MATH 269X - MODEL THEORY FOR ABSTRACT ELEMENTARY CLASSES, SPRING 2018 LECTURE NOTES

SEBASTIEN VASEY

CONTENTS

1. Notation

We use the letter τ for a vocabulary, K for a class of structures. For M a τ-structure, we write $|M|$ for its universe and $\|M\|$ for the cardinality of its universe. We often abuse notation and write for example $a \in M$ instead of $a \in |M|$. We write $M \subseteq N$ for M is a substructure of N.

For I, A sets, we let $^I A$ be the set of functions from I to A (we think of them as I-indexed sequences of elements of A). We write $\bar{a}$ for a sequence of elements. We write $^{<\infty} A$ for $\bigcup_\kappa {}^\alpha A$, where α ranges over all ordinals. For $\bar{a} \in {}^I A$ and $I_0 \subseteq I$, we write $\bar{a} \upharpoonright I_0$ for the restriction of $\bar{a}$ to I_0, $\ell(\bar{a}) = I$ (usually used when I is an ordinal), $\mathrm{dom}(\bar{a}) = I$, and $\mathrm{ran}(\bar{a})$ be the range of $\bar{a}$: the set of elements in the sequence.

For λ a cardinal, we write $[A]^\lambda$ for the subsets of A of cardinality λ. Similarly, $[A]^{<\lambda}$ denotes the subsets of A of cardinality less than λ.

2. Universal classes

We start by studying a simple model-theoretic framework. It was first studied by Tarski under the assumption that the vocabulary is finite [Tar54].

Definition 2.1 (Tarski). A *universal class* is a class K of structures in a fixed vocabulary $\tau = \tau(K)$ that is fixed under isomorphisms, substructures, and unions of chains (according to the substructure relation).

Example 2.2. The class of all fields, of all locally finite groups, of all vector spaces over $\mathbb{Q}$ are universal classes. The class of all algebraically closed fields is not ($\mathbb{Q}$ is a subfield of $\mathbb{C}$ which is not algebraically closed).

In the definition, we could have required closure under directed unions instead of just unions of chains. However it turns out that this follows. This is due to Iwamura [Iwa44]:

Exercise 2.3. Let K be a universal class. Let $\langle M_i : i \in I \rangle$ be a directed (according to substructure) system in K. Then $\bigcup_{i \in I} M_i \in K$.

The following is an important basic result about universal classes. We will see it generalizes (in some sense) to AECs.

Definition 2.4. Call a τ-structure M *finitely-generated* if there exists a *finite* subset $A \subseteq |M|$ such that M is the closure of A under its functions.

Theorem 2.5. Let K be a universal class in a vocabulary τ and let M be a τ-structure. The following are equivalent:

    (1) $M \in K$.
    (2) $M_0 \in K$ for all finitely-generated substructures M_0 of M.

Proof. If $M \in K$, then by closure under substructure any substructure of it is in K as well. Conversely, if all finitely-generated substructures of M, then they form a directed system in K whose union is M, hence by Exercise 2.3 we have $M \in K$. $\square$

There is a correspondence between universal classes and classes axiomatized by universal sentences in infinitary logics. When the vocabulary is finitary (and relational), this was observed by Tarski [Tar54] (in this case universal classes correspond to classes of models of a universal first-order theory). Tarski's proof generalizes.

Definition 2.6. We call an $\mathbb{L}_{\infty,\omega}$-sentence *universal* if it is of the form $\forall x_0 \ldots \forall x_{n-1} \psi(\bar{x})$, where ψ is quantifier-free.

Theorem 2.7 (Tarski's presentation theorem)**.** Let K be a class of structures in some vocabulary τ. The following are equivalent:

(1) There is a set Γ of quantifier-free (first-order) types such that K is the class of all τ-structures omitting Γ.
(2) K is the class of models of a universal $\mathbb{L}_{\infty,\omega}$ theory.
(3) K is a universal class.

Proof.

- (1) implies (2): Assume that K is the class of τ-structures omitting Γ. For each $p(\bar{x}) \in \Gamma$, let ϕ_p be the sentence $\forall \bar{x} \bigvee_{\psi \in p} \neg\psi(\bar{x})$. Let $T := \{\phi_p \mid p \in \Gamma\}$. It is easy to check that K is the class of models of T.
- (2) implies (3): This is straightforward to check.
- (3) implies (1): Let K_0 be the class of τ-structures that are finitely generated and are *not* in K. For each $M_0 \in K_0$, let $p_{M_0}(\bar{x})$ be a type coding it. That is, for any N, if $N \models p[\bar{a}]$, then N is generated by $\bar{a}$ and $N \cong M_0$. Let $\Gamma := \{p_{M_0} \mid M_0 \in K_0\}$. We claim that K is the set of τ-structures omitting Γ. To see this, first notice that any member of K omits Γ by closure under substructure. Conversely, if M omits Γ, then any finitely-generated substructure of M omits Γ, hence is in K. By Theorem 2.5, $M \in K$.

$\square$

Remark 2.8. The proof of Tarski's presentation theorem shows that any universal class K is axiomatized by a universal $\mathbb{L}_{\left(2^{|\tau(\mathbf{K})|+\aleph_0}\right)^+,\omega}$ theory.

The following concept was somewhat implicit in Definition 2.4:

Definition 2.9. Let K be a universal class. For $M \in K$ and $A \subseteq |M|$, let $\mathrm{cl}^M(A)$ be the closure of A under the functions of M. Equivalently, $\mathrm{cl}^M(A)$ is the intersection of all $M_0 \subseteq M$ which contain A. Note that $\mathrm{cl}^M(A)$ is a substructure of M, hence is itself in K.

2.1. **Tameness in universal classes.** It is natural to ask how much of the compactness theorem is lost in the setup of universal classes. We have seen that locally finite groups are universal classes, so clearly we cannot expect the compactness theorem to hold in full generality. However, consider the following interesting consequence of compactness:

Exercise 2.10. Let T be a first-order theory. Let $\mathfrak{C}$ be a monster model for T (i.e. it is λ-saturated, where λ is much bigger than any of the other objects appearing in the statement). Let α be an ordinal and let $\bar{a}, \bar{b} \in {}^\alpha\mathfrak{C}$. The following are equivalent:

(1) $\mathfrak{C} \models \phi[\bar{a}] \leftrightarrow \phi[\bar{b}]$ for all first-order formulas ϕ.

(2) There exists an automorphism of $\mathfrak{C}$ taking $\bar{a}$ to $\bar{b}$.

In other words, syntactic first-order types contain the same information as "semantic" types (defined in terms of orbit of a monster model). Is there a version of such a statement for universal classes? Note that universal classes may fail the amalgamation property (e.g. locally finite groups do [Neu60]), so it may not be possible to build a monster model in this case. Further, first-order types are not the right notion here, since they are not necessarily preserved by substructure. Quantifier-free types should be used and we then have the following result, due to Will Boney, which appears in [Vas17c, 3.7].

Theorem 2.11 (Boney)**.** Let K be a universal class. Let $M_1, M_2 \in K$ and let $\bar{a}_\ell \in {}^\alpha M_\ell$, $\ell = 1, 2$. The following are equivalent:

(1) For any quantifier-free formula ϕ, $M_1 \models \phi[\bar{a}_1]$ if and only if $M_2 \models \phi[\bar{a}_2]$.
(2) There exists $f : \mathrm{cl}^{M_1}(\bar{a}_1) \cong \mathrm{cl}^{M_2}(\bar{a}_2)$ such that $f(\bar{a}_1) = \bar{a}_2$.

Proof. (2) implies (1) is obvious: quantifier-free formulas are preserved by taking substructures and isomorphisms. We show (1) implies (2). For each $I \subseteq \alpha$ and $\ell = 1, 2$, write $M_\ell^I := \mathrm{cl}^{M_\ell}(\bar{a}_\ell \restriction I)$. We will build by induction on $|I|$ maps $f_I : M_1^I \cong M_2^I$ such that $f_I(\bar{a}_1 \restriction I) = \bar{a}_2 \restriction I$. This will clearly be enough: take $I = \alpha$.

This is possible: for I finite, M_ℓ^I is coded by its quantifier-free type, hence such a map exists by equality of the quantifier-free types of $\bar{a}_1 \restriction I$ and $\bar{a}_2 \restriction I$. Now if $|I|$ is infinite, observe that for $I_0 \subseteq J_0 \subseteq I$ with $|I_0| + |J_0| < |I|$, $f_{I_0} \subseteq f_{J_0}$. This is because we know that $f_{J_0}(\bar{a}_1 \restriction I_0) = \bar{a}_2 \restriction I_0 = f_{I_0}(\bar{a}_1 \restriction I_0)$ and for any $b \in M_1^{I_0}$, $b = \sigma(\bar{a}_1 \restriction I_0)$, for σ a term (*this is the key feature of universal classes used in the proof*). Thus $f_{J_0}(b) = \sigma(f_{J_0}(\bar{a}_1 \restriction I_0)) = \sigma(f_{I_0}(\bar{a}_1 \restriction I_0)) = f_{I_0}(\sigma(\bar{a}_1 \restriction I_0)) = f_{I_0}(b)$. Therefore $f_I := \bigcup_{I_0 \subseteq I, |I_0| < |I|} f_{I_0}$ is a directed union of a system of isomorphisms, and therefore and isomorphism itself. By definition, it must take $\bar{a}_1 \restriction I$ to $\bar{a}_2 \restriction I$, as desired. $\qquad\square$

Remark 2.12. We could have added a parameter set A contained in both M_1 and M_2, but this is not needed: one can take $\bar{a}_1$ and $\bar{a}_2$ to include an enumeration of it.

We will later see that this result says in technical terms, than "universal classes are fully $(< \aleph_0)$-tame and short over the empty set". A little less formally, orbital types in universal classes are determined by their finite restrictions.

3. Abstract elementary classes and the presentation theorem

Not all elementary classes are universal (algebraically closed fields are one example). Thus the framework of universal classes is limited. Shelah introduced in the late 70s AECs as a semantic framework encompassing in particular classes of models of $\mathbb{L}_{\infty, \omega}(Q)$ (the paper that introduced them was [She87a], but Shelah lectured on them many years before 1987). We will first give the definition of an abstract class (due to Grossberg).

Definition 3.1. An *abstract class* is a pair $\mathbf{K} = (K, \leq_{\mathbf{K}})$, where K is a class of structures in a fixed vocabulary $\tau = \tau(\mathbf{K})$ and $\leq_{\mathbf{K}}$ is a partial order, $M \leq_{\mathbf{K}} N$

implies $M \subseteq N$, and both K and $\leq_{\mathbf{K}}$ respect isomorphisms. Any abstract class admits a notion of $\mathbf{K}$-*embedding*: these are functions $f : M \to N$ such that $f : M \cong f[M]$ and $f[M] \leq_{\mathbf{K}} N$. We sometimes think of $\mathbf{K}$ as the category whose objects are elements in K and whose morphisms are $\mathbf{K}$-embeddings.

We often do not distinguish between K and $\mathbf{K}$. For λ a cardinal, we will write $\mathbf{K}_\lambda$ for the restriction of $\mathbf{K}$ to models of cardinality λ. Similarly define $\mathbf{K}_{\geq \lambda}$ or more generally $\mathbf{K}_S$, where S is a class of cardinals. We will also use the following notation:

Notation 3.2. For $\mathbf{K}$ an abstract class and $N \in \mathbf{K}$, write $\mathcal{P}_{\mathbf{K}}(N)$ for the set of $M \in \mathbf{K}$ with $M \leq_{\mathbf{K}} N$. Similarly define $\mathcal{P}_{\mathbf{K}_\lambda}(N)$, $\mathcal{P}_{\mathbf{K}_{<\lambda}}(N)$, etc.

For an abstract class $\mathbf{K}$, we denote by $\mathbb{I}(\mathbf{K})$ the number of models in $\mathbf{K}$ up to isomorphism (i.e. the cardinality of $\mathbf{K}/{\cong}$). We write $\mathbb{I}(\mathbf{K}, \lambda)$ instead of $\mathbb{I}(\mathbf{K}_\lambda)$. When $\mathbb{I}(\mathbf{K}) = 1$, we say that $\mathbf{K}$ is *categorical*. We say that $\mathbf{K}$ is *categorical in* λ if $\mathbf{K}_\lambda$ is categorical, i.e. $\mathbb{I}(\mathbf{K}, \lambda) = 1$.

We say that $\mathbf{K}$ has *amalgamation* if for any $M_0 \leq_{\mathbf{K}} M_\ell$, $\ell = 1, 2$ there is $M_3 \in \mathbf{K}$ and $\mathbf{K}$-embeddings $f_\ell : M_\ell \to M_3$, $\ell = 1, 2$. $\mathbf{K}$ has *joint embedding* if any two models can be $\mathbf{K}$-embedded in a common model. $\mathbf{K}$ has *no maximal models* if for any $M \in \mathbf{K}$ there exists $N \in \mathbf{K}$ with $M \leq_{\mathbf{K}} N$ and $M \neq N$ (we write $M <_{\mathbf{K}} N$). Localized concepts such as *amalgamation in* λ mean that $\mathbf{K}_\lambda$ has amalgamation.

Definition 3.3 (Shelah)**.** An *abstract elementary class (AEC)* is an abstract class $\mathbf{K}$ in a finitary vocabulary satisfying:

(1) Coherence: if $M_0, M_1, M_2 \in \mathbf{K}$, $M_0 \subseteq M_1 \leq_{\mathbf{K}} M_2$ and $M_0 \leq_{\mathbf{K}} M_2$, then $M_0 \leq_{\mathbf{K}} M_1$.
(2) Tarski-Vaught axioms: if δ is a limit ordinal, $\langle M_i : i < \delta \rangle$ is a $\leq_{\mathbf{K}}$-increasing chain and $M := \bigcup_{i<\delta} M_i$, then:
  (a) $M \in \mathbf{K}$.
  (b) $M_j \leq_{\mathbf{K}} M$ for all $j < \delta$.
  (c) Smoothness: if $N \in \mathbf{K}$ is such that $M_i \leq_{\mathbf{K}} N$ for all $i < \delta$, then $M \leq_{\mathbf{K}} N$.
(3) Löwenheim-Skolem-Tarski (LST) axiom: there exists a cardinal $\lambda \geq |\tau(\mathbf{K})| + \aleph_0$ such that for any $N \in \mathbf{K}$ and any $A \subseteq |N|$, there exists $M \in \mathcal{P}_{\mathbf{K}_{\lambda+|A|}}(N)$ such that $A \subseteq |M|$ and $M \leq_{\mathbf{K}} N$. We write $\mathrm{LS}(\mathbf{K})$ for the least such λ.

Similarly to Exercise 2.3, the following holds:

Exercise 3.4. Let $\mathbf{K}$ be an AEC. Then the Tarski-Vaught axioms holds for directed systems. That is, let $\langle M_i : i \in I \rangle$ be a $\leq_{\mathbf{K}}$-directed system. Let $M := \bigcup_{i \in I} M_i$. Then:

(1) $M \in \mathbf{K}$.
(2) $M_i \leq_{\mathbf{K}} M$ for all $i \in I$.
(3) Smoothness: if $N \in \mathbf{K}$ is such that $M_i \leq_{\mathbf{K}} N$ for all $i \in I$, then $M \leq_{\mathbf{K}} N$.

Example 3.5.

(1) $\mathbf{K} = (\mathrm{Mod}(T), \preceq)$, where T is any first-order theory, is an AEC with $\mathrm{LS}(\mathbf{K}) = |\tau(T)| + \aleph_0$.

(2) $\mathbf{K} = (K, \subseteq)$, where K is a universal class, is an AEC with $\mathrm{LS}(\mathbf{K}) = |\tau(K)| + \aleph_0$. We may abuse notation and call also such a $\mathbf{K}$ a universal class (or even a universal AEC).

(3) $\mathbf{K} = (\mathrm{Mod}(\psi), \preceq_\Phi)$, where $\psi \in \mathbb{L}_{\infty,\omega}$ and Φ is a fragment containing ψ, is an AEC with $\mathrm{LS}(\mathbf{K}) \leq |\Phi| + |\tau(\psi)| + \aleph_0$.

(4) For a fixed infinite cardinal λ, the class of well-orderings of type at most λ^+ ordered by being an initial segment is an AEC $\mathbf{K}$ with $\mathrm{LS}(\mathbf{K}) = \lambda$.

(5) The class of well-orderings ordered by being an initial segment is not an AEC (it fails the Löwenheim-Skolem-Tarski axiom).

(6) The class of well-orderings ordered by being a subordering is not an AEC (it fails to be closed under chains).

(7) See more examples in [BV17a, §3].

How are AECs related to universal classes? The following result of Shelah says that any AEC is the reduct of a universal class [She09a, 1.9(1)] (the presentation we give combines [Vas17c, §2] and [LRVc, 6.4]):

Theorem 3.6 (Shelah's presentation theorem)**.** Let $\mathbf{K}$ be an AEC with vocabulary $\tau = \tau(\mathbf{K})$. Then there exists a universal class $\mathbf{K}^+$ in an expansion τ^+ of τ with $|\tau^+| = \mathrm{LS}(\mathbf{K})$ and such that the reduct map is a faithful functor from $\mathbf{K}^+$ into $\mathbf{K}$ which is surjective on objects. In other words:

(1) For any $M \in \mathbf{K}$, there exists $M^+ \in \mathbf{K}^+$ such that $M^+ \restriction \tau = M$.

(2) For any $M^+ \subseteq N^+$ both in $\mathbf{K}^+$, letting $M := M^+ \restriction \tau$, $N := N^+ \restriction \tau$, we have that $M, N \in \mathbf{K}$ and $M \leq_{\mathbf{K}} N$.

Corollary 3.7. For any AEC $\mathbf{K}$, there exists a universal $\mathbb{L}_{\left(2^{\mathrm{LS}(\mathbf{K})}\right)^+,\omega}$-sentence ψ in an expansion of $\tau(\mathbf{K})$ such that the models in $\mathbf{K}$ are exactly the $\tau(\mathbf{K})$-reducts of models of ψ.

Proof. By Theorem 3.6, Tarski's presentation Theorem 2.7, and Remark 2.8. $\square$

To prove Theorem 3.6, the following notion will be useful [Vas17c, 2.9]:

Definition 3.8. Let $\mathbf{K}$ be an abstract class and let $N \in K$. We say $\mathcal{F}$ is a *set of Skolem functions for N* if:

(1) $\mathcal{F}$ is a non-empty set, and each element f of $\mathcal{F}$ is a function from N^n to N, for some $n < \omega$.

(2) For all $A \subseteq |N|$, $M := \mathcal{F}[A] := \bigcup \{f[A] \mid f \in \mathcal{F}\}$ is such that $M \leq_{\mathbf{K}} N$ and contains A.

Remark 3.9. Let $\mathbf{K}$ be an AEC, let $N \in \mathbf{K}$, $\mathcal{F}$ be a set of Skolem functions for N, and $A \subseteq |N|$. Then (by the smoothness axiom) the closure of A under the functions in $\mathcal{F}$ is also a $\mathbf{K}$-substructure of N containing A.

Lemma 3.10. Let $\mathbf{K}$ be an AEC. For any $N \in \mathbf{K}$, there exists a set $\mathcal{F}$ of Skolem functions for N with $|\mathcal{F}| = \mathrm{LS}(\mathbf{K})$.

Proof. We build $\langle N_s \mid s \in [N]^{<\aleph_0} \rangle$ such that for each $s, t \in [N]^{<\aleph_0}$:

(1) $N_s \in \mathcal{P}_{\mathbf{K}_{\leq \mathrm{LS}(\mathbf{K})}}(N)$.

(2) $s \subseteq |N_s|$.
(3) $s \subseteq t$ implies $N_s \leq_{\mathbf{K}} N_t$.

This is possible by inductive applications of the LST and coherence axioms. This is enough: for each $s \in [N]^{<\aleph_0}$, let $\{a_i^s : i < \mathrm{LS}(\mathbf{K})\}$ be an enumeration (possibly with repetitions) of N_s. Now for each $n < \omega$, each $i < \mathrm{LS}(\mathbf{K})$, and each $\bar{a} \in {}^n N$, we let $f_i^n(\bar{a})$ be $a_i^{\mathrm{ran}(\bar{a})}$. Let $\mathcal{F} := \{f_i^n : i < \mathrm{LS}(\mathbf{K}), n < \omega\}$. This is as desired: let $A \subseteq |N|$ and let $M := \mathcal{F}[A]$. Then it is easy to check that $M = \bigcup_{s \in [A]^{<\aleph_0}} N_s$. Note that $\langle N_s : s \in [A]^{<\aleph_0} \rangle$ is a directed system and since $N_s \leq_{\mathbf{K}} N$ for all s, it follows from the smoothness axiom that $M \leq_{\mathbf{K}} N$. $\qquad\square$

Proof of Theorem 3.6. Let τ^+ consist of $\tau \cup \{f_i^n : i < \mathrm{LS}(\mathbf{K}), n < \omega\}$, where f_i^n is a new function symbol of arity n. Let K^+ be class of τ^+-structures M^+ such that $M_0^+ \upharpoonright \tau \leq_{\mathbf{K}} M^+ \upharpoonright \tau$ for any $M_0^+ \subseteq M^+$. Let $\mathbf{K}^+ := (K^+, \subseteq)$. It is easy to check that $\mathbf{K}^+$ is a universal class and by definition, (2) is satisfied. To see (1), let $M \in \mathbf{K}$. By Lemma 3.10, M has a set of Skolem functions $\mathcal{F}$. Expand M to $M^+ := (M, g)_{g \in \mathcal{F}}$. Then by definition of Skolem functions, $M^+ \in \mathbf{K}^+$. $\qquad\square$

4. Abstract elementary classes with intersections

The following generalizes Definition 2.9:

Definition 4.1. For $\mathbf{K}$ an AEC, $N \in \mathbf{K}$ and $A \subseteq |N|$, let $\mathrm{cl}^N(A) := \bigcap \{M \in \mathbf{K} \mid M \leq_{\mathbf{K}} N, A \subseteq |M|\}$. We see it as a $\tau(\mathbf{K})$-substructure of N.

Exercise 4.2. Let $\mathbf{K}$ be an AEC, $M \leq_{\mathbf{K}} N$ be in $\mathbf{K}$, and $A, B \subseteq |N|$.

(1) Invariance: If $f : N \cong N'$, then $f[\mathrm{cl}^N(A)] = \mathrm{cl}^{N'}(f[A])$.
(2) Monotonicity 1: $A \subseteq \mathrm{cl}^N(A)$.
(3) Monotonicity 2: $A \subseteq B$ implies $\mathrm{cl}^N(A) \subseteq \mathrm{cl}^N(B)$.
(4) Monotonicity 3: If $A \subseteq |M|$, then $\mathrm{cl}^N(A) \subseteq \mathrm{cl}^M(A)$.
(5) Idempotence: $\mathrm{cl}^N(M) = M$ and $\mathrm{cl}^N(\mathrm{cl}^N(A)) = \mathrm{cl}^N(A)$.

The notion of having (or admitting) intersections is introduced for AECs in [BS08, 1.2] and further studied in [Vas17c, §2].

Definition 4.3. Let $\mathbf{K}$ be an abstract class, $N \in \mathbf{K}$, and $A \subseteq |N|$.

(1) We say that N *has intersections over* A if $\mathrm{cl}^N(A) \leq_{\mathbf{K}} N$.
(2) We say that N *has intersections* if it has intersections over all $A \subseteq |N|$.
(3) We say that $\mathbf{K}$ *has intersections* if all $N \in \mathbf{K}$ have intersections.

Remark 4.4. Formally, $\mathrm{cl}^N(A)$ also depends on $\mathbf{K}$ but usually $\mathbf{K}$ is clear from context. We may write $\mathrm{cl}_{\mathbf{K}}^N(A)$ to make $\mathbf{K}$ explicit.

Exercise 4.5. Let $\mathbf{K}$ be an AEC and let $N \in \mathbf{K}$. The following are equivalent:

(1) N has intersections.
(2) For any non-empty $S \subseteq \mathcal{P}_{\mathbf{K}}(N)$, $\bigcap S \leq_{\mathbf{K}} N$.

Example 4.6. Any universal class has intersections. Algebraically closed fields also have intersections. See more examples in [Vas17c, 2.6]. On the other hand, the

class of dense linear orderings without endpoints (ordered by suborder) does not have intersections. Indeed, working in side $(\mathbb{Q}, <)$, for each $n \in [1, \omega)$, $(\frac{-1}{n}, \frac{1}{n})_{\mathbb{Q}}$ is a dense linear ordering without endpoints, but the intersections is $\{0\}$ which has an endpoint. Now apply Exercise 4.5.

Definition 4.7. Let $\mathbf{K}$ be an AEC. Let $M \in K$ and let $A \subseteq |M|$ be a set. M is *minimal over A* if whenever $M \leq_{\mathbf{K}} N$ and $M' \leq_{\mathbf{K}} N$ contains A, then $M' = M$. M is *minimal over A in N* if $M \leq_{\mathbf{K}} N$ and this holds whenever $N' \leq_{\mathbf{K}} N$.

The following characterization of having intersections is [Vas17c, 2.11]:

Theorem 4.8. Let $\mathbf{K}$ be an AEC and let $N \in \mathbf{K}$. The following are equivalent:

(1) N admits intersections.
(2) There is an operator $\mathrm{cl} := \mathrm{cl}^N : \mathcal{P}(|N|) \to \mathcal{P}(|N|)$ such that for all $A, B \subseteq |N|$ and all $M \leq_{\mathbf{K}} N$:
   (a) $\mathrm{cl}(A) \leq_{\mathbf{K}} N$.
   (b) $A \subseteq \mathrm{cl}(A)$.
   (c) $A \subseteq B$ implies $\mathrm{cl}(A) \subseteq \mathrm{cl}(B)$.
   (d) $\mathrm{cl}(M) = M$.
(3) For each $A \subseteq |N|$, there is a unique minimal model over A in N.
(4) There is a set $\mathcal{F}$ of Skolem functions for N such that:
   (a) $|\mathcal{F}| \leq \mathrm{LS}(K)$.
   (b) For all $M \leq_{\mathbf{K}} N$, we have $\mathcal{F}[M] = M$.

Moreover the operator $\mathrm{cl}^N : \mathcal{P}(|N|) \to \mathcal{P}(|N|)$ with the properties in (2) is unique and if it exists then it has the following characterizations:

- $\mathrm{cl}^N(A) = \bigcap \{M \leq_{\mathbf{K}} N \mid A \subseteq |M|\}$.
- $\mathrm{cl}^N(A) = \mathcal{F}[A]$, for any set of Skolem functions $\mathcal{F}$ for N such that $\mathcal{F}[M] = M$ for all $M \leq_{\mathbf{K}} N$.
- $\mathrm{cl}^N(A)$ is the unique minimal model over A in N.

Proof.

- (1) implies (2): Let $\mathrm{cl}^N(A) := \bigcap \{M \leq_{\mathbf{K}} N \mid A \subseteq |M|\}$. Even without hypotheses on N, (2b), (2c), and (2d) are satisfied. Since N admits intersections, (2a) is also satisfied.
- (2) implies (3): Let $A \subseteq |N|$. Let cl be as given by (2). Let $M := \mathrm{cl}(A)$. By (2a), $M \leq_{\mathbf{K}} N$. By (2b), $A \subseteq |M|$. Moreover if $M' \leq_{\mathbf{K}} N$ contains A, then by (2c), $|M| \subseteq |\mathrm{cl}(M')|$ but by (2d), $\mathrm{cl}(M') = M'$. Thus by coherence and (2a) $M \leq_{\mathbf{K}} M'$. This shows both that M is minimal over A and that it is unique.
- (3) implies (4): We slightly change the proof of Lemma 3.10 as follows: in the construction of the N_s's, let N_s be the unique minimal model over s in N. Now let $\mathcal{F}$ be as obtained by the rest of the construction there. Let $A \subseteq |N|$. We claim that $\mathcal{F}[A]$ is minimal over A in N. This shows in particular that $\mathcal{F}$ is as required.
  Let $M := \mathcal{F}[A]$. Since $\mathcal{F}$ is a set of Skolem functions, $M \leq_{\mathbf{K}} N$ and M contains A. Moreover, $M = \bigcup_{s \in [A]^{<\aleph_0}} N_s$. Now if $M' \leq_{\mathbf{K}} N$ contains

A, then for all $s \in [A]^{<\aleph_0}$, $s \in [M']^{<\aleph_0}$, so as N_s is minimal over s in N, $N_s \leq_{\mathbf{K}} M'$. It follows that $M \leq_{\mathbf{K}} M'$, so $M = M'$.

- (4) implies (1): Let $\mathcal{F}$ be as given by (4). Let $A \subseteq |N|$. Let $M := \mathcal{F}[A]$. By definition of Skolem functions, M contains A and $M \leq_{\mathbf{K}} N$. We claim that $M = \bigcap\{M' \leq_{\mathbf{K}} N \mid A \subseteq |M'|\}$. Indeed, if $M' \leq_{\mathbf{K}} N$ contains A, then by the hypothesis on $\mathcal{F}$, $M = \mathcal{F}[A] \subseteq \mathcal{F}[M'] = M'$.

The moreover part follows from the arguments above. $\qquad\square$

Exercise 4.9 ([Vasa, 3.6])**.** Let $\mathbf{K}$ be an AEC. Show that if N has intersections for all $N \in \mathbf{K}_{\leq \mathrm{LS}(\mathbf{K})}$, then $\mathbf{K}$ has intersections.

We obtain the following properties of the closure operator, which complement Exercise 4.2.

Theorem 4.10. Let $\mathbf{K}$ be an AEC with intersections, let $M \leq_{\mathbf{K}} N$ and let $A \subseteq |M|$.

(1) Monotonicity 3: $\mathrm{cl}^M(A) = \mathrm{cl}^N(A)$.
(2) (Finite character) For any $b \in \mathrm{cl}^N(A)$, there exists a finite $A_0 \subseteq A$ such that $b \in \mathrm{cl}^N(A_0)$.

Proof. Finite character follows from the characterization of cl^N in terms of Skolem functions (Theorem 4.8). For monotonicity 3, let $M_0 := \mathrm{cl}^N(A)$. We have $M_0 \leq_{\mathbf{K}} N$ since N admits intersections over A. Since $M \leq_{\mathbf{K}} N$ contains A, we must have $|M_0| \subseteq |M|$. By coherence, $M_0 \leq_{\mathbf{K}} M$. Now M_0 is the unique minimal model over A in N, so it must be minimal in M as well, and hence $M_0 = \mathrm{cl}^M(A)$. $\qquad\square$

Remark 4.11. There is a generalization of Tarski's presentation Theorem 2.7 to AECs with intersections [BV].

5. μ-AECS AND ACCESSIBLE CATEGORIES

The following naturally generalizes the definition of an AEC to classes that are only closed under sufficiently directed unions:

Definition 5.1 ([BGL$^+$16, 2.2])**.** Let μ be a regular cardinal. A μ-*abstract elementary class* (or μ-*AEC* for short) is an abstract class $\mathbf{K}$ (where we allow here the vocabulary to be $(< \mu)$-ary) satisfying:

(1) Coherence: if $M_0, M_1, M_2 \in \mathbf{K}$, $M_0 \subseteq M_1 \leq_{\mathbf{K}} M_2$ and $M_0 \leq_{\mathbf{K}} M_2$, then $M_0 \leq_{\mathbf{K}} M_1$.
(2) Tarski-Vaught axioms: if $\langle M_i : i \in I \rangle$ is a μ-directed system (where I is μ-directed if every subset of I of size strictly less than μ has a least upper bound) and $M := \bigcup_{i \in I} M_i$, then:
  (a) $M \in \mathbf{K}$.
  (b) $M_i \leq_{\mathbf{K}} M$ for all $i \in I$.
  (c) Smoothness: if $N \in \mathbf{K}$ is such that $M_i \leq_{\mathbf{K}} N$ for all $i \in I$, then $M \leq_{\mathbf{K}} N$.

(3) Löwenheim-Skolem-Tarski (LST) axiom: there exists a cardinal $\lambda \geq |\tau(\mathbf{K})| +$
μ such that $\lambda = \lambda^{<\mu}$ and for any $N \in \mathbf{K}$ and any $A \subseteq |N|$, there exists
$M \in \mathcal{P}_{\mathbf{K}_{\lambda + |A|<\mu}}(N)$ such that $A \subseteq |M|$. We write $\mathrm{LS}(\mathbf{K})$ for the least such
λ.

Remark 5.2. Technically, $\mathrm{LS}(\mathbf{K})$ depends on μ, but this should not cause any
problems, so we remove this from the notation.

Note that, in contrast to Exercise 2.3, asking only that the class be closed under
chains of cofinality at least μ is a significantly weaker condition:

Exercise 5.3 ([AR94, 1.c.(2)]). For $n < \omega$, let P_n be the ordinal $\omega_n + 1$, ordered
as usual. Let $Q := \prod_{1 \leq n < \omega} P_n$ and let P be the subposet of Q consisting of those
sequences $(x_n)_{n<\omega}$ with only finitely many $n < \omega$ so that $x_n = \omega_n$.

(1) Check that Q is a complete lattice.
(2) Check that P is closed (in Q) under joins of chains of uncountable cofinality.
(3) Check that P is not closed under joins of $\aleph_1$-directed sets. *Hint*: Consider
$\prod_{1 \leq n < \omega} \omega_n$.

The coherence axiom also has the following stronger form:

Exercise 5.4. Show that the coherence axiom is equivalent to the following state-
ment: for $M_0, M_1, M_2 \in \mathbf{K}$ with $|M_0| \subseteq |M_1| \subseteq |M_2|$, if $M_0 \leq_{\mathbf{K}} M_2$ and $M_1 \leq_{\mathbf{K}}$
M_2, then $M_0 \leq_{\mathbf{K}} M_1$.

Example 5.5.

(1) AECs are exactly the $\aleph_0$-AECs.
(2) The class of well-orderings ordered by being a suborder is an $\aleph_1$-AEC.
(3) The class of well-founded models of ZFC, ordered by elementary substruc-
ture, is an $\aleph_1$-AEC.
(4) The class of well-orderings ordered by being an initial segment is *not* a
μ-AEC for any μ (the LST axiom fails).
(5) The class of all Banach spaces (ordered by being a closed subspace) is an
$\aleph_1$-AEC.
(6) The class of all μ-complete Boolean algebras (ordered by being a subal-
gebra) is a μ-AEC. However the class of all complete Boolean algebras is
not.
(7) The class of models of any $\mathbb{L}_{\infty,\mu}$ sentence can be made into a μ-AEC by
ordering it with elementarity according to a fragment.
(8) See more examples in [BGL$^+$16, §2].

Accessible categories were introduced by Lair [Lai81] (he called them "catégorie
modelable"). The standard textbooks on them are [MP89, AR94] (see also the
following basic references on category theory [AHS04, Lan98]). One can see them as
axiomatizing the category-theoretic essence of classes of models of $\mathbb{L}_{\infty,\infty}$ sentences:

Definition 5.6. Let $\mathcal{K}$ be a category and let λ be a regular cardinal.

(1) An object M is λ-*presentable* if its hom-functor $\mathcal{K}(M, -) : \mathcal{K} \to \mathrm{Set}$ pre-
serves λ-directed colimits. Put another way, M is λ-presentable if for any
morphism $f : M \to N$ with N a λ-directed colimit $\langle \phi_\alpha : N_\alpha \to N \rangle$ with

diagram maps $\phi_{\alpha\beta} : N_\alpha \to N_\beta$, f factors essentially uniquely through one of the N_α. That is, $f = \phi_\alpha f_\alpha$ for some $f_\alpha : M \to N_\alpha$, and if $f = \phi_\beta f_\beta$ as well, there is $\gamma > \alpha, \beta$ such that $\phi_{\gamma\alpha} f_\alpha = \phi_{\gamma\beta} f_\beta$.

(2) $\mathcal{K}$ is λ-*accessible* if it has λ-directed colimits and $\mathcal{K}$ contains a set S of λ-presentable objects such that every object of $\mathcal{K}$ is isomorphic to a λ-directed colimit of objects in S.

(3) $\mathcal{K}$ is *accessible* if it is λ'-accessible for some regular cardinal λ'.

Intuitively, an accessible category is a category with all sufficiently directed colimits and such that every object can be written as a highly directed colimit of "small" objects. Here "small" is interpreted in terms of *presentability*, a notion of size that makes sense in any (possibly non-concrete) category. In the category of sets, of course, a set is λ-presentable if and only if its cardinality is less than λ; in an AEC $\mathbf{K}$, the same is true for all regular $\lambda > \mathrm{LS}(\mathbf{K})$. More generally:

Exercise 5.7. Let $\mathbf{K}$ be a μ-AEC, let $\lambda = \lambda^{<\mu} \geq \mathrm{LS}(\mathbf{K})$, and let $M \in \mathbf{K}$. Show that M is λ^+-presentable if and only if $\|M\| \leq \lambda$.

When $\lambda < \lambda^{<\mu}$, presentability still gives a natural notion of size in several categories. For example, in Banach spaces it corresponds to the *density character* [LR17, 3.1].

From Exercise 5.7, it is easy to see the following:

Exercise 5.8. Prove that if $\mathbf{K}$ is a μ-AEC, then it is an $\mathrm{LS}(\mathbf{K})^+$-accessible category.

There are examples of accessible categories that are *not* (equivalent to) μ-AECs. The simplest one is the category of sets (where the morphisms are functions). The problem is that the morphisms need not be monomorphisms. If we assume that all morphisms are mono, then we will see (Theorem 5.21) that we do in some sense have a μ-AEC. Before proving this, we take a second look at presentability. First, we prove the following generalization of the fact that a small union of small sets is not too big:

Lemma 5.9. Let $\mathcal{K}$ be a λ-accessible category. Then any λ-directed colimit of at most θ-many λ-presentable objects is $(\theta + \lambda)^+$-presentable.

Proof. Let M be a λ-directed colimit $\langle \phi_i : M_i \to M, i \in I \rangle$, where $|I| \leq \theta$ and each M_i is λ-presentable. Let $\mu := (\theta + \lambda)^+$. Let $f : M \to N$ be a morphism, with N a μ-directed colimit of objects $\langle N_j : j \in J \rangle$. Let $f_i := f\phi_i$. By λ-presentability of M_i, f_i factors (essentially uniquely) through some N_{j_i}, $j_i \in J$. Now there are at most θ-many j_i's, so since J is μ-directed, there is $j \in J$ with $j_i \leq j$ for all $i \in I$. It follows that f must factor through N_j, showing that M is μ-presentable. $\qquad \square$

Recall that a *retract* is a map $f : M \to N$ such that there is $g : N \to M$ so that fg is the identity on N. We also say that N is a *retract* of M. In the category of sets, retracts are exactly the surjections. The following is easy to check:

Exercise 5.10. Prove that if $f_1 : M \to N_1$ and $f_2 : M \to N_2$ are retracts, as witnessed by g_1 and g_2, and $g_1 f_1 = g_2 f_2$, then N_1 and N_2 are isomorphic. Conclude that there is only a set (up to isomorphism) of retracts of any given object M.

The following follows from the definition of λ-presentability and playing with morphisms:

Exercise 5.11. Let $\mathcal{K}$ be a λ-accessible category and let S be a set of λ-presentable objects such that any object in $\mathcal{K}$ is a λ-directed colimit of members of S. Prove that any λ-presentable object is a retract of a member of S. Thus $\mathcal{K}$ has only a set (up to isomorphism) of λ-presentable objects. Conversely, show that a retract of a μ-presentable object is μ-presentable, for any regular $\mu \geq \lambda$.

Toward understanding presentability further, we prove a technical lemma saying when an object resolves into a sufficiently directed colimit. We will use the following definitions:

Definition 5.12. For μ a cardinal, μ^* is μ^+ if μ is successor, and μ if μ is limit.

Definition 5.13. For κ, μ infinite cardinals, we say that μ is κ-*closed* if $\theta^{<\kappa} < \mu$ for all $\theta < \mu$.

Definition 5.14. For λ an uncountable cardinal, we call an object M in a category $\mathcal{K}$ $(< \lambda)$-*presentable* if it is λ_0-presentable for some regular $\lambda_0 < \lambda$.

The following is given by the proof of [MP89, 2.3.10]. It is stated as [LRVb, 3.8].

Lemma 5.15. Let $\kappa < \mu \leq \lambda$ be cardinals with κ and μ regular and $\mathrm{cf}(\lambda) \geq \mu$. Let $\mathcal{K}$ be a category with κ-directed colimits. If $M \in \mathcal{K}$ is a κ-directed colimit of $(< \lambda)$-presentable objects and μ is κ-closed, then M is a μ-directed colimit of $(< \lambda + \mu^*)$-presentable objects.

Proof sketch. Suppose that M is a κ-directed colimit of the $(< \lambda)$-presentable objects $\langle M_i : i \in I \rangle$. Since μ is κ-closed, any subset of I of cardinality strictly less than μ is contained inside a κ-directed subset of I of cardinality strictly less than μ. Thus the set $\mathbb{P}$ of all κ-directed subsets of I of cardinality strictly less than μ is μ-directed. For $s \in \mathbb{P}$, let M_s be the colimit of the M_i's with $i \in s$. Now the induced system $\langle M_s : s \in \mathbb{P} \rangle$ has M as its colimit and:

(1) μ-directed, since $\mathbb{P}$ is μ-directed.
(2) Made of $(< \lambda + \mu^*)$-presentable objects.

$\square$

We deduce several interesting results:

Theorem 5.16. Let $\mathcal{K}$ be a λ-accessible category. If $\mu > \lambda$ is a λ-closed regular cardinal, then $\mathcal{K}$ is μ-accessible.

Proof. Directly from Lemma 5.15. $\square$

Remark 5.17. We cannot in general remove the assumption that μ is λ-closed from Theorem 5.16 (see [AR94, 2.11]). In fact, for $\mu > 2^{<\lambda}$ regular, the statements "μ is λ-closed" and "every λ-accessible category is μ-accessible" are equivalent (see [LR17, 4.11] or [LRVb, 2.6]).

Theorem 5.18. Let $\mathcal{K}$ be an accessible category. Then:

(1) Any object of $\mathcal{K}$ is λ-presentable, for some λ.

(2) For any regular cardinal λ, there is only a set (up to isomorphism) of λ-presentable objects.

Proof. Let μ be such that $\mathcal{K}$ is μ-accessible. Let S be a set of μ-presentable objects so that any object is isomorphic to a μ-directed colimit of members of S. It follows from Lemma 5.9 that any object must be λ-presentable, for some λ. This proves the first item. For the second, Exercise 5.11 shows that there is only a set of μ-presentable objects. By Theorem 5.16, $\mathcal{K}$ is moreover λ-accessible for arbitrarily large λ, so the result follows. $\square$

As mentioned before, in the category of sets, an object is λ-presentable if and only if its cardinality is strictly less than λ. Thus the least cardinal λ such that an object is λ-presentable (we call this the *presentability rank*) is always a successor. The following question of Beke and Rosický [BR12] remains open:

Question 5.19. For a fixed accessible category, is every *high-enough* presentability rank a successor?

We can give the following approximation [LRVb, 3.11]:

Theorem 5.20. Let $\mathcal{K}$ be a λ-accessible category. If $\mu > \lambda$ is weakly inaccessible and λ-closed, then any μ-presentable object is $(< \mu)$-presentable.

Proof. Let M be μ-presentable. By definition, M can be resolved into a λ-directed colimit of λ-presentable objects, hence of $(< \mu)$-presentables. By Lemma 5.15, M can be resolved into a μ-directed colimit of $(< \mu)$-presentable objects. By μ-presentability of M, this means that M is a retract of a $(< \mu)$-presentable object, hence is itself $(< \mu)$-presentable, as desired. $\square$

Note that assuming the singular cardinal hypothesis, every weakly inaccessible above $2^{<\lambda}$ is λ-closed. Since Solovay showed that the singular cardinal hypothesis holds above certain large cardinals (see [Sol74] or [Jec03, 20.8]) it follows that Question 5.19 has a positive answer assuming a large cardinal axiom (a proper class of strongly compact cardinals).

5.1. **From accessible category to μ-AEC.** We now aim to show[1]:

Theorem 5.21 ([BGL+16, 4.5]). For any μ-accessible category $\mathcal{K}$ whose morphisms are monomorphisms, $\mathcal{K}$ is equivalent to a μ-AEC.

Recall that two categories $\mathcal{K}_1$ and $\mathcal{K}_2$ are *equivalent* if there is a functor $F : \mathcal{K}_1 \to \mathcal{K}_2$ which is:

(1) Full: its restriction to sets of the form $\mathrm{Hom}(M, N)$ is onto $\mathrm{Hom}(FM, FN)$.
(2) Faithful: its restriction to sets of the form $\mathrm{Hom}(M, N)$ is injective.
(3) Essentially surjective: any object N in $\mathcal{K}_2$ is isomorphic to FM for some object M in $\mathcal{K}_1$.

---

[1]It was known since Rosický's thesis [Ros83, Ros81] that accessible categories are classes of models of certain $\mathbb{L}_{\infty,\infty}$ sentence, but seeing them as μ-AEC is more direct.

This is weaker than an isomorphism of category, but preserves all reasonable category-theoretic notions. Intuitively, we allow isomorphic objects inside the category to be identified. One example to keep in mind is that the category of a single object with only the identity morphism is equivalent (but not isomorphic) to the category of all singleton sets.

The proof of Theorem 5.21 proceeds in two steps. The first shows that $\mathcal{K}$ is equivalent to a certain accessible category of structures. The second shows that this category must actually be a μ-AEC. Let us implement the first step. For τ a vocabulary, we denote by $\mathrm{Emb}(\tau)$ the category whose objects are τ-structures and whose morphisms are injective homomorphisms.

Lemma 5.22 ([BGL$^+$16, 4.8]). Let $\mathcal{K}$ be a λ-accessible category whose morphisms are monomorphism. Then there is a (finitary) vocabulary τ and a functor $E : \mathcal{K} \to \mathrm{Emb}(\tau)$ which is full and faithful and preserves λ-directed colimits.

Proof. Let $\mathcal{K}_0$ be a small full subcategory of $\mathcal{K}$ containing (up to isomorphism) all the λ-presentable objects. For each $M \in \mathcal{K}_0$, let S_M be a unary relation symbol and for each morphism f in $\mathcal{K}_0$, let $\underline{f}$ be a binary function symbol. The vocabulary τ will consist of all such S_M and $\underline{f}$. Now map each $M \in \mathcal{K}$ to the following τ-structure EM:

   (1) Its universe are the morphisms $g : M_0 \to M$, where $M_0 \in \mathcal{K}_0$.
   (2) For each $M_0 \in \mathcal{K}_0$, $S_{M_0}^{EM}$ is the set of morphisms $g : M_0 \to M$.
   (3) For each morphism $f : M_0 \to M_1$ of $\mathcal{K}_0$, and each $g : M_1 \to M$, $\underline{f}^{EM}(g) = gf$. When $g \notin S_{M_1}^{EM}$, just let $\underline{f}^{EM}(g) = g$.

Map each morphism $f : M \to N$ to the function $\bar{f} : EM \to EN$ given by $\bar{f}(g) = fg$. That E is full and faithful and preserves λ-directed colimits is a long but crucial exercise in diagram chasing (closely related to the Yoneda lemma). For example, to see that E is full, assume first that $M \in \mathcal{K}_0$. Then id_M is a morphism in $\mathcal{K}_0$ so given $g : EM \to EN$, we can let $f := g(\mathrm{id}_M)$ and it turns out that $E(f) = g$. When M is not λ-presentable, resolve it into a λ-directed colimit of λ-presentable objects. □

The second step shows that any coherent abstract class which looks like an accessible category is in fact a μ-AEC. First, it is not too hard to show (using resolutions into directed systems again) that only a weak version of the LST axiom suffices:

Exercise 5.23. Let $\mathbf{K}$ be an abstract class satisfying all the axioms of a μ-AEC except possibly the LST axiom. Let $\theta \geq \mu + |\tau(\mathbf{K})|$ be such that:

   (1) θ is μ-closed.
   (2) $\mathrm{cf}(\theta) \geq \mu$.
   (3) For any $M \in \mathbf{K}$ and any $A \subseteq |M|$ with $|A| < \theta$, there exists $M_0 \in \mathcal{P}_{\mathbf{K}_{<\theta}}(M)$ with $A \subseteq |M_0|$.

Then $\mathbf{K}$ is a μ-AEC with $\mathrm{LS}(\mathbf{K}) \leq \theta$.

Lemma 5.24. Let $\mathbf{K}$ be an abstract class satisfying the coherence axiom and let μ be a regular cardinal. Assume that $\mathbf{K}$ is μ-accessible and further the μ-directed colimits are concrete (given by unions, i.e. they are the same as in $\mathrm{Emb}(\tau(\mathbf{K}))$).

Let C be the set of cardinals λ such that for any $M \in \mathbf{K}$, $\|M\| < \lambda$ if and only if M is $(< \lambda)$-presentable. Then C is closed unbounded. In particular, $\mathbf{K}$ is a μ-AEC.

Proof. C is clearly closed. Now given any cardinal λ, there is (up to isomorphism) only a set of λ^+-presentable objects (Theorem 5.18) and only a set of objects of cardinality λ. Thus there is a cardinal λ' such that any λ^+-presentable object has cardinality strictly less than λ' and any object of cardinality at most λ is $(< \lambda')$-presentable. Thus given any cardinal λ_0, we can build an increasing sequence $\langle \lambda_i : i < \omega \rangle$ such that for any $i < \omega$, any λ_i^+-presentable object has cardinality strictly less than λ_{i+1} and any object of cardinality λ_i is $(< \lambda_{i+1})$-presentable. Now by construction $\sup_{i < \omega} \lambda_i$ is in C. Thus C is unbounded.

To see the "in particular" part, we have to prove the LST axiom. Pick $\theta \in C$ a limit cardinal such that θ is μ-closed and $\mathrm{cf}(\theta) \geq \mu + |\tau(\mathbf{K})|$. Now let $M \in \mathbf{K}$ and let $A \subseteq |M|$ with $|A| < \theta$ be given. Let $\theta_0 := ((|A| + \aleph_0)^{<\mu})^+$. Note that θ_0 is μ-closed so by Theorem 5.16, $\mathbf{K}$ is θ_0-accessible. Thus M is a θ_0-directed colimit of θ_0-presentable objects $\langle M_i : i \in I \rangle$. Since θ_0-directed colimits are concrete, this implies that A is contained inside some M_i. Now by definition of C, M_i has cardinality strictly less than θ. This shows that the hypotheses of Exercise 5.23 are satisfied. $\square$

Proof of Theorem 5.21. Let $\mathcal{K}$ be a μ-accessible category whose morphisms are monomorphisms. By Lemma 5.22, there is a vocabulary τ such that $\mathcal{K}$ is equivalent to a full subcategory of $\mathrm{Emb}(\tau)$ which is closed under μ-directed colimits inside $\mathrm{Emb}(\tau)$. Equivalently, it is closed under μ-directed unions. Closing such a category under isomorphism, we obtain an abstract class $\mathbf{K}$ (the ordering is just substructure) which satisfies the hypotheses of Lemma 5.24, hence is a μ-AEC. $\square$

6. μ-AECs and infinitary logics

Makkai and Paré [MP89, 3.2.3, 3.3.5, 4.3.2] have shown (refining an argument of Rosický) that any λ-accessible category is equivalent to a category of models of an $\mathbb{L}_{\infty,\lambda}$-sentence (the morphisms are homomorphisms). In this section, we prove results around that neighborhood for μ-AECs.

We first review the following semantic characterization of elementary equivalence.

Definition 6.1. Let M and N be τ-structures. We call f a *partial isomorphism* from M to N if:

(1) f is a function from a subset of $|M|$ to a subset of $|N|$.
(2) For any enumeration $\bar{a}$ of the domain of f and any first-order quantifier-free formula ϕ, $M \models \phi[\bar{a}]$ if and only if $N \models \phi[\bar{a}]$.

Definition 6.2. Let M and N be τ-structures and let θ be an infinite cardinal. A *θ-forth system from M to N* is a set $\mathcal{F}$ such that:

(1) $\mathcal{F} \neq \emptyset$.
(2) Any member f of $\mathcal{F}$ is a partial isomorphism from M to N.
(3) For any $f \in \mathcal{F}$, $|\mathrm{dom}(f)| < \theta$.
(4) For any $f \in \mathcal{F}$ and any $A \subseteq \mathrm{dom}(f)$, $f \restriction A \in \mathcal{F}$.

(5) For any $f \in \mathcal{F}$ and any $A \subseteq |M|$ with $|A| < \lambda$, there exists $g \in \mathcal{F}$ with $f \subseteq g$ and $A \subseteq \operatorname{dom}(g)$.

We say that $\mathcal{F}$ is a θ-*back and forth system from M to N* if it is a θ-forth system and $\{f^{-1} \mid f \in \mathcal{F}\}$ is θ-forth system from N to M.

We write $M \equiv^*_{\infty,\theta} N$ if there is a θ-back and forth system from M to N.

The following result is due to Karp for $\mathbb{L}_{\infty,\omega}$, see [Kar65]. A good basic reference on such theorems (and on $\mathbb{L}_{\infty,\infty}$ in general) is [Dic75].

Theorem 6.3. Let M and N be τ-structures and let θ be an infinite cardinal. The following are equivalent:

(1) $M \equiv_{\infty,\theta} N$.
(2) $M \equiv^*_{\infty,\theta} N$.

Proof.

- (1) implies (2): Let $\mathcal{F}$ be the set of partial functions f from $|M|$ to $|N|$ whose domain has cardinality strictly less than θ, and such that for any enumeration $\bar{a}$ of their domain and any $\mathbb{L}_{\infty,\theta}$-formula ϕ, $M \models \phi[\bar{a}]$ if and only if $N \models \phi[f(\bar{a})]$. We claim that $\mathcal{F}$ is as desired. By symmetry, it suffices to show it is a θ-forth system. Since $M \equiv_{\infty,\theta} N$, the empty map is in $\mathcal{F}$, hence $\mathcal{F}$ is not empty. Clearly, any member of $\mathcal{F}$ is a partial isomorphism from M to N whose domain has cardinality strictly less than θ. If $f \in \mathcal{F}$ and $A \subseteq \operatorname{dom}(f)$, then by definition $f \upharpoonright A \in \mathcal{F}$. Now let $f \in \mathcal{F}$ and let $A \subseteq |M|$. Let $\bar{a}$ be an enumeration of A and let $\bar{a}_0$ be an enumeration of $\operatorname{dom}(f)$. For a cardinal μ, let p_μ be the class of formulas $\psi(\bar{x}, \bar{y}) \in \mathbb{L}_{\mu,\theta}$ such that $M \models \psi[\bar{a}, \bar{a}_0]$. We have that $M \models \exists \bar{x} \bigwedge_{\psi \in p_\mu} \psi[\bar{x}, \bar{a}_0]$. Thus $N \models \exists \bar{x} \bigwedge_{\psi \in p_\mu} \psi[\bar{x}, f(\bar{a}_0)]$. Let $\bar{b}^\mu$ be a witness. Now N is a set, so there must exist a proper class C of cardinals such that $\mu, \mu' \in C$ implies $\bar{b} := \bar{b}^\mu = \bar{b}^{\mu'}$. Let g send $\bar{a}$ to $\bar{b}$. It is easy to check that this works.
- (2) implies (1): We show that for any $\mathbb{L}_{\infty,\theta}$-formula $\phi(\bar{x})$, any $\bar{a} \in {}^{<\theta}M$, and any $f \in \mathcal{F}$ whose domain contains $\bar{a}$, $M \models \phi[\bar{a}]$ if and only if $N \models \phi[f(\bar{a})]$. We proceed by induction on ϕ. When ϕ is atomic, this is because f is a partial isomorphism. When ϕ is a conjunction or negation, this is similarly easy. Assume that $\phi = \exists \bar{y} \psi(\bar{y}, \bar{x})$. We show that $M \models \phi[\bar{a}]$ implies $N \models \phi[f(\bar{a})]$, and the converse follows from the symmetric definition of a back and forth system. So let $\bar{b} \in {}^{<\theta}M$ be such that $M \models \psi[\bar{b}, \bar{a}]$. Let $g \in \mathcal{F}$ extend f such that the domain of g contains $\bar{b}$. By the induction hypothesis, $N \models \psi[g(\bar{b}), g(\bar{a})]$. Thus $N \models \phi[g(\bar{a})]$. Since $g(\bar{a}) = f(\bar{a})$, we are done.

$\square$

The proof can be refined to yield:

Exercise 6.4. Show that if θ is regular one can replace (1) by "$M \equiv_{\lambda,\theta} N$", where $\lambda := \left((2 + \|M\| + \|N\|)^{<\theta}\right)^+$.

Exercise 6.5 (Scott)**.** Let θ be regular and let M be a τ-structure. Let $\lambda := \left((2 + \|M\|)^{<\theta}\right)^+$. Show that there exists an $\mathbb{L}_{\lambda,\theta}$-sentence ϕ such that for any τ-structure N, $N \models \phi$ implies $M \equiv_{\infty,\theta} N$.

The following consequence is interesting:

Corollary 6.6. Let θ be an infinite cardinal of cofinality $\aleph_0$ and let M and N be τ-structures of cardinality θ. If $M \equiv_{\infty,\theta} N$, then $M \cong N$.

Proof. By Theorem 6.3, $M \equiv^*_{\infty,\theta} N$. Let $\mathcal{F}$ witness it. Write $|M| = \bigcup_{n<\omega} A_n$, $|N| = \bigcup_{n<\omega} B_n$ with $|A_n| + |B_n| < \theta$. This is possible by the cofinality assumption. Finally, build an increasing chain $\langle f_n : n < \omega \rangle$ of elements of $\mathcal{F}$ such that $A_n \subseteq \operatorname{dom}(f_{n+1})$ and $B_n \subseteq \operatorname{ran}(f_{n+1})$ for all $n < \omega$. This is possible since $\mathcal{F}$ is a θ-back and forth system. $\square$

We can also deduce that AECs are closed under infinitary elementary equivalence. This was observed independently by Kueker [Kue08] and Shelah [She09a, IV.1.11]. First, we prove a lemma:

Lemma 6.7. Let $\mathbf{K}$ be an AEC and let M be a τ-structure. If D is a set such that:

(1) For all $M_0 \in D$, $M_0 \in \mathbf{K}_{\leq \operatorname{LS}(\mathbf{K})}$ and $M_0 \subseteq M$.
(2) For all $M_0 \in D$ and all $A \in [M]^{\leq \operatorname{LS}(\mathbf{K})}$, there is $M_1 \in D$ such that $M_0 \leq_{\mathbf{K}} M_1$ and $A \subseteq |M_1|$.

Then $M \in \mathbf{K}$ and $M_0 \leq_{\mathbf{K}} M$ for all $M_0 \in D$.

Proof. First we show:

Claim: If M_0 and M_1 are in D, there exists $M_2 \in \mathbf{K}$ such that $M_0 \leq_{\mathbf{K}} M_2$ and $M_1 \leq_{\mathbf{K}} M_2$.

Proof of Claim: For $\ell = 0, 1$, we build $\langle M_\ell^i : i < \omega \rangle$ $\leq_{\mathbf{K}}$-increasing in D such that $M_\ell^0 = M_\ell$ and $|M_{1-\ell}^i| \subseteq |M_\ell^{i+1}|$ for all $i < \omega$. This is possible by the assumptions on D. Now let $M_2 := \bigcup_{i<\omega} M_0^i = \bigcup_{i<\omega} M_1^i$. $\dagger_{\text{Claim}}$

Now we build $\langle M_s : s \in [M]^{\leq \operatorname{LS}(\mathbf{K})} \rangle$ a sequence of models in D such that $s \subseteq t$ implies $|M_s| \subseteq |M_t|$ and $s \subseteq |M_s|$ for all $s, t \in [M]^{\leq \operatorname{LS}(\mathbf{K})}$. This is possible by the assumptions on D. Now let $s, t \in [M]^{\leq \operatorname{LS}(\mathbf{K})}$ be such that $s \subseteq t$. Then $|M_s| \subseteq |M_t|$ and by the claim, there is $M' \in \mathbf{K}$ such that $M_s \leq_{\mathbf{K}} M'$ and $M_t \leq_{\mathbf{K}} M'$. By coherence, this implies that $M_s \leq_{\mathbf{K}} M_t$. Thus $\langle M_s : s \in [M]^{\leq \operatorname{LS}(\mathbf{K})} \rangle$ is a directed system in $\mathbf{K}$ whose union is M, so $M \in \mathbf{K}$ and it follows from the proof that $M_0 \leq_{\mathbf{K}} M$ for all $M_0 \in D$. $\square$

Theorem 6.8. Let $\mathbf{K}$ be an AEC and let $M \in \mathbf{K}$. Let N be a $\tau(\mathbf{K})$-structure. If $M \equiv_{\infty,\operatorname{LS}(\mathbf{K})^+} N$, then $N \in \mathbf{K}$.

Proof. By Theorem 6.3, there is an $\operatorname{LS}(\mathbf{K})^+$-back and forth system $\mathcal{F}$ from M to N. Let

$$D := \{ f[M_0] \mid f \in \mathcal{F}, M_0 \in \mathcal{P}_{\mathbf{K}_{\leq \operatorname{LS}(\mathbf{K})}}(M) \}$$

It suffices to observe that D satisfies the hypotheses of Lemma 6.7 (where M there is N here). Indeed, by closure of $\mathbf{K}$ under isomorphisms, any member of D is a member of $\mathbf{K}_{\leq \mathrm{LS}(\mathbf{K})}$. Moreover if $f[M_0] \in D$ and $A \in [N]^{\leq (\mathrm{LS}(\mathbf{K}))}$, we can use the axioms of back and forth to extend f to g whose range contains A, and moreover $M_1 := \mathrm{dom}(g) \leq_{\mathbf{K}} M$. By coherence, $M_0 \leq_{\mathbf{K}} M_1$. By closure of $\leq_{\mathbf{K}}$ under isomorphisms, $f[M_0] \leq_{\mathbf{K}} g[M_1]$, and by definition $g[M_1] \in D$. $\square$

Question 6.9. Does Theorem 6.8 generalize to μ-AECs?

To better understand the relationship between infinitary logics and μ-AECs, the following concept is useful. The idea is to expand the μ-AECs with predicate that "do not add any information" in the sense that the expansion is already uniquely determined by the structure. The definition appears in [Vas16, 3.1].

Definition 6.10. Let $\mathbf{K}$ be an abstract class. A *functorial expansion* of $\mathbf{K}$ is an abstract class $\mathbf{K}^+$ in a vocabulary $\tau(\mathbf{K}^+)$ expanding $\tau(\mathbf{K})$ such that the reduct map is an isomorphism of category from $\mathbf{K}^+$ onto $\mathbf{K}$. That is:

   (1) If $M^+ \leq_{\mathbf{K}^+} N^+$, then $M^+ \restriction \tau(\mathbf{K}) \leq_{\mathbf{K}} N^+ \restriction \tau(\mathbf{K})$.
   (2) If $M \in \mathbf{K}$, there is a unique expansion $M^+ \in \mathbf{K}^+$ such that $M^+ \restriction \tau(\mathbf{K}) = M$.
   (3) If $f : M \to N$ is a $\mathbf{K}$-embedding then the induced map $f^+ : M^+ \to N^+$ also is.

We call a functorial expansion $(< \mu)$-*ary* if its vocabulary is $(< \mu)$-ary.

Remark 6.11. If $\mathbf{K}^+$ is a functorial expansion of $\mathbf{K}$, then $M^+ \leq_{\mathbf{K}^+} N^+$ holds if and only if $M^+ \restriction \tau(\mathbf{K}) \leq_{\mathbf{K}} N^+ \restriction \tau(\mathbf{K})$. Thus a functorial expansion is entirely determined by its class of models.

Remark 6.12. If $\mathbf{K}^+$ is a $(< \mu)$-ary functorial expansion of a μ-AEC $\mathbf{K}$, then $\mathbf{K}^+$ is a μ-AEC with $\mathrm{LS}(\mathbf{K}^+) = \mathrm{LS}(\mathbf{K})$.

Example 6.13.

   (1) $\mathbf{K}$ is a functorial expansion of $\mathbf{K}$.
   (2) If $\mathbf{K}$ is an elementary class (ordered with elementary substructure), we can add a relation symbol for each first-order formula and obtain a functorial expansion, called the *Morleyization of* $\mathbf{K}$.
   (3) The expansion given by Shelah's presentation Theorem 3.6 is not functorial (unless the starting class is a universal class itself). This is because the reduct functor is not necessarily full.

Another example of a functorial expansion, to be defined later, is the *orbital (or Galois) Morleyization*, which consists in adding a relation symbol for each orbital type. In this section, the following functorial expansion will play an important role:

Definition 6.14. Let $\mathbf{K}$ be a μ-AEC. The *substructure functorial expansion* of $\mathbf{K}$ is the abstract class $\mathbf{K}^+$ defined as follows:

   (1) $\tau(\mathbf{K}^+) = \tau(\mathbf{K}) \cup \{P\}$, where P is an $\mathrm{LS}(\mathbf{K})$-ary predicate.
   (2) $M^+ \in \mathbf{K}^+$ if and only if $M^+ \restriction \tau(\mathbf{K}) \in \mathbf{K}$ and for any $\bar{a} \in {}^{\mathrm{LS}(\mathbf{K})} M^+$, $P^{M^+}(\bar{a})$ holds if and only if $\mathrm{ran}(\bar{a}) \leq_{\mathbf{K}} M^+ \restriction \tau(\mathbf{K})$, where we see $\mathrm{ran}(\bar{a})$ as a $\tau(\mathbf{K})$-structure.

(3) For $M^+, N^+ \in \mathbf{K}^+$, $M^+ \leq_{\mathbf{K}^+} N^+$ if and only if $M^+ \upharpoonright \tau(\mathbf{K}) \leq_{\mathbf{K}} N^+ \upharpoonright \tau(\mathbf{K})$.

Exercise 6.15. Check that the substructure functorial expansion is indeed a functorial expansion.

The substructure functorial expansion has a number of nice properties.

Definition 6.16. We call an abstract class $\mathbf{K}$ *model-complete* if for $M, N \in \mathbf{K}$, $M \leq_{\mathbf{K}} N$ if and only if $M \subseteq N$.

Note that a model complete abstract class does *not* have to be closed under substructure (the class of algebraically closed fields is one example).

The following criteria to prove model-completeness is a directed system argument:

Exercise 6.17. Let $\mathbf{K}$ be a μ-AEC and let $M, N \in \mathbf{K}$. Suppose that $M \subseteq N$. The following are equivalent:

(1) $M \leq_{\mathbf{K}} N$.
(2) For any $M_0 \in \mathcal{P}_{\mathbf{K}_{\leq \mathrm{LS}(\mathbf{K})}}(M)$, $M_0 \leq_{\mathbf{K}} N$.

The substructure functorial expansion is model-complete:

Theorem 6.18. Let $\mathbf{K}$ be a μ-AEC. Then the substructure functorial expansion of $\mathbf{K}$ is model-complete.

Proof. Let $\mathbf{K}^+$ be the substructure functorial expansion of $\mathbf{K}$. For $M \in \mathbf{K}$, write M^+ for the expansion of M to $\mathbf{K}^+$. Let $M, N \in \mathbf{K}$ and assume that $M^+ \subseteq N^+$. We have to see that $M \leq_{\mathbf{K}} N$. For this, we use the equivalent condition of Exercise 6.17. Let $M_0 \in \mathcal{P}_{\mathbf{K}_{\leq \mathrm{LS}(\mathbf{K})}} M$. We have to see that $M_0 \leq_{\mathbf{K}} N$. Let $\bar{a}$ be an enumeration of M_0. We have that $M^+ \models P[\bar{a}]$ (where P is the additional predicate in $\tau(\mathbf{K})^+$), so $N^+ \models P[\bar{a}]$ (as M^+ is a substructure of N^+). This means that $M_0 \leq_{\mathbf{K}} N$, as desired. $\square$

The substructure functorial expansion of a μ-AEC can be axiomatized (a variation of this is due to Baldwin and Boney [BB17]). Since the ordering is trivial by the previous result, this gives that any μ-AEC is isomorphic (as a category) to the category of models of an $\mathbb{L}_{\infty,\infty}$ sentence, where the morphisms are injective homomorphisms.

Theorem 6.19. Let $\mathbf{K}$ be a μ-AEC and let $\mathbf{K}^+$ be its substructure functorial expansion. There is an $\mathbb{L}_{\left(2^{\mathrm{LS}(\mathbf{K})}\right)^+, \mathrm{LS}(\mathbf{K})^+}$ sentence ϕ such that $\mathbf{K}^+$ is the class of models of ϕ.

Proof. First note that for each $M_0 \in \mathbf{K}_{\leq \mathrm{LS}(\mathbf{K})}$, there is a sentence $\psi_{M_0}(\bar{x})$ of $\mathbb{L}_{\mathrm{LS}(\mathbf{K})^+, \mathrm{LS}(\mathbf{K})^+}$ coding its isomorphism type, i.e. whenever $M \models \phi[\bar{a}]$, then $\bar{a}$ is an enumeration of an isomorphic copy of M_0. Similarly, whenever M_0, M_1 are in $\mathbf{K}_{\leq \mathrm{LS}(\mathbf{K})}$ with $M_0 \leq_{\mathbf{K}} M_1$, there is $\psi_{M_0, M_1}(\bar{x}, \bar{y})$ that codes that $(\bar{x}, \bar{y})$ is isomorphic to (M_0, M_1) (so in particular $\bar{x} \leq_{\mathbf{K}} \bar{y}$). Let S be a complete set of members of $\mathbf{K}_{\leq \mathrm{LS}(\mathbf{K})}$ (i.e. any other model is isomorphic to it) and let T be a complete set of pairs (M_0, M_1), with each in $\mathbf{K}_{\leq \mathrm{LS}(\mathbf{K})}$, such that $M_0 \leq_{\mathbf{K}} M_1$. Now define the following:

$$\phi_1 = \forall \bar{x} \exists \bar{y} \left(\left(\bigvee_{M_0 \in S} \psi_{M_0}(\bar{y}) \right) \wedge \bar{x} \subseteq \bar{y} \wedge P(\bar{y}) \right)$$

$$\phi_2 = \forall \bar{x} \forall \bar{y} \left((\bar{x} \subseteq \bar{y} \wedge P(\bar{x}) \wedge P(\bar{y})) \rightarrow \bigvee_{(M_0, M_1) \in T} \psi_{M_0, M_1}(\bar{x}, \bar{y}) \right)$$

$$\phi = \phi_1 \wedge \phi_2$$

Where $\bar{x} \subseteq \bar{y}$ abbreviates the obvious formula. This works. First, any $M^+ \in \mathbf{K}^+$ satisfies ϕ_1 by the LST axiom and satisfies ϕ_2 by the coherence axiom. Conversely, if $M \models \phi$, then we can build a μ-directed system $\langle M_s : s \in [M]^{<\mu} \rangle$ in $\mathbf{K}$ such that $s \subseteq |M_s|$ and $M_s \in \mathbf{K}_{\leq \mathrm{LS}(\mathbf{K})}$ for all $s \in [M]^{<\mu}$. We then get that $\bigcup_{s \in [M]^{<\mu}} M_s = M \in \mathbf{K}$ by closure under μ-directed systems. A similar directed system argument shows that $M^{M^+}(\bar{a})$ holds if and only if $\mathrm{ran}(\bar{a}) \leq_{\mathbf{K}} M$, so $M^+ \in \mathbf{K}^+$. $\square$

The following shows that elementary equivalence is preserved when passing to functorial expansions of AECs. This is because back and forth systems are preserved:

Lemma 6.20. Let $\mathbf{K}$ be an AEC. let $\mathbf{K}^+$ be a $(< \mathrm{LS}(\mathbf{K})^+)$-ary functorial expansion of $\mathbf{K}$. Let $M, N \in \mathbf{K}$ and let M^+, N^+ be their respective expansions to $\mathbf{K}^+$. If $\mathcal{F}$ is an $\mathrm{LS}(\mathbf{K})^+$-back and forth system from M to N, then it is an $\mathrm{LS}(\mathbf{K})^+$-back and forth system from M^+ to N^+.

Proof. For any $M_0 \in \mathbf{K}$, write M_0^+ for its expansion to $\mathbf{K}^+$. Let $f \in \mathcal{F}$. Using the axioms of a back and forth system and the LST axiom, one can pick $g \in \mathcal{F}$ such that $f \subseteq g$ and $M_0 := \mathrm{dom}(g) \leq_{\mathbf{K}} M$. Let $N_0 := g[M_0]$. Since $M_0 \cong N_0$, $N_0 \in \mathbf{K}$. Moreover by the proof of Theorem 6.8, $N_0 \leq_{\mathbf{K}} N$. Now by definition of a functorial expansion, we must have $M_0^+ \leq_{\mathbf{K}^+} M^+$ and $N_0^+ \leq_{\mathbf{K}^+} N^+$ and moreover g is a $\mathbf{K}^+$-isomorphism. It follows that f is itself a partial isomorphism from M^+ to N^+. Since f was arbitrary, this shows that $\mathcal{F}$ is indeed a back and forth system from M^+ to N^+. $\square$

As a consequence, we deduce a relationship between the ordering of the class and infinitary elementary equivalence:

Theorem 6.21. Let $\mathbf{K}$ be an AEC. Let $M \in \mathbf{K}$. If $M \preceq_{\mathbb{L}_{\infty, \mathrm{LS}(\mathbf{K})^+}} N$, then $M \leq_{\mathbf{K}} N$.

Proof. By Theorem 6.8, $N \in \mathbf{K}$. We use Exercise 6.17. Let $M_0 \in \mathcal{P}_{\mathbf{K}_{\leq \mathrm{LS}(\mathbf{K})}}(M)$. Let $\bar{a}$ be an enumeration of M_0. We have that $(M, \bar{a}) \equiv_{\infty, \mathrm{LS}(\mathbf{K})^+} (N, \bar{a})$. By Theorem 6.3, there is an $\mathrm{LS}(\mathbf{K})^+$-back and forth system $\mathcal{F}$ from $(M, \bar{a})$ to $(N, \bar{a})$. By Lemma 6.20 it is also a back and forth system from M^+ to N^+, and hence it is easy to check from $(M^+, \bar{a})$ to $(N^+, \bar{a})$, where M^+ and N^+ denote the expansions of M and N in the substructure functorial expansion. By Theorem 6.3 again, this implies that $P^{M^+}(\bar{a})$ holds if and only if $P^{N^+}(\bar{a})$ holds. Since $M_0 \leq_{\mathbf{K}} M$, we have that $P^{M^+}(\bar{a})$, so $P^{N^+}(\bar{a})$, so $M_0 \leq_{\mathbf{K}} N$, as desired. $\square$

There are converses to Theorem 6.21 when M and N are sufficiently saturated. For example, in a first-order theory T, if M and N are saturated of cardinality λ and $M \preceq N$, then $M \preceq_{\mathbb{L}_{\infty,\lambda}} N$ (exercise). The following beautiful argument of Shelah uses Fodor's lemma to provide some kind of analog even when there is no obvious notion of saturated (see [BGL$^+$16, 6.8] for a generalization to certain μ-AECs).

Theorem 6.22 (Shelah, [She09a, IV.1.12(1)]). Let $\mathbf{K}$ be an AEC, let θ be regular and let $\lambda = \lambda^{<\theta} \geq \mathrm{LS}(\mathbf{K})$. Assume that $\mathbf{K}$ is categorical in λ and let $M, N \in \mathbf{K}_{\geq \lambda}$. If $M \leq_{\mathbf{K}} N$, then $M \preceq_{\mathbb{L}_{\infty,\theta}} N$.

Proof. A directed systems argument (exercise) establishes that it suffices to prove it when $M, N \in \mathbf{K}_\lambda$. We now prove by induction on $\phi(\bar{x}) \in \mathbb{L}_{\infty,\theta}$ that for any $M, N \in \mathbf{K}_\lambda$ with $M \leq_{\mathbf{K}} N$ and any $\bar{a} \in {}^{<\theta}M$, $M \models \phi[\bar{a}]$ if and only if $N \models \phi[\bar{a}]$. This is easy when ψ is atomic (since $\leq_{\mathbf{K}}$ extends substructure) and when ϕ is a conjunction or a negation. We prove what happens when $\phi = \exists \bar{y}\psi(\bar{x}, \bar{y})$. If $M \models \phi[\bar{a}]$, then $N \models \phi[\bar{a}]$ as well. Now suppose that $N \models \phi[\bar{a}]$. We build an increasing continuous chain $\langle M_i : i < \lambda^+ \rangle$ and $\langle f_i : i < \lambda^+ \rangle$ such that for all $i < \lambda^+$:

(1) $M_i \in \mathbf{K}_\lambda$.
(2) $f_i : N \cong M_{i+1}$ is such that $f_i[M] = M_i$.

This is possible by categoricity in λ and some renaming. Now let $\bar{a}_i := f_i(\bar{a})$. Note that since $\bar{a} \in {}^{<\theta}M$, we have that $\bar{a}_i \in {}^{<\theta}M_i$. Let $S := \{i < \lambda^+ \mid \mathrm{cf}(i) \geq \theta\}$. This is a stationary set, and for each $i \in S$, there exists $j_i < i$ such that $\bar{a}_i \in {}^{<\theta}M_{j_i}$. Thus the map $i \mapsto j_i$ is regressive so by Fodor's lemma there exists $S_0 \subseteq S$ stationary and $j < \lambda^+$ such that for any $i \in S_0$, $j_i = j$. Since $\lambda = \lambda^{<\theta}$ and $|S_0| = \lambda^+$, there exists $\bar{a}' \in {}^{<\theta}M_j$ and $S_1 \subseteq S_0$ of cardinality λ^+ and such that $i \in S_1$ implies $\bar{a}_i = \bar{a}'$. Let $i \in S_1$. Since $N \models \phi[\bar{a}]$, we have (applying f_i) that $M_{i+1} \models \phi[\bar{a}']$. Thus there exists $\bar{b} \in {}^{<\theta}M_{i+1}$ such that $M_{i+1} \models \psi[\bar{b}, \bar{a}']$. Pick $i' \in S_1$ such that $i + 1 < i'$. By the induction hypothesis, $M_{i'} \models \psi[\bar{b}, \bar{a}']$. Applying $f_{i'}^{-1}$ to this statement (and the definition of S_1), $M \models \psi[f_{i'}^{-1}(\bar{b}), \bar{a}]$, hence $M \models \phi[\bar{a}]$, as desired. $\square$

7. Orbital types

In any abstract class, one can define a semantic notion of type (loosely, this is the finest possible notion of types that preserves $\mathbf{K}$-embeddings). They were introduced by Shelah [She87b]. The name "Galois type" is used a lot in the literature, but we prefer Shelah's terminology of "orbital type" for reasons that will soon become apparent.

Definition 7.1. Let $\mathbf{K}$ be an abstract class. We define an equivalence relation $\equiv$ ($=\equiv^{\mathbf{K}}$) on pairs $(\bar{a}, M)$, where $M \in \mathbf{K}$ and $\bar{a} \in {}^{<\infty}M$ as follows: $\equiv$ is the intersection of all equivalence relations E on such pairs satisfying:

If $f : M \to N$ is a $\mathbf{K}$-embedding, then $(\bar{a}, M)E(f(\bar{a}), N)$.

For $N_1, N_2 \in \mathbf{K}$, $A \subseteq N_1 \cap N_2$, and $\bar{b}_\ell \in {}^{<\infty}N_\ell$, we write $(\bar{a}_1, N_1) \equiv_A (\bar{a}_2, N_2)$ if for some (equivalently, any) enumeration $\bar{a}$ of A, $(\bar{a}_1\bar{a}, N_1) \equiv (\bar{a}_2\bar{a}, N_2)$. For $N \in \mathbf{K}$, $\bar{b} \in {}^{<\infty}N$ and $A \subseteq |N|$, we let $\mathbf{tp}(\bar{b}/A; N)$ denote the $\equiv_A$-equivalence class of $(\bar{b}, N)$. When $\mathbf{K}$ is not clear from context, we may write $\mathbf{tp}_{\mathbf{K}}(\bar{b}/A; N)$.

A more explicit definition is:

Exercise 7.2. Let $\mathbf{K}$ be an abstract class. Show that $\equiv^{\mathbf{K}}$ is the transitive closure of the relation E_{at} defined by $(\bar{b}_1, N_1) E_{\mathrm{at}} (\bar{b}_2, N_2)$ if and only if there exists $N \in \mathbf{K}$, $f_\ell : N_\ell \to N$ such that $f_1(\bar{b}_1) = f_2(\bar{b}_2)$.

From this and a diagram chase, we obtain an easier definition for abstract classes with amalgamation:

Exercise 7.3. Let $\mathbf{K}$ be an abstract class with amalgamation. Show that $\equiv^{\mathbf{K}} = E_{\mathrm{at}}$, where E_{at} is defined in the previous exercise. Deduce that $\mathbf{tp}(\bar{b}_1/A; N_1) = \mathbf{tp}(\bar{b}_2/A; N_2)$ if and only if there exists $N \in \mathbf{K}$ and $f_\ell : N_\ell \underset{A}{\to} N$ such that $f_1(\bar{b}_1) = f_2(\bar{b}_2)$.

One can also prove an easier characterization in AECs with intersections:

Exercise 7.4. Let $\mathbf{K}$ be an AEC with intersections. Show that $\mathbf{tp}(\bar{b}_1/A; N_1) = \mathbf{tp}(\bar{b}_2/A; N_2)$ if and only if there exists $f : \mathrm{cl}^{N_1}(A\bar{b}_1) \cong_A \mathrm{cl}^{N_2}(A\bar{b}_2)$ such that $f(\bar{b}_1) = \bar{b}_2$. *Hint: first show that $(\bar{b}_1, N_1) E_{at}(\bar{b}_2, N_2)$ implies there is $f : \mathrm{cl}^{N_1}(\bar{b}_1) \cong \mathrm{cl}^{N_2}(\bar{b}_2)$ sending $\bar{b}_1$ to $\bar{b}_2$, then use Exercise 7.2.*

Example 7.5.

(1) Let $\mathbf{K}$ be an elementary class (ordered by elementary substructure). Then orbital types coincide with the usual syntactic types. More precisely, if $N_1, N_2 \in \mathbf{K}$, $A \subseteq N_1 \cap N_2$, $\bar{b}_\ell \in {}^{<\infty}N_\ell$, the following are equivalent:
   (a) $\mathbf{tp}(\bar{b}_1/A; N_1) = \mathbf{tp}(\bar{b}_2/A; N_2)$.
   (b) For any $\mathbb{L}_{\omega,\omega}$ formula ϕ, $N_1 \models \phi[\bar{b}_1]$ if and only if $N_2 \models \phi[\bar{b}_2]$.
   This follows from Exercise 2.10. In particular, orbital types are exactly orbits of the monster model under the action of its automorphism group. We will soon generalize this last fact to any AEC with amalgamation.
(2) Let $\mathbf{K}$ be a universal class. By Exercise 7.4 and Theorem 2.11, orbital types are exactly the same as the quantifier-free types.

It will be convenient to have some notation to talk about orbital types.

Definition 7.6. Let $\mathbf{K}$ be an abstract class.

(1) Let $N \in \mathbf{K}$, $A \subseteq |N|$, and α be an ordinal. Define:

$$\mathbf{S}^\alpha(A; N) := \{\mathbf{tp}(\bar{b}/A; N) \mid \bar{b} \in {}^\alpha|N|\}$$

(2) For $M \in \mathbf{K}$ and α an ordinal, let:

$$\mathbf{S}^\alpha(M) := \{p \mid \exists N \in \mathbf{K} : M \leq_{\mathbf{K}} N \text{ and } p \in \mathbf{S}^\alpha(M; N)\}$$

(3) For α an ordinal, let:

$$\mathbf{S}^\alpha(\emptyset) := \bigcup_{N \in \mathbf{K}} \mathbf{S}^\alpha(\emptyset; N)$$

When $\alpha = 1$, we omit it. Similarly define $\mathbf{S}^{<\alpha}$, where α is allowed to be ∞. When $\mathbf{K}$ is not clear from context, we may write $\mathbf{S}^\alpha_{\mathbf{K}}$, etc.

Remark 7.7. When α is an ordinal, $\mathbf{S}^\alpha(M)$ and $\mathbf{S}^\alpha(\emptyset)$ could a priori be proper classes. However in reasonable cases (e.g. when $\mathbf{K}$ is a μ-AEC) they are sets. For example when $\mathbf{K}$ is a μ-AEC, an upper bound for $|\mathbf{S}^\alpha(M)|$ is $2^{(\|M\|+\alpha+\mathrm{LS}(\mathbf{K}))^{<\mu}}$.

Definition 7.8. Let $\mathbf{K}$ be an abstract class and let p be an orbital type.

(1) Let $\ell(p)$ and $\mathrm{dom}(p)$ be the unique α and A such that there exists $N \in \mathbf{K}$ so that $p \in \mathbf{S}^\alpha(A; N)$.
(2) We say that p is *realized in N (by $\bar{b}$)* if $p = \mathbf{tp}(\bar{b}/\mathrm{dom}(p); N)$. Similarly define type omission.
(3) For $A \subseteq \mathrm{dom}(p)$, we let $p \restriction A$ be $\mathbf{tp}(\bar{b}/A; N)$ for some (any) $\bar{b}$ and N such that p is realized by $\bar{b}$ in N.
(4) We say that an orbital type q is an *extension* of p if $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$ and $q \restriction \mathrm{dom}(p) = p$.
(5) If $p = \mathbf{tp}(\bar{b}/M; N)$, $M \leq_{\mathbf{K}} N$, and $f : M \cong M'$, we let $f(p)$ be $\mathbf{tp}(g(\bar{b})/M'; N')$ for some (any) extension $g : N \cong N'$ of f.

7.1. Model-homogeneous and universal models. Even without a notion of type, one can make the following definitions:

Definition 7.9. Let $\mathbf{K}$ be an abstract class, let $M \in \mathbf{K}$, and let λ be an infinite cardinal.

(1) M is *λ-universal* if any $N \in \mathbf{K}_{<\lambda}$ $\mathbf{K}$-embeds into M. When $\lambda = \|M\|^+$, we omit it.
(2) M is *λ-model-homogeneous* if for any $M_0 \leq_{\mathbf{K}} N_0$ both in $\mathbf{K}_{<\lambda}$, if $M_0 \leq_{\mathbf{K}} M$ then there exists $f : N_0 \xrightarrow[M_0]{} N$. When $\lambda = \|M\|$, we omit it.

Let us note for later use that there is a weaker definition of being model-homogeneous which suffices:

Exercise 7.10. Let $\mathbf{K}$ be an AEC with amalgamation. Let $M \in \mathbf{K}$ and let $\lambda > \mathrm{LS}(\mathbf{K})$. The following are equivalent:

(1) M is λ-model-homogeneous.
(2) For any $M_0 \in \mathcal{P}_{\mathbf{K}_{<\lambda}}(M)$ and any $N_0 \in \mathbf{K}_{\|M_0\|+\mathrm{LS}(\mathbf{K})}$ with $M_0 \leq_{\mathbf{K}} N_0$, there exists $f : N_0 \xrightarrow[M_0]{} N$.
(3) For any $M_0 \in \mathcal{P}_{\mathbf{K}_{<\lambda}}(M)$ and any $N_0 \in \mathbf{K}_{\leq\lambda}$ with $M_0 \leq_{\mathbf{K}} N_0$, there exists $f : N_0 \xrightarrow[M_0]{} N$.

In an AEC with amalgamation and joint embedding, it is reasonably easy to create such models via a general exhaustion argument:

Exercise 7.11. Let $\mathbf{K}$ be an AEC with amalgamation and let $M \in \mathbf{K}$. Let $\lambda > \mathrm{LS}(\mathbf{K})$.

(1) For any $\theta \geq \|M\| + 2$ with $\theta = \theta^{<\lambda}$, there exists a λ-model-homogeneous $N \in \mathbf{K}$ with $M \leq_{\mathbf{K}} N$.
(2) If $\mathbf{K}$ has joint embedding, any λ-model-homogeneous model is λ^+-universal.

Moreover, the model-homogeneous universal model is unique (in a fixed cardinality) if if exists:

Exercise 7.12. Let $\mathbf{K}$ be an AEC with amalgamation. Let $M, N \in \mathbf{K}$ be model-homogeneous of the same cardinality $\lambda > \mathrm{LS}(\mathbf{K})$. Let $M_0 \in \mathcal{P}_{\mathbf{K}_{<\lambda}}(M)$ and let $f : M_0 \to N$. Then there exists an isomorphism $g : M \cong N$ extending f.

Let us call a *monster model* in an AEC $\mathbf{K}$ a proper class-sized $\tau(\mathbf{K})$-structure $\mathfrak{C}$ such that there exists $\langle \mathfrak{C}_i : i \in \mathrm{OR} \rangle$ increasing in $\mathbf{K}$ with $\mathfrak{C}_i$ $(|i| + \mathrm{LS}(\mathbf{K})^+)$-model-homogeneous and $(|i| + \mathrm{LS}(\mathbf{K})^+)$-universal for all $i \in \mathrm{OR}$. Note that if it exists, $\mathfrak{C}$ must be unique up to isomorphism. We abuse notation and think of $\mathfrak{C}$ as a member of $\mathbf{K}$.

Exercise 7.13. Let $\mathbf{K}$ be an AEC. Then $\mathbf{K}$ has a monster model if and only if $\mathbf{K}$ has amalgamation, joint embedding, and arbitrarily large models.

Orbital types are actually orbits (under the action of an automorphism group) when their equality is computed inside a model-homogeneous model (in particular in the monster model).

Exercise 7.14. Let $\mathbf{K}$ be an AEC with amalgamation. Let $M \in \mathbf{K}$ be model-homogeneous and let $\bar{b}_1, \bar{b}_2 \in {}^\alpha M$ with $\alpha < \|M\|$. Then $\mathbf{tp}(\bar{b}_1/\emptyset; M) = \mathbf{tp}(\bar{b}_2/\emptyset; M)$ if and only if there is an automorphism of M sending $\bar{b}_1$ to $\bar{b}_2$.

7.2. **Model-homogeneous is equivalent to saturated.** Using orbital types, one can define a notion related to being model-homogeneous:

Definition 7.15. Let $\mathbf{K}$ be an AEC with amalgamation, let $M \in \mathbf{K}$ and let $\lambda > \mathrm{LS}(\mathbf{K})$. We say that M is λ-*saturated* if for any $M_0 \in \mathcal{P}_{\mathbf{K}_{<\lambda}}(M)$, any $p \in \mathbf{S}(M_0)$ is realized inside M.

Exercise 7.16. Show that in an AEC with amalgamation, any λ-model-homogeneous model is λ-saturated.

We will prove the following converse, due to Shelah [She09a, II.1.14] (originally proven in [She87b]). This provides some justification for using orbital types, as it tells us that model-homogeneous models can be built "element by element".

Theorem 7.17. Let $\mathbf{K}$ be an AEC with amalgamation. Let $\lambda > \mathrm{LS}(\mathbf{K})$ and let $M \in \mathbf{K}$. If M is λ-saturated, then M is λ-model-homogeneous.

Proof. By Exercise 7.10, it suffices to show that for all $M_0 \in \mathcal{P}_{\mathbf{K}_{<\lambda}}(M)$ and all $N \in \mathbf{K}_{\|M_0\| + \mathrm{LS}(\mathbf{K})}$ with $M_0 \leq_{\mathbf{K}} N$, there is $f : N \xrightarrow[M_0]{} M$. Let $\mu := \|N\| + \mathrm{LS}(\mathbf{K})$ and let $\langle a_i : i < \mu \rangle$ be an enumeration of $|N|$ (possibly with repetitions). We build $\langle N_i^0 : i \leq \mu \rangle$, $\langle N_i^1 : i \leq \mu \rangle$ increasing continuous in $\mathbf{K}_{\leq \mu}$ and $\langle f_i : i \leq \mu \rangle$ increasing continuous such that for all $i < \mu$:

(1) $f_i : N_i^0 \to M$.
(2) $N_0^0 = M_0$, $N_0^1 = N$, $f_0 = \mathrm{id}_{M_0}$.
(3) $N_i^0 \leq_{\mathbf{K}} N_i^1$.
(4) $a_i \in N_{i+1}^0$.

<u>This is enough</u>: By (4), we have that $|N| \subseteq |N_\mu^0|$. Since $N \leq_{\mathbf{K}} N_\mu^1$ and $N_\mu^0 \leq_{\mathbf{K}} N_\mu^1$, coherence implies that $N \leq_{\mathbf{K}} N_\mu^0$. Let $f := f_\mu \restriction N$. Then f is the desired $\mathbf{K}$-embedding of N inside M fixing M_0.

This is possible: The base case has already been specified and at limits we take unions. Suppose now that $i = j + 1$ and stage j has been implemented. Since $N \leq_{\mathbf{K}} N_j^1$, $a_i \in N_j^1$. Let $q_i := \mathbf{tp}(a_i/N_j^0; N_j^1)$. Let $M_j := f_j[N_j^0]$ and let $g : N_j^1 \cong M_j'$ be an extension of f_j. Let $p_i := \mathbf{tp}(g(a_i)/M_j; M_j')$ (so $p_i = \bar{f}(q_i)$, see Definition 7.8). By assumption, p_i is realized in M, say by b_i. Thus there exists $M_j'' \in \mathbf{K}_{\leq \mu}$ with $M_j' \leq_{\mathbf{K}} M_{j''}$ and $h : M \xrightarrow[M_j]{} M_j''$ such that $h(b_i) = g(a_i)$. Let $g' : N_i^1 \cong M_j''$ be an extension of g. Let $M_i \in \mathcal{P}_{\mathbf{K}_{\leq \mu}}(M)$ be such that $M_j \leq_{\mathbf{K}} M_i$ and $b_i \in M_i$. Let $N_i^0 := (g')^{-1}h[M_i]$. Let $f_i := h^{-1}g' \upharpoonright N_i^0$.

$\square$

Remark 7.18. It suffices to assume that $\mathbf{K}_{<(\lambda + \|M\|)}$ has amalgamation.

We deduce a more local technical lemma which will have several other interesting consequences:

Definition 7.19. For $\mathbf{K}$ an abstract class and $M, N \in \mathbf{K}$, we say that N is *universal over* M if $M \leq_{\mathbf{K}} N$ and whenever $M' \in \mathbf{K}$ is such that $M \leq_{\mathbf{K}} M'$ and $\|M'\| = \|M\|$, there is $f : M' \xrightarrow[M]{} N$.

Lemma 7.20 (The universal extension construction lemma)**.** Let $\mathbf{K}$ be an abstract class satisfying all the axioms of AECs except perhaps the LST axiom. Assume that $\mathbf{K}$ has amalgamation. Let λ be a cardinal and let $\langle M_i : i \leq \lambda \rangle$ be an increasing continuous chain in $\mathbf{K}$ such that $\lambda = \|M_0\|$. If for any $i < \lambda$, any $p \in \mathbf{S}(M_i)$ is realized in M_{i+1}, then M_λ is universal over M_0.

Proof. Exactly as in the proof of Theorem 7.17. We require that $f_i : N_i^0 \cong M_i$. $\square$

Definition 7.21. Let $\mathbf{K}$ be an AEC and let $\lambda \geq \mathrm{LS}(\mathbf{K})$. We say that $\mathbf{K}$ is *stable in* λ if $|\mathbf{S}(M)| \leq \lambda$ for all $M \in \mathbf{K}_\lambda$.

Corollary 7.22. Let $\mathbf{K}$ be an AEC and let $\lambda \geq \mathrm{LS}(\mathbf{K})$. Assume that $\mathbf{K}_\lambda$ has amalgamation and $\mathbf{K}$ is stable in λ. For any $M \in \mathbf{K}_\lambda$, there exists $N \in \mathbf{K}_\lambda$ such that N is universal over M.

Proof. Build an increasing continuous chain $\langle M_i : i \leq \lambda \rangle$ in $\mathbf{K}_\lambda$ such that $M_0 = M$ and M_{i+1} realizes all types over M_i. This is possible by stability in λ. This is enough: by Lemma 7.20, M_λ is universal over M_0. $\square$

8. Tameness

Tameness is a locality property for orbital types first isolated by Grossberg and VanDieren [GV06]. Type-shortness is a generalization introduced by Will Boney [Bon14b]. We only give two variations here.

Definition 8.1. Let $\mathbf{K}$ be an abstract class and let κ be an infinite cardinal.

(1) $\mathbf{K}$ is $(< \kappa)$-*tame* if for any two distinct orbital types $p, q \in \mathbf{S}(M)$ there exists $A \in [M]^{<\kappa}$ such that $p \upharpoonright A \neq q \upharpoonright A$.

(2) $\mathbf{K}$ is $(< \kappa)$-*short* if for any two $M_1, M_2 \in \mathbf{K}$, $\bar{b}_\ell \in {}^\alpha M_\ell$, if $\mathbf{tp}(\bar{b}_1/\emptyset; M_1) \neq \mathbf{tp}(\bar{b}_2/\emptyset; M_2)$, then there exists $I \subseteq \alpha$ with $|I| < \kappa$ such that $\mathbf{tp}(\bar{b}_1 \upharpoonright I/\emptyset; M_1) \neq \mathbf{tp}(\bar{b}_2 \upharpoonright I/\emptyset; M_2)$.

When we omit the κ, we meant "for some κ".

The different between tameness and shortness is the length of the types involved and their domains (tameness is for types of length one over models). In the literature, $(< \kappa)$-short is called "fully $(< \kappa)$-tame and type-short over $\emptyset$". The following is not difficult to show:

Exercise 8.2. If $\mathbf{K}$ is $(< \kappa)$-short, then $\mathbf{K}$ is $(< \kappa)$-tame.

We have seen (Exercise 2.10 and Theorem 2.11) that elementary and universal classes are both $(< \aleph_0)$-short. The following is a trivial non-example:

Example 8.3. Let $\mathbf{K} = (K, \leq_{\mathbf{K}})$ be defined by $K := \{M \mid M \cong (\mathbb{Q}, <)\}$ and $M \leq_{\mathbf{K}} N$ if and only if $M, N \in K$ and $M = N$. Then $\mathbf{K}$ is *not* $(< \aleph_0)$-short, since $\mathbf{tp}(1/(0,1); \mathbb{Q}) \neq \mathbf{tp}(2/(0,1); \mathbb{Q})$ (there is no automorphism of $\mathbb{Q}$ sending 1 to 2 fixing $(0,1)$) but all the finite restrictions of these types are equal.

There are various less trivial examples of non-tameness [BS08, BK09]. The following is due to Will Boney [Bon14b]:

Theorem 8.4. Let $\mathbf{K}$ be an AEC and let $\kappa > \mathrm{LS}(\mathbf{K})$ be a strongly compact cardinal. Then $\mathbf{K}$ is $(< \kappa)$-short.

Boney's proof uses closure of AECs under sufficiently-complete ultraproducts (which follows from the presentation theorem and the fact that reducts commute with ultraproducts). Later Lieberman and Rosický [LR16, 5.2] found a different proof using an older category-theoretic result of Makkai and Paré. We present here yet another proof (unpublished) which uses compactness of $\mathbb{L}_{\kappa,\kappa}$ directly. The proof actually generalizes to μ-AECs, just like the ones mentioned earlier (see also [BGL$^+$16, §5]).

Proof of Theorem 8.4. We more generally prove the statement for any μ-AEC. We will assume for notational simplicity that $\mathbf{K}$ has amalgamation (more precisely that $\equiv^{\mathbf{K}} = E_{\mathrm{at}}$) but if $\mathbf{K}$ does not have amalgamation a similar proof (with more coding) also gives the result. Since shortness is invariant under taking functorial expansions, we may assume without loss of generality (Theorems 6.18 and 6.19) that $\mathbf{K}$ is axiomatized by an $\mathbb{L}_{\kappa,\kappa}$-sentence ϕ and $\mathbf{K}$ is model-complete. Let $\tau := \tau(\mathbf{K})$.

Let $M_1, M_2 \in \mathbf{K}$. Let $\bar{b}_\ell \in {}^\alpha M_\ell$. Suppose that $(\bar{b}_1 \restriction I, M_1) \equiv (\bar{b}_2 \restriction I, M_2)$ for all $I \in [\alpha]^{<\kappa}$. Without loss of generality, $M_1 \cap M_2 = \emptyset$. Let τ_ℓ be τ expanded with new constants symbols $\langle c_a : a \in M_\ell \rangle$. Let M_ℓ^+ be the expansions of M_ℓ to τ_ℓ. Let T_ℓ be the $\mathbb{L}_{\kappa,\kappa}$-quantifier-free diagram of M_ℓ^+. Let B_ℓ be the range of $\bar{b}_\ell$ and let f be a map sending $\bar{b}_1$ to $\bar{b}_2$. Let T be the $\mathbb{L}_{\kappa,\kappa}$-theory $\{\phi\} \cup T_1 \cup T_2 \cup \{c_b = c_{f(b)} \mid b \in B_1\}$. It suffices to prove that T is consistent. By the compactness theorem for $\mathbb{L}_{\kappa,\kappa}$, it suffices to prove that T is $(< \kappa)$-consistent. This is given by the assumption that $\mathbf{tp}(\bar{a}_1 \restriction I; M_1) = \mathbf{tp}(\bar{a}_2 \restriction I; M_2)$ for any $I \in [\alpha]^{<\kappa}$: any M witnessing this will (in a suitable expansion) model T. $\square$

Recently, Boney and Unger [BU17] (building on earlier work of Shelah [She]) found an example of an AEC $\mathbf{K}$ which is tame if and only if there is an (almost) strongly compact above $\mathrm{LS}(\mathbf{K})$. Thus the statement "every AEC is tame" is a large cardinal axiom.

The following characterization of shortness in terms of functorial expansion appears in [Vas16]. We first expand **K** with a symbol for each orbital type:

Definition 8.5. Let **K** be an abstract class. The $(< \kappa)$-*orbital Morlyeization* of **K** is given by adding an $\ell(p)$-ary relation symbol R_p for each $p \in \mathbf{S}^{<\kappa}(\emptyset)$ and expanding each $M \in \mathbf{K}$ to M^+ with $R_p^{M^+}(\bar{b})$ holding if and only if $\mathbf{tp}(\bar{b}/\emptyset; M) = p$.

Exercise 8.6. Prove that the $(< \kappa)$-orbital Morleyization of **K** is a functorial expansion.

Exercise 8.7. Let **K** be an abstract class. The following are equivalent:

(1) **K** is $(< \kappa)$-short.
(2) The map sending each $p = \mathbf{tp}(\bar{b}/\emptyset; M) \in \mathbf{S}^{<\infty}(\emptyset)$ to the quantifier-free type of $\bar{b}$ inside M^+ is an injection, where M^+ is the expansion of M in the $(< \kappa)$-orbital Morleyization.

9. Amalgamation from diamond

The following result is due to Shelah [She87a].

Theorem 9.1. Let **K** be an AEC and let $\lambda \geq \mathrm{LS}(\mathbf{K})$. Assume $2^\lambda < 2^{\lambda^+}$. If **K** is categorical in λ and $\mathbb{I}(\mathbf{K}, \lambda^+) < 2^{\lambda^+}$, then $\mathbf{K}_\lambda$ has amalgamation.

Here, $\mathbb{I}(\mathbf{K}, \lambda^+)$ denotes the number of models of cardinality λ^+ up to isomorphism. The hypothesis that $2^\lambda < 2^{\lambda^+}$ is in general needed: there is an example with $\lambda = \aleph_0$ where Martin's axiom plus $\aleph_1 < 2^{\aleph_0}$ implies that the example is categorical in both $\aleph_0$ and $\aleph_1$ yet fails amalgamation [She09a, §I.6]. We will prove Theorem 9.1 using a stronger hypothesis than $2^\lambda < 2^{\lambda^+}$ known as the diamond principle:

Definition 9.2. For an uncountable regular cardinal λ, $\Diamond_\lambda$ is the statement that there exists a sequence $\langle A_i : i < \lambda \rangle$ such that $A_i \subseteq i$ and for any $X \subseteq \lambda$, the set $\{i < \lambda \mid X \cap i = A_i\}$ is stationary.

If $V = L$, $\Diamond_\lambda$ holds for any uncountable regular λ (this is due to Jensen, who also introduced $\Diamond$; see [Kun80, VI.5.2]). On the other hand, $\Diamond_\lambda$ implies that $2^{<\lambda} = \lambda$ (since any bounded subset of λ must be equal to some A_i). Thus $\Diamond_\lambda$ is independent of ZFC (at least when λ is a successor cardinal). We will use the following form of $\Diamond$:

Exercise 9.3. Let λ be an uncountable regular cardinal. Then $\Diamond_\lambda$ is equivalent to:

There are $\{\eta_\alpha, \nu_\alpha : \alpha \to 2 \mid \alpha < \lambda\}, \{g_\alpha : \alpha \to \alpha \mid \alpha < \lambda\}$ such that for all $\eta, \nu : \lambda \to 2, g : \lambda \to \lambda, \{\alpha < \lambda \mid \eta_\alpha = \eta \restriction \alpha, \nu_\alpha = \nu \restriction \alpha, g_\alpha = g \restriction \alpha\}$ is stationary.

Before proving Theorem 9.1, we need one more fact:

Exercise 9.4. Let **K** be an AEC and let $\lambda > \mathrm{LS}(\mathbf{K})$ be a regular cardinal. Let $M, N \in \mathbf{K}_\lambda$ and let $f : M \cong N$. Let $\langle M_i : i \leq \lambda \rangle$, $\langle N_i : i \leq \lambda \rangle$ be increasing continuous resolutions of M, N respectively (in particular, $M_\lambda = M$, $N_\lambda = N$, $\|M_i\| + \|N_i\| < \lambda$ for all $i < \lambda$). Then the set of ordinals $\alpha < \lambda$ such that $f \restriction M_\alpha : M_\alpha \cong N_\alpha$ is a club.

Proof of Theorem 9.1 assuming $\Diamond_{\lambda^+}$. Suppose that $\mathbf{K}_\lambda$ fails to have amalgamation. Using categoricity in λ, it is easy to see that $\mathbf{K}_\lambda$ has no maximal models. Fix $\langle \eta_\alpha, \nu_\alpha, g_\alpha : \alpha < \lambda^+ \rangle$ as given by Exercise 9.3 (where λ there stands for λ^+ here)

Build a strictly increasing continuous tree $\{M_\eta \mid \eta \in {}^{\leq \lambda^+} 2\}$ such that:

(1) $|M_\eta| \subseteq \lambda^+$ for all $\eta \in {}^{\leq \lambda^+} 2$
(2) If $|M_{\eta_\delta}| = \delta$, $\eta_\delta \neq \nu_\delta$, and $g_\delta : M_{\eta_\delta} \cong M_{\nu_\delta}$ is an isomorphism, then it cannot be extended to an embedding of $M_{\eta_\delta \frown i}$ into M_ν for all $\nu \supseteq \nu_\delta \frown j$, $\nu \in {}^{<\lambda^+} 2$, for all $i, j \in 2$.

<u>This is enough:</u> We show that $M_\eta \not\cong M_\nu$ for $\eta \neq \nu \in {}^{\lambda^+} 2$. Suppose for a contradiction that $f : M_\eta \cong M_\nu$ is an isomorphism. Note that $\{\alpha < \lambda^+ \mid |M_\alpha| = \alpha\}$ is club, and so is $\{\alpha < \lambda^+ \mid f \restriction M_{\eta \restriction \alpha} : M_{\eta \restriction \alpha} \cong M_{\nu \restriction \alpha}\}$. Thus using diamond, there is a stationary set of $\delta < \lambda^+$ such that $\eta \restriction \delta \neq \nu \restriction \delta$, $\eta_\delta = \eta \restriction \delta, \nu_\delta = \nu \restriction \delta$, $g_\delta = f \restriction \delta$, $\delta = |M_{\eta_\delta}| = |M_{\nu_\delta}|$, and $g_\delta : M_{\eta_\delta} \cong M_{\nu_\delta}$. But f extends g_δ and restricts to an embedding of $M_{\eta \frown \eta(\delta)}$ into $M_{\nu \restriction \gamma}$, for some $\lambda^+ > \gamma > \delta$ sufficiently large. This contradicts the first property of the construction.

<u>This is possible:</u> Take any $M_{<>} \in \mathbf{K}$ with $|M_{<>}| = \lambda$ for the base case, and take unions at limits. Now if one wants to define $M_{\eta \frown l}$ for $\eta \in {}^{\delta} 2$ (assuming by induction that M_ν for all $\nu \in {}^{\leq \delta} 2$ have been defined) take any two strict extensions, unless $|M_\eta| = \delta$, $\eta_\delta \neq \nu_\delta$, $g_\delta : M_{\eta_\delta} \cong M_{\nu_\delta}$ is an isomorphism, and either $\eta = \eta_\delta$, or $\eta = \nu_\delta$. We show what to do when $\eta = \eta_\delta$. The other case is symmetric.

By failure of amalgamation and categoricity, we know that there exists M^1, M^2 extensions of M_{η_δ}, M_{ν_δ} respectively such that there is *no* $N \in \mathbf{K}$ and $f_\ell : M^\ell \to N$ commuting with g_δ. Now let let $M_{\eta_\delta \frown l}, M_{\nu_\delta \frown l}$ be two copies of M^1, M^2 respectively. $\square$

10. Existence from successive categoricity

The goal of this section is to prove:

Theorem 10.1 (Shelah, [She87a])**.** Let $\psi \in \mathbb{L}_{\omega_1, \omega}$. If ψ is categorical in $\aleph_0$ and $\aleph_1$, then ψ has a model of cardinality $\aleph_2$.

This result has a long history, recalled e.g. at the beginning of [She09a, Chapter I]. There is also an exposition by Makowsky [BFB85, Chapter XX].

The proof and some more work also answers negatively Baldwin's question: is there an $\mathbb{L}(Q)$-formula with exactly one uncountable model?

We will more generally prove the result for any $\mathrm{PC}_{\aleph_0}$-AEC:

Definition 10.2. An AEC $\mathbf{K}$ is $\mathrm{PC}_{\aleph_0}$ if $\mathrm{LS}(\mathbf{K}) = \aleph_0$ and there is a countable expansion τ^+ of $\tau(\mathbf{K})$ and a universal $\mathbb{L}_{\omega_1, \omega}(\tau^+)$-sentence ϕ such that $\mathbf{K} = \mathrm{Mod}(\phi) \restriction \tau(\mathbf{K})$ and if $N^+ \models \phi$ and $M^+ \subseteq N^+$, then $M^+ \restriction \tau(\mathbf{K}) \leq_{\mathbf{K}} N^+ \restriction \tau(\mathbf{K})$.

By Skolemizing, we get that for any $\mathbb{L}_{\omega_1, \omega}$ formula ψ and any countable fragment Φ containing ψ, the AEC $(\mathrm{Mod}(\psi), \preceq_\Phi)$ is $\mathrm{PC}_{\aleph_0}$. This is the main example to keep in mind but there are others. In fact [BL16, 3.3]:

Fact 10.3. An AEC $\mathbf{K}$ with $\mathrm{LS}(\mathbf{K}) = \aleph_0$ is $\mathrm{PC}_{\aleph_0}$ if and only if $\{(M, N) \mid M, N \in \mathbf{K}_{\leq \aleph_0}, M \leq_{\mathbf{K}} N\}$ is analytic (when seen as a set of reals).

Our goal is:

Theorem 10.4. Let $\mathbf{K}$ be a $\mathrm{PC}_{\aleph_0}$ AEC. If $\mathbf{K}$ is categorical in $\aleph_0$ and $\aleph_1$, then $\mathbf{K}$ has a model of cardinality $\aleph_2$.

Note that it is open whether for an arbitrary AEC $\mathbf{K}$ and a $\lambda \geq \mathrm{LS}(\mathbf{K})$, categoricity in λ and λ^+ implies existence in λ^{++}. The best approximation is due to Shelah [She01] who proved assuming some set-theoretic hypothesis that categoricity in *three* (not two) successive cardinals implies existence in the next.

To prove Theorem 10.4, we need some sufficient conditions for existence of models. The following is basic:

Exercise 10.5. Let $\mathbf{K}$ be an AEC and let $\lambda \geq \mathrm{LS}(\mathbf{K})$. Assume that $\mathbf{K}$ is categorical in λ. The following are equivalent:

  (1) $\mathbf{K}_{\lambda^+} \neq \emptyset$.
  (2) There exists $M, N \in \mathbf{K}_\lambda$ such that $M <_{\mathbf{K}} N$.
  (3) $\mathbf{K}_\lambda$ has no maximal models.

We will apply Exercise 10.5 with $\lambda = \aleph_1$. What we need now is a criteria in $\aleph_0$ to ensure that there exists a pair (M, N) in $\aleph_1$ with $M <_{\mathbf{K}} N$. The following does the trick:

Definition 10.6. Let $\mathbf{K}$ be an abstract class.

  (1) We call an orbital type p of length one *algebraic* if whenever $p = \mathbf{tp}(a/M; N)$, $a \in |M|$. We write $\mathbf{S}^{\mathrm{na}}(M)$ for the class of non-algebraic types over M, and $\mathbf{K}^{3,\mathrm{na}}$ for the class of non-algebraic triples: triples (a, M, N) such that $M \leq_{\mathbf{K}} N$ and $a \in |N| \backslash |M|$.
  (2) We say that a triple $(a, M, N) \in \mathbf{K}^{3,\mathrm{na}}$ has the *extension property* if for any $M' \in \mathbf{K}$ with $M \leq_{\mathbf{K}} M'$, there exists $q \in \mathbf{S}(M')$ such that $q \upharpoonright M = \mathbf{tp}(a/M; N)$.
  (3) For $(a, M, N), (b, M', N') \in \mathbf{K}^{3,\mathrm{na}}$, write $(a, M, N) <_{\mathbf{K}} (b, M', N')$ if $a = b$, $N \leq_{\mathbf{K}} N'$, and $M <_{\mathbf{K}} M'$.
  (4) We say that $(a, M, N) \in \mathbf{K}^{3,\mathrm{na}}$ has the *weak extension property* if it is *not* $<_{\mathbf{K}}$-maximal in $\mathbf{K}^{3,\mathrm{na}}$.
  (5) We say that $\mathbf{K}$ has the *[weak] extension property* if every triple in $\mathbf{K}^{3,\mathrm{na}}$ has the [weak] extension property.

We write $\mathbf{K}_\lambda^{3,\mathrm{na}}$ for $\mathbf{K}_\lambda{}^{3,\mathrm{na}}$.

The weak extension property is all we will use. However we mentioned the extension property as well, since it seems more natural. We have the following relationship between the two:

Exercise 10.7. Let $\mathbf{K}$ be an abstract class with no maximal models. Show that the extension property implies the weak extension property.

Lemma 10.8. Let $\mathbf{K}$ be an AEC and let $\lambda \geq \mathrm{LS}(\mathbf{K})$. If $\mathbf{K}_\lambda^{3,\mathrm{na}} \neq \emptyset$ and $\mathbf{K}_\lambda$ has the weak extension property, then there exists $M, N \in \mathbf{K}_{\lambda^+}$ such that $M <_{\mathbf{K}} N$.

Proof. Build a $<_{\mathbf{K}}$-increasing chain $\langle(a, M_i, N_i) : i < \lambda^+\rangle$ in $\mathbf{K}_\lambda^{3,\mathrm{na}}$. Take any member of $\mathbf{K}_\lambda^{3,\mathrm{na}}$ for the base case. At successors, use the weak extension property and at limits take unions. In the end, let $M := \bigcup_{i<\lambda^+} M_i$, $N := \bigcup_{i<\lambda^+} N_i$. Since the chain was strictly increasing, both M and N are in $\mathbf{K}_{\lambda^+}$. By definition of $\mathbf{K}_\lambda^{3,\mathrm{na}}$, $a \in N_i \backslash M_i$ for any $i < \lambda^+$. Thus $a \in N \backslash M$, so $M <_{\mathbf{K}} N$, as desired. $\square$

A partial converse to Lemma 10.8 holds: if $\mathbf{K}$ is categorical in λ and has no maximal models in λ^+, then $\mathbf{K}_\lambda$ has the weak extension property [She09b, VI.1.9]. In particular, if $\mathbf{K}$ is categorical in λ and λ^+, then $\mathbf{K}_{\lambda^{++}} \neq \emptyset$ if and only if $\mathbf{K}_\lambda$ has the weak extension property.

We will be done once we have proven the following:

Theorem 10.9. Let $\mathbf{K}$ be a $\mathrm{PC}_{\aleph_0}$ AEC. Assume that $\mathbf{K}$ is categorical in $\aleph_0$ and $\mathbf{K}_{\aleph_0}$ has no maximal models. If $\mathbf{K}_{\aleph_0}$ does *not* have the weak extension property, then $\mathbb{I}(\mathbf{K}, \aleph_1) = 2^{\aleph_1}$.

Proof of Theorem 10.4. By Exercise 10.5 (with λ there standing for $\aleph_0$ here), $\mathbf{K}_{\aleph_0}$ has no maximal models. By Theorem 10.9, $\mathbf{K}_{\aleph_0}$ must have the weak extension property. By Lemma 10.8, there exists $M, N \in \mathbf{K}_{\aleph_1}$ such that $M <_{\mathbf{K}} N$. By Exercise 10.5 (with λ there standing for $\aleph_1$ here), this implies that $\mathbf{K}_{\aleph_2} \neq \emptyset$. $\square$

There are two keys to Theorem 10.9. The first (and the only place where we use that $\mathbf{K}$ is $\mathrm{PC}_{\aleph_0}$) is a consequence of the undefinability of well-orderings in such classes:

Lemma 10.10 (The magic lemma). Let $\mathbf{K}$ be a $\mathrm{PC}_{\aleph_0}$ AEC. If $\mathbf{K}_{\aleph_1} \neq \emptyset$, then there exists a *strictly decreasing continuous* chain $\langle M_i : i \leq \omega\rangle$ in $\mathbf{K}_{\aleph_0}$. That is, $M_j <_{\mathbf{K}} M_i$ for all $i < j \leq \omega$ and $M_\omega = \bigcap_{i<\omega} M_i$.

The second key to Theorem 10.9 is Solovay's splitting theorem [Jec03, 8.10].

Fact 10.11. Let λ be a regular uncountable cardinal and let S be a stationary subset of λ. Then there exists λ-many pairwise disjoint stationary subsets of S.

Exercise 10.12. Show that for any regular cardinal λ and any stationary set S of λ, there exists $\langle S_i : i < 2^\lambda\rangle$ stationary subsets of S such that $i \neq j$ implies that $S_i \Delta S_j$ is stationary. *Hint: let $\langle T_j : j < \lambda\rangle$ be pairwise disjoint stationary subsets of λ and for a non-empty $A \subseteq \lambda$, consider $S_A := \bigcup_{j\in A} T_j$.*

Assuming the magic lemma for now, let us prove the theorem.

Proof of Theorem 10.9. For each stationary subset S of $\aleph_1$, we build an increasing continuous chain $\langle M_i^S : i \leq \aleph_1\rangle$ and $\langle a_i^S : i \in S\rangle$ such that for all $i < \aleph_1$:

(1) $M_i^S \in \mathbf{K}_{\aleph_0}$.
(2) If $i \in S$, then $(a_i^S, M_i^S, M_{i+1}^S) \in \mathbf{K}_{\aleph_0}^{3,\mathrm{na}}$ and is $<_{\mathbf{K}}$-maximal.
(3) If $i \notin S$, then there exists a strictly decreasing continuous chain $\langle N_j : j \leq \omega\rangle$ in $\mathbf{K}_{\aleph_0}$ such that $M_i^S = N_\omega$ and $M_{i+1}^S = N_0$.

This is possible: At limits, take unions. For $i = 0$, take any $M_0^S \in \mathbf{K}_{\aleph_0}$. Assume now that $\langle M_j^S : j \leq i\rangle$ and $\langle a_j^S : j < i\rangle$ have been defined. We show how to define M_{i+1}^S and a_i^S. There are two cases.

- If $i \in S$, fix $(a, M, N) \in \mathbf{K}_{\aleph_0}^{3,\mathrm{na}}$ which is $<_{\mathbf{K}}$-maximal (exists by assumption). Let $f : M \cong M_i^S$, and extend it to $g : N \cong M_{i+1}^S$. Let $a_i^S := f(a)$.
- If $i \notin S$, then by Lemma 10.10 and categoricity, we can get a strictly decreasing continuous chain $\langle N_j : j \leq \omega \rangle$ in $\mathbf{K}_{\aleph_0}$ such that $N_\omega = M_i^S$. Let $M_{i+1}^S := N_0$.

This is enough: We claim that if S and T are stationary subsets of $\aleph_1$ such that $S \Delta T$ is stationary, then $M_{\aleph_1}^S \not\cong M_{\aleph_1}^T$. This suffices by Exercise 10.12. So let S and T be such that $S \Delta T$ is stationary, and suppose for a contradiction that $f : M_{\aleph_1}^S \cong M_{\aleph_1}^T$. By Exercise 9.4, there exists a club C such that $i \in C$ implies that $f_i := f \upharpoonright M_i^S$ is an isomorphism from M_i^S onto M_i^T. Suppose without loss of generality that $S \backslash T$ is stationary, and let $i \in (S \backslash T) \cap C$.

Since $i \in S$, we have that $(a_i^S, M_i^S, M_{i+1}^S)$ is in $\mathbf{K}_{\aleph_0}^{3,\mathrm{na}}$ and $<_{\mathbf{K}}$-maximal. Let $i' < \aleph_1$ be such that $f[M_{i+1}^S] \leq_{\mathbf{K}} M_{i'}^T$ and $i' > i + 1$. Since $i \notin T$, one can fix a strictly decreasing continuous chain $\langle N_j : j \leq \omega \rangle$ such that $M_i^T = N_\omega$, $M_{i+1}^T = N_1$, and $M_{i'}^T = N_0$. Since $f(a_i^S) \in M_{i'}^T$ and $f(a_i^S) \notin M_i^T = \bigcap_{j < \omega} N_j$, there must exist a least $\ell < \omega$ such that $f(a_i^S) \in N_\ell \backslash N_{\ell+1}$. This shows that $(f(a_i^S), M_i^T, f[M_{i+1}^S]) <_{\mathbf{K}} (f(a_i^S), N_{\ell+1}, M_{i'}^T)$, so $(f(a_i^S), M_i^T, f[M_{i+1}^S])$ is not $<_{\mathbf{K}}$-maximal, so $(a_i^S, M_i^S, M_{i+1}^S)$ is not $<_{\mathbf{K}}$-maximal, a contradiction. $\square$

It remains to prove the magic lemma. For this we will use undefinability of uncountable well-orderings in $\mathbb{L}_{\omega_1, \omega}$, due to Lopez-Escobar and (independently) Morley. To state it in a general setup, we introduce some notation.

Definition 10.13. For λ an infinite cardinal and $\mu \geq 1$, let $\delta(\lambda, \mu)$ be the least ordinal δ such that whenever $T \subseteq \mathbb{L}_{\lambda^+, \omega}$ is such that:

1. $|T| \leq \mu$ and $|\tau(T)| \leq \lambda$.
2. $\tau(T)$ contains a predicate P and a binary relation $<$ such that in any model M of T, $(P^M, <^M \upharpoonright P^M)$ is a linear order.
3. For every $\alpha < \delta$, there exists $M \models T$ with α an initial segment of $(P^M, <^M \upharpoonright P^M)$.

Then there exists $M \models T$ where $(P^M, <^M \upharpoonright P^M)$ is ill-founded.

We let $\delta(\lambda) := \delta(\lambda, 1)$.

The proofs of the following facts can all be found in [She90, §VIII.5]:

Fact 10.14.

1. $\lambda^+ \leq \delta(\lambda, \mu) \leq (2^\lambda)^+$ and $\delta(\lambda) < (2^\lambda)^+$.
2. $\delta(\lambda) = \lambda^+$ whenever λ is a strong limit of cofinality $\aleph_0$. In particular, $\delta(\aleph_0) = \omega_1$.
3. $\delta(\lambda) > \lambda^+$ whenever $\mathrm{cf}(\lambda) > \aleph_0$

Note that Theorem 10.4 generalizes to any AEC $\mathbf{K}$ with $\mathrm{LS}(\mathbf{K}) = \lambda$ which is the reduct of a universal $\mathbb{L}_{\lambda^+, \omega}$-sentence with $\delta(\lambda) = \lambda^+$.

Proof of Lemma 10.10. Let $\langle N_i : i < \aleph_1 \rangle$ be strictly increasing continuous in $\mathbf{K}_{\aleph_0}$. This exists: take a resolution of the model in $\mathbf{K}_{\aleph_1}$. Now let χ be a "big" cardinal,

and let τ^+ be a countable expansion of the vocabulary of set theory (it should have "enough" symbols, in a sense that the proof will tell us). Let $\mathfrak{B}$ be a τ^+-expansion of $(V_\chi, \in)$ (V_χ is the χth level of the cumulative hierarchy). In particular, we require that there is a binary function symbol $f \in \tau^+$ where $f^{\mathfrak{B}} : \omega_1 \to \mathbf{K}$, $f(i) := N_i$. We also ask that $\mathfrak{B}$ contains the definition of the AEC $\mathbf{K}$, that the natural numbers of $\mathfrak{B}$ are well-founded, and that τ^+ also contains bijections from the natural number to every N_i. Code all this data into an $\mathbb{L}_{\omega_1, \omega}(\tau^+)$-sentence ϕ. Now since $\delta(\aleph_0) = \omega_1$, there exists $\mathfrak{B}^*$ an *ill-founded* model of set theory such that $\mathfrak{B}^* \equiv \mathfrak{B}$, and $\mathfrak{B}^* \models \phi$. Let $\langle \alpha_n : n < \omega \rangle$ be a strictly decreasing sequence of "countable ordinals" in $\mathfrak{B}^*$. Let $I := \{\alpha \in \mathfrak{B}^* \mid \forall n < \omega : \alpha < \alpha_n\}$. This is a linear order, hence a directed system. Let $M_n := f^{\mathfrak{B}^*}(\alpha_n)$. Then since enough of the definition of $\mathbf{K}$ is reflected, $M_n \in \mathbf{K}$ and $M_{n+1} <_{\mathbf{K}} M_n$ for $n < \omega$. For $\alpha \in I$, let $M_\alpha := f(\alpha)$. We then also have that $M_\alpha \in \mathbf{K}$ and $M_\alpha \leq_{\mathbf{K}} M_n$ for all $n < \omega$. Let $M := \bigcup_{\alpha \in I} M_\alpha$. Clearly, $M \leq_{\mathbf{K}} M_n$ for all $n < \omega$. Now if $a \in \bigcap_{n < \omega} M_n$, then $\mathfrak{B}^*$ thinks there is a least ordinal β such that $a \in f(\beta)$. Now $\beta < \alpha_n$ for all $n < \omega$, so $\beta \in I$, hence $a \in M_\beta$, so $a \in M$. This means that $M = \bigcap_{n < \omega} M_n$, as desired. □

11. Ehrenfeucht-Mostowski models and stability

Everywhere in this section, $\mathbf{K}$ is an AEC.

We go very fast on the material related to EM models, since there is already a lot of details about them in [Bal09], and Will Boney covered them extensively in the past semester. Most of the results from this section can be found in [Vas17b], see there for attribution.

We will not use EM models too much in this course. The goal of this section is to use them to prove that (in an AEC with amalgamation) categoricity implies stability and failure of the order property. In this section, we will use EM models to prove a local superstability condition. Once we have proven these properties, we will start out by assuming them and develop a local classification theory.

We will use the notation from [She09a, Chapter IV]:

Definition 11.1. [She09a, Definition IV.0.8] Let $\mathbf{K}$ be an AEC. For $\mu \geq \mathrm{LS}(\mathbf{K})$, let $\Upsilon_\mu[\mathbf{K}]$ be the set of Φ proper for linear orders (that is, Φ is a set $\{p_n : n < \omega\}$, where p_n is an n-variable quantifier-free type in a fixed vocabulary $\tau(\Phi)$ and the types in Φ can be used to generate a $\tau(\Phi)$-structure $\mathrm{EM}(I, \Phi)$ for each linear order I; that is, $\mathrm{EM}(I, \Phi)$ is the closure under the functions of $\tau(\Phi)$ of the universe of I and for any $i_0 < \ldots < i_{n-1}$ in I, $i_0 \ldots i_{n-1}$ realizes p_n) with:

(1) $|\tau(\Phi)| \leq \mu$.
(2) If I is a linear order of cardinality λ, $\mathrm{EM}_{\tau(\mathbf{K})}(I, \Phi) \in \mathbf{K}_{\lambda + |\tau(\Phi)| + \mathrm{LS}(\mathbf{K})}$, where $\tau(\mathbf{K})$ is the vocabulary of $\mathbf{K}$ and $\mathrm{EM}_{\tau(\mathbf{K})}(I, \Phi)$ denotes the reduct of $\mathrm{EM}(I, \Phi)$ to $\tau(\mathbf{K})$. Here we are implicitly also assuming that $\tau(\mathbf{K}) \subseteq \tau(\Phi)$.
(3) For $I \subseteq J$ linear orders, $\mathrm{EM}_{\tau(\mathbf{K})}(I, \Phi) \leq_{\mathbf{K}} \mathrm{EM}_{\tau(\mathbf{K})}(J, \Phi)$.

We call Φ as above an *EM blueprint*.

While the definitions are somewhat technical, it turns out the only thing that really matters is that the map $I \mapsto \mathrm{EM}_\tau(I, \Phi)$ is a faithful functor from the category of linear orders into $\mathbf{K}$.

The following follows from Shelah's presentation theorem (Theorem 3.6) and Morley's omitting type theorem. We will use it without explicit mention. See for example [Bal09, Appendix A]

Notation 11.2. $h(\mu) := \beth_{(2^\mu)^+}$.

Fact 11.3. Let $\mathbf{K}$ be an AEC. The following are equivalent:

(1) $\mathbf{K}$ has arbitrarily large models.
(2) $\mathbf{K}_{\geq \mu} \neq \emptyset$ for all $\mu < h(\mathrm{LS}(\mathbf{K}))$.
(3) For any $\mu \geq \mathrm{LS}(\mathbf{K})$, $\Upsilon_\mu[\mathbf{K}] \neq \emptyset$.
(4) For some $\mu \geq \mathrm{LS}(\mathbf{K})$, $\Upsilon_\mu[\mathbf{K}] \neq \emptyset$.

We have the following important property of types computed inside EM models:

Lemma 11.4. Let Φ be an EM blueprint. Let $M_\ell := \mathrm{EM}_\tau(I_\ell, \Phi)$. Let $\bar{a}_\ell \in {}^{<\infty} I_\ell$. If $\bar{a}_1$ and $\bar{a}_2$ are isomorphic as linear orders, then for any sequence of $\tau(\Phi)$-term $\bar{\rho}$, $(\bar{\rho}(\bar{a}_1), M_1) \equiv^{\mathbf{K}} (\bar{\rho}(\bar{a}_2), M_2)$.

Proof. We have that $\mathrm{EM}(\bar{a}_1; \Phi)$ and $\mathrm{EM}(\bar{a}_2; \Phi)$ are isomorphic, hence $\mathrm{EM}_\tau(\bar{a}_1; \Phi)$ and $\mathrm{EM}_\tau(\bar{a}_2; \Phi)$ also are. $\square$

The following concept is key:

Definition 11.5. Let Φ be an EM blueprint. Let I, J be a linear orders, let δ be a limit ordinal and let $\langle \bar{a}_j : j \in J \rangle$ be a sequence. We say that $\langle \bar{a}_j : j \in J \rangle$ is (Φ, I)-*strictly indiscernible* if:

(1) J is infinite.
(2) For some α, for all $j \in J$, $\bar{a}_j \in {}^\alpha \mathrm{EM}_\tau(I, \Phi)$.
(3) There exists a sequence $\langle \bar{a}'_j : j \in J \rangle$ and a sequence of terms $\bar{\rho}$ such that $\bar{a}_j = \bar{\rho}(\bar{a}'_j)$ for all $j \in J$ and $\langle \bar{a}'_j : j \in J \rangle$ is quantifier-free indiscernible in the vocabulary of linear orders inside I.

We call $\langle \bar{a}_j : j \in J \rangle$ (Φ, I)-*strictly indiscernible over A* if $\langle \bar{a}_j \bar{a} : j \in J \rangle$ is (Φ, I)-strictly indiscernible for some (any) enumeration $\bar{a}$ of A.

Using compactness for linear orders, one can prove that strictly indiscernible sequences can be extended:

Exercise 11.6. If $\langle \bar{a}_j : j \in J \rangle$ is a (Φ, I)-strictly indiscernible, then for any $J' \supseteq J$, there exists $I' \supseteq I$ and $\langle \bar{a}_j : j \in J' \rangle$ which is (Φ, I')-strictly indiscernible.

We prove that inside EM models generated by well-ordered sets, one can extract strict indiscernibles. This appears as [She99, Claim 4.15]:

Theorem 11.7 (Strict indiscernible extraction)**.** Let $\mathbf{K}$ be an AEC with arbitrarily large models and let $\mathrm{LS}(\mathbf{K}) < \theta \leq \lambda$ be cardinals with θ regular. Let $\kappa < \theta$ be a (possibly finite) cardinal. Let $\Phi \in \Upsilon_{\mathrm{LS}(\mathbf{K})}[\mathbf{K}]$ be an EM blueprint for $\mathbf{K}$.

Let $N := \mathrm{EM}_{\tau(\mathbf{K})}(\lambda, \Phi)$. Let $M \in \mathbf{K}_{\leq \mathrm{LS}(\mathbf{K})}$ be such that $M \leq_{\mathbf{K}} N$. Let $\langle \bar{a}_i : i < \theta \rangle$ be a sequence of distinct elements such that for all $i < \theta$, $\bar{a}_i \in {}^\kappa |N|$.

If $\theta_0^\kappa < \theta$ for all $\theta_0 < \theta$, then there exists $w \subseteq \theta$ with $|w| = \theta$ such that $\langle \bar{a}_i : i \in w \rangle$ is (Φ, λ)-strictly indiscernible over M.

Proof. First we claim that one can assume without loss of generality that $\kappa <$ LS($\mathbf{K}$). Assume that the statement of the lemma has been proven for that case. If $\kappa >$ LS($\mathbf{K}$) one can replace $\mathbf{K}$ with $\mathbf{K}_{>\kappa}$ (and increase M) so assume that $\kappa \leq$ LS($\mathbf{K}$). Now if $\kappa =$ LS($\mathbf{K}$), then $2^{\text{LS}(\overline{\mathbf{K}})} = \kappa^{\kappa} < \theta$ so we can replace $\mathbf{K}$ by $\mathbf{K}_{\geq \text{LS}(\mathbf{K})^+}$ and work there. Thus assume without loss of generality that $\kappa <$ LS($\mathbf{K}$).

Pick $u \subseteq \lambda$ such that $|u| = \theta$, $M \leq_{\mathbf{K}} N_0 := \text{EM}_{\tau(\mathbf{K})}(u, \Phi)$, and $\bar{a}_i \in {}^{\kappa}|N_0|$ for all $i < \theta$. Increasing M if necessary, we can assume without loss of generality that $M = \text{EM}_{\tau(\mathbf{K})}(u', \Phi)$ for some $u' \subseteq u$ with $|u'| = \text{LS}(\mathbf{K})$.

For each $i < \theta$, we can also pick $u_i \subseteq u$ with $|u_i| < \kappa^+ + \aleph_0$ such that $\bar{a}_i \in {}^{\kappa}|\text{EM}_{\tau(\mathbf{K})}(u_i, \Phi)|$. Without loss of generality $u = u' \cup \bigcup_{i<\theta} u_i$. By the pigeonhole principle, we can without loss of generality fix an ordinal $\alpha < \kappa^+ + \aleph_0$ such that $\text{otp}(u_i) = \alpha$ for all $i < \theta$. List u_i in increasing order as $\bar{u}_i := \langle u_{i,j} : j < \alpha \rangle$. By pruning further (using that $\text{LS}(\mathbf{K})^{\kappa} < \theta$), we can assume without loss of generality that for each $i, i' < \theta$ and $j < \alpha$, the u'-cut of $u_{i,j}$ and $u_{i',j}$ are the same (i.e. for any $\gamma \in u'$, $\gamma < u_{i,j}$ if and only if $\gamma < u_{i',j}$).

Pruning again with the Δ-system lemma, we can assume without loss of generality that $\langle u_i : i < \theta \rangle$ forms a Δ-system (see Definition II.1.4 and Theorem II.1.6 in [Kun80]; at that point we are using that $\theta_0^{\kappa} < \theta$ for all $\theta_0 < \theta$). In fact, the proof of the Δ-system lemma shows that we can make sure that in the end $\langle \bar{u}_i : i < \theta \rangle$ is indiscernible over u' in the vocabulary of linear orders.

Now list $\bar{a}_i$ as $\langle a_{i,j} : j < \kappa \rangle$. Fix $i < \theta$. Since $\bar{a}_i \in {}^{\kappa}|\text{EM}_{\tau(\mathbf{K})}(u_i)|$, for each $j < \kappa$ there exists a $\tau(\Phi)$-term $\rho_{i,j}$ of arity $n := n_{i,j}$ and $j_0^{i,j} < \ldots < j_{n-1}^{i,j} < \alpha$ such that $a_{i,j} = \rho_{i,j} \left(u_{i,j_0^{i,j}} \ldots u_{i,j_{n-1}^{i,j}} \right)$. By the pigeonhole principle applied to the map $i \mapsto \langle (\rho_{i,j}, n_{i,j}, j_0^{i,j}, \ldots, j_{n_{i,j}-1}^{i,j}) : j < \kappa \rangle$ (using that $\text{LS}(\mathbf{K})^{\kappa} < \theta$), we can assume without loss of generality that these depend only on j, i.e. $\rho_{i,j} = \rho_j$, $n_{i,j} = n_j$, and $j_{\ell}^{i,j} = j_{\ell}^j$.

Let $\bar{u}'$ be an enumeration of u', and let $\bar{a}_i' := \bar{u}_i \bar{u}'$. Then $\langle \bar{a}_i' : i < \theta \rangle$ witnesses the strict indiscernibility of $\langle \bar{a}_i : i < \theta \rangle$. $\qquad \square$

We deduce stability inside an EM model generated by a well-ordering:

Corollary 11.8. Let Φ be an EM blueprint. Let $\lambda > $ LS($\mathbf{K}$), $\kappa > 0$ be a cardinal, and let $M := \text{EM}_{\tau}(\lambda, \Phi)$. For any $A \subseteq |M|$, $|\mathbf{S}^{\kappa}(A; M)| \leq (|A| + \text{LS}(\mathbf{K}))^{\kappa}$.

Proof. Let $\theta := ((|A| + \text{LS}(\mathbf{K}))^{\kappa})^+$. Let $\langle \bar{a}_i : i < \theta \rangle$ be an arbitrary sequence of elements of ${}^{\kappa}M$. By Theorem 11.7 and Lemma 11.4, there exists in particular $i < j < \theta$ such that $\mathbf{tp}(\bar{a}_i/A; M) = \mathbf{tp}(\bar{a}_j/A; M)$. This implies that $|\mathbf{S}^{\kappa}(A; M)| < \theta$, as desired. $\qquad \square$

Corollary 11.9. Let $\mathbf{K}$ be an AEC with arbitrarily large models. Let $\lambda > $ LS($\mathbf{K}$). If $\mathbf{K}_{<\lambda}$ has amalgamation and no maximal models and $\mathbf{K}$ is categorical in λ, then $\mathbf{K}$ is stable in every $\mu \in [\text{LS}(\mathbf{K}), \lambda)$.

Proof. Let $\Phi \in \Upsilon_{\text{LS}(\mathbf{K})}[\mathbf{K}]$ be an EM blueprint for $\mathbf{K}$. Let $\mu \in [\text{LS}(\mathbf{K}), \lambda)$. Let $M_0 \in \mathbf{K}_{\mu}$. Let $\langle p_i : i < \mu^+ \rangle$ be types over M_0. By amalgamation, we can make sure they are all realized inside a fixed $M_1 \in \mathbf{K}_{\leq \mu^+}$ with $M_0 \leq_{\mathbf{K}} M_1$. Since $\mathbf{K}$

has no maximal models, there exists $M_2 \in \mathbf{K}_\lambda$ with $M_1 \leq_{\mathbf{K}} M_2$. By categoricity, $M_2 \cong \mathrm{EM}_\tau(\lambda, \Phi)$. Applying Corollary 11.8 and taking an isomorphic image, we get that $|\mathbf{S}(M_0; M_2)| \leq \mu$, hence that there must exist $i < j$ so that $p_i = p_j$. This proves that $|\mathbf{S}(M_0)| \leq \mu$. $\qquad \square$

Another interesting application concerns the order property:

Definition 11.10 (Order property, Definition 4.3 in [She99])**.**

(1) Let $N \in \mathbf{K}$ and let I be a linear order. We say that N *has the (κ, μ)-order property of type (or of length) I* if there exists a sequence $\langle \bar{a}_i : i \in I \rangle$ and a set A such that $\bar{a}_i \in {}^\kappa |N|$ for every $i \in I$, $A \subseteq |N|$, $|A| \leq \mu$, and for every $i_0 < i_1$, $j_0 < j_1$ in I, $\mathbf{tp}(\bar{a}_{i_0} \bar{a}_{i_1}/A; N) \neq \mathbf{tp}(\bar{a}_{j_1} \bar{a}_{j_0}/A; N)$.

(2) We say that $\mathbf{K}$ *has the (κ, μ)-order property of type I* if some $N \in \mathbf{K}$ has it.

(3) We say that $\mathbf{K}$ *has the (κ, μ)-order property* if it has the (κ, μ)-order property of type I for every linear order I.

(4) When $\mu = 0$, we omit it and talk of the κ-order property.

Remark 11.11. For T a first-order theory and $\mathbf{K}$ its corresponding AEC of models, the following are equivalent:

(1) T is unstable.

(2) $\mathbf{K}$ has the $(\kappa, 0)$-order property, for some $\kappa < \aleph_0$.

(3) $\mathbf{K}$ has the (κ, μ)-order property, for some cardinals κ and μ.

The following can be obtained by building an order property indexed by a linear order I with a dense subset I_0 smaller than I (for example, given λ, let μ be least such that $\lambda = 2^{<\mu} < 2^\mu$ and let $I_0 := {}^{<\mu}2$, $I := {}^\mu 2$, both ordered by the lexicographical ordering):

Exercise 11.12. [BGKV16, Fact 5.13] If $\mathbf{K}$ has the (κ, μ)-order property, then there exists $M \in \mathbf{K}$ and $A \subseteq |M|$ such that $|A| = \mu$ but $|\mathbf{S}^\kappa(A; M)| > \mu$.

An easy consequence of Theorem 11.7 is that if a long-enough order property holds, then we can assume that the sequence witnessing it is strictly indiscernible, and hence extend it:

Corollary 11.13. Let $\mathbf{K}$ be an AEC with arbitrarily large models and let $\mathrm{LS}(\mathbf{K}) < \lambda$. Let $\kappa < \lambda$ be a (possibly finite) cardinal. Let $\Phi \in \Upsilon_{\mathrm{LS}(\mathbf{K})}[\mathbf{K}]$ be an EM blueprint for $\mathbf{K}$.

Let $N := \mathrm{EM}_{\tau(\mathbf{K})}(\lambda, \Phi)$. If N has the $(\kappa, \mathrm{LS}(\mathbf{K}))$-order property of length $(\mathrm{LS}(\mathbf{K})^\kappa)^+$ and $\mathrm{LS}(\mathbf{K})^\kappa < \lambda$, then $\mathbf{K}$ has the $(\kappa, \mathrm{LS}(\mathbf{K}))$-order property (of *any* length).

Proof. Set $\theta := (\mathrm{LS}(\mathbf{K})^\kappa)^+$. Fix $\langle \bar{a}_i : i < \theta \rangle$ and A witnessing that N has the $(\kappa, \mathrm{LS}(\mathbf{K}))$-order property of length θ. Using the Löwenheim-Skolem-Tarski axiom, pick $M \in \mathbf{K}_{\mathrm{LS}(\mathbf{K})}$ such that $A \subseteq |M|$ and $M \leq_{\mathbf{K}} N$. By Theorem 11.7, there exists $w \subseteq \theta$ such that $|w| = \theta$ and $\langle \bar{a}_i : i \in w \rangle$ is (Φ, λ)-strictly indiscernible over M. Now check that any extension of that sequence (Exercise 11.6) witnesses the order property. $\qquad \square$

Remark 11.14. By appending an enumeration of the base set to each element of the sequence, we get that the (κ, μ)-order property implies the $(\kappa + \mu)$-order property. However Corollary 11.13 applies more easily to the (κ, μ)-order property: think for example of the case $\kappa < \aleph_0$, when we always have that $\mathrm{LS}(\mathbf{K})^\kappa = \mathrm{LS}(\mathbf{K}) < \lambda$.

Corollary 11.15. Let $\mathbf{K}$ be an AEC with arbitrarily large models. Let $\lambda > \mathrm{LS}(\mathbf{K})$ and assume that $\mathbf{K}_{<\lambda}$ has amalgamation, no maximal models, and $\mathbf{K}$ is categorical in λ. Let $\mu \in [\mathrm{LS}(\mathbf{K}), \lambda)$ and let $\kappa > 0$ be a cardinal. If $\mu^\kappa < \lambda$, then $\mathbf{K}$ does *not* have the (κ, μ)-order property of length $(\mu^\kappa)^+$.

Proof. Let $\Phi \in \Upsilon_{\mathrm{LS}(\mathbf{K})}[\mathbf{K}]$ be an EM blueprint. Replacing μ by μ^κ if necessary, we can assume without loss of generality that $\mu = \mu^\kappa$. Replacing $\mathbf{K}$ by $\mathbf{K}_{\geq\mu}$ if necessary, we can also assume without loss of generality that $\mu = \mathrm{LS}(\mathbf{K})$. Suppose that $\mathbf{K}$ has the (κ, μ)-order property of length μ^+. Then there exists $M \in \mathbf{K}_\lambda$ which has the (κ, μ)-order property of length μ^+. By categoricity, we can assume without loss of generality that $M = \mathrm{EM}_\tau(\lambda, \Phi)$. By Theorem 11.13, $\mathbf{K}$ has the (κ, μ)-order property. By Exercise 11.12, $\mathbf{K}$ must be unstable in μ, contradicting Corollary 11.9. $\square$

We summarize the results of this section in one corollary:

Corollary 11.16. Let $\mathbf{K}$ be an AEC with arbitrarily large models. Let $\lambda > \mathrm{LS}(\mathbf{K})$. Assume that $\mathbf{K}_{<\lambda}$ has amalgamation and no maximal models, and $\mathbf{K}$ is categorical in λ. Let $\mu \in [\mathrm{LS}(\mathbf{K}), \lambda)$. Then:

(1) $\mathbf{K}$ is stable in μ.
(2) $\mathbf{K}$ does *not* have the $(2, \mu)$-order property of length μ^+.

12. Superstability from categoricity

Everywhere in this section, $\mathbf{K}$ is still an AEC. Most of the material here is derived from [BGVV17]. We want to start studying independence notions. The following definition is a starting point:

Definition 12.1 ([She99, 3.2])**.** $p \in \mathbf{S}(N)$ λ-*splits* over M if there exists $N_\ell \in \mathcal{P}_{\mathbf{K}_\lambda}(N)$ such that $M \leq_{\mathbf{K}} N_\ell \leq_{\mathbf{K}} N$, ℓ $1, 2$ and $f : N_1 \cong_M N_2$ such that $f(p \restriction N_1) \neq p \restriction N_2$. When $\lambda = \|N\| = \|M\|$, we may omit it.

The following are basic properties of splitting:

Exercise 12.2. Let $\mathbf{K}$ be an AEC. Let $\lambda \geq \mathrm{LS}(\mathbf{K})$. Assume that $\mathbf{K}_\lambda$ has amalgamation. Let $M \leq_{\mathbf{K}} M' \leq_{\mathbf{K}} N$ all be in $\mathbf{K}_\lambda$.

(1) Invariance: if $p \in \mathbf{S}(N)$ does not split over M and $f : N \cong N'$, then $f(p)$ does not split over $f[M]$.
(2) Monotonicity: if $p \in \mathbf{S}(N)$ does not split over M, then $p \restriction M'$ does not split over M and p does not split over M'.

We are interested in studying the following local character properties that splitting may have:

Definition 12.3. Let $\mathbf{K}$ be an AEC, $\lambda \geq \mathrm{LS}(\mathbf{K})$.

(1) For a limit ordinal $\alpha < \lambda^+$, splitting has *weak universal local character at* α *in* $\mathbf{K}_\lambda$ if for any increasing continuous sequence $\langle M_i \in \mathbf{K}_\lambda \mid i \leq \alpha \rangle$ and any type $p \in \mathbf{S}(M_\alpha)$, if M_{i+1} is universal over M_i for each $i < \alpha$, then there is some $i_0 < \alpha$ such that $p \restriction M_{i_0+1}$ does not split over M_{i_0}.

(2) For a limit ordinal $\alpha < \lambda^+$, splitting has *strong universal local character at* α *in* $\mathbf{K}_\lambda$ if for any increasing continuous sequence $\langle M_i \in \mathbf{K}_\lambda \mid i \leq \alpha \rangle$ and any type $p \in \mathbf{S}(M_\alpha)$, if M_{i+1} is universal over M_i for each $i < \alpha$, then there is some $i_0 < \alpha$ such that p does not split over M_{i_0}.

Weak universal local character at some α can be obtained by building a tree of types. The argument was presented by Will Boney last semester. It is due to Shelah [She99, I.3.3].

Fact 12.4. Let $\mathbf{K}$ be an AEC. Let $\lambda \geq \mathrm{LS}(\mathbf{K})$. Assume that $\mathbf{K}_\lambda$ has amalgamation. If $\mathbf{K}$ is stable in λ, then in $\mathbf{K}_\lambda$ splitting has weak universal local character at every cardinal $\sigma < \lambda^+$ such that $2^\sigma > \lambda$.

At this point, the reader may forget the definition of splitting. All that we will use are the properties just listed.

We want to derive the strong version of universal local character. In fact, we give a name to setups where it holds.

Definition 12.5. We call an AEC $\mathbf{K}$ λ-*superstable* if:

(1) $\lambda \geq \mathrm{LS}(\mathbf{K})$.
(2) $\mathbf{K}_\lambda$ is not empty, has amalgamation, joint embedding, and no maximal models.
(3) $\mathbf{K}$ is stable in λ.
(4) In $\mathbf{K}_\lambda$, splitting has strong universal local character at every limit ordinal $\alpha < \lambda^+$.

We will show the following result, essentially due to Shelah and Villavecès [SV99] (with some small gaps fixed in [BGVV17]):

Theorem 12.6. Let $\mathbf{K}$ be an AEC. Let $\lambda \geq \mathrm{LS}(\mathbf{K})$. If:

(1) $\mathbf{K}_\lambda$ has amalgamation, joint embedding, and no maximal models.
(2) $\mathbf{K}$ is stable in λ.
(3) There is an EM blueprint $\Phi \in \Upsilon_{\leq \lambda}[\mathbf{K}]$ such that any $\mathrm{EM}_\tau(\lambda^+, \Phi)$ is universal in $\mathbf{K}_{\lambda^+}$.

Then $\mathbf{K}$ is λ-superstable.

Remark 12.7. In fact, stability in λ follows from the other hypotheses by Corollary 11.8.

Corollary 12.8 (Superstability from categoricity)**.** Let $\mathbf{K}$ be an AEC with arbitrarily large models. Let $\lambda > \mathrm{LS}(\mathbf{K})$. Assume that $\mathbf{K}_{<\lambda}$ has amalgamation and no maximal models. If $\mathbf{K}$ is categorical in λ, then $\mathbf{K}$ is μ-superstable for any $\mu \in [\mathrm{LS}(\mathbf{K}), \lambda)$.

Proof. Let $\mu \in [\mathrm{LS}(\mathbf{K}), \lambda)$. By Corollary 11.9, $\mathbf{K}$ is stable in μ. By categoricity in λ, it is easy to see that $\mathbf{K}_\mu$ is not empty and has joint embedding. To apply Theorem 12.6, it suffices to find an EM blueprint Φ such that $\mathrm{EM}_\tau(\mu^+, \Phi)$ is universal in $\mathbf{K}_{\mu^+}$. First fix an EM blueprint $\Psi \in \Upsilon_{\mathrm{LS}(\mathbf{K})}[\mathbf{K}]$. Note that any $M \in \mathbf{K}_{\mu^+}$ embeds into $\mathrm{EM}_\tau(\lambda, \Psi)$ by categoricity. Thus it embeds into some $\mathrm{EM}_\tau(I, \Psi)$, with $I \subseteq \lambda$ $|I| \leq \mu^+$. Thus it embeds into $\mathrm{EM}_\tau(\alpha, \Psi)$, where $\alpha := \mathrm{otp}(I) < \mu^{++}$. Now note that $^{<\omega}\mu^+$ (ordered by the lexicographical ordering) contains a copy of α for each $\alpha < \mu^{++}$, see [Bal09, Claim 15.5]. Thus $\mathrm{EM}_\tau(^{<\omega}\mu^+, \Psi)$ is universal in $\mathbf{K}_{\mu^+}$. We can now change the blueprint Ψ to a new blueprint Φ (in expanded vocabulary) such that $|\tau(\Phi)| \leq \mu$ and $\mathrm{EM}_\tau(I, \Phi) = \mathrm{EM}_\tau(^{<\omega}I, \Psi)$ for any linear order I. In particular, $\mathrm{EM}_\tau(\mu^+, \Phi)$ is universal in $\mathbf{K}_{\mu^+}$, as desired. $\qquad\square$

12.1. **The proof of Theorem 12.6.** For the rest of this section, we assume:

Hypothesis 12.9.

(1) $\mathbf{K}$ is an AEC.
(2) $\lambda \geq \mathrm{LS}(\mathbf{K})$.
(3) $\mathbf{K}_\lambda$ is not empty, has joint embedding, amalgamation, and no maximal models.
(4) $\mathbf{K}$ is stable in λ.

Recall that this implies (Corollary 7.22) that we can construct universal extensions in $\mathbf{K}_\lambda$. The following definition is crucial and will be explored much more later:

Definition 12.10. Let $\delta < \lambda^+$ be a limit ordinal. We call M (λ, δ)-*limit* over M_0 if there exists an increasing continuous chain $\langle N_i : i \leq \delta \rangle$ in $\mathbf{K}_\lambda$ such that $N_0 = M_0$, $N_\delta = N$, and N_{i+1} is universal over N_i for all $i < \delta$. We say that M is (λ, δ)-*limit* if it is (λ, δ)-limit over some M_0. When we just say "limit" we mean "(λ, δ)-limit" for some λ and δ (but λ is fixed in this section).

An easy back and forth argument gives:

Exercise 12.11. Let M_ℓ be (λ, δ_ℓ)-limit, $\ell = 1, 2$. Assume that $\mathrm{cf}(\delta_1) = \mathrm{cf}(\delta_2)$. Then:

(1) $M_1 \cong M_2$.
(2) If there is M_0 such that M_ℓ is (λ, δ_ℓ)-limit over M_0 for $\ell = 1, 2$, then $M_1 \cong_{M_0} M_2$.

We start by stating more local character properties that splitting may have:

Definition 12.12. Let $\alpha < \lambda^+$ be a limit ordinal.

(1) Splitting has *universal continuity* at α if for any increasing continuous sequence $\langle M_i \in \mathbf{K}_\lambda \mid i \leq \alpha \rangle$ and any type $p \in \mathbf{S}(M_\alpha)$, if for each $i < \alpha$ M_{i+1} is universal over M_i and $p \restriction M_i$ does not split over M_0, then p does not split over M_0.
(2) For $\delta < \lambda^+$ a limit, splitting has *no δ-limit alternations at* α if for any increasing continuous sequence $\langle M_i \in \mathbf{K}_\lambda \mid i \leq \alpha \rangle$ with M_{i+1} (λ, δ)-limit over M_i for all $i < \alpha$ and any type $p \in \mathbf{S}(M_\alpha)$, there exists $i < \alpha$ such that the following fails: $p \restriction M_{2i+1}$ splits over M_{2i} and $p \restriction M_{2i+2}$ does not split over M_{2i+1}. If this fails, we say that splitting has δ-limit alternations at α.

We start by showing that having nice EM models implies that splitting has these two properties. We will then show completely locally that universal continity, no alternations, and weak universal local character imply strong universal local character.

Lemma 12.13. Assume $\mathbf{K}$ has an EM blueprint Φ with $|\tau(\Phi)| \leq \lambda$ such that every $M \in \mathbf{K}_{[\lambda,\lambda^+]}$ embeds inside $\mathrm{EM}_\tau(\lambda^+, \Phi)$. Let $\alpha < \lambda^+$ be a regular cardinal. Then:

(1) Splitting has universal continuity at α.
(2) If in addition $\alpha < \lambda$, then for any limit $\gamma < \mu^+$, splitting has no γ-limit alternations at α.

Proof. Let $\langle M_i \mid i \leq \alpha \rangle$ and p be as in the definition of universal continuity or γ-limit alternations. For simplicity, let us assume that in the case of universal continuity M_0 is (λ, γ)-limit and that M_{i+1} is (λ, γ)-limit over M_i for all $i < \alpha$. The general case can be obtained from this after some renaming.

Let $S_\alpha^{\lambda^+} := \{ \delta < \lambda^+ \mid \mathrm{cf}(\delta) = \alpha \}$. We say that $\bar{C} = \langle C_\delta \mid \delta \in S_\alpha^{\lambda^+} \rangle$ is an $S_\alpha^{\lambda^+}$-*club sequence* if each $C_\delta \subseteq \delta$ is club. Clearly, club sequences exist: just take $C_\delta := \delta$ (this will be enough for proving universal continuity). Shelah [She94] proves the existence of club-guessing club sequences in ZFC under various hypotheses (the specific result that we use will be stated later, see Fact 12.14). We will describe a construction of a sequence of models $\bar{N}(\bar{C})$ based on a club sequence and then plug in the necessary club sequence in each case.

Given an $S_\alpha^{\lambda^+}$-club sequence $\bar{C}$, enumerate $C_\delta \cup \{\delta\}$ in increasing order as $\langle \beta_{\delta,j} \mid j \leq \alpha \rangle$.

Claim: Let $\gamma < \lambda^+$ be a limit ordinal. We can build increasing, continuous $\bar{N}(\bar{C}) = \langle N_i \in \mathbf{K}_\lambda \mid i < \lambda^+ \rangle$ such that for all $i < \lambda^+$:

(1) N_{i+1} is (λ, γ)-limit over N_i;
(2) when $i \in S_\alpha^{\lambda^+}$, there is $g_i : M_\alpha \cong N_i$ such that $g_i[M_j] = N_{\beta_{i,j}}$ for all $j \leq \alpha$; and:
(3) when $i \in S_\alpha^{\lambda^+}$, there is $a_i \in N_{i+1}$ that realizes $g_i(p)$.

Proof of Claim: Build the increasing continuous chain of models as follows: start with any $N_0 \in \mathbf{K}_\lambda$. Given an N_i, build N_{i+1} to be (λ, γ)-limit over it. At limits, take unions. At limits i of cofinality α, use the uniqueness of (λ, γ)-limits models to find the desired isomorphisms: the weak version gives $M_0 \cong M_{\beta_{i,0}}$, and the strong (over the base) version allows this isomorphism to be extended to get an isomorphism g_i between $\langle M_j \mid j \leq \alpha \rangle$ and $\langle N_{\beta_{i,j}} \mid j \leq \alpha \rangle$ as described. Since N_{i+1} is universal over N_i, we there is some $a_i \in N_{i+1}$ that realizes $g_i(p)$. $\dagger_{\text{Claim}}$

By assumption, we may assume that $N := \bigcup_{i < \lambda^+} N_i \leq_{\mathbf{K}} \mathrm{EM}_\tau(\lambda^+, \Phi)$. Thus, we can write $a_i = \rho_i(\gamma_1^i, \ldots, \gamma_{n(i)}^i)$ with:

$$ \gamma_1^i < \cdots < \gamma_{m(i)}^i < i \leq \gamma_{m(i)+1}^i < \cdots < \gamma_{n(i)}^i < \lambda^+ $$

Now we begin to prove each part of the lemma. In each, we will find $i_1 < i_2 \in S_\alpha^{\lambda^+}$ such that $\mathbf{tp}(a_{i_1}/N_{i_1}; N)$ and $\mathbf{tp}(a_{i_2}/N_{i_1}; N)$ are both the same (because of the EM

structure) and different (because they exhibit different splitting behavior), which is our contradiction.

(1) Assume that $p \restriction M_j$ does not split over M_0, for all $j < \alpha$.

Let $\bar{C}$ be an $S_\alpha^{\lambda^+}$-club sequence, and set $\langle N_i \in \mathbf{K}_\lambda \mid i < \lambda^+ \rangle = \bar{N}(\bar{C})$ as in the Claim (the value of γ doesn't matter here, e.g. take $\gamma := \omega$). By Fodor's Lemma, there is a stationary subset $S^* \subseteq S_\alpha^{\lambda^+}$, a term ρ_*, $m_*, n_* < \omega$ and ordinals $\gamma_0^*, \ldots \gamma_{n_*}^*, \beta_{*,0}$ such that:

For every $i \in S^*$, we have $\rho_i = \rho_*$; $n(i) = n_*$; $m(i) = m_*$; $\gamma_j^i = \gamma_j^*$ for $j \leq m_*$; and $\beta_{i,0} = \beta_{*,0}$.

Set $E := \{\delta < \lambda^+ \mid \delta \text{ is limit and } N_\delta \leq_\mathbf{K} \mathrm{EM}_\tau(\delta, \Phi)\}$. This is a club. Let $i_1 < i_2$ both be in $S^* \cap E$. Then we have:

$$\mathbf{tp}\left(a_{i_1}/N_{i_1}\right) = \mathbf{tp}\left(\rho_*(\gamma_1^*, \ldots, \gamma_{m_*}^*, \gamma_{m_*+1}^{i_1}, \ldots, \gamma_{n_*}^{i_1})/N \cap \mathrm{EM}_\tau(i_1, \Phi)\right)$$
$$= \mathbf{tp}\left(\rho_*(\gamma_1^*, \ldots, \gamma_{m_*}^*, \gamma_{m_*+1}^{i_2}, \ldots, \gamma_{n_*}^{i_2})/N \cap \mathrm{EM}_\tau(i_1, \Phi)\right)$$
$$= \mathbf{tp}\left(a_{i_2}/N_{i_1}\right)$$

where all the types are computed inside N. This is because the only differences between a_{i_1} and a_{i_2} lie entirely above i_1.

We have that $g_{i_1} : (N_{i_1}, N_{\beta_{*,0}}) \cong (M_\alpha, M_0)$ and that p splits over M_0. Thus, $\mathbf{tp}(a_{i_1}/N_{i_1}) = g_{i_1}(p)$ splits over $N_{\beta_{*,0}}$. On the other hand, C_{i_2} is cofinal in i_2, so there is $j < \alpha$ such that $\beta_{i_2,j} > i_1$ and, thus, $N_{i_1} \leq_\mathbf{K} N_{\beta_{i_2,j}}$. Again, $g_{i_2} : (N_{\beta_{i_2,j}}, N_{\beta_{*,0}}) \cong (M_j, M_0)$ and $p \restriction M_j$ does not split over M_0 by assumption. Thus, $\mathbf{tp}(a_{i_2}/N_{\beta_{i_2,j}}) = g_{i_2}(p \restriction M_j)$ does not split over $N_{\beta_{*,0}}$. By monotonicity, $\mathbf{tp}(a_{i_2}/N_{i_1})$ does not split over $N_{\beta_{*,0}}$. Thus, $\mathbf{tp}(a_{i_1}/N_{i_1}) \neq \mathbf{tp}(a_{i_2}/N_{i_1})$, a contradiction.

(2) Let χ be a big-enough cardinal and create an increasing, continuous elementary chain of models of set theory $\langle \mathfrak{B}_i \mid i < \lambda^+ \rangle$ such that for all $i < \lambda^+$:
   (a) $\mathfrak{B}_i \prec (H(\chi), \in)$;
   (b) $\|\mathfrak{B}_i\| = \lambda$;
   (c) $\mathfrak{B}_0$ contains, as elements[2], Φ, $\mathrm{EM}(\lambda^+, \Phi)$, $\langle g_i \mid i < \lambda^+ \rangle$, λ^+, $\langle N_i \mid i < \lambda^+ \rangle$, $S_\alpha^{\lambda^+}$, $\langle a_i \mid i \in S_\alpha^{\lambda^+} \rangle$, and each $f \in \tau(\Phi)$; and
   (d) $\mathfrak{B}_i \cap \lambda^+$ is an ordinal.

We will use the following fact which was originally proven in [She94, III.2] (or see [AM10, Theorem 2.17] for a short proof).

Fact 12.14. Let λ be a cardinal such that $\mathrm{cf}(\lambda) \geq \theta^{++}$ for some regular θ and let $S \subseteq S_\theta^\lambda$ be stationary. Then there is a S-club sequence $\langle C_\delta \mid \delta \in S \rangle$ such that, if $E \subseteq \lambda$ is club, then there are stationarily many $\delta \in S$ such that $C_\delta \subseteq E$.

We have that $\alpha < \lambda$, so we can apply Fact 12.14 with λ, θ, S there standing for $\lambda^+, \alpha, S_\alpha^{\lambda^+}$ here. Let $\bar{C}$ be the $S_\alpha^{\lambda^+}$-club sequence that the fact gives. Let $\langle N_i \in \mathbf{K}_\lambda \mid i < \lambda^+ \rangle = \bar{N}(\bar{C})$ be as in the Claim. Note that

---

[2]When we say that $\mathfrak{B}_0$ contains a sequence as an element, we mean that it contains the function that maps an index to its sequence element.

$E := \{i < \lambda^+ \mid \mathfrak{B}_i \cap \lambda^+ = i\}$ is a club. By the conclusion of Fact 12.14, there is some $i_2 \in S_\alpha^{\lambda^+}$ such that $C_{i_2} \subseteq E$. We have $a_{i_2} = \rho_{i_2}(\gamma_1^{i_2}, \ldots, \gamma_{n(i_2)}^{i_2})$, with:

$$\gamma_1^{i_2} < \cdots < \gamma_{m(i_2)}^{i_2} < i_2 \le \gamma_{m(i_2)+1}^{i_2} < \cdots < \gamma_{n(i_2)}^{i_2}$$

Since the $\beta_{i_2,j}$'s enumerate a cofinal sequence in i_2, we can find $j < \alpha$ such that $\gamma_{m(i_2)}^{i_2} < \beta_{i_2,2j+1} < i_2$. Recall that we have $p \upharpoonright M_{2j+2}$ does not split over M_{2j+1} by assumption. Then $(H(\chi), \in)$ satisfies the following formulas with parameters exactly the objects listed in item (2c) above and ordinals below $\beta_{i_2,2j+2}$:

$$\exists x, y_{m(i_2)+1}, \ldots, y_{n(i)}.(\text{``}x \in S_\alpha^{\lambda^+}\text{''}$$
$$\wedge \text{``}x > \beta_{i_2,2j+1}\text{''} \wedge \text{``}y_k \in (x, \lambda^+) \text{ are increasing ordinals''}$$
$$\wedge \text{``}a_x = \rho_{i_2}(\gamma_1^{i_2}, \ldots, \gamma_{m(i_2)}^{i_2}, y_{m(i_2)+1}, \ldots, y_{n(i_2)})\text{''}$$
$$\wedge \text{``}N_x \subseteq \mathrm{EM}(x, \Phi)\text{''})$$

This is witnessed by $x = i_2$ and $y_k = \gamma_k^{i_2}$. By elementarity, $\mathfrak{B}_{\beta_{i_2,2j+2}}$ satisfies this formula as it contains all the parameters. Let $i_1 \in (\beta_{i_2,2j+1}, \lambda^+) \cap \mathfrak{B}_{\beta_{i_2,2j+2}} = (\beta_{i_2,2j+1}, \beta_{i_2,2j+2})$[3] witness this, along with $\gamma'_{m(i_2)+1} < \cdots < \gamma'_{n(i_2)} < \lambda^+$. Then we have:

$$a_{i_1} = \rho_{i_2}(\gamma_1^{i_2}, \ldots, \gamma_{m(i_2)}^{i_2}, \gamma'_{m(i_2)+1}, \ldots, \gamma'_{n(i_2)})$$

with $\beta_{i_2,2j+1} < \gamma'_{m(i_2)+1}$. We want to compare $\mathbf{tp}(a_{i_2}/N_{i_1})$ and $\mathbf{tp}(a_{i_1}/N_{i_1})$.
- From the elementarity, we get that $N_{i_1} \subseteq \mathrm{EM}_\tau(i_1, \Phi)$. We also know that $i_1 < \beta_{i_2,2j+2} < \gamma_{m(i_2)+1}^{i_2}, \gamma'_{m(i_2)+1}$. Thus, as before, the types are equal.
- We know that $p \upharpoonright M_{2j+2}$ does not split over M_{2j+1}. Thus, $\mathbf{tp}(a_{i_2}/N_{\beta_{i_2,2j+2}})$ does not split over $N_{\beta_{i_2,2j+1}}$. Since we have $N_{\beta_{i_2,2j+1}} \le_\mathbf{K} N_{i_1} \le_\mathbf{K} N_{\beta_{i_2,2j+2}}$, this gives $\mathbf{tp}(a_{i_2}/N_{i_1})$ does not split over $N_{\beta_{i_2,2j+1}}$.
- We have $\beta_{i_2,2j+1} < i_1$, so there is some $k < \alpha$ such that $\beta_{i_2,2j+1} < \beta_{i_1,k} < i'$. By assumption, p splits over M_k. Thus $g_{i_1}(p)$ splits over $N_{\beta_{i_1,k}}$. Therefore $\mathbf{tp}(a_{i_1}/N_{i_1})$ splits over $N_{\beta_{i_2,2j+1}} \le_\mathbf{K} N_{\beta_{i_1,k}}$.
As before, these three statements contradict each other.

$\square$

In Lemma 12.13, we are missing γ-limit alternations at λ. This is fixed by:

Lemma 12.15. For any limit $\delta < \lambda^+$, splitting has no δ-limit alternations at λ.

Proof. By Fact 12.4, splitting has weak universal local character at λ. Fix $\langle M_i : i \le \lambda \rangle$, δ, p as in the definition of having no δ-limit alternations and apply weak universal local character to the chain $\langle M_{2i} : i \le \lambda \rangle$. $\square$

---

[3]The equality here is the key use of club guessing.

We now outline how we are going to prove strong universal local character. We already have weak universal local character, continuity, and no alternations. Three important basic results are

- It suffices to prove strong universal local character at regular cardinals (Lemma 12.16.(1)).
- Continuity together with weak local character imply strong local character at *regular length* (Lemma 12.16.(2)); and
- It does not matter whether in the definition of weak and strong universal local character we require "M_{i+1} limit over M_i" or "M_{i+1} universal over M_i," and the length of the limit models does not matter (Lemma 12.16.(3)).

The second of these is proven by contradiction, and the first and third are straightforward.

Assume for a moment we have strong universal local character at some limit length γ. Let us try to prove weak universal local character at (say) ω (then we can use the second basic result to get the strong version, assuming continuity). By the third basic result, we can assume we are given an increasing continuous sequence $\langle M_n : n \leq \omega \rangle$ with M_{n+1} (λ, γ)-limit over M_n for all $n < \omega$ and $p \in \mathbf{S}(M_\omega)$. By the strong universal local character assumption we know that $p \restriction M_{n+1}$ does not split over some intermediate model between M_n and M_{n+1}, so if we assume that $p \restriction M_{n+1}$ splits over M_n for all $n < \omega$, we will end up getting alternations. This is the essence of Lemma 12.16.(6).

Thus to prove strong universal local character at *all* cardinals, it is enough to obtain it at *some* cardinal. Fortunately, we already know weak universal local character holds at some σ (Fact 12.4). If σ is regular we are done by the second basic result, but unfortunately σ could be singular (if λ is singular and $2^{<\lambda} = \lambda$, e.g. $\lambda = \beth_\omega$). In this case Lemma 12.16.(5) (using Lemma 12.16.(4) as an auxiliary claim) shows that failure of strong universal local character at σ implies alternations, even when σ is singular.

Lemma 12.16. Let $\alpha < \mu^+$ be a regular cardinal, $\sigma < \mu^+$ be a (not necessarily regular) cardinal, and $\delta < \mu^+$ be a limit ordinal.

(1) If splitting has strong universal character at all regular cardinals below λ^+, then splitting has strong universal character at all limits below λ^+.
(2) If splitting has universal continuity at α and weak universal local character at α, then splitting has strong universal local character at α.
(3) We obtain an equivalent definition of weak [strong] universal local character at σ, if we ask in addition that "M_{i+1} is (μ, δ)-limit over M_i" for all $i < \sigma$.
(4) Assume that splitting has weak universal local character at σ. Let $\langle M_i : i \leq \sigma \rangle$ be increasing continuous in $\mathbf{K}_\lambda$ with M_{i+1} universal over M_i for all $i < \sigma$. For any $p \in \mathbf{S}(M_\sigma)$ there exists a *successor* $i < \sigma$ such that $p \restriction M_{i+1}$ does not split over M_i.
(5) If splitting has universal continuity at σ, weak universal local character at σ, and no δ-limit alternations at ω, then splitting has strong universal local character at σ.

(6) Assume that splitting has strong universal local character at σ. If splitting does *not* have weak universal local character at α, then splitting has σ-limit alternations at α.

Proof.

(1) Straightforward by cofinality consideration and the monotonicity of splitting.
(2) Suppose that $\langle M_i : i \leq \alpha \rangle$, p is a counterexample.

   Claim: For each $i < \alpha$, there exists $j_i \in (i, \alpha)$ such that $p \restriction M_{j_i}$ splits over M_i.

   Proof of Claim: If $i < \alpha$ is such that for all $j \in (i, \alpha)$, $p \restriction M_j$ does not split over M_i, then applying universal continuity at α on the chain $\langle M_k : k \in [i, \alpha] \rangle$ we would get that p does not split over M_i, contradicting the choice of $\langle M_i : i \leq \alpha \rangle$, p. $\dagger_{\text{Claim}}$

   Now define inductively for $i \leq \alpha$, $k_0 := 0$, $k_{i+1} := j_{k_i}$, and when i is limit $k_i := \sup_{j<i} k_j$. Note that $\langle k_i : i \leq \alpha \rangle$ is strictly increasing continuous and $i < \alpha$ implies $k_i < \alpha$ (this uses regularity of α; when α is singular, see (5)).
   Apply weak universal local character to the chain $\langle M_{k_i} : i \leq \alpha \rangle$ and the type p. We get that there exists $i < \alpha$ such that $p \restriction M_{k_{i+1}}$ does not split over M_{k_i}. This is a contradiction since $k_{i+1} = j_{k_i}$ and we chose j_{k_i} so that $p \restriction M_{j_{k_i}}$ splits over M_{k_i}.
(3) We prove the result for weak universal local character, and the proof for the strong version is similar. Fix $\langle M_i^0 : i \leq \sigma \rangle$, p witnessing failure of weak universal local character at σ. We build a witness of failure $\langle M_i : i \leq \sigma \rangle$, p such that $M_\sigma = M_\sigma^0$, and M_{i+1} is (λ, δ)-limit over M_i for each $i < \alpha$. Using existence of universal extensions, we can extend each M_i^0 to M_i^* that is (λ, δ)-limit over M_i^0. Since M_{i+1}^0 is universal over M_i^0, we can find $f_i : M_{i+1}^* \xrightarrow[M_i^0]{} M_{i+1}^0$. Now set $M_i^1 := M_i^0$ for $i \leq \sigma$ limit or 0 and $M_{i+1}^1 := f_i[M_{i+1}^*]$. This is an increasing continuous chain with M_{i+1}^1 (λ, δ)-limit over M_i^1. Let $M_i := M_{2i}^1$.
   This works: if there was an $i < \sigma$ such that $p \restriction M_{i+1}$ does not split over M_i, this would mean that $p \restriction M_{2i+2}^1$ does not split over M_{2i}^1, but since $M_{2i}^1 \leq_{\mathbf{K}} M_{2i+1}^0 \leq_{\mathbf{K}} M_{2i+2}^0 \leq_{\mathbf{K}} M_{2i+2}^1$, we have by monotonicity that $p \restriction M_{2i+2}^0$ does not split over M_{2i+1}^0, a contradiction.
(4) Apply weak universal local character to the chain $\langle M_{2i} : i < \sigma \rangle$ to get $j < \sigma$ such that $p \restriction M_{2j+2}$ does not split over M_{2j}. By monotonicity, this implies that $p \restriction M_{2j+2}$ does not split over M_{2j+1}. Let $i := 2j + 1$.
(5) Suppose not, and let $\langle M_i : i \leq \sigma \rangle, p$ be a counterexample. By (3), without loss of generality M_{i+1} is (λ, δ)-limit over M_i for all $i < \delta$. As in the proof of (2), for each $\beta < \sigma$, there exists $j_i \in [i, \beta)$ such that $p \restriction M_{j_i}$ splits over M_i. On the other hand, applying (4) to the chain $\langle M_j : j \in [j_i, \beta] \rangle$, for each $i < \beta$, there exists a *successor* ordinal $k_i \geq j_i$ such that $p \restriction M_{k_i+1}$ does not split over M_{k_i}. Define by induction on $n \leq \omega$, $m_0 := 0$, $m_{2n+1} := k_{m_{2n}}$, $m_{2n+2} := k_{m_{2n}} + 1$, and $m_\omega := \sup_{n<\omega} m_n$. By construction, the sequence

$\langle M_{m_n} : n \leq \omega \rangle$ witnesses that splitting has δ-limit alternations at ω (we use that k_i is a successor to see that each model is (λ, γ)-limit over the previous ones), a contradiction.

(6) Let $\gamma := \sigma \cdot \sigma$. By (3), there exists $\langle M_i : i \leq \alpha \rangle$, p witnessing failure of weak universal local character at α such that for all $i < \alpha$, M_{i+1} is (λ, γ)-limit over M_i. Let $\langle M_{i,j} : j \leq \gamma \rangle$ witness that M_{i+1} is (λ, γ)-limit over M_i (i.e. it is increasing continuous with $M_{i,j+1}$ universal over $M_{i,j}$ for all $j < \gamma$, $M_{i,0} = M_i$, and $M_{i,\delta} = M_{i+1}$). By strong universal local character at σ, for all $i < \alpha$, there exists $j_i < \gamma$ such that $p \restriction M_{i+1}$ does not fork over M_{i,j_i}. By replacing j_i by $j_i + \sigma$ if necessary we can assume without loss of generality that $\mathrm{cf}(j_i) = \mathrm{cf}(\sigma)$.

Observe also that for any $i < \alpha$, $p \restriction M_{i+1,j_i}$ splits over M_i (using monotonicity and the assumption that $p \restriction M_{i+1}$ splits over M_i). Therefore $\langle M_0, M_{1,j_1}, M_2, M_{3,j_3}, \ldots \rangle$, p witness that splitting has σ-limit alternations at α.

$\square$

Proof of Theorem 12.6. By Fact 12.4, splitting has weak universal local character at some $\sigma < \lambda^+$. By lemma 12.13 and 12.15, splitting also has continuity and no δ-limit alternations for every limit $\alpha, \delta < \lambda^+$.

By Lemma 12.16(5), splitting has strong universal character at σ. By Lemma 12.16(6), splitting has weak universal character at every regular cardinal $\alpha < \lambda^+$. By Lemma 12.16(2), splitting has strong universal character at every regular cardinal $\alpha < \lambda^+$. This suffices by Lemma 12.16(1). $\square$

13. SUPERSTABILITY AND UNIQUENESS

The material of this section is mostly from [Vas17a].

Recall the definition of λ-superstability (Definition 12.5). We want to study the following independence notion:

Definition 13.1. Let $\mathbf{K}$ be a λ-superstable AEC. For $M \leq_{\mathbf{K}} N$ both in $\mathbf{K}_\lambda$, $p \in \mathbf{S}(N)$, we say that p *does not λ-fork over* (M_0, M) if $M_0 \in \mathbf{K}_\lambda$ is such that M is universal over M_0 and p does not λ-split over M_0. We say that p *does not λ-fork over* M if there exists M_0 such that p does not λ-fork over (M_0, M). When λ is clear from context, we omit it.

It turns out nonforking is much better behaved than nonsplitting. In fact in all known cases it defines a canonical superstable-like independence notion. We will see that in the elementary case it coincides with first-order nonforking, at least over limit models.

The following will prove very useful and will be used without comments:

Exercise 13.2. Let $\mathbf{K}$ be a λ-superstable AEC and let $M_0 \leq_{\mathbf{K}} M_1 \leq_{\mathbf{K}} M_2$ all be in $\mathbf{K}_\lambda$. If M_1 is universal over M_0, then M_2 is universal over M_0. If M_2 is universal over M_1, then M_2 is universal over M_0.

The following are basic properties of nonforking:

Lemma 13.3. Let $\mathbf{K}$ be a λ-superstable AEC.

(1) (Invariance) If $p \in \mathbf{S}(N)$ does not fork over (M_0, M) and $f : N \cong N'$, then $f(p)$ does not fork over $(f[M_0], f[M])$.
(2) (Monotonicity) If $p \in \mathbf{S}(N)$ does not fork over (M_0, M), then:
   (a) Whenever $M_0 \leq_{\mathbf{K}} M' \leq_{\mathbf{K}} N$ is such that M' is universal over M_0, then p does not fork over (M_0, M').
   (b) Whenever $M \leq_{\mathbf{K}} N' \leq_{\mathbf{K}} N$, then $p \restriction N'$ does not fork over (M_0, M).
   (c) Whenever $M_0 \leq_{\mathbf{K}} M_0' \leq_{\mathbf{K}} M$ is such that M is universal over M_0', then p does not fork over (M_0', M).
(3) (Universal local character) If $\delta < \lambda^+$ is limit, $\langle M_i : i \leq \delta \rangle$ are increasing continuous in $\mathbf{K}_\lambda$ with M_{i+1} universal over M_i for all $i < \delta$, then for any $p \in \mathbf{S}(M_\delta)$, there exists $i < \delta$ such that p does not fork over (M_i, M_{i+1}).
(4) (Weak uniqueness) If $p, q \in \mathbf{S}(N)$ both do not fork over (M_0, M) and $p \restriction M = q \restriction M$, then $p = q$.
(5) (Weak extension) If $p \in \mathbf{S}(N)$ does not fork over (M_0, M) and $N' \geq_{\mathbf{K}} N$ is in $\mathbf{K}_\lambda$, then there exists $q \in \mathbf{S}(N')$ that extends p and does not fork over (M_0, M).

Proof. The first three properties are immediate from the definition. For weak uniqueness, we know that both p and q do not split over M_0. By universality of M over M_0, fix $f : N \xrightarrow[M_0]{} M$. Let $N_1 := N$, $N_2 := f[N]$. Using the definition of nonsplitting, we must have that $f(p) = p \restriction N_2$ and $f(q) = q \restriction N_2$. Since $N_2 \leq_{\mathbf{K}} M$ and $p \restriction M = q \restriction M$, we have that $p \restriction N_2 = q \restriction N_2$. Thus $f(p) = f(q)$, and so $p = q$.

For weak extension, we first prove it when M is (λ, ω)-limit over M_0. In this case we can fix $\langle M_i : i \leq \omega \rangle$ increasing continuous such that M_{i+1} is universal over M_i and $M_\omega = M$. Fix $f : N' \xrightarrow[M_1]{} N$ and let $q := f^{-1}(p \restriction f[N'])$. By invariance, q does not split over M_0, hence does not fork over (M_0, M). It remains to see that $q \restriction N = p$. Indeed by monotonicity, $q \restriction N$ does not fork over (M_0, M), hence over (M_0, M_1). Also, p does not fork over (M_0, M_1), and since f fixes M_1, $q \restriction M_1 = p \restriction M_1$. By weak uniqueness, $q \restriction N = p$.

Now if M is not (λ, ω)-limit over M_0, use universality to find $M' \in \mathbf{K}_\lambda$, $M_0 \leq_{\mathbf{K}} M' \leq_{\mathbf{K}} M$ such that M' is (λ, ω)-limit over M_0. By monotonicity, p does not fork over (M_0, M'). By the previous case, there is an extension $q \in \mathbf{S}(N')$ of p that does not fork over (M_0, M'), hence by monotonicity over (M_0, M). $\qquad\square$

In superstable AECs, increasing sequences of types have upper bounds. This is an improvement on:

Exercise 13.4 ([Bal09, 11.1]). Let $\mathbf{K}$ be an AEC and let $\lambda \geq \mathrm{LS}(\mathbf{K})$. Let $\langle M_i : i \leq \omega \rangle$ be an increasing continuous chain in $\mathbf{K}_\lambda$. Let $\langle p_i : i < \omega \rangle$ be an increasing chain of types with $p_i \in \mathbf{S}(M_i)$ for all $i < \omega$. If $\mathbf{K}_\lambda$ has amalgamation, then there exists $p_\omega \in \mathbf{S}(M_\omega)$ such that p_ω extends p_i for all $i < \omega$. *Hint: draw a picture of what the definition of orbital type gives, and take a direct limit.*

Lemma 13.5. Assume that $\mathbf{K}$ is λ-superstable. Let $\delta < \lambda^+$ be a limit ordinal and let $\langle M_i : i \leq \delta \rangle$ be increasing continuous in $\mathbf{K}_\lambda$ with M_{i+1} universal over M_i for all $i < \delta$. Suppose we are given an increasing chain of types $\langle p_i : i < \delta \rangle$ such

that $p_i \in \mathbf{S}(M_i)$ for all $i < \delta$. Then there exists a unique $p_\delta \in \mathbf{S}(M_\delta)$ such that $p_\delta \restriction M_i = p_i$ for all $i < \delta$.

Proof. Without loss of generality, δ is regular. To see uniqueness, suppose that $p, q \in \mathbf{S}(M_\delta)$ both extend all the p_i's. By universal local character and monotonicity, there exists $i < \delta$ such that both p and q do not fork over (M_i, M_{i+1}). Since $p \restriction M_{i+1} = q \restriction M_{i+1}$, we must have $p = q$. Let us now prove existence. If $\delta = \omega$, the conclusion is given by Exercise 13.4, so assume that $\delta > \omega$. Using universal local character, for each limit $i < \delta$ there exists $j_i < i$ such that p_i does not fork over M_{j_i}. By Fodor's lemma, there exists a stationary $S \subseteq \delta$ and a $j < \delta$ such that p_i does not fork over M_j for all $i \in S$. Since S is unbounded and the p_i's are increasing, p_i does not fork over M_j for *all* $i \in [j, \delta)$. Let $q \in \mathbf{S}(M_\delta)$ be an extension of p_{j+1} that does not fork over (M_j, M_{j+1}). By weak uniqueness, $q \restriction M_i = p_i$ for all $i \in [j+1, \delta)$, as desired. $\qquad\square$

The following property will be crucial in many places. It will be improved later.

Definition 13.6. We say that two orbital types $p \in \mathbf{S}(M)$ and $q \in \mathbf{S}(N)$ are *conjugate over* A if $A \subseteq M \cap N$ and there exists $f : M \cong_A N$ such that $f(p) = q$.

Lemma 13.7 (The conjugation property). Let $\mathbf{K}$ be a λ-superstable AEC. Let $\delta < \lambda^+$ be a limit ordinal. Let $M_0 \leq_{\mathbf{K}} M \leq_{\mathbf{K}} N$ be such that M is (λ, δ)-limit over M_0 and N is (λ, δ)-limit over M. If $p \in \mathbf{S}(N)$ does not fork over (M_0, M), then p and $p \restriction M$ are conjugates over M_0.

Proof. Since M is (λ, δ)-limit over M_0, there exists $M_1 \leq_{\mathbf{K}} M$ which is universal over M_0 and such that M is (λ, δ)-limit over M_1. It is easy to check that N is also (λ, δ)-limit over M_1. By uniqueness of limit models of the same length, there exists $f : N \cong_{M_1} M$. Now $f(p) \in \mathbf{S}(M)$ and $f(p)$ does not fork over $(M_0, f[M])$, hence (by monotonicity) over (M_0, M_1). We also have that $p \restriction M$ does not fork over (M_0, M_1) and $f(p) \restriction M_1 = p \restriction M_1$, since f fixes M_1. Thus $p \restriction M = f(p)$, as desired. $\qquad\square$

We aim to get rid of the witness M_0 in the definition of nonforking, at least when working over limit models. For this, we want to prove the following result (due to the author):

Theorem 13.8 (The uniqueness theorem). Let $\mathbf{K}$ be a λ-superstable AEC. Let $M \leq_{\mathbf{K}} N$ both be limit models in $\mathbf{K}_\lambda$. If $p, q \in \mathbf{S}(N)$ do not fork over M and $p \restriction M = q \restriction M$, then $p = q$.

To see the difficulty, expand the definition: if p does not fork over M, there is M_p such that p does not fork over (M_p, M). Similarly, there is M_q such that q does not fork over (M_q, M). If we knew that $M_p = M_q$, or at least that one was contained in the other, then we would be able to use weak uniqueness and be done. However there is no reason this should be the case (think of the case of the AEC of all sets, M is some infinite sets, and $M = M_p \cup M_q$, where M_p and M_q are disjoint). Thus we will prove the uniqueness theorem by contradiction: a type that does not satisfy the conclusion of the uniqueness theorem will be called *bad*, and the theory of bad types will turn out to be so nice that there cannot be any bad type.

Throughout the rest of this section, we assume:

Hypothesis 13.9. $\mathbf{K}$ is a λ-superstable AEC.

For technical reason, we close the concept of "failing the conclusion of the uniqueness theorem" under nonforking extensions. This leads to the following technical concept:

Definition 13.10. Let $M \in \mathbf{K}_\lambda$ be limit. We define by induction on $n < \omega$ what it means for a type $p \in \mathbf{S}(M)$ to be n-*bad*:

(1) p is 0-*bad* if there exists a limit model $N \in \mathbf{K}_\lambda$ with $M \leq_{\mathbf{K}} N$ and $q_1, q_2 \in \mathbf{S}(N)$ such that:
   (a) Both q_1 and q_2 extend p.
   (b) $q_1 \neq q_2$.
   (c) Both q_1 and q_2 do not fork over M.
(2) For $n < \omega$, p is $(n+1)$-*bad* if there exists a limit model $M_0 \in \mathbf{K}_\lambda$ with $M_0 \leq_{\mathbf{K}} M$ such that $p \restriction M_0$ is n-bad and p does not fork over M_0.
(3) p is *bad* if p is n-bad for some $n < \omega$.

The following is an easy consequence of the definition (in fact the definition is tailored exactly to make this work):

Remark 13.11. Let $M \leq_{\mathbf{K}} N$ both be limit in $\mathbf{K}_\lambda$. If $p \in \mathbf{S}(N)$ does not fork over M and $p \restriction M$ is bad, then p is bad.

We now proceed to develop some the theory of bad types. In the end, we will conclude that this contradicts stability in λ, hence there cannot be any bad types. The next two lemmas are crucial: bad types are closed under unions of universal chains, and any bad type has two distinct bad extensions.

Lemma 13.12. Assume that $\mathbf{K}$ is λ-superstable. Let $\delta < \lambda^+$ be a limit ordinal. Let $\langle M_i : i \leq \delta \rangle$ be an increasing continuous chain of limit models in $\mathbf{K}_\lambda$ with M_{i+1} limit over M_i for all $i < \delta$. Let $\langle p_i : i \leq \delta \rangle$ be an increasing chain of types, with $p_i \in \mathbf{S}(M_i)$ for all $i < \delta$. If p_i is bad for all $i < \delta$, then p_δ is bad.

Proof. By universal local character, there exists $i < \delta$ such that p_δ does not fork over M_i. By assumption, $p \restriction M_i$ is bad, so by Remark 13.11 p_δ is also bad, as desired. $\square$

Lemma 13.13. Let $M \in \mathbf{K}_\lambda$ be a limit model. If $p \in \mathbf{S}(M)$ is bad, then there exists a limit model N in $\mathbf{K}_\lambda$ with $M \leq_{\mathbf{K}} N$ and $q_1, q_2 \in \mathbf{S}(N)$ such that:

(1) Both q_1 and q_2 extend p.
(2) $q_1 \neq q_2$.
(3) Both q_1 and q_2 are bad.

Proof. By definition, p is n-bad for some $n < \omega$. We proceed by induction on n.

- If $n = 0$, this is the definition of being 0-bad (note that q_1 and q_2 from Definition 13.10 are bad because they are nonforking extensions of the bad type p, see Remark 13.11)

- If $n = m + 1$, let $M_0 \in \mathbf{K}_\lambda$ be a limit model such that $M_0 \leq_{\mathbf{K}} M$, p does not fork over M_0, and $p \restriction M_0$ is m-bad. Pick M_0' such that p does not fork over (M_0', M_0). Let M_1' be (μ, ω)-limit over M_0' with $M_1' \leq_{\mathbf{K}} M_0$. By monotonicity, p does not fork over (M_0', M_1'). Let M^* be (λ, ω)-limit over M (hence over M_1'). Let $q \in \mathbf{S}(M^*)$ be an extension of p that does not fork over (M_0', M), hence over (M_0', M_1'). By Lemma 13.7, q and $p \restriction M_1'$ are conjugate over M_0'. Now by the induction hypothesis, there exists a limit model N^* extending M_0 and two distinct bad extensions of $p \restriction M_0$ to N^*. These are also extensions of $p \restriction M_1'$, so the result follows from the fact that q and $p \restriction M_1'$ are conjugate over M_0'.

$\square$

The following nominally stronger version of Lemma 13.13 (where N is fixed first) is the one that we will use to show that there are no bad types:

Lemma 13.14. Let M be a limit model in $\mathbf{K}_\lambda$ and let N be limit over M. If $p \in \mathbf{S}(M)$ is bad, then there exists $q_1, q_2 \in \mathbf{S}(N)$ such that:

(1) Both q_1 and q_2 extend p.
(2) $q_1 \neq q_2$.
(3) Both q_1 and q_2 are bad.

Proof. By Lemma 13.13, there exists $N' \in \mathbf{K}_\lambda$ limit with $M \leq_{\mathbf{K}} N'$ and $q_1', q_2' \in \mathbf{S}(N')$ distinct bad extensions of p. Use universality of N to pick $f : N' \xrightarrow[M]{} N$. For $\ell = 1, 2$, let $q_\ell'' := f(q_\ell')$. Clearly, q_1'', q_2'' are still distinct bad extensions of p. Now for $\ell = 1, 2$, let $q_\ell \in \mathbf{S}(N)$ be an extension of q_ℓ'' that does not fork over $f[N']$ (use universal local character and extension). Then q_1 and q_2 are as desired (they are bad because they are nonforking extensions of the bad types q_1'', q_2'', see Remark 13.11). $\square$

Lemma 13.15. If $\mathbf{K}$ is λ-superstable, then there are no bad types.

Proof. Suppose for a contradiction that there is a limit model M in $\mathbf{K}_\lambda$ and a bad type $p \in \mathbf{S}(M)$. Fix an increasing continuous chain $\langle M_i : i \leq \lambda \rangle$ with $M_0 = M$ and M_{i+1} limit over M_i for all $i < \lambda$. We build a tree of types $\langle p_\eta : \eta \in {}^{\leq \lambda} 2 \rangle$ satisfying:

(1) $p_{<>} = p$.
(2) For all $\eta \in {}^{\leq \lambda} 2$, $p_\eta \in \mathbf{S}(M_{\ell(\eta)})$.
(3) For all $\nu \leq \eta \in {}^{\leq \lambda} 2$, p_η is an extension of p_ν.
(4) For all $\eta \in {}^{\leq \lambda} 2$, p_η is bad.
(5) For all $\eta \in {}^{< \lambda} 2$, $p_{\eta \frown 0} \neq p_{\eta \frown 1}$.

This is enough: The requirements give that for all $\eta, \nu \in {}^\lambda 2$, $\eta \neq \nu$ implies $p_\eta \neq p_\nu$. Therefore $|\mathbf{S}(M_\lambda)| = 2^\lambda > \lambda$, contradicting stability in λ.

This is possible: We proceed by induction on $\ell(\eta)$. The base case has already been specified. At limits, we use Lemma 13.5 and Lemma 13.12. At successors, we use Lemma 13.14. $\square$

Proof of Theorem 13.8. Otherwise, this would mean that $p \restriction M$ is 0-bad, contradicting Lemma 13.15. □

We now restate the properties of nonforking over limit models, without the witness.

Corollary 13.16. Let $\mathbf{K}$ be a λ-superstable AEC. We have the following properties of nonforking:

(1) Invariance: if $M \leq_{\mathbf{K}} N$ are both limit models in $\mathbf{K}_\lambda$ and $p \in \mathbf{S}(N)$ does not fork over M, then if $f : N \cong N'$, $f(p)$ does not fork over $f[M]$.

(2) Monotonicity: if $M \leq_{\mathbf{K}} M' \leq_{\mathbf{K}} N' \leq_{\mathbf{K}} N$ are all limit models in $\mathbf{K}_\lambda$, $p \in \mathbf{S}(N)$ does not fork over M, then $p \restriction N'$ does not fork over M'.

(3) Universal local character: if $\delta < \lambda^+$, $\langle M_i : i \leq \delta \rangle$ is an increasing continuous chain of limit models in $\mathbf{K}_\lambda$, with M_{i+1} universal over M_i for all $i < \delta$, then for any $p \in \mathbf{S}(M_\delta)$, there exists $i < \delta$ such that p does not fork over M_i.

(4) Uniqueness: if $M \leq_{\mathbf{K}} N$ are both limit models in $\mathbf{K}_\lambda$ and $p, q \in \mathbf{S}(N)$ do not fork over M, then $p \restriction M = q \restriction M$ implies $p = q$.

(5) Existence: Let $M \leq_{\mathbf{K}} N$ both be limit models in $\mathbf{K}_\lambda$. If $p \in \mathbf{S}(M)$, then there exists $q \in \mathbf{S}(N)$ such that q does not fork over M and q extends p.

(6) Transitivity: Let $M_0 \leq_{\mathbf{K}} M_1 \leq_{\mathbf{K}} M_2$ all be limit models in $\mathbf{K}_\lambda$. Let $p \in \mathbf{S}(M_2)$ and assume that p does not fork over M_1 and $p \restriction M_1$ does not fork over M_0. Then p does not fork over M_0.

(7) Continuity: Let $\delta < \lambda^+$ be a limit ordinal and let $\langle M_i : i \leq \delta \rangle$ be an increasing continuous chain of limit models with M_{i+1} universal over M_i. Let $\langle p_i : i < \delta \rangle$ be given such that for all $i < \delta$, $p_i \in \mathbf{S}(M_i)$, and p_i is a nonforking extension of p_0. Then there exists a unique $p_\delta \in \mathbf{S}(M_\delta)$ such that p_δ does not fork over M_0. In particular, p_δ extends each p_i.

(8) Disjointness: If $M \leq_{\mathbf{K}} N$ are both limit models in $\mathbf{K}_\lambda$ and $p \in \mathbf{S}(N)$ does not fork over M, then p is algebraic if and only if $p \restriction M$ is algebraic (recall Definition 10.6)

Proof. The first five follow from Lemma 13.3 and Theorem 13.8. For transitivity, let $q \in \mathbf{S}(M_2)$ be a nonforking extension of $p \restriction M_0$. By monotonicity, $q \restriction M_1$ does not fork over M_0, hence by uniqueness $p \restriction M_1 = q \restriction M_1$. By monotonicity again, q does not fork over M_1. By uniqueness again, $q = p$, so p does not fork over M_0.

For continuity, let $p_\delta \in \mathbf{S}(M_\delta)$ be the nonforking extension of p_0. By uniqueness, p_δ also extends each p_i.

For disjointness, observe that if $p \restriction M$ is algebraic, then any extension is also algebraic, so p is algebraic. Conversely, assume that p is algebraic. Using local character and transitivity, pick M_0 and δ such that M is (λ, δ)-limit over M_0 and p does not fork over M_0. Now pick N' which is (λ, δ)-limit over N, hence over M_0. Let $q \in \mathbf{S}(N')$ be a nonforking extension of p. Since p is algebraic, q is algebraic. By conjugation (Lemma 13.7), q and $p \restriction M$ are conjugates over M_0. This means that one is algebraic if and only if the other is, so $p \restriction M$ is also algebraic. □

14. Canonicity of forking

In this section, we show that the exact definition of forking we take is immaterial: at least over limit models, any two definitions satisfying some basic properties must be

the same. In particular, nonforking in the sense defined here must in a superstable elementary class coincide (over limit models) with the usual nonforking.

Theorem 14.1 (The canonicity theorem). Let $\mathbf{K}$ be a λ-superstable AEC. Assume we have a relation "p is free over M" for a type $p \in \mathbf{S}(N)$ and $M, N \in \mathbf{K}_\lambda$ *limit models*. If this relation satisfies invariance, monotonicity, universal local character, uniqueness, and extension (in the sense of Corollary 13.16), then p is free over M if and only if p does not fork over M (in the sense of Definition 13.1).

Proof. We first show that if $M \leq_{\mathbf{K}} N$ are both limit models in $\mathbf{K}_\lambda$ and p is free over M, then p does not split over M. Indeed, let $N_1, N_2 \in \mathbf{K}_\lambda$ with $M \leq_{\mathbf{K}} N_1, N_2 \leq_{\mathbf{K}} N$ and let $f : N_1 \cong_M N_2$. Since p is free over M, both $p \restriction N_1$ and $p \restriction N_2$ are free over M. By invariance, $f(p \restriction N_1)$ is also free over M, and $f(p \restriction N_1) \restriction M = p \restriction M$ because f fixes M. Thus by uniqueness $f(p \restriction N_1) = p \restriction N_2$, as desired.

It follows that if $M \leq_{\mathbf{K}} N$ are both limit models in $\mathbf{K}_\lambda$, then whenever $p \in \mathbf{S}(N)$ is free over M, we have that p does not fork over M. Indeed, we can write $M = \bigcup_{i < \delta} M_i$, with $\langle M_i : i < \delta \rangle$ an increasing continuous chain of limit models in $\mathbf{K}_\lambda$, M_{i+1} universal over M_i for all $i < \delta$. By universal local character, there exists $i < \delta$ such that $p \restriction M$ is free over M_i. By transitivity (which follows from uniqueness and extension), we must also have that p is free over M_i. Thus p does not split over M_i, hence p does not fork over M.

Assume now that $p \in \mathbf{S}(N)$ does not fork over M (where as before $M \leq_{\mathbf{K}} N$ and $M, N \in \mathbf{K}_\lambda$ are limit models). Note that $p \restriction M$ is free over M (by universal local character and monotonicity). Pick $q \in \mathbf{S}(N)$ such that q extends $p \restriction M$ and q is free over M. By what has been said before, q does not fork over M. By uniqueness of nonforking $p = q$. Thus p is free over M, as desired. $\qquad\square$

Remark 14.2. The proof of Lemma 13.3 shows that extension (over limit models) follows abstractly from the other properties. Thus it is not necessary to assume that freeness satisfies extension in Theorem 14.1.

15. Superstability and symmetry

The following auxiliary property of forking in a superstable AEC will be useful:

Lemma 15.1. Let $\mathbf{K}$ be a λ-superstable AEC. Let $M \leq_{\mathbf{K}} N \leq_{\mathbf{K}} N'$ all be limits in $\mathbf{K}_\lambda$. Let $a \in N'$ and let $\bar{b}_1, \bar{b}_2 \in {}^{\leq\lambda}N$. If $\mathbf{tp}(a/N; N')$ does not fork over M and $\mathbf{tp}(\bar{b}_1/M; N) = \mathbf{tp}(\bar{b}_2/M; N)$, then $\mathbf{tp}(a\bar{b}_1/M; N') = \mathbf{tp}(a\bar{b}_2/M; N')$.

Proof. Extending N if necessary (using extension and transitivity), we can assume without loss of generality that N is limit over M, and in fact limit over an extension of M containing $\bar{b}_1\bar{b}_2$. We can then find $f \in \operatorname{Aut}_M(N)$ such that $f(\bar{b}_1) = \bar{b}_2$. Let $p := \mathbf{tp}(a/N; N')$. Since p does not fork over M, it does not split over M, hence $f(p) = p$. Assume without loss of generality that N' is limit over N and f extends to an automorphism g of N', we then have that $\mathbf{tp}(a/N; N') = \mathbf{tp}(g(a)/N; N')$, hence there exists an automorphism h of N' such that $h(g(a)) = a$ and h fixes N. Now if $h' := hg$, we have that $h'(\bar{b}_1) = \bar{b}_2$ and $h'(a) = a$. Thus $\mathbf{tp}(a\bar{b}_1/M; N') = \mathbf{tp}(h'(a)h'(\bar{b}_1)/M; N') = \mathbf{tp}(a\bar{b}_2/M; N')$, as desired. $\qquad\square$

We aim to investigate the following property:

Definition 15.2. Let $\mathbf{K}$ be a λ-superstable AEC. We say that $\mathbf{K}$ has λ-*symmetry* if for any two limit model $M \leq_{\mathbf{K}} N$ in $\mathbf{K}_\lambda$ and any $a, b \in N$, the following are equivalent:

   (1) There exists $M_b \leq_{\mathbf{K}} N_b$ in $\mathbf{K}_\lambda$ both limits such that $N \leq_{\mathbf{K}} N_b$, $M \leq_{\mathbf{K}} M_b$, $b \in M_b$, and $\mathbf{tp}(a/M_b; N_b)$ does not fork over M.
   (2) There exists $M_a \leq_{\mathbf{K}} N_a$ in $\mathbf{K}_\lambda$ both limits such that $N \leq_{\mathbf{K}} N_a$, $M \leq_{\mathbf{K}} M_a$, $a \in M_a$, and $\mathbf{tp}(b/M_a; N_a)$ does not fork over M.

The first-order version of this symmetry property would say that $\mathrm{tp}(a/Mb)$ does not fork over M if and only if $\mathrm{tp}(b/Ma)$ does not fork over M (Shelah established that this holds in any stable theory). Playing with monotonicity a little bit, one sees that Definition 15.2 says essentially this. We will see that symmetry is useful in order to understand limit models better. The following is open:

Question 15.3. Does λ-symmetry follow from λ-superstability?

We will establish that it follows from categoricity by showing that failure of symmetry implies an instance of the order property [VV17, 5.7]:

Theorem 15.4. Let $\mathbf{K}$ be a λ-superstable AEC which does *not* have λ-symmetry. Then $\mathbf{K}$ has the $(2, \lambda)$-order property of length λ^+.

Proof. To simplify the notation, let us work inside a saturated model $\mathfrak{C}$ of cardinality λ^+. We assume that all the models we work with are inside $\mathfrak{C}$ and all the types are also computed inside $\mathfrak{C}$. We will establish that $\mathfrak{C}$ has the $(2, \lambda)$-order property of length λ^+. Fix $M \in \mathbf{K}_\lambda$ limit, and $a, b \in \mathfrak{C}$ witnessing failure of symmetry. Say there exists $M_b \in \mathbf{K}_\lambda$ limit such that $M \leq_{\mathbf{K}} M_b$, $b \in M_b$, and $\mathbf{tp}(a/M_b)$ does not fork over M, but there is *no* $M_a \in \mathbf{K}_\lambda$ limit such that $M \leq_{\mathbf{K}} M_a$, $a \in M_a$, and $\mathbf{tp}(b/M_a)$ does not fork over M.

We build increasing continuous $\langle N_\alpha : \alpha < \lambda^+ \rangle$ and $\langle a_\alpha, b_\alpha, N'_\alpha : \alpha < \lambda^+ \rangle$ by induction so that for all $i < \lambda^+$:

   (1) $N_i, N'_i \in \mathbf{K}_\lambda$ are limits.
   (2) N_0 is limit over M_b and $a \in N_0$.
   (3) $\mathbf{tp}(a_i/M_b) = \mathbf{tp}(a/M_b)$ and $a_i \in N_{i+1}$.
   (4) $\mathbf{tp}(b_i/M) = \mathbf{tp}(b/M)$ and $b_i \in N'_i$.
   (5) N'_i is limit over N_i and N_{i+1} is limit over N'_i.
   (6) $\mathbf{tp}(a_i/N'_i)$ and $\mathbf{tp}(b_i/N_i)$ both do not fork over M.

This is possible. Let N_0 be any model in $\mathbf{K}_\lambda$ containing a that is limit over M_b. At i limits, let $N_i := \bigcup_{j<i} N_j$. Now assume inductively that N_j has been defined for $j \leq i$, and a_j, b_j, N_j have been defined for $j < i$. By extension, find $q \in \mathbf{S}(N_i)$ that does not fork over M and extends $\mathbf{tp}(b/M)$. Let b_i realize q and pick N'_i limit over N_i containing b_i. Now by extension again, find $q' \in \mathbf{S}(N'_i)$ that does not fork over M and extends $\mathbf{tp}(a/M)$. Let a_i realize q' and pick N_{i+1} limit over N'_i containing a_i.

This is enough. We show that for $i, j < \lambda^+$:

(1) $\mathbf{tp}(ab_j/M) \neq \mathbf{tp}(ab/M)$
(2) $\mathbf{tp}(a_i b/M) = \mathbf{tp}(ab/M)$
(3) If $i < j$, $\mathbf{tp}(a_i b_j/M) = \mathbf{tp}(ab_j/M)$.
(4) If $i \geq j$, $\mathbf{tp}(a_i b_j/M) = \mathbf{tp}(a_i b/M)$.

For (1), observe that $a \in N_0 \subseteq N_j$ and $\mathbf{tp}(b_j/N_j)$ does not fork over M. Therefore by monotonicity N_j witnesses that there exists $M_a \in \mathbf{K}_\lambda$ limit containing a so that $\mathbf{tp}(b_j/M_a)$ does not fork over M. By failure of symmetry and invariance, we must have that $\mathbf{tp}(ab_j/M) \neq \mathbf{tp}(ab/M)$.

For (2), use the assumption that $b \in M_b$ together with clause (3) of the construction.

For (3), suppose $i < j$. We know that $\mathbf{tp}(b_j/N_j)$ does not fork over M. Since $i < j$, $a, a_i \in N_j$, and $\mathbf{tp}(a/M) = \mathbf{tp}(a_i/M)$, we must have by Lemma 15.1 that $\mathbf{tp}(a_i b_j/M) = \mathbf{tp}(ab_j/M)$.

For (4), suppose $i \geq j$. We know that $\mathbf{tp}(a_i/N_i')$ does not fork over M. Since $i \geq j$, $b, b_j \in N_i'$, and $\mathbf{tp}(b/M) = \mathbf{tp}(b_j/M)$, we must have by Lemma 15.1 again that $\mathbf{tp}(a_i b_j/M) = \mathbf{tp}(a_i b/M)$.

Now by (1) and (3), $\mathbf{tp}(a_i b_j/M) \neq \mathbf{tp}(ab/M)$ when $i < j$. Similarly, by (2) and (4), $\mathbf{tp}(a_i b_j/M) = \mathbf{tp}(ab/M)$ when $i \geq j$. Letting $\bar{c}_i := a_i b_i$ for $i < \lambda^+$, this tells us that the sequence $\langle \bar{c}_i : i < \lambda^+ \rangle$ witnesses the $(2, \lambda)$-order property of length λ^+. $\square$

Remark 15.5. The converse of Theorem 15.4 is also true, see [LRVa, 9.7].

Corollary 15.6 ([Vas17b, 4.8])**.** Let $\mathbf{K}$ be an AEC with arbitrarily large models and let $\mu > \mathrm{LS}(\mathbf{K})$. If $\mathbf{K}_{<\mu}$ has amalgamation and no maximal models and $\mathbf{K}$ is categorical in μ, then for every $\lambda \in [\mathrm{LS}(\mathbf{K}), \mu)$, $\mathbf{K}$ is λ-superstable and has λ-symmetry.

Proof. Fix $\lambda \in [\mathrm{LS}(\mathbf{K}), \mu)$. By Theorem 12.8, $\mathbf{K}$ is λ-superstable. By Corollary 11.16, $\mathbf{K}$ does not have the $(2, \lambda)$-order property of length λ^+. By Theorem 15.4, $\mathbf{K}$ has λ-symmetry. $\square$

Limit models are, in a sense, a local analog of saturated models. Indeed, at least when λ is a regular cardinal, a (λ, λ)-limit model will be (say in an AEC with amalgamation and $\lambda > \mathrm{LS}(\mathbf{K})$) saturated. It is natural to ask whether a (λ, δ)-limit model is also saturated, for $\delta < \lambda$. That is, are we building the same saturated models if we only take a chain of universal extension of length δ instead of going all the way to λ? More precisely:

Definition 15.7. Let $\mathbf{K}$ be a λ-superstable AEC. We say that $\mathbf{K}$ has *uniqueness of limit models in* λ if whenever $M_0, M_1, M_2 \in \mathbf{K}_\lambda$ are such that M_1 is limit over M_0 and M_2 is limit over M_0, then $M_1 \cong_{M_0} M_2$.

The interesting part of the definition is when M_1 is (λ, δ_1)-limit over M_0, M_2 is (λ, δ_2)-limit over M_0 and $\mathrm{cf}(\delta_1) \neq \mathrm{cf}(\delta_2)$. In this case it is no longer clear how to carry out a back and forth argument to build the isomorphism. We will prove the following result, essentially due to VanDieren [Van16a] and Shelah [She09a, II.4.8] in slightly different contexts:

Theorem 15.8. Let $\mathbf{K}$ be a λ-superstable AEC. If $\mathbf{K}$ has λ-symmetry, then $\mathbf{K}$ has uniqueness of limit models in λ.

It is again not known whether the assumption of λ-symmetry is necessary. It is known that in a first-order stable theory, uniqueness of limit models holds if and only if the theory is superstable [GVV16, 6.1].

Before proving Theorem 15.8, we give several applications. First, we improve on Lemma 13.7:

Theorem 15.9 (The conjugation property, improved)**.** Let $\mathbf{K}$ be a λ-superstable AEC with uniqueness of limit models in λ. Let $M \leq_{\mathbf{K}} N$ both be limit models in $\mathbf{K}_\lambda$. If $p \in \mathbf{S}(N)$ does not fork over M, then p is conjugate with $p \upharpoonright M$.

Proof. Assume first that N is limit over M. Then using local character and transitivity, pick M_0 such that M is limit over M_0 and p does not fork over M_0. Since N is limit over M, N is also limit over M_0. By uniqueness of limit models, there exists $f : N \cong_{M_0} M$. Now $f(p)$ does not fork over M_0 by invariance, and $f(p) \upharpoonright M_0 = p \upharpoonright M_0$, so by uniqueness $f(p) = p \upharpoonright M$, as desired.

In case N is not necessarily limit over M, pick N' limit over N (hence over N) and let $q' \in \mathbf{S}(N')$ be a nonforking extension of q. By the previous case (used twice), q' is conjugate with p and with $p \upharpoonright M$, thus since being conjugate is an equivalence relation, p is conjugate with $p \upharpoonright M$. $\qquad\square$

The following result connects uniqueness of limit models to chains of saturated models. It was first proven by VanDieren [Van16b] but the proof we give is from [Vasb, 3.3].

Theorem 15.10. Let $\mathbf{K}$ be a λ-superstable AEC. If $\mathbf{K}$ is λ^+-superstable and has uniqueness of limit models in λ^+, then the union of any increasing chain of λ^+-saturated models is λ^+-saturated.

Proof. We first prove:

<u>Claim</u>: If $M \leq_{\mathbf{K}} N$ are both in $\mathbf{K}_{\lambda^+}$ with M saturated and $M^0 \in \mathcal{P}_{\mathbf{K}_\lambda}(M)$, $N^0 \in \mathcal{P}_{\mathbf{K}_\lambda}(N)$ are both limits such that $M^0 \leq_{\mathbf{K}} N^0$ and $p \in \mathbf{S}(N^0)$ does not fork over M^0, then p is realized inside M. That is, there exists $a \in M$ such that $p = \mathbf{tp}(a/N^0; N)$.

<u>Proof of Claim</u>: Suppose not. We build $\langle M_i : i \leq \omega \rangle$, $\langle N_i : i \leq \omega \rangle$ increasing continuous such that for all $i < \omega$:

(1) $M_0 = M^0$, $N_0 = M$.
(2) $M_i \in \mathbf{K}_\lambda$, M_i is limit, $N_i \in \mathbf{K}_{\lambda^+}$, N_i is saturated, $M_i \leq_{\mathbf{K}} N_i$.
(3) N_{i+1} is universal over N_i.
(4) The nonforking extension of p to M_{i+1} is *not* realized in N_i.

(In what follows, we write $p_{M'}$ for the nonforking extension of p to some fixed $M' \geq_{\mathbf{K}} M$.)

This is enough: N_ω is (λ^+, ω)-limit, hence by uniqueness of limit models in λ^+, it is also (λ^+, λ^+)-limit, hence saturated. Thus, there must exist $a \in N_\omega$ such that

$p_{M_\omega} = \mathbf{tp}(a/M_\omega; N_\omega)$. But then there exists $i < \omega$ such that $a \in N_i$ so p_{M_ω}, hence $p_{M_{i+1}}$, is realized in N_i, a contradiction.

This is possible: the base case is already specified. Then let $M_1 := N^0$ and pick $N_1 \in \mathbf{K}_{\lambda^+}$ saturated such that N_1 is universal over N. Now given $\langle M_j, N_j : j \leq i + 1 \rangle$, note that by saturation N_i and N_{i+1} are isomorphic over M_i. Pick $f : N_i \cong_{M_i} N_{i+1}$. Let $g : N_{i+1} \cong N'_{i+2}$ be an extension of f (so in particular $N_{i+1} \leq_{\mathbf{K}} N'_{i+2}$). Since $p_{M_{i+1}}$ is not realized inside N_i, $g(p_{M_{i+1}})$ is not realized inside N_{i+1}, and note that $g(p_{M_{i+1}})$ is the nonforking extension of p_{M_i}, hence of p, by uniqueness and the fact that g fixes M_i. Pick $M_{i+2} \in \mathcal{P}_{\mathbf{K}_\lambda}(N'_{i+2})$ limit such that $g[M_{i+1}] \leq_{\mathbf{K}} M_{i+2}$ and M_{i+2} is universal over M_{i+1}. Finally, pick some $N_{i+2} \in \mathbf{K}_{\lambda^+}$ saturated such that N_{i+2} is universal over N'_{i+2}. †$_{\text{Claim}}$

Now let $\langle M_i : i < \delta \rangle$ be an increasing chain of λ^+-saturated models. Let $M_\delta := \bigcup_{i < \delta} M_i$. We can assume without loss of generality that $\delta = \mathrm{cf}(\delta)$. If $\delta \geq \lambda^+$, then an easy cofinality argument gives the result, so assume that $\delta < \lambda^+$. It is enough to see that every submodel of M_δ of cardinality λ^+ is saturated, so assume without loss of generality again that $M_i \in \mathbf{K}_{\lambda^+}$ (so in particular is saturated) for all $i < \delta$. Now let $M \in \mathcal{P}_{\mathbf{K}_\lambda}(M_\delta)$ and let $p \in \mathbf{S}(M)$. Find $\langle M_i^0 : i < \delta \rangle$ an increasing chain of limit models in $\mathbf{K}_\lambda$ such that for all $i < \delta$, M_{i+1}^0 is universal over M_i^0, $M_i^0 \leq_{\mathbf{K}} M_i$, and M_i^0 contains $M \cap M_i$. Let $M_\delta^0 := \bigcup_{i < \delta} M_i^0$. Then by coherence $M \leq_{\mathbf{K}} M_\delta^0$. Let $q \in \mathbf{S}(M_\delta^0)$ be any extension of p. By universal local character, there is $i < \delta$ such that q does not fork over M_i^0. By the claim, q is realized inside M_i, hence inside M_δ, hence p is also realized in M_δ, as desired. □

Modulo Theorem 15.8, we deduce the following structural properties of categorical AECs with amalgamation [Vas17b, 5.7]:

Corollary 15.11. Let $\mathbf{K}$ be an AEC with arbitrarily large models and let $\mu > \mathrm{LS}(\mathbf{K})$. Assume that $\mathbf{K}_{<\mu}$ has amalgamation and no maximal models. If $\mathbf{K}$ is categorical in μ, then:

(1) For any $\lambda \in (\mathrm{LS}(\mathbf{K}), \mu]$, the union of any increasing chain of λ-saturated models is λ-saturated. Moreover, there is a saturated model of cardinality λ.

(2) The model of cardinality μ is saturated.

Proof. The second part obviously follows from the first (taking $\lambda = \mu$). Now let $\lambda \in [\mathrm{LS}(\mathbf{K}), \mu)$. By Corollary 15.6, $\mathbf{K}$ is λ-superstable and has λ-symmetry. By Theorem 15.8, this implies that $\mathbf{K}$ has uniqueness of limit models in λ. Thus if $\lambda > \mathrm{LS}(\mathbf{K})$ and λ is a successor, Theorem 15.10 implies that unions of chains of λ-saturated models are saturated. If λ is limit, then the same result holds, as a model is λ-saturated if and only if it is λ_0^+-saturated for all $\lambda_0 < \lambda$. We also have that any limit model is saturated, hence in this case there is a saturated model in λ

Next, we show that the union of an increasing chain of μ-saturated models is μ-saturated. If μ is limit, this again follows from what was previously established, so assume that μ is a successor. In this case, it is straightforward to establish that there is a saturated model of cardinality μ: simply use stability below μ to build $\langle M_i : i < \mu \rangle$ increasing continuous in $\mathbf{K}_{<\mu}$ with M_{i+1} universal over M_i, and

observe that $\bigcup_{i<\mu} M_i$ is saturated by regularity of μ and has cardinality μ. Once we know that the model of cardinality μ is saturated, it follows that any model of larger cardinality is μ-saturated as well, and hence in particular the union of any chain of μ-saturated models is μ-saturated.

It remains to establish that the model of cardinality μ is saturated when μ is limit. But this is straightforward given what we have established already: simply build $\langle M_i : i < \mu \rangle$ increasing continuous in $\mathbf{K}_{<\mu}$ such that each M_i is saturated. Then the union will be μ_0-saturated for each $\mu_0 < \mu$, hence (since μ is limit) saturated. $\square$

16. The uniqueness of limit models

The goal of this section is to prove Theorem 15.8. We follow the proof from [GVV16, Van16a], but bring a lot of simplifications, with ideas from Shelah [She09a, Chapter II]. All throughout, we assume:

Hypothesis 16.1. $\mathbf{K}$ is a λ-superstable AEC with λ-symmetry.

We work inside $\mathbf{K}_\lambda$: except if said otherwise, all models come from there. The following consequence of symmetry will be crucial. The idea is that we can make sure that two elements are independent "in a uniform way".

Lemma 16.2 (Nonforking amalgamation)**.** Let $M_0 \leq_{\mathbf{K}} M_\ell$, $\ell = 1, 2$, be limit models. Let $a_\ell \in M_\ell$. There exists f_1, f_2, M_3 such that M_3 is limit and $f_\ell : M_\ell \xrightarrow[M_0]{} M_3$ is such that $\mathbf{tp}(f_\ell(a_\ell)/f_{3-\ell}(M_{3-\ell}); M_3)$ does not fork over M_0 for $\ell = 1, 2$.

Proof. Work inside a saturated $\mathfrak{C}$ of cardinality λ^+ (so in particular without loss of generality $M_\ell \leq_{\mathbf{K}} \mathfrak{C}$ for $\ell = 1, 2$). First find g_1 an automorphism of $\mathfrak{C}$ fixing M_0 such that $\mathbf{tp}(g_1(a_2)/M_1)$ does not fork over M_0 (do this by fixing a realization of the nonforking extension of $\mathbf{tp}(a_2/M_0)$ and sending a_2 to it). By symmetry, there exists N_2 limit containing $g_1(a_2)$ and extending M_0 such that $\mathbf{tp}(a_1/N_2)$ does not fork over M_0. Pick $N_2' \leq_{\mathbf{K}} \mathfrak{C}$ limit containing M_1, $g_1[M_2]$, and N_2. Find an automorphism g_2 of $\mathfrak{C}$ fixing N_2 such that $\mathbf{tp}(a_1/g_2[N_2'])$ does not fork over N_2 (possible using existence of nonforking extensions with some renaming). Now by transitivity, $\mathbf{tp}(a_1/g_2[N_2'])$ does not fork over M_0. Let f_1 be the identity on M_1, and let $f_2 := g_2 g_1$. We claim that this works. First, $f_1(a_1) = a_1$ and $f_2[M_2] = g_2 g_1[M_2] \leq_{\mathbf{K}} g_2[N_2']$, so $\mathbf{tp}(a_1/f_2[M_2])$ does not fork over M_0 by monotonicity. Second, $f_1[M_1] = M_1$ and since g_2 fixes N_2 which contains $g_1(a_2)$, $f_2(a_2) = g_2(g_1(a_2)) = g_1(a_2)$, and $\mathbf{tp}(g_1(a_2)/M_1)$ does not fork over M_0, as desired. $\square$

Towers will be key in the proof of uniqueness of limit models. To define them, we first introduce some notation:

Notation 16.3. For $I = (I, <)$ a well-ordering, let I^- be the initial segment of I which is isomorphic to I if I is isomorphic to a limit ordinal or zero, or α if I is isomorphic to $\alpha + 1$. For $i \in I^-$, let $i + 1$ denote the successor of i in I.

Definition 16.4. A *tower* $\mathcal{T}$ consists of $\langle M_i : i \in I \rangle \frown \langle a_i : i \in I^- \rangle$, where:

(1) I is a well-ordering of cardinality at most λ.

(2) $\langle M_i : i \in I \rangle$ is an increasing chain of limit models, not necessarily continuous.
(3) $a_i \in M_{i+1} \backslash M_i$ for each $i \in I^-$.

We call I the *length* (or *index set*) of the tower. We call $\mathcal{T}$ *continuous* if $\langle M_i : i \in I \rangle$ is continuous. We say that $\mathcal{T}$ is *universal* if M_{i+1} is universal over M_i for each $i \in I^-$. We may often identify a tower $\mathcal{T}$ indexed by I with the tower indexed by the ordinal $\text{otp}(I)$.

Definition 16.5. For $\mathcal{T} = \langle M_i : i \in I \rangle \frown \langle a_i : i \in I^- \rangle$ and $I_0 \subseteq I$, we let $\mathcal{T} \upharpoonright I_0$ be the sequences $\langle M_i : i \in I_0 \rangle \frown \langle a_i : i \in I_0 \rangle$.

Remark 16.6. If $\mathcal{T}$ is a tower indexed by I and $I_0 \subseteq I$, then $\mathcal{T} \upharpoonright I_0$ is a tower indexed by I_0.

Towers exist. More precisely:

Exercise 16.7.

(1) The two empty sequences form a tower of length zero.
(2) Let I be a well-ordering of cardinality at most λ and let $\mathcal{T} = \langle M_i : i \in I \rangle \frown \langle a_i : i \in I^- \rangle$ be such that $\mathcal{T} \upharpoonright I_0$ is a tower for all $I_0 \subseteq I$ with $|I_0| \leq 2$. Then $\mathcal{T}$ is a tower.
(3) Let $\mathcal{T} = \langle M_i : i < \alpha \rangle \frown \langle a_i : i + 1 < \alpha \rangle$ be a tower of length α. Let $i < \alpha$ and let $p \in \mathbf{S}(M_i)$. There exists M_α and a such that $\mathbf{tp}(a_\alpha / M_\alpha)$ is a nonforking extension of p and $\langle M_i : i \leq \alpha \rangle \frown \langle a_i : i + 1 < \alpha \rangle \frown \langle a \rangle$ is a tower $\mathcal{T}'$ of length $\alpha + 1$ with $\mathcal{T}' \upharpoonright \alpha = \mathcal{T}$.
(4) Let $I_0 \subseteq I$ be two well-orderings. Let $\mathcal{T} = \langle M_i : i \in I_0 \rangle \frown \langle a_i : i \in I_0^- \rangle$ be a *universal* tower. Then there exists a universal tower $\mathcal{T}'$ of index I such that $\mathcal{T}' \upharpoonright I_0 = \mathcal{T}$.

Definition 16.8 (Ordering on towers)**.** For $\mathcal{T}^\ell = \langle M_i^\ell : i \in I^\ell \rangle \frown \langle a_i^\ell : i \in (I^\ell)^- \rangle$, $\ell = 1, 2$, two towers, we write $\mathcal{T}^1 \lhd \mathcal{T}^2$ if:

(1) $I^1 \subseteq I^2$.
(2) M_i^2 is universal over M_i^1 for all $i \in I^1$.
(3) $a_i^1 = a_i^2$ for all $i \in (I^1)^-$.
(4) $\mathbf{tp}(a_i^1 / M_i^2 ; M_{i+1}^2)$ does not fork over M_i^1 for all $i \in (I^1)-$.

We write $\mathcal{T}^1 \trianglelefteq \mathcal{T}^2$ if $\mathcal{T}^1 = \mathcal{T}^2$ or $\mathcal{T}^1 \lhd \mathcal{T}^2$.

Remark 16.9. $\trianglelefteq$ is a partial order on the class of all towers. Moreover, if $\mathcal{T}^1$ and $\mathcal{T}^2$ have index sets $I^1 \subseteq I^2$ respectively, then $\mathcal{T}^1 \trianglelefteq \mathcal{T}^2$ if and only if $\mathcal{T}^1 \trianglelefteq \mathcal{T}^2 \upharpoonright I^1$.

Definition 16.10. Let $\delta < \lambda^+$ be a limit ordinal and let $\langle \mathcal{T}^j : j < \delta \rangle$ be a $\trianglelefteq$-increasing chain of towers. Assume that $\mathcal{T}^j = \langle M_i^j : i \in I^j \rangle \frown \langle a_i^j : i \in (I^j)^- \rangle$. We define $\mathcal{T}^\delta := \langle M_i^\delta : i \in I^\delta \rangle \frown \langle a_i^\delta : i \in I^\delta \rangle$ as follows:

(1) $I^\delta = \bigcup_{j < \delta} I^j$.
(2) $a_i^\delta = a_i^j$ for some (any) $j < \delta$ such that $i \in I^j$.
(3) $M_i^\delta = \bigcup_{j < \delta} M_i^j$.

We write $\bigcup_{j < \delta} \mathcal{T}^j$ for $\mathcal{T}^\delta$.

Exercise 16.11. If $\langle \mathcal{T}^j : j < \delta \rangle$ is a $\trianglelefteq$-increasing chain of towers, where $\mathcal{T}^j$ is indexed by I^j, and $\bigcup_{j<\delta} I^j$ is a well-ordering, then $\bigcup_{j<\delta} \mathcal{T}^j$ is a tower and $\mathcal{T}^k \trianglelefteq \bigcup_{j<\delta} \mathcal{T}^j$ for every $k < \delta$.

Definition 16.12. If a $\trianglelefteq$-increasing chain of towers $\langle \mathcal{T}^j : j < \gamma \rangle$ is such that for every limit $j < \gamma$, $\mathcal{T}^j = \bigcup_{k<j} \mathcal{T}^k$, we call the chain *continuous*.

The following is an interesting property of towers. It says that any extension must be "as disjoint as possible".

Definition 16.13. A tower $\mathcal{T} = \langle M_i : i < \alpha \rangle \frown \langle a_i : i < \alpha \rangle$ is called *reduced* if whenever $\mathcal{T}' = \langle M'_i : i < \alpha \rangle \frown \langle a_i : i < \alpha \rangle$ extends $\mathcal{T}$, we have that $M'_i \cap M_j = M_i$ for all $i < \alpha$ and $j \in [i, \alpha)$.

Exercise 16.14. If $\langle \mathcal{T}^j : j < \delta \rangle$ is a chain of towers and $\mathcal{T}^j$ is reduced for all $j < \delta$, then $\bigcup_{j<\delta} \mathcal{T}^j$ is reduced (provided that its index is a well-ordering).

Reduced towers exist: any tower has a reduced extension. To prove this, we will use:

Exercise 16.15. Suppose that $M \leq_{\mathbf{K}} N$ are in $\mathbf{K}_{\lambda^+}$. Let $\langle M_i : i < \lambda^+ \rangle$, $\langle N_i : i < \lambda^+ \rangle$ be continuous resolutions in $\mathbf{K}_\lambda$ of M and N respectively. Then there exists a club $C \subseteq \lambda^+$ such that for $i \in C$, $M_i \leq_{\mathbf{K}} N_i$ and $M \cap N_i = M_i$.

Lemma 16.16 (Density of reduced towers)**.** For any tower $\mathcal{T}$ of length α, there exists a reduced tower $\mathcal{T}'$ of length α such that $\mathcal{T} \trianglelefteq \mathcal{T}'$.

Proof. Suppose not. Let α be the length of $\mathcal{T}$. We build $\langle \mathcal{T}^j : j < \lambda^+ \rangle$ an increasing continuous chain of towers of length α such that for all $j < \lambda^+$, writting $\mathcal{T}^j = \langle M_i^j : i < \alpha \rangle \frown \langle a_i : i + 1 < \alpha^j \rangle$:

(1) $\mathcal{T}^0 = \mathcal{T}$.
(2) For some $i < i' < \alpha$, $M_i^{j+1} \cap M_{i'}^j \neq M_i^j$.

This is possible since by assumption $\mathcal{T}^j$ cannot be reduced. This is enough: for each $i < \alpha$, by Exercise 16.15 (where M, N there stand for $\bigcup_{j<\lambda^+} M_i^j$, $\bigcup_{j<\lambda^+} \bigcup_{i'<\alpha} M_{i'}^j$ here) there is a club C_i such that $M_i^{j+1} \cap \bigcup_{i'<\alpha} M_{i'}^j = M_i^j$ for all $j \in C_i$. Let $C := \bigcap_{i<\alpha} C_i$. C is club, and any $j \in C$ witnesses that requirement (2) fails. $\square$

So far, we haven't shown that towers have any nontrivial $\trianglelefteq$-extensions. We do this now. In fact, we prove more using nonforking amalgamation.

Lemma 16.17 (Existence of extensions of towers)**.** Let $\mathcal{T} = \langle M_i : i < \alpha \rangle \frown \langle a_i : i + 1 < \alpha \rangle$ be a tower.

(1) There exists a universal tower $\mathcal{T}'$ of length α such that $\mathcal{T} \vartriangleleft \mathcal{T}'$.
(2) Assume in addition that $\mathcal{T}$ is universal and continuous. If $q \in \mathbf{S}(M_0)$. Then there exists a tower $\mathcal{T}' = \langle M'_i : i < \alpha \rangle \frown \langle a_i : i + 1 < \alpha \rangle$ and $b \in M'_0$ such that:
    (a) $\mathcal{T} \vartriangleleft \mathcal{T}'$.
    (b) $\mathbf{tp}(b/M_0; M'_0) = q$.
    (c) $\mathbf{tp}(b/\bigcup_{i<\alpha} M_i; \bigcup_{i<\alpha} M'_i)$ does not fork over M_0.

Proof. If we are in setup (2), pick b and N such that $q = \mathbf{tp}(b/M_0; N)$. We build $\langle N_i, f_i : i \leq \alpha \rangle$ such that:

(1) $\langle N_i : i \leq \alpha \rangle$, $\langle N_i : i \leq \alpha \rangle$ are increasing.
(2) $f_i : M_i \to N_i$ is a $\mathbf{K}$-embedding for all $i \leq \alpha$.
(3) For all $i < \alpha$, N_i is limit and N_{i+1} is limit over N_i if $i + 1 < \alpha$.
(4) N_i is limit over M_i for all $i < \alpha$.
(5) f_0 is the identity on N_0.
(6) For all $i + 1 < \alpha$, $\mathbf{tp}(f_{i+1}(a_i)/N_i; N_{i+1})$ does not fork over M_i.
(7) If we are in setup (2), $N \leq_{\mathbf{K}} N_0$ and $\mathbf{tp}(b/f_i[M_i]; N_i)$ does not fork over M_0 for all $i < \alpha$.

This is possible: At $i = 0$, what to do has been specified. For i limit, let $N_i^* := \bigcup_{j<i} N_j$ and $g_i := \bigcup_{j<i} f_j$. Pick N_i limit over N_i^* and use amalgamation to extend $g_i : \bigcup_{j<i} M_j \to N_i^*$ to $f_i : M_i \to N_i$. If we are in setup (2), use continuity of non-forking (and the fact that $\mathcal{T}$ is universal and continuous) to check that requirement (7) is preserved. At successors, we use the existence property of forking and (if we are in setup (2)) Lemma 16.2.

This is enough: Extend f_α to $g : M_\alpha' \cong N_\alpha$. For $i < \alpha$, let $M_i' := g^{-1}[N_i]$ and let $\mathcal{T}' := \langle M_i' : i < \alpha \rangle \frown \langle a_i : i + 1 < \alpha \rangle$. This works: note that since for $i + 1 < \alpha$, $\mathbf{tp}(a_i/M_i'; M_{i+1}')$ does not fork over M_i and $a_i \notin M_i$, we have by disjointness (Corollary 13.16) that $a_i \notin M_i'$, so $\mathcal{T}'$ is indeed a tower. The rest of the properties follow directly from the construction. $\qquad\square$

The following technical consequence will be used in the proof of the next theorem.

Lemma 16.18. Let $\delta < \lambda^+$ be a limit ordinal. Let $\mathcal{T} = \langle M_i : i \leq \delta \rangle \frown \langle a_i : i < \delta \rangle$ be a tower such that $\mathcal{T} \restriction \delta$ is universal and continuous. Let $b \in M_\delta$. If $\mathbf{tp}(b/\bigcup_{i<\delta} M_i; M_\delta)$ does not fork over M_0, then there exists a tower $\mathcal{T}' = \langle M_i' : i \leq \delta \rangle \frown \langle a_i : i < \delta \rangle$ such that $\mathcal{T} \triangleleft \mathcal{T}'$ and $b \in M_0'$.

Proof. Write $M_\delta^0 := \bigcup_{i<\delta} M_i$. Let $q := \mathbf{tp}(b/M_0; M_\delta)$. By Lemma 16.17 applied to the tower $\mathcal{T} \restriction \delta$, there exists $\mathcal{T}^* = \langle M_i^* : i < \delta \rangle \frown \langle a_i : i < \delta \rangle$ and $b^* \in M_0^*$ such that $\mathcal{T} \restriction \delta \triangleleft \mathcal{T}^*$, $\mathbf{tp}(b^*/M_0; M_0^*) = q$, and $\mathbf{tp}(b^*/M_\delta^0; M_\delta^*)$ does not fork over M_0 (we have set $M_\delta^* := \bigcup_{i<\delta} M_i^*$). Note that we have used that δ is a limit ordinal to make sure that all the a_i's are still in $\mathcal{T} \restriction \delta$. By uniqueness, $\mathbf{tp}(b^*/M_\delta^0; M_\delta^*) = \mathbf{tp}(b/M_\delta^0; M_\delta)$. Pick M_δ' limit over M_δ and $f : M_\delta^* \xrightarrow[M_\delta^0]{} M_\delta'$ such that $f(b^*) = b$. Let $M_i' := f[M_i^*]$ for $i < \delta$. $\qquad\square$

We obtain the following powerful tool to build continuous towers:

Theorem 16.19. Any reduced tower is continuous.

Proof. Suppose not. Let α be the least length of a reduced non-continuous tower. Then it is easy to see that $\alpha = \delta + 1$, where δ is a limit ordinal. Let $\mathcal{T} = \langle M_i : i \leq \delta \rangle \frown \langle a_i : i < \delta \rangle$ be such a reduced non-continuous tower. Thus $\bigcup_{i<\delta} M_i \neq M_\delta$. Pick $b \in M_\delta \backslash \bigcup_{i<\delta} M_i$.

Claim: There is *no* $k < \delta$ and no tower $\mathcal{T}' = \langle M_i' : i \in [k, \delta] \rangle \frown \langle a_i : i \in [k\delta] \rangle$ such that $\mathcal{T} \restriction [k, \delta] \triangleleft \mathcal{T}'$ and $b \in \bigcup_{i \in [k,\delta)} M_i'$.

<u>Proof of Claim</u>: Suppose $\mathcal{T}'$ is such a tower and fix $i < \delta$ such that $b \in M_i'$. Then $b \in M_i' \cap M_\delta$ but $b \notin M_i$, so $M_i' \cap M_\delta \neq M_i$. Moreover, one can extend $\mathcal{T}'$ to a tower $\mathcal{T}''$ of length $\delta + 1$ so that $\mathcal{T}'' \upharpoonright [k, \delta) = \mathcal{T}'$ and $\mathcal{T} \trianglelefteq \mathcal{T}''$ (use universality of M_k' over M_k). This implies that $\mathcal{T}$ is not reduced, contradiction. $\dagger_{\text{Claim}}$

We aim to build a tower as in the claim to get a contradiction. Build $\langle \mathcal{T}^j : j \leq \delta \rangle$ an $\vartriangleleft$-increasing continuous chain of reduced towers such that $\mathcal{T}^0 = \mathcal{T}$. Write $\mathcal{T}^j = \langle M_i^j : i \leq \delta \rangle \frown \langle a_i : i < \delta \rangle$. Now consider the diagonal tower $\mathcal{T}^* := \langle M_i^i : i \leq \delta \rangle \frown \langle a_i : i < \delta \rangle$. It is easy to check that $\mathcal{T}^*$ is indeed a tower, and further it is universal. Since δ was minimal, $\mathcal{T}^j \upharpoonright \delta$ is continuous for all $j \leq \delta$, and hence $\mathcal{T}^* \upharpoonright \delta$ is also continuous. Further, it is easy to check that $\mathcal{T}^* \upharpoonright \delta$ is universal. By local character, there exists $i < \delta$ such that $\mathbf{tp}(b/\bigcup_{k<\delta} M_k^k; M_\delta^\delta)$ does not fork over M_i^i. Let $\mathcal{T}^{**} := \langle M_i^i : i \in [k+1, \delta] \rangle \frown \langle a_i : i \in [k+1, \delta] \rangle$. By Lemma 16.18, where $\mathcal{T}$ there stands for $\mathcal{T}^{**}$ here, there exists a tower $\mathcal{T}'$ such that $\mathcal{T}^{**} \vartriangleleft \mathcal{T}'$ and $\mathcal{T}' \upharpoonright \delta$ contains b. Since $\mathcal{T}'$ also extends $\mathcal{T} \upharpoonright [k, \delta]$, this contradicts the claim. $\qquad\square$

We now want to give conditions under which a tower $\langle M_i : i < \alpha \rangle \frown \langle a_i : i+1 < \alpha \rangle$ is such that $\bigcup_{i<\alpha} M_i$ is limit over M_0. Of course, being a universal tower suffices but it is not clear whether this property is closed under unions. Instead, we will rely on the following weakening:

Definition 16.20. A tower $\mathcal{T} = \langle M_i : i \in I \rangle \frown \langle a_i : i \in I^- \rangle$ is *full* if:

(1) $\operatorname{otp}(I) \cdot \lambda = \operatorname{otp}(I)$.
(2) For any $i \in I$ and any $p \in \mathbf{S}^{\mathrm{na}}(M_i)$ (recall Definition 10.6), there exists $k \geq i$ such that the order type of $[i, k]$ is less than λ^2 and $\mathbf{tp}(a_k/M_k; M_{k+1})$ is the nonforking extension of p.

Intuitively, full towers are those for which the a_i's realize all the types.

Lemma 16.21. If $\mathcal{T} = \langle M_i : i < \delta \rangle \frown \langle a_i : i+1 < \delta \rangle$ is a full tower, then $\bigcup_{i<\delta} M_i$ is (λ, δ)-limit over M_0.

Proof. Fix $i < \delta$. By Lemma 7.20 applied to $\langle M_{i+j\cdot\lambda\cdot\lambda} : j \leq \lambda \rangle$, we obtain that $M_{i+\lambda^3}$ is universal over M_i. Since i was arbitrary, we obtain that $\langle M_{i\cdot\lambda^3} : i < \delta \rangle$ is the desired witness that $\bigcup_{i<\delta} M_i$ is (λ, δ)-limit over M_0. $\qquad\square$

Full towers are also preserved by unions:

Lemma 16.22. Let $\delta < \lambda^+$ be a limit ordinal. Let $\langle \mathcal{T}^j : j < \delta \rangle$ be an increasing chain of full towers. Suppose that $\mathcal{T}^j$ is indexed by I^j, and $I^\delta := \bigcup_{j<\delta} I^j$ is a well-ordering. Then $\bigcup_{j<\delta} \mathcal{T}^j$ is full.

Proof. Let $\mathcal{T}^\delta := \bigcup_{j<\delta} \mathcal{T}^j$. For $j \leq \delta$, write $\mathcal{T}^j = \langle M_i^j : i \in I^j \rangle \frown \langle a_i : i \in (I^j)^- \rangle$. It is easy to check that $\operatorname{otp}(I^\delta) \cdot \lambda = \operatorname{otp}(I^\delta)$. Now let $i \in I^\delta$ and let $p \in \mathbf{S}^{\mathrm{na}}(M_i^\delta)$. Pick $j_0 < \delta$ such that p does not fork over M_i^j and $i \in I^{j_0}$.

<u>Claim</u>: For any $j \in [j_0, \delta)$ and any $k \in I^j$, if $\mathbf{tp}(a_k/M_k^j)$ is the nonforking extension of $p \upharpoonright M_i^j$, then $\mathbf{tp}(a_k/M_k^\delta; M_{k+1}^\delta)$ is the nonforking extension of p.

<u>Proof of Claim</u>: By definition of extension of tower, $\mathbf{tp}(a_k/M_k^\delta; M_{k+1}^\delta)$ does not fork over M_k^j. By transitivity of forking and uniqueness, $\mathbf{tp}(a_k/M_k^\delta; M_{k+1}^\delta)$ is the nonforking extension of $p \restriction M_i^j$, hence of p. $\dagger_{\text{Claim}}$

Now let $k \in I^\delta$ be minimal such that $\mathbf{tp}(a_k/M_k^\delta; M_{k+1}^\delta)$ is the nonforking extension of p (k exists by the claim and the fact that $\mathcal{T}^j$ is full for $j < \delta$). For $j \le \delta$, let $\alpha_j \in I^j$ be such that $\text{otp}([i, \alpha_j)) = \lambda^2$. We have to see that $k < \alpha_\delta$. For any $j \in [j_0, \delta)$, we must have by the claim and the fact that $\mathcal{T}^j$ is full that $k < \alpha_j$. Now observe that $j < j'$ implies that $\alpha_{j'} \le \alpha_j$. Since I^δ is a well-ordering, it cannot contain an infinite decreasing sequence so there must exist $j_1 \in [j_0, \delta)$ such that for any $j \ge j_1$, $\alpha_{j_1} = \alpha_j$. In particular, $\alpha_\delta = \alpha_{j_1}$. Since $k < \alpha_{j_1}$, we also have that $k < \alpha_\delta$, as desired. $\square$

Finally, full towers can be built as follows:

Exercise 16.23. Let $\delta < \lambda^+$ be a limit ordinal with $\delta = \delta \cdot \lambda$. Let $\langle M_i : i < \delta \rangle$ be increasing continuous with M_{i+1} universal over M_i. Then there exists $\langle a_i : i < \delta \rangle$ such that $\langle M_i : i < \delta \rangle \frown \langle a_i : i < \delta \rangle$ is a full tower.

Exercise 16.24. Let $\mathcal{T} = \langle M_i : i \in I \rangle \frown \langle a_i : i \in I^- \rangle$ be a tower. Assume that one can write $I = I^1 \cup I^2$, where I^1 and I^2 are disjoint, I^1 is an initial segment of I, and so I^2 is an end-segment of I. If $\mathcal{T} \restriction I^1$ and $\mathcal{T} \restriction I^2$ are full towers, then $\mathcal{T}$ is a full tower.

We are now ready to prove the uniqueness of limit models. The idea is to build an increasing chain of full towers, interleaved with reduced towers. At the end, the union will be both full and reduced, hence continuous. Thus the last model in the tower will have the cofinality of the union, as well as the cofinality of the length of the tower. Since these cofinalities can be chosen arbitrarily, this means that we have built a model that is limit of two different lengths, hence that any limit of these two different lengths must be isomorphic. In details, we will show:

Lemma 16.25. Let $M \in \mathbf{K}_\lambda$ and let $\delta < \lambda^+$ be a limit ordinal. Then there exists $N \in \mathbf{K}_\lambda$ that is both (λ, δ)-limit over M and (λ, ω)-limit over M.

Proof of Theorem 15.8. Let $M_0, M_1, M_2 \in \mathbf{K}_\lambda$ be such that M_1 and M_2 are both limit over M_0. Say M_ℓ is (λ, δ_ℓ)-limit, $\ell = 1, 2$. By Lemma 16.25 applied twice, there exists for $\ell = 1, 2$, $N_\ell \in \mathbf{K}_\lambda$ which is both (λ, δ_ℓ)-limit over M_0 and (λ, ω)-limit over M_0. By uniqueness of limit models of the same length, $M_\ell \cong_{M_0} N_\ell$. Since N_1 and N_2 are both (λ, ω)-limit over M, $N_1 \cong_{M_0} N_2$. Thus $M_1 \cong_{M_0} M_2$, as desired. $\square$

Proof of Lemma 16.25. Without loss of generality, $\delta = \text{cf}(\delta)$. Let $\gamma < \lambda^+$ be such that $\text{cf}(\gamma) = \delta$ and $\gamma \cdot \lambda = \gamma$. For $j < \omega$, let $J^j := \gamma \times (\gamma \cdot (j + 1))$, ordered lexicographically. Let $J^\omega := \bigcup_{j < \omega} J^j = \gamma \times (\gamma \cdot \omega)$, and let I^ω be $J^\omega \frown \{\rho\}$, where ρ is bigger than any member of J^ω. Finally, let $I^j := J^j \cup \{\rho\}$. Note that $\text{cf}(J^j) = \delta$ for all $j \le \omega$.

We build $\langle \mathcal{T}^j : j \le \omega \rangle$ a $\lhd$-increasing continuous chain of towers such that, writing $\mathcal{T}^j = \langle M_i^j : i \in I_0^j \rangle \frown \langle a_i : i \in (I_0^j)^- \rangle$:

(1) M_0^0 extends M.
(2) $I_0^{2j} = I^j$, $I_0^{2j+1} = I_0^{2j}$.
(3) For $j < \omega$ odd, $\mathcal{T}^j$ is reduced.
(4) For $j < \omega$ non-zero and even, $\mathcal{T}^j$ is full.

This is enough: Let $N := \bigcup_{i \in (I^\omega)^-} M_i^\omega$. By Lemma 16.22, $\mathcal{T}^\omega$ is full. By Lemma 16.21, N is (λ, δ)-limit over $M_{\min(I^\omega)}^\omega$, hence over M. By Exercise 16.14, $\mathcal{T}^\omega$ is reduced. By Theorem 16.19, $\mathcal{T}^\omega$ is continuous. This means that $N = M_\rho^\omega$. Now by definition of $\lhd$, M_ρ^ω is (λ, ω)-limit over M_ρ^0, hence over M, as desired.

This is possible: At $j = 0$, take any tower indexed by I^0 where M_0^0 extends M. For $j < \omega$, odd, use the density of reduced towers (Lemma 16.16). For $j < \omega$ even and not zero, first use Lemma 16.17 to get $\mathcal{T}$ a $\lhd$-extension of $\mathcal{T}^{j-1}$ which is universal and indexed by I_0^{j-1}. Now find $\mathcal{T}'$ indexed by I_0^j such that $\mathcal{T}' \upharpoonright I_0^{j-1} = \mathcal{T}$ and between each element of I^0, there is a full tower. By properties of full towers, $\mathcal{T}'$ is also a full tower. Let $\mathcal{T}^j := \mathcal{T}'$. $\qquad\square$

17. Good frames

So far, we have given a local definition of superstability, using splitting, defined a notion of nonforking from splitting, and studied the properties of forking. The rough idea is that forking is well-behaved over limit models. We now want to axiomatize this setup, and more precisely axiomatize a setup where forking is well-behaved over all models of the AEC. In fact, we will require full local character, not just local character for universal chains. This approach was pioneered by Shelah [She09a, Chapter II].

Definition 17.1. Let $\mathcal{F} = [\lambda, \theta)$ be an interval of cardinals with $\lambda < \theta$ (we allow $\theta = \infty$). A (type-full[4]) *good $\mathcal{F}$-frame* consists of $\mathfrak{s} = (\mathbf{K}, F)$, where:

(1) $\mathbf{K}$ is an AEC satisfying the following properties:
    (a) $\lambda \geq \mathrm{LS}(\mathbf{K})$.
    (b) $\mathbf{K}_\lambda \neq \emptyset$.
    (c) $\mathbf{K}_\mathcal{F}$ has amalgamation, joint embedding, and no maximal models.
    (d) $\mathbf{K}$ is stable in every $\mu \in \mathcal{F}$.
(2) F is a binary relation between a type $p \in \mathbf{S}(N)$ and a model M, with $M, N \in \mathbf{K}_\mathcal{F}$ and $M \leq_{\mathbf{K}} N$. We write p *does not fork over* M (or p *does not $\mathfrak{s}$-fork over* M) instead of $F(p, M)$. We require:
    (a) Invariance: if $f : N \cong N'$ and $p \in \mathbf{S}(N)$ does not fork over M, then $f(p)$ does not fork over M.
    (b) Monotonicity: If $p \in \mathbf{S}(N)$ does not fork over M and $M \leq_{\mathbf{K}} M' \leq_{\mathbf{K}} N$, then $p \upharpoonright M'$ does not fork over M and p does not fork over M'.
    (c) Disjointness: If $p \in \mathbf{S}(N)$ does not fork over M, then p is algebraic if and only if $p \upharpoonright M$ is algebraic.
    (d) Existence: If $M, N \in \mathbf{K}_\mathcal{F}$ are such that $M \leq_{\mathbf{K}} N$ and $p \in \mathbf{S}(M)$ is given, then there exists $q \in \mathbf{S}(N)$ such that q extends p and q does not fork over M.

---

[4]Shelah has a more general definition, where the nonforking relation is defined only for certain types, the basic types. He calls a frame where the nonforking relation is defined for all types type-full. We do not adopt this approach here, so will never mention the "type-full".

(e) Uniqueness: If $p, q \in \mathbf{S}(N)$ do not fork over M and $p \upharpoonright M = q \upharpoonright M$, then $p = q$.

(f) Local character: if δ is limit, $\langle M_i : i \leq \delta \rangle$ is an increasing continuous chain in $\mathbf{K}_{\mathcal{F}}$, and $p \in \mathbf{S}(M_\delta)$, then there exists $i < \delta$ such that p does not fork over M_i.

(g) Symmetry: Let $M \leq_{\mathbf{K}} N$ both be in $\mathbf{K}_{\mathcal{F}}$ and $a, b \in |N|$. The following are equivalent:
  (i) There exists $M_b, N_b \in \mathbf{K}_{\mathcal{F}}$ such that $M \leq_{\mathbf{K}} M_b \leq_{\mathbf{K}} N_b$, $N \leq_{\mathbf{K}} N_b$, $b \in |M_b|$, and $\mathbf{tp}(a/M_b; N_b)$ does not fork over M.
  (ii) There exists $M_a, N_a \in \mathbf{K}_{\mathcal{F}}$ such that $M \leq_{\mathbf{K}} M_a \leq_{\mathbf{K}} N_a$, $N \leq_{\mathbf{K}} N_a$, $a \in |M_a|$, and $\mathbf{tp}(b/M_a; N_a)$ does not fork over M.

We call a good $\mathcal{F}$-frame $\mathfrak{s} = (\mathbf{K}, F)$ *categorical* if $\mathbf{K}$ is categorical in λ. When $\theta = \lambda^+$, we write *good λ-frame* instead of *good $[\lambda, \lambda^+)$-frame*.

The same basic properties derived in the previous sections hold in good frames. For example:

Exercise 17.2. Let $\mathfrak{s} = (\mathbf{K}, F)$ be a good $\mathcal{F}$-frame. Then:

(1) Transitivity: If $M_0 \leq_{\mathbf{K}} M_1 \leq_{\mathbf{K}} M_2$ are in $\mathbf{K}_{\mathcal{F}}$ and $p \in \mathbf{S}(M_2)$ is such that p does not fork over M_1 and $p \upharpoonright M_1$ does not fork over M_0, then p does not fork over M_0.

(2) Continuity: if $\langle M_i : i \leq \delta \rangle$ is increasing continuous in $\mathbf{K}_{\mathcal{F}}$ and $\langle p_i : i < \delta \rangle$ is an increasing sequence of types with $p_i \in \mathbf{S}(M_i)$ and p_i does not fork over M_0 for all $i < \delta$, then there exists a unique $p_\delta \in \mathbf{S}(M_\delta)$ such that $p_i \leq p_\delta$ for all $i < \delta$. Moreover, p does not fork over M_0.

We also have the natural restriction properties:

Definition 17.3. For $\mathfrak{s} = (\mathbf{K}, F)$ a good $\mathcal{F}$-frame and $\mathcal{F}_0 \subseteq \mathcal{F}$ not empty, we write $\S_{\mathcal{F}_0}$ for the pair $(\mathbf{K}, F \upharpoonright \mathcal{F}_0)$, where $F \upharpoonright \mathcal{F}_0$ is the natural restriction of F to models in $\mathbf{K}_{\mathcal{F}_0}$.

Exercise 17.4. If $\mathfrak{s}$ is a good $\mathcal{F}$-frame and $\mathcal{F}_0 \subseteq \mathcal{F}$ is not empty, then $\mathfrak{s}_{\mathcal{F}_0}$ is a good $\mathcal{F}_0$-frame. *Hint: the only non-trivial part is proving that $\mathbf{K}_{\mathcal{F}_0} \neq \emptyset$, for this use no maximal models repeatedly.*

Good frames are a stronger notion than superstability. In fact:

Lemma 17.5. Let $\mathfrak{s} = (\mathbf{K}, F)$ satisfy all the properties of good λ-frames except perhaps for disjointness, existence, and symmetry. Then $\mathbf{K}$ is λ-superstable and $\mathfrak{s}$-nonforking coincides with λ-nonforking over limit models. If in addition $\mathfrak{s}$ satisfies symmetry, then $\mathbf{K}$ has λ-symmetry.

Proof. By Theorem 14.1 and the remark following it. □

Thus we obtain that if $\mathfrak{s} = (\mathbf{K}, F)$ is a good λ-frame, then $\mathbf{K}$ has uniqueness of limit models in λ. Consequently,

Theorem 17.6 (The conjugation property for good λ-frames)**.** If $\mathfrak{s} = (\mathbf{K}, F)$ is a *categorical* good λ-frame and $p \in \mathbf{S}(N)$ does not $\mathfrak{s}$-fork over M, then p and $p \upharpoonright M$ are conjugates.

Proof. By Theorem 15.9 and Lemma 17.5, noting that by categoricity the model of size λ must be limit. $\qquad\square$

Building good frames is a nontrivial matter. We will use the following variations of tameness:

Definition 17.7. Let $\mathbf{K}$ be an AEC, $\kappa \leq \mu$, with $\mu > \mathrm{LS}(\mathbf{K})$. Assume that $\mathbf{K}_{<\mu}$ has amalgamation. We say that $\mathbf{K}$ is *weakly $(< \kappa, \mu)$-tame* if for any *saturated* $M \in \mathbf{K}_\mu$ and any $p, q \in \mathbf{S}(M)$, $p \neq q$ implies there exists $A \subseteq |M|$ with $|A| < \kappa$ such that $p \restriction A \neq q \restriction A$. We also define the following variations:

(1) *weakly (κ, μ)-tame* means weakly $(< \kappa^+, \mu)$-tame.
(2) *weakly $(< \kappa, < \mu)$-tame* means weakly $(< \kappa, \mu_0)$-tame for all $\mu_0 \in [\kappa, \mu)$, similarly for *weakly $(\kappa, < \mu)$-tame*.

We will use as a blackbox the following result (the core of the argument is due to Shelah), whose proof was covered by Will Boney last semester.

Fact 17.8 ([Vas17b, 5.7]). Let $\mathbf{K}$ be an AEC and let $\mu \geq \beth_{(2^{\mathrm{LS}(\mathbf{K})})^+}$. If $\mathbf{K}$ is categorical in μ and $\mathbf{K}_{<\mu}$ has amalgamation and no maximal models, then there exists $\chi < \beth_{(2^{\mathrm{LS}(\mathbf{K})})^+}$ such that $\mathbf{K}$ is weakly $(\chi, < \mu)$-tame.

Using weak tameness, we can build a good frame as follows [VV17, 6.4]:

Theorem 17.9 (The good frame construction theorem). Let $\mathbf{K}$ be an AEC and let $\lambda \geq \mathrm{LS}(\mathbf{K})$. If:

(1) $\mathbf{K}$ is λ-superstable.
(2) $\mathbf{K}$ is λ^+-superstable and has λ^+-symmetry.
(3) $\mathbf{K}$ is weakly (λ, λ^+)-tame.

Then there is a categorical good λ^+-frame $\mathfrak{s}$ with underlying AEC the class $\mathbf{K}^{\lambda^+\text{-sat}}$ of λ^+-saturated models of $\mathbf{K}$ (ordered by $\leq_{\mathbf{K}}$).

Proof. Using Theorems 15.8 and 15.10, it is easy to check that $\mathbf{K}^{\lambda^+\text{-sat}}$ is indeed an AEC with Löwenheim-Skolem-Tarski number λ^+. By definition of λ^+-superstability, $\mathbf{K}_{\lambda^+}$ is not empty, has amalgamation, joint embedding, and no maximal models. It is straightforward to see that these properties carry out to $\mathbf{K}^{\lambda^+\text{-sat}}_{\lambda^+}$. Similarly, $\mathbf{K}^{\lambda^+\text{-sat}}$ is stable in λ^+ and is categorical in λ^+. We now let $\mathfrak{s} := (\mathbf{K}^{\lambda^+\text{-sat}}, F)$, where $F(p, M)$ holds if and only if $p \in \mathbf{S}(N)$, $M \leq_{\mathbf{K}} N$ are both in $\mathbf{K}^{\lambda^+\text{-sat}}_{\lambda^+}$, and there exists $M_0 \in \mathcal{P}_{\mathbf{K}_\lambda}(M)$ limit such that for all $N_0 \in \mathcal{P}_{\mathbf{K}_\lambda}(N)$ limit, if $M_0 \leq_{\mathbf{K}} N_0$, then $p \restriction N_0$ does not λ-fork over M_0. We claim that this works.

As is easy to see, F satisfies invariance and monotonicity. For uniqueness, suppose that $p, q \in \mathbf{S}(N)$ do not $\mathfrak{s}$-fork over M and $p \restriction M = q \restriction M$. By definition of $\mathfrak{s}$-nonforking and monotonicity, there is $M_0 \in \mathcal{P}_{\mathbf{K}_\lambda}(M)$ limit such that both $p \restriction N_0$ and $q \restriction N_0$ do not λ-fork over M_0 for $M_0 \leq_{\mathbf{K}} N_0$ in $\mathbf{K}_\lambda$. If $p \neq q$, then by weak tameness there exists $N_0 \in \mathbf{K}_\lambda$ such that $p \restriction N_0 \neq q \restriction N_0$. We might as well assume that $M_0 \leq_{\mathbf{K}} N_0$ and N_0 is limit, so $p \restriction N_0$ and $q \restriction N_0$ both do not fork over M_0. But by uniqueness for λ-nonforking, this implies that $p \restriction N_0 = q \restriction N_0$, a contradiction.

To prove local character, we first show:

<u>Claim</u>: For any $M \in \mathbf{K}_{\lambda+}^{\lambda^+\text{-sat}}$ saturated and any $p \in \mathbf{S}(M)$, there exists $M_0 \in \mathbf{K}_\lambda$ limit such that $p \upharpoonright N_0$ does not λ-fork over M_0 for any limit $N_0 \in \mathcal{P}_{\mathbf{K}_\lambda}(M)$ with $M_0 \leq_{\mathbf{K}} N_0$.

<u>Proof of Claim</u>: Suppose not. Build $\langle M_i : i \leq \omega \rangle$ an increasing continuous chain of limit models in $\mathbf{K}_\lambda$ such that for all $i < \omega$, M_{i+1} is universal over M_i, $M_i \leq_{\mathbf{K}} M$, and $p \upharpoonright M_{i+1}$ λ-forks over M_i. This is possible by assumption, but contradicts universal local character in λ. †$_{\mathrm{Claim}}$

We now prove local character in general. Let $\langle M_i : i \leq \delta \rangle$ be an increasing continuous chain of saturated models in $\mathbf{K}_{\lambda^+}$ and let $p \in \mathbf{S}(M_\delta)$. Without loss of generality, $\delta = \mathrm{cf}(\delta)$. If $\delta = \lambda^+$, the Claim and monotonicity imply that there is $i < \delta$ such that p does not $\mathfrak{s}$-fork over M_i, so assume that $\delta < \lambda^+$. Suppose for a contradiction that there is *no* $i < \delta$ such that p does not $\mathfrak{s}$-fork over M_i. We build $\langle N_i^\ell : i \leq \delta \rangle$, $\ell = 0, 1$, increasing continuous such that for all $i < \delta$:

(1) $N_i^\ell \in \mathbf{K}_\lambda$ and is limit, for $\ell = 0, 1$.
(2) $N_i^0 \leq_{\mathbf{K}} M_i$.
(3) $N_i^1 \leq_{\mathbf{K}} M_\delta$ and $N_i^0 \leq_{\mathbf{K}} N_i^1$.
(4) N_{i+1}^ℓ is universal over N_i^ℓ for $\ell = 0, 1$.
(5) $|N_i^1| \cap |M_i| \subseteq |N_{i+1}^0|$.
(6) $p \upharpoonright N_{i+1}^1$ λ-forks over N_i^0.

This is enough: at the end, $N_\delta^0 = N_\delta^1$. By universal local character in λ, there is $i < \delta$ such that $p \upharpoonright N_\delta^1$ does not λ-fork over N_i^0. In particular, $p \upharpoonright N_{i+1}^1$ does not λ-fork over N_i^0. This contradicts (6).

This is possible: take any limit $N_0^0 = N_0^1 \leq_{\mathbf{K}} M_0$. At limits, take unions. At successors, given N_i^0, N_i^1, we know that p $\mathfrak{s}$-forks over M_i, so in particular there exists a limit $N' \in \mathcal{P}_{\mathbf{K}_\lambda}(M_\delta)$ such that $N_i^0 \leq_{\mathbf{K}} N'$ and $p \upharpoonright N'$ λ-forks over N_i^0. Extend N' to N_{i+1}^1 that is universal over N_i^1. Now take $N_{i+1}^0 \in \mathcal{P}_{\mathbf{K}_\lambda}(M_{i+1})$ limit, universal over N_i^0, containing $|N_i^1| \cap |M_i|$.

Now that we have local character, we can apply Lemma 17.5: $\mathfrak{s}$-nonforking coincides with λ^+-nonforking, so by λ^+-superstability, λ^+-superstability, and categoricity of $\mathbf{K}_{\lambda+}^{\lambda^+\text{-sat}}$ in λ^+, $\mathfrak{s}$ must have existence, disjointness, and λ^+-superstability. $\square$

Corollary 17.10. Let $\mathbf{K}$ be an AEC and let $\mu \geq \beth_{\left(2^{\mathrm{LS}(\mathbf{K})}\right)^+}$. If $\mathbf{K}$ is categorical in μ and $\mathbf{K}_{<\mu}$ has amalgamation and no maximal models, then there is $\chi < \beth_{\left(2^{\mathrm{LS}(\mathbf{K})}\right)^+}$ such that for every $\lambda \in [\chi, \mu)$, there is a good λ^+-frame with underlying class $\mathbf{K}^{\lambda^+\text{-sat}}$.

Proof. By Corollary 15.6, $\mathbf{K}$ is λ-superstable and has λ-symmetry for every $\lambda \in [\mathrm{LS}(\mathbf{K}), \mu)$. Now apply Fact 17.8 and Theorem 17.9. $\square$

Exercise 17.11. Generalize the conclusion of Corollary 17.10 to the limit case. That is, show under the assumptions there that for every $\lambda \in (\chi, \mu)$, there is a good λ-frame with underlying class $\mathbf{K}^{\lambda\text{-sat}}$.

18. Weak amalgamation and the frame extension theorem

How do we build a good $\mathcal{F}$-frame, when $\mathcal{F}$ does not contain just one cardinal? This can be done assuming tameness and a little bit of amalgamation in $\mathcal{F}$. The "little bit of amalgamation" is just enough to ensure that equality of types is witnessed by some kind of amalgam. We present the definition now:

Definition 18.1 ([Vas17c, 4.11])**.** An abstract class $\mathbf{K}$ has *weak amalgamation* if whenever $\mathbf{tp}(a_1/M; N_1) = \mathbf{tp}(a_2/M; N_2)$, there exists $N_2' \leq_{\mathbf{K}} N_3$ and f such that $M \leq_{\mathbf{K}} N_2' \leq_{\mathbf{K}} N_2$, $a \in |N_2'|$, $N_2' \leq_{\mathbf{K}} N_3$, and $f : N_1 \xrightarrow[M]{} N_3$ such that $f(a_1) = a_2$.
In other words, $(a_1, M, N_1)E_{\mathrm{at}}(a_2, M, N_2')$.

Intuitively, weak amalgamation say we can amalgamate at least singletons over a base model. We have the following two sufficient conditions for weak amalgamation.

Exercise 18.2. Let $\mathbf{K}$ be an abstract class.

(1) If $\mathbf{K}$ has amalgamation, then $\mathbf{K}$ has weak amalgamation.
(2) If $\mathbf{K}$ has intersections, then $\mathbf{K}$ has weak amalgamation.

Further, weak amalgamation (as opposed to having intersections) is preserved by passing to a subinterval:

Exercise 18.3. Let $\mathbf{K}$ be an AEC. If $\mathbf{K}$ has weak amalgamation, then $\mathbf{K}_{[\lambda,\theta)}$ has weak amalgamation for any $\lambda < \theta$ (we allow $\theta = \infty$).

When does weak amalgamation imply amalgamation? It turns out it is enough to be able to extend types:

Definition 18.4. An abstract class $\mathbf{K}$ has the *type extension property* if for any $M \leq_{\mathbf{K}} N$ and $p \in \mathbf{S}(M)$, there exists $q \in \mathbf{S}(N)$ such that q extends p.

Lemma 18.5. Let $\mathbf{K}$ be an AEC and let $\lambda \geq \mathrm{LS}(\mathbf{K})$. The following are equivalent:

(1) $\mathbf{K}_\lambda$ has weak amalgamation and the type extension property.
(2) $\mathbf{K}_\lambda$ has amalgamation.

Proof. We leave (2) implies (1) to the reader and prove (1) implies (2). So assume that $\mathbf{K}_\lambda$ has weak amalgamation and the type extension property. We show:

<u>Claim</u>: Let $M_0, M_1, M_2 \in \mathbf{K}_\lambda$, with $M_0 <_{\mathbf{K}} M_1$ and $f : M_0 \to M_2$. There exists $M_1', M_2' \in \mathbf{K}_\lambda$ and g such that $M_0 <_{\mathbf{K}} M_1' \leq_{\mathbf{K}} M_1$, $M_2 \leq_{\mathbf{K}} M_2'$, and $g : M_1' \to M_2'$ extends f.

<u>Proof of Claim</u>: Pick $a \in M_1 \backslash M_0$ and let $p := \mathbf{tp}(a/M_0; M_1)$. By the type extension property, we can extend p to $q \in \mathbf{S}(M_2)$. Write $q = \mathbf{tp}(b/M_2; M)$. By weak amalgamation and some renaming, there exists $M_1' \in \mathbf{K}_\lambda$ containing a with $M_0 \leq_{\mathbf{K}} M_1'$ (so since $a \in M_1' \backslash M_0$, $M_0 <_{\mathbf{K}} M_1'$), $M_2' \in \mathbf{K}_\lambda$ with $M \leq_{\mathbf{K}} M_2'$, and $g : M_1' \to M_2'$ such that g extends f such that $g(a) = b$. This is as desired. $\dagger_{\mathrm{Claim}}$

Now let $M_0, M_1, M_2 \in \mathbf{K}_\lambda$ be given such that $M_0 \leq_{\mathbf{K}} M_1$ and $f : M_0 \to M_2$. We want to amalgamate M_1 and M_2. If $M_0 = M_1$, we are done, so assume that $M_0 <_{\mathbf{K}} M_1$, and assume for a contradiction we cannot amalgamate them. We build $\langle M_\ell^i : i < \lambda^+ \rangle$ increasing continuous in $\mathbf{K}_\lambda$ and $\langle f_i : i < \lambda^+ \rangle$ increasing continuous such that for all $i < \lambda^+$:

(1) $M_1^0 = M_0$, $M_2^0 = M_2$, $f_0 = f$.
(2) $f_i : M_1^i \to M_2^i$.
(3) $M_1^i <_{\mathbf{K}} M_1^{i+1}$.
(4) $M_1^i <_{\mathbf{K}} M_1$.

This is enough: then $\bigcup_{i<\lambda^+} M_1^i$ has cardinality λ^+ but is contained in M_1, which has cardinality λ, a contradiction. This is possible by the claim and the assumption that we cannot amalgamate M_1 and M_2 over M_0. $\square$

We then have [Bon14a, BV17b] (see [Vas17c, 4.16] for the version with weak amalgamation):

Theorem 18.6 (The frame extension theorem)**.** Let $\mathfrak{s} = (\mathbf{K}, F)$ be a good λ-frame and let $\mathcal{F} = [\lambda, \theta)$ be an interval of cardinals with $\theta > \lambda$. If $\mathbf{K}_{\mathcal{F}}$ is λ-tame and has weak amalgamation, then there is a good $\mathcal{F}$-frame $\mathfrak{t}$ such that $\mathfrak{t}_\lambda = \mathfrak{s}$.

Proof. We define $\mathfrak{t}$-nonforking as in the proof of the frame construction theorem: for $M, N \in \mathbf{K}_{\mathcal{F}}$, say $p \in \mathbf{S}(N)$ does not $\mathfrak{t}$-fork over M if there exists $M_0 \in \mathcal{P}_{\mathbf{K}_\lambda}(M)$ such that for all $N_0 \in \mathbf{K}_\lambda$ with $M_0 \leq_{\mathbf{K}} N_0$, $p \restriction N_0$ does not $\mathfrak{s}$-fork over M_0. As before, it is easy to check that $\mathfrak{t}$-nonforking satisfies monotonicity and invariance. Further, one can also check that it satisfies disjointness. Exactly as in the proof of the good frame construction theorem, $\mathfrak{t}$ satisfies local character and uniqueness (using tameness). Now working inductively, we can assume that $\mathfrak{t}_{[\lambda,\theta_0)}$ is a good $[\lambda, \theta_0)$-frame for all $\theta_0 \in [\lambda^+, \theta)$. If θ is a limit cardinal, we are done, so assume that $\theta = \theta_0^+$.

We prove that $\mathfrak{t}$ satisfies the existence property. Let $M \leq_{\mathbf{K}} N$ both be in $\mathbf{K}_{\mathcal{F}}$ and let $p \in \mathbf{S}(M)$. First, by local character for $\mathfrak{t}$ (which we just proved), there exists $M_0 \in \mathbf{K}_\lambda$ such that p does not $\mathfrak{t}$-fork over M_0. If we can find a nonforking extension of $p \restriction M_0$ to N, then by uniqueness it will also be a nonforking extension of p to N. Thus without loss of generality we already have that $M = M_0$, i.e. $M \in \mathbf{K}_\lambda$. If $N \in \mathbf{K}_{<\theta_0}$, then we can find a nonforking extension of p to N by the induction hypothesis, so assume that $N \in \mathbf{K}_{\theta_0}$. Let $\delta := \mathrm{cf}(\theta_0)$. Find an increasing continuous chain $\langle M_i : i \leq \delta \rangle$ such that $M_0 = M$, $M_\delta = N$, and $M_i \in \mathbf{K}_{<\theta_0}$ for all $i < \delta$. For $i < \delta$, let $p_i \in \mathbf{S}(M_i)$ be the nonforking extension of p (exists by the induction hypothesis). We build $\langle a_i : i < \delta \rangle$, $\langle N_i : i < \delta \rangle$, $\langle f_{i,j} : i < j < \delta \rangle$ such that for all $i < j < k < \delta$:

(1) $p_i = \mathbf{tp}(a_i/M_i; N_i)$.
(2) $f_{i,j} : N_i \to N_j$.
(3) $f_{j,k} f_{i,j} = f_{i,k}$.
(4) $f_{i,j}(a_i) = a_j$.
(5) For i limit, $(N_i, f_{i_0,i})_{i_0<i}$ is the direct limit of the system $(N_{i_0}, f_{i_0,i_1})_{i_0<i_1<i}$.

This is possible by the uniqueness property and is enough since the direct limit of this system is the desired nonforking extension by local character.

Now that we have existence, $\mathbf{K}_{\theta_0}$ has the type extension property. Thus by Lemma 18.5, $\mathbf{K}_{\theta_0}$ has amalgamation, so $\mathbf{K}_{\mathcal{F}}$ has amalgamation. Since $\mathbf{K}_\lambda$ has joint embedding and $\mathbf{K}_{\mathcal{F}}$ has amalgamation, $\mathbf{K}_{\mathcal{F}}$ has joint embedding (exercise). To see

that $\mathbf{K}_{\mathcal{F}}$ has no maximal models[5], start with $M \in \mathbf{K}_{\mathcal{F}}$. Find $M_0 \in \mathcal{P}_{\mathbf{K}_\lambda}(M)$. If $M \in \mathbf{K}_\lambda$, then by assumption it has a proper extension so assume that $M \in \mathbf{K}_{>\lambda}$. Then in particular there is $a \in M \backslash M_0$. Let $p := \mathbf{tp}(a/M_0; M)$. By existence, find $q \in \mathbf{S}(M)$ the nonforking extension of p. Since p was not algebraic, q is not algebraic by disjointness, so in particular it must be realized in a proper extension of M. Since M was arbitrary, $\mathbf{K}_{\mathcal{F}}$ has no maximal models. To see stability, we know inductively that $\mathbf{K}$ is stable in every $\lambda' \in [\lambda, \theta_0)$. Let's see stability in θ_0. Let $M \in \mathbf{K}_{\theta_0}$ and let $\langle p_i : i < \theta_0^+ \rangle$ be types over M. Write M as an increasing continuous union $\bigcup_{i < \theta_0} M_i$, with $M_i \in \mathbf{K}_{<\theta_0}$. For each $i < \theta_0^+$, there exists $j_i < \theta_0$ such that p_i does not fork over M_{j_i}. By the pigeonhole principle, there exists $S \subseteq \theta_0^+$ unbounded and a $j < \theta_0^+$ such that $i \in S$ implies $j_i = j$. By stability below θ_0, $|\mathbf{S}(M_j)| < \theta_0$, so by the pigeonhole principle again, there is $S' \subseteq S$ unbounded in θ_0^+ such that $i, i' \in S'$ imply that $p_i \restriction M_j = p_{i'} \restriction M_j$. This implies by uniqueness that $p_i = p_{i'}$, hence showing that there must be at most θ_0 types over M, as desired.

The proof of symmetry is more complex and we omit it. See [BV17b]. $\square$

The converse of the frame extension theorem is also true, as is not too hard to check:

Exercise 18.7. Let $\mathfrak{s} = (\mathbf{K}, F)$ be a good $[\lambda, \theta)$-frame. Prove that $\mathbf{K}_{[\lambda, \theta)}$ is λ-tame.

We conclude with the specialization of the frame extension theorem to universal classes:

Corollary 18.8. Let $\mathbf{K}$ be a universal class and let $\mathfrak{s} = (\mathbf{K}_{\geq \lambda}, F)$ be a good λ-frame. Then there exists a good $[\lambda, \infty)$-frame $\mathfrak{t}$ such that $\mathfrak{t}_\lambda = \mathfrak{s}$. In particular, $\mathbf{K}_{\geq \lambda}$ has amalgamation, joint embedding, and no maximal models.

Proof. We have seen (Theorem 2.11) that universal classes are $(< \aleph_0)$-tame. They also have intersections. By Exercises 18.2 and 18.3, $\mathbf{K}_{\geq \lambda}$ has weak amalgamation. Now apply Theorem 18.6. $\square$

19. UNIDIMENSIONALITY

For this section, assume:

Hypothesis 19.1. $\mathfrak{s} = (\mathbf{K}, F)$ is a categorical good λ-frame.

We have seen in the previous sections that this is a reasonable assumption. Is there a condition in λ equivalent to categoricity in λ^+?

There are many answers to this question. In the first-order case, this is given by the notion of a Vaughtian pair (a pair (M, N) in $\mathbf{K}_\lambda$ and a formula ϕ such that $\phi(M) = \phi(N)$). To state an answer for AEC, we will use the notion of a minimal type.

Definition 19.2. Let $M \in \mathbf{K}_\lambda$. A type $p \in \mathbf{S}(M)$ is *minimal* if it is nonalgebraic and whenever $M \leq_{\mathbf{K}} N$ is such that $N \in \mathbf{K}_\lambda$, p has at most one nonalgebraic extension to N.

---

[5]The argument is due to Marcos Mazari Armida [Arm].

Remark 19.3. The nonforking extension of p will be this nonalgebraic extension.

Intuitively, minimal types are simple in that their only nontrivial (i.e. nonalgebraic) extensions are the nonforking extension. They exist:

Lemma 19.4 (Density of minimal types)**.** Let $M \in \mathbf{K}_\lambda$. For any nonalgebraic $p \in \mathbf{S}(M)$, there exists an extension $q \in \mathbf{S}(N)$ of p which is minimal.

Proof. Suppose not. We build $\langle M_i : i \leq \omega \rangle$ increasing continuous in $\mathbf{K}_\lambda$ and $\langle p_i : i \leq \omega \rangle$ types over M_i extending p such that $M_0 = M$, $i < j$ implies that p_j extends p_i, p_i is not minimal for any $i \leq \omega$, and p_{i+1} forks over M_i for all $i < \omega$. This is enough since it contradicts the local character property of forking. This is possible: take $M_0 = M$, $p_0 = p$. Now given $i < \omega$, p_i, by assumption it is not minimal, so there exists $M_{i+1} \in \mathbf{K}_\lambda$ extending M_i and $p_{i+1} \in \mathbf{S}(M_{i+1})$ which is not algebraic, extends p_i, but is *not* the nonforking extension of p_i (which is *another* nonalgebraic extension of p_i). In particular, p_{i+1} forks over M_i. Take any p_ω extending all the p_i's (exists by a straightforward directed system argument). $\square$

Definition 19.5. We say that $\mathfrak{s}$ is *unidimensional* if for every minimal type $p \in \mathbf{S}(M)$, p is realized in *any* $N \in \mathbf{K}_\lambda$ with $M <_\mathbf{K} N$.

Lemma 19.6. If $\mathbf{K}$ is categorical in λ^+, then $\mathfrak{s}$ is unidimensional.

Proof. Suppose not, and let $p \in \mathbf{S}(M)$ be a minimal type that is omitted in $N \in \mathbf{K}_\lambda$, $M <_\mathbf{K} N$. We build $\langle M_i : i \leq \lambda^+ \rangle$ strictly increasing continuous such that for any $i < \lambda^+$:

(1) $M_0 = M$.
(2) p is omitted in M_i.

This is enough: then M_{λ^+} omits p, hence is not saturated, contradicting categoricity in λ^+.

This is possible: the base case is already specified. At limits, take unions. Given M_i, let $q \in \mathbf{S}(M_i)$ be the nonforking extension of p. By the conjugation property, q is conjugate to p, so it has all the same properties as p. In particular, it is minimal and omitted in some strict extension M_{i+1} of M_i. If p was realized inside M_{i+1}, then it would be realized by some $b \in M_{i+1} \backslash M_i$ (we are assuming it is not realized in M_i). This means that $\mathbf{tp}(b/M_i; M_{i+1})$ is nonalgebraic. However, we are assuming that b does not realize q, so $\mathbf{tp}(b/M_i; M_{i+1}) \neq q$. Thus p has two different nonalgebraic extensions to M_i, contradicting its minimality. $\square$

For the converse, we will use:

Lemma 19.7 (Fodor's lemma for AECs; [JS13, 1.0.30])**.** The following is impossible: There are increasing continuous chains in $\mathbf{K}_\lambda$ $\langle M_i : i < \lambda^+ \rangle$, $\langle N_i : i < \lambda^+ \rangle$, $\langle f_i : i < \lambda^+ \rangle$, and $S \subseteq \lambda^+$ stationary such that:

(1) $f_i : M_i \to N_i$.
(2) For any $i \in S$, there is $a \in M_{i+1} \backslash M_i$ such that $f_{i+1}(a) \in N_i$.

Proof. Suppose such things exist. Let $M := \bigcup_{i<\lambda^+} M_i$, $N := \bigcup_{i<\lambda^+} N_i$, and $f := \bigcup_{i<\lambda^+} f_i$. By the second requirement, $\|M\| = \|N\| = \lambda^+$. First, find a

club C such that for $i \in C$, $f_i[M_i] = N_i \cap f[M]$ for all $i \in C$. Find $i \in C \cap S$. Then in particular $f_i[M_i] = N_i \cap f_{i+1}[M_{i+1}]$. But by the second requirement, there is $a \in M_{i+1} \backslash M_i$ such that $f_{i+1}(a) \in N_i$. This means that $f_{i+1}(a) \notin f_i[M_i]$ but $f_i(a) \in N_i \cap f_{i+1}[M_{i+1}]$, a contradiction. $\square$

Lemma 19.8. If $\mathfrak{s}$ is unidimensional, then $\mathbf{K}$ is categorical in λ^+.

Proof. Let $M \in \mathbf{K}_{\lambda^+}$ and let $\langle M_i : i < \lambda^+ \rangle$ be strictly increasing continuous in $\mathbf{K}_\lambda$ with $M = \bigcup_{i<\lambda^+} M_i$. It suffices to show that M is universal over M_0. Let N be an extension of M_0 in $\mathbf{K}_\lambda$. Assume for a contradiction that N cannot be embedded into M over M_0. We build $\langle N_i : i < \lambda^+ \rangle$ increasing continuous in $\mathbf{K}_\lambda$ and $\langle f_i : i < \lambda^+ \rangle$ increasing continuous such that for all $i < \lambda^+$:

(1) $f_i : M_i \to N_i$.
(2) $N_0 = N$, $f_0 = \mathrm{id}_{M_0}$.
(3) There is $a \in M_{i+1} \backslash M_i$ such that $f_{i+1}(a) \in N_i$.

This is enough, as it contradicts Lemma 19.7. This is possible: the base case is specified and at limits take unions. If $i = j + 1$ and we are given N_j and f_j, then let $p \in \mathbf{S}(M_j)$ be a minimal type. By unidimensionality, p is realized in M_i, say by a. Since we are assuming that N_0 cannot be embedded into M over M_0, f_j cannot be an isomorphism. Thus $f_j[M_j] <_{\mathbf{K}} N_j$. By unidimensionality, $f_j(p)$ is realized inside N_j, say by b. Now use the definition of equality of types and amalgamation to find f_{j+1}, N_{j+1} such that $f_{j+1} : M_{j+1} \to N_{j+1}$ extends f_j and $f_{j+1}(a) = b$. These are as desired. $\square$

Theorem 19.9. $\mathfrak{s}$ is unidimensional if and only if $\mathbf{K}$ is categorical in λ^+.

Proof. By Lemmas 19.6 and 19.8. $\square$

References

[AHS04]   Jiří Adámek, Horst Herrlich, and George E. Strecker, *Abstract and concrete categories*, online edition ed., 2004, Available from `http://katmat.math.uni-bremen.de/acc/`.

[AM10]   Uri Abraham and Menachem Magidor, *Cardinal arithmetic*, Handbook of set theory (Matthew Foreman and Akihiro Kanamori, eds.), Springer, 2010, pp. 1149–1227.

[AR94]   Jiří Adamek and Jiří Rosický, *Locally presentable and accessible categories*, London Math. Society Lecture Notes, Cambridge University Press, 1994.

[Arm]   Marcos Mazari Armida, *Non-forking w-good frames*, Preprint. URL: `https://arxiv.org/abs/1803.01679v2`.

[Bal09]   John T. Baldwin, *Categoricity*, University Lecture Series, vol. 50, American Mathematical Society, 2009.

[BB17]   John T. Baldwin and Will Boney, *Hanf numbers and presentation theorems in AECs*, Beyond first order model theory (José Iovino, ed.), CRC Press, 2017, pp. 327–352.

[BFB85]   Jon Barwise, Solomon Feferman, and Andreas Baudisch (eds.), *Model-theoretic logics*, Perspectives in mathematical logic, Springer-Verlag, 1985.

[BGKV16] Will Boney, Rami Grossberg, Alexei Kolesnikov, and Sebastien Vasey, *Canonical forking in AECs*, Annals of Pure and Applied Logic **167** (2016), no. 7, 590–613.

[BGL+16] Will Boney, Rami Grossberg, Michael J. Lieberman, Jiří Rosický, and Sebastien Vasey, *μ-Abstract elementary classes and other generalizations*, The Journal of Pure and Applied Algebra **220** (2016), no. 9, 3048–3066.

[BGVV17] Will Boney, Rami Grossberg, Monica VanDieren, and Sebastien Vasey, *Superstability from categoricity in abstract elementary classes*, Annals of Pure and Applied Logic **168** (2017), no. 7, 1383–1395.

[BK09]   John T. Baldwin and Alexei Kolesnikov, *Categoricity, amalgamation, and tameness*,
          Israel Journal of Mathematics **170** (2009), 411–443.
[BL16]   John T. Baldwin and Paul B. Larson, *Iterated elementary embeddings and the model
          theory of infinitary logic*, Annals of Pure and Applied Logic **167** (2016), 309–334.
[Bon14a] Will Boney, *Tameness and extending frames*, Journal of Mathematical Logic **14**
          (2014), no. 2, 1450007.
[Bon14b] _____, *Tameness from large cardinal axioms*, The Journal of Symbolic Logic **79**
          (2014), no. 4, 1092–1119.
[BR12]   Tibor Beke and Jiří Rosický, *Abstract elementary classes and accessible categories*,
          Annals of Pure and Applied Logic **163** (2012), 2008–2017.
[BS08]   John T. Baldwin and Saharon Shelah, *Examples of non-locality*, The Journal of Sym-
          bolic Logic **73** (2008), 765–782.
[BU17]   Will Boney and Spencer Unger, *Large cardinal axioms from tameness in AECs*, Pro-
          ceedings of the American Mathematical Society **145** (2017), 4517–4532.
[BV]   Will Boney and Sebastien Vasey, *Structural logic and abstract elementary classes with
          intersections*, Preprint. URL: `http://arxiv.org/abs/1801.01908v2`.
[BV17a]   _____, *A survey on tame abstract elementary classes*, Beyond first order model the-
          ory (José Iovino, ed.), CRC Press, 2017, pp. 353–427.
[BV17b]   _____, *Tameness and frames revisited*, The Journal of Symbolic Logic **82** (2017),
          no. 3, 995–1021.
[Dic75]   M. A. Dickmann, *Large infinitary languages*, Studies in logic and the foundations of
          mathematics, vol. 83, North-Holland, 1975.
[GV06]   Rami Grossberg and Monica VanDieren, *Galois-stability for tame abstract elementary
          classes*, Journal of Mathematical Logic **6** (2006), no. 1, 25–49.
[GVV16]   Rami Grossberg, Monica VanDieren, and Andrés Villaveces, *Uniqueness of limit mod-
          els in classes with amalgamation*, Mathematical Logic Quarterly **62** (2016), 367–382.
[Iwa44]   Tsurane Iwamura, *A lemma on directed sets*, Zenkoku Shijo Sugaku Danwakai **262**
          (1944), 107–111.
[Jec03]   Thomas Jech, *Set theory*, 3rd ed., Springer-Verlag, 2003.
[JS13]   Adi Jarden and Saharon Shelah, *Non-forking frames in abstract elementary classes*,
          Annals of Pure and Applied Logic **164** (2013), 135–191.
[Kar65]   Carol Karp, *Finite-quantifier equivalence*, The theory of models (Proc. 1963 Symp. at
          Berkeley) (J. W. Addison, L. Henkin, and A. Tarski, eds.), Studies in logic and the
          foundations of mathematics, North-Holland, 1965, pp. 407–412.
[Kue08]   David W. Kueker, *Abstract elementary classes and infinitary logics*, Annals of Pure
          and Applied Logic **156** (2008), 274–286.
[Kun80]   Kenneth Kunen, *Set theory: an introduction to independence proofs*, Studies in logic
          and the foundations of mathematics, vol. 102, Elsevier, 1980.
[Lai81]   Christian Lair, *Catégories modelables et catégories esquissables*, Diagrammes **6** (1981),
          L1–L20.
[Lan98]   Saunders Mac Lane, *Categories for the working mathematician*, second ed., Graduate
          Texts in Mathematics, vol. 5, Springer, 1998.
[LR16]   Michael J. Lieberman and Jiří Rosický, *Classification theory for accessible categories*,
          The Journal of Symbolic Logic **81** (2016), no. 1, 151–165.
[LR17]   _____, *Metric abstract elementary classes as accessible categories*, The Journal of
          Symbolic Logic **82** (2017), no. 3, 1022–1040.
[LRVa]   Michael J. Lieberman, Jiří Rosický, and Sebastien Vasey, *Forking independence from
          the categorical point of view*, Preprint. URL: `http://arxiv.org/abs/1801.09001v2`.
[LRVb]   _____, *Internal sizes in μ-abstract elementary classes*, Preprint. URL: `http://
          arxiv.org/abs/1708.06782v2`.
[LRVc]   _____, *Universal abstract elementary classes and locally multipresentable categories*,
          Preprint. URL: `http://arxiv.org/abs/1707.09005v2`.
[MP89]   Michael Makkai and Robert Paré, *Accessible categories: The foundations of categorical
          model theory*, Contemporary Mathematics, vol. 104, American Mathematical Society,
          1989.
[Neu60]   B. H. Neumann, *On amalgams of periodic groups*, Proceedings of the Royal Society
          (London), Series A **255** (1960), 477–489.

[Ros81] Jiří Rosický, *Categories of models of languages $L_{\kappa,\lambda}$*, Abstracts of the 9th winter school on abstract analysis, Mathematics institute Prague, 1981, pp. 153–157.

[Ros83] ———, *Representace konkrétních kategorií*, 1983, Doctoral thesis, University of Brno.

[She] Saharon Shelah, *Maximal failure of sequence locality in A.E.C.*, Preprint. URL: `http://arxiv.org/abs/0903.3614v3`.

[She87a] ———, *Classification of non elementary classes II. Abstract elementary classes*, Classification Theory (Chicago, IL, 1985) (John T. Baldwin, ed.), Lecture Notes in Mathematics, vol. 1292, Springer-Verlag, 1987, pp. 419–497.

[She87b] ———, *Universal classes*, Classification theory (Chicago, IL, 1985) (John T. Baldwin, ed.), Lecture Notes in Mathematics, vol. 1292, Springer-Verlag, 1987, pp. 264–418.

[She90] ———, *Classification theory and the number of non-isomorphic models*, 2nd ed., Studies in logic and the foundations of mathematics, vol. 92, North-Holland, 1990.

[She94] ———, *Cardinal arithmetic*, Oxford Logic Guides, no. 29, Clarendon Press, 1994.

[She99] ———, *Categoricity for abstract classes with amalgamation*, Annals of Pure and Applied Logic **98** (1999), no. 1, 261–294.

[She01] ———, *Categoricity of an abstract elementary class in two successive cardinals*, Israel Journal of Mathematics **126** (2001), 29–128.

[She09a] ———, *Classification theory for abstract elementary classes*, Studies in Logic: Mathematical logic and foundations, vol. 18, College Publications, 2009.

[She09b] ———, *Classification theory for abstract elementary classes 2*, Studies in Logic: Mathematical logic and foundations, vol. 20, College Publications, 2009.

[Sol74] Robert M. Solovay, *Strongly compact cardinals and the GCH*, Proceedings of the Tarski Symposium, Proc. Sympos. Pure Math., vol. 25, American Mathematical Society, 1974, pp. 365–372.

[SV99] Saharon Shelah and Andrés Villaveces, *Toward categoricity for classes with no maximal models*, Annals of Pure and Applied Logic **97** (1999), 1–25.

[Tar54] Alfred Tarski, *Contributions to the theory of models I*, Indagationes Mathematicae **16** (1954), 572–581.

[Van16a] Monica VanDieren, *Superstability and symmetry*, Annals of Pure and Applied Logic **167** (2016), no. 12, 1171–1183.

[Van16b] ———, *Symmetry and the union of saturated models in superstable abstract elementary classes*, Annals of Pure and Applied Logic **167** (2016), no. 4, 395–407.

[Vasa] Sebastien Vasey, *Quasiminimal abstract elementary classes*, Archive for Mathematical Logic, To appear. DOI: `10.1007/s00153-017-0570-7`. URL: `https://arxiv.org/abs/1611.07380v7`.

[Vasb] ———, *Tameness from two successive good frames*, Preprint. URL: `http://arxiv.org/abs/1707.09008v3`.

[Vas16] ———, *Infinitary stability theory*, Archive for Mathematical Logic **55** (2016), 567–592.

[Vas17a] ———, *On the uniqueness property of forking in abstract elementary classes*, Mathematical Logic Quarterly **63** (2017), no. 6, 598–604.

[Vas17b] ———, *Saturation and solvability in abstract elementary classes with amalgamation*, Archive for Mathematical Logic **56** (2017), 671–690.

[Vas17c] ———, *Shelah's eventual categoricity conjecture in universal classes: part I*, Annals of Pure and Applied Logic **168** (2017), no. 9, 1609–1642.

[VV17] Monica VanDieren and Sebastien Vasey, *Symmetry in abstract elementary classes with amalgamation*, Archive for Mathematical Logic **56** (2017), no. 3, 423–452.

E-mail address: `sebv@math.harvard.edu`

URL: `http://math.harvard.edu/~sebv/`

Department of Mathematics, Harvard University, Cambridge, Massachusetts, USA