MATH 269X - MODEL THEORY FOR ABSTRACT ELEMENTARY CLASSES, SPRING 2018 LECTURE NOTES

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1. Notation

We use the letter τ for a vocabulary, K for a class of structures. For M a τ-structure, we write $|M|$ for its universe and $\|M\|$ for the cardinality of its universe. We often abuse notation and write for example $a \in M$ instead of $a \in |M|$. We write $M \subseteq N$ for M is a substructure of N.

For I, A sets, we let ${}^I A$ be the set of functions from I to A (we think of them as I-indexed sequences of elements of A). We write $\bar{a}$ for a sequence of elements. We write ${}^{<\infty} A$ for $\bigcup_\kappa {}^\alpha A$, where α ranges over all ordinals. For $\bar{a} \in {}^I A$ and $I_0 \subseteq I$, we write $\bar{a} \upharpoonright I_0$ for the restriction of $\bar{a}$ to I_0, $\ell(\bar{a}) = I$ (usually used when I is an ordinal), $\mathrm{dom}(\bar{a}) = I$, and $\mathrm{ran}(\bar{a})$ be the range of $\bar{a}$: the set of elements in the sequence.

For λ a cardinal, we write $[A]^\lambda$ for the subsets of A of cardinality λ. Similarly, $[A]^{<\lambda}$ denotes the subsets of A of cardinality less than λ.

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2. Universal classes

We start by studying a simple model-theoretic framework. It was first studied by Tarski under the assumption that the vocabulary is finite [Tar54].

Definition 2.1 (Tarski)**.** A *universal class* is a class K of structures in a fixed vocabulary $\tau = \tau(K)$ that is fixed under isomorphisms, substructures, and unions of chains (according to the substructure relation).

Example 2.2. The class of all fields, of all locally finite groups, of all vector spaces over $\mathbb{Q}$ are universal classes. The class of all algebraically closed fields is not ($\mathbb{Q}$ is a subfield of $\mathbb{C}$ which is not algebraically closed).

In the definition, we could have required closure under directed unions instead of just unions of chains. However it turns out that this follows. This is due to Iwamura [Iwa44]:

Exercise 2.3. Let K be a universal class. Let $\langle M_i : i \in I \rangle$ be a directed (according to substructure) system in K. Then $\bigcup_{i \in I} M_i \in K$.

The following is an important basic result about universal classes. We will see it generalizes (in some sense) to AECs.

Definition 2.4. Call a τ-structure M *finitely-generated* if there exists a *finite* subset $A \subseteq |M|$ such that M is the closure of A under its functions.

Theorem 2.5. Let K be a universal class in a vocabulary τ and let M be a τ-structure. The following are equivalent:

(1) $M \in K$.
(2) $M_0 \in K$ for all finitely-generated substructures M_0 of M.

Proof. If $M \in K$, then by closure under substructure any substructure of it is in K as well. Conversely, if all finitely-generated substructures of M, then they form a directed system in K whose union is M, hence by Exercise 2.3 we have $M \in K$. $\square$

There is a correspondence between universal classes and classes axiomatized by universal sentences in infinitary logics. When the vocabulary is finitary (and relational), this was observed by Tarski [Tar54] (in this case universal classes correspond to classes of models of a universal first-order theory). Tarski's proof generalizes.

Definition 2.6. We call an $\mathbb{L}_{\infty,\omega}$-sentence *universal* if it is of the form $\forall x_0 \ldots \forall x_{n-1} \psi(\bar{x})$, where ψ is quantifier-free.

Theorem 2.7 (Tarski's presentation theorem)**.** Let K be a class of structures in some vocabulary τ. The following are equivalent:

(1) There is a set Γ of quantifier-free (first-order) types such that K is the class of all τ-structures omitting Γ.
(2) K is the class of models of a universal $\mathbb{L}_{\infty,\omega}$ theory.
(3) K is a universal class.

Proof.

- (1) implies (2): Assume that K is the class of τ-structures omitting Γ. For each $p(\bar{x}) \in \Gamma$, let ϕ_p be the sentence $\forall \bar{x} \bigvee_{\psi \in p} \neg \psi(\bar{x})$. Let $T := \{\phi_p \mid p \in \Gamma\}$. It is easy to check that K is the class of models of T.
- (2) implies (3): This is straightforward to check.
- (3) implies (1): Let K_0 be the class of τ-structures that are finitely generated and are *not* in K. For each $M_0 \in K_0$, let $p_{M_0}(\bar{x})$ be a type coding it. That is, for any N, if $N \models p[\bar{a}]$, then N is generated by $\bar{a}$ and $N \cong M_0$. Let $\Gamma := \{p_{M_0} \mid M_0 \in K_0\}$. We claim that K is the set of τ-structures omitting Γ. To see this, first notice that any member of K omits Γ by closure under substructure. Conversely, if M omits Γ, then any finitely-generated substructure of M omits Γ, hence is in K. By Theorem 2.5, $M \in K$.

$\square$

Remark 2.8. The proof of Tarski's presentation theorem shows that any universal class K is axiomatized by a universal $\mathbb{L}_{\left(2^{|\tau(\mathbf{K})|+\aleph_0}\right)^+,\omega}$ theory.

The following concept was somewhat implicit in Definition 2.4:

Definition 2.9. Let K be a universal class. For $M \in K$ and $A \subseteq |M|$, let $\mathrm{cl}^M(A)$ be the closure of A under the functions of M. Equivalently, $\mathrm{cl}^M(A)$ is the intersection of all $M_0 \subseteq M$ which contain A. Note that $\mathrm{cl}^M(A)$ is a substructure of M, hence is itself in K.

2.1. **Tameness in universal classes.** It is natural to ask how much of the compactness theorem is lost in the setup of universal classes. We have seen that locally finite groups are universal classes, so clearly we cannot expect the compactness theorem to hold in full generality. However, consider the following interesting consequence of compactness:

Exercise 2.10. Let T be a first-order theory. Let $\mathfrak{C}$ be a monster model for T (i.e. it is λ-saturated, where λ is much bigger than any of the other objects appearing in the statement). Let α be an ordinal and let $\bar{a}, \bar{b} \in {}^{\alpha}\mathfrak{C}$. The following are equivalent:

(1) $\mathfrak{C} \models \phi[\bar{a}] \leftrightarrow \phi[\bar{b}]$ for all first-order formulas ϕ.
(2) There exists an automorphism of $\mathfrak{C}$ taking $\bar{a}$ to $\bar{b}$.

In other words, syntactic first-order types contain the same information as "semantic" types (defined in terms of orbit of a monster model). Is there a version of such a statement for universal classes? Note that universal classes may fail the amalgamation property (e.g. locally finite groups do [Neu60]), so it may not be possible to build a monster model in this case. Further, first-order types are not the right notion here, since they are not necessarily preserved by substructure. Quantifier-free types should be used and we then have the following result, due to Will Boney, which appears in [Vas17, 3.7].

Theorem 2.11 (Boney)**.** Let K be a universal class. Let $M_1, M_2 \in K$ and let $\bar{a}_\ell \in {}^{\alpha}M_\ell$, $\ell = 1, 2$. The following are equivalent:

(1) For any quantifier-free formula ϕ, $M_1 \models \phi[\bar{a}_1]$ if and only if $M_2 \models \phi[\bar{a}_2]$.
(2) There exists $f : \mathrm{cl}^{M_1}(\bar{a}_1) \cong \mathrm{cl}^{M_2}(\bar{a}_2)$ such that $f(\bar{a}_1) = \bar{a}_2$.

Proof. (2) implies (1) is obvious: quantifier-free formulas are preserved by taking substructures and isomorphisms. We show (1) implies (2). For each $I \subseteq \alpha$ and $\ell = 1, 2$, write $M_\ell^I := \mathrm{cl}^{M_\ell}(\bar{a}_\ell \upharpoonright I)$. We will build by induction on $|I|$ maps $f_I : M_1^I \cong M_2^I$ such that $f_I(\bar{a}_1 \upharpoonright I) = \bar{a}_2 \upharpoonright I$. This will clearly be enough: take $I = \alpha$.

This is possible: for I finite, M_ℓ^I is coded by its quantifier-free type, hence such a map exists by equality of the quantifier-free types of $\bar{a}_1 \upharpoonright I$ and $\bar{a}_2 \upharpoonright I$. Now if $|I|$ is infinite, observe that for $I_0 \subseteq J_0 \subseteq I$ with $|I_0| + |J_0| < |I|$, $f_{I_0} \subseteq f_{J_0}$. This is because we know that $f_{J_0}(\bar{a}_1 \upharpoonright I_0) = \bar{a}_2 \upharpoonright I_0 = f_{I_0}(\bar{a}_1 \upharpoonright I_0)$ and for any $b \in M_1^{I_0}$, $b = \sigma(\bar{a}_1 \upharpoonright I_0)$, for σ a term (*this is the key feature of universal classes used in the proof*). Thus $f_{J_0}(b) = \sigma(f_{J_0}(\bar{a}_1 \upharpoonright I_0)) = \sigma(f_{I_0}(\bar{a}_1 \upharpoonright I_0)) = f_{I_0}(\sigma(\bar{a}_1 \upharpoonright I_0)) = f_{I_0}(b)$. Therefore $f_I := \bigcup_{I_0 \subseteq I, |I_0| < |I|} f_{I_0}$ is a directed union of a system of isomorphisms, and therefore and isomorphism itself. By definition, it must take $\bar{a}_1 \upharpoonright I$ to $\bar{a}_2 \upharpoonright I$, as desired. □

Remark 2.12. We could have added a parameter set A contained in both M_1 and M_2, but this is not needed: one can take $\bar{a}_1$ and $\bar{a}_2$ to include an enumeration of it.

We will later see that this result says in technical terms, than "universal classes are fully $(< \aleph_0)$-tame and short over the empty set". A little less formally, orbital types in universal classes are determined by their finite restrictions.

3. Abstract elementary classes and the presentation theorem

Not all elementary classes are universal (algebraically closed fields are one example). Thus the framework of universal classes is limited. Shelah introduced in the late 70s AECs as a semantic framework encompassing in particular classes of models of $\mathbb{L}_{\infty,\omega}(Q)$ (the paper that introduced them was [She87a], but Shelah lectured on them many years before 1987). We will first give the definition of an abstract class (due to Grossberg).

Definition 3.1. An *abstract class* is a pair $\mathbf{K} = (K, \leq_{\mathbf{K}})$, where K is a class of structures in a fixed vocabulary $\tau = \tau(\mathbf{K})$ and $\leq_{\mathbf{K}}$ is a partial order, $M \leq_{\mathbf{K}} N$ implies $M \subseteq N$, and both K and $\leq_{\mathbf{K}}$ respect isomorphisms. Any abstract class admits a notion of $\mathbf{K}$-*embedding*: these are functions $f : M \to N$ such that $f : M \cong f[M]$ and $f[M] \leq_{\mathbf{K}} N$. We sometimes think of $\mathbf{K}$ as the category whose objects are elements in K and whose morphisms are $\mathbf{K}$-embeddings.

We often do not distinguish between K and $\mathbf{K}$. For λ a cardinal, we will write $\mathbf{K}_\lambda$ for the restriction of $\mathbf{K}$ to models of cardinality λ. Similarly define $\mathbf{K}_{\geq \lambda}$ or more generally $\mathbf{K}_S$, where S is a class of cardinals. We will also use the following notation:

Notation 3.2. For $\mathbf{K}$ an abstract class and $N \in \mathbf{K}$, write $\mathcal{P}_{\mathbf{K}}(N)$ for the set of $M \in \mathbf{K}$ with $M \leq_{\mathbf{K}} N$. Similarly define $\mathcal{P}_{\mathbf{K}_\lambda}(N)$, $\mathcal{P}_{\mathbf{K}_{<\lambda}}(N)$, etc.

For an abstract class $\mathbf{K}$, we denote by $\mathbb{I}(\mathbf{K})$ the number of models in $\mathbf{K}$ up to isomorphism (i.e. the cardinality of $\mathbf{K}/_{\cong}$). We write $\mathbb{I}(\mathbf{K}, \lambda)$ instead of $\mathbb{I}(\mathbf{K}_\lambda)$. When $\mathbb{I}(\mathbf{K}) = 1$, we say that $\mathbf{K}$ is *categorical*. We say that $\mathbf{K}$ is *categorical in* λ if $\mathbf{K}_\lambda$ is categorical, i.e. $\mathbb{I}(\mathbf{K}, \lambda) = 1$.

We say that $\mathbf{K}$ has *amalgamation* if for any $M_0 \leq_{\mathbf{K}} M_\ell$, $\ell = 1, 2$ there is $M_3 \in \mathbf{K}$ and $\mathbf{K}$-embeddings $f_\ell : M_\ell \to M_3$, $\ell = 1, 2$. $\mathbf{K}$ has *joint embedding* if any two models can be $\mathbf{K}$-embedded in a common model. $\mathbf{K}$ has *no maximal models* if for any $M \in \mathbf{K}$ there exists $N \in \mathbf{K}$ with $M \leq_{\mathbf{K}} N$ and $M \neq N$ (we write $M <_{\mathbf{K}} N$). Localized concepts such as *amalgamation in* λ mean that $\mathbf{K}_\lambda$ has amalgamation.

Definition 3.3 (Shelah)**.** An *abstract elementary class (AEC)* is an abstract class $\mathbf{K}$ in a finitary vocabulary satisfying:

(1) Coherence: if $M_0, M_1, M_2 \in \mathbf{K}$, $M_0 \subseteq M_1 \leq_{\mathbf{K}} M_2$ and $M_0 \leq_{\mathbf{K}} M_2$, then $M_0 \leq_{\mathbf{K}} M_1$.
(2) Tarski-Vaught axioms: if δ is a limit ordinal, $\langle M_i : i < \delta \rangle$ is a $\leq_{\mathbf{K}}$-increasing chain and $M := \bigcup_{i<\delta} M_i$, then:
   (a) $M \in \mathbf{K}$.
   (b) $M_j \leq_{\mathbf{K}} M$ for all $j < \delta$.
   (c) Smoothness: if $N \in \mathbf{K}$ is such that $M_i \leq_{\mathbf{K}} N$ for all $i < \delta$, then $M \leq_{\mathbf{K}} N$.
(3) Löwenheim-Skolem-Tarski (LST) axiom: there exists a cardinal $\lambda \geq |\tau(\mathbf{K})| + \aleph_0$ such that for any $N \in \mathbf{K}$ and any $A \subseteq |N|$, there exists $M \in \mathcal{P}_{\mathbf{K}_{\lambda+|A|}}(N)$ such that $A \subseteq |M|$ and $M \leq_{\mathbf{K}} N$. We write $\text{LS}(\mathbf{K})$ for the least such λ.

Similarly to Exercise 2.3, the following holds:

Exercise 3.4. Let $\mathbf{K}$ be an AEC. Then the Tarski-Vaught axioms holds for directed systems. That is, let $\langle M_i : i \in I \rangle$ be a $\leq_{\mathbf{K}}$-directed system. Let $M := \bigcup_{i \in I} M_i$. Then:

(1) $M \in \mathbf{K}$.
(2) $M_i \leq_{\mathbf{K}} M$ for all $i \in I$.
(3) Smoothness: if $N \in \mathbf{K}$ is such that $M_i \leq_{\mathbf{K}} N$ for all $i \in I$, then $M \leq_{\mathbf{K}} N$.

Example 3.5.

(1) $\mathbf{K} = (\text{Mod}(T), \preceq)$, where T is any first-order theory, is an AEC with $\text{LS}(\mathbf{K}) = |\tau(T)| + \aleph_0$.
(2) $\mathbf{K} = (K, \subseteq)$, where K is a universal class, is an AEC with $\text{LS}(\mathbf{K}) = |\tau(K)| + \aleph_0$. We may abuse notation and call also such a $\mathbf{K}$ a universal class (or even a universal AEC).
(3) $\mathbf{K} = (\text{Mod}(\psi), \preceq_\Phi)$, where $\psi \in \mathbb{L}_{\infty,\omega}$ and Φ is a fragment containing ψ, is an AEC with $\text{LS}(\mathbf{K}) \leq |\Phi| + |\tau(\psi)| + \aleph_0$.
(4) For a fixed infinite cardinal λ, the class of well-orderings of type at most λ^+ ordered by being an initial segment is an AEC $\mathbf{K}$ with $\text{LS}(\mathbf{K}) = \lambda$.
(5) The class of well-orderings ordered by being an initial segment is not an AEC (it fails the Löwenheim-Skolem-Tarski axiom).
(6) The class of well-orderings ordered by being a subordering is not an AEC (it fails to be closed under chains).
(7) See more examples in [BV17, §3].

How are AECs related to universal classes? The following result of Shelah says that any AEC is the reduct of a universal class [She09, 1.9(1)] (the presentation we give combines [Vas17, §2] and [LRVb, 6.4]):

Theorem 3.6 (Shelah's presentation theorem)**.** Let $\mathbf{K}$ be an AEC with vocabulary $\tau = \tau(\mathbf{K})$. Then there exists a universal class $\mathbf{K}^+$ in an expansion τ^+ of τ with $|\tau^+| = \text{LS}(\mathbf{K})$ and such that the reduct map is a faithful functor from $\mathbf{K}^+$ into $\mathbf{K}$ which is surjective on objects. In other words:

(1) For any $M \in \mathbf{K}$, there exists $M^+ \in \mathbf{K}^+$ such that $M^+ \upharpoonright \tau = M$.
(2) For any $M^+ \subseteq N^+$ both in $\mathbf{K}^+$, letting $M := M^+ \upharpoonright \tau$, $N := N^+ \upharpoonright \tau$, we have that $M, N \in \mathbf{K}$ and $M \leq_{\mathbf{K}} N$.

Corollary 3.7. For any AEC $\mathbf{K}$, there exists a universal $\mathbb{L}_{\left(2^{\text{LS}(\mathbf{K})}\right)^+,\omega}$-sentence ψ in an expansion of $\tau(\mathbf{K})$ such that the models in $\mathbf{K}$ are exactly the $\tau(\mathbf{K})$-reducts of models of ψ.

Proof. By Theorem 3.6, Tarski's presentation Theorem 2.7, and Remark 2.8. $\square$

To prove Theorem 3.6, the following notion will be useful [Vas17, 2.9]:

Definition 3.8. Let $\mathbf{K}$ be an abstract class and let $N \in K$. We say $\mathcal{F}$ is a *set of Skolem functions for* N if:

(1) $\mathcal{F}$ is a non-empty set, and each element f of $\mathcal{F}$ is a function from N^n to N, for some $n < \omega$.
(2) For all $A \subseteq |N|$, $M := \mathcal{F}[A] := \bigcup\{f[A] \mid f \in \mathcal{F}\}$ is such that $M \leq_{\mathbf{K}} N$ and contains A.

Remark 3.9. Let $\mathbf{K}$ be an AEC, let $N \in \mathbf{K}$, $\mathcal{F}$ be a set of Skolem functions for N, and $A \subseteq |N|$. Then (by the smoothness axiom) the closure of A under the functions in $\mathcal{F}$ is also a $\mathbf{K}$-substructure of N containing A.

Lemma 3.10. Let $\mathbf{K}$ be an AEC. For any $N \in \mathbf{K}$, there exists a set $\mathcal{F}$ of Skolem functions for N with $|\mathcal{F}| = \text{LS}(\mathbf{K})$.

Proof. We build $\langle N_s \mid s \in [N]^{<\aleph_0} \rangle$ such that for each $s, t \in [N]^{<\aleph_0}$:

(1) $N_s \in \mathcal{P}_{\mathbf{K}_{\leq \text{LS}(\mathbf{K})}}(N)$.
(2) $s \subseteq |N_s|$.
(3) $s \subseteq t$ implies $N_s \leq_{\mathbf{K}} N_t$.

This is possible by inductive applications of the LST and coherence axioms. This is enough: for each $s \in [N]^{<\aleph_0}$, let $\{a_i^s : i < \text{LS}(\mathbf{K})\}$ be an enumeration (possibly with repetitions) of N_s. Now for each $n < \omega$, each $i < \text{LS}(\mathbf{K})$, and each $\bar{a} \in {}^n N$, we let $f_i^n(\bar{a})$ be $a_i^{\text{ran}(\bar{a})}$. Let $\mathcal{F} := \{f_i^n : i < \text{LS}(\mathbf{K}), n < \omega\}$. This is as desired: let $A \subseteq |N|$ and let $M := \mathcal{F}[A]$. Then it is easy to check that $M = \bigcup_{s \in [A]^{<\aleph_0}} N_s$. Note that $\langle N_s : s \in [A]^{<\aleph_0} \rangle$ is a directed system and since $N_s \leq_{\mathbf{K}} N$ for all s, it follows from the smoothness axiom that $M \leq_{\mathbf{K}} N$. $\square$

Proof of Theorem 3.6. Let τ^+ consist of $\tau \cup \{f_i^n : i < \text{LS}(\mathbf{K}), n < \omega\}$, where f_i^n is a new function symbol of arity n. Let K^+ be class of τ^+-structures M^+ such that $M_0^+ \upharpoonright \tau \leq_{\mathbf{K}} M^+ \upharpoonright \tau$ for any $M_0^+ \subseteq M^+$. Let $\mathbf{K}^+ := (K^+, \subseteq)$. It is easy to check that $\mathbf{K}^+$ is a universal class and by definition, (2) is satisfied. To see (1), let $M \in \mathbf{K}$. By Lemma 3.10, M has a set of Skolem functions $\mathcal{F}$. Expand M to $M^+ := (M, g)_{g \in \mathcal{F}}$. Then by definition of Skolem functions, $M^+ \in \mathbf{K}^+$. $\square$

4. Abstract elementary classes with intersections

The following generalizes Definition 2.9:

Definition 4.1. For $\mathbf{K}$ an AEC, $N \in \mathbf{K}$ and $A \subseteq |N|$, let $\mathrm{cl}^N(A) := \bigcap\{M \in \mathbf{K} \mid M \leq_{\mathbf{K}} N, A \subseteq |M|\}$. We see it as a $\tau(\mathbf{K})$-substructure of N.

Exercise 4.2. Let $\mathbf{K}$ be an AEC, $M \leq_{\mathbf{K}} N$ be in $\mathbf{K}$, and $A, B \subseteq |N|$.

(1) Invariance: If $f : N \cong N'$, then $f[\mathrm{cl}^N(A)] = \mathrm{cl}^{N'}(f[A])$.
(2) Monotonicity 1: $A \subseteq \mathrm{cl}^N(A)$.
(3) Monotonicity 2: $A \subseteq B$ implies $\mathrm{cl}^N(A) \subseteq \mathrm{cl}^N(B)$.
(4) Monotonicity 3: If $A \subseteq |M|$, then $\mathrm{cl}^N(A) \subseteq \mathrm{cl}^M(A)$.
(5) Idempotence: $\mathrm{cl}^N(M) = M$ and $\mathrm{cl}^N(\mathrm{cl}^N(A)) = \mathrm{cl}^N(A)$.

The notion of having (or admitting) intersections is introduced for AECs in [BS08, 1.2] and further studied in [Vas17, §2].

Definition 4.3. Let $\mathbf{K}$ be an abstract class, $N \in \mathbf{K}$, and $A \subseteq |N|$.

(1) We say that N *has intersections over* A if $\mathrm{cl}^N(A) \leq_{\mathbf{K}} N$.
(2) We say that N *has intersections* if it has intersections over all $A \subseteq |N|$.
(3) We say that $\mathbf{K}$ *has intersections* if all $N \in \mathbf{K}$ have intersections.

Remark 4.4. Formally, $\mathrm{cl}^N(A)$ also depends on $\mathbf{K}$ but usually $\mathbf{K}$ is clear from context. We may write $\mathrm{cl}^N_{\mathbf{K}}(A)$ to make $\mathbf{K}$ explicit.

Exercise 4.5. Let $\mathbf{K}$ be an AEC and let $N \in \mathbf{K}$. The following are equivalent:

(1) N has intersections.
(2) For any non-empty $S \subseteq \mathcal{P}_{\mathbf{K}}(N)$, $\bigcap S \leq_{\mathbf{K}} N$.

Example 4.6. Any universal class has intersections. Algebraically closed fields also have intersections. See more examples in [Vas17, 2.6]. On the other hand, the class of dense linear orderings without endpoints (ordered by suborder) does not have intersections. Indeed, working in side $(\mathbb{Q}, <)$, for each $n \in [1, \omega)$, $(\frac{-1}{n}, \frac{1}{n})_{\mathbb{Q}}$ is a dense linear ordering without endpoints, but the intersections is $\{0\}$ which has an endpoint. Now apply Exercise 4.5.

Definition 4.7. Let $\mathbf{K}$ be an AEC. Let $M \in K$ and let $A \subseteq |M|$ be a set. M is *minimal over* A if whenever $M \leq_{\mathbf{K}} N$ and $M' \leq_{\mathbf{K}} N$ contains A, then $M' = M$. M is *minimal over* A *in* N if $M \leq_{\mathbf{K}} N$ and this holds whenever $N' \leq_{\mathbf{K}} N$.

The following characterization of having intersections is [Vas17, 2.11]:

Theorem 4.8. Let $\mathbf{K}$ be an AEC and let $N \in \mathbf{K}$. The following are equivalent:

(1) N admits intersections.
(2) There is an operator $\mathrm{cl} := \mathrm{cl}^N : \mathcal{P}(|N|) \to \mathcal{P}(|N|)$ such that for all $A, B \subseteq |N|$ and all $M \leq_{\mathbf{K}} N$:
 (a) $\mathrm{cl}(A) \leq_{\mathbf{K}} N$.
 (b) $A \subseteq \mathrm{cl}(A)$.
 (c) $A \subseteq B$ implies $\mathrm{cl}(A) \subseteq \mathrm{cl}(B)$.
 (d) $\mathrm{cl}(M) = M$.

(3) For each $A \subseteq |N|$, there is a unique minimal model over A in N.
(4) There is a set $\mathcal{F}$ of Skolem functions for N such that:
   (a) $|\mathcal{F}| \leq \text{LS}(K)$.
   (b) For all $M \leq_{\mathbf{K}} N$, we have $\mathcal{F}[M] = M$.

Moreover the operator $\text{cl}^N : \mathcal{P}(|N|) \to \mathcal{P}(|N|)$ with the properties in (2) is unique and if it exists then it has the following characterizations:

- $\text{cl}^N(A) = \bigcap\{M \leq_{\mathbf{K}} N \mid A \subseteq |M|\}$.
- $\text{cl}^N(A) = \mathcal{F}[A]$, for any set of Skolem functions $\mathcal{F}$ for N such that $\mathcal{F}[M] = M$ for all $M \leq_{\mathbf{K}} N$.
- $\text{cl}^N(A)$ is the unique minimal model over A in N.

Proof.

- (1) implies (2): Let $\text{cl}^N(A) := \bigcap\{M \leq_{\mathbf{K}} N \mid A \subseteq |M|\}$. Even without hypotheses on N, (2b), (2c), and (2d) are satisfied. Since N admits intersections, (2a) is also satisfied.
- (2) implies (3): Let $A \subseteq |N|$. Let cl be as given by (2). Let $M := \text{cl}(A)$. By (2a), $M \leq_{\mathbf{K}} N$. By (2b), $A \subseteq |M|$. Moreover if $M' \leq_{\mathbf{K}} N$ contains A, then by (2c), $|M| \subseteq |\text{cl}(M')|$ but by (2d), $\text{cl}(M') = M'$. Thus by coherence and (2a) $M \leq_{\mathbf{K}} M'$. This shows both that M is minimal over A and that it is unique.
- (3) implies (4): We slightly change the proof of Lemma 3.10 as follows: in the construction of the N_s's, let N_s be the unique minimal model over s in N. Now let $\mathcal{F}$ be as obtained by the rest of the construction there. Let $A \subseteq |N|$. We claim that $\mathcal{F}[A]$ is minimal over A in N. This shows in particular that $\mathcal{F}$ is as required.

  Let $M := \mathcal{F}[A]$. Since $\mathcal{F}$ is a set of Skolem functions, $M \leq_{\mathbf{K}} N$ and M contains A. Moreover, $M = \bigcup_{s \in [A]^{<\aleph_0}} N_s$. Now if $M' \leq_{\mathbf{K}} N$ contains A, then for all $s \in [A]^{<\aleph_0}$, $s \in [M']^{<\aleph_0}$, so as N_s is minimal over s in N, $N_s \leq_{\mathbf{K}} M'$. It follows that $M \leq_{\mathbf{K}} M'$, so $M = M'$.
- (4) implies (1): Let $\mathcal{F}$ be as given by (4). Let $A \subseteq |N|$. Let $M := \mathcal{F}[A]$. By definition of Skolem functions, M contains A and $M \leq_{\mathbf{K}} N$. We claim that $M = \bigcap\{M' \leq_{\mathbf{K}} N \mid A \subseteq |M'|\}$. Indeed, if $M' \leq_{\mathbf{K}} N$ contains A, then by the hypothesis on $\mathcal{F}$, $M = \mathcal{F}[A] \subseteq \mathcal{F}[M'] = M'$.

The moreover part follows from the arguments above. □

Exercise 4.9 ([Vas, 3.6])**.** Let $\mathbf{K}$ be an AEC. Show that if N has intersections for all $N \in \mathbf{K}_{\leq \text{LS}(\mathbf{K})}$, then $\mathbf{K}$ has intersections.

We obtain the following properties of the closure operator, which complement Exercise 4.2.

Theorem 4.10. Let $\mathbf{K}$ be an AEC with intersections, let $M \leq_{\mathbf{K}} N$ and let $A \subseteq |M|$.

(1) Monotonicity 3: $\text{cl}^M(A) = \text{cl}^N(A)$.
(2) (Finite character) For any $b \in \text{cl}^N(A)$, there exists a finite $A_0 \subseteq A$ such that $b \in \text{cl}^N(A_0)$.

Proof. Finite character follows from the characterization of cl^N in terms of Skolem functions (Theorem 4.8). For monotonicity 3, let $M_0 := \mathrm{cl}^N(A)$. We have $M_0 \leq_{\mathbf{K}} N$ since N admits intersections over A. Since $M \leq_{\mathbf{K}} N$ contains A, we must have $|M_0| \subseteq |M|$. By coherence, $M_0 \leq_{\mathbf{K}} M$. Now M_0 is the unique minimal model over A in N, so it must be minimal in M as well, and hence $M_0 = \mathrm{cl}^M(A)$. $\square$

Remark 4.11. There is a generalization of Tarski's presentation Theorem 2.7 to AECs with intersections [BV].

5. μ-AECs and accessible categories

The following naturally generalizes the definition of an AEC to classes that are only closed under sufficiently directed unions:

Definition 5.1 ([BGL+16, 2.2])**.** Let μ be a regular cardinal. A *μ-abstract elementary class* (or *μ-AEC* for short) is an abstract class $\mathbf{K}$ (where we allow here the vocabulary to be $(<\mu)$-ary) satisfying:

1. Coherence: if $M_0, M_1, M_2 \in \mathbf{K}$, $M_0 \subseteq M_1 \leq_{\mathbf{K}} M_2$ and $M_0 \leq_{\mathbf{K}} M_2$, then $M_0 \leq_{\mathbf{K}} M_1$.
2. Tarski-Vaught axioms: if $\langle M_i : i \in I \rangle$ is a μ-directed system (where I is μ-directed if every subset of I of size strictly less than μ has a least upper bound) and $M := \bigcup_{i \in I} M_i$, then:
   (a) $M \in \mathbf{K}$.
   (b) $M_i \leq_{\mathbf{K}} M$ for all $i \in I$.
   (c) Smoothness: if $N \in \mathbf{K}$ is such that $M_i \leq_{\mathbf{K}} N$ for all $i \in I$, then $M \leq_{\mathbf{K}} N$.
3. Löwenheim-Skolem-Tarski (LST) axiom: there exists a cardinal $\lambda \geq |\tau(\mathbf{K})| + \mu$ such that $\lambda = \lambda^{<\mu}$ and for any $N \in \mathbf{K}$ and any $A \subseteq |N|$, there exists $M \in \mathcal{P}_{\mathbf{K}_{\lambda+|A|^{<\mu}}}(N)$ such that $A \subseteq |M|$. We write $\mathrm{LS}(\mathbf{K})$ for the least such λ.

Remark 5.2. Technically, $\mathrm{LS}(\mathbf{K})$ depends on μ, but this should not cause any problems, so we remove this from the notation.

Note that, in contrast to Exercise 2.3, asking only that the class be closed under *chains* of cofinality at least μ is a significantly weaker condition:

Exercise 5.3 ([AR94, 1.c.(2)])**.** For $n < \omega$, let P_n be the ordinal $\omega_n + 1$, ordered as usual. Let $Q := \prod_{1 \leq n < \omega} P_n$ and let P be the subposet of Q consisting of those sequences $(x_n)_{n<\omega}$ with only finitely many $n < \omega$ so that $x_n = \omega_n$.

1. Check that Q is a complete lattice.
2. Check that P is closed (in Q) under joins of chains of uncountable cofinality.
3. Check that P is not closed under joins of $\aleph_1$-directed sets. *Hint*: Consider $\prod_{1 \leq n < \omega} \omega_n$.

The coherence axiom also has the following stronger form:

Exercise 5.4. Show that the coherence axiom is equivalent to the following statement: for $M_0, M_1, M_2 \in \mathbf{K}$ with $|M_0| \subseteq |M_1| \subseteq |M_2|$, if $M_0 \leq_{\mathbf{K}} M_2$ and $M_1 \leq_{\mathbf{K}} M_2$, then $M_0 \leq_{\mathbf{K}} M_1$.

Example 5.5.

(1) AECs are exactly the $\aleph_0$-AECs.
(2) The class of well-orderings ordered by being a suborder is an $\aleph_1$-AEC.
(3) The class of well-founded models of ZFC, ordered by elementary substructure, is an $\aleph_1$-AEC.
(4) The class of well-orderings ordered by being an initial segment is *not* a μ-AEC for any μ (the LST axiom fails).
(5) The class of all Banach spaces (ordered by being a closed subspace) is an $\aleph_1$-AEC.
(6) The class of all μ-complete Boolean algebras (ordered by being a subalgebra) is a μ-AEC. However the class of all complete Boolean algebras is not.
(7) The class of models of any $\mathbb{L}_{\infty,\mu}$ sentence can be made into a μ-AEC by ordering it with elementarity according to a fragment.
(8) See more examples in [BGL+16, §2].

Accessible categories were introduced by Lair [Lai81] (he called them "catégorie modelable"). The standard textbooks on them are [MP89, AR94] (see also the following basic references on category theory [AHS04, Lan98]). One can see them as axiomatizing the category-theoretic essence of classes of models of $\mathbb{L}_{\infty,\infty}$ sentences:

Definition 5.6. Let $\mathcal{K}$ be a category and let λ be a regular cardinal.

(1) An object M is λ-*presentable* if its hom-functor $\mathcal{K}(M,-) : \mathcal{K} \to \text{Set}$ preserves λ-directed colimits. Put another way, M is λ-presentable if for any morphism $f : M \to N$ with N a λ-directed colimit $\langle \phi_\alpha : N_\alpha \to N \rangle$ with diagram maps $\phi_{\alpha\beta} : N_\alpha \to N_\beta$, f factors essentially uniquely through one of the N_α. That is, $f = \phi_\alpha f_\alpha$ for some $f_\alpha : M \to N_\alpha$, and if $f = \phi_\beta f_\beta$ as well, there is $\gamma > \alpha, \beta$ such that $\phi_{\gamma\alpha} f_\alpha = \phi_{\gamma\beta} f_\beta$.
(2) $\mathcal{K}$ is λ-*accessible* if it has λ-directed colimits and $\mathcal{K}$ contains a set S of λ-presentable objects such that every object of $\mathcal{K}$ is isomorphic to a λ-directed colimit of objects in S.
(3) $\mathcal{K}$ is *accessible* if it is λ'-accessible for some regular cardinal λ'.

Intuitively, an accessible category is a category with all sufficiently directed colimits and such that every object can be written as a highly directed colimit of "small" objects. Here "small" is interpreted in terms of *presentability*, a notion of size that makes sense in any (possibly non-concrete) category. In the category of sets, of course, a set is λ-presentable if and only if its cardinality is less than λ; in an AEC $\mathbf{K}$, the same is true for all regular $\lambda > \text{LS}(\mathbf{K})$. More generally:

Exercise 5.7. Let $\mathbf{K}$ be a μ-AEC, let $\lambda = \lambda^{<\mu} \geq \text{LS}(\mathbf{K})$, and let $M \in \mathbf{K}$. Show that M is λ^+-presentable if and only if $\|M\| \leq \lambda$.

When $\lambda < \lambda^{<\mu}$, presentability still gives a natural notion of size in several categories. For example, in Banach spaces it corresponds to the *density character* [LR17, 3.1].

From Exercise 5.7, it is easy to see the following:

Exercise 5.8. Prove that if $\mathbf{K}$ is a μ-AEC, then it is an $\text{LS}(\mathbf{K})^+$-accessible category.

There are examples of accessible categories that are *not* (equivalent to) μ-AECs. The simplest one is the category of sets (where the morphisms are functions). The problem is that the morphisms need not be monomorphisms. If we assume that all morphisms are mono, then we will see (Theorem 5.21) that we do in some sense have a μ-AEC. Before proving this, we take a second look at presentability. First, we prove the following generalization of the fact that a small union of small sets is not too big:

Lemma 5.9. Let $\mathcal{K}$ be a λ-accessible category. Then any λ-directed colimit of at most θ-many λ-presentable objects is $(\theta + \lambda)^+$-presentable.

Proof. Let M be a λ-directed colimit $\langle \phi_i : M_i \to M, i \in I \rangle$, where $|I| \leq \theta$ and each M_i is λ-presentable. Let $\mu := (\theta + \lambda)^+$. Let $f : M \to N$ be a morphism, with N a μ-directed colimit of objects $\langle N_j : j \in J \rangle$. Let $f_i := f\phi_i$. By λ-presentability of M_i, f_i factors (essentially uniquely) through some N_{j_i}, $j_i \in J$. Now there are at most θ-many j_i's, so since J is μ-directed, there is $j \in J$ with $j_i \leq j$ for all $i \in I$. It follows that f must factor through N_j, showing that M is μ-presentable. $\square$

Recall that a *retract* is a map $f : M \to N$ such that there is $g : N \to M$ so that fg is the identity on N. We also say that N is a *retract* of M. In the category of sets, retracts are exactly the surjections. The following is easy to check:

Exercise 5.10. Prove that if $f_1 : M \to N_1$ and $f_2 : M \to N_2$ are retracts, as witnessed by g_1 and g_2, and $g_1 f_1 = g_2 f_2$, then N_1 and N_2 are isomorphic. Conclude that there is only a set (up to isomorphism) of retracts of any given object M.

The following follows from the definition of λ-presentability and playing with morphisms:

Exercise 5.11. Let $\mathcal{K}$ be a λ-accessible category and let S be a set of λ-presentable objects such that any object in $\mathcal{K}$ is a λ-directed colimit of members of S. Prove that any λ-presentable object is a retract of a member of S. Thus $\mathcal{K}$ has only a set (up to isomorphism) of λ-presentable objects. Conversely, show that a retract of a μ-presentable object is μ-presentable, for any regular $\mu \geq \lambda$.

Toward understanding presentability further, we prove a technical lemma saying when an object resolves into a sufficiently directed colimit. We will use the following definitions:

Definition 5.12. For μ a cardinal, μ^* is μ^+ if μ is successor, and μ if μ is limit.

Definition 5.13. For κ, μ infinite cardinals, we say that μ is *κ-closed* if $\theta^{<\kappa} < \mu$ for all $\theta < \mu$.

Definition 5.14. For λ an uncountable cardinal, we call an object M in a category $\mathcal{K}$ *$(< \lambda)$-presentable* if it is λ_0-presentable for some regular $\lambda_0 < \lambda$.

The following is given by the proof of [MP89, 2.3.10]. It is stated as [LRVa, 3.8].

Lemma 5.15. Let $\kappa < \mu \leq \lambda$ be cardinals with κ and μ regular and $\mathrm{cf}(\lambda) \geq \mu$. Let $\mathcal{K}$ be a category with κ-directed colimits. If $M \in \mathcal{K}$ is a κ-directed colimit of $(< \lambda)$-presentable objects and μ is κ-closed, then M is a μ-directed colimit of $(< \lambda + \mu^*)$-presentable objects.

Proof sketch. Suppose that M is a κ-directed colimit of the $(< \lambda)$-presentable objects $\langle M_i : i \in I \rangle$. Since μ is κ-closed, any subset of I of cardinality strictly less than μ is contained inside a κ-directed subset of I of cardinality strictly less than μ. Thus the set $\mathbb{P}$ of all κ-directed subsets of I of cardinality strictly less than μ is μ-directed. For $s \in \mathbb{P}$, let M_s be the colimit of the M_i's with $i \in s$. Now the induced system $\langle M_s : s \in \mathbb{P} \rangle$ has M as its colimit and:

(1) μ-directed, since $\mathbb{P}$ is μ-directed.
(2) Made of $(< \lambda + \mu^*)$-presentable objects.

□

We deduce several interesting results:

Theorem 5.16. Let $\mathcal{K}$ be a λ-accessible category. If $\mu > \lambda$ is a λ-closed regular cardinal, then $\mathcal{K}$ is μ-accessible.

Proof. Directly from Lemma 5.15. □

Remark 5.17. We cannot in general remove the assumption that μ is λ-closed from Theorem 5.16 (see [AR94, 2.11]). In fact, for $\mu > 2^{<\lambda}$ regular, the statements "μ is λ-closed" and "every λ-accessible category is μ-accessible" are equivalent (see [LR17, 4.11] or [LRVa, 2.6]).

Theorem 5.18. Let $\mathcal{K}$ be an accessible category. Then:

(1) Any object of $\mathcal{K}$ is λ-presentable, for some λ.
(2) For any regular cardinal λ, there is only a set (up to isomorphism) of λ-presentable objects.

Proof. Let μ be such that $\mathcal{K}$ is μ-accessible. Let S be a set of μ-presentable objects so that any object is isomorphic to a μ-directed colimit of members of S. It follows from Lemma 5.9 that any object must be λ-presentable, for some λ. This proves the first item. For the second, Exercise 5.11 shows that there is only a set of μ-presentable objects. By Theorem 5.16, $\mathcal{K}$ is moreover λ-accessible for arbitrarily large λ, so the result follows. □

As mentioned before, in the category of sets, an object is λ-presentable if and only if its cardinality is strictly less than λ. Thus the least cardinal λ such that an object is λ-presentable (we call this the *presentability rank*) is always a successor. The following question of Beke and Rosický [BR12] remains open:

Question 5.19. For a fixed accessible category, is every *high-enough* presentability rank a successor?

We can give the following approximation [LRVa, 3.11]:

Theorem 5.20. Let $\mathcal{K}$ be a λ-accessible category. If $\mu > \lambda$ is weakly inaccessible and λ-closed, then any μ-presentable object is $(< \mu)$-presentable.

Proof. Let M be μ-presentable. By definition, M can be resolved into a λ-directed colimit of λ-presentable objects, hence of $(< \mu)$-presentables. By Lemma 5.15,

M can be resolved into a μ-directed colimit of $(<\mu)$-presentable objects. By μ-presentability of M, this means that M is a retract of a $(<\mu)$-presentable object, hence is itself $(<\mu)$-presentable, as desired. $\square$

Note that assuming the singular cardinal hypothesis, every weakly inaccessible above $2^{<\lambda}$ is λ-closed. Since Solovay showed that the singular cardinal hypothesis holds above certain large cardinals (see [Sol74] or [Jec03, 20.8]) it follows that Question 5.19 has a positive answer assuming a large cardinal axiom (a proper class of strongly compact cardinals).

5.1. From accessible category to μ-AEC.

We now aim to show[1]:

Theorem 5.21 ([BGL$^+$16, 4.5])**.** For any μ-accessible category $\mathcal{K}$ whose morphisms are monomorphisms, $\mathcal{K}$ is equivalent to a μ-AEC.

Recall that two categories $\mathcal{K}_1$ and $\mathcal{K}_2$ are *equivalent* if there is a functor $F : \mathcal{K}_1 \to \mathcal{K}_2$ which is:

(1) Full: its restriction to sets of the form $\mathrm{Hom}(M, N)$ is onto $\mathrm{Hom}(FM, FN)$.
(2) Faithful: its restriction to sets of the form $\mathrm{Hom}(M, N)$ is injective.
(3) Essentially surjective: any object N in $\mathcal{K}_2$ is isomorphic to FM for some object M in $\mathcal{K}_1$.

This is weaker than an isomorphism of category, but preserves all reasonable category-theoretic notions. Intuitively, we allow isomorphic objects inside the category to be identified. One example to keep in mind is that the category of a single object with only the identity morphism is equivalent (but not isomorphic) to the category of all singleton sets.

The proof of Theorem 5.21 proceeds in two steps. The first shows that $\mathcal{K}$ is equivalent to a certain accessible category of structures. The second shows that this category must actually be a μ-AEC. Let us implement the first step. For τ a vocabulary, we denote by $\mathrm{Emb}(\tau)$ the category whose objects are τ-structures and whose morphisms are injective homomorphisms.

Lemma 5.22 ([BGL$^+$16, 4.8])**.** Let $\mathcal{K}$ be a λ-accessible category whose morphisms are monomorphism. Then there is a (finitary) vocabulary τ and a functor $E : \mathcal{K} \to \mathrm{Emb}(\tau)$ which is full and faithful and preserves λ-directed colimits.

Proof. Let $\mathcal{K}_0$ be a small full subcategory of $\mathcal{K}$ containing (up to isomorphism) all the λ-presentable objects. For each $M \in \mathcal{K}_0$, let S_M be a unary relation symbol and for each morphism f in $\mathcal{K}_0$, let $\underline{f}$ be a binary function symbol. The vocabulary τ will consist of all such S_M and $\underline{f}$. Now map each $M \in \mathcal{K}$ to the following τ-structure EM:

(1) Its universe are the morphisms $g : M_0 \to M$, where $M_0 \in \mathcal{K}_0$.
(2) For each $M_0 \in \mathcal{K}_0$, $S_{M_0}^{EM}$ is the set of morphisms $g : M_0 \to M$.
(3) For each morphism $f : M_0 \to M_1$ of $\mathcal{K}_0$, and each $g : M_1 \to M$, $\underline{f}^{EM}(g) = gf$. When $g \notin S_{M_1}^{EM}$, just let $\underline{f}^{EM}(g) = g$.

[1]It was known since Rosický's thesis [Ros83, Ros81] that accessible categories are classes of models of certain $\mathbb{L}_{\infty,\infty}$ sentence, but seeing them as μ-AEC is more direct.

Map each morphism $f : M \to N$ to the function $\bar{f} : EM \to EN$ given by $\bar{f}(g) = fg$. That E is full and faithful and preserves λ-directed colimits is a long but crucial exercise in diagram chasing (closely related to the Yoneda lemma). For example, to see that E is full, assume first that $M \in \mathcal{K}_0$. Then id_M is a morphism in $\mathcal{K}_0$ so given $g : EM \to EN$, we can let $f := g(\mathrm{id}_M)$ and it turns out that $E(f) = g$. When M is not λ-presentable, resolve it into a λ-directed colimit of λ-presentable objects. $\square$

The second step shows that any coherent abstract class which looks like an accessible category is in fact a μ-AEC. First, it is not too hard to show (using resolutions into directed systems again) that only a weak version of the LST axiom suffices:

Exercise 5.23. Let $\mathbf{K}$ be an abstract class satisfying all the axioms of a μ-AEC except possibly the LST axiom. Let $\theta \geq \mu + |\tau(\mathbf{K})|$ be such that:

(1) θ is μ-closed.
(2) $\mathrm{cf}(\theta) \geq \mu$.
(3) For any $M \in \mathbf{K}$ and any $A \subseteq |M|$ with $|A| < \theta$, there exists $M_0 \in \mathcal{P}_{\mathbf{K}_{<\theta}}(M)$ with $A \subseteq |M_0|$.

Then $\mathbf{K}$ is a μ-AEC with $\mathrm{LS}(\mathbf{K}) \leq \theta$.

Lemma 5.24. Let $\mathbf{K}$ be an abstract class satisfying the coherence axiom and let μ be a regular cardinal. Assume that $\mathbf{K}$ is μ-accessible and further the μ-directed colimits are concrete (given by unions, i.e. they are the same as in $\mathrm{Emb}(\tau(\mathbf{K}))$). Let C be the set of cardinals λ such that for any $M \in \mathbf{K}$, $\|M\| < \lambda$ if and only if M is $(< \lambda)$-presentable. Then C is closed unbounded. In particular, $\mathbf{K}$ is a μ-AEC.

Proof. C is clearly closed. Now given any cardinal λ, there is (up to isomorphism) only a set of λ^+-presentable objects (Theorem 5.18) and only a set of objects of cardinality λ. Thus there is a cardinal λ' such that any λ^+-presentable object has cardinality strictly less than λ' and any object of cardinality at most λ is $(< \lambda')$-presentable. Thus given any cardinal λ_0, we can build an increasing sequence $\langle \lambda_i : i < \omega \rangle$ such that for any $i < \omega$, any λ_i^+-presentable object has cardinality strictly less than λ_{i+1} and any object of cardinality λ_i is $(< \lambda_{i+1})$-presentable. Now by construction $\sup_{i<\omega} \lambda_i$ is in C. Thus C is unbounded.

To see the "in particular" part, we have to prove the LST axiom. Pick $\theta \in C$ a limit cardinal such that θ is μ-closed and $\mathrm{cf}(\theta) \geq \mu + |\tau(\mathbf{K})|$. Now let $M \in \mathbf{K}$ and let $A \subseteq |M|$ with $|A| < \theta$ be given. Let $\theta_0 := ((|A| + \aleph_0)^{<\mu})^+$. Note that θ_0 is μ-closed so by Theorem 5.16, $\mathbf{K}$ is θ_0-accessible. Thus M is a θ_0-directed colimit of θ_0-presentable objects $\langle M_i : i \in I \rangle$. Since θ_0-directed colimits are concrete, this implies that A is contained inside some M_i. Now by definition of C, M_i has cardinality strictly less than θ. This shows that the hypotheses of Exercise 5.23 are satisfied. $\square$

Proof of Theorem 5.21. Let $\mathcal{K}$ be a μ-accessible category whose morphisms are monomorphisms. By Lemma 5.22, there is a vocabulary τ such that $\mathcal{K}$ is equivalent to a full subcategory of $\mathrm{Emb}(\tau)$ which is closed under μ-directed colimits inside $\mathrm{Emb}(\tau)$. Equivalently, it is closed under μ-directed unions. Closing such a category under isomorphism, we obtain an abstract class $\mathbf{K}$ (the ordering is just substructure) which satisfies the hypotheses of Lemma 5.24, hence is a μ-AEC. $\square$

6. μ-AECs and infinitary logics

Makkai and Paré [MP89, 3.2.3, 3.3.5, 4.3.2] have shown (refining an argument of Rosický) that any λ-accessible category is equivalent to a category of models of an $\mathbb{L}_{\infty,\lambda}$-sentence (the morphisms are homomorphisms). In this section, we prove results around that neighborhood for μ-AECs.

We first review the following semantic characterization of elementary equivalence.

Definition 6.1. Let M and N be τ-structures. We call f a *partial isomorphism* from M to N if:

(1) f is a function from a subset of $|M|$ to a subset of $|N|$.
(2) For any enumeration $\bar{a}$ of the domain of f and any first-order quantifier-free formula ϕ, $M \models \phi[\bar{a}]$ if and only if $N \models \phi[\bar{a}]$.

Definition 6.2. Let M and N be τ-structures and let θ be an infinite cardinal. A *θ-forth system from M to N* is a set $\mathcal{F}$ such that:

(1) $\mathcal{F} \neq \emptyset$.
(2) Any member f of $\mathcal{F}$ is a partial isomorphism from M to N.
(3) For any $f \in \mathcal{F}$, $|\operatorname{dom}(f)| < \theta$.
(4) For any $f \in \mathcal{F}$ and any $A \subseteq \operatorname{dom}(f)$, $f \upharpoonright A \in \mathcal{F}$.
(5) For any $f \in \mathcal{F}$ and any $A \subseteq |M|$ with $|A| < \lambda$, there exists $g \in \mathcal{F}$ with $f \subseteq g$ and $A \subseteq \operatorname{dom}(g)$.

We say that $\mathcal{F}$ is a *θ-back and forth system from M to N* if it is a θ-forth system and $\{f^{-1} \mid f \in \mathcal{F}\}$ is θ-forth system from N to M.

We write $M \equiv^*_{\infty,\theta} N$ if there is a θ-back and forth system from M to N.

The following result is due to Karp for $\mathbb{L}_{\infty,\omega}$, see [Kar65]. A good basic reference on such theorems (and on $\mathbb{L}_{\infty,\infty}$ in general) is [Dic75].

Theorem 6.3. Let M and N be τ-structures and let θ be an infinite cardinal. The following are equivalent:

(1) $M \equiv_{\infty,\theta} N$.
(2) $M \equiv^*_{\infty,\theta} N$.

Proof.

- <u>(1) implies (2)</u>: Let $\mathcal{F}$ be the set of partial functions f from $|M|$ to $|N|$ whose domain has cardinality strictly less than θ, and such that for any enumeration $\bar{a}$ of their domain and any $\mathbb{L}_{\infty,\theta}$-formula ϕ, $M \models \phi[\bar{a}]$ if and only if $N \models \phi[f(\bar{a})]$. We claim that $\mathcal{F}$ is as desired. By symmetry, it suffices to show it is a θ-forth system. Since $M \equiv_{\infty,\theta} N$, the empty map is in $\mathcal{F}$, hence $\mathcal{F}$ is not empty. Clearly, any member of $\mathcal{F}$ is a partial isomorphism from M to N whose domain has cardinality strictly less than θ. If $f \in \mathcal{F}$ and $A \subseteq \operatorname{dom}(f)$, then by definition $f \upharpoonright A \in \mathcal{F}$. Now let $f \in \mathcal{F}$ and let

$A \subseteq |M|$. Let $\bar{a}$ be an enumeration of A and let $\bar{a}_0$ be an enumeration of $\text{dom}(f)$. For a cardinal μ, let p_μ be the class of formulas $\psi(\bar{x}, \bar{y}) \in \mathbb{L}_{\mu,\theta}$ such that $M \models \psi[\bar{a}, \bar{a}_0]$. We have that $M \models \exists \bar{x} \bigwedge_{\psi \in p_\mu} \psi[\bar{x}, \bar{a}_0]$. Thus $N \models \exists \bar{x} \bigwedge_{\psi \in p_\mu} \psi[\bar{x}, f(\bar{a}_0)]$. Let $\bar{b}^\mu$ be a witness. Now N is a set, so there must exist a proper class C of cardinals such that $\mu, \mu' \in C$ implies $\bar{b} := \bar{b}^\mu = \bar{b}^{\mu'}$. Let g send $\bar{a}$ to $\bar{b}$. It is easy to check that this works.

- (2) implies (1): We show that for any $\mathbb{L}_{\infty,\theta}$-formula $\phi(\bar{x})$, any $\bar{a} \in {}^{<\theta}M$, and any $f \in \mathcal{F}$ whose domain contains $\bar{a}$, $M \models \phi[\bar{a}]$ if and only if $N \models \phi[f(\bar{a})]$. We proceed by induction on ϕ. When ϕ is atomic, this is because f is a partial isomorphism. When ϕ is a conjunction or negation, this is similarly easy. Assume that $\phi = \exists \bar{y} \psi(\bar{y}, \bar{x})$. We show that $M \models \phi[\bar{a}]$ implies $N \models \phi[f(\bar{a})]$, and the converse follows from the symmetric definition of a back and forth system. So let $\bar{b} \in {}^{<\theta}M$ be such that $M \models \psi[\bar{b}, \bar{a}]$. Let $g \in \mathcal{F}$ extend f such that the domain of g contains $\bar{b}$. By the induction hypothesis, $N \models \psi[g(\bar{b}), g(\bar{a})]$. Thus $N \models \phi[g(\bar{a})]$. Since $g(\bar{a}) = f(\bar{a})$, we are done.

□

The proof can be refined to yield:

Exercise 6.4. Show that if θ is regular one can replace (1) by "$M \equiv_{\lambda,\theta} N$", where $\lambda := \left((2 + \|M\| + \|N\|)^{<\theta}\right)^+$.

Exercise 6.5 (Scott)**.** Let θ be regular and let M be a τ-structure. Let $\lambda := \left((2 + \|M\|)^{<\theta}\right)^+$. Show that there exists an $\mathbb{L}_{\lambda,\theta}$-sentence ϕ such that for any τ-structure N, $N \models \phi$ implies $M \equiv_{\infty,\theta} N$.

The following consequence is interesting:

Corollary 6.6. Let θ be an infinite cardinal of cofinality $\aleph_0$ and let M and N be τ-structures of cardinality θ. If $M \equiv_{\infty,\theta} N$, then $M \cong N$.

Proof. By Theorem 6.3, $M \equiv^*_{\infty,\theta} N$. Let $\mathcal{F}$ witness it. Write $|M| = \bigcup_{n<\omega} A_n$, $|N| = \bigcup_{n<\omega} B_n$ with $|A_n| + |B_n| < \theta$. This is possible by the cofinality assumption. Finally, build an increasing chain $\langle f_n : n < \omega \rangle$ of elements of $\mathcal{F}$ such that $A_n \subseteq \text{dom}(f_{n+1})$ and $B_n \subseteq \text{ran}(f_{n+1})$ for all $n < \omega$. This is possible since $\mathcal{F}$ is a θ-back and forth system. □

We can also deduce that AECs are closed under infinitary elementary equivalence. This was observed independently by Kueker [Kue08] and Shelah [She09, IV.1.11]. First, we prove a lemma:

Lemma 6.7. Let $\mathbf{K}$ be an AEC and let M be a τ-structure. If D is a set such that:

(1) For all $M_0 \in D$, $M_0 \in \mathbf{K}_{\leq \text{LS}(\mathbf{K})}$ and $|M_0| \subseteq |M|$.
(2) For all $M_0 \in D$ and all $A \in [M]^{\leq \text{LS}(\mathbf{K})}$, there is $M_1 \in D$ such that $M_0 \leq_{\mathbf{K}} M_1$ and $A \subseteq |M_1|$.

Then $M \in \mathbf{K}$ and $M_0 \leq_{\mathbf{K}} M$ for all $M_0 \in D$.

Proof. First we show:

Claim: If M_0 and M_1 are in D, there exists $M_2 \in \mathbf{K}$ such that $M_0 \leq_{\mathbf{K}} M_2$ and $M_1 \leq_{\mathbf{K}} M_2$.

Proof of Claim: For $\ell = 0, 1$, we build $\langle M_\ell^i : i < \omega \rangle$ $\leq_{\mathbf{K}}$-increasing in D such that $M_\ell^0 = M_\ell$ and $|M_{1-\ell}^i| \subseteq |M_\ell^{i+1}|$ for all $i < \omega$. This is possible by the assumptions on D. Now let $M_2 := \bigcup_{i<\omega} M_0^i = \bigcup_{i<\omega} M_1^i$. $\dagger_{\text{Claim}}$

Now we build $\langle M_s : s \in [M]^{\leq \text{LS}(\mathbf{K})} \rangle$ a sequence of models in D such that $s \subseteq t$ implies $|M_s| \subseteq |M_t|$ and $s \subseteq |M_s|$ for all $s, t \in [M]^{\leq \text{LS}(\mathbf{K})}$. This is possible by the assumptions on D. Now let $s, t \in [M]^{\leq \text{LS}(\mathbf{K})}$ be such that $s \subseteq t$. Then $|M_s| \subseteq |M_t|$ and by the claim, there is $M' \in \mathbf{K}$ such that $M_s \leq_{\mathbf{K}} M'$ and $M_t \leq_{\mathbf{K}} M'$. By coherence, this implies that $M_s \leq_{\mathbf{K}} M_t$. Thus $\langle M_s : s \in [M]^{\leq \text{LS}(\mathbf{K})} \rangle$ is a directed system in $\mathbf{K}$ whose union is M, so $M \in \mathbf{K}$ and it follows from the proof that $M_0 \leq_{\mathbf{K}} M$ for all $M_0 \in D$. $\square$

Theorem 6.8. Let $\mathbf{K}$ be an AEC and let $M \in \mathbf{K}$. Let N be a $\tau(\mathbf{K})$-structure. If $M \equiv_{\infty, \text{LS}(\mathbf{K})^+} N$, then $N \in \mathbf{K}$.

Proof. By Theorem 6.3, there is an $\text{LS}(\mathbf{K})^+$-back and forth system $\mathcal{F}$ from M to N. Let

$$D := \{f[M_0] \mid f \in \mathcal{F}, M_0 \in \mathcal{P}_{\mathbf{K}_{\leq \text{LS}(\mathbf{K})}}(M)\}$$

It suffices to observe that D satisfies the hypotheses of Lemma 6.7 (where M there is N here). Indeed, by closure of $\mathbf{K}$ under isomorphisms, any member of D is a member of $\mathbf{K}_{\leq \text{LS}(\mathbf{K})}$. Moreover if $f[M_0] \in D$ and $A \in [N]^{\leq (\text{LS}(\mathbf{K}))}$, we can use the axioms of back and forth to extend f to g whose range contains A, and moreover $M_1 := \text{dom}(g) \leq_{\mathbf{K}} M$. By coherence, $M_0 \leq_{\mathbf{K}} M_1$. By closure of $\leq_{\mathbf{K}}$ under isomorphisms, $f[M_0] \leq_{\mathbf{K}} g[M_1]$, and by definition $g[M_1] \in D$. $\square$

Question 6.9. Does Theorem 6.8 generalize to μ-AECs?

To better understand the relationship between infinitary logics and μ-AECs, the following concept is useful. The idea is to expand the μ-AECs with predicate that "do not add any information" in the sense that the expansion is already uniquely determined by the structure. The definition appears in [Vas16, 3.1].

Definition 6.10. Let $\mathbf{K}$ be an abstract class and let μ be a regular cardinal. A *functorial expansion* of $\mathbf{K}$ is an abstract class $\mathbf{K}^+$ in a vocabulary $\tau(\mathbf{K}^+)$ expanding $\tau(\mathbf{K})$ such that the reduct map is an isomorphism of category from $\mathbf{K}^+$ onto $\mathbf{K}$. That is:

(1) If $M^+ \leq_{\mathbf{K}^+} N^+$, then $M^+ \upharpoonright \tau(\mathbf{K}) \leq_{\mathbf{K}} N^+ \upharpoonright \tau(\mathbf{K})$.
(2) If $M \in \mathbf{K}$, there is a unique expansion $M^+ \in \mathbf{K}^+$ such that $M^+ \upharpoonright \tau(\mathbf{K}) = M$.
(3) If $f : M \to N$ is a $\mathbf{K}$-embedding then the induced map $f^+ : M^+ \to N^+$ also is.

We call a functorial expansion $(< \mu)$-*ary* if its vocabulary is $(< \mu)$-ary.

Remark 6.11. If $\mathbf{K}^+$ is a functorial expansion of $\mathbf{K}$, then $M^+ \leq_{\mathbf{K}^+} N^+$ holds if and only if $M^+ \upharpoonright \tau(\mathbf{K}) \leq_{\mathbf{K}} N^+ \upharpoonright \tau(\mathbf{K})$. Thus a functorial expansion is entirely determined by its class of models.

Remark 6.12. If $\mathbf{K}^+$ is a $(< \mu)$-ary functorial expansion of a μ-AEC $\mathbf{K}$, then $\mathbf{K}^+$ is a μ-AEC with $\mathrm{LS}(\mathbf{K}^+) = \mathrm{LS}(\mathbf{K})$.

Example 6.13.

(1) $\mathbf{K}$ is a functorial expansion of $\mathbf{K}$.
(2) If $\mathbf{K}$ is an elementary class (ordered with elementary substructure), we can add a relation symbol for each first-order formula and obtain a functorial expansion, called the *Morleyization of* $\mathbf{K}$.
(3) The expansion given by Shelah's presentation Theorem 3.6 is not functorial (unless the starting class is a universal class itself). This is because the reduct functor is not necessarily full.

Another example of a functorial expansion, to be defined later, is the *orbital (or Galois) Morleyization*, which consists in adding a relation symbol for each orbital type. In this section, the following functorial expansion will play an important role:

Definition 6.14. Let $\mathbf{K}$ be a μ-AEC. The *substructure functorial expansion* of $\mathbf{K}$ is the abstract class $\mathbf{K}^+$ defined as follows:

(1) $\tau(\mathbf{K}^+) = \tau(\mathbf{K}) \cup \{P\}$, where P is an $\mathrm{LS}(\mathbf{K})$-ary predicate.
(2) $M^+ \in \mathbf{K}^+$ if and only if $M^+ \upharpoonright \tau(\mathbf{K}) \in \mathbf{K}$ and for any $\bar{a} \in {}^{\mathrm{LS}(\mathbf{K})}M^+$, $P^{M^+}(\bar{a})$ holds if and only if $\mathrm{ran}(\bar{a}) \leq_{\mathbf{K}} M^+ \upharpoonright \tau(\mathbf{K})$, where we see $\mathrm{ran}(\bar{a})$ as a $\tau(\mathbf{K})$-structure.
(3) For $M^+, N^+ \in \mathbf{K}^+$, $M^+ \leq_{\mathbf{K}^+} N^+$ if and only if $M^+ \upharpoonright \tau(\mathbf{K}) \leq_{\mathbf{K}} N^+ \upharpoonright \tau(\mathbf{K})$.

Exercise 6.15. Check that the substructure functorial expansion is indeed a functorial expansion.

The substructure functorial expansion has a number of nice properties.

Definition 6.16. We call an abstract class $\mathbf{K}$ *model-complete* if for $M, N \in \mathbf{K}$, $M \leq_{\mathbf{K}} N$ if and only if $M \subseteq N$.

Note that a model complete abstract class does *not* have to be closed under substructure (the class of algebraically closed fields is one example).

The following criteria to prove model-completeness is a directed system argument:

Exercise 6.17. Let $\mathbf{K}$ be a μ-AEC and let $M, N \in \mathbf{K}$. Suppose that $M \subseteq N$. The following are equivalent:

(1) $M \leq_{\mathbf{K}} N$.
(2) For any $M_0 \in \mathcal{P}_{\mathbf{K}_{\leq \mathrm{LS}(\mathbf{K})}}(M)$, $M_0 \leq_{\mathbf{K}} N$.

The substructure functorial expansion is model-complete:

Theorem 6.18. Let $\mathbf{K}$ be a μ-AEC. Then the substructure functorial expansion of $\mathbf{K}$ is model-complete.

Proof. Let $\mathbf{K}^+$ be the substructure functorial expansion of $\mathbf{K}$. For $M \in \mathbf{K}$, write M^+ for the expansion of M to $\mathbf{K}^+$. Let $M, N \in \mathbf{K}$ and assume that $M^+ \subseteq N^+$. We have to see that $M \leq_{\mathbf{K}} N$. For this, we use the equivalent condition of Exercise 6.17. Let $M_0 \in \mathcal{P}_{\mathbf{K}_{\leq \mathrm{LS}(\mathbf{K})}} M$. We have to see that $M_0 \leq_{\mathbf{K}} N$. Let $\bar{a}$ be an enumeration of M_0. We have that $M^+ \models P[\bar{a}]$ (where P is the additional predicate in $\tau(\mathbf{K})^+$), so $N^+ \models P[\bar{a}]$ (as M^+ is a substructure of N^+). This means that $M_0 \leq_{\mathbf{K}} N$, as desired. $\square$

The substructure functorial expansion of a μ-AEC can be axiomatized (a variation of this is due to Baldwin and Boney [BB17]). Since the ordering is trivial by the previous result, this gives that any μ-AEC is isomorphic (as a category) to the category of models of an $\mathbb{L}_{\infty,\infty}$ sentence, where the morphisms are injective homomorphisms.

Theorem 6.19. Let $\mathbf{K}$ be a μ-AEC and let $\mathbf{K}^+$ be its substructure functorial expansion. There is an $\mathbb{L}_{\left(2^{\mathrm{LS}(\mathbf{K})}\right)^+, \mathrm{LS}(\mathbf{K})^+}$ sentence ϕ such that $\mathbf{K}^+$ is the class of models of ϕ.

Proof. First note that for each $M_0 \in \mathbf{K}_{\leq \mathrm{LS}(\mathbf{K})}$, there is a sentence $\psi_{M_0}(\bar{x})$ of $\mathbb{L}_{\infty, \mathrm{LS}(\mathbf{K})^+}$ coding its isomorphism type, i.e. whenever $M \models \phi[\bar{a}]$, then $\bar{a}$ is an enumeration of an isomorphic copy of M_0. Similarly, whenever M_0, M_1 are in $\mathbf{K}_{\leq \mathrm{LS}(\mathbf{K})}$ with $M_0 \leq_{\mathbf{K}} M_1$, there is $\psi_{M_0, M_1}(\bar{x}, \bar{y})$ that codes that $(\bar{x}, \bar{y})$ is isomorphic to (M_0, M_1) (so in particular $\bar{x} \leq_{\mathbf{K}} \bar{y}$). Let S be a complete set of members of $\mathbf{K}_{\leq \mathrm{LS}(\mathbf{K})}$ (i.e. any other model is isomorphic to it) and let T be a complete set of pairs (M_0, M_1), with each in $\mathbf{K}_{\leq \mathrm{LS}(\mathbf{K})}$, such that $M_0 \leq_{\mathbf{K}} M_1$. Now define the following:

$$\phi_1 = \forall \bar{x} \exists \bar{y} \left(\left(\bigvee_{M_0 \in S} \psi_{M_0}(\bar{y}) \right) \wedge \bar{x} \subseteq \bar{y} \wedge P(\bar{y}) \right)$$

$$\phi_2 = \forall \bar{x} \forall \bar{y} \left((\bar{x} \subseteq \bar{y} \wedge P(\bar{x}) \wedge P(\bar{y})) \to \bigvee_{(M_0, M_1) \in T} \psi_{M_0, M_1}(\bar{x}, \bar{y}) \right)$$

$$\phi = \phi_1 \wedge \phi_2$$

Where $\bar{x} \subseteq \bar{y}$ abbreviates the obvious formula. This works. First, any $M^+ \in \mathbf{K}^+$ satisfies ϕ_1 by the LST axiom and satisfies ϕ_2 by the coherence axiom. Conversely, if $M \models \phi$, then we can build a μ-directed system $\langle M_s : s \in [M]^{<\mu} \rangle$ in $\mathbf{K}$ such that $s \subseteq |M_s|$ and $M_s \in \mathbf{K}_{\leq \mathrm{LS}(\mathbf{K})}$ for all $s \in [M]^{<\mu}$. We then get that $\bigcup_{s \in [M]^{<\mu}} M_s = M \in \mathbf{K}$ by closure under μ-directed systems. $\square$

The following shows that elementary equivalence is preserved when passing to functorial expansions of AECs. This is because back and forth systems are preserved:

Lemma 6.20. Let $\mathbf{K}$ be an AEC. let $\mathbf{K}^+$ be a $(< \mathrm{LS}(\mathbf{K})^+)$-ary functorial expansion of $\mathbf{K}$. Let $M, N \in \mathbf{K}$ and let M^+, N^+ be their respective expansions to $\mathbf{K}^+$. If $\mathcal{F}$ is an $\mathrm{LS}(\mathbf{K})^+$-back and forth system from M to N, then it is an $\mathrm{LS}(\mathbf{K})^+$-back and forth system from M^+ to N^+.

Proof. For any $M_0 \in \mathbf{K}$, write M_0^+ for its expansion to $\mathbf{K}^+$. Let $f \in \mathcal{F}$. Using the axioms of a back and forth system and the LST axiom, one can pick $g \in \mathcal{F}$ such that $f \subseteq g$ and $M_0 := \text{dom}(g) \leq_{\mathbf{K}} M$. Let $N_0 := g[M_0]$. Since $M_0 \cong N_0$, $N_0 \in \mathbf{K}$. Moreover by the proof of Theorem 6.8, $N_0 \leq_{\mathbf{K}} N$. Now by definition of a functorial expansion, we must have $M_0^+ \leq_{\mathbf{K}^+} M^+$ and $N_0^+ \leq_{\mathbf{K}^+} N^+$ and moreover g is a $\mathbf{K}^+$-isomorphism. It follows that f is itself a partial isomorphism from M^+ to N^+. Since f was arbitrary, this shows that $\mathcal{F}$ is indeed a back and forth system from M^+ to N^+. □

As a consequence, we deduce a relationship between the ordering of the class and infinitary elementary equivalence:

Theorem 6.21. Let $\mathbf{K}$ be an AEC. Let $M \in \mathbf{K}$. If $M \preceq_{\mathbb{L}_{\infty,\text{LS}(\mathbf{K})^+}} N$, then $M \leq_{\mathbf{K}} N$.

Proof. By Theorem 6.8, $N \in \mathbf{K}$. We use Exercise 6.17. Let $M_0 \in \mathcal{P}_{\mathbf{K}_{\leq \text{LS}(\mathbf{K})}}(M)$. Let $\bar{a}$ be an enumeration of M_0. We have that $(M, \bar{a}) \equiv_{\infty,\text{LS}(\mathbf{K})^+} (N, \bar{a})$. By Theorem 6.3, there is an $\text{LS}(\mathbf{K})^+$-back and forth system $\mathcal{F}$ from $(M, \bar{a})$ to $(N, \bar{a})$. By Lemma 6.20 it is also a back and forth system from $(M^+, \bar{a})$ to $(N^+, \bar{a})$, where M^+ and N^+ denote the expansions of M and N in the substructure functorial expansion. By Theorem 6.3 again, this implies that $P^{M^+}(\bar{a})$ holds if and only if $P^{N^+}(\bar{a})$ holds. Since $M_0 \leq_{\mathbf{K}} M$, we have that $P^{M^+}(\bar{a})$, so $P^{N^+}(\bar{a})$, so $M_0 \leq_{\mathbf{K}} N$, as desired. □

There are converses to Theorem 6.21 when M and N are sufficiently saturated. For example, in a first-order theory T, if M and N are saturated of cardinality λ and $M \preceq N$, then $M \preceq_{\mathbb{L}_{\infty,\lambda}} N$ (exercise). The following beautiful argument of Shelah uses Fodor's lemma to provide some kind of analog even when there is no obvious notion of saturated (see [BGL+16, 6.8] for a generalization to certain μ-AECs).

Theorem 6.22 (Shelah, [She09, IV.1.12(1)])**.** Let $\mathbf{K}$ be an AEC, let θ be regular and let $\lambda = \lambda^{<\theta} \geq \text{LS}(\mathbf{K})$. Assume that $\mathbf{K}$ is categorical in λ and let $M, N \in \mathbf{K}_{\geq \lambda}$. If $M \leq_{\mathbf{K}} N$, then $M \preceq_{\mathbb{L}_{\infty,\theta}} N$.

Proof. A directed systems argument (exercise) establishes that it suffices to prove it when $M, N \in \mathbf{K}_\lambda$. We now prove by induction on $\phi(\bar{x}) \in \mathbb{L}_{\infty,\theta}$ that for any $M, N \in \mathbf{K}_\lambda$ with $M \leq_{\mathbf{K}} N$ and any $\bar{a} \in {}^{<\theta}M$, $M \models \phi[\bar{a}]$ if and only if $N \models \phi[\bar{a}]$. This is easy when ψ is atomic (since $\leq_{\mathbf{K}}$ extends substructure) and when ϕ is a conjunction or a negation. We prove what happens when $\phi = \exists \bar{y} \psi(\bar{x}, \bar{y})$. If $M \models \phi[\bar{a}]$, then $N \models \phi[\bar{a}]$ as well. Now suppose that $N \models \phi[\bar{a}]$. We build an increasing continuous chain $\langle M_i : i < \lambda^+ \rangle$ and $\langle f_i : i < \lambda^+ \rangle$ such that for all $i < \lambda^+$:

(1) $M_i \in \mathbf{K}_\lambda$.
(2) $f_i : N \cong M_{i+1}$ is such that $f_i[M] = M_i$.

This is possible by categoricity in λ and some renaming. Now let $\bar{a}_i := f_i(\bar{a})$. Note that since $\bar{a} \in {}^{<\theta}M$, we have that $\bar{a}_i \in {}^{<\theta}M_i$. Let $S := \{i < \lambda^+ \mid \text{cf}(i) \geq \theta\}$. This is a stationary set, and for each $i \in S$, there exists $j_i < i$ such that $\bar{a}_i \in {}^{<\theta}M_{j_i}$. Thus the map $i \mapsto j_i$ is regressive so by Fodor's lemma there exists $S_0 \subseteq S$ stationary and

$j < \lambda^+$ such that for any $i \in S_0$, $j_i = j$. Since $\lambda = \lambda^{<\theta}$ and $|S_0| = \lambda^+$, there exists $\bar{a}' \in {}^{<\theta}M_j$ and $S_1 \subseteq S_0$ of cardinality λ^+ and such that $i \in S_1$ implies $\bar{a}_i = \bar{a}'$. Let $i \in S_1$. Since $N \models \phi[\bar{a}]$, we have (applying f_i) that $M_{i+1} \models \phi[\bar{a}']$. Thus there exists $\bar{b} \in {}^{<\theta}M_{i+1}$ such that $M_{i+1} \models \psi[\bar{b}, \bar{a}']$. Pick $i' \in S_1$ such that $i + 1 < i'$. By the induction hypothesis, $M_{i'} \models \psi[\bar{b}, \bar{a}']$. Applying $f_{i'}^{-1}$ to this statement (and the definition of S_1), $M \models \psi[f_{i'}^{-1}(\bar{b}), \bar{a}]$, hence $M \models \phi[\bar{a}]$, as desired. $\square$

7. Orbital types

In any abstract class, one can define a semantic notion of type (loosely, this is the finest possible notion of types that preserves **K**-embeddings). They were introduced by Shelah [She87b]. The name "Galois type" is used a lot in the litterature, but we prefer Shelah's terminology of "orbital type" for reasons that will soon become apparent.

Definition 7.1. Let **K** be an abstract class. We define an equivalence relation $\equiv$ $(=\equiv^{\mathbf{K}})$ on pairs $(\bar{a}, M)$, where $M \in \mathbf{K}$ and $\bar{a} \in {}^{<\infty}M$ as follows: $\equiv$ is the intersection of all equivalence relations E on such pairs satisfying:

If $f : M \to N$ is a **K**-embedding, then $(\bar{a}, M)E(f(\bar{a}), N)$.

For $N_1, N_2 \in \mathbf{K}$, $A \subseteq N_1 \cap N_2$, and $\bar{b}_\ell \in {}^{<\infty}N_\ell$, we write $(\bar{a}_1, N_1) \equiv_A (\bar{a}_2, N_2)$ if for some (equivalently, any) enumeration $\bar{a}$ of A, $(\bar{a}_1\bar{a}, N_1) \equiv (\bar{a}_2\bar{a}, N_2)$. For $N \in \mathbf{K}$, $\bar{b} \in {}^{<\infty}N$ and $A \subseteq |N|$, we let $\mathbf{tp}(\bar{b}/A; N)$ denote the $\equiv_A$-equivalence class of $(\bar{b}, N)$. When **K** is not clear from context, we may write $\mathbf{tp}_{\mathbf{K}}(\bar{b}/A; N)$.

A more explicit definition is:

Exercise 7.2. Let **K** be an abstract class. Show that $\equiv^{\mathbf{K}}$ is the transitive closure of the relation E_{at} defined by $(\bar{b}_1, N_1)E_{\text{at}}(\bar{b}_2, N_2)$ if and only if there exists $N \in \mathbf{K}$, $f_\ell : N_\ell \to N$ such that $f_1(\bar{b}_2) = \bar{b}_2$.

From this and a diagram chase, we obtain an easier definition for abstract classes with amalgamation:

Exercise 7.3. Let **K** be an abstract class with amalgamation. Show that $\equiv^{\mathbf{K}} = E_{\text{at}}$, where E_{at} is defined in the previous exercise. Deduce that $\mathbf{tp}(\bar{b}_1/A; N_1) = \mathbf{tp}(\bar{b}_2/A; N_2)$ if and only if there exists $N \in \mathbf{K}$ and $f_\ell : N_\ell \underset{A}{\longrightarrow} N$ such that $f_1(\bar{b}_1) = f_2(\bar{b}_2)$.

One can also prove an easier characterization in AECs with intersections:

Exercise 7.4. Let **K** be an AEC with intersections. Show that $\mathbf{tp}(\bar{b}_1/A; N_1) = \mathbf{tp}(\bar{b}_2/A; N_2)$ if and only if there exists $f : \text{cl}^{N_1}(A\bar{b}_1) \cong_A \text{cl}^{N_2}(A\bar{b}_2)$ such that $f(\bar{b}_1) = \bar{b}_2$. *Hint: first show that $(\bar{b}_1, N_1)E_{at}(\bar{b}_2, N_2))$ implies there is $f : \text{cl}^{N_1}(\bar{b}_1) \cong \text{cl}^{N_2}(\bar{b}_2)$ sending $\bar{b}_1$ to $\bar{b}_2$, then use Exercise 7.2.*

Example 7.5.

(1) Let **K** be an elementary class (ordered by elementary substructure). Then orbital types coincide with the usual syntactic types. More precisely, if $N_1, N_2 \in \mathbf{K}$, $A \subseteq N_1 \cap N_2$, $\bar{b}_\ell \in {}^{<\infty}N_\ell$, the following are equivalent:

   (a) $\mathbf{tp}(\bar{b}_1/A; N_1) = \mathbf{tp}(\bar{b}_2/A; N_2)$.
   (b) For any $\mathbb{L}_{\omega,\omega}$ formula ϕ, $N_1 \models \phi[\bar{b}_1]$ if and only if $N_2 \models \phi[\bar{b}_2]$.

   This follows from Exercise 2.10. In particular, orbital types are exactly orbits of the monster model under the action of its automorphism group. We will soon generalize this last fact to any AEC with amalgamation.
(2) Let $\mathbf{K}$ be a universal class. By Exercise 7.4 and Theorem 2.11, orbital types are exactly the same as the quantifier-free types.

It will be convenient to have some notation to talk about orbital types.

Definition 7.6. Let $\mathbf{K}$ be an abstract class.

(1) Let $N \in \mathbf{K}$, $A \subseteq |N|$, and α be an ordinal. Define:
$$\mathbf{S}^\alpha(A; N) := \{\mathbf{tp}(\bar{b}/A; N) \mid \bar{b} \in {}^\alpha|N|\}$$
(2) For $M \in \mathbf{K}$ and α an ordinal, let:
$$\mathbf{S}^\alpha(M) := \{p \mid \exists N \in \mathbf{K} : M \leq_{\mathbf{K}} N \text{ and } p \in \mathbf{S}^\alpha(M; N)\}$$
(3) For α an ordinal, let:
$$\mathbf{S}^\alpha(\emptyset) := \bigcup_{N \in \mathbf{K}} \mathbf{S}^\alpha(\emptyset; N)$$

When $\alpha = 1$, we omit it. Similarly define $\mathbf{S}^{<\alpha}$, where α is allowed to be ∞. When $\mathbf{K}$ is not clear from context, we may write $\mathbf{S}^\alpha_{\mathbf{K}}$, etc.

Remark 7.7. When α is an ordinal, $\mathbf{S}^\alpha(M)$ and $\mathbf{S}^\alpha(\emptyset)$ could a priori be proper classes. However in reasonable cases (e.g. when $\mathbf{K}$ is a μ-AEC) they are sets. For example when $\mathbf{K}$ is a μ-AEC, an upper bound for $|\mathbf{S}^\alpha(M)|$ is $2^{(\|M\|+\alpha+\mathrm{LS}(\mathbf{K}))^{<\mu}}$.

Definition 7.8. Let $\mathbf{K}$ be an abstract class and let p be an orbital type.

(1) Let $\ell(p)$ and $\mathrm{dom}(p)$ be the unique α and A such that there exists $N \in \mathbf{K}$ so that $p \in \mathbf{S}^\alpha(A; N)$.
(2) We say that p is *realized in N (by $\bar{b}$)* if $p = \mathbf{tp}(\bar{b}/\mathrm{dom}(p); N)$. Similarly define type omission.
(3) For $A \subseteq \mathrm{dom}(p)$, we let $p \upharpoonright A$ be $\mathbf{tp}(\bar{b}/A; N)$ for some (any) $\bar{b}$ and N such that p is realized by $\bar{b}$ in N.
(4) We say that an orbital type q is an *extension* of p if $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$ and $q\,\mathrm{dom}(p) = p$.
(5) If $p = \mathbf{tp}(\bar{b}/M; N)$, $M \leq_{\mathbf{K}} N$, and $f : M \cong M'$, we let $f(p)$ be $\mathbf{tp}(g(\bar{b})/M'; N')$ for some (any) extension $g : N \cong N'$ of f.

7.1. **Model-homogeneous and universal models.** Even without a notion of type, one can make the following definitions:

Definition 7.9. Let $\mathbf{K}$ be an abstract class, let $M \in \mathbf{K}$, and let λ be an infinite cardinal.

(1) M is *λ-universal* if any $N \in \mathbf{K}_{<\lambda}$ $\mathbf{K}$-embeds into M. When $\lambda = \|M\|^+$, we omit it.

(2) M is λ-*model-homogeneous* if for any $M_0 \leq_{\mathbf{K}} N_0$ both in $\mathbf{K}_{<\lambda}$, if $M_0 \leq_{\mathbf{K}} M$ then there exists $f : N_0 \xrightarrow[M_0]{} N$. When $\lambda = \|M\|$, we omit it.

Let us note for later use that there is a weaker definition of being model-homogeneous which suffices.

Exercise 7.10. Let $\mathbf{K}$ be an AEC with amalgamation. Let $M \in \mathbf{K}$ and let $\lambda > \text{LS}(\mathbf{K})$. The following are equivalent:

(1) M is λ-model-homogeneous.
(2) For any $M_0 \in \mathcal{P}_{\mathbf{K}_{<\lambda}}(M)$ and any $N_0 \in \mathbf{K}_{\|M_0\|+\text{LS}(\mathbf{K})}$ with $M_0 \leq_{\mathbf{K}} N_0$, there exists $f : N_0 \xrightarrow[M_0]{} N$.
(3) For any $M_0 \in \mathcal{P}_{\mathbf{K}_{<\lambda}}(M)$ and any $N_0 \in \mathbf{K}_{\leq\lambda}$ with $M_0 \leq_{\mathbf{K}} N_0$, there exists $f : N_0 \xrightarrow[M_0]{} N$.

In an AEC with amalgamation and joint embedding, it is reasonably easy to create such models via a general exhaustion argument:

Exercise 7.11. Let $\mathbf{K}$ be an AEC with amalgamation and let $M \in \mathbf{K}$. Let $\lambda > \text{LS}(\mathbf{K})$.

(1) For any $\theta \geq \|M\|$ with $\theta = \theta^{<\lambda}$, there exists a λ-model-homogeneous $N \in \mathbf{K}$ with $M \leq_{\mathbf{K}} N$.
(2) If $\mathbf{K}$ has joint embedding, any λ-model-homogeneous model is λ^+-universal.

Moreover, the model-homogeneous universal model is unique (in a fixed cardinality) if if exists:

Exercise 7.12. Let $\mathbf{K}$ be an AEC with amalgamation. Let $M, N \in \mathbf{K}$ be model-homogeneous of the same cardinality $\lambda > \text{LS}(\mathbf{K})$. Let $M_0 \in \mathcal{P}_{\mathbf{K}_{<\lambda}}(M)$ and let $f : M_0 \to N$. Then there exists an isomorphism $g : M \cong N$ extending f.

Let us call a *monster model* in an AEC $\mathbf{K}$ a proper class-sized $\tau(\mathbf{K})$-structure $\mathfrak{C}$ such that there exists $\langle \mathfrak{C}_i : i \in \text{OR} \rangle$ increasing in $\mathbf{K}$ with $\mathfrak{C}_i$ $(|i| + \text{LS}(\mathbf{K})^+)$-model-homogeneous and $(|i| + \text{LS}(\mathbf{K})^+)$-universal for all $i \in \text{OR}$. We abuse notation and think of $\mathfrak{C}$ as a member of $\mathbf{K}$.

Exercise 7.13. Let $\mathbf{K}$ be an AEC. Then $\mathbf{K}$ has a monster model if and only if $\mathbf{K}$ has amalgamation, joint embedding, and arbitrarily large models.

Orbital types are actually orbits (under the action of an automorphism group) when their equality is computed inside a model-homogeneous model (in particular in the monster model).

Exercise 7.14. Let $\mathbf{K}$ be an AEC with amalgamation. Let $M \in \mathbf{K}$ be model-homogeneous and let $\bar{b}_1, \bar{b}_2 \in {}^{\alpha}M$ with $\alpha < \|M\|$. Then $\mathbf{tp}(\bar{b}_1/\emptyset; M) = \mathbf{tp}(\bar{b}_2/\emptyset; M)$ if and only if there is an automorphism of M sending $\bar{b}_1$ to $\bar{b}_2$.

7.2. **Model-homogeneous is equivalent to saturated.** Using orbital types, one can define a notion related to being model-homogeneous:

Definition 7.15. Let $\mathbf{K}$ be an AEC with amalgamation, let $M \in \mathbf{K}$ and let $\lambda > \mathrm{LS}(\mathbf{K})$. We say that M is λ-*saturated* if for any $M_0 \in \mathcal{P}_{\mathbf{K}_{<\lambda}}(M)$, any $p \in \mathbf{S}(M_0)$ is realized inside M.

Exercise 7.16. Show that in an AEC with amalgamation, any λ-model-homogeneous model is λ-saturated.

We will prove the following converse, due to Shelah [She09, II.1.14] (originally proven in [She87b]). This provides some justification for using orbital types, as it tells us that model-homogeneous models can be built "element by element".

Theorem 7.17. Let $\mathbf{K}$ be an AEC with amalgamation. Let $\lambda > \mathrm{LS}(\mathbf{K})$ and let $M \in \mathbf{K}$. If M is λ-saturated, then M is λ-model-homogeneous.

Proof. By Exercise 7.10, it suffices to show that for all $M_0 \in \mathcal{P}_{\mathbf{K}_{<\lambda}}(M)$ and all $N \in \mathbf{K}_{\|M_0\|+\mathrm{LS}(\mathbf{K})}$ with $M_0 \leq_{\mathbf{K}} N$, there is $f : N \xrightarrow[M_0]{} M$. Let $\mu := \|N\| + \mathrm{LS}(\mathbf{K})$ and let $\langle a_i : i < \mu \rangle$ be an enumeration of $|N|$ (possibly with repetitions). We build $\langle N_i^0 : i \leq \mu \rangle$, $\langle N_i^1 : i \leq \mu \rangle$ increasing continuous in $\mathbf{K}_{\leq \mu}$ and $\langle f_i : i \leq \mu \rangle$ increasing continuous such that for all $i < \mu$:

(1) $f_i : N_i^0 \to M$.
(2) $N_0^0 = M_0$, $N_0^1 = N$, $f_0 = \mathrm{id}_{M_0}$.
(3) $N_i^0 \leq_{\mathbf{K}} N_i^1$.
(4) $a_i \in N_{i+1}^0$.

<u>This is enough</u>: By (4), we have that $|N| \subseteq |N_\mu^0|$. Since $N \leq_{\mathbf{K}} N_\mu^1$ and $N_\mu^0 \leq_{\mathbf{K}} N_\mu^1$, coherence implies that $N \leq_{\mathbf{K}} N_\mu^0$. Let $f := f_\mu \upharpoonright N$. Then f is the desired $\mathbf{K}$-embedding of N inside M fixing M_0.

<u>This is possible</u>: The base case has already been specified and at limits we take unions. Suppose now that $i = j + 1$ and stage j has been implemented. Since $N \leq_{\mathbf{K}} N_j^1$, $a_i \in N_j^1$. Let $q_i := \mathbf{tp}(a_i / N_j^0; N_j^1)$. Let $M_j := f_j[N_j^0]$ and let $g : N_j^1 \cong M_j'$ be an extension of f_j. Let $p_i := \mathbf{tp}(g(a_i) / M_j; M_j')$ (so $p_i = f(q_i)$, see Definition 7.8). By assumption, p_i is realized in M, say by b_i. Thus there exists $M_j'' \in \mathbf{K}_{\leq \mu}$ with $M_j' \leq_{\mathbf{K}} M_{j''}$ and $h : M \xrightarrow[M_j]{} M_j''$ such that $h(b_i) = g(a_i)$. Let $g' : N_i^1 \cong M_j''$ be an extension of g. Let $M_i \in \mathcal{P}_{\mathbf{K}_{\leq \mu}}(M)$ be such that $M_j \leq_{\mathbf{K}} M_i$ and $b_i \in M_i$. Let $N_i^0 := (g')^{-1} h[M_i]$. Let $f_i := h^{-1} g' \upharpoonright N_i^0$.

$\square$

Remark 7.18. It suffices to assume that $\mathbf{K}_{<(\lambda + \|M\|)}$ has amalgamation.

We deduce a more local technical lemma which will have several other interesting consequences:

Definition 7.19. For $\mathbf{K}$ an abstract class and $M, N \in \mathbf{K}$, we say that N is *universal over* M if $M \leq_{\mathbf{K}} N$ and whenever $M' \in \mathbf{K}$ is such that $M \leq_{\mathbf{K}} M'$ and $\|M'\| = \|M\|$, there is $f : M' \xrightarrow[M]{} N$.

Lemma 7.20 (The universal extension construction lemma)**.** Let $\mathbf{K}$ be an abstract class satisfying all the axioms of AECs except perhaps the LST axiom. Assume that $\mathbf{K}$ has amalgamation. Let λ be a cardinal and let $\langle M_i : i \leq \lambda \rangle$ be an increasing continuous chain in $\mathbf{K}$ such that $\lambda = \|M_0\|$. If for any $i < \lambda$, any $p \in \mathbf{S}(M_i)$ is realized in M_{i+1}, then M_λ is universal over M_0.

Proof. Exactly as in the proof of Theorem 7.17. □

Definition 7.21. Let $\mathbf{K}$ be an AEC and let $\lambda \geq \mathrm{LS}(\mathbf{K})$. We say that $\mathbf{K}$ is *stable in* λ if $|\mathbf{S}(M)| \leq \lambda$ for all $M \in \mathbf{K}_\lambda$.

Corollary 7.22. Let $\mathbf{K}$ be an AEC and let $\lambda \geq \mathrm{LS}(\mathbf{K})$. Assume that $\mathbf{K}_\lambda$ has amalgamation and $\mathbf{K}$ is stable in λ. For any $M \in \mathbf{K}_\lambda$, there exists $N \in \mathbf{K}_\lambda$ such that N is universal over M.

Proof. Build an increasing continuous chain $\langle M_i : i \leq \lambda \rangle$ in $\mathbf{K}_\lambda$ such that $M_0 = M$ and M_{i+1} realizes all types over M_i. This is possible by stability in λ. This is enough: by Lemma 7.20, M_λ is universal over M_0. □

8. Tameness

Tameness is a locality property for orbital types first isolated by Grossberg and VanDieren [GV06]. Type-shortness is a generalization introduced by Will Boney [Bon14]. We only give two variations here.

Definition 8.1. Let $\mathbf{K}$ be an abstract class and let κ be an infinite cardinal.

(1) $\mathbf{K}$ is $(<\kappa)$-*tame* if for any two distinct orbital types $p, q \in \mathbf{S}(M)$ there exists $A \in [M]^{<\kappa}$ such that $p \upharpoonright A \neq q \upharpoonright A$.
(2) $\mathbf{K}$ is $(<\kappa)$-*short* if for any two $M_1, M_2 \in \mathbf{K}$, $\bar{b}_\ell \in {}^\alpha M_\ell$, if $\mathbf{tp}(\bar{b}_1/\emptyset; M_1) \neq \mathbf{tp}(\bar{b}_2/\emptyset; M_2)$, then there exists $I \subseteq \alpha$ with $|I| < \kappa$ such that $\mathbf{tp}(\bar{b}_1 \upharpoonright I/\emptyset; M_1) \neq \mathbf{tp}(\bar{b}_2 \upharpoonright I/\emptyset; M_2)$.

When we omit the κ, we meant "for some κ".

The different between tameness and shortness is the length of the types involved and their domains (tameness is for types of length one over models). In the litterature, $(<\kappa)$-short is called "fully $(<\kappa)$-tame and type-short over $\emptyset$". The following is not difficult to show:

Exercise 8.2. If $\mathbf{K}$ is $(<\kappa)$-short, then $\mathbf{K}$ is $(<\kappa)$-tame.

We have seen (Exercise 2.10 and Theorem 2.11) that elementary and universal classes are both $(<\aleph_0)$-short. The following is a trivial non-example:

Example 8.3. Let $\mathbf{K} = (K, \leq_{\mathbf{K}})$ be defined by $K := \{M \mid M \cong (\mathbb{Q}, <)\}$ and $M \leq_{\mathbf{K}} N$ if and only if $M, N \in K$ and $M = N$. Then $\mathbf{K}$ is *not* $(<\aleph_0)$-short, since $\mathbf{tp}(1/(0,1); \mathbb{Q}) \neq \mathbf{tp}(2/(0,1); \mathbb{Q})$ (there is no automorphism of $\mathbb{Q}$ sending 1 to 2 fixing $(0,1)$) but all the finite restrictions of these types are equal.

There are various less trivial examples of non-tameness [BS08, BK09]. The following is due to Will Boney [Bon14]:

Theorem 8.4. Let $\mathbf{K}$ be an AEC and let $\kappa > \mathrm{LS}(\mathbf{K})$ be a strongly compact cardinal. Then $\mathbf{K}$ is $(<\kappa)$-short.

Boney's proof uses closure of AECs under sufficiently-complete ultraproducts (which follows from the presentation theorem and the fact that reducts commute with ultraproducts). Later Lieberman and Rosický [LR16, 5.2] found a different proof using an older category-theoretic result of Makkai and Paré. We present here yet another proof (unpublished) which uses compactness of $\mathbb{L}_{\kappa,\kappa}$ directly. The proof actually generalizes to μ-AECs (see also [BGL$^+$16, §5]).

Proof of Theorem 8.4. We more generally prove the statement for any μ-AEC. We will assume for notational simplicity that $\mathbf{K}$ has amalgamation (more precisely that $\equiv^{\mathbf{K}} = E_{\text{at}}$) but if $\mathbf{K}$ does not have amalgamation a similar proof (with more coding) also gives the result. Since shortness is invariant under taking functorial assumption, we may assume without loss of generality (Theorems 6.18 and 6.19) that $\mathbf{K}$ is axiomatized by an $\mathbb{L}_{\kappa,\kappa}$-sentence ϕ and $\mathbf{K}$ is model-complete. Let $\tau := \tau(\mathbf{K})$.

Let $M_1, M_2 \in \mathbf{K}$. Let $\bar{b}_\ell \in {}^\alpha M_\ell$. Suppose that $(\bar{b}_1 \upharpoonright I, M_1) \equiv (\bar{b}_2 \upharpoonright I, M_2)$ for all $I \in [\alpha]^{<\kappa}$. Without loss of generality, $M_1 \cap M_2 = \emptyset$. Let τ_ℓ be τ expanded with new constants symbols $\langle c_a : a \in M_\ell \rangle$. Let M_ℓ^+ be the expansions of M_ℓ to τ_ℓ. Let T_ℓ be the $\mathbb{L}_{\kappa,\kappa}$-quantifier-free diagram of M_ℓ^+. Let B_ℓ be the range of $\bar{b}_\ell$ and let f be a map sending $\bar{b}_1$ to $\bar{b}_2$. Let T be the $\mathbb{L}_{\kappa,\kappa}$-theory $\{\phi\} \cup T_1 \cup T_2 \cup \{c_b = c_{f(b)} \mid b \in B_1\}$. It suffices to prove that T is consistent. By the compactness theorem for $\mathbb{L}_{\kappa,\kappa}$, it suffices to prove that T is $(< \kappa)$-consistent. This is given by the assumption that $\mathbf{tp}(\bar{a}_1 \upharpoonright I; M_1) = \mathbf{tp}(\bar{a}_2 \upharpoonright I; M_2)$ for any $I \in [\alpha]^{<\kappa}$: any M witnessing this will (in a suitable expansion) model T. $\square$

Recently, Boney and Unger [BU17] (building on earlier work of Shelah [She]) found an example of an AEC $\mathbf{K}$ which is tame if and only if there is an (almost) strongly compact above $\text{LS}(\mathbf{K})$. Thus the statement "every AEC is tame" is a large cardinal axiom.

The following characterization of shortness in terms of functorial expansion appears in [Vas16]. We first expand $\mathbf{K}$ with a symbol for each orbital type:

Definition 8.5. Let $\mathbf{K}$ be an abstract class. The $(< \kappa)$*-orbital Morlyeization* of $\mathbf{K}$ is given by adding an $\ell(p)$-ary relation symbol R_p for each $p \in \mathbf{S}^{<\kappa}(\emptyset)$ and expanding each $M \in \mathbf{K}$ to M^+ with $R_p^{M^+}(\bar{b})$ holding if and only if $\mathbf{tp}(\bar{b}/\emptyset; M) = p$.

Exercise 8.6. Prove that the $(< \kappa)$-orbital Morleyization of $\mathbf{K}$ is a functorial expansion.

Exercise 8.7. Let $\mathbf{K}$ be an abstract class. The following are equivalent:

(1) $\mathbf{K}$ is $(< \kappa)$-short.
(2) The map sending each $p = \mathbf{tp}(\bar{b}/\emptyset; M) \in \mathbf{S}^{<\infty}(\emptyset)$ to the quantifier-free type of $\bar{b}$ inside M^+ is an injection, where M^+ is the expansion of M in the $(< \kappa)$-orbital Morleyization.

9. Amalgamation from diamond

The following result is due to Shelah [She87a].

Theorem 9.1. Let $\mathbf{K}$ be an AEC and let $\lambda \geq \mathrm{LS}(\mathbf{K})$. Assume $2^\lambda < 2^{\lambda^+}$. If $\mathbf{K}$ is categorical in λ and $\mathbb{I}(\mathbf{K}, \lambda^+) < 2^{\lambda^+}$, then $\mathbf{K}_\lambda$ has amalgamation.

Recall that $\mathbb{I}(\mathbf{K}, \lambda^+)$ is the number of models of cardinality λ^+ up to isomorphism. The hypothesis that $2^\lambda < 2^{\lambda^+}$ is in general needed: there is an example with $\lambda = \aleph_0$ where Martin's axiom plus $\aleph_1 < 2^{\aleph_0}$ implies that the example is categorical in both $\aleph_0$ and $\aleph_1$ yet fails amalgamation [She09, §I.6]. We will prove Theorem 9.1 using a stronger hypothesis than $2^\lambda < 2^{\lambda^+}$ known as the diamond principle:

Definition 9.2. For an uncountable regular cardinal λ, $\diamondsuit_\lambda$ is the statement that there exists a sequence $\langle A_i : i < \lambda \rangle$ such that $A_i \subseteq i$ and for any $X \subseteq \lambda$, the set $\{i < \lambda \mid X \cap i = A_i\}$ is stationary.

If $V = L$, $\diamondsuit_\lambda$ holds for any uncountable regular λ (this is due to Jensen, who also introduced $\diamondsuit$). On the other hand, $\diamondsuit_\lambda$ implies that $2^{<\lambda} = \lambda$ (since any bounded subset of λ must be equal to some A_i). Thus $\diamondsuit_\lambda$ is independent of ZFC. We will use the following form of $\diamondsuit$:

Exercise 9.3. Let λ be an uncountable regular cardinal. Then $\diamondsuit_\lambda$ is equivalent to:

There are $\{\eta_\alpha, \nu_\alpha : \alpha \to \alpha \mid \alpha < \lambda\}, \{g_\alpha : \alpha \to \alpha \mid \alpha < \lambda\}$ such that for all $\eta, \nu : \lambda \to 2$, $g : \lambda \to \lambda$, $\{\alpha < \lambda \mid \eta_\alpha = \eta \restriction \alpha, \nu_\alpha = \nu \restriction \alpha, g_\alpha = g \restriction \alpha\}$ is stationary.

Our goal will be to show:

Theorem 9.4. Let $\mathbf{K}$ be an AEC and let $\lambda \geq \mathrm{LS}(\mathbf{K})$. Assume $\diamondsuit_{\lambda^+}$. Assume that $\mathbf{K}$ is categorical in λ. If $\mathbf{K}_\lambda$ does *not* have amalgamation, then there exists a family $\{M_S \mid S \subseteq \lambda^+\}$ of models in $\mathbf{K}$ of cardinality λ^+ such that if S_1 and S_2 are stationary subset of λ^+ and $S_1 \Delta S_2$ is stationary, then $M_{S_1} \ncong M_{S_2}$.

Before proving Theorem 9.1, we need one more fact:

Exercise 9.5. Let $\mathbf{K}$ be an AEC and let $\lambda > \mathrm{LS}(\mathbf{K})$ be a regular cardinal. Let $M, N \in \mathbf{K}_\lambda$ and let $f : M \cong N$. Let $\langle M_i : i \leq \lambda \rangle$, $\langle N_i : i \leq \lambda \rangle$ be increasing continuous resolutions of M, N respectively (in particular, $M_\lambda = M$, $N_\lambda = N$, $\|M_i\| + \|N_i\| < \lambda$ for all $i < \lambda$). Then the set of ordinals $\alpha < \lambda$ such that $f \restriction M_\alpha : M_\alpha \cong N_\alpha$ is a club.

Proof of Theorem 9.1 assuming $\diamondsuit_{\lambda^+}$. Suppose that $\mathbf{K}_\lambda$ fails to have amalgamation. Fix $\eta_\alpha, \nu_\alpha, g_\alpha$ as given by Exercise 9.3 (where λ there stand for λ^+ here)

Build a strictly increasing continuous tree $\{M_\eta \mid \eta \in {}^{\leq \lambda^+}2\}$ such that:

1. $|M_\eta| \subseteq \lambda^+$ for all $\eta \in {}^{\leq \lambda^+}2$
2. $M_\eta \neq M_\nu$ whenever $\eta \neq \nu$.
3. If $|M_{\eta_\delta}| = \delta$, $\eta_\delta \neq \nu_\delta$, and $g_\delta : M_{\eta_\delta} \cong M_{\nu_\delta}$ is an isomorphism, then it cannot be extended to an embedding of $M_{\eta_\delta \frown i}$ into M_ν for all $\nu \supseteq \nu_\delta \frown j$, $\nu \in {}^{<\lambda^+}2$, for all $i, j \in 2$.

<u>This is enough</u>: We show that $M_\eta \ncong M_\nu$ for $\eta \neq \nu \in {}^{\lambda^+}2$. Suppose for a contradiction that $f : M_\eta \cong M_\nu$ is an isomorphism. Note that $\{\alpha < \lambda^+ \mid |M_\alpha| = \alpha\}$ is

club, and so is $\{\alpha < \lambda^+ \mid f \restriction M_{\eta\restriction\alpha} : M_{\eta\restriction\alpha} \cong M_{\nu\restriction\alpha}\}$. Thus using diamond, there is a stationary set of $\delta < \lambda^+$ such that $\eta \restriction \delta \neq \nu \restriction \delta$, $\eta_\delta = \eta \restriction \delta, \nu_\delta = \nu \restriction \delta$, $g_\delta = f \restriction \delta$, $\delta = |M_{\eta_\delta}| = |M_{\nu_\delta}|$, and $g_\delta : M_{\eta_\delta} \cong M_{\nu_\delta}$. But f extends g_δ and restricts to an embedding of $M_{\eta\frown\eta(\delta)}$ into $M_{\nu\restriction\gamma}$, for some $\lambda^+ > \gamma > \delta$ sufficiently large. This contradicts the first property of the construction.

This is possible: Take any $M_{<>} \in \mathbf{K}$ with $|M_{<>}| = \lambda$ for the base case, and take unions at limits. Now if one wants to define $M_{\eta\frown l}$ for $\eta \in {}^{\delta}2$ (assuming by induction that M_ν for all $\nu \in {}^{\leq\delta}2$ have been defined) take any two different extensions, unless $|M_\eta| = \delta$, $\eta_\delta \neq \nu_\delta$, $g_\delta : M_{\eta_\delta} \cong M_{\nu_\delta}$ is an isomorphism, and either $\eta = \eta_\delta$, or $\eta = \nu_\delta$. We show what to do when $\eta = \eta_\delta$. The other case is symmetric.

By failure of amalgamation and categoricity, we know that there exists M^1, M^2 extensions of M_{η_δ}, M_{ν_δ} respectively such that there is *no* $N \in \mathbf{K}$ and $f_\ell : M^\ell \to N$ commuting with g_δ. Now let let $M_{\eta_\delta\frown l}, M_{\nu_\delta\frown l}$ be two different copies of M^1, M^2 respectively. $\square$

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