ABSTRACT ELEMENTARY CLASSES STABLE IN $\aleph_0$

SAHARON SHELAH AND SEBASTIEN VASEY

ABSTRACT. We study abstract elementary classes (AECs) that, in $\aleph_0$, have amalgamation, joint embedding, no maximal models and are stable (in terms of the number of orbital types). Assuming a locality property for types, we prove that such classes exhibit superstable-like behavior at $\aleph_0$. More precisely, there is a superlimit model of cardinality $\aleph_0$ and the class generated by this superlimit has a type-full good $\aleph_0$-frame (a local notion of nonforking independence) and a superlimit model of cardinality $\aleph_1$. We also give a supersimplicity condition under which the locality hypothesis follows from the rest.

1. INTRODUCTION

1.1. **Motivation.** In [She87a] (a revised version of which appears as [She09a, Chapter I], from which we cite), the first author introduced *abstract elementary classes* (AECs): a semantic framework generalizing first-order model theory and also encompassing logics such as $\mathbb{L}_{\omega_1,\omega}(Q)$. The first author studied $\mathrm{PC}_{\aleph_0}$-representable AECs (roughly, AECs which are reducts of a class of models of a first-order theory omitting a countable set of types) and generalized and improved some of his earlier results on $\mathbb{L}_{\omega_1,\omega}$ [She83a, She83b] and $\mathbb{L}_{\omega_1,\omega}(Q)$ [She75].

For example, fix a $\mathrm{PC}_{\aleph_0}$-representable AEC $\mathfrak{K}$ and assume that it is categorical in $\aleph_0$. Assuming $2^{\aleph_0} < 2^{\aleph_1}$ and $1 \leq \mathbb{I}(\mathfrak{K}, \aleph_1) < 2^{\aleph_1}$, the first author shows [She09a, I.3.8] that $\mathfrak{K}$ has amalgamation in $\aleph_0$. Further, [She09a, §I.4, §I.5], it has a lot of structure in $\aleph_0$ and assuming more set-theoretic assumptions as well as few models in $\aleph_2$, $\mathfrak{K}$ has a superlimit model in $\aleph_1$ [She09a, I.5.34, I.5.40]. This means roughly (see Section 2) that there is a saturated model in $\aleph_1$ and that the union of an increasing chain of type ω consisting of saturated models of cardinality $\aleph_1$ is saturated.

The reader can think of the existence of a superlimit in $\aleph_1$ as a step toward showing that the models of cardinality $\aleph_1$ behave in a "superstable-like" way. Indeed several recent works [Van16b, VV17, BV17a, GV] have connected superlimits with other definitions of superstability in AECs, including uniqueness of limit models and local character of orbital splitting.

Another notable consequence of the existence of a superlimit in $\aleph_1$ is that it implies that there is a model of cardinality $\aleph_2$. This ties back to a result of the first author: [She09a, I.3.11]: for a $\text{PC}_{\aleph_0}$-AEC, categoricity in $\aleph_0$ and $\aleph_1$ implies the existence of a model in $\aleph_2$. The argument first establishes, using only categoricity in $\aleph_0$ and few models in $\aleph_1$, that there is a pair M, N of models in $\mathfrak{K}_{\aleph_1}$ such that $M <_{\mathfrak{K}} N$ and then uses, in essence, that (by $\aleph_1$-categoricity) these models are superlimits. In general, the existence of a superlimit model in $\aleph_1$ can be thought of as a key conceptual step toward proving existence of models in higher cardinality.

A difficulty in [She09a, §I.5] is the lack of stability: one can only get that there are $\aleph_1$-many orbital types over countable models. The workaround there is to redefine the ordering (but not the class of models) to obtain a stable class, see [She09a, I.5.29]. If the AEC is "nicely-presented", e.g. a class of models of an $\mathbb{L}_{\omega_1,\omega}(Q)$-sentence or a finitary AEC [HK06], then this difficulty does not occur (see [BL16]): $\aleph_0$-stability follows from few models in $\aleph_1$ and $2^{\aleph_0} < 2^{\aleph_1}$. One can also obtain $\aleph_0$-stability by starting with only countably-many models in $\aleph_1$ [BLS15, 3.18]. Finally, it is worth noting that (assuming amalgamation and joint embedding in $\aleph_0$), $\aleph_0$-stability is upward absolute for $\text{PC}_{\aleph_0}$-AECs [LS].

1.2. **Main results.** The bottom line is that $\aleph_0$-stability holds in several cases of interests. In fact, there are no known examples which (under $2^{\aleph_0} < 2^{\aleph_1}$) are categorical in $\aleph_0$, have few models in $\aleph_1$, and are not $\aleph_0$-stable (see [BLS15, Question 3.15]). Thus in the present paper, we *start* with stability in $\aleph_0$ (and amalgamation and categoricity in $\aleph_0$). Our goal is to say as much as we can on the structure of the class, in particular to get superstable-like behavior in $\aleph_0$ and $\aleph_1$, *without* assuming a non-ZFC hypothesis or $\mathbb{I}(\mathfrak{K}, \aleph_1) < 2^{\aleph_1}$. Instead, we assume for our main result that orbital types over countable models are determined by their finite restrictions (we call this $(< \aleph_0, \aleph_0)$-locality here, this is often called tameness elsewhere, see e.g. [GV06]). This is known to hold for several classes of interests, including classes of models of an $\mathbb{L}_{\omega_1,\omega}$-sentence and more generally finitary AECs, see Example 5.10. We prove the following:

Theorem 1.1. Let $\mathfrak{K}$ be an AEC with $\text{LS}(\mathfrak{K}) = \aleph_0$ and countable vocabulary. Assume that $\mathfrak{K}$ is categorical in $\aleph_0$, $\mathfrak{K}$ is $(< \aleph_0, \aleph_0)$-local, $\mathfrak{K}$ has amalgamation and no maximal models in $\aleph_0$ and $\mathfrak{K}$ is stable in $\aleph_0$. Then:

(1) There is a good $\aleph_0$-frame on $\mathfrak{K}_{\aleph_0}$.
(2) There is a superlimit model of cardinality $\aleph_1$.

The good $\aleph_0$-frame (or the superlimit in $\aleph_1$) imply the nontrivial corollary that $\mathfrak{K}$ has a model of cardinality $\aleph_2$ [She09a, II.4.13]. This consequence also follows from a theorem of the second author [Vasb, 12.1] (which however does *not* give a good $\aleph_0$-frame or a superlimit in $\aleph_1$). The conclusion that there is a superlimit model in $\aleph_1$ seems new, even for finitary AECs.

Note that Theorem 1.1 does *not* need to assume that $\mathfrak{K}$ is $\text{PC}_{\aleph_0}$ (rather, we can prove it from the hypotheses, see Theorem 4.2 and Corollary 4.7).

We also show that the assumption of categoricity in $\aleph_0$ is not really needed: without assuming it (and also without assuming locality), one can find a superlimit in $\aleph_0$ and change to the class generated by that superlimit, which will be categorical

in $\aleph_0$. In fact, we prove (Theorem 4.4) that one can characterize brimmed models (also called limit models in the literature) as those that are homogeneous for orbital types. This has as immediate consequence that the brimmed model of cardinality $\aleph_0$ is superlimit (Corollary 4.6). This sheds light on an argument of Lessmann [Les05] and answers a question of Fred Drueck (see footnote 3 on [Dru13, p. 25]. The argument works more generally assuming only density of amalgamation bases, as in [SV99].

Finally, it is natural to ask whether the locality hypothesis in Theorem 1.1 is really needed[1]. In fact we do not even know whether the existence of a good $\aleph_0$-frame implies $(< \aleph_0, \aleph_0)$-locality:

Question 1.2. Let $\mathfrak{K}$ be an AEC with $\mathrm{LS}(\mathfrak{K}) = \aleph_0$. If there is a good $\aleph_0$-frame on $\mathfrak{K}$, is $\mathfrak{K}$ $(< \aleph_0, \aleph_0)$-local?

In Section 6, we give a partial answer: any AEC that is $\aleph_0$-stable, $\aleph_0$-categorical, and *supersimple* (in a sense generalizing that of homogeneous model theory [BL03]) is $(< \aleph_0, \aleph_0)$-local. This generalizes the proof of [BHH$^+$14, 5.2], which shows that quasiminimal pregeometry classes are $(< \aleph_0, \aleph_0)$-local (see also [Vasa]). Supersimple $\aleph_0$-stable AECs are also much more general than the Finite U-rank (FUR) classes of Hyttinen and Kangas [HK16] (we thank Will Boney for pointing us to that reference).

1.3. **Notes.** This paper was started while the second author was working on a Ph.D. thesis under the direction of Rami Grossberg at Carnegie Mellon University and he would like to thank Professor Grossberg for his guidance and assistance in his research in general and in this work specifically. We also thank Will Boney, Marcos Mazari Armida, and the referee, for their valuable comments on earlier versions of this paper.

Note that at the beginning of several sections, we make global hypotheses assumed throughout the section.

2. Preliminaries

We assume familiarity with the basics of AECs, as presented for example in [Gro02, Bal09], or the first three sections of Chapter I together with the first section of Chapter II in [She09a]. We also assume familiarity with good frames (see [She09a, Chapter II] or [BV17b]; it would help the reader to have a copy of both available during the reading of Section 3). This section mostly fixes the notation that we will use.

Given a τ-structure M, we write $|M|$ for its universe and $\|M\|$ for its cardinality. We may abuse notation and write e.g $a \in M$ instead of $a \in |M|$. We may even write $\bar{a} \in M$ instead of $\bar{a} \in {}^{<\omega}|M|$.

We write $\mathfrak{K} = (K, \leq_{\mathfrak{K}})$ for an AEC. We may abuse notation and write $M \in \mathfrak{K}$ instead of $M \in K$. For a cardinal λ, we write $\mathfrak{K}_\lambda$ for the AEC restricted to its

---

models of size λ. As shown in [She09a, II.1], any AEC is uniquely determined by its restriction $\mathfrak{K}_{\leq \mathrm{LS}(\mathfrak{K})}$.

When we say that $M \in \mathfrak{K}$ is an *amalgamation base*, we mean (as in [SV99]) that it is an amalgamation base in $\mathfrak{K}_{\|M\|}$, i.e. we do *not* require that larger models can be amalgamated.

For $M_0 \in \mathfrak{K}$, we say that M is *universal over M_0* if $M_0 \leq_{\mathfrak{K}} M$ and for any $N \in \mathfrak{K}$ with $M_0 \leq_{\mathfrak{K}} N$, if $\|N\| \leq \|M_0\| + \mathrm{LS}(\mathfrak{K})$, there exists $f : N \xrightarrow[M_0]{} M$ (usually we will require also that $\|M\| = \|M_0\|$). We say that M is *(λ, δ)-brimmed over M_0* (often also called (λ, δ)-limit e.g. in [SV99, GVV16]) if $\delta < \lambda^+$ is a limit ordinal, $M_0 \in \mathfrak{K}_\lambda$, and there exists an increasing continuous chain $\langle N_i : i \leq \delta \rangle$ of members of $\mathfrak{K}_\lambda$ such that N_0 is universal over M_0, $N_\delta = M$, and N_{i+1} is universal over N_i for all $i < \delta$. We say that M is *brimmed over M_0* if it is $(\|M\|, \delta)$-brimmed over M_0 for some limit $\delta < \|M\|^+$. We say that M is *brimmed* if it is brimmed over some M_0.

We say that $M \in \mathfrak{K}$ is *superlimit* [She09a, I.3.3] if, letting $\lambda := \|M\|$, we have that $\lambda \geq \mathrm{LS}(\mathfrak{K})$, M is universal in $\mathfrak{K}_\lambda$ (i.e. any $M' \in \mathfrak{K}_\lambda$ embeds into M), M is not maximal, and whenever $\delta < \lambda^+$ is limit, $\langle M_i : i < \delta \rangle$ is increasing with $M_i \cong M$ for all $i < \delta$, then $\bigcup_{i < \delta} M_i \cong M$.

The following notion of types already appears in [She87b]. It is called Galois types by many, but we prefer the term *orbital types* here. They are the same types that are defined in [She09a, II.1.9], but we also define them over sets. As pointed out in [Vas16b, Section 2], this causes no additional difficulties. The following technical point is important: when the AEC does *not* have amalgamation, we may want to compute orbital types only in the subclass of *amalgamation bases* in $\mathfrak{K}$ (as in [SV99]). Thus we allow orbital types to be computed in a subclass of $\mathfrak{K}$ in the definition.

Definition 2.1. Fix an AEC $\mathfrak{K}$ and a subclass $\mathfrak{K}^*$ of $\mathfrak{K}$, closed under isomorphisms.

  (1) We say $(A, N_1, \bar{b}_1) E_{\mathrm{at}}^{\mathfrak{K}^*} (A, N_2, \bar{b}_2)$ if:
      (a) For $\ell = 1, 2$, $N_\ell \in \mathfrak{K}^*$, $A \subseteq |N_\ell|$, and $\bar{b}_\ell \in {}^{<\infty}|N_\ell|$.
      (b) There exists $N \in \mathfrak{K}^*$ and $f_\ell : N_\ell \xrightarrow[A]{} N$, $\ell = 1, 2$, such that $f_1(\bar{b}_1) = \bar{b}_2$.
  (2) $E_{\mathrm{at}}^{\mathfrak{K}^*}$ is a reflexive and symmetric relation. Let $E^{\mathfrak{K}^*}$ be its transitive closure.
  (3) Let $\mathbf{ortp}_{\mathfrak{K}^*}(\bar{b}, A, N)$ be the $E^{\mathfrak{K}^*}$-equivalence class of $(\bar{b}, A, N)$.
  (4) Define $\mathscr{S}_{\mathfrak{K}^*}(A, N)$, $\mathscr{S}_{\mathfrak{K}^*}(M)$, $\mathscr{S}_{\mathfrak{K}^*}^{<\omega}(M)$, etc. for the Stone spaces of orbital types. See for example [Vas16b, Section 2].
  (5) When $\mathfrak{K}^* = \mathfrak{K}$, we omit it.

Let us say that an AEC $\mathfrak{K}$ is *stable in λ* if for any $M \in \mathfrak{K}_\lambda$, $|\mathscr{S}(M)| \leq \lambda$. This makes sense in any AEC, and is quite well-behaved assuming amalgamation and no maximal models (since then it is known that one can build universal extensions). We will often work in the following axiomatic setup, a slight weakening where full amalgamation is not assumed. This comes from the context derived in [SV99]:

Definition 2.2. Let $\mathfrak{K}$ be an AEC and let λ be a cardinal. We say that $\mathfrak{K}$ is *nicely stable in λ* (or *nicely λ-stable*) if:

  (1) $\mathrm{LS}(\mathfrak{K}) \leq \lambda$.
  (2) $\mathfrak{K}_\lambda \neq \emptyset$.

(3) $\mathfrak{K}$ has joint embedding in λ.

(4) Density of amalgamation bases: For any $M \in \mathfrak{K}_\lambda$, there exists $N \in \mathfrak{K}_\lambda$ such that $M \leq_{\mathfrak{K}} N$ and N is an amalgamation base (in $\mathfrak{K}_\lambda$).

(5) Existence of universal extensions: For any amalgamation base $M \in \mathfrak{K}_\lambda$, there exists an amalgamation base $N \in \mathfrak{K}_\lambda$ such that $M <_{\mathfrak{K}} N$ and N is universal over M.

(6) Any brimmed model in $\mathfrak{K}_\lambda$ is an amalgamation base.

We say that $\mathfrak{K}$ is *very nicely stable in* λ if in addition it has amalgamation in λ.

Remark 2.3. An AEC $\mathfrak{K}$ is very nicely stable in λ if and only if $\mathrm{LS}(\mathfrak{K}) \leq \lambda$, $\mathfrak{K}_\lambda \neq \emptyset$, $\mathfrak{K}$ is stable in λ, and $\mathfrak{K}_\lambda$ has amalgamation, joint embedding, and no maximal models. In particular, stability is a consequence of the existence of universal extensions in Definition 2.2.

We will repeatedly use the following fact [SV99, 1.3.6].

Fact 2.4. Let $\mathfrak{K}$ be nicely stable in λ and let $M_0, M_1, M_2 \in \mathfrak{K}_\lambda$. Let $\delta_1, \delta_2 < \lambda^+$ be limit ordinals such that $\mathrm{cf}(\delta_1) = \mathrm{cf}(\delta_2)$.

(1) If M_ℓ is (λ, δ_ℓ)-brimmed over M_0, for $\ell = 1, 2$, then $M_1 \cong_{M_0} M_2$.

(2) If M_ℓ is (λ, δ_ℓ)-brimmed, for $\ell = 1, 2$, then $M_1 \cong M_2$.

Proof. The first is a straightforward back and forth argument and the second follows from the first using joint embedding. $\square$

Remark 2.5. Uniqueness of brimmed models when $\mathrm{cf}(\delta_1) \neq \mathrm{cf}(\delta_2)$ is a much harder property to establish, akin to superstability. See for example [SV99, Van06, GVV16, Van16a]. However when $\lambda = \aleph_0$ we automatically have that $\mathrm{cf}(\delta_1) = \mathrm{cf}(\delta_2) = \omega$.

Good frames were first defined by the first author in [She09a, Chapter II]. The idea is to provide a localized (i.e. only for base models of a given size λ) axiomatization of a forking-like notion for a "nice enough" set of 1-types. These axioms are similar to the properties of first-order forking in a superstable theory. Jarden and the first author (in [JS13]) later gave a slightly more general definition, not assuming the existence of a superlimit model and dropping some of the redundant clauses. We will make use of good frames for types of finite length (not just length one). Their definition is just like for types of length one, we call them *good* $(< \omega, \lambda)$-*frames*. For the convenience of the reader, we give the full definition from [BV17b, 3.8] here:

Definition 2.6. Let λ be an infinite cardinal. A *good* $(< \omega, \lambda)$-*frame* is a triple $(\mathfrak{K}, \underset{\smile}{\downarrow}, \mathscr{S}_{\mathfrak{s}}^{\mathrm{bs}})$ satisfying, where:

(1) $\mathfrak{K}$ is an abstract elementary class with $\lambda \geq \mathrm{LS}(\mathfrak{K})$, $\mathfrak{K}_\lambda \neq \emptyset$.

(2) $\mathscr{S}_{\mathfrak{s}}^{\mathrm{bs}} \subseteq \bigcup_{M \in \mathfrak{K}_\lambda} \mathscr{S}^{<\omega}(M)$. Moreover, if $\mathbf{ortp}(\langle a_i : i < n \rangle / M; N) \in \mathscr{S}_{\mathfrak{s}}^{\mathrm{bs}}(M)$, then $a_i \notin M$ for all $i < n$.

(3) $\underset{\smile}{\downarrow}$ is a relation on quadruples of the form $(M_0, M_1, \bar{a}, N)$, where $M_0 \leq_{\mathfrak{K}} M_1 \leq_{\mathfrak{K}} N$, $\bar{a} \in {}^{<\omega}N$, and M_0, M_1, N are all in K_λ. We write $\underset{\smile}{\downarrow}(M_0, M_1, \bar{a}, N)$ or $\bar{a} \overset{N}{\underset{M_0}{\downarrow}} M_1$ instead of $(M_0, M_1, a, N) \in \underset{\smile}{\downarrow}$.

(4) The following properties hold:

(a) <u>Invariance</u>: If $f : N \cong N'$ and $\bar{a} \underset{M_0}{\overset{N}{\downarrow}} M_1$, then $f(\bar{a}) \underset{f[M_0]}{\overset{N'}{\downarrow}} f[M_1]$. If
$\mathbf{ortp}(\bar{a}/M_1; N) \in \mathscr{S}_{\mathfrak{s}}^{\mathrm{bs}}(M_1)$, then $\mathbf{ortp}(f(\bar{a})/f[M_1]; N') \in \mathscr{S}_{\mathfrak{s}}^{\mathrm{bs}}(f[M_1])$.

(b) <u>Monotonicity</u>: If $\bar{a} \underset{M_0}{\overset{N}{\downarrow}} M_1$, $\bar{a}'$ is a subsequence of $\bar{a}$, $M_0 \leq_{\mathfrak{K}} M_0' \leq_{\mathfrak{K}} M_1' \leq_{\mathfrak{K}} M_1 \leq_{\mathfrak{K}} N' \leq_{\mathfrak{K}} N \leq_{\mathfrak{K}} N''$ with $\bar{a}' \in N'$, and $N'' \in K_{\mathcal{F}}$, then $\bar{a}' \underset{M_0'}{\overset{N'}{\downarrow}} M_1'$ and $\bar{a}' \underset{M_0'}{\overset{N''}{\downarrow}} M_1'$. If $\mathbf{ortp}(\bar{a}/M_1; N) \in \mathscr{S}_{\mathfrak{s}}^{\mathrm{bs}}(M_1)$ and $\bar{a}'$ is a subsequence of $\bar{a}$, then $\mathbf{ortp}(\bar{a}'/M_1; N) \in \mathscr{S}_{\mathfrak{s}}^{\mathrm{bs}}(M_1)$. [This property and the previous one show that $\underset{}{\downarrow}$ is really a relation on types. Thus if $p \in \mathscr{S}^{<\omega}(M_1)$ is a type, we say p *does not fork over* M_0 if $\bar{a} \underset{M_0}{\overset{N}{\downarrow}} M_1$ for some (equivalently any) $\bar{a}$ and N such that $p = \mathbf{ortp}(\bar{a}/M_1; N)$. Note that this depends on $\mathfrak{s}$, but $\mathfrak{s}$ will always be clear from context.]

(c) <u>Nonforking types are basic</u>: If $\bar{a} \underset{M}{\overset{N}{\downarrow}} M$, then $\mathbf{ortp}(\bar{a}/M; N) \in \mathscr{S}_{\mathfrak{s}}^{\mathrm{bs}}(M)$.

(d) K_λ has amalgamation, joint embedding, and no maximal models.

(e) bs-Stability: $|\mathscr{S}_{\mathfrak{s}}^{\mathrm{bs}}(M)| \leq \|M\|$ for all $M \in K_\lambda$.

(f) <u>Density of basic types</u>: If $M <_{\mathfrak{K}} N$ are in K_λ, then there is $a \in N$ such that $\mathbf{ortp}(a/M; N) \in \mathscr{S}_{\mathfrak{s}}^{\mathrm{bs}}(M)$.

(g) <u>Existence of nonforking extension</u>: If $m \leq n < \omega$, $p \in \mathscr{S}_{\mathfrak{s}}^{\mathrm{bs}}(M) \cap \mathscr{S}^m(M)$, $N \geq_{\mathfrak{K}} M$ is in K_λ, then there is some $q \in \mathscr{S}_{\mathfrak{s}}^{\mathrm{bs}}(N) \cap \mathscr{S}^n(M)$ that does not fork over M and extends p.

(h) <u>Uniqueness</u>: If $p, q \in \mathscr{S}^{<\omega}(N)$ do not fork over M and $p \restriction M = q \restriction M$, then $p = q$.

(i) <u>Symmetry</u>: If $\bar{a}_1 \underset{M_0}{\overset{N}{\downarrow}} M_2$, $\bar{a}_2 \in {}^{<\alpha}M_2$, and $\mathbf{ortp}(\bar{a}_2/M_0; N) \in \mathscr{S}_{\mathfrak{s}}^{\mathrm{bs}}(M_0)$, then there is M_1 containing $\bar{a}_1$ and $N' \geq_{\mathfrak{K}} N$ such that $\bar{a}_2 \underset{M_0}{\overset{N'}{\downarrow}} M_1$.

(j) <u>Local character</u>: If δ is a regular cardinal, $\langle M_i \in K_\lambda : i \leq \delta \rangle$ is increasing continuous, and $p \in \mathscr{S}_{\mathfrak{s}}^{\mathrm{bs}}(M_\delta)$ is such that $\ell(p) < \delta$, then there exists $i < \delta$ such that p does not fork over M_i.

(k) <u>Continuity</u>: If δ is a limit ordinal, $\langle M_i \in K_\lambda : i \leq \delta \rangle$ and $\langle \alpha_i < \alpha : i \leq \delta \rangle$ are increasing and continuous, and $p_i \in \mathscr{S}_{\mathfrak{s}}^{\mathrm{bs}}(M_i)$ for $i < \delta$ are such that $j < i < \delta$ implies $p_j = p_i \restriction M_j$, then there is some $p \in \mathscr{S}_{\mathfrak{s}}^{\mathrm{bs}}(M_\delta)$ such that for all $i < \delta$, $p_i = p \restriction M_i$. Moreover, if each p_i does not fork over M_0, then neither does p.

(l) <u>Transitivity</u>: If $M_0 \leq_{\mathfrak{K}} M_1 \leq_{\mathfrak{K}} M_2$, $p \in \mathscr{S}(M_2)$ does not fork over M_1 and $p \restriction M_1$ does not fork over M_0, then p does not fork over M_0.

A *good λ-frame* is defined similarly, except we require all types to be types of one element (i.e. they are in $\mathscr{S}(M)$ instead of $\mathscr{S}^{<\omega}(M)$).

We say that an AEC $\mathfrak{K}$ *has a good $(< \omega, \lambda)$-frame* if there is a good $(< \alpha, \mathcal{F})$-frame where K is the underlying AEC (and similarly for good$^-$).

Notation 2.7. If $\mathfrak{s} = (\mathfrak{K}, \underset{\mathfrak{s}}{\downarrow}, \mathscr{S}_\mathfrak{s}^{\mathrm{bs}})$ is a good-$(< \omega, \lambda)$-frame (or a good λ-frame), write $\downarrow := \underset{\mathfrak{s}}{\downarrow}$. Also write $\mathfrak{K}^\mathfrak{s}$ for $\mathfrak{K}$ and $\mathfrak{K}_\mathfrak{s} = \mathfrak{K}_\lambda$. We will also write $M \leq_\mathfrak{s} N$ as a shortcut for $M \leq_\mathfrak{K} N$ and $M, N \in \mathfrak{K}_\mathfrak{s}$ $(= \mathfrak{K}_\lambda)$.

Remark 2.8. The reader might wonder about the reasons for having a special class of basic types. Following [She09a, Definition III.9.2], let us call a good frame *type-full* if the basic types are all the nonalgebraic types. There are no known examples of a good λ-frame which which cannot be extended to a type-full one. However a construction of good frames of the first author [She09a, II.3] builds a non type-full good frame and it is not clear that it can be extended to a type-full one until a lot more machinery has been developed. Thus it can be easier to build a good frame than to build a type-full one, and most results about frames already hold in the non-type-full context. That being said, readers would not miss the essence of the present paper if they assumed that all the frames here were type-full.

Remark 2.9. Any good λ-frame (i.e. for types of length one) extends to a good $(< \omega, \lambda)$-frame (using independent sequences, see [She09a, III.9.4]) or [BV17b, 5.8]. This frame will however *not* be type-full.

3. Weak nonforking amalgamation

In this section, we work in a good λ-frame and study a natural weak version of nonforking amalgamation, $\mathrm{LWNF}_\mathfrak{s}$. The goal is to obtain a natural criteria for proving the existence of a superlimit in $\aleph_1$ and also prepare the ground for the proof of symmetry in the good frame built in Section 5. The main results are the existence property (Theorem 3.11) and how the symmetry property of $\mathrm{LWNF}_\mathfrak{s}$ is connected to $\mathfrak{s}$ being good$^+$ (Theorem 3.15). Throughout this section, we assume:

Hypothesis 3.1.

(1) $\mathfrak{s} = (\mathfrak{K}, \downarrow, \mathscr{S}_\mathfrak{s}^{\mathrm{bs}})$ is a fixed good $(< \omega, \lambda)$-frame, except that it may not satisfy the symmetry axiom.
(2) $\mathfrak{K}$ is categorical in λ.

Remark 3.2. In this section, λ is allowed to be uncountable. However the case $\lambda = \aleph_0$ is the one that will interest us in the next sections.

The reason for not assuming symmetry is that we will use some of the results of this section to *prove* that the symmetry axiom holds of a certain nonforking relation in Section 5.

We will use:

Fact 3.3 (II.4.3 in [She09a]). Let $\delta < \lambda^+$ be a limit ordinal divisible by λ. Let $\langle M_i : i \leq \delta \rangle$ be increasing continuous in $\mathfrak{K}_\mathfrak{s}$. If for any $i < \delta$ and any $p \in \mathscr{S}_\mathfrak{s}^{\mathrm{bs}}(M_i)$, there exists λ-many $j \in [i, \delta)$ such that the nonforking extension of p to M_j is realized in M_{j+1}, then M_δ is brimmed over M_0.

To understand the definition below, it may be helpful to think of $\mathfrak{s}$ as type-full. Then $\mathrm{LWNF}_\mathfrak{s}(M_0, M_1, M_2, M_3)$ holds if and only if the type of any finite subsequences of M_1 over M_2 does not fork over M_0 (M_3 is the ambient model). Thus $\mathrm{LWNF}_\mathfrak{s}$ is an attempt to extend nonforking to types of infinite sequences so that it keeps

a strong finite character property. In the present paper, $\text{LWNF}_{\mathfrak{s}}$ will be a helpful technical tool but it is not clear that it has the uniqueness property (in contrast with the relation NF from [She09a, §II.6] or [JS13, §5], which will have the uniqueness property but requires more assumptions on the good frame). If $\text{LWNF}_{\mathfrak{s}}$ *does* have the uniqueness property, this has strong consequence on the structure of the frame, see Theorem 3.18.

Definition 3.4. Define the following 4-ary relations on $\mathfrak{K}_{\mathfrak{s}}$:

(1) $\text{LWNF}_{\mathfrak{s}}(M_0, M_1, M_2, M_3)$ if and only if $M_0 \leq_{\mathfrak{s}} M_\ell \leq_{\mathfrak{s}} M_3$ for $\ell = 1, 2$ and for any $\bar{b} \in {}^{<\omega}|M_1|$, if $\mathbf{ortp}(\bar{b}, M_2, M_3)$ and $\mathbf{ortp}(\bar{b}, M_0, M_3)$ are basic then $\mathbf{ortp}(\bar{b}, M_2, M_3)$ does not fork over M_0.
(2) $\text{RWNF}_{\mathfrak{s}}(M_0, M_1, M_2, M_3)$ if and only if $\text{LWNF}_{\mathfrak{s}}(M_0, M_2, M_1, M_3)$.
(3) $\text{WNF}_{\mathfrak{s}}(M_0, M_1, M_2, M_3)$ if and only if both $\text{LWNF}_{\mathfrak{s}}(M_0, M_1, M_2, M_3)$ and $\text{RWNF}_{\mathfrak{s}}(M_0, M_1, M_2, M_3)$.

When $\mathfrak{s}$ is clear from context, we write LWNF, RWNF, and WNF.

Remark 3.5. WNF stands for weak nonforking amalgamation, and LWNF, RWNF stand for left (respectively right) weak nonforking amalgamation.

The following result often comes in handy.

Lemma 3.6. Let $\delta < \lambda^+$ be a limit ordinal. Let $\langle M_i : i \leq \delta \rangle$, $\langle N_i : i \leq \delta \rangle$ be increasing continuous in $\mathfrak{K}_{\mathfrak{s}}$. Assume that for each $i \leq j < \delta$, we have that $\text{LWNF}(M_i, N_i, M_j, N_j)$. If for each $i < \delta$, N_i realizes all the basic types over M_i, then N_δ realizes all the basic types over M_δ.

Proof. Let $p \in \mathscr{S}_{\mathfrak{s}}^{\text{bs}}(M_\delta)$. By local character, there exists $i < \delta$ such that p does not fork over M_i. By assumption, there exists $a \in |N_i|$ such that $p \upharpoonright M_i = \mathbf{ortp}(a, M_i, N_i)$. Because for all $j \in [i, \delta)$, $\text{LWNF}(M_i, N_i, M_j, N_j)$, we have by continuity that $\mathbf{ortp}(a, M_\delta, N_\delta)$ does not fork over M_i, hence by uniqueness it must be equal to p. Therefore a realizes p, as needed. $\square$

Next, we investigate the properties of LWNF. We are especially interested in the symmetry property: whether LWNF is equal to RWNF. To understand it better, we consider the following ordering, defined similarly to $\leq_{\lambda^+}^*$ from [She09a, II.7.2]:

Definition 3.7. For $R \in \{\text{LWNF}, \text{RWNF}, \text{WNF}\}$, define a relation $\leq_R$ on $\mathfrak{K}_{\lambda^+}$ as follows. For $M^0, M^1 \in \mathfrak{K}_{\lambda^+}$, $M^0 \leq_R M^1$ if and only if there exists increasing continuous resolutions $\langle M_i^\ell \in \mathfrak{K}_\lambda : i < \lambda^+ \rangle$ of M^ℓ for $\ell = 0, 1$ such that for all $i < j < \lambda^+$, $R(M_i^0, M_i^1, M_j^0, M_j^1)$.

The following is a straightforward "catching your tail argument", see the proof of [Vas17, 4.6]. Roughly, it says that if $M \leq_{\mathfrak{K}} N$ ($\leq_{\mathfrak{K}}$ is the usual order on $\mathfrak{K}$), then we can find a resolution of M and N so that the pieces are in *left* weak nonforking amalgamation.

Fact 3.8. Let $M, N \in \mathfrak{K}_{\lambda^+}$. If $M \leq_{\mathfrak{K}} N$, then $M \leq_{\text{LWNF}} N$.

Whether $M \leq_{\text{RWNF}} N$ can be concluded as well seems to be a much more complicated question, and in fact is equivalent to $\mathfrak{s}$ being good$^+$ (Theorem 3.15), a

weakening of symmetry. Observe that an increasing union of a $\leq_{\mathrm{RWNF}}$-increasing chain of saturated models is saturated:

Lemma 3.9. Let $\delta < \lambda^{++}$ be a limit ordinal. If $\langle M_i : i < \delta \rangle$ is a $\leq_{\mathrm{RWNF}}$-increasing sequence of saturated models in $\mathfrak{K}_{\lambda^+}$, then $\bigcup_{i < \delta} M_i$ is saturated.

Proof. Without loss of generality, $\delta = \mathrm{cf}(\delta) < \lambda^+$. Let $M_\delta := \bigcup_{i < \delta} M_i$. We build $\langle M_{i,j} : i \leq \delta, j \leq \lambda^+ \rangle$ such that:

  (1) For any $i \leq \delta$, $M_{i,\lambda^+} = M_i$.
  (2) For any $i < \delta, j < \lambda^+$, $M_{i,j} \in \mathfrak{K}_{\mathfrak{s}}$.
  (3) For any $i \leq \delta$, $\langle M_{i,j} : j < \lambda^+ \rangle$ is increasing and continuous.
  (4) For any $j \leq \lambda^+$, $\langle M_{i,j} : i < \delta \rangle$ is increasing and $M_{\delta,j} = \bigcup_{i < \delta} M_{i,j}$.
  (5) For any $i_1 < i_2 \leq \delta$, $j_1 < j_2 \leq \lambda^+$, M_{i_2,j_2} realizes all the types in $\mathscr{S}_{\mathfrak{s}}^{\mathrm{bs}}(M_{i_1,j_1})$.

This is easy to do. Now for each $i_1 < i_2 < \delta$, we have by assumption that $M_{i_1} \leq_{\mathrm{RWNF}} M_{i_2}$. Thus the set C_{i_1,i_2} of $j < \lambda^+$ such that for all $j' \in [j, \lambda^+)$, $\mathrm{RWNF}(M_{i_1,j}, M_{i_2,j}, M_{i_1,j'}, M_{i_2,j'})$ is a club (that it is closed follows from the local character and continuity axioms of good frames). Therefore $C := \bigcap_{i_1 < i_2 < \delta} C_{i_1,i_2}$ is also a club. Hence by renaming without loss of generality for all $i_1 < i_2 < \delta$ and all $j \leq j' < \lambda^+$, $\mathrm{RWNF}(M_{i_1,j}, M_{i_2,j}, M_{i_1,j'}, M_{i_2,j'})$.

Now let $N \leq_{\mathfrak{K}} M_\delta$ be such that $N \in \mathfrak{K}_\lambda$. We want to see that any type over N is realized in M_δ. By Fact 3.3, it is enough to show that any *basic* type over N is realized in M_δ.

Let $j < \lambda^+$ be big-enough such that $N \leq_{\mathfrak{K}} M_{\delta,j}$. It is enough to see that any basic type over $M_{\delta,j}$ is realized in $M_{\delta,j+1}$. To see this, use Lemma 3.6 with $\langle M_i : i \leq \delta \rangle$, $\langle N_i : i \leq \delta \rangle$ there standing for $\langle M_{i,j} : i \leq \delta \rangle$, $\langle M_{i,j+1} : i \leq \delta \rangle$ here. We know that for each $i \leq i' < \delta$, $\mathrm{RWNF}(M_{i,j}, M_{i',j}, M_{i,j+1}, M_{i',j+1})$ and therefore $\mathrm{LWNF}(M_{i,j}, M_{i,j+1}, M_{i',j}, M_{i',j+1})$. Thus the hypotheses of Lemma 3.6 are satisfied. $\square$

The next fact will be used to prove the existence property of LWNF. Its proof is a direct limit argument similar to e.g. [GVV16, 5.3]. Note that the symmetry axiom is *not* needed.

Fact 3.10. Let $\alpha < \lambda^+$. Let $\langle M_i : i \leq \alpha \rangle$ be $\leq_{\mathfrak{s}}$-increasing continuous and let $\langle \bar{a}_i : i < \alpha \rangle$ be given such that $\bar{a}_i \in M_{i+1}$ for all $i < \alpha$ and $\mathbf{ortp}(\bar{a}_i, M_i, M_{i+1}) \in \mathscr{S}_{\mathfrak{s}}^{\mathrm{bs}}(M_i)$.

There exists $\langle N_i : i \leq \alpha \rangle \leq_{\mathfrak{s}}$-increasing continuous such that:

  (1) $M_i <_{\mathfrak{s}} N_i$ for all $i \leq \alpha$.
  (2) $\mathbf{ortp}(\bar{a}_i, N_i, N_{i+1})$ does not fork over M_i.

We can now list some basic properties of weak nonforking amalgamation.

Theorem 3.11. Let $R \in \{\mathrm{LWNF}, \mathrm{RWNF}, \mathrm{WNF}\}$.

  (1) Invariance: If $R(M_0, M_1, M_2, M_3)$ and $f : M_3 \cong M_3'$, then $R(f[M_0], f[M_1], f[M_2], M_3')$.
  (2) Monotonicity: If $R(M_0, M_1, M_2, M_3)$ and $M_0 \leq_{\mathfrak{s}} M_\ell' \leq_{\mathfrak{s}} M_\ell$ for $\ell = 1, 2$, then $R(M_0, M_1', M_2', M_3)$.

(3) Ambient monotonicity: If $R(M_0, M_1, M_2, M_3)$ and $M_3 \leq_{\mathfrak{s}} M_3'$, then $R(M_0, M_1, M_2, M_3')$. If $M_3'' \leq_{\mathfrak{s}} M_3$ contains $|M_1| \cup |M_2|$, then $R(M_0, M_1, M_2, M_3'')$.

(4) Continuity: If $\delta < \lambda^+$ is a limit ordinal and $\langle M_i^\ell : i \leq \delta \rangle$ are increasing continuous for $\ell < 4$ with $R(M_i^0, M_i^1, M_i^2, M_i^3)$ for each $i < \delta$, then $R(M_\delta^0, M_\delta^1, M_\delta^2, M_\delta^3)$.

(5) Long transitivity: If $\alpha < \lambda^+$ is an ordinal, $\langle M_i : i \leq \alpha \rangle$, $\langle N_i : i \leq \alpha \rangle$ are increasing continuous and $\mathrm{LWNF}(M_i, N_i, M_{i+1}, N_{i+1})$ for all $i < \alpha$, then $\mathrm{LWNF}(M_0, N_0, M_\alpha, N_\alpha)$.

(6) Existence: If $R \neq \mathrm{WNF}$, $M_0 \leq_{\mathfrak{s}} M_\ell$, $\ell = 1, 2$, then there exists $M_3 \in \mathfrak{K}_\lambda$ and $f_\ell : M_\ell \xrightarrow[M_0]{} M_3$ such that $R(M_0, f_1[M_1], f_2[M_2], M_3)$.

Proof. Invariance and the monotonicity properties are straightforward to prove. Continuity and long transitivity follow directly from the local character, continuity, and transitivity properties of good frames. We prove existence via the following claim:

<u>Claim</u>: There exists $N_0, N_1, N_2, N_3 \in \mathfrak{K}_{\mathfrak{s}}$ such that $\mathrm{LWNF}(N_0, N_1, N_2, N_3)$ and N_ℓ is brimmed over N_0 for $\ell = 1, 2$.

Existence easily follows from the claim: given $M_0 \leq_{\mathfrak{s}} M_\ell$, $\ell = 1, 2$, there is (by categoricity in λ) an isomorphism $f : M_0 \cong N_0$ and (by universality of brimmed models) embeddings $f_\ell : M_\ell \to N_\ell$ extending f for $\ell = 1, 2$. After some renaming, we obtain the desired LWNF-amalgam. To obtain an RWNF-amalgam, reverse the role of M_1 and M_2.

<u>Proof of Claim</u>: Let $\delta := \lambda \cdot \lambda$. We choose $(\bar{M}^\alpha, \bar{a}^\alpha)$ by induction on $\alpha \leq \delta$ such that:

(1) $\bar{M}^\alpha = \langle M_i^\alpha : i \leq \alpha \rangle$ is $\leq_{\mathfrak{s}}$-increasing continuous.

(2) $\bar{a}^\alpha = \langle \bar{a}_i : i < \alpha \rangle$, and $\bar{a}_i \in M_{i+1}^\alpha$ for all $i < \alpha$.

(3) For all $i < \alpha$, $\mathbf{ortp}(\bar{a}_i^\alpha, M_i^\alpha, M_{i+1}^\alpha) \in \mathscr{S}_{\mathfrak{s}}^{\mathrm{bs}}(M_i^\alpha)$.

(4) For each $i \leq \delta$, $\langle M_i^\alpha : \alpha \in [i, \delta] \rangle$ is $<_{\mathfrak{s}}$-increasing continuous.

(5) For each $i < \delta$ and each $\alpha \in (i, \delta]$, $\mathbf{ortp}(\bar{a}_i, M_i^\alpha, M_{i+1}^\alpha)$ does not fork over M_i^i.

(6) If $p \in \mathscr{S}_{\mathfrak{s}}^{\mathrm{bs}}(M_i^\alpha)$ for $i \leq \alpha < \delta$, then for λ-many $\beta \in [\alpha, \delta)$, $\mathbf{ortp}(\bar{a}_\beta, M_\beta^{\beta+1}, M_{\beta+1}^{\beta+1})$ is a nonforking extension of p.

(7) If $i < \alpha < \delta$ and $\mathbf{ortp}(\bar{a}, M_0^\alpha, M_{i+1}^\alpha) \in \mathscr{S}_{\mathfrak{s}}^{\mathrm{bs}}(M_0^\alpha)$, then for some $\beta \in (\alpha, \delta)$ exactly one of the following occurs:

  (a) $\mathbf{ortp}(\bar{a}, M_0^{\beta+1}, M_{i+1}^{\beta+1})$ forks over M_0^α.
  (b) There is no $\langle M_j^* : j \leq i+1 \rangle$ $\leq_{\mathfrak{s}}$-increasing continuous such that:
   (i) $M_j^\beta \leq_{\mathfrak{s}} M_j^*$ for all $j \leq i+1$.
   (ii) $\mathbf{ortp}(\bar{a}_j, M_j^*, M_{j+1}^*)$ does not fork over M_j^β for all $j < i+1$.
   (iii) $\mathbf{ortp}(\bar{a}, M_0^*, M_{i+1}^*)$ forks over M_0^β.

  <u>This is possible</u>: Along the construction, we also build an enumeration $\overline{\langle (\bar{b}_j^\gamma, k_j^\gamma, i_j^\gamma, \alpha_j^\gamma) : j < \lambda, \gamma < \lambda \rangle}$ such that for any $\gamma \in (0, \lambda)$, any $\alpha < \lambda \cdot \gamma$, any $i < \alpha$, any $k \leq i$, and any $\bar{a} \in {}^{<\omega}M_{i+1}^\alpha$, if $\mathbf{ortp}(\bar{a}, M_k^\alpha, M_{i+1}^\alpha) \in \mathscr{S}_{\mathfrak{s}}^{\mathrm{bs}}(M_k^\alpha)$, then there exists $j < \lambda$ so that $\bar{b}_j^\gamma = \bar{a}$, $i_j^\gamma = i$, $k_j^\gamma = k$, and $\alpha_j^\gamma = \alpha$. We require that always $k_j^\gamma \leq i_j^\gamma < \alpha_j^\gamma < \lambda \cdot \gamma$ and the triple

$(\bar{b}_j^\gamma, M_{k_j^\gamma}^{\alpha_j^\gamma}, M_{i_j^\gamma+1}^{\alpha_j^\gamma})$ represents a basic type. We make sure that at stage $\lambda \cdot (\gamma + 1)$ of the construction below, $\bar{b}_j^{\gamma'}, k_j^{\gamma'}, i_j^{\gamma'}, \alpha_j^{\gamma'}$ are defined for all $j < \lambda$, $\gamma' \leq \gamma$.

For $\alpha = 0$, take any $M_0^0 \in \mathfrak{K}_\mathfrak{s}$. For α limit, let $M_i^\alpha := \bigcup_{\beta \in [i, \alpha)} M_i^\beta$ for $i < \alpha$ and $M_\alpha^\alpha := \bigcup_{i < \alpha} M_i^\alpha$. Now assume that $\bar{M}^\alpha$, $\bar{a}^\alpha$ have been defined for $\alpha < \delta$. We define $\bar{M}^{\alpha+1}$ and $\bar{a}_\alpha$. Fix ρ and $j < \lambda$ such that $\alpha = \lambda \cdot \rho + j$. We consider two cases.

- Case 1: ρ is zero or a limit: Use Fact 3.10 to get $\langle M_i^{\alpha+1} : i \leq \alpha \rangle$ $<_\mathfrak{s}$-increasing continuous such that $M_i^\alpha <_\mathfrak{s} M_i^{\alpha+1}$ for all $i \leq \alpha$, and for all $i < \alpha$, $\mathbf{ortp}(\bar{a}_i, M_i^{\alpha+1}, M_{i+1}^{\alpha+1})$ does not fork over M_i^α. Pick any $M_{\alpha+1}^{\alpha+1}$ with $M_\alpha^{\alpha+1} <_\mathfrak{s} M_{\alpha+1}^{\alpha+1}$ and any $\bar{a}_\alpha \in {}^{<\omega}M_{\alpha+1}^{\alpha+1}$ such that $\mathbf{ortp}(\bar{a}_\alpha, M_\alpha^{\alpha+1}, M_{\alpha+1}^{\alpha+1}) \in \mathscr{S}_\mathfrak{s}^{\mathrm{bs}}(M_\alpha^{\alpha+1})$.

- Case 2: ρ is a successor: Say $\rho = \gamma + 1$. Let $\bar{a} := \bar{b}_j^\gamma$, $\alpha_0 := \alpha_j^\gamma$, $k_0 := k_j^\gamma$, $i_0 := i_j^\gamma$. There are two subcases. It is possible that $k_0 \neq 0$ or $k_0 = 0$ and (7b) holds with i, α, β there standing for i_0, α_0, α here. In this case, we proceed as in Case 1 to define $\langle M_i^{\alpha+1} : i \leq \alpha \rangle$. Then we pick $\bar{a}_\alpha$, $M_{\alpha+1}^{\alpha+1}$ such that $\mathbf{ortp}(\bar{a}_\alpha, M_\alpha^\alpha, M_{\alpha+1}^{\alpha+1})$ is the nonforking extension of $\mathbf{ortp}(\bar{a}, M_{i_0}^{\alpha_0}, M_{i_0+1}^{\alpha_0})$.
  On the other hand, it is possible that $k_0 = 0$ and (7b) fails. In this case let $\langle M_j^* : j \leq i_0 + 1 \rangle$ witness the failure and set $M_j^{\alpha+1} := M_j^*$ for $j \leq i_0 + 1$. Then continue as in Case 1 and define $\bar{a}_\alpha$, $M_{\alpha+1}^{\alpha+1}$ as before.

This is enough: We choose $\bar{M}^* = \langle M_i^* : i \leq \delta \rangle$ increasing continuous such that M_0^* is brimmed over M_0^δ, $M_i^\delta \leq_\mathfrak{s} M_i^*$ for all $i \leq \delta$, and $\mathbf{ortp}(\bar{a}_i, M_i^*, M_{i+1}^*)$ does not fork over M_i^δ. This is possible, see case 1 above. Now:

- M_0^* is brimmed over M_0^δ.
  [Why? By construction].

- If $p \in \mathscr{S}_\mathfrak{s}^{\mathrm{bs}}(M_i^\delta)$ for $i < \delta$, then for λ-many $\beta \in [i, \delta)$, $\mathbf{ortp}(\bar{a}_\beta, M_\beta^\delta, M_{\beta+1}^\delta)$ is a nonforking extension of p.
  [Why? Pick $i' \in (i, \delta)$ such that p does not fork over $M_i^{i'}$. By (6), we know that for λ-many $\beta \in [i', \delta)$, the nonforking extension of $p \upharpoonright M_i^{i'}$ to $M_\beta^{\beta+1}$ is realized in $M_\beta^{\beta+1}$ by $\bar{a}_\beta$. But by (5) we also have that $\mathbf{ortp}(\bar{a}_\beta, M_\beta^\delta, M_{\beta+1}^\delta)$ does not fork over M_β^β. In particular by uniqueness $\bar{a}_\beta$ also realizes p.]

- M_δ^δ is brimmed over M_0^δ.
  [Why? We apply Fact 3.3 to the chain $\langle M_i^\delta : i \leq \delta \rangle$, using the previous step.].

- $\mathrm{LWNF}(M_0^\delta, M_0^\delta, M_0^*, M_\delta^*)$.
  [Why? Pick $\bar{a} \in {}^{<\omega}M_\delta^\delta$ such that $\mathbf{ortp}(\bar{a}, M_0^\delta, M_\delta^\delta)$ is basic. By local character, there exists $\alpha < \delta$ such that $\mathbf{ortp}(\bar{a}, M_0^\delta, M_\delta^\delta)$ does not fork over M_0^α. Further, we can increase α if necessary and pick $i < \alpha$ such that $\bar{a} \in {}^{<\omega}M_{i+1}^\alpha$. We now apply Clause (7). We know that (7a) fails for all $\beta \in (\alpha, \delta)$ by the choice of α, therefore (7b) must hold for all $\beta \in (\alpha, \delta)$. Now if $\mathbf{ortp}(\bar{a}, M_0^*, M_\delta^*)$ forks over M_0^δ, then it must fork over M_0^β for all high-enough β, but then $\langle M_j^* : j \leq i + 1 \rangle$ would

contradict Clause (7b). Therefore $\mathbf{ortp}(\bar{a}, M_0^*, M_\delta^*)$ does not fork over M_0^δ, as desired.]

Therefore we can take $(M_0, M_1, M_2, M_3) := (M_0^\delta, M_\delta^\delta, M_0^*, M_\delta^*)$. †$_{\text{Claim}}$

$\square$

The following properties of LWNF may or may not hold in general (we have no examples for the failure of symmetry, but uniqueness fails in the last good frame of the Hart-Shelah example, see [HS90, BV]):

Definition 3.12. Let $R \in \{\text{LWNF}, \text{RWNF}, \text{WNF}\}$.

(1) We say that R has the *symmetry property* if $R(M_0, M_1, M_2, M_3)$ implies $R(M_0, M_2, M_1, M_3)$.
(2) We say that R has the *uniqueness property* if whenever $R(M_0, M_1, M_2, M_3)$ and $R(M_0, M_1, M_2, M_3')$, there exists M_3'' with $M_3' \leq_{\mathfrak{s}} M_3''$ and $f : M_3 \xrightarrow[|M_1| \cup |M_2|]{} M_3''$.

The following are trivial observations about the definitions:

Remark 3.13.

(1) WNF has the symmetry property, and LWNF has the symmetry property if and only if RWNF has the symmetry property if and only if LWNF = RWNF = WNF.
(2) LWNF has the uniqueness property if and only RWNF has it.

Recall from [She09a, III.1.3]:

Definition 3.14. $\mathfrak{s}$ is *good*$^+$ when the following is *impossible*:

There exists an increasing continuous $\langle M_i : i < \lambda^+ \rangle$, $\langle N_i : i < \lambda^+ \rangle$, a basic type $p \in \mathscr{S}_{\mathfrak{s}}^{\text{bs}}(M_0)$, and $\langle \bar{a}_i : i < \lambda^+ \rangle$ such that for any $i < \lambda^+$:

(1) $M_i \leq_{\mathfrak{s}} N_i$.
(2) $\bar{a}_{i+1} \in |M_{i+2}|$ and $\mathbf{ortp}(\bar{a}_{i+1}, M_{i+1}, M_{i+2})$ is a nonforking extension of p, but $\mathbf{ortp}(\bar{a}_{i+1}, N_0, N_{i+2})$ forks over M_0.
(3) $\bigcup_{j < \lambda^+} M_j$ is saturated.

We now show that being good$^+$ is a consequence of symmetry for LWNF. Moreover, good$^+$ allows us to build a superlimit in λ^+.

Theorem 3.15. $(1) \Rightarrow (2) \Leftrightarrow (3) \Rightarrow (4)$, where:

(1) LWNF has the symmetry property.
(2) $\mathfrak{s}$ is good$^+$.
(3) For $M, N \in \mathfrak{K}_{\lambda^+}$ both saturated, $M \leq_{\mathfrak{K}} N$ implies $M \leq_{\text{WNF}} N$.
(4) There is a superlimit model in $\mathfrak{K}_{\lambda^+}$.

Proof.

- (3) implies (4): This follows from Lemma 3.9 and the fact that the saturated model in λ^+ is universal and has a proper extension [She09a, II.4.13].

- $\neg(2)$ implies $\neg(3)$: Fix a witness $\langle M_i : i < \lambda^+ \rangle$, $\langle N_i : i < \lambda^+ \rangle$, $\langle \bar{a}_i : i < \lambda^+ \rangle$, p to the failure of being good$^+$. Write $M_{\lambda^+} := \bigcup_{i < \lambda^+} M_i$, $N_{\lambda^+} := \bigcup_{i < \lambda^+} N_i$. By assumption, M_{λ^+} is saturated. Clearly, increasing the N_i's will not change that we have a witness so without loss of generality N_{λ^+} is also saturated. We claim that $M_{\lambda^+} \not\leq_{\mathrm{RWNF}} N_{\lambda^+}$. We show this by proving that for any $i < \lambda^+$ and any $j \leq i + 1$, $\neg \mathrm{RWNF}(M_j, N_j, M_{i+2}, N_{i+2})$. Indeed, $\mathbf{ortp}(\bar{a}_{i+1}, N_j, N_{i+2})$ forks over M_j: if not, then by transitivity $\mathbf{ortp}(\bar{a}_{i+1}, N_j, N_{i+2})$ does not fork over M_0, and hence $\mathbf{ortp}(\bar{a}_{i+1}, N_0, N_{i+2})$ does not fork over M_0, and we know that this is not the case of the witness we selected.

- $\neg(3)$ implies $\neg(2)$: Fix M, N saturated in $\mathfrak{K}_{\lambda^+}$ such that $M \leq_{\mathfrak{K}} N$ but $M \not\leq_{\mathrm{RWNF}} N$.

  Claim: For any $A \subseteq |M|$ of size λ, there exists $M_0 \leq_{\mathfrak{s}} M_1 \leq_{\mathfrak{K}} M$ and $N_0 \leq_{\mathfrak{s}} N_1 \leq_{\mathfrak{K}} N$ such that $M_0 \leq_{\mathfrak{s}} N_0$, $M_1 \leq_{\mathfrak{s}} N_1$, $A \subseteq |M_0|$, but $\neg \mathrm{RWNF}(M_0, N_0, M_1, N_1)$.

  Proof of Claim: If not, we can use failure of the claim and continuity of RWNF to build increasing continuous resolution $\langle M_i : i \leq \lambda^+ \rangle$, $\langle N_i : i \leq \lambda^+ \rangle$ of M and N respectively such that $\mathrm{RWNF}(M_i, N_i, M_j, N_j)$ for all $i < j < \lambda^+$. Thus $M \leq_{\mathrm{RWNF}} N$, contradicting the assumption. $\dagger_{\mathrm{Claim}}$

  Build $\langle M_i^* : i \leq \lambda^+ \rangle$, $\langle N_i^* : i \leq \lambda^+ \rangle$ increasing continuous resolutions of M, N respectively such that for all $i < \lambda^+$, $M_i^* \leq_{\mathfrak{s}} N_i^*$ and $\neg \mathrm{RWNF}(M_{i+1}^*, N_{i+1}^*, M_{i+2}^*, N_{i+2}^*)$. This is possible by the claim. Let $\bar{a}_{i+1}^* \in |M_{i+2}^*|$ witness the RWNF-forking, i.e. $\mathbf{ortp}(\bar{a}_{i+1}^*, N_{i+1}^*, N_{i+2}^*)$ forks over M_{i+1}^*. By Fodor's lemma, local character, and stability, there exists a stationary set S, $i_0 < \lambda^+$ and $p \in \mathscr{S}_{\mathfrak{s}}^{\mathrm{bs}}(M_{i_0}^*)$ such that for all $i \in S$, $\mathbf{ortp}(\bar{a}_{i+1}^*, M_i^*, M_{i+2}^*)$ is the nonforking extension of p. Without loss of generality, i_0 is limit and all elements of S are also limit ordinals.

  Now build an increasing continuous sequence of ordinals $\langle j_i : i < \lambda^+ \rangle$ as follows. Let $j_0 := i_0$. For i limit, let $j_i := \sup_{k < i} j_k$. For i successor, pick any $j_i \in S$ with $j_i > j_{i-1}$.

  Now for i not the successor of a limit, let $M_i := M_{j_i}^*$, $N_i := N_{j_i}^*$, $\bar{a}_i := \bar{a}_{j_i}^*$. For $i = k + 1$ with k a limit, set $M_i := M_{j_k}^*$, $N_i := N_{j_k}^*$, $\bar{a}_i := \bar{a}_{j_k}^*$. This gives a witness to the failure of being good$^+$.

- (1) implies (3): If LWNF has the symmetry property, then by Remark 3.13, $\overline{\mathrm{LWNF}} = \mathrm{RWNF} = \mathrm{WNF}$. By Fact 3.8, it follows that $M \leq_{\mathfrak{K}} N$ implies $M \leq_{\mathrm{WNF}} N$ for any $M, N \in \mathfrak{K}_{\lambda^+}$, so (3) holds.

$\square$

Question 3.16. Are the conditions in Theorem 3.15 all equivalent?

Question 3.17. Is there a good λ-frame $\mathfrak{s}$ such that LWNF$_{\mathfrak{s}}$ does *not* have the symmetry property?

The next result shows that the uniqueness property has strong consequences. The first author has given conditions under which when $\lambda = \aleph_0$, failure of uniqueness implies nonstructure [She09b, VII.4.16].

Theorem 3.18. Assume that $\mathfrak{s}$ is a good $(< \omega, \lambda)$-frame (so it satisfies symmetry). If LWNF has the uniqueness property, then LWNF has the symmetry property and $\mathfrak{s}$ is successful good$^+$ (see [She09a, III.1.1]).

Proof. By [Vas17, 3.11] (used with the pre-$(\leq \lambda, \lambda)$-frame induced by LWNF, recalling Fact 3.8) $\mathfrak{s}$ is weakly successful. This implies that there is a relation $\mathrm{NF} = \mathrm{NF}_{\mathfrak{s}}$ that is a nonforking relation respecting $\mathfrak{s}$ (see [She09a, II.6.1], in particular it has all the properties listed in Theorem 3.11, as well as uniqueness and symmetry). Now as NF respects $\mathfrak{s}$, we must have that $\mathrm{NF}(M_0, M_1, M_2, M_3)$ implies $\mathrm{LWNF}(M_0, M_1, M_2, M_3)$. Since LWNF has the uniqueness property and NF has the existence property, it follows from [BGKV16, 4.1] that $\mathrm{LWNF} = \mathrm{NF}$. In particular, LWNF has the symmetry property.

To see that $\mathfrak{s}$ is successful good$^+$, it is enough to show that for $M, N \in \mathfrak{K}_{\lambda^+}$, $M \leq_{\mathfrak{K}} N$ implies $M \leq_{\mathrm{NF}} N$ (where $\leq_{\mathrm{NF}}$ is defined as in Definition 3.7). This is immediate from Fact 3.8 and $\mathrm{LWNF} = \mathrm{NF}$. $\qquad\square$

To prepare for the proof of symmetry in the $\lambda = \aleph_0$ case, we end this section by introducing yet another notion of nonforking amalgamation (VWNF stands for "very weak nonforking amalgamation"). In this case, we look at finite sequences both on the left *and* the right hand side. We show that if $\mathfrak{s}$ is a good frame, then VWNF has the symmetry property and locality of types implies that $\mathrm{VWNF} = \mathrm{LWNF}$. Thus in this case LWNF has the symmetry property too.

Definition 3.19.

(1) For $M \leq_{\mathfrak{s}} N$, $B \subseteq |N|$, $\bar{a} \in {}^{<\omega}N$, we say that $\mathbf{ortp}(\bar{a}, B, N)$ *does not fork over* M if there exist M', N' with $N \leq_{\mathfrak{s}} N'$, $M \leq_{\mathfrak{s}} M' \leq_{\mathfrak{s}} N'$, and $B \subseteq |M'|$ such that $\mathbf{ortp}(\bar{a}, M', N')$ does not fork over M_0.

(2) We define a 4-ary relation $\mathrm{VWNF}_{\mathfrak{s}} = \mathrm{VWNF}$ on $\mathfrak{K}_{\mathfrak{s}}$ by $\mathrm{VWNF}(M_0, M_1, M_2, M_3)$ if and only if $M_0 \leq_{\mathfrak{s}} M_\ell \leq_{\mathfrak{s}} M_3$, $\ell = 1, 2$ and for any $\bar{a} \in {}^{<\omega}M_1$ and any finite $B \subseteq |M_2|$, if $\mathbf{ortp}(\bar{a}, M_0, M_3)$ and $\mathbf{ortp}(\bar{a}, M_2, M_3)$ are both basic, then $\mathbf{ortp}(\bar{a}, B, M_3)$ does not fork over M_0.

Theorem 3.20. Assume that $\mathfrak{s}$ is a type-full good $(< \omega, \lambda)$-frame.

(1) VWNF has the symmetry property: $\mathrm{VWNF}(M_0, M_1, M_2, M_3)$ if and only if $\mathrm{VWNF}(M_0, M_2, M_1, M_3)$.

(2) If for any $M \in \mathfrak{K}_{\mathfrak{s}}$ and any $p \neq q \in \mathscr{S}_{\mathfrak{s}}^{\mathrm{bs}}(M)$ there exists $B \subseteq |M|$ finite such that $p \restriction B \neq q \restriction B$, then $\mathrm{VWNF} = \mathrm{LWNF}$. In particular, LWNF has the symmetry property.

Proof.

(1) By the symmetry axiom of good frames.

(2) This is observed in [Vas16a, 4.5]. In details, it suffices to show that for $M \leq_{\mathfrak{s}} N$, $p \in \mathscr{S}_{\mathfrak{s}}^{\mathrm{bs}}(N)$ does not fork over M if and only if $p \restriction B$ does not fork over M for all finite $B \subseteq |N|$. Let $q \in \mathscr{S}_{\mathfrak{s}}^{\mathrm{bs}}(N)$ be the nonforking extension of $p \restriction M$. For any finite $B \subseteq |N|$, we have that $q \restriction B = p \restriction B$, by the uniqueness property for (the extended notion of) forking, see [BGKV16, 5.4]. Therefore by the assumption we must have $p = q$, as desired.

$\square$

4. Building a superlimit

In this section, we work in $\aleph_0$ and show assuming stability and amalgamation that $\mathfrak{K}$ is $\mathrm{PC}_{\aleph_0}$ (Theorem 4.2) and has a superlimit (Corollary 4.6).

Hypothesis 4.1. $\mathfrak{K} = (K, \leq_{\mathfrak{K}})$ is an AEC with $\mathrm{LS}(\mathfrak{K}) = \aleph_0$ (and countable vocabulary).

We will use without comments Fact 2.4 and Remark 2.5. The essence of it is that since $\lambda = \aleph_0$ all brimmed models have the same length, and hence are isomorphic (and the isomorphism fixes any common base they may have).

First note that if $\mathfrak{K}$ is stable and has few models, we can say something about its definability:

Theorem 4.2. Assume that $\mathbb{I}(\mathfrak{K}, \aleph_0) \leq \aleph_0$.

(1) The set $\{M \in K_{\aleph_0} : |M| \subseteq \omega\}$ is Borel.
(2) If $\mathfrak{K}$ has amalgamation in $\aleph_0$ and is stable in $\aleph_0$, then the set $\{(M, N) : M \leq_{\mathfrak{K}} N \text{ and } |N| \subseteq \omega\}$ is $\mathbf{\Sigma_1^1}$.

In particular if $\mathfrak{K}$ has amalgamation in $\aleph_0$ and is stable in $\aleph_0$, then $\mathfrak{K}$ is a $\mathrm{PC}_{\aleph_0}$-representable AEC.

Proof.

(1) Fix $M \in \mathfrak{K}_{\aleph_0}$. By Scott's isomorphism theorem, there exists a formula ϕ_M of $\mathbb{L}_{\aleph_1, \aleph_0}(\tau_{\mathfrak{K}})$ such that $N \models \phi_M$ if and only if $M \cong N$. Now observe that the set

$$\{N : N \text{ is a } \tau_{\mathfrak{K}}\text{-structure with } |N| \subseteq \omega \text{ and } N \models \phi_M\}$$

is Borel and use that $\mathbb{I}(\mathfrak{K}, \aleph_0) \leq \aleph_0$.
(2) For $M, N \in \mathfrak{K}_{\aleph_0}$ with $M \leq_{\mathfrak{K}} N$, let us say that N is *almost brimmed over* M if either N is brimmed over M, or N is $\leq_{\mathfrak{K}}$-maximal. Using amalgamation, it is easy to check that if N, N' are both almost brimmed over M, then $N \cong_M N'$ (as in Fact 2.4, recalling that the chains witnessing the brimmedness must have cofinality ω). Moreover there always exists an almost brimmed model over any $M \in \mathfrak{K}_{\aleph_0}$.

Fix $\langle M_n^* : n < \omega \rangle$ such that for any $M \in \mathfrak{K}_{\aleph_0}$ there exists $n < \omega$ such that $M \cong M_n^*$ (possible as $\mathbb{I}(\mathfrak{K}, \aleph_0) \leq \aleph_0$). For each $n < \omega$, fix $N_n^* \in \mathfrak{K}_{\aleph_0}$ almost brimmed over M_n^*. We have:

$\circledast_1$ For $M, N \in \mathfrak{K}_{\aleph_0}$:
  (a) There is $n < \omega$ and an isomorphism $f : M_n^* \cong M$.
  (b) If N is almost brimmed over M, then any such f extends to $g : N_n^* \cong N$.

$\circledast_2$ For $M_1, M_2 \in \mathfrak{K}_{\aleph_0}$, $M_1 \leq_{\mathfrak{K}} M_2$ if and only if $M_1 \subseteq M_2$ and for some $n_1, n_2 < \omega$, for some (N, f_1, f_2) we have: $M_1 \subseteq M_2 \subseteq N$ and for $\ell = 1, 2$, f_ℓ is an isomorphism from $(M_{n_\ell}^*, N_{n_\ell}^*)$ onto (M_ℓ, N).

[Why? The implication "if" holds by the coherence axiom of AECs. The implication "only if" holds as there is $N \in \mathfrak{K}_{\aleph_0}$ which is almost brimmed over M_2 (and so $M_2 \leq_{\mathfrak{K}} N$) hence N is almost brimmed over M_1 and use $\circledast_1$ above.]

The result now follows from $\circledast_2$.

By [BL16, 3.3], it follows that $\mathfrak{K}$ is $\mathrm{PC}_{\aleph_0}$. $\hfill\square$

We now study homogeneous models and show that they coincide with brimmed models. Note that the homogeneity here is with respect to a set D of *orbital* types.

Definition 4.3. Let D be a set of orbital types over the empty set and let $M \in \mathfrak{K}$. We say that M is $(D, \aleph_0)$-*homogeneous* if it realizes all the types in D and whenever $p \in D$ is the type of an $(n + m)$-elements sequence and $\bar{a} \in {}^nM$ realizes p^n (the restriction of p to its first n "variables"), there exists a sequence $\bar{b} \in {}^mM$ such that $\bar{a}\bar{b}$ realizes p.

Theorem 4.4. Assume that $\mathfrak{K}$ is nicely stable in $\aleph_0$ and let $\mathfrak{K}^*$ be the class of amalgamation bases in $\mathfrak{K}_{\aleph_0}$. Let $M \in \mathfrak{K}_{\aleph_0}$. The following are equivalent:

(1) M is brimmed.
(2) M is $(\mathscr{S}_{\mathfrak{K}^*}^{<\omega}(\emptyset), \aleph_0)$-homogeneous.

Proof. Let $M \in \mathfrak{K}_{\aleph_0}$. First we show:

<u>Claim</u>: If M is brimmed, $\bar{a}, \bar{b} \in M$, then $\mathbf{ortp}_{\mathfrak{K}^*}(\bar{a}/\emptyset; M) = \mathbf{ortp}_{\mathfrak{K}^*}(\bar{b}/\emptyset; M)$ if and only if there is an automorphism f of M sending $\bar{a}$ to $\bar{b}$.

<u>Proof of Claim</u>: The right to left direction is clear. Now assume that $\mathbf{ortp}_{\mathfrak{K}^*}(\bar{a}/\emptyset; M) = \mathbf{ortp}_{\mathfrak{K}^*}(\bar{b}/\emptyset; M)$. Say $\langle M_i : i < \omega \rangle$ witness that M is brimmed, and without loss of generality assume that $\bar{a}, \bar{b} \in M_0$. Then $\mathbf{ortp}_{\mathfrak{K}^*}(\bar{a}/\emptyset; M_0) = \mathbf{ortp}_{\mathfrak{K}^*}(\bar{b}/\emptyset; M_0)$. Since $\mathfrak{K}^*$ has amalgamation, there exists $M_0' \in \mathfrak{K}^*$ with $M_0 \leq_{\mathfrak{K}} M_0'$ and $f : M_0 \to M_0'$ so that $f(\bar{a}) = \bar{b}$. Since M_1 is universal over M_0, we can assume without loss of generality that $M_0' = M_1$. Now extend f to an automorphism of M using a back and forth argument.

From the claim, it follows directly that if M is brimmed, then it is $(\mathscr{S}_{\mathfrak{K}^*}^{<\omega}(\emptyset), \aleph_0)$-homogeneous. Conversely, the countable $(\mathscr{S}_{\mathfrak{K}^*}^{<\omega}(\emptyset), \aleph_0)$-homogeneous model is unique (standard back and forth argument) and so it must also be brimmed. $\hfill\square$

Remark 4.5. By adding constants to the language, we can also characterized brimmed models over M_0 as those that are homogeneous for orbital types of finite sequences over M_0.

Corollary 4.6. If $\mathfrak{K}$ is nicely stable in $\aleph_0$, then there is a superlimit model of cardinality $\aleph_0$.

Proof. The brimmed model works (use Theorem 4.4 to see that the conditions of the definition of a superlimit model are satisfied). $\hfill\square$

We have justified assuming amalgamation in the following sense:

Corollary 4.7. If $\mathfrak{K}$ is nicely stable in $\aleph_0$, then there exists an AEC $\mathfrak{K}' = (K', \leq_{\mathfrak{K}'})$ such that:

(1) $\mathrm{LS}(\mathfrak{K}') = \aleph_0$.
(2) $\mathfrak{K}'_{<\aleph_0} = \emptyset$.
(3) $\tau_{\mathfrak{K}'} = \tau_{\mathfrak{K}}$.
(4) $K' \subseteq K$ and for $M, N \in K'$, $M \leq_{\mathfrak{K}'} N$ if and only if $M \leq_{\mathfrak{K}} N$.
(5) For any $M \in \mathfrak{K}_{\aleph_0}$ there exists $M' \in \mathfrak{K}'_{\aleph_0}$ with $M \leq_{\mathfrak{K}} M'$.
(6) $\mathfrak{K}'$ is categorical in $\aleph_0$.
(7) $\mathfrak{K}'$ is very nicely stable in $\aleph_0$. In particular it has amalgamation in $\aleph_0$.
(8) $\mathfrak{K}'$ is $\mathrm{PC}_{\aleph_0}$.

Proof. Let $M \in \mathfrak{K}_{\aleph_0}$ be superlimit (exists by Corollary 4.6). Let $K'_{\aleph_0} := \{N \in K : N \cong M\}$. Now let $\mathfrak{K}'$ be the AEC generated by $(K'_{\aleph_0}, \leq_{\mathfrak{K}})$ (in the sense of [She09a, II.23]). One can easily check that $\mathfrak{K}'$ is nicely stable in $\aleph_0$ and from categoricity in $\aleph_0$ we get amalgamation in $\aleph_0$, hence (7) holds. As for (8), it follows from Theorem 4.2. $\qquad\square$

5. Building a good $\aleph_0$-frame

The aim of this section is to build a good $\aleph_0$-frame from nice $\aleph_0$-stability. By Corollary 4.7, we may restrict the class to a superlimit so that it is categorical in $\aleph_0$. As before, we assume:

Hypothesis 5.1. $\mathfrak{K} = (K, \leq_{\mathfrak{K}})$ is an AEC with $\mathrm{LS}(\mathfrak{K}) = \aleph_0$ (and countable vocabulary).

The nonforking relation of the frame will be nonsplitting:

Definition 5.2. For $M \in \mathfrak{K}_{\aleph_0}$ and $A \subseteq |M|$, $p \in \mathscr{S}^{<\omega}(M)$ *splits over* A if whenever $p = \mathbf{ortp}(\bar{a}/M; N)$, there exists $\bar{b}_1, \bar{b}_2 \in M$ such that $\mathbf{ortp}(\bar{b}_1/A; M) = \mathbf{ortp}(\bar{b}_1/A; M)$ but $\mathbf{ortp}(\bar{a}\bar{b}_1/A; N) \neq \mathbf{ortp}(\bar{a}\bar{b}_2/A; N)$.

The following is proven in [She09a, I.5.6]. Similar proofs appear in [HK06, 3.16] or [BHH$^+$14, 4.2].

Fact 5.3. Assume that $\mathfrak{K}$ is nicely stable in $\aleph_0$ and categorical in $\aleph_0$. If $M \in \mathfrak{K}_{\aleph_0}$ and $p \in \mathscr{S}^{<\omega}(M)$, then there exists $A \subseteq |M|$ finite such that p does not split over A.

The following result about nonsplitting will also come in handy. It appears in various forms in the literature, see e.g. [BV17a, 4.8].

Lemma 5.4 (Weak uniqueness)**.** Assume that $\mathfrak{K}$ is nicely stable in $\aleph_0$ and categorical in $\aleph_0$. Let $M \leq_{\mathfrak{K}} N$ both be in $\mathfrak{K}_{\aleph_0}$, $p, q \in \mathscr{S}^{<\omega}(N)$. If both p and q do not split over a finite subset of M and $p \restriction A = q \restriction A$ for all finite $A \subseteq |M|$, then $p \restriction B = q \restriction B$ for all finite $B \subseteq |N|$.

Proof. Let N' be brimmed over N. Let $\bar{b}_1, \bar{b}_2 \in N'$ realize p and q respectively. Fix $A \subseteq |M|$ finite such that p and q do not split over A. Let $B \subseteq |N|$ be finite and let $\bar{b}$ be an enumeration of B. Since M is brimmed, there exists $\bar{b}' \in M$ such that $\mathbf{ortp}(\bar{b}/A; N) = \mathbf{ortp}(\bar{b}'/A; N)$. By nonsplitting, $\mathbf{ortp}(\bar{b}_\ell\bar{b}/A; N') = \mathbf{ortp}(\bar{b}_\ell\bar{b}'/A; N')$ for $\ell = 1, 2$. Now since $\bar{b}' \in M$ we have by assumption that $p \restriction A\bar{b}' = q \restriction A\bar{b}'$. Therefore $\mathbf{ortp}(\bar{b}_1\bar{b}'/A; N') = \mathbf{ortp}(\bar{b}_2\bar{b}'/A; N')$. Putting these

equalities together, $\mathbf{ortp}(\bar{b}_1\bar{b}/A; N') = \mathbf{ortp}(\bar{b}_2\bar{b}/A; N')$, so $p \restriction B = q \restriction B$, as desired. $\square$

Definition 5.5. Let $\mathfrak{K}$ be nicely stable in $\aleph_0$ and categorical in $\aleph_0$. We define a triple $\mathfrak{s} = (\mathfrak{K}^\mathfrak{s}, \underset{\smile}{\smile}, \mathscr{S}^{\mathrm{bs}}_\mathfrak{s})$ by:

(1) $\mathfrak{K}^\mathfrak{s} = \mathfrak{K}$.

(2) For $M_0 \leq_\mathfrak{K} M \leq_\mathfrak{K} N$ all in $\mathfrak{K}_{\aleph_0}$, $n < \omega$, $\bar{a} = \langle a_i : i < n \rangle \in {}^n N$, $\bar{a} \underset{M_0}{\overset{N}{\smile}} M$
   holds if and only if $a_i \notin M$ for all $i < n$ and there exists a finite $A \subseteq |M_0|$ so that $\mathbf{ortp}(\bar{a}/M; N)$ does not split over A.

(3) For $M \in \mathfrak{K}_{\aleph_0}$, $\mathscr{S}^{\mathrm{bs}}_\mathfrak{s}(M)$ is the set of all types of finite sequences $\langle a_i : i < n \rangle$ over M such that for all $i < n$, $a_i \notin M$.

In order to prove that $\mathfrak{s}$ is a good $\aleph_0$-frame, we will make an additional locality hypothesis. See Example 5.10 and the next section for setups where it holds.

Definition 5.6. $\mathfrak{K}$ is $(< \aleph_0, \aleph_0)$-*local* if for any $M \in \mathfrak{K}$ $p, q \in \mathscr{S}^{<\omega}(M)$, $p \restriction A = q \restriction A$ for all finite $A \subseteq |M|$ implies $p = q$. We say that $\mathfrak{K}$ is *weakly* $(< \aleph_0, \aleph_0)$-*local* if this holds for a superlimit M.

Remark 5.7. The definition of locality includes types of any finite length, not just of length one. This will be used to prove the symmetry property of $\mathrm{LWNF}_\mathfrak{s}$, via Theorem 3.20.

We now prove, assuming nice stability, categoricity, and locality, that the pre-frame defined above is a good $\aleph_0$-frame.

Theorem 5.8. Assume that $\mathfrak{K}$ is nicely stable in $\aleph_0$ and categorical in $\aleph_0$. If $\mathfrak{K}$ is $(< \aleph_0, \aleph_0)$-local, then $\mathfrak{s}$ (Definition 5.5) is a type-full good $(< \omega, \aleph_0)$-frame. Moreover $\mathrm{LWNF}_\mathfrak{s}$ has the symmetry property (recall Definitions 3.4 and 3.12). In particular, $\mathfrak{s}$ is good^+.

Proof. Once we have shown that $\mathfrak{s}$ is a type-full good frame, the moreover part follows from Theorem 3.20. The last sentence is by Theorem 3.15.

Now except for symmetry, the axioms of good frames are easy to check (see the proof of [She09a, II.3.4]). For example:

- Local character: Let $\langle M_i : i \leq \delta \rangle$ be increasing continuous in $\mathfrak{K}_\mathfrak{s}$. Let $p \in \mathscr{S}^{\mathrm{bs}}_\mathfrak{s}(M_\delta)$. By Fact 5.3, there exists a finite $A \subseteq |M_\delta|$ such that p does not split over A. Pick $i < \delta$ such that $A \subseteq |M_i|$. Then p does not fork over M_i.
- Uniqueness: by Lemma 5.4 and locality.
- Extension: follows on general grounds, see [Vasb, 3.5].

Symmetry is the hardest to prove, and is done as in [She09a, I.5.30]. We give a full proof for the convenience of the reader.

Suppose that $\mathbf{ortp}(\bar{b}, N_2, N_3)$ does not fork over N_0 and let $\bar{c} \in {}^{<\omega}N_2 \backslash N_1$. We want to find N_1, N'_3 such that $N_0 \leq_\mathfrak{s} N_1 \leq_\mathfrak{s} N'_3$, $N_3 \leq_\mathfrak{s} N'_3$, $\bar{b} \in {}^{<\omega}N_1$ and $\mathbf{ortp}(\bar{c}, N_1, N'_3)$ does not fork over N_0. Assume for a contradiction that there is no such N_1. Using existence for $\mathrm{LWNF}_\mathfrak{s} = \mathrm{LWNF}$ (see Theorem 3.11), as well as the

extension property for nonforking, we can increase N_2 and N_3 if necessary and find N_1 such that $\mathrm{LWNF}(N_0, N_1, N_2, N_3)$, N_ℓ is brimmed over N_0, and N_3 is brimmed over N_ℓ for $\ell = 1, 2$. By assumption, $p := \mathbf{ortp}(\bar{c}, N_1, N_3)$ forks over N_0.

<u>Claim 1</u>: Let I be the linear order $[0, \infty) \cap \mathbb{Q}$. There exists an increasing chain $\langle M_s : s \in I \rangle$ such that for any $s < t$ in I, M_s, M_t are in $\mathfrak{K}_{\aleph_0}$ and M_t is brimmed over M_s.

<u>Proof of Claim 1</u>: Fix $\langle M_i^* : i < \omega_1 \rangle$ increasing continuous in $\mathfrak{K}_{\aleph_0}$ such that M_{i+1}^* is brimmed over M_i^* for all $i < \omega_1$. Using undefinability of well-ordering, pick a countable ill-founded model of set theory $\mathfrak{B} = (A, E, \langle M_s : s \in I^* \rangle)$ elementarily equivalent to $(H(\aleph_2), \in, \langle M_i^* : i < \omega_1 \rangle)$. Now I^* contains a copy of the rationals by a general argument on ill-founded models of set theory, see [Fri73, Section 3]). Recalling that $\mathfrak{K}$ is $\mathrm{PC}_{\aleph_0}$ (see Corollary 4.7) and the syntactic characterization of brimmed models (Theorem 4.4), the result follows. $\dagger_{\mathrm{Claim}\ 1}$

Fix I, $\langle M_s : s \in I \rangle$ as in Claim 1. Fix N_0' such that N_0 is brimmed over N_0' and $p \restriction N_0$ does not fork over N_0'.

For any fixed infinite $J \subseteq I$, write $M_J := \bigcup_{s \in J} M_s$. Assume now that M_I is brimmed over M_J. Let $N_0^J := M_J$, $N_1^J := M_I$. Let N_3^J be brimmed over N_1^J. By categoricity and uniqueness of brimmed models, there exists $f_0 : N_0' \cong M_0$, $f_0^J : N_0 \cong N_0^J$, $f_1^J : N_1 \cong N_1^J$, and $f_3^J : N_3 \cong N_3^J$ such that $f_0 \subseteq f_0^J \subseteq f_1^J \subseteq f_3^J$. Let $f_2^J := f_3^J \restriction N_2$ and let $N_2^J := f_2^J[N_2]$. Note that $\mathrm{LWNF}(N_0^J, N_1^J, N_2^J, N_3^J)$ holds.

Let $p_J := \mathbf{ortp}(f_3^J(\bar{c}), f_3^J[N_1], f_3^J[N_3]) = \mathbf{ortp}(f_3^J(\bar{c}), M_I, N_3^J)$. Since we are assuming that $\mathbf{ortp}(\bar{c}, N_1, N_3)$ forks over N_0, we have that p_J forks over N_0^J. Moreover $p_J \restriction N_0^J$ does not fork over M_0.

<u>Claim 2</u>: If J has no last elements, $I \setminus J$ has no first elements, and $t \in I \setminus J$, then $p_J \restriction M_t$ forks over N_0^J.

<u>Proof of Claim 2</u>: Suppose that $p_J \restriction M_t$ does not fork over N_0^J. Note that M_t is brimmed over M_J. Find N_1' such that $N_0 \leq_{\mathfrak{s}} N_1' \leq_{\mathfrak{s}} N_1$, N_1' is brimmed over N_1, and $f_1^J : N_1' \cong M_t$. Let $\bar{b}' \in {}^{<\omega} N_1'$ be such that $\mathbf{ortp}(\bar{b}', N_0, N_1') = \mathbf{ortp}(\bar{b}, N_0, N_1)$. Since $\mathrm{LWNF}(N_0, N_1, N_2, N_3)$, we know that $\mathbf{ortp}(\bar{b}', N_2, N_3)$ does not fork over N_0, hence by uniqueness $\mathbf{ortp}(\bar{b}, N_2, N_3) = \mathbf{ortp}(\bar{b}', N_2, N_3)$. But we have assumed that $\mathbf{ortp}(\bar{c}, N_1', N_3)$ does not fork over N_0 and $\bar{b}' \in {}^{<\omega_1} N_1'$, hence by a simple renaming we obtain a contradiction to our hypothesis that symmetry failed. $\dagger_{\mathrm{Claim}\ 2}$

<u>Claim 3</u>: If $J_1 \subsetneq J_2$ are both proper initial segments of I with no last elements and $J_2 \setminus J_1$ has no first elements, then $p_{J_1} \neq p_{J_2}$.

<u>Proof of Claim 3</u>: Fix $t \in J_2 \setminus J_1$. By Claim 2, $p_{J_1} \restriction M_t$ forks over $N_0^{J_1}$. We claim that $p_{J_2} \restriction M_t$ does not fork over $N_0^{J_1}$. Indeed recall that $N_0^{J_2} = M_{J_2}$ and by assumption $p_{J_2} \restriction N_0^{J_2}$ does not fork over M_0. Therefore by monotonicity also $p_{J_2} \restriction M_t$ does not fork over $M_{J_1} = N_0^{J_1}$. $\dagger_{\mathrm{Claim}\ 3}$

To finish, observe that there are $2^{\aleph_0}$ cuts of I as in Claim 3. Therefore stability fails, a contradiction. $\square$

The next corollary does not assume categoricity, but uses amalgamation in $\aleph_0$, rather than just density of amalgamation bases.

Corollary 5.9. If $\mathfrak{K}$ is *very* nicely stable in $\aleph_0$ and weakly $(<\aleph_0, \aleph_0)$-local, then $\mathfrak{K}$ has a superlimit of cardinality $\aleph_1$.

Proof. By Corollary 4.6, $\mathfrak{K}$ has a superlimit N_0 in $\aleph_0$. Let $\mathfrak{K}'$ be the class generated by this superlimit, as described by the proof of Corollary 4.7. Then $\mathfrak{K}'$ is categorical in $\aleph_0$ and nicely stable in $\aleph_0$, hence we can apply Theorem 5.8 and get a type-full good$^+$ $\aleph_0$-frame with underlying class $\mathfrak{K}'_{\aleph_0}$. By Theorem 3.15, $\mathfrak{K}'$ has a superlimit model in $\aleph_1$. This is also a superlimit in $\mathfrak{K}$: the only nontrivial property to check is universality. Let $M \in \mathfrak{K}_{\aleph_1}$. Fix any $M_0 \in \mathfrak{K}_{\aleph_0}$ with $M_0 \leq_{\mathfrak{K}} M$. By universality of N_0, there exists $f : M_0 \to N_0$. Now let $N \in \mathfrak{K}'$ be superlimit in $\aleph_1$ with $N_0 \leq_{\mathfrak{K}} N$. Using amalgamation (amalgamation in $\aleph_0$ suffices for this, see [She09a, I.2.11]), we can find $g : M \to N$ extending f, as needed. $\qquad\square$

Example 5.10.

(1) Let $\mathfrak{K}$ be a finitary AEC (see [HK06], this encompasses classes of models of $\mathbb{L}_{\omega_1, \omega}$-sentences) and assume that $\mathfrak{K}$ is stable in $\aleph_0$ (finitary AECs have amalgamation and no maximal models by definition). By [HK06, 4.11], $\mathfrak{K}$ is $(<\aleph_0, \aleph_0)$-local. By Corollary 5.9, $\mathfrak{K}$ has a superlimit in $\aleph_1$. This seems to be new.

(2) Finitary AECs are not the only setup where $(<\aleph_0, \aleph_0)$-locality holds. For example, it is known in quasiminimal pregeometry classes (that may not be finitary [Kir13, Theorem 2]), see [BHH$^+$14, 5.2], and more generally in the finite U-rank (FUR) classes of Hyttinen and Kangas [HK16, 2.17]. Corollary 5.9 seems to be new also for FUR classes. In the next section, we give a much more general framework than FUR classes which still has locality.

6. LOCALITY FROM SUPERSIMPLICITY

In this section, we give a sufficient condition for locality. As before, we assume:

Hypothesis 6.1. $\mathfrak{K} = (K, \leq_{\mathfrak{K}})$ is an AEC with $\mathrm{LS}(\mathfrak{K}) = \aleph_0$ (and countable vocabulary).

In the context of a nicely $\aleph_0$-stable AEC, the following definition generalizes that of a supersimple homogeneous model [BL03, 2.5(iv)]. The idea is that we want to have a nice notion of nonforking available for all finite sets (not only models). However we do not require that forking over finite sets satisfies any uniqueness requirement. Thus it is not a-priori clear that supersimplicity implies the existence of a good frame (although this will follow from Theorems 5.8 and 6.8).

Definition 6.2. Assume that $\mathfrak{K}$ is nicely $\aleph_0$-stable and categorical in $\aleph_0$. We say that $\mathfrak{K}$ is *supersimple* if there exists a 4-ary relation $\underset{}{\downarrow}$ such that:

(1) $\underset{}{\downarrow}(A, B, C, N)$ implies that $N \in \mathfrak{K}_{\aleph_0}$, $A \cup B \cup C \subseteq |N|$ and A, B, C are all finite or countable. We write $B \underset{A}{\overset{N}{\downarrow}} C$ instead of $\underset{}{\downarrow}(A, B, C, N)$. Below, we

may abuse notation and write e.g. $A_1 A_2$ instead of $A_1 \cup A_2$, or $\bar{b} \underset{\bar{a}}{\overset{N}{\smile}} \bar{c}$ instead of $B \underset{A}{\overset{N}{\smile}} C$, where A, B, C stand for the ranges of $\bar{a}$, $\bar{b}$, and $\bar{c}$ respectively.

(2) Normality: $B \underset{A}{\overset{N}{\smile}} C$ if and only if $AB \underset{A}{\overset{N}{\smile}} AC$.

(3) Disjointness: $B \underset{A}{\overset{N}{\smile}} C$ implies $B \cap C \subseteq A$.

(4) Invariance under $\mathfrak{K}$-embeddings: If $f : N \to N'$ and $A \cup B \cup C \subseteq |N|$, then $B \underset{A}{\overset{N}{\smile}} C$, if and only if $f[B] \underset{f[A]}{\overset{N'}{\smile}} f[C]$. This shows that $\smile$ is really a relation on types, so we say $\mathbf{ortp}(\bar{b}/C; N)$ *does not fork over* A if $\bar{b} \underset{A}{\overset{N}{\smile}} C$.

(5) Monotonicity: If $B \underset{A}{\overset{N}{\smile}} C$ and $A \subseteq A' \subseteq B' \subseteq B$ then $B' \underset{A'}{\overset{N}{\smile}} C$.

(6) Symmetry: If $B \underset{A}{\overset{N}{\smile}} C$, then $C \underset{A}{\overset{N}{\smile}} B$.

(7) Local character: If $M \in \mathfrak{K}_{\aleph_0}$ and $p \in \mathscr{S}^{<\omega}(M)$, then there exists $A \subseteq |M|$ finite such that p does not fork over A.

(8) Extension: If $p \in \mathscr{S}^{<\omega}(C; N)$ does not fork over $A \subseteq C$, then there is $q \in \mathscr{S}^{<\omega}(N)$ such that q extends p and q does not fork over A.

(9) Transitivity: If $B \underset{A}{\overset{N}{\smile}} C$ and $B \underset{C}{\overset{N}{\smile}} D$ with $A \subseteq C \subseteq D$, then $B \underset{A}{\overset{N}{\smile}} D$.

(10) Relationship with splitting: If $M \leq_{\mathfrak{K}} N$ are both in $\mathfrak{K}_{\aleph_0}$, $p \in \mathscr{S}^{<\omega}(N)$, and p does not fork over M, then there is $A \subseteq |M|$ finite such that p does not split over A.

We first show that if a type p over a finite set does not fork over a subset A of a countable model M, then the type is realized inside M.

Lemma 6.3. Assume that $\mathfrak{K}$ is nicely $\aleph_0$-stable, supersimple, and categorical in $\aleph_0$. Let $M \leq_{\mathfrak{K}} N$ both be in $\mathfrak{K}_{\aleph_0}$, and let $B \subseteq |N|$ be finite. Let $p \in \mathscr{S}^{<\omega}(B; N)$. If p does not fork over $B \cap |M|$, then p is realized in M.

Proof. Extending N if necessary, we can assume without loss of generality that N is brimmed over M. Let $q \in \mathscr{S}^{<\omega}(N)$ be a nonforking extension of p. Let $\bar{b}$ be an enumeration of B. Fix $A' \subseteq |M|$ finite and big-enough such that $B \cap |M| \subseteq A'$, $\mathbf{ortp}(\bar{b}/M; N)$ does not split and does not fork over A' and q does not split over A' (recall Fact 5.3). Let $M_0 \leq_{\mathfrak{K}} M$ be such that M is brimmed over M_0 and M_0 contains A'. Let $\bar{d} \in M$ be such that $\bar{d}$ realizes $q \upharpoonright M_0$. We have that $\bar{b} \underset{M_0}{\overset{N}{\smile}} M$ so by symmetry and monotonicity, $\bar{d} \underset{M_0}{\overset{N}{\smile}} \bar{b}$. By extension, we can pick M_0' such that $M_0 \leq_{\mathfrak{K}} M_0' \leq_{\mathfrak{K}} N$, M_0' contains $\bar{b}$, and $\bar{d} \underset{M_0}{\overset{N}{\smile}} M_0'$. Now as $q \upharpoonright M_0'$ does not split over a finite subset of M_0 and $\bar{d}$ realizes $q \upharpoonright M_0$, we must have by Lemma 5.4 that $\bar{d}$ realizes $q \upharpoonright B = p$, as desired. $\square$

We will prove locality in supersimple $\aleph_0$-stable AECs by a back and forth argument. More precisely, we start with $M \leq_{\mathfrak{K}} N$, N brimmed over M, and elements $\bar{a}_1, \bar{a}_2 \in N$ whose types over every finite subset of M match. First, we will do a back and forth argument to find an automorphism of N sending $\bar{a}_1$ to $\bar{a}_2$ and fixing M *setwise* (Lemma 6.6). We will then use this automorphism and nonsplitting to build another automorphism that fixes M *pointwise* (Theorem 6.8).

The next lemma starts setting up the stage by making sure that we can map an element of M to an element of M.

Lemma 6.4. Assume that $\mathfrak{K}$ is nicely $\aleph_0$-stable, supersimple, and categorical in $\aleph_0$. Let $M \leq_{\mathfrak{K}} N$ both be in $\mathfrak{K}_{\aleph_0}$. Let $\bar{b}_1, \bar{b}_2 \in N$, $\bar{a}_1, \bar{a}_2, \bar{c}_1 \in M$ be such that $\mathbf{ortp}(\bar{a}_1\bar{b}_1; N) = \mathbf{ortp}(\bar{a}_2\bar{b}_2; N)$. If $\bar{b}_1 \underset{\bar{a}_1}{\overset{N}{\downarrow}} \bar{c}_1$, then there exists $\bar{c}_2 \in M$ such that $\mathbf{ortp}(\bar{a}_1\bar{b}_1\bar{c}_1; N) = \mathbf{ortp}(\bar{a}_2\bar{b}_2\bar{c}_2; N)$.

Proof. Extending N if necessary, we can assume without loss of generality that N is brimmed over M. By symmetry, $\bar{c}_1 \underset{\bar{a}_1}{\overset{N}{\downarrow}} \bar{b}_1$. Let f be an automorphism of N sending $\bar{a}_1\bar{b}_1$ to $\bar{a}_2\bar{b}_2$. By invariance, $f(\bar{c}_1) \underset{\bar{a}_2}{\overset{N}{\downarrow}} \bar{b}_2$ (but we do *not* know that $f(\bar{c}_1) \in M$). Let $q := \mathbf{ortp}(f(\bar{c}_1)/\bar{a}_2\bar{b}_2; N)$. We have to show that q is realized in M. Since q does not fork over $\bar{a}_2 \in M$, this is exactly what Lemma 6.3 tells us. $\square$

Our main lemma in the back and forth argument will be:

Lemma 6.5 (The back and forth lemma). Assume that $\mathfrak{K}$ is nicely $\aleph_0$-stable, supersimple, and categorical in $\aleph_0$. Let $M \leq_{\mathfrak{K}} N$ both be in $\mathfrak{K}_{\aleph_0}$ with N brimmed over M. Let $\bar{b}_1, \bar{b}_2 \in N$, $\bar{a}_1, \bar{a}_2 \in M$ be such that $\mathbf{ortp}(\bar{a}_1\bar{b}_1; N) = \mathbf{ortp}(\bar{a}_2\bar{b}_2; N)$ and $\bar{b}_\ell \underset{\bar{a}_\ell}{\overset{N}{\downarrow}} M$ for $\ell = 1, 2$. If $\bar{d}_1 \in N\backslash M$, $\bar{d}_1' \in M$, there exists $\bar{c}_1, \bar{c}_2, \bar{d}_2' \in M$ and $\bar{d}_2 \in N\backslash M$ such that:

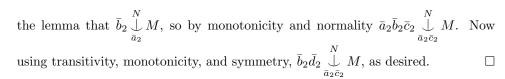　(1) $\mathbf{ortp}(\bar{a}_1\bar{b}_1\bar{c}_1\bar{d}_1'\bar{d}_1; N) = \mathbf{ortp}(\bar{a}_2\bar{b}_2\bar{c}_2\bar{d}_2'\bar{d}_2; N)$.
　(2) $\bar{b}_\ell\bar{d}_\ell \underset{\bar{a}_\ell\bar{c}_\ell\bar{d}_\ell'}{\overset{N}{\downarrow}} M$ for $\ell = 1, 2$.

Proof. Using Lemma 6.4, we can enlarge $\bar{a}_1$ and $\bar{a}_2$ if necessary to assume without loss of generality that $\bar{d}_1'$ is empty. Now using local character, fix $\bar{c}_1 \in M$ such that $\bar{b}_1\bar{d}_1 \underset{\bar{a}_1\bar{c}_1}{\overset{N}{\downarrow}} M$. By Lemma 6.4, there exists $\bar{c}_2 \in M$ such that $\mathbf{ortp}(\bar{a}_1\bar{b}_1\bar{c}_1; N) = \mathbf{ortp}(\bar{a}_2\bar{b}_2\bar{c}_2; N)$. Let f be an automorphism of N witnessing this. By extension and monotonicity, pick $\bar{d}_2 \in N$ such that $\mathbf{ortp}(\bar{d}_2/\bar{a}_2\bar{b}_2\bar{c}_2; N) = \mathbf{ortp}(f(\bar{d}_1)/\bar{a}_2\bar{b}_2\bar{c}_2; N)$ and $\bar{d}_2 \underset{\bar{a}_2\bar{b}_2\bar{c}_2}{\overset{N}{\downarrow}} M\bar{b}_2$. First observe that $\bar{d}_2 \in N\backslash M$ by disjointness and invariance. It remains to see that $\bar{b}_2\bar{d}_2 \underset{\bar{a}_2\bar{c}_2}{\overset{N}{\downarrow}} M$. We do this using a standard nonforking calculus argument: by normality, $\bar{b}_2\bar{d}_2 \underset{\bar{a}_2\bar{b}_2\bar{c}_2}{\overset{N}{\downarrow}} M\bar{b}_2$ and we also know from the hypotheses of

the lemma that $\bar{b}_2 \underset{\bar{a}_2}{\overset{N}{\underset{\smile}{\downarrow}}} M$, so by monotonicity and normality $\bar{a}_2\bar{b}_2\bar{c}_2 \underset{\bar{a}_2\bar{c}_2}{\overset{N}{\underset{\smile}{\downarrow}}} M$. Now

using transitivity, monotonicity, and symmetry, $\bar{b}_2\bar{d}_2 \underset{\bar{a}_2\bar{c}_2}{\overset{N}{\underset{\smile}{\downarrow}}} M$, as desired. $\qquad\square$

We can now build the desired automorphism which fixes M setwise.

Lemma 6.6. Assume that $\mathfrak{K}$ is nicely $\aleph_0$-stable, supersimple, and categorical in $\aleph_0$. Let $M \in \mathfrak{K}_{\aleph_0}$, $A \subseteq |M|$ finite, $p, q \in \mathscr{S}^{<\omega}(M)$ such that p and q do not fork over A. If $p \restriction A = q \restriction A$, then there exists an automorphism f of M fixing A such that $f(p) = q$.

Proof. Let $N \in \mathfrak{K}_{\aleph_0}$ be brimmed over M. Say $p = \mathbf{ortp}(\bar{b}_1/M; N)$, $q = \mathbf{ortp}(\bar{b}_2/M; N)$. Let $\bar{a}$ be an enumeration of A and let $\bar{a}_\ell := \bar{a}$, $\ell = 1, 2$. Now apply Lemma 6.5 repeatedly in a back and forth argument to build an automorphism g of N fixing A such that $g(\bar{b}_1) = \bar{b}_2$ and $g[M] = M$. Let $f := g \restriction M$. $\qquad\square$

We now show that we can actually build an automorphism fixing M *pointwise*. The next lemma is the main argument for this:

Lemma 6.7. Assume that $\mathfrak{K}$ is nicely $\aleph_0$-stable, supersimple, and categorical in $\aleph_0$. Let $M \in \mathfrak{K}_{\aleph_0}$ and let N be brimmed over M. Let $\bar{b}_1, \bar{b}_2 \in N$ be such that $\mathbf{ortp}(\bar{b}_1/A; N) = \mathbf{ortp}(\bar{b}_2/A; N)$ for all finite $A \subseteq |M|$. For any $\bar{d}_1 \in N$, there exists $\bar{d}_2 \in N$ such that $\mathbf{ortp}(\bar{b}_1\bar{d}_1/A; N) = \mathbf{ortp}(\bar{b}_2\bar{d}_2/A; N)$ for all finite $A \subseteq |M|$.

Proof. Let $p := \mathbf{ortp}(\bar{b}_1/M; N)$, $q := \mathbf{ortp}(\bar{b}_2/M; N)$. Fix $A \subseteq |M|$ finite such that both p and q do not fork over A and $\mathbf{ortp}(\bar{b}_1\bar{d}_1/M; N)$ does not split over A. By Lemma 6.6, there exists an automorphism f of M fixing A such that $f(p) = q$. Let g be an automorphism of N extending f such that $g(\bar{b}_1) = \bar{b}_2$ and let $\bar{d}_2 := g(\bar{d}_1)$. We claim that this works. Fix $C \subseteq |M|$ finite and let $\bar{c}_2$ be an enumeration of C. Let $\bar{c}_1 := g^{-1}(\bar{c}_2)$. We know that $\mathbf{ortp}(\bar{c}_1/A; N) = \mathbf{ortp}(\bar{c}_2/A; N)$, so by nonsplitting, $\mathbf{ortp}(\bar{b}_1\bar{d}_1\bar{c}_1/A; N) = \mathbf{ortp}(\bar{b}_1\bar{d}_1\bar{c}_2/A; N)$. Applying g, we have that $\mathbf{ortp}(\bar{b}_1\bar{d}_1\bar{c}_1/A; N) = \mathbf{ortp}(\bar{b}_2\bar{d}_2\bar{c}_2/A; N)$. Putting the two equalities together, $\mathbf{ortp}(\bar{b}_1\bar{d}_1\bar{c}_2/A; N) = \mathbf{ortp}(\bar{b}_2\bar{d}_2\bar{c}_2/A; N)$, so $\mathbf{ortp}(\bar{b}_1\bar{d}_1/C; N) = \mathbf{ortp}(\bar{b}_2\bar{d}_2/C; N)$, as desired. $\qquad\square$

We have arrived to the main theorem of this section:

Theorem 6.8. If $\mathfrak{K}$ is nicely $\aleph_0$-stable, supersimple, and categorical in $\aleph_0$, then $\mathfrak{K}$ is $(< \aleph_0, \aleph_0)$-local (recall Definition 5.6).

Proof. Let $M \in \mathfrak{K}_{\aleph_0}$ and let $p, q \in \mathscr{S}^{<\omega}(M)$ be such that $p \restriction A = q \restriction A$ for all finite $A \subseteq |M|$. Let N be brimmed over M and let $\bar{b}_1, \bar{b}_2 \in N$ realize p and q respectively. Now apply Lemma 6.7 in a back and forth argument to get an automorphism of N fixing M taking $\bar{b}_1$ to $\bar{b}_2$. $\qquad\square$

References

[Bal09]   John T. Baldwin, *Categoricity*, University Lecture Series, vol. 50, American Mathematical Society, 2009.

[BGKV16] Will Boney, Rami Grossberg, Alexei Kolesnikov, and Sebastien Vasey, *Canonical fork-ing in AECs*, Annals of Pure and Applied Logic **167** (2016), no. 7, 590–613.

[BHH$^+$14] Martin Bays, Bradd Hart, Tapani Hyttinen, Meeri Kesälä, and Jonathan Kirby, *Quasi-minimal structures and excellence*, Bulletin of the London Mathematical Society **46** (2014), no. 1, 155–163.

[BL03] Steven Buechler and Olivier Lessmann, *Simple homogeneous models*, Journal of the American Mathematical Society **16** (2003), no. 1, 91–121.

[BL16] John T. Baldwin and Paul B. Larson, *Iterated elementary embeddings and the model theory of infinitary logic*, Annals of Pure and Applied Logic **167** (2016), 309–334.

[BLS15] John T. Baldwin, Paul B. Larson, and Saharon Shelah, *Almost Galois ω-stable classes*, The Journal of Symbolic Logic **80** (2015), no. 3, 763–784.

[BV] Will Boney and Sebastien Vasey, *Good frames in the Hart-Shelah example*, Preprint. URL: `http://arxiv.org/abs/1607.03885v3`.

[BV17a] ———, *Chains of saturated models in AECs*, Archive for Mathematical Logic **56** (2017), no. 3, 187–213.

[BV17b] ———, *Tameness and frames revisited*, The Journal of Symbolic Logic **82** (2017), no. 3, 995–1021.

[Dru13] Fred Drueck, *Limit models, superlimit models, and two cardinal problems in abstract elementary classes*, Ph.D. thesis, University of Illinois at Chicago, 2013, Available online. URL: `http://homepages.math.uic.edu/~drueck/thesis.pdf`.

[Fri73] Harvey M. Friedman, *Countable models of set theories*, Cambridge Summer School in Mathematical Logic (A.R.D. Mathias and H. Rogers, eds.), Springer-Verlag, 1973, pp. 539–573.

[Gro02] Rami Grossberg, *Classification theory for abstract elementary classes*, Contemporary Mathematics **302** (2002), 165–204.

[GV] Rami Grossberg and Sebastien Vasey, *Equivalent definitions of superstability in tame abstract elementary classes*, The Journal of Symbolic Logic, To appear. URL: `http://arxiv.org/abs/1507.04223v5`.

[GV06] Rami Grossberg and Monica VanDieren, *Galois-stability for tame abstract elementary classes*, Journal of Mathematical Logic **6** (2006), no. 1, 25–49.

[GVV16] Rami Grossberg, Monica VanDieren, and Andrés Villaveces, *Uniqueness of limit models in classes with amalgamation*, Mathematical Logic Quarterly **62** (2016), 367–382.

[HK06] Tapani Hyttinen and Meeri Kesälä, *Independence in finitary abstract elementary classes*, Annals of Pure and Applied Logic **143** (2006), 103–138.

[HK16] Tapani Hyttinen and Kaisa Kangas, *Quasiminimal structures, groups and Zariski-like geometries*, Annals of Pure and Applied Logic **167** (2016), no. 6, 457–505.

[HS90] Bradd Hart and Saharon Shelah, *Categoricity over P for first order T or categoric-ity for $\phi \in L_{\omega_1,\omega}$ can stop at $\aleph_k$ while holding for $\aleph_0, \ldots, \aleph_{k-1}$*, Israel Journal of Mathematics **70** (1990), 219–235.

[JS13] Adi Jarden and Saharon Shelah, *Non-forking frames in abstract elementary classes*, Annals of Pure and Applied Logic **164** (2013), 135–191.

[Kir13] Jonathan Kirby, *A note on the axioms for Zilber's pseudo-exponential fields*, Notre Dame Journal of Formal Logic **54** (2013), 509–520.

[Les05] Olivier Lessmann, *Upward categoricity from a successor cardinal for tame abstract elementary classes with amalgamation*, The Journal of Symbolic Logic **70** (2005), no. 2, 639–660.

[LS] Paul B. Larson and Saharon Shelah, *On the absoluteness of orbital ω-stability*, Preprint. URL: `http://shelah.logic.at/files/1073.pdf`.

[She75] Saharon Shelah, *Categoricity in $\aleph_1$ of sentences in $L_{\omega_1,\omega}(Q)$*, Israel Journal of Mathematics **20** (1975), no. 2, 127–148.

[She83a] ———, *Classification theory for non-elementary classes I: The number of uncountable models of $\psi \in L_{\omega_1,\omega}$. Part A*, Israel Journal of Mathematics **46** (1983), no. 3, 214–240.

[She83b] ———, *Classification theory for non-elementary classes I: The number of uncountable models of $\psi \in L_{\omega_1,\omega}$. Part B*, Israel Journal of Mathematics **46** (1983), no. 4, 241–273.

[She87a] ———, *Classification of non elementary classes II. Abstract elementary classes*, Classification Theory (Chicago, IL, 1985) (John T. Baldwin, ed.), Lecture Notes in Mathematics, vol. 1292, Springer-Verlag, 1987, pp. 419–497.

[She87b] _____ , *Universal classes*, Classification theory (Chicago, IL, 1985) (John T. Baldwin, ed.), Lecture Notes in Mathematics, vol. 1292, Springer-Verlag, 1987, pp. 264–418.

[She09a] _____ , *Classification theory for abstract elementary classes*, Studies in Logic: Mathematical logic and foundations, vol. 18, College Publications, 2009.

[She09b] _____ , *Classification theory for abstract elementary classes 2*, Studies in Logic: Mathematical logic and foundations, vol. 20, College Publications, 2009.

[SV99] Saharon Shelah and Andrés Villaveces, *Toward categoricity for classes with no maximal models*, Annals of Pure and Applied Logic **97** (1999), 1–25.

[Van06] Monica VanDieren, *Categoricity in abstract elementary classes with no maximal models*, Annals of Pure and Applied Logic **141** (2006), 108–147.

[Van16a] _____ , *Superstability and symmetry*, Annals of Pure and Applied Logic **167** (2016), no. 12, 1171–1183.

[Van16b] _____ , *Symmetry and the union of saturated models in superstable abstract elementary classes*, Annals of Pure and Applied Logic **167** (2016), no. 4, 395–407.

[Vasa] Sebastien Vasey, *Quasiminimal abstract elementary classes*, Archive for Mathematical Logic, To appear. DOI: `10.1007/s00153-017-0570-7` . URL: `https://arxiv.org/abs/1611.07380v7`.

[Vasb] _____ , *Toward a stability theory of tame abstract elementary classes*, Preprint. URL: `http://arxiv.org/abs/1609.03252v4`.

[Vas16a] _____ , *Building independence relations in abstract elementary classes*, Annals of Pure and Applied Logic **167** (2016), no. 11, 1029–1092.

[Vas16b] _____ , *Infinitary stability theory*, Archive for Mathematical Logic **55** (2016), 567–592.

[Vas17] _____ , *Downward categoricity from a successor inside a good frame*, Annals of Pure and Applied Logic **168** (2017), no. 3, 651–692.

[VV17] Monica VanDieren and Sebastien Vasey, *Symmetry in abstract elementary classes with amalgamation*, Archive for Mathematical Logic **56** (2017), no. 3, 423–452.

E-mail address: `shelah@math.huji.ac.il`

URL: `http://shelah.logic.at`

Einstein Institute of Mathematics, Edmond J. Safra Campus, Givat Ram, The Hebrew University of Jerusalem, Jerusalem, 91904, Israel, and Department of Mathematics, Hill Center - Busch Campus, Rutgers, The State University of New Jersey, 110 Frelinghuysen Road, Piscataway, NJ 08854-8019, USA

E-mail address: `sebv@math.harvard.edu`

URL: `http://math.harvard.edu/~sebv/`

Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, Pennsylvania, USA

Current address: Department of Mathematics, Harvard University, Cambridge, Massachusetts, USA