INDISCERNIBLE EXTRACTION AND MORLEY SEQUENCES

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Abstract. We present a new proof of the existence of Morley sequences in simple theories. We avoid using the Erdős-Rado theorem and instead use Ramsey's theorem. The proof shows that the basic theory of forking in simple theories can be developed inside $\left\langle H((2^{2^{|T|}})^+), \in \right\rangle$, answering a question of Grossberg, Iovino and Lessmann, as well as a question of Baldwin.

1. Introduction

Shelah [She80, Lemma 9.3] has shown that, in a simple first-order theory T, Morley sequences exist for every type. The proof proceeds by first building an independent sequence of length $\beth_{\left(2^{|T|}\right)^+}$ for the given type and then using the Erdős-Rado theorem together with Morley's method to extract the desired indiscernibles.

After slightly improving on the length of the original independent sequence [GIL02, Appendix A], Grossberg, Iovino and Lessmann observed that, in contrast, most of the theory of forking in a stable first-order theory T can be carried out inside $\langle H(\chi), \in \rangle$ for $\chi := \left(2^{2^{|T|}}\right)^+$. The authors then asked whether the same could be said about simple theories, and so in particular whether there was another way to build Morley sequences there.

Baldwin (see [Bal10] and [Bal13, Question 3.1.9]) similarly asked whether the equivalence between forking and dividing in simple theories can be proven without using the axiom of replacement.

We answer those questions in the affirmative by showing how to build a Morley sequence from any infinite independent sequence. We avoid using cardinals like $\beth_{\left(2^{|T|}\right)^+}$, whose existence need powerful set-theoretic principles, and use only axioms from "ordinary mathematics".

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Our construction relies on a property of forking we call *dual finite character*. We show it holds in simple theories, and present Itay Kaplan's proof (based on an argument of Chernikov) that the converse is also true.

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2. Preliminaries

For the rest of this paper, fix a complete first-order theory T in a language $L(T)$ and work inside its monster model $\mathfrak{C}$. We write $|T|$ for $|L(T)| + \aleph_0$. We denote by $\mathrm{Fml}(L(T))$ the set of first-order formulas in the language $L(T)$. If A is a set, we say a formula is *over* A if it has parameters from A. For a tuple $\bar{a}$ in $\mathfrak{C}$ and ϕ a formula, we write $\models \phi[\bar{a}]$ instead of $\mathfrak{C} \models \phi[\bar{a}]$.

When I is a linearly ordered set, $(\bar{a}_i)_{i \in I}$ are tuples, and $i \in I$, we write $\bar{a}_{<i}$ for $(\bar{a}_j)_{j<i}$. It is often assumed without comments that all the $\bar{a}_i$s have the same (finite) arity.

One could, in the spirit of reverse mathematics [Fri74], try to find the exact proof-theoretic strength of some of the facts of simple theories, like Harnik did for stable theories [Har85, Har87]. We do not attempt this here and focus on the model theory: For simplicity, we will state and prove our results in ZFC.

We point out, however, that all the results of this paper (except for Fact 5, Corollary 6, and Proposition 7 which are background facts not used anywhere else) can be formalized in ZC^0, a much weaker theory which we now define[1]:

Definition 1 (ZC^0)**.** Work in the language of set theory with a constant symbol Θ (intended to denote $|T|$). There is a version of ZFC for that language, which we will also denote by ZFC. Following the terminology of [Kun80], let ZC denote ZFC without replacement (but with full comprehension). Let $\mathrm{ZC} - P$ be ZC without power set. Let ZC^0 be $\mathrm{ZC} - P$, together with the following two axioms:

[1]Although we have not verified this, we suspect ZC^0 could be replaced by a suitably modified version of Woodin's ZFC^* [Woo10].

(1) Θ is an infinite cardinal.
(2) For any set X of size $\leq \Theta$, $\mathcal{P}(\mathcal{P}(X))$ exists.

Note that the universe of "ordinary mathematics" $\langle V(\omega+\omega), \in, \aleph_0 \rangle$ is a model of ZC, hence of ZC^0. Letting $\chi := \left(2^{2^{|T|}}\right)^+$, $\langle H(\chi), \in, |T| \rangle$ is also a model of ZC^0. When formalizing our statements in ZC^0, some obvious changes have to be made. For example, we need to work inside local monster models, assume $|T| \leq \Theta$, and similarly bound the sizes of other sets.

We assume the reader is familiar with forking. As a brief reminder, forking is an independence notion, originally developed by Shelah to solve the stability spectrum problem, which has turned out to be central in classification theory. We will use the following definition:

Definition 2 (Forking)**.** Let $\bar{b}$ be a tuple, let $k < \omega$, and let A be a set. A formula $\phi(\bar{x}, \bar{b})$ *k-divides over A* if there is a sequence $\langle \bar{b}_i \mid i < \omega \rangle$ such that $\text{tp}(\bar{b}_i/A) = \text{tp}(\bar{b}/A)$ for all $i < \omega$, and any k-elements subset of $\{\phi(\bar{x}, \bar{b}_i) \mid i < \omega\}$ is inconsistent. $\phi(\bar{x}, \bar{b})$ *divides over A* if it k-divides over A for some $k < \omega$.

A formula $\phi(\bar{x}, \bar{b})$ *forks over A* if there are formulas $\phi_0(\bar{x}, \bar{b}_0), \ldots, \phi_{m-1}(\bar{x}, \bar{b}_{m-1})$, each dividing over A, such that $\phi(\bar{x}, \bar{b}) \vdash \bigvee_{i<m} \phi_i(\bar{x}, \bar{b}_i)$.

A type p *forks over A* if $p \vdash \phi(\bar{x}, \bar{b})$ for some formula $\phi(\bar{x}, \bar{b})$ that forks over A.

It turns out most of the results of this paper do not rely on this exact definition, but only on abstract properties of forking such as invariance, extension, or symmetry (in simple theories).

The following concepts are central:

Definition 3 (Morley sequence)**.** Let I be a linearly ordered set. Let $\mathbf{I} := \langle \bar{a}_i \mid i \in I \rangle$ be a sequence of finite tuples of the same arity. Let $A \subseteq B$ be sets, and let $p \in S(B)$ be a type that does not fork over A.

$\mathbf{I}$ is said to be an *independent sequence for p over A* if:

(1) For all $i \in I$, $\bar{a}_i \models p$.
(2) For all $i \in I$, $\text{tp}(\bar{a}_i / B\bar{a}_{<i})$ does not fork over A.

$\mathbf{I}$ is said to be a *Morley sequence for p over A* if:

(1) $\mathbf{I}$ is an independent sequence for p over A.
(2) $\mathbf{I}$ is indiscernible over B.

The following fact about forking (which holds in all complete first-order theories) follows from the extension property and the compactness theorem:

Fact 4 (Existence of independent sequences)**.** Let $A \subseteq B$ be sets, and let $p \in S(B)$ be a type that does not fork over A. Let I be a linearly ordered set. Then there is an independent sequence $\mathbf{I} := \langle \bar{a}_i \mid i \in I \rangle$ for p over A.

3. Indiscernible extraction

Fact 4 tells us it is easy to build independent sequences. What about Morley sequences? If a sufficiently long sequence always contains an indiscernible subsequence, the existence of Morley sequences follows from Fact 4. This is the case in stable theories, but not in general: There are counterexamples among both simple [She85, p. 209] and dependent [KS] theories. Thus a different approach is needed in the unstable case. Shelah observed [She80, Lemma 9.3] the following:

Fact 5 (The indiscernible extraction theorem)**.** Let A be a set, and let I be a linearly ordered set. Let $\gamma := \left(2^{|T|+|A|}\right)^+$, $\mu := \beth_\gamma$, and let $\langle \bar{a}_j \mid j < \mu \rangle$ be a sequence of finite tuples of the same arity. Then there exists a sequence $\mathbf{I} := \left\langle \bar{b}_i \mid i \in I \right\rangle$, indiscernible over A such that:

For any $i_0 < \ldots < i_{n-1}$ in I, there exists $j_0 < \ldots < j_{n-1} < \mu$ so that $\text{tp}(\bar{b}_{i_0} \ldots \bar{b}_{i_{n-1}}/A) = \text{tp}(\bar{a}_{j_0} \ldots \bar{a}_{j_{n-1}}/A)$.

Corollary 6 (Existence of Morley sequences in arbitrary theories)**.** Let $A \subseteq B$. Let $p \in S(B)$ be a type that does not fork over A. Let I be a linearly ordered set. Then there is a Morley sequence $\mathbf{I} := \left\langle \bar{b}_i \mid i \in I \right\rangle$ for p over A.

Proof sketch. Build a long-enough independent sequence for p over A using Fact 4, then use the monotonicity, finite character and invariance properties of forking to see that the extracted sequence $\mathbf{I}$ given by Fact 5 is as desired. $\square$

It is shown in [GIL02, Theorem A.2] that the length μ of the original sequence in Fact 5 can be decreased to $\beth_{\delta(|T|+|A|)}$, where $\delta(\lambda)$ is defined to be the least ordinal not definable in the logic $L_{\lambda^+,\omega}$. The proof is set-theoretic. The main idea is to use the Erdős–Rado theorem infinitely many times inside an ill-founded model of (a large fragment of) set theory.

Observe that in the indiscernible extraction theorem, the bound $\mu := \beth_{\delta(|T|+|A|)}$ is optimal:

Proposition 7. For every infinite cardinal λ, and every $\mu < \beth_{\delta(\lambda)}$, there is a theory T with $|T| = \lambda$ such that the indiscernible extraction theorem (with $A = \emptyset$) fails for sequences of length μ.

Proof. Fix λ. Pick $\mu < \beth_{\delta(\lambda)}$. By building on Morley's idea for lower bounds of Hanf numbers (see [She90], Theorem VII.5.4), we can get a complete theory T of size λ with built-in Skolem functions, and a type p such that $\mathrm{EC}(T, p)$ (the class of all models of T omitting the type p) contains a model M of size $\geq \mu$, but no model of size $\beth_{\delta(\lambda)}$. Without loss of generality, $||M|| = \mu$. Let $\{a_j \mid j < \mu\}$ enumerate M. Work in the monster model for T. By assumption, a_j does not realize p, for all $j < \mu$.

Assume for a contradiction that there exists a sequence of indiscernibles $\mathbf{I} := \left\langle b_i \mid i < \beth_{\delta(\lambda)} \right\rangle$ satisfying the conclusion of the indiscernible extraction theorem. Then in particular, b_i does not realize p for any $i < \beth_{\delta(\lambda)}$. Let N be the Skolem hull of $\mathbf{I}$. Then $||N|| = \beth_{\delta(\lambda)}$, so by construction, $N \notin \mathrm{EC}(T, p)$. This means there is $i_0 < \ldots < i_n < \beth_{\delta(\lambda)}$ and a term τ such that $\tau(b_{i_0}, \ldots, b_{i_n}) \models p$, so $b_{i_0} \ldots b_{i_n} \models q$, where $q(\bar{x}) := p(\tau(\bar{x}))$. But then for some $j_0 < \ldots < j_n < \mu$, $a_{j_0} \ldots a_{j_n} \models q$, i.e. $\tau(a_{j_0}, \ldots, a_{j_n}) \models p$. But $\tau(a_{j_0}, \ldots, a_{j_n}) \in M \in \mathrm{EC}(T, p)$, so $\tau(a_{j_0}, \ldots, a_{j_n})$ does not realize p, a contradiction. $\square$

Proposition 7 does not rule out a smaller upper bound μ for particular classes of theories: As was hinted at earlier, if T is stable $\mu := \left(2^{|T|+|A|}\right)^+$ is enough (see [She90, Theorem I.2.8]). We do not know if there is also a smaller bound for simple theories. Restricting the initial sequence to be independent may also give additional information.

We emphasize once again that Fact 5, Corollary 6, and Proposition 7 are ZFC results. *They will not be used in the rest of this paper.*

Recall from the introduction that we aim to prove Corollary 6 (for simple theories) in ZC^0. In particular, we cannot assume the existence of cardinals like $\beth_{\delta(|T|)}$, so a different approach is needed: we will use the following weaker version of the indiscernible extraction theorem which holds for $\mu = \omega$. As the proof makes clear, this is really just a slight variation on the Ehrenfeucht-Mostowski theorem.

Theorem 8 (The weak indiscernible extraction theorem)**.** Let A be a set, and let I be a linearly ordered set. Let $\mathbf{J} := \langle \bar{a}_j \mid j < \omega \rangle$ be a sequence of finite tuples of the same arity. Then there exists a sequence $\mathbf{I} := \langle \bar{b}_i \mid i \in I \rangle$, indiscernible over A such that:

For any $i_0 < \ldots < i_{n-1}$ in I, for all *finite* $q \subseteq \text{tp}(\bar{b}_{i_0} \ldots \bar{b}_{i_{n-1}}/A)$, there exists $j_0 < \ldots < j_{n-1} < \omega$ so that $\bar{a}_{j_0} \ldots \bar{a}_{j_{n-1}} \models q$.

Proof. By adding new constant symbols to T, we can without loss of generality assume that $A = \emptyset$.

Using the terminology of [TZ12, Definition 5.1.2], let Γ be the Ehrenfeucht-Mostowski type of $\mathbf{J}$. That is:

$$\Gamma := \{\phi(\bar{x}_0, \ldots, \bar{x}_{n-1}) \in \text{Fml}(L(T)) \mid \quad \models \phi[\bar{a}_{j_0}, \ldots, \bar{a}_{j_{n-1}}] \text{ for all } j_0 < \ldots < j_{n-1} < \omega\}$$

Using Ramsey's theorem (see [TZ12, Lemma 5.1.3]), there is a sequence of indiscernibles $\mathbf{I} := \langle \bar{b}_i \mid i \in I \rangle$ realizing Γ, i.e. for any $n < \omega$, any $\phi(\bar{x}_0, \ldots, \bar{x}_{n-1}) \in \Gamma$, and any $i_0 < \ldots < i_{n-1}$ in I, we have $\models \phi[\bar{b}_{i_0}, \ldots, \bar{b}_{i_{n-1}}]$.

Now, observe that $\mathbf{I}$ is as desired: Let $i_0 < \ldots < i_{n-1}$ in I be given, and let q be a finite subset of $\text{tp}(\bar{b}_{i_0} \ldots \bar{b}_{i_{n-1}}/\emptyset)$. Without loss of generality, q consists of a single formula $\phi(\bar{x})$. Assume for a contradiction there is no $j_0 < \ldots < j_{n-1} < \omega$ such that $\models \phi[\bar{a}_{j_0}, \ldots, \bar{a}_{j_{n-1}}]$. By definition, this implies $\neg\phi(\bar{x}) \in \Gamma$, and hence we must have $\models \neg\phi[\bar{b}_{i_0}, \ldots \bar{b}_{i_{n-1}}]$, a contradiction. $\square$

The reader should be wary of concluding the existence of Morley sequences directly from Theorem 8 and the finite character of forking. Indeed, Theorem 8 does not give us enough invariance to imitate the proof of Corollary 6. In fact, we suspect that a sequence extracted from an independent sequence using Theorem 8 need not in general be Morley.

4. Extracting Morley sequences in simple theories

Next, we investigate the following property of forking:

Definition 9 (Dual finite character)**.** Forking is said to have *dual finite character (DFC)* if whenever $\text{tp}(\bar{c}/A\bar{b})$ forks over A, there is a formula $\phi(\bar{x}, \bar{y})$ over A such that:

- $\models \phi[\bar{c}, \bar{b}]$, and:
- $\models \phi[\bar{c}, \bar{b}']$ implies $\text{tp}(\bar{c}/A\bar{b}')$ forks over A.

Notice that this immediately implies something stronger:

Proposition 10. Assume forking has DFC. Assume $p := \mathrm{tp}(\bar{c}/A\bar{b})$ forks over A, and $\phi(\bar{x}, \bar{y})$ is as given by Definition 9. Then $\mathrm{tp}(\bar{c}'/A) = \mathrm{tp}(\bar{c}/A)$ and $\models \phi[\bar{c}', \bar{b}']$ imply $\mathrm{tp}(\bar{c}'/A\bar{b}')$ forks over A.

Proof. Assume $\mathrm{tp}(\bar{c}'/A) = \mathrm{tp}(\bar{c}/A)$. Let f be an automorphism of $\mathfrak{C}$ fixing A such that $f(\bar{c}') = \bar{c}$. Assume $\models \phi[\bar{c}', \bar{b}']$. Applying f, $\models \phi[\bar{c}, f(\bar{b}')]$. Since ϕ witnesses DFC, $\mathrm{tp}(\bar{c}/Af(\bar{b}'))$ forks over A. Applying f^{-1} and using invariance of forking, $\mathrm{tp}(\bar{c}'/A\bar{b}')$ forks over A. $\square$

Dual finite character is a sort of local definability of forking. It says that forking is witnessed by a formula, in a way that lets us change the *domain* of the type under consideration. This allows us to complete the proof of existence of Morley sequences:

Theorem 11. Assume forking has DFC. Let $A \subseteq B$ be sets. Let $p \in S(B)$ be a type that does not fork over A. Let I be a linearly ordered set. Then there is a Morley sequence $\mathbf{I} := \left\langle \bar{b}_i \mid i \in I \right\rangle$ for p over A.

Proof. Use Fact 4 to build an independent sequence $\mathbf{J} := \langle \bar{a}_j \mid j < \omega \rangle$ for p over A. Let $\mathbf{I} := \left\langle \bar{b}_i \mid i \in I \right\rangle$ be indiscernible over B, as described by Theorem 8. We claim $\mathbf{I}$ is as required.

It is indiscernible over B, and for every $i \in I$, every $\bar{b}_i$ realizes p: If $\bar{b}_i \not\models p$, fix a formula $\phi(\bar{x}, \bar{b}) \in p$ so that $\models \neg\phi[\bar{b}_i, \bar{b}]$. By the defining property of $\mathbf{I}$, there exists $j < \omega$ so that $\models \neg\phi[\bar{a}_j, \bar{b}]$, so $\bar{a}_j \not\models p$, a contradiction.

It remains to see that for every $i \in I$, $p_i := \mathrm{tp}(\bar{b}_i/B\bar{b}_{<i})$ does not fork over A. Assume not, and fix $i \in I$ so that p_i forks over A. Fix $\bar{b} \in B$ and $i_0 < \ldots < i_{n-1} < i$ such that $p_i' := \mathrm{tp}(\bar{b}_i/A\bar{b}_{i_0} \ldots \bar{b}_{i_{n-1}}\bar{b})$ forks over A. Fix $\phi(\bar{x}, \bar{b}_{i_0} \ldots \bar{b}_{i_{n-1}}\bar{b}) \in p_i'$ a formula over A witnessing DFC.

Find $j_0 < \ldots < j_n < \omega$ such that $\models \phi[\bar{a}_{j_n}, \bar{a}_{j_0} \ldots \bar{a}_{j_{n-1}}\bar{b}]$. Since it has already been observed that $\mathrm{tp}(\bar{a}_{j_n}/A) = \mathrm{tp}(\bar{b}_i/A) = p \restriction A$, Proposition 10 implies that $\mathrm{tp}(\bar{a}_{j_n}/A\bar{a}_{j_0} \ldots \bar{a}_{j_{n-1}}\bar{b})$ forks over A, contradicting the independence of $\mathbf{J}$. $\square$

A variation of DFC appears as property A.7' in [Mak84], but we haven't found any other occurrence in the literature. Makkai observed that forking symmetry implies DFC, so this is how we will define simplicity:

Definition 12. A first-order theory T is *simple* if its forking has the symmetry property, i.e. whenever $\mathrm{tp}(\bar{c}/A\bar{b})$ forks over A, $\mathrm{tp}(\bar{b}/A\bar{c})$ forks over A.

This is equivalent to T not having the tree property, or to forking having local character [Kim01, Theorem 2.4]. Moreover, the methods of [Adl09] show that the equivalence can be proven in ZC^0, without using Morley sequences. As an example, we outline why symmetry follows from local character: The key is [Adl09, Theorem 3.6], which shows (without using Morley sequences) that if the D-rank is bounded, then symmetry holds. One can then use [Adl09, Lemma 4.1], which says that the D-rank is bounded if and only if forking has local character.

Lemma 13. Assume T is simple. Then forking has DFC.

Proof. Assume $p := \mathrm{tp}(\bar{c}/A\bar{b})$ fork over A. By symmetry, $q := \mathrm{tp}(\bar{b}/A\bar{c})$ forks over A. Fix $\psi(\bar{y}, \bar{x})$ over A such that $\psi(\bar{y}, \bar{c}) \in q$ witnesses forking, i.e. if $\models \psi[\bar{b}', \bar{c}]$ then $\mathrm{tp}(\bar{b}'/A\bar{c})$ forks over A.

Let $\phi(\bar{x}, \bar{y}) := \psi(\bar{y}, \bar{x})$. Then $\phi(\bar{x}, \bar{b}) \in p$, and if $\models \phi[\bar{c}, \bar{b}']$, then $\models \psi[\bar{b}', \bar{c}]$, so $\mathrm{tp}(\bar{b}'/A\bar{c})$ forks over A, so by symmetry, $\mathrm{tp}(\bar{c}/A\bar{b}')$ forks over A. This shows $\phi(\bar{x}, \bar{y})$ witnesses DFC. $\square$

Corollary 14 (Existence of Morley sequences in simple theories)**.** Assume T is simple. Let $A \subseteq B$ be sets. Let $p \in S(B)$ be a type that does not fork over A. Let I be a linearly ordered set. Then there is a Morley sequence $\mathbf{I} := \left\langle \bar{b}_i \mid i \in I \right\rangle$ for p over A.

Proof. Combine Lemma 13 and Theorem 11. $\square$

We can now answer the questions mentioned in the introduction:

(1) [GIL02, Question A.1] asked whether the first three sections of [GIL02] could be formalized inside $\langle H(\chi), \in \rangle$. We have already observed that $\langle H(\chi), \in, |T| \rangle$ is a model of ZC^0, and Corollary 14 establishes that the existence of Morley sequences ([GIL02, Theorem 1.14]) can be formalized in ZC^0. The only other problematic results are the characterization of forking using the D-rank ([GIL02, Corollary 3.21]), and [GIL02, Corollary 3.20] preceding it. To formalize these in ZC^0, one can use the (equivalent) definition of the D-rank given in [Wag00, Definition 2.3.4], together with the proof of [Wag00, Proposition 2.3.9].
(2) [Bal13, Question 3.1.9] and [Bal10] asked whether the equivalence between forking and dividing in simple theories could be formalized in ZC. ZC^0 is a subtheory of ZC, and the equivalence between forking and dividing is proven in [GIL02, Theorem 2.5]. Thus (1) also answers Baldwin's question.

We end by closing the loop on our study of DFC. Observe first that DFC *can* fail:

Example 15. Let T be the first-order theory of dense linear orderings. Let A be the set of nonzero rational numbers. Let $b_0 < c < b_1$ be positive infinitesimals (i.e. greater than zero but smaller than any positive rational). Then $p := \mathrm{tp}(c/Ab_0b_1)$ divides over A, and for any $\phi(x, b_0, b_1) \in p$, one can find nonzero rationals b'_0, b'_1 such that $\models \phi[c, b'_0, b'_1]$. But then $\mathrm{tp}(c/Ab'_0b'_1) = \mathrm{tp}(c/A)$ does not fork over A. Thus forking does not have DFC in T.

Itay Kaplan pointed out (in a personal communication) that in fact, this example generalizes to any nonsimple theory. Definition 17 and (2) implies (3) implies (1) in Theorem 18 below are due to Kaplan, and I am grateful to him for allowing me to include them here.

The key is to observe that symmetry fails very badly when the theory is not simple:

Fact 16. Assume T is *not* simple. Then there is a model M and tuples $\bar{b}, \bar{c}$ such that $\mathrm{tp}(\bar{b}/M\bar{c})$ is finitely satisfiable in M, but $\mathrm{tp}(\bar{c}/M\bar{b})$ divides over M.

Proof. See [Che14, Lemma 6.16]. □

We are now ready to prove that forking has DFC exactly when the theory is simple. In fact, we only need the following version of DFC:

Definition 17. Forking is said to have *weak dual finite character (weak DFC)* if whenever M is a model and $\mathrm{tp}(\bar{c}/M\bar{b})$ divides over M, there is a formula $\phi(\bar{x}, \bar{y})$ over M such that:

- $\models \phi[\bar{c}, \bar{b}]$, and:
- $\models \phi[\bar{c}, \bar{b}']$ implies $\mathrm{tp}(\bar{c}/M\bar{b}')$ is not finitely satisfiable in M.

Theorem 18. The following are equivalent:

(1) T is simple.
(2) Forking has DFC.
(3) Forking has weak DFC.

Proof. (1) implies (2) is Lemma 13, and (2) implies (3) is because finite satisfiability implies nonforking. We show (3) implies (1). Assume T is not simple. Fix M and $\bar{b}$, $\bar{c}$ as given by Fact 16. In particular, $p := \mathrm{tp}(\bar{c}/M\bar{b})$ divides over M. Let $\phi(\bar{x}, \bar{y})$ be a formula over M such that $\models \phi[\bar{c}, \bar{b}]$. By assumption, $\mathrm{tp}(\bar{b}/M\bar{c})$ is finitely satisfiable in M, so in particular there is $\bar{b}' \in M$ such that $\models \phi[\bar{c}, \bar{b}']$. Thus $\mathrm{tp}(\bar{c}/M\bar{b}') = \mathrm{tp}(\bar{c}/M)$ must be finitely satisfiable over M, hence $\phi(\bar{x}, \bar{y})$ cannot witness weak DFC for p. Since ϕ was arbitrary, this shows weak DFC fails. $\square$

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