amalgamation bases. In particular, there exists $M_0 \in \mathbf{K}_\mu$ such that M_0 is universal over a model containing $\mathrm{EM}_\tau(\alpha, \Phi)$. Now pick $u \subseteq \gamma$ with $|u| = \mu$ such that $M_0 \leq_{\mathbf{K}} \mathrm{EM}_\tau(u, \Phi)$. Then $\mathrm{EM}_\tau(u, \Phi)$ is also universal over a model containing $\mathrm{EM}_\tau(\alpha, \Phi)$, and we can apply Lemma 4.3. □

We can now prove that categoricity in $\mu^{(n+1)}$ implies Φ-nice μ^{+m}-superstability for every $m \leq n$:

Theorem 4.7. Let $n < \omega$ and assume that $\mathbf{K}$ is nicely stable in μ^{+m} for every $m \leq n$. Let Φ be an EM blueprint with $|\tau(\Phi)| \leq \mu$. If there exists an ordinal γ such that $\mathrm{EM}_\tau(\gamma, \Phi)$ is $(\mu^{+n}, \mu^{+(n+1)})$-limit (so in particular if $\mathbf{K}$ is categorical in $\mu^{+(n+1)}$), then $\mathbf{K}$ is Φ-nicely superstable in μ^{+m} for every $m \leq n$.

Proof. Let $m \leq n$. By Lemma 4.6 (with μ there standing for μ^{+m} here), $\mathbf{K}$ is Φ-nicely superstable in μ^{+m}, as desired. $\square$

As a corollary, we can characterize Φ-nice μ-superstability in terms of the behavior of EM models in μ^+:

Corollary 4.8. Let $\mu \geq \mathrm{LS}(\mathbf{K})$ and let Φ be an EM blueprint for $\mathbf{K}$ with $|\tau(\Phi)| \leq \mu$. The following are equivalent:

(1) $\mathbf{K}$ is Φ-nicely superstable in μ.
(2) $\mathbf{K}$ is nicely stable in μ and $\mathrm{EM}_\tau(\mu^+, \Phi)$ is (μ, μ^+)-limit.
(3) $\mathbf{K}$ is nicely stable in μ and $\mathrm{EM}_\tau(\gamma, \Phi)$ is (μ, μ^+)-limit for some $\gamma \in [\mu^+, \mu^{++})$.

Proof. (1) implies (2) is straightforward from the definition (recalling Lemma 4.2), and (2) implies (3) is trivial. Finally, (3) implies (1) is given by Theorem 4.7 (used with $n := 0$). $\square$

We also obtain that Φ-nice μ-superstability implies the notion of superstability derived from categoricity in [SV99, Theorem 2.2.1] (see e.g. the definition in [Van, Definition 4]; this will only be used marginally, so we do not repeat it here):

Corollary 4.9. If $\mathbf{K}$ is Φ-nicely superstable in μ, then $\mathbf{K}$ is superstable in μ (in the sense of [Van, Definition 4]).

Proof. It follows from Corollary 4.8 that $M := \mathrm{EM}_\tau(\mu^+, \Phi)$ is universal for models in $\mathbf{K}_{[\mu,\mu^+]}$ (i.e. every element in $\mathbf{K}_{[\mu,\mu^+]}$ embeds into M). Therefore by the proof of [SV99, Theorem 2.2.1] (see [BGVV] for more details), $\mathbf{K}$ is superstable in μ. $\square$

We now move toward the main result of this section: the consequences of Φ-nice μ-superstability on the uniqueness of limit models:

Lemma 4.10. Let $n \in [1, \omega)$ and assume that $\mathbf{K}$ is nicely stable in μ^{+m} for every $m \leq n$. Assume also that $\mathbf{K}$ is Φ-nicely superstable in

μ^+. If for every linear order I of cardinality $\mu^{+(n+1)}$, $\mathrm{EM}_\tau(I, \Phi)$ is dense with μ-amalgamation bases and saturated over μ-amalgamation bases, then any two limit models in $\mathbf{K}_{\mu^+}$ are isomorphic.

Proof. Let $\delta < \mu^{++}$ be a limit ordinal. Using the definition of nice stability in (μ^+, Φ), we can pick $\alpha \in [\mu^+, \mu^{++})$ such that $M := \mathrm{EM}_\tau(\alpha, \Phi)$ is (μ^+, δ)-limit. We show that M is also (μ^+, μ^+)-limit, which is enough by uniqueness of limit models of the same length. By Facts 3.2 and 3.4, it is enough to show that M is dense with μ-amalgamation bases and saturated over μ-amalgamation bases.

First, M is dense with μ-amalgamation bases by Fact 3.3. It remains to see saturation. We proceed as in the proof of [Bal09, Lemma 10.11] (full amalgamation is assumed there, so we repeat the argument here to convince the reader that this is not needed). Let $J_0 := \mu^+$, $J := \alpha$. Let $\lambda := \mu^{+(n+1)}$. Insert a copy J_0' of λ in J directly after the end of J_0. This gives a new linear order $I \supseteq J$ of size λ. By assumption we have that $N := \mathrm{EM}_\tau(I, \Phi)$ is dense with μ-amalgamation bases and saturated over μ-amalgamation bases. Let $M_0 \in \mathbf{K}_\mu$ be an amalgamation base with $M_0 \leq_{\mathbf{K}} M$ and let $p \in \mathrm{gS}(M_0)$. We show that p is realized in M. We know that p is realized in N, say by a.

Write $a = \rho(\bar{i}, \bar{j})$, where ρ is a $\tau(\Phi)$-term, $\bar{i} = (i_0, \ldots, i_{n-1})$ is a strictly increasing tuple from J, and $\bar{j} = (j_0, \ldots, j_{m-1})$ is a strictly increasing tuple from J_0' (the copy of λ).

Pick $X \subseteq J$ such that $|X| \leq \mu$ and $|M_0| \cup \mathrm{ran}(\bar{i}) \subseteq |\mathrm{EM}_\tau(X, \Phi)|$. Now by cofinality consideration, one can pick an increasing tuple $\bar{j}'$ in J_0 such that $X\bar{i}\bar{j}$ and $X\bar{i}\bar{j}'$ have the same order type. In particular, letting $b := \rho(\bar{i}, \bar{j}')$, $p = \mathrm{gtp}(a/M_0; N) = \mathrm{gtp}(b/M_0; M)$ so M realizes p, as desired. $\square$

We have arrived to the main theorem:

Theorem 4.11. Let $n \in [1, \omega)$ and assume that $\mathbf{K}$ is nicely stable in μ^{+m} for every $m \leq n$. Let Φ be an EM blueprint with $|\tau(\Phi)| \leq \mu$. If for every linear order I of cardinality $\mu^{+(n+1)}$, $\mathrm{EM}_\tau(I, \Phi)$ is $(\mu^{+n}, \mu^{+(n+1)})$-limit (so in particular if $\mathbf{K}$ is categorical in $\mu^{+(n+1)}$), then for any $m \in [1, n]$ and any $M_0, M_1, M_2 \in \mathbf{K}_{\mu^{+m}}$, if M_1 and M_2 are both limit over M_0, then $M_1 \cong_{M_0} M_2$.

Proof. By Theorem 4.7, $\mathbf{K}$ is Φ-nicely superstable in μ^{+m} for every $m \leq n$. Now fix $m \in [1, n]$. By Lemma 4.10 (where μ, n there stand for $\mu^{+(m-1)}, n-(m-1)$ here), any two limit models in $\mathbf{K}_{\mu^{+m}}$ are isomorphic. Now why can we find an isomorphism that also fixes the base?

This follows from an earlier result of the second author, more specifically from the proof of [Van, Theorem 1]. Note that the superstability hypothesis holds by Corollary 4.9. □

A more set-theoretic version of the previous theorem is:

Corollary 4.12. Let $n \in [1, \omega)$ and let $\mu \geq \mathrm{LS}(\mathbf{K})$. Let $\lambda := \mu^{+(n+1)}$. Assume that $\mathrm{CHWD}_{[\mu, \lambda)}$ holds (see Notation 2.1).

Assume that $\mathbf{K}$ has arbitrarily large model, is categorical in λ, and $\mathbf{K}_{<\lambda}$ has no maximal models. For any $m \in [1, n]$, and any $M_0, M_1, M_2 \in \mathbf{K}_{\mu^{+m}}$, if M_1 and M_2 are both limit over M_0, then $M_1 \cong_{M_0} M_2$.

Proof. By Fact 2.10, $\mathbf{K}$ is nicely stable in μ^{+m} for every $m \leq n$. Now apply Theorem 4.11. □

5. A categoricity transfer

This section aims to prove a categoricity transfer in AECs with no maximal models (using non-ZFC set-theoretic hypotheses). We will rely on the theory of good frames developed in [She09], and assume some familiarity with it (although we will not rely on the exact definitions, so the reader can simply black box terms like "good frames" or "successful" and check the facts that we quote).

We say that an AEC $\mathbf{K}$ has a *good μ-frame* $\mathfrak{s}$ (or that $\mathfrak{s}$ *is a good μ-frame on* $\mathbf{K}$) if there is a good μ-frame with underlying class $\mathbf{K}_\mu$ (see [She09, Definition II.2.1]). In this paper, all the good frames that we consider will be type-full (i.e. their basic types are all the nonalgebraic types), so we will just say "good frame" instead of "type-full good frame". Note that if $\mathbf{K}$ has a good μ-frame, then $\mathbf{K}_\mu \neq \emptyset$, $\mathbf{K}_\mu$ has amalgamation, joint embedding, no maximal models, and $\mathbf{K}$ is stable in μ. Thus in particular, $\mathbf{K}$ is nicely stable in μ.

A good frame $\mathfrak{s}$ is *weakly successful* if it has the existence property for uniqueness triples and it is *successful* if it is weakly successful and in addition a certain ordering $\leq^*_{\lambda^+}$ is well-behaved (see [She09, Definition III.1.1]). Given a successful good μ-frame $\mathfrak{s}$ on the AEC $\mathbf{K}$, Shelah shows [She09, Definition III.1.7] how to define the successor $\mathfrak{s}^+$ of $\mathfrak{s}$: it is a good μ^+-frame on an AEC $\mathbf{K}^*$ with underlying class the μ^+-saturated models in $\mathbf{K}$ and with ordering $\leq^*_{\lambda^+}$ mentioned earlier in this paragraph. Shelah also says that $\mathfrak{s}$ is *good*$^+$ if it satisfies a certain technical condition [She09, Definition III.1.3], which turns out to be equivalent to saying that $\leq^*_{\lambda^+}$ is just $\leq_{\mathbf{K}}$ on the μ^+-saturated models (see [BVa, Fact 2.15]). Therefore in case $\mathfrak{s}$ is successful good$^+$, $\mathbf{K}^*$ will be the AEC of μ^+-saturated models of $\mathbf{K}$ (ordered by the restriction of $\leq_{\mathbf{K}}$). In [She09, Claim III.1.9], Shelah shows that if $\mathfrak{s}$ is a successful good μ-frame, then $\mathfrak{s}^+$ is a good$^+$ μ-frame. Thus the order of the class need only change once.

It is natural to ask whether $\mathfrak{s}^+$ is also successful. If this is the case, we can take the successor of $\mathfrak{s}^+$, obtaining a new frame $\mathfrak{s}^{++}$. Shelah [She09, Definition III.1.12] defines by induction what it means for a good frame to be *n-successful*, and what $\mathfrak{s}^{+n}$, the nth successor of $\mathfrak{s}$ would then be. This is done so that $\mathfrak{s}$ is $(n+1)$-successful if and only if $\mathfrak{s}^{+n}$ is successful. Finally, $\mathfrak{s}$ is said to be *ω-successful* if it is n-successful for all $n < \omega$.

At the end of [She09, Chapter III], Shelah announces the following result. A proof should appear in [She]:

Claim 5.1. Assume $2^{\mu^{+n}} < 2^{\mu^{+(n+1)}}$ for all $n < \omega$. Let $\mathfrak{s}$ be an ω-successful good$^+$ μ-frame on $\mathbf{K}$. If $\mathbf{K}$ is categorical in μ and in *some* $\lambda > \mu^{+\omega}$, then $\mathbf{K}$ is categorical in *all* $\lambda' \geq \mu$.

Remark 5.2. The case $\lambda = \mu^{+\omega}$ can probably also be dealt with, but as the Claim as not yet been proven, we prefer to avoid introducing more variations. Further, here we really care only about eventual categoricity so the “low” cases are peripheral for us.

We want to use Claim 5.1 in AECs with no maximal models. Thus we have to build an ω-successful good$^+$ frame. We will use several facts. The following appears in the proof of [She09, Theorem IV.7.12] (see $\odot_4$ there). A detailed proof is in [Vasb, Theorem E.8].

Fact 5.3. Assume $2^{\mu} < 2^{\mu^+}$. Let $\mathfrak{s}$ be a good μ-frame on $\mathbf{K}$. If $\mathbf{K}$ is categorical in μ and for any saturated $M \in \mathbf{K}_{\mu^+}$ there exists $N \in \mathbf{K}_{\mu^+}$ universal over M, then $\mathfrak{s}$ is weakly successful.

The next fact is due to Adi Jarden [Jar16, Theorem 7.9] (see the proof of [Vasa, Fact 3.3] for why the hypotheses here implies those in Jarden's paper). To state it, we need a definition:

Definition 5.4.

(1) We say that a Galois type p is *μ-tame* if it is determined by its μ-sized restrictions, i.e. if $p \neq q$, then there exists $M \in \mathbf{K}_{\leq \mu}$ such that $p \restriction M \neq q \restriction M$.
(2) We say that $\mathbf{K}$ *is μ-tame* if for any $M \in \mathbf{K}$, any $p \in \text{gS}(M)$ is μ-tame. We say that $\mathbf{K}$ is *tame* if it is μ-tame for some μ.

(3) When $\mathbf{K}$ is nicely stable in μ, we say that $\mathbf{K}$ is (μ, μ^+)-*weakly tame* if for every (μ, μ^+)-limit model M, every Galois type over M is μ-tame.

Remark 5.5. Tameness was isolated by Grossberg and VanDieren [GV06b] and has been studied in numerous papers. See [BVb] for a survey.

Fact 5.6. Let $\mathfrak{s}$ be a good μ-frame on $\mathbf{K}$. If:

(1) $\mathbf{K}$ is categorical in μ.
(2) $\mathfrak{s}$ is weakly successful.
(3) Every saturated model in $\mathbf{K}_{\mu^+}$ is an amalgamation base.
(4) $\mathbf{K}$ is (μ, μ^+)-weakly tame.

Then $\mathfrak{s}$ is successful good$^+$.

We will use the following corollary of the two previous facts:

Lemma 5.7. Assume $2^\mu < 2^{\mu^+}$. Let $\mathfrak{s}$ be a good μ-frame on $\mathbf{K}$. If $\mathbf{K}$ is categorical in μ, nicely stable in μ^+, and (μ, μ^+)-weakly tame, then $\mathfrak{s}$ is successful good$^+$.

Proof. The hypotheses of Facts 5.3 and 5.6 hold. $\square$

In [She09, Chapter IV], Shelah introduces a way to build good frames from categoricity in very high cardinals. The following notion is key:

Definition 5.8. Let $\mathbf{K}^*$ be a class of τ-structures, ordered by a partial order $\leq_{\mathbf{K}^*}$. Let θ be an infinite cardinal. We say that $\mathbf{K}^*$ *is* $\mathbb{L}_{\infty,\theta}$-*syntactically characterizable* if for any $M, N \in \mathbf{K}^*$, $M \leq_{\mathbf{K}^*} N$ implies $M \preceq_{\mathbb{L}_{\infty,\theta}} N$.

Note that Kueker [Kue08, Theorem 7.2] and Shelah [She09, Fact IV.1.10] (independently) show that, if $\mathbf{K}$ is an AEC and $\theta > \text{LS}(\mathbf{K})$, then $M \preceq_{\mathbb{L}_{\infty,\theta}} N$ implies $M \leq_{\mathbf{K}} N$. Further, a straightforward directed system argument (see the proof of [BGL$^+$16, Theorem 6.8]) shows that if $\mathbf{K}_\mu$ is $\mathbb{L}_{\infty,\theta}$-syntactically characterizable, then $\mathbf{K}_{\geq\mu}$ also is.

Shelah shows [She09, Claim IV.1.12] that if $\lambda = \lambda^{<\theta}$ and $\mathbf{K}$ is categorical in λ, then $\mathbf{K}_\lambda$ is $\mathbb{L}_{\infty,\theta}$-syntactically characterizable. In [She09, Section IV.2], he asserts that an AEC categorical in sufficiently-many cardinals will eventually be $\mathbb{L}_{\infty,\text{LS}(\mathbf{K})^+}$-syntactically characterizable. We do not need this here, because when we assume nice stability matters are easier:

Lemma 5.9. If $\mathbf{K}$ is nicely stable in μ and categorical in μ, then $\mathbf{K}_\mu$ is $\mathbb{L}_{\infty,\mu}$-syntactically characterizable.

Proof. Let ϕ be an $\mathbb{L}_{\infty,\mu}$-formula. We prove by induction on ϕ that for any $M, N \in \mathbf{K}_\mu$ with $M \leq_{\mathbf{K}} N$ and any $\bar{a} \in {}^{<\mu}|M|$, $M \models \phi[\bar{a}]$ if and only if $N \models \phi[\bar{a}]$.

So fix $M, N \in \mathbf{K}_\mu$ with $M \leq_{\mathbf{K}} N$ and $\bar{a} \in {}^{<\mu}|M|$. If ϕ is atomic, the result is clear (because by the axioms of AECs, $M \leq_{\mathbf{K}} N$ implies $M \subseteq N$). If ϕ is a conjunction or a negation, the result follows from the induction hypothesis so assume that $\phi(\bar{y}) = \exists \bar{x} \psi(\bar{x}, \bar{y})$. If $M \models \phi[\bar{a}]$, then by the induction hypothesis we immediately have $N \models \phi[\bar{a}]$. Assume now that $N \models \phi[\bar{a}]$, say $N \models \psi[\bar{b}, \bar{a}]$. Let $\chi := |\ell(\bar{a})|^+ + \aleph_0$. Note that $\chi \leq \mu$. By categoricity, we know that M is a (μ, χ)-limit model, as witnessed by $\langle M_i : i \leq \chi \rangle$. Pick $i < \chi$ such that $\bar{a} \in {}^{<\mu}|M_i|$. We have that M is universal over M_i, so there exists $f : N \underset{M_i}{\longrightarrow} M$. By invariance, $f[N] \models \psi[f(\bar{b}), \bar{a}]$. By the induction hypothesis (applied to $f[N]$ and M), $M \models \psi[f(\bar{b}), \bar{a}]$. Therefore $f(\bar{b})$ witnesses that $M \models \phi[\bar{a}]$, as desired. $\square$

We will work with the following class:

Definition 5.10. Let $\mathbf{K}$ be AEC with arbitrarily large models and let Φ be an EM blueprint for $\mathbf{K}$. $\mathbf{K}^\Phi$ is the class of models $M \in \mathbf{K}$ such that there exists a linear order I with $\mathrm{EM}_\tau(I, \Phi) \cong M$. We order it with the restriction of $\leq_{\mathbf{K}}$.

Note that $\mathbf{K}^\Phi$ may not be an AEC. We will use the following powerful fact. To simplify the notation, we will use the following terminology:

Definition 5.11. We say a cardinal μ is a *nice fixed point* if both $\mathrm{cf}(\mu) = \aleph_0$ and for unboundedly-many $\chi < \mu$, $\chi = \beth_\chi$.

Remark 5.12. If μ is a nice fixed point, then $\mu = \beth_\mu$.

Fact 5.13 ([She09], Theorem IV.4.10)**.** Let $\mathrm{LS}(\mathbf{K}) < \mu < \lambda$ be given, with μ a nice fixed point. Let Φ be an EM blueprint with $|\tau(\Phi)| < \mu$. Suppose that:

(1) $\mathbf{K}$ is categorical in λ.
(2) $\mathbf{K}_\lambda$ is $\mathbb{L}_{\infty,\mu}$-syntactically characterizable.

Then there is a good μ-frame $\mathfrak{s}$ with underlying class $\mathbf{K}^\Phi_\mu$. Moreover $\mathbf{K}^\Phi_\mu$ is categorical in μ.

In particular, this says that $\mathbf{K}^\Phi_\mu$ is an AEC in μ.

We can combine these facts to build an ω-successful frame in any tame categorical AEC that is nicely stable everywhere. This is a generalization of the corresponding result in AECs with full amalgamation [She09, Theorem IV.7.12]:

Theorem 5.14. Let $\mathrm{LS}(\mathbf{K}) < \mu < \lambda$ be given, with μ a nice fixed point. Assume:

(1) $\mathbf{K}$ is categorical in λ.
(2) $\mathbf{K}$ is nicely stable in every $\theta \in [\mu, \lambda]$.

Then for any EM blueprint Φ for $\mathbf{K}$ with $|\tau(\Phi)| < \mu$, $\mathbf{K}^\Phi_\mu$ is categorical in μ and there is a type-full good μ-frame $\mathfrak{s}$ with underlying class $\mathbf{K}^\Phi_\mu$. Moreover:

(1) If $2^\mu < 2^{\mu^+}$, then $\mathfrak{s}$ is weakly successful.
(2) If $\lambda \geq \mu^{+n}$ for $n \in [1, \omega)$ and for all $m < n$, $2^{\mu^{+m}} < 2^{\mu^{+(m+1)}}$ and $\mathbf{K}$ is $(\mu^{+m}, \mu^{+(m+1)})$-weakly tame, then $\mathfrak{s}$ is n-successful good$^+$.
(3) If in addition to the hypotheses of (2) holding for all $n < \omega$, Claim 5.1 holds and $\lambda > \mu^{+\omega}$, then $\mathbf{K}$ is categorical in all $\lambda' \geq \min(\lambda, \beth_{(2^\mu)^+})$.

Proof. By Lemma 5.9, $\mathbf{K}_\lambda$ is $\mathbb{L}_{\infty,\lambda}$-syntactically characterizable (and hence $\mathbb{L}_{\infty,\mu}$-syntactically characterizable). Fix an EM blueprint Φ with $|\tau(\Phi)| < \mu$. By Fact 5.13, we obtain the desired good λ-frame $\mathfrak{s}$ on $\mathbf{K}^\Phi_\mu$.

Now let $\mathbf{K}^*$ denote the AEC generated by $\mathbf{K}^\Phi_\mu$ (this need not be the same as $\mathbf{K}^\Phi$). Note that the saturated model of size μ^+ in $\mathbf{K}^*$ is a (μ, μ^+)-limit (in $\mathbf{K}^*$ and therefore in $\mathbf{K}$). Therefore $\mathbf{K}^*$ is also nicely stable in μ^+ (and weakly (μ, μ^+)-tame if $\mathbf{K}$ is). Moreover if $\mathfrak{s}$ happens to be successful good$^+$, then the underlying AEC of $\mathfrak{s}^+$ will be the AEC generated by the (μ, μ^+)-limits of $\mathbf{K}$.

(1) Apply Fact 5.3.
(2) By Lemma 5.7 (recalling that $\mathbf{K}^*$ is nicely stable in μ^+ and weakly (μ, μ^+)-tame), $\mathfrak{s}$ is successful good$^+$. Now if $n > 1$ apply this again to $\mathfrak{s}^+$ (recalling that the underlying class of $\mathfrak{s}^+$ is the class of (μ, μ^+)-limits) and continue (formally we should use induction).

(3) By the previous part, $\mathfrak{s}$ is ω-successful good$^+$. By Claim 5.1 applied to the frame $\mathfrak{s}^+$, the class $\mathbf{K}^{**}$ of (μ, μ^+)-limits is categorical in all $\lambda' \geq \mu^+$. Now by a standard omitting type argument (see for example [Bal09, Lemma 14.2]), it follows that $\mathbf{K}$ is categorical in all $\lambda' \geq \min(\lambda, \beth_{(2^\mu)^+})$.

$\square$

Note that while (in part (2) and therefore in part (3)) we assume tameness, we do *not* assume full amalgamation (just nice stability). We had initially hoped to prove the result without using tameness at all (deriving the amount we need from the assumption) but have been unable to do so (however tameness is only used at one step).

Question 5.15. Can one derive the tameness hypothesis from the other assumptions in (2) of Theorem 5.14?

Towards this question, we note the following obvious consequence of syntactic characterization (Definition 5.8). If the implication could be reversed, at least for sufficiently large, saturated models, then we would have an affirmative answer.

Remark 5.16. Suppose that $\mathbf{K}$ is $\mathbb{L}_{\infty,\theta}$-syntactically characterizable. Then:

$$\mathrm{gtp}(a/M; N_1) = \mathrm{gtp}(b/M; N_2) \Rightarrow \mathrm{tp}_{\mathbb{L}_{\infty,\theta}}(a/M; N_1) = \mathrm{tp}_{\mathbb{L}_{\infty,\theta}}(b/M; N_2)$$

We can now state and prove the main result of this section:

Corollary 5.17. Assume Claim 5.1. Let $\mathrm{LS}(\mathbf{K}) < \mu < \mu^{+\omega} < \lambda_1 < \lambda_2$ be given, with μ a nice fixed point. Assume that $\mathrm{CHWD}_{[\mu,\lambda_1]}$ holds (recall Notation 2.1).

If:

(1) $\mathbf{K}_{<\lambda_2}$ has no maximal models.
(2) $\mathbf{K}$ is categorical in both λ_1 and λ_2.
(3) $\mathbf{K}$ is μ-tame.

Then $\mathbf{K}$ is categorical in *all* $\lambda' \geq \min(\lambda_1, \beth_{(2^\mu)^+})$.

Proof. Since $\lambda_1 \geq \mu \geq \beth_{\left(2^{\mathrm{LS}(\mathbf{K})}\right)^+}$ and $\mathbf{K}$ has a model of cardinality λ_1, $\mathbf{K}$ has arbitrarily large models. By Fact 2.10, $\mathbf{K}$ is nicely stable in every $\mu' \in [\mu, \lambda_1]$. Now apply Theorem 5.14.(3), where λ there stands for λ_1 here. $\square$

Corollary 5.18. Assume Claim 5.1 and GCHWD hold (see Notation 2.1). If $\mathbf{K}$ is tame, has no maximal models, and is categorical in a proper class of cardinals, then $\mathbf{K}$ is categorical on a tail of cardinals.

Proof. Follows directly from Corollary 5.17. □

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