DOWNWARD CATEGORICITY FROM A SUCCESSOR INSIDE A GOOD FRAME

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ABSTRACT. We use orthogonality calculus to prove a downward transfer from categoricity in a successor in abstract elementary classes (AECs) that have a good frame (a forking-like notion for types of singletons) on an interval of cardinals:

Theorem 0.1. Let $\mathbf{K}$ be an AEC and let $\mathrm{LS}(\mathbf{K}) \leq \lambda < \theta$ be cardinals. If $\mathbf{K}$ has a type-full good $[\lambda, \theta]$-frame and $\mathbf{K}$ is categorical in both λ and θ^+, then $\mathbf{K}$ is categorical in all $\lambda' \in [\lambda, \theta]$.

We deduce improvements on the threshold of several categoricity transfers that do not mention frames. For example, we can combine our theorem with the upward transfer of Grossberg and VanDieren to obtain Shelah's categoricity conjecture from a successor for tame AECs with amalgamation:

Corollary 0.2. Let $\mathbf{K}$ be a $\mathrm{LS}(\mathbf{K})$-tame AEC with amalgamation. If $\mathbf{K}$ is categorical in *some successor* $\lambda \geq \beth_{\left(2^{\mathrm{LS}(\mathbf{K})}\right)^+}$, then $\mathbf{K}$ is categorical in *all* $\lambda' \geq \beth_{\left(2^{\mathrm{LS}(\mathbf{K})}\right)^+}$.

Assuming that the AEC has primes over sets of the form $M \cup \{a\}$, the successor hypothesis can be removed from the previous corollary. It can also be removed by heavily using results of Shelah and assuming the weak generalized continuum hypothesis:

Corollary 0.3. Assume $2^\theta < 2^{\theta^+}$ for every cardinal θ, as well as an unpublished claim of Shelah. Let $\mathbf{K}$ be a $\mathrm{LS}(\mathbf{K})$-tame AEC with amalgamation. If $\mathbf{K}$ is categorical in *some* $\lambda \geq \beth_{\left(2^{\mathrm{LS}(\mathbf{K})}\right)^+}$, then $\mathbf{K}$ is categorical in *all* $\lambda' \geq \beth_{\left(2^{\mathrm{LS}(\mathbf{K})}\right)^+}$.

Contents

1. Introduction

In his two volume book [She09a, She09b] on classification theory for abstract elementary classes (AECs), Shelah introduces the notion of a *good λ-frame* [She09a, II.2.1]. Roughly, a good λ-frame is a local notion of independence for types of length one over models of size λ. Good frames are the central concept of the book. In Chapter II and III, Shelah discusses the following three questions regarding frames:

Question 1.1.

(1) Given an AEC $\mathbf{K}$, when does there exist a good λ-frame $\mathfrak{s}$ whose underlying AEC $\mathbf{K}_{\mathfrak{s}}$ is $\mathbf{K}_{\lambda}$ (or some subclass of saturated models in $\mathbf{K}_{\lambda}$)?
(2) Given a good λ-frame, under what conditions can it be extended to a good λ^+-frame?
(3) Once one has a good frame, how can one prove categoricity transfers?

Shelah's answers (see for example II.3.7, III.1, and III.2 in [She09a]) involve a mix of set-theoretic hypotheses (such as the weak generalized continuum hypothesis: $2^{\theta} < 2^{\theta^+}$ for all cardinals θ) and strong local model-theoretic hypotheses (such as few models in λ^{++}). While Shelah's approach is very powerful (for example in [She09a, Chapter IV], Shelah proves the eventual categoricity conjecture in AECs with amalgamation, see more below), most of his results do not hold in ZFC.

An alternate approach is to make *global* model-theoretic assumptions. In [GV06b], Grossberg and VanDieren introduced *tameness*, a locality property which says that Galois types are determined by their small restrictions. In [Bon14a], Boney showed that in an AEC which is λ-tame for types of length two and has amalgamation, a good λ-frame can be extended to all models of size at least λ (we call the resulting object a good $(\geq \lambda)$-frame, and similarly define good $[\lambda, \theta]$-frame for $\theta > \lambda$ a cardinal). In [BVb], tameness for types of length two was improved to tameness for types of length one. In particular, the answer to Question 1.1.(2) is always positive in tame AECs with amalgamation. As for existence (Question 1.1.(1)), we showed in [Vas16] how to build good frames in tame AECs with amalgamation assuming categoricity in a cardinal of high-enough cofinality. Further improvements were made in [Vasa, BVa, VV]. This gives answers to Questions 1.1.(1),(2) in tame AECs with amalgamation:

Fact 1.2. Let $\mathbf{K}$ be an AEC with amalgamation and let $\lambda \geq \mathrm{LS}(\mathbf{K})$ be such that $\mathbf{K}$ is λ-tame.

(1) [BVb, Corollary 6.9] If there is a good λ-frame $\mathfrak{s}$ with $\mathbf{K}_{\mathfrak{s}} = \mathbf{K}_{\lambda}$, then $\mathfrak{s}$ can be extended to a good $(\geq \lambda)$-frame (with underlying class $\mathbf{K}$).
(2) [VV, Corollary 6.14] If $\mathbf{K}$ has no maximal models and is categorical in some $\mu > \lambda$, then there is a type-full good λ^+-frame with underlying class the Galois saturated models of $\mathbf{K}$ of size λ^+.

In this paper, we study Question 1.1.(3) in the global setting: assuming the existence of a good frame together with some global model-theoretic properties, what can we say about the categoricity spectrum? From the two results above, it is natural to assume that we are already working inside a type-full good $(\geq \lambda)$-frame (this implies properties such as λ-tameness and amalgamation). It is then known how to transfer categoricity with the additional assumption that the class has primes over sets of the form $M \cup \{a\}$. This has been used to prove Shelah's eventual categoricity conjecture for universal classes [Vasd, Vase].

Definition 1.3 (III.3.2 in [She09a]). An AEC **K** *has primes* if for any nonalgebraic Galois type $p \in \mathrm{gS}(M)$ there exists a triple (a, M, N) such that $p = \mathrm{gtp}(a/M; N)$ and for every $N' \in \mathbf{K}$, $a' \in |N'|$, such that $p = \mathrm{gtp}(a'/M; N')$, there exists $f : N \xrightarrow[M]{} N'$ with $f(a) = a'$.

Fact 1.4 (Theorem 2.16 in [Vasc]). Assume that there is a type-full good $[\lambda, \theta)$-frame on the AEC **K**. Assume that **K** has primes and is categorical in λ. If **K** is categorical in *some* $\mu \in (\lambda, \theta]$, then **K** is categorical in *all* $\mu' \in (\lambda, \theta]$.

What if we do not assume existence of primes? The main result of this paper is a downward categoricity transfer for global good frames categorical in a successor:

Theorem 6.14. Assume that there is a type-full good $[\lambda, \theta]$-frame on the AEC **K**. Assume that **K** is categorical in λ. If **K** is categorical in θ^+, then **K** is categorical in *all* $\mu \in [\lambda, \theta]$.

The proof of Theorem 6.14 develops orthogonality calculus in this setup (versions of some of our results have been independently derived by Villaveces and Zambrano [VZ14]). We were heavily inspired from Shelah's development of orthogonality calculus in successful good λ-frames [She09a, Section III.6], and use it to define a notion of unidimensionality similar to what is defined in [She09a, Section III.2]. We show unidimensionality in λ is equivalent to categoricity in λ^+ and use orthogonality calculus to transfer unidimensionality across cardinals. While we work in a more global setup than Shelah's, we do *not* assume that the good frames we work with are successful [She09a, Definition III.1.1], so we do *not* assume that the forking relation is defined for types of models (it is only defined for types of elements). To get around this difficulty, we use the theory of *independent sequences* introduced by Shelah for good λ-frames in [She09a, Section III.5] and developed in [BVb] for global good frames.

Let us discuss the hypotheses of Theorem 6.14. We are assuming that the good frame is *type-full*: the basic types are all the nonalgebraic types. This is a natural assumption to make if we are only interested in tame AECs: by Fact 1.2, type-full good frames exist under natural conditions there. Moreover by [GV, Remark 4.9], if a tame AECs has a good frame, then it has a type-full one (possibly with a different class of models). We do not know if the type-full assumption is necessary; our argument uses it when dealing with minimal types (we do not know in general whether minimal types are basic; if this is the case for the frame we are working with then it is not necessary to assume that it is type-full).

What about categoricity in λ? This is assumed in order to have some starting degree of saturation (namely all the models of size λ are limit models, see Definition 2.2). We do not see it as a strong assumption: in applications, we will take the AEC to be a class of λ-saturated models, where this automatically holds. Still, we do not know if it is necessary.

Another natural question is whether one really needs to assume the existence of a global good frame at all. The Hart-Shelah example [HS90, BK09] shows that it is *not true* that any AEC **K** categorical in LS(**K**) and in a successor $\lambda > $ LS(**K**) is categorical everywhere (even if **K** has amalgamation). One strengthening of Theorem 6.14 assumes only that we have a good frame for *saturated* models. Appendix A states this precisely and outlines a proof. Consequently, most of the result stated above hold assuming only *weak tameness* instead of tameness: that is, only types over saturated models are required to be determined by their small restrictions. In Appendix B we mention in which results of the paper tameness can be replaced by weak tameness.

In the second part of this paper, we give several applications of Theorem 6.14 to Shelah's eventual categoricity conjecture, the central test problem in the classification theory of non-elementary classes (see the introduction of [Vasd] for a history):

Conjecture 1.5 (Conjecture N.4.2 in [She09a]). An AEC that is categorical in a high-enough cardinal is categorical on a tail of cardinals.

For an AEC **K** we will call *Shelah's categoricity conjecture for* **K** the statement that if **K** is categorical in *some* $\lambda \geq \beth_{\left(2^{\mathrm{LS}(\mathbf{K})}\right)^+}$, then **K** is categorical in *all* $\lambda' \geq \beth_{\left(2^{\mathrm{LS}(\mathbf{K})}\right)^+}$ (that is, we explicitly require the "high-enough" threshold to be the first Hanf number).

Shelah [She99] has proven a *downward* categoricity transfer from a successor in AECs with amalgamation where the threshold is the second Hanf number. Complementing it, Grossberg and VanDieren have established an *upward* transfer assuming tameness:

Fact 1.6 ([GV06c, GV06a])**.** Let $\mathbf{K}$ be a $\mathrm{LS}(\mathbf{K})$-tame AEC with amalgamation and arbitrarily large models. Let $\lambda > \mathrm{LS}(\mathbf{K})^+$ be a successor cardinal. If $\mathbf{K}$ is categorical in λ, then $\mathbf{K}$ is categorical in all $\lambda' \geq \lambda$.

Grossberg and VanDieren concluded that Shelah's *eventual* categoricity conjecture from a successor holds in tame AECs with amalgamation. Baldwin [Bal09, Problem D.1.(5)] has asked whether the threshold in Shelah's downward transfer can be lowered to the first Hanf number. The answer is not known, but we show here that tameness is the only obstacle: assuming $\mathrm{LS}(\mathbf{K})$-tameness, the threshold becomes the first Hanf number, and so using Fact 1.6, we obtain Shelah's categoricity conjecture from a successor in tame AECs with amalgamation:

Corollary 10.6. Let $\mathbf{K}$ be a $\mathrm{LS}(\mathbf{K})$-tame AEC with amalgamation and arbitrarily large models. If $\mathbf{K}$ is categorical in *some successor* $\lambda > \mathrm{LS}(\mathbf{K})^+$, then $\mathbf{K}$ is categorical in *all* $\lambda' \geq \min(\lambda, \beth_{(2^{\mathrm{LS}(\mathbf{K})})^+})$.

This can be seen as a generalization (see [Bon14b]) of the corresponding result of Makkai and Shelah [MS90] for classes of models of an $\mathbb{L}_{\kappa,\omega}$-theory, κ a strongly compact cardinal. It is a central open question whether tameness follows from categoricity in AECs with amalgamation (see [GV06a, Conjecture 1.5]).

We can use (the proof of) Theorem 6.14 to give alternate arguments for Shelah's downward transfer [She99] (see Corollary B.6 in the appendix) and for the Grossberg-VanDieren upward transfer (see Corollary D.6 in the appendix). We also prove a local categoricity transfer that does not mention frames:

Corollary 10.3. Let $\mathbf{K}$ be a $\mathrm{LS}(\mathbf{K})$-tame AEC with amalgamation and arbitrarily large models. Let[1] $\mathrm{LS}(\mathbf{K}) < \lambda_0 < \lambda$. If λ is a successor cardinal and $\mathbf{K}$ is categorical in both λ_0 and λ, then $\mathbf{K}$ is categorical in all $\lambda' \in [\lambda_0, \lambda]$

Remark 1.7. We believe that the methods of [She99] are not sufficient to prove Corollary 10.3 (indeed, Shelah uses models of set theory to prove the transfer of "no Vaughtian pairs", and hence uses that the starting categoricity cardinal is above the Hanf number, see $(*)_9$ in the proof of [She99, Theorem II.2.7], or [Bal09, Theorem 14.12]), but we

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[1]On the case $\lambda_0 = \mathrm{LS}(\mathbf{K})$, see Remark 10.4.

noticed after posting a first draft of this paper that they *are* enough to prove Corollary 0.2. We sketch the details in Section 9.

Beyond categoricity in a successor, we can appeal to Theorem 6.14 to give improvements on the threshold of our previous categoricity transfer in AECs with primes [Vasc]: there the threshold was the second Hanf number (see Fact 10.8) and here we show that the first Hanf number suffices:

Corollary 10.9. Let $\mathbf{K}$ be a $\mathrm{LS}(\mathbf{K})$-tame AEC with amalgamation and arbitrarily large models. Assume that $\mathbf{K}$ has primes. If $\mathbf{K}$ is categorical in *some* $\lambda > \mathrm{LS}(\mathbf{K})$, then $\mathbf{K}$ is categorical in *all* $\lambda' \geq \min(\lambda, \beth_{(2^{\mathrm{LS}(\mathbf{K})})^+})$.

Remark 1.8. Compared to Fact 1.6 and Corollary 10.6, The case $\lambda = \mathrm{LS}(\mathbf{K})^+$ is allowed. We can also deal with it assuming the weak generalized continuum hypothesis instead, see below.

Finally, a natural question is how to deal with categoricity in a limit cardinal *without* assuming the existence of prime models. In [She09a, Theorem IV.7.12], Shelah claims assuming the weak generalized continuum hypothesis that if $\mathbf{K}$ is an AEC with amalgamation[2], then[3] categoricity in some $\lambda \geq \beth_{\left(2^{\aleph_{\mathrm{LS}(\mathbf{K})^+}}\right)^+}$ implies categoricity in all $\lambda' \geq \beth_{\left(2^{\aleph_{\mathrm{LS}(\mathbf{K})^+}}\right)^+}$. Shelah's argument relies on an unpublished claim (whose proof should appear in [She]), as well as PCF theory and long constructions of linear orders from [She09a, Sections IV.5,IV.6]. We have not fully checked it. In [Vasa], we gave a way to work around the use of PCF theory and the construction of linear orders (though still using Shelah's unpublished claim) by using the locality assumption of full tameness and shortness (a stronger assumption than tameness introduced by Will Boney in his Ph.D. thesis, see [Bon14b, Definition 3.3]).

In Section 11, we give an exposition of Shelah's proof that does not use PCF or the construction of linear orders. This uses a recent result of VanDieren and the author [VV, Corollary 7.4], showing that a model at a high-enough categoricity cardinal must have some degree of saturation (regardless of the cofinality of the cardinal). We deduce (using the aforementioned unpublished claim of Shelah) that Shelah's eventual categoricity conjecture is consistent assuming the existence of

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[2]Shelah only assumes some instances of amalgamation and no maximal models at specific cardinals, see the discussion in Section 11.

[3]Shelah gives a stronger, erroneous statement (it contradicts Morley's categoricity theorem) but this is what his proof gives.

a proper class of *measurable* cardinals (this was implicit in [She09a, Chapter IV] but we give the details). Furthermore we give an explicit upper bound on the categoricity threshold (see Theorem 11.14). This partially answers [She00, Question 6.14.(1)].

Using Theorem 6.14, we also give an improvement on the categoricity threshold of $\beth_{\left(2^{\aleph_{\mathrm{LS}(\mathbf{K})^+}}\right)^+}$ if the AEC is tame (this is Corollary 0.3 from the abstract):

Corollary 11.9. Assume the weak generalized continuum hypothesis and an unpublished claim of Shelah (Claim 11.2). Let $\mathbf{K}$ be a LS($\mathbf{K}$)-tame AEC with amalgamation and arbitrarily large models. If $\mathbf{K}$ is categorical in *some* $\lambda > \mathrm{LS}(\mathbf{K})$, then $\mathbf{K}$ is categorical in *all* $\lambda' \geq \min(\lambda, \beth_{\left(2^{\mathrm{LS}(\mathbf{K})}\right)^+})$.

Moreover, we give two ZFC consequences of a lemma in Shelah's proof (which obtains weak tameness from categoricity in certain cardinals below the Hanf number): an improvement on the Hanf number for constructing good frames (Theorem 8.5) and a nontrivial restriction on the categoricity spectrum *below* the Hanf number of an AEC with amalgamation and no maximal models (Theorem 8.6).

For clarity, we emphasize once again that Corollary 11.9 is due to Shelah when the threshold is $\beth_{\left(2^{\aleph_{\mathrm{LS}(\mathbf{K})^+}}\right)^+}$ (and then tameness is not needed). The main contribution of Section 11 is a clear outline of Shelah's proof that avoids several of his harder arguments.

In conclusion, the aim of the second part of this paper is to clarify the status of Shelah's eventual categoricity conjecture by simplifying existing proofs and improving several thresholds. We give a table summarizing the known results on the conjecture in Section 12.

The background required to read this paper is a solid knowledge of AECs (at minimum Baldwin's book [Bal09]) together with some familiarity with good frames (e.g. the first four sections of [She09a, Chapter II]). As mentioned before, the paper has two parts: The first gives a proof of the main Theorem (Theorem 6.14), and the second gives applications. If one is willing to take Theorem 6.14 as a black box, the second part can be read independently from the first part. The first part relies on [BVb] and the second relies on several other recent preprints (e.g. [Vas16, Vasc, VV], as well as on parts of [She09a, Chapter IV] (we only use results for which Shelah gives a full proof). We

have tried to state all background facts as black boxes that can be used with little understanding of the underlying machinery.

We warn the reader: at the beginning of most sections, we state a *global* hypothesis which applies to any result stated in the section.

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Part 1. **The main theorem**

2. Background

We assume that the reader is familiar with the definition of an AEC and notions such as amalgamation, joint embedding, Galois types, and Ehrenfeucht-Mostowski models (see for example [Bal09]). The notation we use is standard and is described in details at the beginning of [Vasb]. For example, we write $\mathrm{gtp}(\bar{b}/M; N)$ for the Galois type of $\bar{b}$ over M, as computed in N. Everywhere in this paper and unless mentioned otherwise, we are working inside a fixed AEC $\mathbf{K}$.

2.1. **Good frames.** In [She09a, Definition II.2.1][4], Shelah introduces *good frames*, a local notion of independence for AECs. This is the central concept of his book and has seen several other applications, such as a proof of Shelah's eventual categoricity conjecture for universal classes [Vasd, Vase]. A *good λ-frame* is a triple $\mathfrak{s} = (\mathbf{K}_\lambda, \underset{\smile}{\smile}, \mathrm{gS}^{\mathrm{bs}})$ where:

(1) $\mathbf{K}$ is a nonempty AEC which has amalgamation in λ, joint embedding in λ, no maximal models in λ, and is stable in λ.
(2) For each $M \in \mathbf{K}_\lambda$, $\mathrm{gS}^{\mathrm{bs}}(M)$ (called the set of *basic types* over M) is a set of nonalgebraic Galois types over M satisfying (among others) the *density property*: if $M <_{\mathbf{K}} N$ are in $\mathbf{K}_\lambda$, there exists $a \in |N|\backslash|M|$ such that $\mathrm{gtp}(a/M; N) \in \mathrm{gS}^{\mathrm{bs}}(M)$.
(3) $\underset{\smile}{\smile}$ is an (abstract) independence relation on types of length one over models in $\mathbf{K}_\lambda$ satisfying the basic properties of first-order

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[4]The definition here is simpler and more general than the original: We will *not* use Shelah's axiom (B) requiring the existence of a superlimit model of size λ. Several papers (e.g. [JS13]) define good frames without this assumption.

forking in a superstable theory: invariance, monotonicity, extension, uniqueness, transitivity, local character, and symmetry (see [She09a, Definition II.2.1]).

As in [She09a, Definition II.6.35], we say that a good λ-frame $\mathfrak{s}$ is *type-full* if for each $M \in \mathbf{K}_\lambda$, $\mathrm{gS}^{\mathrm{bs}}(M)$ consists of *all* the nonalgebraic types over M. We focus on type-full good frames in this paper and hence just write $\mathfrak{s} = (\mathbf{K}_\lambda, \bigcup)$. For notational simplicity, we extend forking to algebraic types by specifying that algebraic types do not fork over their domain. Given a type-full good μ-frame $\mathfrak{s} = (\mathbf{K}_\lambda, \bigcup)$ and $M_0 \leq_{\mathbf{K}} M$ both in $\mathbf{K}_\lambda$, we say that a nonalgebraic type $p \in \mathrm{gS}(M)$ *does not $\mathfrak{s}$-fork over M_0* if it does not fork over M_0 according to the abstract independence relation $\bigcup$ of $\mathfrak{s}$. When $\mathfrak{s}$ is clear from context, we omit it and just say that *p does not fork over M_0*. We write $\mathbf{K}_\mathfrak{s}$ for the underlying class (containing only models of size λ) of $\mathfrak{s}$. We say that a good λ-frame $\mathfrak{s}$ is *on $\mathbf{K}_\lambda$* if $\mathbf{K}_\mathfrak{s} = \mathbf{K}_\lambda$. We might also just say that $\mathfrak{s}$ is *on $\mathbf{K}$*.

We more generally look at frames where the forking relation works over larger models. For $\mathcal{F} = [\lambda, \theta)$ an interval with $\theta \geq \lambda$ a cardinal or ∞, we define a *type-full good $\mathcal{F}$-frame* similarly to a type-full good λ-frame but require forking to be defined over models in $\mathbf{K}_\mathcal{F}$ (similarly, the good properties hold of the class $\mathbf{K}_\mathcal{F}$, e.g. $\mathbf{K}$ is stable in every $\mu \in \mathcal{F}$). See [Vas16, Definition 2.21] for a precise definition. For a type-full good $\mathcal{F}$-frame $\mathfrak{s} = (\mathbf{K}_\mathcal{F}, \bigcup)$ and $\mathbf{K}'$ a subclass of $\mathbf{K}_\mathcal{F}$, we define the restriction $\mathfrak{s} \upharpoonright \mathbf{K}'$ of $\mathfrak{s}$ to $\mathbf{K}'$ in the natural way (see [Vasa, Notation 3.17.(2)]).

At one point in the paper (Section 3) we will look at frames defined over types longer than one element. This was first defined in [BVb, Definition 3.1] but we use [Vasa, Definition 3.1]. We require in addition that it satisfies the base monotonicity property and that the underlying class is an AEC.

Definition 2.1. A type-full pre-$(\leq \lambda, \lambda)$-frame is a pair $\mathfrak{t} := (K_\lambda, \bigcup)$, where $\mathbf{K}$ is an AEC with amalgamation in λ and $\bigcup$ is a relation on types of length at most λ over models in $\mathbf{K}_\lambda$ satisfying invariance and monotonicity (including base monotonicity, see [Vasa, Definition 3.12.(4)]).

For a type-full good λ-frame $\mathfrak{s}$, we say that $\mathfrak{t}$ *extends* $\mathfrak{s}$ if they have the same underlying class and forking in $\mathfrak{s}$ and $\mathfrak{t}$ coincide.

2.2. **Saturated and limit models.** We will make heavy use of limit models (see [GVV] for history and motivation). Here we give a global

definition, where we permit the limit model and the base to have different sizes.

Definition 2.2. Let $M_0 \leq_{\mathbf{K}} M$ be models in $\mathbf{K}_{\geq \mathrm{LS}(\mathbf{K})}$. M is *limit over* M_0 if there exists a limit ordinal δ and a strictly increasing continuous sequence $\langle N_i : i \leq \delta \rangle$ such that:

(1) $N_0 = M_0$.
(2) $N_\delta = M$.
(3) For all $i < \delta$, N_{i+1} is universal over N_i (that is, for any $N \in \mathbf{K}_{\|N_i\|}$ with $N_0 \leq_{\mathbf{K}} N$, there exists $f : N \xrightarrow[N_i]{} N_{i+1}$).

We say that M is *limit* if it is limit over some $M' \leq_{\mathbf{K}} M$.

Definition 2.3. Assume that $\mathbf{K}$ has amalgamation.

(1) For $\lambda > \mathrm{LS}(\mathbf{K})$, $\mathbf{K}^{\lambda\text{-sat}}$ is the class of λ-saturated (in the sense of Galois types) models in $\mathbf{K}_{\geq \lambda}$. We order it with the strong substructure relation inherited from $\mathbf{K}$.
(2) We also define $\mathbf{K}^{\mathrm{LS}(\mathbf{K})\text{-sat}}$ to be the class of models $M \in \mathbf{K}_{\geq \mathrm{LS}(\mathbf{K})}$ such that for all $A \subseteq |M|$ with $|A| \leq \mathrm{LS}(\mathbf{K})$, there exists a limit model $M_0 \leq_{\mathbf{K}} M$ with $M_0 \in \mathbf{K}_{\mathrm{LS}(\mathbf{K})}$ and $A \subseteq |M_0|$. We order $\mathbf{K}^{\mathrm{LS}(\mathbf{K})\text{-sat}}$ with the strong substructure relation inherited from $\mathbf{K}$.

Remark 2.4. If $\mathbf{K}$ has amalgamation and is stable in $\mathrm{LS}(\mathbf{K})$, then $\mathbf{K}_{\mathrm{LS}(\mathbf{K})}^{\mathrm{LS}(\mathbf{K})\text{-sat}}$ is the class of limit models in $\mathbf{K}_{\mathrm{LS}(\mathbf{K})}$.

We will repeatedly use the uniqueness of limit models inside a good frame, first proven by Shelah in [She09a, Claim II.4.8] (see also [Bon14a, Theorem 9.2]).

Fact 2.5. Let $\mathfrak{s}$ be a type-full good λ-frame.

Let $M_0, M_1, M_2 \in \mathbf{K}_{\mathfrak{s}}$.

(1) If M_1 and M_2 are limit models, then $M_1 \cong M_2$.
(2) If in addition M_1 and M_2 are both limit over M_0, then $M_1 \cong_{M_0} M_2$.

For global frames, we can combine this with a result of VanDieren [Van16] to obtain that limit models are saturated and closed under unions:

Fact 2.6. Let $\mathfrak{s}$ be a type-full good $[\lambda, \theta)$-frame on the AEC $\mathbf{K}$. Let $\mu \in [\lambda, \theta)$.

(1) $M \in \mathbf{K}_\mu^{\mu\text{-sat}}$ if and only if M is limit.
(2) If $\mu > \lambda$, then $\mathbf{K}_{[\mu,\theta)}^{\mu\text{-sat}}$ is the initial segment of an AEC with Löwenheim-Skolem-Tarski number λ.

Proof.

(1) This is trivial if $\lambda = \mathrm{LS}(\mathbf{K})$, so assume that $\lambda > \mathrm{LS}(\mathbf{K})$. If M is limit, then by uniqueness of limit models, M is saturated. Conversely if M is saturated, then it must be unique, hence isomorphic to a limit model.
(2) By uniqueness of limit models (Fact 2.5) and [Van16, Theorem 1]. Note that there a condition called μ-superstability (see Definition 7.1) rather than the existence of a good μ-frame is used. However the existence of a type-full good μ-frame implies μ-superstability (see Fact 7.2).

$\square$

We will use Facts 2.5 and 2.6 freely.

3. Domination and uniqueness triples

In this section, we assume:

Hypothesis 3.1. $\mathfrak{s} = (\mathbf{K}_\lambda, \underset{\smile}{\perp})$ is a type-full good λ-frame.

Remark 3.2. In this section, we do not need the frame to be type-full but adopt this hypotheses anyway for notational convenience.

Our aim (for the next sections) is to develop some orthogonality calculus as in [She09a, Section III.6]. There Shelah works in a good λ-frame that is *successful* (see [She09a, Definition III.1.1]). Note that by [She09a, Claim III.9.6] such a good frame can be extended to a type-full one. Thus the framework of this section is more general (although there are no known examples separating the two).

One of the main component of the definition of successful is the existence property for uniqueness triples (see [She09a, Definition III.1.1], [JS13, Definition 4.1.(5)], or here Definition E.6). We showed in [Vasa, Lemma 11.7] that this property is equivalent to a version of domination assuming the existence of a global independence relation. Using an argument of Makkai and Shelah [MS90, Proposition 4.22], we showed [Vasa, Lemma 11.12] that this version of domination satisfies a natural existence property. We give a slight improvement on this result here by

working only locally in λ (i.e. using limit models rather than saturated ones).

Crucial in this section is the uniqueness of limit models (Fact 2.5). A consequence is the following conjugation property [She09a, Claim 1.21]. It is stated for M, N superlimit but the proof goes through if M and N are limit models.

Fact 3.3 (The conjugation property). If $M \leq_{\mathbf{K}} N$ are limit models in $\mathbf{K}_\lambda$ and $p \in \mathrm{gS}^{\mathrm{bs}}(N)$ does not fork over M, then there exists $f : N \cong M$ so that $f(p) = p \restriction M$.

The next definition is modeled on [Vasa, Definition 11.5]:

Definition 3.4. Let $\mathfrak{t}$ be a pre-$(\leq \lambda, \lambda)$-frame extending $\mathfrak{s}$ (see Definition 2.1, we write $\underset{\smile}{}$ for the nonforking relation of both $\mathfrak{s}$ and $\mathfrak{t}$). (a, M, N) is a *domination triple for* $\mathfrak{t}$ if $M, N \in K_\lambda$, $M \leq_{\mathbf{K}} N$, $a \in |N| \backslash |M|$ and for any $N' \geq_{\mathbf{K}} N$ and $M' \leq_{\mathbf{K}} N$ with $M \leq_{\mathbf{K}} M'$ and $M', N' \in K_\lambda$, if $a \overset{N'}{\underset{M}{\smile}} M'$, then $N \overset{N'}{\underset{M}{\smile}} M'$.

Remark 3.5. In this paper, we will take $\mathfrak{t}$ to be 1-forking (Definition 4.2), so the reader who wants a concrete example may substitute it for $\mathfrak{t}$ throughout this section.

Domination triples are related to Shelah's uniqueness triples (see [She09a, Definition III.1.1] or [JS13, Definition 4.1.(5)]) by the following result (this will not be used outside of this section):

Fact 3.6 (Lemma 11.7 in [Vasa]). Let $\mathfrak{t}$ be a pre-$(\leq \lambda, \lambda)$-frame extending $\mathfrak{s}$. If $\mathfrak{t}$ has the uniqueness property, then any domination triple for $\mathfrak{t}$ is a uniqueness triple for $\mathfrak{s}$.

We now want to show the existence property for domination triples: For any type $p \in \mathrm{gS}(M)$, there exists a domination triple (a, M, N) with $p = \mathrm{gtp}(a/M; N)$. We manage to do it when M is a limit model. The proof is a local version of [Vasa, Lemma 11.12] (which adapted [MS90, Proposition 4.22]). We will consider the following properties that $\mathfrak{t}$ may have:

Definition 3.7. Let $\mathfrak{t}$ be a pre-$(\leq \lambda, \lambda)$-frame extending $\mathfrak{s}$.

(1) We say that $\mathfrak{t}$ *satisfies* $(*)$ if whenever $\langle M_i : i < \lambda^+ \rangle$ is increasing in $\mathbf{K}_\lambda$ and $p \in \mathrm{gS}^{\leq \lambda}(\bigcup_{i < \lambda^+} M_i)$, there exists $i < \lambda^+$ such that for any $j \in [i, \lambda^+)$, $p \restriction M_j$ does not fork over M_i.

(2) We say that $\mathfrak{t}$ *satisfies* $(**)$ if whenever $\langle M_i : i < \lambda^+ \rangle$, $\langle N_i : i < \lambda^+ \rangle$ are increasing continuous in $\mathbf{K}_\lambda$ so that for all $i < \lambda^+$, $M_i \leq_{\mathbf{K}} N_i$, there exists $i < \lambda^+$ such that $N_i \overset{N_{i+1}}{\underset{M_i}{\smile}} M_{i+1}$.

(3) [Vasa, Definition 3.12.(9)] For $\kappa \geq 2$, we say that $\mathfrak{t}$ *has the left* $(< \kappa)$-*witness property* if for any $M_0 \leq_{\mathbf{K}} M \leq_{\mathbf{K}} N$ in $\mathbf{K}_\lambda$ and any $A \subseteq |N|$, $A \overset{N}{\underset{M_0}{\smile}} M$ holds if and only if $A_0 \overset{N}{\underset{M_0}{\smile}} M$ for all $A_0 \subseteq A$ with $|A_0| < \kappa$.

Note that $(*)$ together with the witness property imply $(**)$.

Lemma 3.8. Let $\mathfrak{t}$ be a pre-$(\leq \lambda, \lambda)$-frame extending $\mathfrak{s}$. If $\mathfrak{t}$ satisfies $(*)$ and has the left $(< \kappa)$-witness property for some regular $\kappa \leq \lambda$, then $\mathfrak{t}$ satisfies $(**)$.

Proof. Fix $\langle M_i : i < \lambda^+ \rangle$, $\langle N_i : i < \lambda^+ \rangle$ as in the definition of $(**)$. By $(*)$, for each $i < \lambda^+$, there exists $j_i < \lambda^+$ such that $N_i \overset{N_j}{\underset{M_{j_i}}{\smile}} M_j$ for all $j \geq j_i$. Let $i^* < \lambda^+$ be such that $\mathrm{cf}(i^*) = \kappa$ and $j_i < i^*$ for all $i < i^*$. Then it is easy to check using the left $(< \kappa)$-witness property that $N_{i^*} \overset{N_j}{\underset{M_{i^*}}{\smile}} M_j$ for all $j \geq i^*$, which is as needed. $\square$

Theorem 3.9. Let $\mathfrak{t}$ be a pre-$(\leq \lambda, \lambda)$-frame extending $\mathfrak{s}$. Assume that $\mathfrak{t}$ satisfies $(**)$.

Let $M \in K_\lambda$ be a limit model. For each nonalgebraic $p \in \mathrm{gS}(M)$, there exists a domination triple (a, M, N) for $\mathfrak{t}$ such that $p = \mathrm{gtp}(a/M; N)$.

Proof. Assume not. We know that M is a limit model. Therefore by local character there exists $M^* \in K_\lambda$ such that M is limit over M^* and p does not fork over M^*.

<u>Claim</u>. For any limit $M' \geq_{\mathbf{K}} M$ with $M' \in \mathbf{K}_\lambda$, if $q \in \mathrm{gS}(M')$ is the nonforking extension of p, then there is no domination triple (b, M', N') such that $q = \mathrm{gtp}(b/M'; N')$.

<u>Proof of claim</u>. By the conjugation property (Fact 3.3), there exists $f : M' \cong M$ such that $f(q) = p$. Now use that domination triples are invariant under isomorphisms. $\dagger_{\mathrm{Claim}}$

We construct a, $\langle M_i : i < \lambda^+ \rangle$, $\langle N_i : i < \lambda^+ \rangle$ increasing continuous such that for all $i < \lambda^+$:

(1) $M_0 = M$.

(2) $M_i \leq_{\mathbf{K}} N_i$ are both in $\mathbf{K}_\lambda$.

(3) M_{i+1} is limit over M_i and N_{i+1} is limit over N_i.

(4) $\mathrm{gtp}(a/M_i; N_i)$ is the nonforking extension of p. In particular,
$$a \underset{M_0}{\overset{N_i}{\downarrow}} M_i.$$

(5) $N_i \underset{M_i}{\overset{N_{i+1}}{\not\downarrow}} M_{i+1}$.

This is enough, since then we get a contradiction to $(**)$. This is possible: If $i = 0$, let $N_0 \in \mathbf{K}_\lambda$ and $a \in |N_0|$ be such that $p = \mathrm{gtp}(a/M; N_0)$. At limits, take unions. Now assume everything up to i has been constructed. By the claim, (a, M_i, N_i) cannot be a domination triple. This means there exists $M_i' \geq_{\mathbf{K}} M_i$ and $N_i' \geq_{\mathbf{K}} N_i$ all in $\mathbf{K}_\lambda$ such that $a \underset{M_i}{\overset{N_i'}{\downarrow}} M_i'$ but $N_i \underset{M_i}{\overset{N_i'}{\not\downarrow}} M_i'$. By the extension property of forking, pick $M_{i+1} \in \mathbf{K}_\lambda$ limit over M_i containing M_i' and $N_{i+1} \geq_{\mathbf{K}} N_i'$ such that N_{i+1} is limit over N_i and $a \underset{M_i}{\overset{N_{i+1}}{\downarrow}} M_{i+1}$. $\square$

The next corollary will not be used in the rest of this paper. It improves on [Vasa, Theorem 11.13] by working exclusively in λ (so there is no need to assume the existence of a good frame below λ).

Corollary 3.10. Assume that $\mathbf{K}$ is categorical in λ. If there exists a pre-$(\leq \lambda, \lambda)$-frame $\mathfrak{t}$ extending $\mathfrak{s}$ and satisfying $(**)$ and uniqueness, then $\mathfrak{s}$ has the existence property for uniqueness triples (i.e. it is weakly successful, see [She09a, Definition III.1.1] or Definition E.6).

Proof. By Theorem 3.9, $\mathfrak{t}$ has the existence property for domination triples. By Fact 3.6, any domination triple is a uniqueness triple. $\square$

Remark 3.11. We can use this in the proof of [Vasd, Lemma A.14] to get an ω-successful good λ-frame instead of an ω-successful good λ^+-frame.

Question 3.12. Is the converse true? Namely if $\mathbf{K}$ is categorical in λ and $\mathfrak{s}$ is a weakly successful good λ-frame on $\mathbf{K}$, does there exist a pre-$(\leq \lambda, \lambda)$-frame $\mathfrak{t}$ that extends $\mathfrak{s}$ and satisfies uniqueness and $(**)$?

It is known (see [She09a, Section II.6] and [Vasa, Section 12]) that weakly successful good λ-frame can be extended to pre-$(\leq \lambda, \lambda)$-frame with several good properties, including uniqueness, but it is not clear that $(**)$ holds there.

4. Local orthogonality

Hypothesis 4.1. $\mathfrak{s} = (\mathbf{K}_\lambda, \bot)$ is a type-full good λ-frame.

The next definition is what we will use for the $\mathfrak{t}$ of the previous section. One can see it as a replacement for a global definition of forking. It already plays a role in [MS90] (see Lemma 4.17 there) and [BVa] (see Definition 3.10 there). A similar notion is called "smooth independence" in [VZ14].

Definition 4.2. For $M \in \mathbf{K}_\lambda$ and $p \in \mathrm{gS}^{<\infty}(M)$, we say that p *does not 1-$\mathfrak{s}$-fork* over M_0 if $M_0 \leq_{\mathbf{K}} M$ and for $I \subseteq \ell(p)$ with $|I| = 1$, we have that p^I does not $\mathfrak{s}$-fork over M_0. We see 1-forking as inducing a pre-$(\leq \lambda, \lambda)$-frame (Definition 2.1).

Notation 4.3. We write $[A]^1 \overset{N}{\underset{M_0}{\bigcup}} M$ if for some (any) enumeration $\bar{a}$ of A, $\mathrm{gtp}(\bar{a}/M; N)$ does not 1-$\mathfrak{s}$-fork over M_0. That is, $a \overset{N}{\underset{M_0}{\bigcup}} M$ for all $a \in A$.

Remark 4.4 (Disjointness)**.** Because nonforking extensions of nonalgebraic types are nonalgebraic, if $[A]^1 \overset{N}{\underset{M_0}{\bigcup}} M$, then $|M| \cap A \subseteq |M_0|$.

Definition 4.5. (a, M, N) is a *weak domination triple in* $\mathfrak{s}$ if it is a domination triple (Definition 3.4) for the pre-frame induced by 1-forking. That is, $M, N \in K_\lambda$, $M \leq_{\mathbf{K}} N$, $a \in |N| \backslash |M|$ and for any $N' \geq_{\mathbf{K}} N$ and $M' \leq_{\mathbf{K}} N$ with $M \leq_{\mathbf{K}} M'$ and $M', N' \in K_\lambda$, if $a \overset{N'}{\underset{M}{\bigcup}} M'$, then $[N]^1 \overset{N'}{\underset{M}{\bigcup}} M'$.

From the results of the previous section, we deduce the existence property for weak uniqueness triples:

Lemma 4.6. 1-forking satisfies $(**)$ (see Definition 3.7).

Proof. It is clear that 1-forking has the left (< 2)-witness property (Definition 3.7) so by Lemma 3.8, it is enough to show that 1-forking satisfies $(*)$.

Let $\langle M_i : i < \lambda^+ \rangle$ be increasing in $\mathbf{K}_\lambda$. Write $M_{\lambda^+} := \bigcup_{i < \lambda^+} M_i$. Let $p \in \mathrm{gS}^\alpha(M_{\lambda^+})$ with $\alpha \leq \lambda$. Say $p = \mathrm{gtp}(\bar{a}/M_{\lambda^+}; N)$. For each $k < \alpha$, we will write p^k for $\mathrm{gtp}(\bar{a}(i)/M_{\lambda^+}; N)$ (the type of a singleton). First we show:

<u>Claim</u>. For each $k < \alpha$ there exists $i_k < \lambda^+$ so that $p^k \upharpoonright M_j$ does not fork over M_{i_k} for all $j \geq i_k$.

This is enough because then taking $i := \sup_{k<\alpha} i_k$ we obtain that $p \upharpoonright M_j$ does not fork over M_i for all $j \geq i$.

<u>Proof of claim</u>. Fix $k < \alpha$. By local character (in $\mathfrak{s}$), for each limit $j < \lambda^+$, there exists $\gamma_j < j$ so that $p^k \upharpoonright M_j$ does not fork over M_{γ_j}. By Fodor's lemma, there exists $i_k < \lambda^+$ such that for unboundedly many $j < \lambda^+$, $\gamma_j = i_k$. By monotonicity of forking, i_k is as desired. $\qquad\square$

Theorem 4.7. Let $M \in K_\lambda$ be a limit model. For each nonalgebraic $p \in \mathrm{gS}(M)$, there exists a weak domination triple (a, M, N) such that $p = \mathrm{gtp}(a/M; N)$.

Proof. By Lemma 4.6 and Theorem 3.9. $\qquad\square$

We now give a definition of orthogonality in terms of independent sequences.

Definition 4.8 (Independent sequence, III.5.2 in [She09a])**.** Let α be an ordinal.

(1) $\langle a_i : i < \alpha \rangle \frown \langle M_i : i \leq \alpha \rangle$ is said to be *independent in* (M, M', N) when:
   (a) $(M_i)_{i \leq \alpha}$ is increasing continuous in $\mathbf{K}_\lambda$.
   (b) $M \leq_{\mathbf{K}} M' \leq_{\mathbf{K}} M_0$ and $M, M' \in \mathbf{K}_\lambda$.
   (c) $M_\alpha \leq_{\mathbf{K}} N$ and $N \in \mathbf{K}_\lambda$.
   (d) For every $i < \alpha$, $a_i \underset{M}{\overset{M_{i+1}}{\downarrow}} M_i$.
   $\langle a_i : i < \alpha \rangle \frown \langle M_i : i \leq \alpha \rangle$ is said to be *independent over* M when it is independent in (M, M_0, M_α).
(2) $\bar{a} := \langle a_i : i < \alpha \rangle$ is said to be *independent in* (M, M_0, N) when $M \leq_{\mathbf{K}} M_0 \leq_{\mathbf{K}} N$, $\bar{a} \in {}^\alpha|N|$, and for some $\langle M_i : i \leq \alpha \rangle$ and a model N^+ such that $M_\alpha \leq_{\mathbf{K}} N^+$, $N \leq_{\mathbf{K}} N^+$, and $\langle a_i : i < \alpha \rangle \frown \langle M_i : i \leq \alpha \rangle$ is independent over M. When $M = M_0$, we omit it and just say that $\bar{a}$ is independent in (M, N).

Remark 4.9. We will use the definition above when $\alpha = 2$. In this case, we have that $\langle ab \rangle$ is independent in (M, N) if and only if $a \underset{M}{\overset{N}{\downarrow}} b$ (technically, the right hand side of the $\downarrow$ relation must be a model but we can remedy this by extending the nonforking relation in the natural way, as in the definition of the minimal closure in [BGKV, Definition 3.4]).

Definition 4.10. Let $M \in \mathbf{K}_\lambda$ and let $p, q \in \mathrm{gS}(M)$ be nonalgebraic. We say that p is *weakly orthogonal to* q and write $p \overset{\mathfrak{s}}{\underset{\mathrm{wk}}{\perp}} q$ (or just $p \underset{\mathrm{wk}}{\perp} q$ if $\mathfrak{s}$ is clear from context) if for all $N \geq_\mathbf{K} M$ and all $a, b \in |N|$ such that $\mathrm{gtp}(a/M; N) = p$ and $\mathrm{gtp}(b/M; N) = q$, $\langle ab \rangle$ is independent in (M, N).

We say that p *is orthogonal to* q (written $p \overset{\mathfrak{s}}{\perp} q$, or just $p \perp q$ if $\mathfrak{s}$ is clear from context) if for every $N \in \mathbf{K}_\lambda$ with $N \geq_\mathbf{K} M$, $p' \underset{\mathrm{wk}}{\perp} q'$, where p', q' are the nonforking extensions to N of p and q respectively.

Remark 4.11. Definition 4.10 is equivalent to Shelah's ([She09a, Definition III.6.2]), see [She09a, Claim III.6.4.(2)] assuming that $\mathfrak{s}$ is *successful*. By a similar proof (and assuming that $\mathbf{K}_\mathfrak{s}$ has primes), it is also equivalent to the definition in terms of primes in [Vasc, Definition 2.2].

We will use the following consequence of symmetry:

Fact 4.12 (Theorem 4.2 in [JS12]). For any $M_0 \leq_\mathbf{K} M \leq_\mathbf{K} N$ all in $\mathbf{K}_\lambda$, if $a, b \in |N| \backslash |M_0|$, then $\langle ab \rangle$ is independent in (M_0, M, N) if and only if $\langle ba \rangle$ is independent in (M_0, M, N).

Lemma 4.13. Let $M \in \mathbf{K}_\lambda$. Let $p, q \in \mathrm{gS}(M)$ be nonalgebraic.

  (1) If M is limit, then $p \perp q$ if and only if $p \underset{\mathrm{wk}}{\perp} q$.

  (2) $p \underset{\mathrm{wk}}{\perp} q$ if and only if $q \underset{\mathrm{wk}}{\perp} p$.

  (3) If $p \underset{\mathrm{wk}}{\perp} q$, then whenever (a, M, N) is a weak domination triple representing q, p is omitted in N. In particular, if M is limit, there exists $N \in \mathbf{K}_\lambda$ with $M <_\mathbf{K} N$ so that p is omitted in N.

Proof.

  (1) By the conjugation property (Fact 3.3). See the proof of [Vasc, Lemma 2.6].
  (2) By Fact 4.12.
  (3) Let $N' \geq_\mathbf{K} N$ be in $\mathbf{K}_\lambda$ and let $b \in |N'|$ realize p. We have that $\langle ab \rangle$ is independent in (M, N'). Therefore there exists $N'' \geq_\mathbf{K} N$ in $\mathbf{K}_\lambda$, $M' \in \mathbf{K}_\lambda$ so that $M \leq_\mathbf{K} M' \leq_\mathbf{K} N''$, $b \in |M'|$, and $a \overset{N''}{\underset{M}{\downarrow}} M'$. By domination, $[N]^1 \overset{N''}{\underset{M}{\downarrow}} M'$, so by disjointness (Remark 4.4), $b \notin |N|$. The last sentence follows from the existence property for weak domination triple (Theorem 4.7).

□

5. Unidimensionality

Hypothesis 5.1. $\mathfrak{s} = (\mathbf{K}_\lambda, \perp)$ is a type-full good λ-frame on $\mathbf{K}$ and $\mathbf{K}$ is categorical in λ.

In this section we give a definition of unidimensionality similar to the ones in [She90, Definition V.2.2] or [She09a, Section III.2]. We show that $\mathfrak{s}$ is unidimensional if and only if $\mathbf{K}$ is categorical in λ^+ (this uses categoricity in λ). In the next section, we will show how to transfer unidimensionality across cardinals, hence getting the promised categoricity transfer. In [She09a, Section III.2], Shelah gives several different definitions of unidimensionality and also shows (see [She09a, III.2.3, III.2.9]) that the so-called "weak-unidimensionality" is equivalent to categoricity in λ^+ (hence our definition is equivalent to Shelah's weak unidimensionality) but it is unclear how to transfer it across cardinals without assuming that the frame is successful.

Note that the hypothesis of categoricity in λ implies that the model of size λ is limit, hence weak orthogonality and orthogonality coincide, see Lemma 4.13.

Rather than defining what it means to be unidimensional, we find it clearer to define what it means to *not* be unidimensional:

Definition 5.2. $\mathfrak{s}$ is *unidimensional* if the following is *false*: for every $M \in \mathbf{K}_\lambda$ and every nonalgebraic $p \in \mathrm{gS}(M)$, there exists $M' \in \mathbf{K}_\lambda$ with $M' \geq_{\mathbf{K}} M$ and nonalgebraic $p', q \in \mathrm{gS}(M')$ so that p' extends p and $p' \perp q$.

We first give an equivalent definition using minimal types:

Definition 5.3. For $M \in \mathbf{K}_\lambda$, a type $p \in \mathrm{gS}(M)$ is *minimal* if for every $M' \geq_{\mathbf{K}} M$ with $M' \in \mathbf{K}_\lambda$, p has a unique nonalgebraic extension to $\mathrm{gS}(M')$.

Remark 5.4. If $M \leq_{\mathbf{K}} N$ are in $\mathbf{K}_\lambda$ and $p \in \mathrm{gS}(N)$ is nonalgebraic such that $p \restriction M$ is minimal, then p does not fork over M (because the nonforking extension of $p \restriction M$ has to be p).

By the proof of $(*)_5$ in [She99, Theorem II.2.7]:

Fact 5.5 (Density of minimal types)**.** For any $M \in \mathbf{K}_\lambda$ and nonalgebraic $p \in \mathrm{gS}(M)$, there exists $M' \in \mathbf{K}_\lambda$ and $p' \in \mathbf{K}_\lambda$ such that $M \leq_{\mathbf{K}} M'$, p' extends p, and p' is minimal.

Lemma 5.6. The following are equivalent:

(1) $\mathfrak{s}$ is not unidimensional.
(2) For every $M \in \mathbf{K}_\lambda$ and every minimal $p \in \mathrm{gS}(M)$, there exists $M' \geq_{\mathbf{K}} M$ with $M' \in \mathbf{K}_\lambda$ and $p', q \in \mathrm{gS}(M')$ nonalgebraic so that $p' \perp q$.
(3) For every $M \in \mathbf{K}_\lambda$ and every minimal $p \in \mathrm{gS}(M)$, there exists a nonalgebraic $q \in \mathrm{gS}(M)$ with $p \perp q$.

Proof. (1) implies (2) because (2) is a special case of (1). Conversely, (2) implies (1): given $M \in \mathbf{K}_\lambda$ and $p \in \mathrm{gS}(M)$, first use density of minimal types to extend p to a minimal $p' \in \mathrm{gS}(M')$ (so $M' \in \mathbf{K}_\lambda$ $M \leq_{\mathbf{K}} M'$). Then apply (2).

Also, if (3) holds, then (2) holds with $M = M'$. Conversely, assume that (2) holds. Let $p \in \mathrm{gS}(M)$ be minimal and let p', q, M' witness (2), i.e. $p', q \in \mathrm{gS}(M')$, p' extends p and $p' \perp q$. By Remark 5.4, p' does not fork over M. By the conjugation property (Fact 3.3), there exists $f : M' \cong M$ so that $f(p') = p$. Thus $p \perp f(q)$, hence (2) holds. $\square$

We use the characterization to show that unidimensionality implies categoricity in λ^+. This is similar to [MS90, Proposition 4.25] but the proof is slightly more involved since our definition of unidimensionality is weaker. We start with a version of density of minimal types inside a fixed model. We will use the following fact, whose proof is a straightforward direct limit argument:

Fact 5.7 (Claim 0.32.(1) in [She01a]). Let $\langle M_i : i \leq \omega \rangle$ be an increasing continuous chain in $\mathbf{K}_\lambda$ and for each $i < \omega$, let $p_i \in \mathrm{gS}(M_i)$ be such that $j < i$ implies $p_i \upharpoonright M_j = p_j$. Then there exists $p \in \mathrm{gS}(M_\omega)$ so that $p \upharpoonright M_i = p_i$ for all $i < \omega$.

Lemma 5.8. Let $M_0 \leq_{\mathbf{K}} M$ with $M_0 \in \mathbf{K}_\lambda$ and $M \in \mathbf{K}_{>\lambda}$. Let $p \in \mathrm{gS}(M_0)$. Then there exists $M_1 \in \mathbf{K}_\lambda$ with $M_0 \leq_{\mathbf{K}} M_1 \leq_{\mathbf{K}} M$ and $q \in \mathrm{gS}(M_1)$ so that q extends p and for all $M' \leq_{\mathbf{K}} M$ with $M' \in \mathbf{K}_\lambda$, $M_1 \leq_{\mathbf{K}} M'$, any extension of q to $\mathrm{gS}(M')$ does not fork over M_1.

Proof. Suppose not. Build $\langle N_i : i < \omega \rangle$ increasing in $\mathbf{K}_\lambda$ and $\langle q_i : i < \omega \rangle$ such that for all $i < \omega$:

(1) $N_0 = M_0$, $q_0 = p$.
(2) $N_i \leq_{\mathbf{K}} M$.
(3) $q_i \in \mathrm{gS}(N_i)$ and q_{i+1} extends q_i.
(4) q_{i+1} forks over N_i.

This is possible since we assumed that the lemma failed. This is enough: let $N_\omega := \bigcup_{i<\omega} N_i$. Let $q \in \mathrm{gS}(N_\omega)$ extend each q_i (exists by Fact 5.7). By local character, there exists $i < \omega$ such that q does not fork over N_i, so $q \upharpoonright N_{i+1} = q_{i+1}$ does not fork over N_i, contradiction. $\square$

Lemma 5.9. If $\mathfrak{s}$ is unidimensional, then $\mathbf{K}$ is categorical in λ^+.

Proof. Assume that $\mathbf{K}$ is *not* categorical in λ^+. We show that (2) of Lemma 5.6 holds so $\mathfrak{s}$ is not unidimensional. Let $M_0 \in \mathbf{K}_\lambda$ and let $p \in \mathrm{gS}(M_0)$ be minimal. We consider two cases:

<u>Case 1</u>. There exists $M \in \mathbf{K}_{\lambda^+}$, $M_1 \in \mathbf{K}_\lambda$ with $M_0 \leq_{\mathbf{K}} M_1 \leq_{\mathbf{K}} M$ and an extension $p' \in \mathrm{gS}(M_1)$ of p so that p' is omitted in M.

Let $c \in |M| \backslash |M_1|$. Fix $M' \leq_{\mathbf{K}} M$ in $\mathbf{K}_\lambda$ containing c so that $M_1 \leq_{\mathbf{K}} M'$ and let $q := \mathrm{gtp}(c/M_1; M')$. We claim that $q \underset{\mathrm{wk}}{\perp} p'$ (and so by Lemma 4.13, $p' \perp q$, as needed). Let $N \in \mathbf{K}_\lambda$ be such that $N \geq_{\mathbf{K}} M_1$ and let $a, b \in |N|$ be such that $p' = \mathrm{gtp}(b/M_1; N)$, $q = \mathrm{gtp}(a/M_1; N)$. We want to see that $\langle ba \rangle$ is independent in (M_1, N). We have that $\mathrm{gtp}(a/M_1; N) = \mathrm{gtp}(c/M_1; M')$, so let $N' \in \mathbf{K}_\lambda$ with $M' \leq_{\mathbf{K}} N'$ and $f : N \xrightarrow[M_1]{} N'$ witness it, i.e. $f(a) = c$. Let $b' := f(b)$. We have that $\mathrm{gtp}(b'/M'; N')$ extends p', and $b' \notin |M'|$ since p' is omitted in M, hence by minimality $\mathrm{gtp}(b'/M'; N')$ does not fork over M_1. In particular, $\langle cb' \rangle$ is independent in (M_1, N'). By invariance and monotonicity, $\langle ba \rangle$ is independent in (M_1, N).

<u>Case 2</u>. Not Case 1: For every $M \in \mathbf{K}_{\lambda^+}$, every $M_1 \in \mathbf{K}_\lambda$ with $M_0 \leq_{\mathbf{K}} M_1 \leq_{\mathbf{K}} M$, every extension $p' \in \mathrm{gS}(M_1)$ of p is realized in M.

By categoricity in λ and non-categoricity in λ^+, we can find $M \in \mathbf{K}_{\lambda^+}$ with $M_0 \leq_{\mathbf{K}} M$ and $q_0 \in \mathrm{gS}(M_0)$ omitted in M. Let $M_1 \in \mathbf{K}_\lambda$, $M_0 \leq_{\mathbf{K}} M_1 \leq_{\mathbf{K}} M$ and $q \in \mathrm{gS}(M_1)$ extend q_0 so that any extension of q to a model $M' \leq_{\mathbf{K}} M$ in $\mathbf{K}_\lambda$ does not fork over M_1 (this exists by Lemma 5.8). Let $p' \in \mathrm{gS}(M_1)$ be a nonalgebraic extension of p. By assumption, p is realized by some $c \in |M|$. Now by the same argument as above (reversing the roles of p' and q), $p' \underset{\mathrm{wk}}{\perp} q$, hence $p' \perp q$, as desired. $\square$

Remark 5.10. In fact, the second case cannot happen. Otherwise, we could use the conjugation property to show that $\mathbf{K}$ has no (p, λ)-Vaughtian pair (in the sense of [GV06c, Definition 3.1]) and apply [GV06c, Theorem 4.1] to get that $\mathbf{K}$ is categorical in λ^+ in that case.

For the converse of Lemma 5.9, we will use:

Fact 5.11 (Theorem 6.1 in [GV06a])**.** Assume that $\mathbf{K}$ is categorical in λ^+. Then there exists $M \in \mathbf{K}_\lambda$ and a minimal type $p \in \mathrm{gS}(M)$ which is realized in every $N \in \mathbf{K}_\lambda$ with $M <_{\mathbf{K}} N$.

Remark 5.12. The proof of Fact 5.11 uses categoricity in λ in a strong way (it uses that the union of an increasing chain of limit models is limit).

Lemma 5.13. If $\mathbf{K}$ is categorical λ^+, then $\mathfrak{s}$ is unidimensional.

Proof. By Fact 5.11, there exists $M \in \mathbf{K}_\lambda$ and a minimal $p \in \mathrm{gS}(M)$ so that p is realized in every $N >_{\mathbf{K}} M$. Now assume for a contradiction that $\mathbf{K}$ is *not* unidimensional. Then by Lemma 5.6, there exists a nonalgebraic $q \in \mathrm{gS}(M)$ such that $p \perp q$. By Lemma 4.13.(3) (note that M is limit by categoricity in λ), there exists $N \in \mathbf{K}_\lambda$ with $N >_{\mathbf{K}} M$ so that p is omitted in N, a contradiction to the choice of p. $\qquad\square$

Theorem 5.14. $\mathfrak{s}$ is unidimensional if and only if $\mathbf{K}$ is categorical in λ^+.

Proof. By Lemmas 5.9 and 5.13. $\qquad\square$

6. Global orthogonality

Hypothesis 6.1.

   (1) $\mathbf{K}$ is an AEC.
   (2) $\theta > \mathrm{LS}(\mathbf{K})$ is a cardinal or ∞. We set $\mathcal{F} := [\mathrm{LS}(\mathbf{K}), \theta)$.
   (3) $\mathfrak{s} = (\mathbf{K}_{\mathcal{F}}, \downarrow)$ is a type-full good $\mathcal{F}$-frame.

We start developing the theory of orthogonality and unidimensionality in a more global context (with no real loss, the reader can think of $\theta = \infty$ as being the main case). The main problem is to show that for M sufficiently saturated, if $p, q \in \mathrm{gS}(M)$ do not fork over M_0, then $p \perp q$ if and only if $p \restriction M_0 \perp q \restriction M_0$. This can be done with the conjugation property in case $\|M_0\| = \|M\|$ but in general one needs to use more tools from the study of independent sequences. We will use Fact 2.6 without further mention. We will also use a few facts about independent sequences:

Fact 6.2 (Corollary 6.10 in [BVb])**.** Independent sequences of length two satisfy the axioms of a good $\mathcal{F}$-frame. For example:

   (1) Monotonicity: If $\langle ab \rangle$ is independent in (M_0, M, N) and $M_0 \leq_{\mathbf{K}} M_0' \leq_{\mathbf{K}} M' \leq_{\mathbf{K}} M \leq_{\mathbf{K}} N \leq_{\mathbf{K}} N'$, then $\langle ab \rangle$ is independent in (M_0', M', N').

(2) Continuity: If $\langle M_i : i \leq \delta \rangle$ is increasing continuous, $M_\delta \leq_{\mathbf{K}} N$, and $\langle ab \rangle$ is independent in (M_0, M_i, N) for all $i < \delta$, then $\langle ab \rangle$ is independent in (M_0, M_δ, N).

Remark 6.3. Inside which frame do we work in when we say that $\langle ab \rangle$ is independent, the global frame $\mathfrak{s}$ or its restriction? By monotonicity, the answer does not matter, i.e. the independent sequences are the same either way. Similarly, if $\lambda > \mathrm{LS}(\mathbf{K})$ and $M_0, M, N \in \mathbf{K}_{[\lambda, \theta)}^{\lambda\text{-sat}}$, then $\langle ab \rangle$ is independent in (M_0, M, N) with respect to $\mathfrak{s}$ if and only if it is independent in (M_0, N) with respect to $\mathfrak{s} \restriction \mathbf{K}_{[\lambda, \theta)}^{\lambda\text{-sat}}$ (i.e. we can require the models witnessing the independence to be saturated). This is a simple consequence of the extension property.

We now define global orthogonality.

Definition 6.4. Let $M \in \mathbf{K}_{\mathcal{F}}$. For $p, q \in \mathrm{gS}(M)$ nonalgebraic, we write $p \perp q$ for $p \overset{\mathfrak{s} \restriction \mathbf{K}_{\|M\|}}{\perp} q$, and $p \underset{\mathrm{wk}}{\perp} q$ for $p \overset{\mathfrak{s} \restriction \mathbf{K}_{\|M\|}}{\underset{\mathrm{wk}}{\perp}} q$ (recall Definition 4.10).

Note that a priori we need not have that if $p \perp q$ and p', q' are nonforking extensions of p and q to big models, then $p' \perp q'$. This will be proven first.

Lemma 6.5. Let δ be a limit ordinal. Let $\langle M_i : i \leq \delta \rangle$ be increasing continuous in $\mathbf{K}_{\mathcal{F}}$. Let $p, q \in \mathrm{gS}(M_\delta)$ be nonalgebraic and assume that $p \restriction M_i \underset{\mathrm{wk}}{\perp} q \restriction M_i$ for all $i < \delta$. Then $p \underset{\mathrm{wk}}{\perp} q$.

Proof. By the continuity property of independent sequences (Fact 6.2). $\square$

Lemma 6.6. Let δ be a limit ordinal. Let $\langle M_i : i \leq \delta \rangle$ be increasing continuous in $\mathbf{K}_{\mathcal{F}}$. Let $p, q \in \mathrm{gS}(M_\delta)$ be nonalgebraic and assume that $p \restriction M_i \perp q \restriction M_i$ for all $i < \delta$. Then $p \perp q$.

Proof. By local character, there exists $i < \delta$ so that both p and q do not fork over M_i. Without loss of generality, $i = 0$. Let $\lambda := \|M_\delta\|$. If there exists $i < \delta$ so that $\lambda = \|M_i\|$, then the result follows from the definition of orthogonality. So assume that $\|M_i\| < \lambda$ for all $i < \delta$. Let $M' \geq_{\mathbf{K}} M_\delta$ be in $\mathbf{K}_\lambda$ and let p', q' be the nonforking extensions to M' of p, q respectively. We want to see that $p' \underset{\mathrm{wk}}{\perp} q'$. Let $\langle M_i' : i \leq \delta \rangle$ be an increasing continuous resolution of M' such that $M_i \leq_{\mathbf{K}} M_i'$ and $\|M_i'\| = \|M_i\|$ for all $i < \delta$. We know that $p' \restriction M_i'$ does not fork

over M_0, hence over M_i and similarly $q' \restriction M_i'$ does not fork over M_i. Therefore by definition of orthogonality, $p' \restriction M_i' \underset{\text{wk}}{\perp} q' \restriction M_i'$. By Lemma 6.5, $p' \underset{\text{wk}}{\perp} q'$. $\square$

Lemma 6.7. Let $M_0 \leq_{\mathbf{K}} M$ be both in $\mathbf{K}_{\mathcal{F}}$. Let $p, q \in \mathrm{gS}(M)$ be nonalgebraic so that both do not fork over M_0. If $p \restriction M_0 \perp q \restriction M_0$, then $p \perp q$.

Proof. Let $\delta := \mathrm{cf}(\|M\|)$. Build $\langle N_i : i \leq \delta \rangle$ increasing continuous such that $N_0 = M_0$, $N_\delta = M$, and $p \restriction N_i \perp q \restriction N_i$ for all $i \leq \delta$. This is easy: at successor steps, we require $\|N_i\| = \|N_{i+1}\|$ and use the definition of orthogonality. At limit steps, we use Lemma 6.6. Then $p \restriction N_\delta \perp q \restriction N_\delta$, but $N_\delta = M$ so $p \perp q$. $\square$

Question 6.8. Is the converse true? That is if $M_0 \leq_{\mathbf{K}} M$ are in $\mathbf{K}_{\mathcal{F}}$, $p, q \in \mathrm{gS}(M)$ do not fork over M_0 and $p \perp q$, do we have that $p \restriction M_0 \perp q \restriction M_0$?

An answer to this question would be useful in order to transfer unidimensionality up in a more conceptual way than below. With a very mild additional hypothesis, we give a positive answer in Theorem D.4 of the appendix.

We now go back to studying unidimensionality.

Definition 6.9. For $\lambda \in \mathcal{F}$, we say that $\mathfrak{s}$ is λ-*unidimensional* if the following is *false*: for every limit $M \in \mathbf{K}_\lambda$ and every nonalgebraic $p \in \mathrm{gS}(M)$, there exists a limit $M' \geq_{\mathbf{K}} M$ in $\mathbf{K}_\lambda$ and $p', q \in \mathrm{gS}(M')$ so that p' extends p and $p' \perp q$.

Remark 6.10. When $\lambda > \mathrm{LS}(\mathbf{K})$, $\mathfrak{s}$ is λ-unidimensional if and only if $\mathfrak{s} \restriction \mathbf{K}_\lambda^{\lambda\text{-sat}}$ is unidimensional (see Definition 5.2). If $\mathbf{K}$ is categorical in $\mathrm{LS}(\mathbf{K})$, this also holds when $\lambda = \mathrm{LS}(\mathbf{K})$ (if $\mathbf{K}$ is not categorical in $\mathrm{LS}(\mathbf{K})$, we do not know that $\mathbf{K}^{\mathrm{LS}(\mathbf{K})\text{-sat}}$ is an AEC).

Our next goal is to prove (assuming categoricity in $\mathrm{LS}(\mathbf{K})$) that λ-unidimensionality is equivalent to μ-unidimensionality for every $\lambda, \mu \in \mathcal{F}$. We will use another characterization of λ-unidimensionality when $\lambda > \mathrm{LS}(\mathbf{K})$:

Lemma 6.11. Let $\lambda > \mathrm{LS}(\mathbf{K})$ be in $\mathcal{F}$. The following are equivalent:

(1) $\mathfrak{s}$ is *not* λ-unidimensional.
(2) There exists a saturated $M \in \mathbf{K}_\lambda$ and nonalgebraic types $p, q \in \mathrm{gS}(M)$ such that p is minimal and $p \perp q$.

Proof.

- (1) implies (2): Assume that $\mathfrak{s}$ is not λ-unidimensional. Let $M \in \mathbf{K}_\lambda^{\lambda\text{-sat}}$ and let $p \in \mathrm{gS}(M)$ be minimal (exists by density of minimal types and uniqueness of saturated models). By Lemma 5.6, there exists $q \in \mathrm{gS}(M)$ so that $p \perp q$.
- (2) implies (1): Let $M \in \mathbf{K}_\lambda^{\lambda\text{-sat}}$ and let $p, q \in \mathrm{gS}(M)$ be non-algebraic so that p is minimal and $p \perp q$. We show that $\mathbf{K}^{\lambda\text{-sat}}$ is *not* categorical in λ^+, which is enough by Theorem 5.14. Fix $N \leq_{\mathbf{K}} M$ in $\mathbf{K}_{\mathrm{LS}(\mathbf{K})}$ so that p does not fork over N. Build a strictly increasing continuous chain $\langle M_i : i \leq \lambda^+ \rangle$ such that for all $i < \lambda^+$:

  (1) $M_i \in \mathbf{K}_\lambda^{\lambda\text{-sat}}$.

  (2) $M_0 = M$.

  (3) p is omitted in M_i.

  This is enough, since then p is omitted in M_{λ^+} so $M_{\lambda^+} \in \mathbf{K}_{\lambda^+}$ cannot be saturated. This is possible: at limits we take unions and for $i = 0$ we set $M_0 := M$. Now let $i = j + 1$ be given. Let $p' \in \mathrm{gS}(M_j)$ be the nonforking extension of p. By uniqueness of saturated models, there exists $f : M_j \cong_N M_0$. By uniqueness of nonforking extension, $f(p') = p$. By Lemma 4.13.(3), there exists $M' \geq_{\mathbf{K}} M_0$ in $\mathbf{K}_\lambda^{\lambda\text{-sat}}$ so that p is omitted in M'. Let $M_{j+1} := f^{-1}[M']$. Then p' is omitted in M_{j+1}. Since p is minimal, p is omitted in $|M_{j+1}| \backslash |M_j|$, and hence by induction in M_{j+1}.

$\square$

An issue in transferring unidimensionality up is that we do not have a converse to Lemma 6.7 (see Question 6.8), so we will "cheat" and use the following transfer which follows from the proof of [GV06a, Theorem 6.3]:

Fact 6.12. If $\mathbf{K}$ is categorical in $\mathrm{LS}(\mathbf{K})$ and $\mathrm{LS}(\mathbf{K})^+$, then $\mathbf{K}$ is categorical in all $\mu \in [\mathrm{LS}(\mathbf{K}), \theta]$.

Theorem 6.13. Assume that $\mathbf{K}$ is categorical in $\mathrm{LS}(\mathbf{K})$. Let λ and μ both be in $\mathcal{F}$. Then $\mathfrak{s}$ is λ-unidimensional if and only if $\mathfrak{s}$ is μ-unidimensional.

Proof. Without loss of generality, $\mu < \lambda$. We first show that if $\mathfrak{s}$ is *not* μ-unidimensional, then $\mathfrak{s}$ is *not* λ-unidimensional. Assume that $\mathfrak{s}$ is not μ-unidimensional. Let $M_0 \in \mathbf{K}_\mu^{\mu\text{-sat}}$ and let $p \in \mathrm{gS}(M_0)$ be minimal (exists by density of minimal types). By definition (and the

proof of Lemma 5.6), there exists $q \in \mathrm{gS}(M_0)$ so that $p \perp q$. Now let $M \in \mathbf{K}_\lambda^{\lambda\text{-sat}}$ be such that $M_0 \leq_{\mathbf{K}} M$. Let p', q' be the nonforking extensions to M of p and q respectively. By Lemma 6.7, $p' \perp q'$. By Lemma 6.11, $\mathfrak{s}$ is not λ-unidimensional.

Conversely, assume that $\mathfrak{s}$ is μ-unidimensional. By the first part, $\mathfrak{s}$ is $\mathrm{LS}(\mathbf{K})$-unidimensional. By Theorem 5.14, $\mathbf{K}$ is categorical in $\mathrm{LS}(\mathbf{K})^+$. By Fact 6.12, $\mathbf{K}$, and hence $\mathbf{K}^{\lambda\text{-sat}}$, is categorical in λ^+. By Theorem 5.14 again, $\mathfrak{s}$ is λ-unidimensional. $\qquad\square$

We obtain the promised categoricity transfer. Note that it suffices to assume that $\mathbf{K}^{\lambda\text{-sat}}$ (instead of $\mathbf{K}$) is categorical in λ^+.

Theorem 6.14. Assume that $\mathbf{K}$ is categorical in $\mathrm{LS}(\mathbf{K})$ and let $\lambda \in \mathcal{F}$. If $\mathbf{K}^{\lambda\text{-sat}}$ is categorical in λ^+, then $\mathbf{K}$ is categorical in every $\mu \in [\mathrm{LS}(\mathbf{K}), \theta]$.

Proof. Assume that $\mathbf{K}^{\lambda\text{-sat}}$ is categorical in λ^+. We prove by induction on $\mu \in [\mathrm{LS}(\mathbf{K}), \theta]$ that $\mathbf{K}$ is categorical in μ. By assumption, $\mathbf{K}$ is categorical in $\mathrm{LS}(\mathbf{K})$. Now let $\mu \in (\mathrm{LS}(\mathbf{K}), \theta]$ and assume that $\mathbf{K}$ is categorical in every $\mu_0 \in [\mathrm{LS}(\mathbf{K}), \mu)$. If μ is limit, then it is easy to see that every model of size μ must be saturated, hence $\mathbf{K}$ is categorical in μ. Now assume that μ is a successor, say $\mu = \mu_0^+$ for $\mu_0 \in \mathcal{F}$. By assumption, $\mathbf{K}^{\lambda\text{-sat}}$ is categorical in λ^+. By Theorem 5.14, $\mathfrak{s}$ is λ-unidimensional. By Theorem 6.13, $\mathfrak{s}$ is μ_0-unidimensional. By Theorem 5.14, $\mathbf{K}^{\mu_0\text{-sat}}$ is categorical in μ_0^+. By the induction hypothesis, $\mathbf{K}$ is categorical in μ_0, hence $\mathbf{K}^{\mu_0\text{-sat}} = \mathbf{K}_{\geq \mu_0}$, so $\mathbf{K}$ is categorical in $\mu_0^+ = \mu$, as desired. $\qquad\square$

Part 2. Applications

7. Background

The definition of superstability below is already implicit in [SV99] and several variants were studied in, e.g. [Van06, GVV, Vasa, BVa, GV, VV]. We will use the statement from [Vasa, Definition 10.1].

Definition 7.1. $\mathbf{K}$ is μ-*superstable* (or *superstable in* μ) if:

(1) $\mu \geq \mathrm{LS}(\mathbf{K})$.
(2) $\mathbf{K}_\mu$ is nonempty, has amalgamation, joint embedding, and no maximal models.
(3) $\mathbf{K}$ is stable in μ, and:

(4) μ-splitting in $\mathbf{K}$ satisfies the following locality property: for every limit ordinal $\delta < \mu^+$ and every increasing continuous sequence $\langle M_i : i \leq \delta \rangle$ in $\mathbf{K}_\mu$ with M_{i+1} universal over M_i for all $i < \delta$, if $p \in \mathrm{gS}(M_\delta)$, then there exists $i < \delta$ so that p does not μ-split over M_i.

We will use the following without comments. See [VV, Fact 4.8.(2)] for a proof.

Fact 7.2. If $\mathfrak{s}$ is a type-full good λ-frame on $\mathbf{K}$, then $\mathbf{K}$ is λ-superstable.

In the setup of this paper, superstability follows from categoricity. If (as will be the case in most of this paper) the AEC is categorical in a successor, this is due to Shelah and appears as [She99, Lemma 6.3]. The heart of the proof in the general case appears as [SV99, Theorem 2.2.1] and the result is stated for classes with amalgamation in [GV, Theorem 6.3].

Fact 7.3 (The Shelah-Villaveces theorem)**.** If $\mathbf{K}$ has amalgamation, no maximal models, and is categorical in a $\lambda > \mathrm{LS}(\mathbf{K})$, then $\mathbf{K}$ is $\mathrm{LS}(\mathbf{K})$-superstable.

Together with superstability, a powerful tool is the symmetry property for splitting, first isolated by VanDieren [Van]:

Definition 7.4. Let $\mu \geq \mathrm{LS}(\mathbf{K})$ and assume that $\mathbf{K}$ has amalgamation in μ. $\mathbf{K}$ exhibits *symmetry for μ-splitting* (or *μ-symmetry* for short) if whenever models $M, M_0, N \in \mathbf{K}_\mu$ and elements a and b satisfy the conditions 1-4 below, then there exists M^b a limit model over M_0, containing b, so that $\mathrm{gtp}(a/M^b)$ does not μ-split over N.

(1) M is universal over M_0 and M_0 is a limit model over N.
(2) $a \in M \backslash M_0$.
(3) $\mathrm{gtp}(a/M_0)$ is non-algebraic and does not μ-split over N.
(4) $\mathrm{gtp}(b/M)$ is non-algebraic and does not μ-split over M_0.

When the class is tame, symmetry follows from superstability [VV, Corollary 6.9] and superstability transfers upward [Vasa, Proposition 10.10] hence they both hold everywhere:

Fact 7.5. If $\mathbf{K}$ has amalgamation, is $\mathrm{LS}(\mathbf{K})$-tame, and is $\mathrm{LS}(\mathbf{K})$-superstable, then $\mathbf{K}$ is superstable and has symmetry in every $\mu \geq \mathrm{LS}(\mathbf{K})$.

One consequence of the symmetry property is given by the following more precise statement of Fact 2.6 (see [VV, Lemma 2.20] and [Van16, Theorem 1]):

Fact 7.6. Assume that $\mathbf{K}$ has amalgamation. Let $\chi > \mathrm{LS}(\mathbf{K})$. If for every $\mu \in [\mathrm{LS}(\mathbf{K}), \chi)$, $\mathbf{K}$ is superstable in μ and μ^+ and has symmetry in μ^+, then $\mathbf{K}^{\chi\text{-sat}}$ is an AEC with $\mathrm{LS}(\mathbf{K}^{\chi\text{-sat}}) = \chi$.

We will use the following consequences of categoricity in a suitable cardinal. The notation from Chapter 14 of [Bal09] will come in handy:

Notation 7.7. For λ an infinite cardinal, let $h(\lambda) := \beth_{(2^\lambda)^+}$. When $\mathbf{K}$ is fixed, we write H_1 for $h(\mathrm{LS}(\mathbf{K}))$ and H_2 for $h(H_1) = h(h(\mathrm{LS}(\mathbf{K})))$.

Fact 7.8. Assume that $\mathbf{K}$ has amalgamation and no maximal models. Let $\lambda > \mathrm{LS}(\mathbf{K})$ be such that $\mathbf{K}$ is categorical in λ.

  (1) [VV, Corollary 7.2] If $\lambda \geq H_1$ or the model of size λ is $\mathrm{LS}(\mathbf{K})^+$-saturated (e.g. if $\mathrm{cf}(\lambda) > \mathrm{LS}(\mathbf{K})$), then $\mathbf{K}$ has $\mathrm{LS}(\mathbf{K})$-symmetry.
  (2) [VV, Corollary 7.4] If $\lambda \geq h(\mathrm{LS}(\mathbf{K})^+)$, then the model of size λ is $\mathrm{LS}(\mathbf{K})^+$-saturated.

As a special case, we obtain the following result that is already stated in [She99, Claim I.6.7].

Corollary 7.9. Assume that $\mathbf{K}$ has amalgamation and no maximal models. Let $\lambda > \mu > \mathrm{LS}(\mathbf{K})$ be such that $\mathbf{K}$ is categorical in λ. If the model of size λ is μ^+-saturated, then $\mathbf{K}^{\mu\text{-sat}}$ is an AEC with $\mathrm{LS}(\mathbf{K}^{\mu\text{-sat}}) = \mu$.

Proof. By Fact 7.3, $\mathbf{K}$ is superstable in every $\chi \in [\mathrm{LS}(\mathbf{K}), \lambda)$. By Fact 7.8, $\mathbf{K}$ has symmetry in every $\chi \in [\mathrm{LS}(\mathbf{K}), \mu]$. Now apply Fact 7.6. $\square$

The following fact tells us that we can often assume without loss of generality that a categorical AECs with amalgamation also has no maximal models. The proof is folklore (see e.g. [Vasa, Proposition 10.13]).

Fact 7.10. Assume that $\mathbf{K}$ has amalgamation. Let $\lambda \geq \mathrm{LS}(\mathbf{K})$ be such that $\mathbf{K}$ has joint embedding in λ. Then there exists $\chi < \beth_{(2^{\mathrm{LS}(\mathbf{K})})^+}$ and an AEC $\mathbf{K}^*$ such that:

  (1) $\mathbf{K}^* \subseteq \mathbf{K}$ and $\mathbf{K}^*$ has the same strong substructure relation as $\mathbf{K}$.
  (2) $\mathrm{LS}(\mathbf{K}^*) = \mathrm{LS}(\mathbf{K})$.
  (3) $\mathbf{K}^*$ has amalgamation, joint embedding, and no maximal models.

(4) $\mathbf{K}_{\geq \min(\lambda, \chi)} = (\mathbf{K}^*)_{\geq \min(\lambda, \chi)}$.

Let us also recall the definition of tameness (first isolated in [GV06b]) and weak tameness (already implicit in [She99]). We use the notation from [Bal09, Definition 11.6]

Definition 7.11 (Tameness). Let χ, μ be cardinals with $\mathrm{LS}(\mathbf{K}) \leq \chi \leq \mu$. Assume that $\mathbf{K}_{[\chi, \mu]}$ has amalgamation. $\mathbf{K}$ is (χ, μ)-*tame* if for any $M \in \mathbf{K}_\mu$, any $p, q \in \mathrm{gS}(M)$, if $p \neq q$, there exists $M_0 \in \mathbf{K}_\chi$ with $M_0 \leq_{\mathbf{K}} M$ and $p \upharpoonright M_0 \neq q \upharpoonright M_0$. For $\theta \geq \mu$, $\mathbf{K}$ is $(\chi, < \theta)$-*tame* if it is (χ, μ)-tame for every $\mu \in [\chi, \theta)$. $(\chi, \leq \theta)$-tame means $(\chi, < \theta^+)$-weakly tame. Finally, $\mathbf{K}$ is χ-*tame* if it is (χ, μ)-weakly tame for every $\mu \geq \chi$. We similarly define variations such as $(< \chi, \mu)$-tame.

Let us also define $\mathbf{K}$ is (χ, μ)-*weakly tame* to mean that for any *saturated* $M \in \mathbf{K}_\mu$, any $p, q \in \mathrm{gS}(M)$, if $p \neq q$, there exists $M_0 \in \mathbf{K}_\chi$ with $M_0 \leq_{\mathbf{K}} M$ and $p \upharpoonright M_0 \neq q \upharpoonright M_0$. Define variations such as $(\chi, < \mu)$-weakly tame as above.

Remark 7.12. Tameness says that type over *any* models are determined by their small restrictions. Weak tameness says that only types over *saturated* models have this property. While there is no known example of an AEC that is weakly tame but not tame, it is known that weak tameness follows from categoricity in a suitable cardinal (but the corresponding result for non-weak tameness is open, see [GV06a, Conjecture 1.5]), see Section 8.

It was noticed in [Vas16] (and further improvements in [Vasa, Section 10] or [VV, Corollary 6.14]) that tameness can be combined with superstability to build a good frame. This can also be done using only weak tameness:

Fact 7.13 (Theorem 6.4 in [VV]). Let $\lambda > \mathrm{LS}(\mathbf{K})$. Assume that $\mathbf{K}$ is superstable in every $\mu \in [\mathrm{LS}(\mathbf{K}), \lambda]$ and has λ-symmetry.

If $\mathbf{K}$ is $(\mathrm{LS}(\mathbf{K}), \lambda)$-weakly tame, then there exists a type-full good λ-frame with underlying class $\mathbf{K}_\lambda^{\lambda\text{-sat}}$ (so in particular, $\mathbf{K}_\lambda^{\lambda\text{-sat}}$ is the initial segment of an AEC).

Once we have a good λ-frame, we can enlarge it so that the forking relation works over larger models.

Fact 7.14 (Corollary 6.9 in [BVb]). Let $\theta > \lambda \geq \mathrm{LS}(\mathbf{K})$. Let $\mathcal{F} := [\lambda, \theta)$. Assume that $\mathbf{K}_{\mathcal{F}}$ has amalgamation. Let $\mathfrak{s}$ be a type-full good λ-frame on $\mathbf{K}_\lambda$. If $\mathbf{K}$ is $(\lambda, < \theta)$-tame, then there exists a type-full good $\mathcal{F}$-frame $\mathfrak{s}'$ extending $\mathfrak{s}$: $\mathfrak{s}' \upharpoonright \mathbf{K}_\lambda = \mathfrak{s}$.

Assuming only weak tameness, we can show that if $\mathfrak{s}$ is a (type-full) good μ-frame and $\mathfrak{s}'$ is a good λ-frame with $\mu < \lambda$ and the underlying class of $\mathfrak{s}'$ is the saturated models in the underlying class of $\mathfrak{s}$, then forking in $\mathfrak{s}'$ can be described in terms of forking in $\mathfrak{s}$. This is proven as Theorem A.5 in the appendix and is used to replace tameness by weak tameness in the main theorem.

8. Weak tameness from categoricity

We quote a result of Shelah from [She09a, Chapter IV] on deriving weak tameness from categoricity and combine it with the corresponding results in [VV]. We derive a small improvement on some of the Hanf numbers, positively answering a question of Baldwin [Bal09, Question 11.16] (see also [Bal09, Remark 14.15]) which asked whether it was possible to obtain χ-weak tameness for some $\chi < H_1$ rather than $(< H_1)$-weak tameness. We give two applications in AECs with amalgamation and no maximal models that are categorical in a high-enough cardinal that is still *below* the Hanf number: the construction of a good frame and a non-trivial restriction on the categoricity spectrum.

The following appears as [She99, Main Claim II.2.3] (a simplified and improved argument is in [Bal09, Theorem 11.15]):

Fact 8.1. Assume that $\mathbf{K}$ has amalgamation. Let $\lambda > \mu \geq H_1$. Assume that $\mathbf{K}$ is categorical in λ, and the model of cardinality λ is μ^+-saturated. Then there exists $\chi < H_1$ such that $\mathbf{K}$ is (χ, μ)-weakly tame.

As opposed to Fact 8.1, the following also applies when the categoricity cardinal is below the Hanf number.

Fact 8.2 (Claim IV.7.2 in [She09a]). Let $\mu > \mathrm{LS}(\mathbf{K})$. If:

   (1) $\mathbf{K}_{<\mu}$ has amalgamation.
   (2) $\mathrm{cf}(\mu) > \mathrm{LS}(\mathbf{K})$.
   (3) Φ is a proper for linear orders, and if $\theta \in (\mathrm{LS}(\mathbf{K}), \mu)$, I is a θ-wide[5] linear order, then $\mathrm{EM}_{\tau(\mathbf{K})}(I, \Phi)$ is θ-saturated.

Then there exists $\chi \in (\mathrm{LS}(\mathbf{K}), \mu)$ such that $\mathbf{K}$ is $(\chi, < \mu)$-weakly tame.

Condition (3) in Fact 8.2 can be derived from categoricity if the model in the categoricity cardinal is sufficiently saturated. This is implicit in [She99] and appears as [Bal09, Lemma 10.11].

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[5]A linear order is θ-*wide* if for every $\theta_0 < \theta$, I contains an increasing sequence of length θ_0^+, see [She09a, Definition IV.0.14.(1)].

Fact 8.3. If $\mathbf{K}$ has amalgamation and no maximal models, $\mu > \mathrm{LS}(\mathbf{K})$, $\mathbf{K}$ is categorical in a $\lambda \geq \mu$, so that the model of size λ is μ-saturated, then for every Φ proper for linear orders, if $\theta \in (\mathrm{LS}(\mathbf{K}), \mu)$ and I is a θ-wide linear order, we have that $\mathrm{EM}_{\tau(\mathbf{K})}(I, \Phi)$ is θ-saturated.

Combining these facts, we obtain the following result. Note that the second part is a slight improvement on Fact 8.1, as the model of size λ is allowed to be H_1-saturated. Moreover the amount of weak tameness χ can be chosen independently of μ:

Theorem 8.4. Assume that $\mathbf{K}$ has amalgamation. Let $\lambda > \mathrm{LS}(\mathbf{K})$ be such that $\mathbf{K}$ is categorical in λ.

   (1) Let μ be a limit cardinal such that $\mathrm{cf}(\mu) > \mathrm{LS}(\mathbf{K})$. If $\mathbf{K}$ has no maximal models and the model of size λ is μ-saturated, then there exists $\chi < \mu$ such that $\mathbf{K}$ is $(\chi, < \mu)$-weakly tame.
   (2) If the model of size λ is H_1-saturated, then there exists $\chi < H_1$ such that whenever $\mu \geq H_1$ is so that the model of size λ is μ-saturated, we have that $\mathbf{K}$ is $(\chi, < \mu)$-weakly tame.

Proof.

   (1) By Fact 8.2 (using Fact 8.3 to see that (3) is satisfied).
   (2) Without loss of generality (Fact 7.10), $\mathbf{K}$ has no maximal models. By the first part (with μ there standing for H_1 here), there exists $\chi < H_1$ such that $\mathbf{K}$ is $(\chi, < H_1)$-weakly tame. Now assume that the model of size λ is μ-saturated, for $\mu > H_1$. Let $\mu' \in [H_1, \mu)$. We show that $\mathbf{K}$ is (χ, μ')-weakly tame. By Fact 8.1 (with μ there standing for μ' here), there exists $\chi' < H_1$ such that $\mathbf{K}$ is (χ', μ')-weakly tame. In particular, $\mathbf{K}$ is (H_1, μ')-weakly tame. Now by Corollary 7.9, $\mathbf{K}^{H_1\text{-sat}}$ is an AEC with $\mathrm{LS}(\mathbf{K}^{H_1\text{-sat}}) = H_1$. Thus we can combine (χ, H_1)-weak and (H_1, μ')-weak tameness to get (χ, μ)-weak tameness, as desired.

$\hfill\square$

We give two applications of (the first part of) Theorem 8.4. First, we obtain an improvement on the Hanf number for the construction of a good frame in [VV, Corollary 7.9] (μ below can be less than H_1, e.g. $\mu = \aleph_{\mathrm{LS}(\mathbf{K})^+}$).

Theorem 8.5. Assume that $\mathbf{K}$ has amalgamation and no maximal models. Let μ be a limit cardinal such that $\mathrm{cf}(\mu) > \mathrm{LS}(\mathbf{K})$ and assume that $\mathbf{K}$ is categorical in a $\lambda \geq \mu$. If the model of size λ is μ-saturated,

then there exists $\chi < \mu$ such that for all $\mu_0 \in [\chi, \mu)$, there is a good μ_0-frame on $\mathbf{K}^{\mu_0\text{-sat}}$.

Proof. By Fact 7.8, $\mathbf{K}$ has μ_0-symmetry for every $\mu_0 < \chi$. By Fact 7.3, $\mathbf{K}$ is also superstable in every $\mu_0 \in [\mathrm{LS}(K), \lambda)$. Now by Theorem 8.4, there exists $\chi < \mu$ so that $\mathbf{K}$ is $(\chi, < \mu)$-weakly tame. We finish by applying Fact 7.13.   □

Second, we can study the categoricity spectrum *below* the Hanf number of an AEC with amalgamation and no maximal models. While it is known that the categoricity spectrum in such AECs must be a closed set (see the proof of [GV06c, Corollary 4.3]), we show (in ZFC) that there are other restrictions:

Theorem 8.6. Assume that $\mathbf{K}$ has amalgamation and arbitrarily large models. Let μ be a limit cardinal such that $\mathrm{cf}(\mu) > \mathrm{LS}(\mathbf{K})$. If $\mathbf{K}$ is categorical in unboundedly many successor cardinals below μ, then there exists $\mu_0 < \mu$ such that $\mathbf{K}$ is categorical in every $\lambda \in [\mu_0, \mu]$.

In particular (setting $\mu := \aleph_{\mathrm{LS}(\mathbf{K})^+}$), if $\mathbf{K}$ is categorical in $\aleph_{\alpha+1}$ for unboundedly many $\alpha < \mathrm{LS}(\mathbf{K})^+$, then there exists $\alpha_0 < \mathrm{LS}(\mathbf{K})^+$ such that $\mathbf{K}$ is categorical in $\aleph_\beta$ for every $\beta \in [\alpha_0, \mathrm{LS}(\mathbf{K})^+]$.

Before starting the proof, we make a remark:

Remark 8.7. Fact 1.6 generalizes to AECs that are only $\mathrm{LS}(\mathbf{K})$-weakly tame, or just $(\mathrm{LS}(\mathbf{K}), < \theta)$-weakly tame (in the second case, we can only conclude categoricity up to and including θ). This is implicit in [GV06c, GV06a] and stated explicitly in Chapter 13 of [Bal09].

Proof of Theorem 8.6. Without loss of generality (Fact 7.10), $\mathbf{K}$ has no maximal models. By amalgamation, every model of size μ is saturated. In particular $\mathbf{K}$ is categorical in μ. By Theorem 8.4 (with λ, μ there standing for μ, μ here), there exists $\mu_0' < \mu$ such that $\mathbf{K}$ is $(\mu_0', < \mu)$-weakly tame. By making μ_0' bigger if necessary, we can assume without loss of generality that $\mu_0' > \mathrm{LS}(\mathbf{K})$ and $\mathbf{K}$ is categorical in $\mu_0 := (\chi_0')^+$. By the upward categoricity transfer of Grossberg and Vandieren (Fact 1.6, keeping in mind Remark 8.7), $\mathbf{K}$ is categorical in every $\lambda \in [\mu_0, \mu]$.   □

9. Shelah's omitting type theorem

In this section, we give a nonlocal proof of Corollary 0.2 using the methods of [She99]. We will present a more powerful local proof in

the next sections. We also give several partial categoricity transfers in AECs with amalgamation, including Theorem 9.8 which says that in a tame AEC with amalgamation, categoricity in *some* cardinal (above the tameness cardinal) implies categoricity in a proper class of cardinals. The main tool is a powerful generalization of Morley's omitting type theorem (Fact 9.3), an early form of which appears in [MS90].

All throughout, we assume:

Hypothesis 9.1. $\mathbf{K}$ is an AEC with amalgamation.

As a motivation, we first state Morley's omitting type theorem for AECs [She99, II.1.10]. We state a slightly stronger conclusion (replacing H_1 by some $\chi < H_1$) that is implicit e.g. in [She99] but to the best of our knowledge, a proof of this stronger result has not appeared in print before. We include a proof (similar to the proof of [BG, Theorem 5.4], though there is an additional step involved) for the convenience of the reader.

Fact 9.2 (Morley's omitting type theorem for AECs)**.** Let $\lambda > \mathrm{LS}(\mathbf{K})$. If every model in $\mathbf{K}_\lambda$ is $\mathrm{LS}(\mathbf{K})^+$-saturated, then there exists $\chi < H_1$ such that every model in $\mathbf{K}_{\geq \chi}$ is $\mathrm{LS}(\mathbf{K})^+$-saturated.

Proof sketch. Without loss of generality (Fact 7.10), $\mathbf{K}$ has no maximal models. Suppose the conclusion fails. Then for every $\chi \in [\mathrm{LS}(\mathbf{K}), H_1)$, there exists $M_\chi \in \mathbf{K}_\chi$ which is not $\mathrm{LS}(\mathbf{K})^+$-saturated. Pick witnesses $M_{0,\chi} \leq_{\mathbf{K}} M_\chi$ and $p_\chi \in \mathrm{gS}(M_{0,\chi})$ such that $\|M_{0,\chi}\| = \mathrm{LS}(\mathbf{K})$ and M_χ omits p_χ. Now there are only $2^{\mathrm{LS}(K)}$ isomorphism types of Galois types over models of size $\mathrm{LS}(\mathbf{K})$, and $\mathrm{cf}(H_1) = \left(2^{\mathrm{LS}(\mathbf{K})}\right)^+ > 2^{\mathrm{LS}(\mathbf{K})}$, so there exists $N \in \mathbf{K}_{\mathrm{LS}(\mathbf{K})}$, $p \in \mathrm{gS}(M)$, and an unbounded $S \subseteq [\mathrm{LS}(\mathbf{K}), H_1)$ such that for all $\chi \in S$, p_χ is isomorphic to p (in the natural sense). Look at the AEC $\mathbf{K}_{\neg p}$ of all the models of $\mathbf{K}$ omitting p, with constants added for N (see e.g. the definition of $\mathbf{K}^+$ in the proof of [BG, Theorem 5.4]). For each $\chi \in S$, an appropriate expansion of a copy of M_χ is in $K_{\neg p}$. $\mathbf{K}_{\neg p}$ has Löwenheim-Skolem-Taski number $\mathrm{LS}(\mathbf{K})$, so by Shelah's presentation theorem and Morley's omitting type theorem (for first-order theories), $\mathbf{K}_{\neg p}$ has arbitrarily large models, contradicting the assumptions on λ. $\square$

A generalization of Fact 9.2 is what we call Shelah's omitting type theorem. The statement appears (in a more general form) in [She99, Lemma II.1.6], but the full proof (for models of an $\mathbb{L}_{\kappa,\omega}$ theory, κ a strongly compact cardinal) can already be found in [MS90, Proposition 3.3] (see also Will Boney's note [Bona]). We state a simplified version:

Fact 9.3 (Shelah's omitting type theorem)**.** Let $M_0 \leq_{\mathbf{K}} M$ both be in $\mathbf{K}_{\geq \mathrm{LS}(\mathbf{K})}$ and let $p \in \mathrm{gS}(M_0)$. Assume that M omits $p/E_{\mathrm{LS}(\mathbf{K})}$. That is, for every $a \in |M|$, there is $M_0' \leq_{\mathbf{K}} M_0$ with $\|M_0'\| = \mathrm{LS}(\mathbf{K})$ such that $\mathrm{gtp}(a/M_0'; M) \neq p \upharpoonright M_0'$.

If $\beth_{(2^{\mathrm{LS}(\mathbf{K})})^+}(\|M_0\|) \leq \|M\|$, then there is a non $\mathrm{LS}(\mathbf{K})^+$-saturated model in every cardinal.

Note that taking $\|M_0\| = \mathrm{LS}(\mathbf{K})$, we recover Morley's omitting type theorem for AECs. Note also that when $\mathbf{K}$ is $\mathrm{LS}(\mathbf{K})$-tame, M above omits $p/E_{\mathrm{LS}(\mathbf{K})}$ if and only if M omits p. The following two direct consequences are implicit in [She99].

Lemma 9.4. Let $\mathrm{LS}(\mathbf{K}) < \lambda$ and let $\mathrm{LS}(\mathbf{K}) \leq \chi < \mu$. Assume that:

(1) $\beth_{(2^{\mathrm{LS}(\mathbf{K})})^+}(\chi) \leq \mu$.
(2) $\mathbf{K}$ is $(\mathrm{LS}(\mathbf{K}), \leq \chi)$-weakly tame.
(3) For every $\chi_0 \in [\mathrm{LS}(\mathbf{K})^+, \chi]$, $\mathbf{K}^{\chi_0\text{-sat}}$ is an AEC with $\mathrm{LS}(\mathbf{K}^{\chi_0\text{-sat}}) = \chi_0$.

If every model in $\mathbf{K}_\lambda$ is $\mathrm{LS}(\mathbf{K})^+$-saturated, then every model in $\mathbf{K}_\mu$ is χ^+-saturated.

Proof. Assume that every model in $\mathbf{K}_\lambda$ is $\mathrm{LS}(\mathbf{K})^+$-saturated. We prove that for all $\chi_0 \in [\mathrm{LS}(\mathbf{K}), \chi]$, every model in $\mathbf{K}_\mu$ is χ_0^+-saturated. We proceed by induction on χ_0. If $\chi_0 = \mathrm{LS}(\mathbf{K})$, take $M \in \mathbf{K}_\mu$ and $M_0 \leq_{\mathbf{K}} M$ of size χ_0. If there is a type over M_0 omitted in $\mathbf{K}_\mu$, then by Fact 9.3, there is a non $\mathrm{LS}(\mathbf{K})^+$-saturated model of size λ, a contradiction. Therefore every model in $\mathbf{K}_\mu$ is $\mathrm{LS}(\mathbf{K})^+$-saturated.

Now let $\chi_0 > \mathrm{LS}(\mathbf{K})$ and assume inductively that every model in $\mathbf{K}_\mu$ is χ_0-saturated. Let $M \in \mathbf{K}_\mu$ and let $M_0 \leq_{\mathbf{K}} M$ have size χ_0. Since $\mathrm{LS}(\mathbf{K}^{\chi_0\text{-sat}}) = \chi_0$ and M is χ_0-saturated, we take enlarge M_0 if necessary to assume without loss of generality that M_0 is χ_0-saturated. Let $p \in \mathrm{gS}(M_0)$. Then by the weak tameness hypothesis $p/E_{\mathrm{LS}(\mathbf{K})} = p$. So if M omits p, then by Fact 9.3, $\mathbf{K}_\lambda$ has a non $\mathrm{LS}(\mathbf{K})^+$-saturated model, a contradiction, so M realizes p, as needed. $\square$

Remark 9.5. Instead of assuming above that $\mathbf{K}^{\chi_0\text{-sat}}$ is an AEC with $\mathrm{LS}(\mathbf{K}^{\chi_0\text{-sat}}) = \chi_0$, it is enough to assume only that $\mathrm{LS}(\mathbf{K}^{\chi_0\text{-sat}}) = \chi_0$. That is, we only use that for $M \in \mathbf{K}^{\chi_0\text{-sat}}$ and $A \subseteq |M|$, there exists a χ_0-saturated $M_0 \leq_{\mathbf{K}} M$ such that $\|M_0\| \leq |A| + \chi_0$.

Lemma 9.6. Let $\mathrm{LS}(\mathbf{K}) < \mu < \lambda$. Assume that:

(1) $\mathbf{K}$ is categorical in λ.

(2) $\mu = \beth_\delta$, for some limit ordinal δ divisible by $\left(2^{\mathrm{LS}(\mathbf{K})}\right)^+$.
(3) $\mathbf{K}$ is $(\mathrm{LS}(\mathbf{K}), < \mu)$-weakly tame.

If the model of size λ is μ-saturated, then $\mathbf{K}$ is categorical in μ.

Proof. Without loss of generality (by Fact 7.10), $\mathbf{K}$ has joint embedding and no maximal models. Thus it is enough to show that every model of size μ is saturated. Let $\chi \in [\mathrm{LS}(\mathbf{K}), \mu)$. We have to check that the hypotheses of Lemma 9.4 hold. The only problematic part is to see that $\mathbf{K}^{\chi\text{-sat}}$ is an AEC with $\mathrm{LS}(\mathbf{K}^{\chi\text{-sat}}) = \chi$ (when $\chi > \mathrm{LS}(\mathbf{K})$). But this holds by Corollary 7.9. $\square$

We obtain the following downward transfer result that slightly improves on [VV, Corollary 7.7]:

Corollary 9.7. Let $\mathrm{LS}(\mathbf{K}) < \mu < \lambda$ be such that:

(1) $\mathbf{K}$ is categorical in λ.
(2) $\mu = \beth_\delta$, for some limit ordinal δ divisible by H_1.

If the model of size λ is μ-saturated (e.g. if $\mathrm{cf}(\lambda) \geq \mu$ or by Fact 7.8 if $\lambda \geq \sup_{\mu_0 < \mu} h(\mu_0^+)$), then $\mathbf{K}$ is categorical in μ.

Proof. By Theorem 8.4, there exists $\chi < H_1$ such that $\mathbf{K}$ is $(\chi, < \mu)$-weakly tame. Now apply Lemma 9.6 to $\mathbf{K}_{\geq \chi}$. $\square$

When tameness holds instead of weak tameness, we obtain the following generalization of the second main result of [MS90]:

Theorem 9.8. If $\mathbf{K}$ is $\mathrm{LS}(\mathbf{K})$-tame, has arbitrarily large models, and is categorical in a $\lambda > \mathrm{LS}(\mathbf{K})$, then $\mathbf{K}$ is categorical in all cardinals of the form $\beth_\delta$, where $\left(2^{\mathrm{LS}(\mathbf{K})}\right)^+$ divides δ.

Proof. Without loss of generality (Fact 7.10), $\mathbf{K}$ has no maximal models. Let δ be a limit ordinal divisible by $\left(2^{\mathrm{LS}(\mathbf{K})}\right)^+$. Let $\mu := \beth_\delta$. We prove that every model in $\mathbf{K}_\mu$ is saturated. Observe first that for every $\chi \geq \mathrm{LS}(\mathbf{K})^+$, $\mathbf{K}^{\chi\text{-sat}}$ is an AEC with $\mathrm{LS}(\mathbf{K}^{\chi\text{-sat}}) = \chi$. This follows from Facts 7.3, 7.5, and 7.6. In particular, the model of size λ is $\mathrm{LS}(\mathbf{K})^+$-saturated. Therefore for each $\chi < \mu$, Lemma 9.4 tells us that every model in $\mathbf{K}_\mu$ is χ^+-saturated. Thus every model in $\mathbf{K}_\mu$ is saturated. $\square$

Remark 9.9. We could rely on fewer background facts by directly using Fact 9.3. In that case, all that one needs to show is that the model of size λ is $\mathrm{LS}(\mathbf{K})^+$-saturated. This holds by combining Fact

7.3 (telling us that $\mathbf{K}$ is $\mathrm{LS}(\mathbf{K})$-superstable) and [Vas16, Theorem 5.6] (saying that $\mathbf{K}$ is stable in λ).

We can derive a proof of Corollary 0.2 using the methods of [She99].

Proof sketch for Corollary 0.2. By Theorem 9.8, $\mathbf{K}$ is categorical in H_1. Now proceed as in the proof of the main result of [She99], see e.g. [Bal09, Theorem 14.14]. $\qquad\square$

Note that this method does not allow us to go lower than the Hanf number, even if we know for example that $\mathbf{K}$ is categorical below it (Shelah's argument for transferring Vaughtian pairs is not local enough). See Corollary 10.6 for another proof.

10. Categoricity at a successor or with primes

We apply Theorem 6.14 to categorical tame AECs with amalgamation when the categoricity cardinal is a successor or the AEC has primes (recall Definition 1.3). All throughout, we assume:

Hypothesis 10.1. $\mathbf{K}$ is an AEC with amalgamation.

Lemma 10.2 (Main lemma)**.** Assume that $\mathbf{K}$ has no maximal models and is $\mathrm{LS}(\mathbf{K})$-tame. Let $\lambda > \mathrm{LS}(\mathbf{K})^+$ be a successor cardinal. If $\mathbf{K}$ is categorical in *some successor* $\lambda > \mathrm{LS}(\mathbf{K})^+$, then $\mathbf{K}^{\mathrm{LS}(\mathbf{K})^+\text{-sat}}$ is categorical in all $\lambda' \geq \mathrm{LS}(\mathbf{K})^+$.

Proof. By Fact 7.3, $\mathbf{K}$ is $\mathrm{LS}(\mathbf{K})$-superstable. By Fact 7.5, $\mathbf{K}$ is superstable and has symmetry in every $\mu \geq \mathrm{LS}(\mathbf{K})$. By Facts 7.13 and 7.14, there exists a type-full good $(\geq \mathrm{LS}(\mathbf{K})^+)$-frame $\mathfrak{s}$ with underlying class $\mathbf{K}_{\mathfrak{s}} := \mathbf{K}^{\mathrm{LS}(\mathbf{K})^+\text{-sat}}$. Moreover, $\mathbf{K}_{\mathfrak{s}}$ is categorical in $\mathrm{LS}(\mathbf{K})^+$. Thus we can apply Theorem 6.14 where $\mathbf{K}, \mathrm{LS}(\mathbf{K}), \theta$ there stand for $\mathbf{K}^{\mathrm{LS}(\mathbf{K})^+\text{-sat}}, \mathrm{LS}(\mathbf{K})^+, \infty$ here. $\qquad\square$

Corollary 10.3. Assume that $\mathbf{K}$ has arbitrarily large models and is $\mathrm{LS}(\mathbf{K})$-tame. Let $\mathrm{LS}(\mathbf{K}) < \lambda_0 < \lambda$. If λ is a successor and $\mathbf{K}$ is categorical in λ_0 and λ, then $\mathbf{K}$ is categorical in all $\lambda' \geq \lambda_0$.

Proof. By Fact 7.10, we can assume without loss of generality that $\mathbf{K}$ has no maximal models. By Lemma 10.2, $\mathbf{K}^{\mathrm{LS}(\mathbf{K})^+\text{-sat}}$ is categorical in all $\lambda' \geq \mathrm{LS}(\mathbf{K})^+$. Moreover by the proof of Lemma 10.2, $\mathbf{K}$ is stable in every $\mu \in [\mathrm{LS}(\mathbf{K})^+, \lambda)$, hence the model of size λ_0 is saturated. Therefore $\mathbf{K}^{\mathrm{LS}(\mathbf{K})^+\text{-sat}}_{\geq \lambda_0} = \mathbf{K}_{\geq \lambda_0}$, and the result follows. $\qquad\square$

Remark 10.4. We can allow $\lambda_0 = \mathrm{LS}(\mathbf{K})$ but then the proof is more complicated: we do not know how to build a good $\mathrm{LS}(\mathbf{K})$-frame so have to work with $\mathrm{LS}(\mathbf{K})$-splitting.

Remark 10.5. The case $\lambda' \geq \lambda$ in Lemma 10.2 and Corollary 10.3 is Fact 1.6. The contribution of this paper is the case $\lambda' < \lambda$.

We deduce another proof of Corollary 0.2. We prove a slightly stronger result:

Corollary 10.6. Assume that $\mathbf{K}$ is $\mathrm{LS}(\mathbf{K})$-tame and has arbitrarily large models. If $\mathbf{K}$ is categorical in *some successor* $\lambda > \mathrm{LS}(\mathbf{K})^+$, then there exists $\chi < H_1$ such that $\mathbf{K}$ is categorical in all $\lambda' \geq \min(\chi, \lambda)$.

Proof. By Lemma 10.2, $\mathbf{K}^{\mathrm{LS}(\mathbf{K})^+\text{-sat}}$ is categorical in all $\mu \geq \mathrm{LS}(\mathbf{K})^+$. Since λ is regular, the model of size λ is saturated hence $\mathrm{LS}(\mathbf{K})^+$-saturated, so every model in $\mathbf{K}_{\geq \lambda}$ is $\mathrm{LS}(\mathbf{K})^+$-saturated. By Fact 9.2, there exists $\chi < H_1$ such that every model in $\mathbf{K}_{\geq \chi}$ is $\mathrm{LS}(\mathbf{K})^+$-saturated. Thus every model in $\mathbf{K}_{\geq \min(\chi, \lambda)}$ is $\mathrm{LS}(\mathbf{K})^+$-saturated, that is:

$$\mathbf{K}^{\mathrm{LS}(\mathbf{K})^+\text{-sat}}_{\geq \min(\chi, \lambda)} = \mathbf{K}_{\geq \min(\chi, \lambda)}$$

The result follows. $\qquad\qquad\square$

Remark 10.7. An alternate proof (which just deduces categoricity in all $\lambda' \geq \min(\lambda, H_1)$) goes as follows. By Theorem 9.8, $\mathbf{K}$ is categorical in H_1. By Fact 1.6, $\mathbf{K}$ is categorical in all $\lambda' \geq \lambda$. By Corollary 10.3 (where λ_0, λ_1 there stand for $\min(\lambda, H_1), \max(\lambda^+, H_1^+)$ here), we get that $\mathbf{K}$ is categorical in every $\lambda' \geq \min(\lambda, H_1)$.

We can similarly deduce several consequences on tame AECs with primes. One of the main result of [Vasc] was (the point compared to Shelah's downward categoricity transfer [She99] is that λ need *not* be a successor):

Fact 10.8 (Theorem 3.8 in [Vasc]). Assume that $\mathbf{K}$ has no maximal models, is H_2-tame, and $\mathbf{K}_{\geq H_2}$ has primes. If $\mathbf{K}$ is categorical in some $\lambda > H_2$, then it is categorical in all $\lambda' \geq H_2$.

We show that we can obtain categoricity in more cardinals provided that $\mathbf{K}$ has more tameness:

Corollary 10.9. Assume that $\mathbf{K}$ is $\mathrm{LS}(\mathbf{K})$-tame and has arbitrarily large models. Assume also that $\mathbf{K}$ has primes (or just that $\mathbf{K}_{\geq \mu}$ has prime, for some μ). If $\mathbf{K}$ is categorical in *some* $\lambda > \mathrm{LS}(\mathbf{K})$, then $\mathbf{K}$ is categorical in *all* $\lambda' \geq \min(\lambda, H_1)$.

Proof. Without loss of generality (Fact 7.10), $\mathbf{K}$ has no maximal models. By Theorem 9.8, $\mathbf{K}$ is categorical in a proper class of cardinals. By Fact 10.8 (applied to $\mathbf{K}_{\geq \mu}$, where μ is such that $\mathbf{K}_{\geq \mu}$ has primes), $\mathbf{K}$ is categorical in a successor cardinal. By Corollary 10.6, $\mathbf{K}$ is categorical in all $\lambda' \geq H_1$. By Corollary 10.3 (with λ_0, λ there standing for λ, H_1^+ here), $\mathbf{K}$ is categorical also in all $\lambda' \in [\lambda, H_1^+]$. $\square$

Remark 10.10. Similarly to Corollary 10.6, we get that there is a $\chi < H_1$ such that $\mathbf{K}$ is categorical in all $\lambda' \geq \chi$.

Specializing to universal classes, we can improve some of the Hanf number bounds in [Vasd], obtaining in particular the full categoricity conjecture (i.e. the Hanf number is H_1) *assuming amalgamation.*

Corollary 10.11. Let $\mathbf{K}$ be a universal class with amalgamation and arbitrarily large models. If $\mathbf{K}$ is categorical in *some* $\lambda > \mathrm{LS}(\mathbf{K})$, then $\mathbf{K}$ is categorical in *all* $\lambda' \geq \min(\lambda, H_1)$.

Proof. By a result of Boney [Bonb] (a full proof appears as [Vasd, Theorem 3.7]), $\mathbf{K}$ is $\mathrm{LS}(\mathbf{K})$-tame. Also [Vasd, Remark 5.3], $\mathbf{K}$ has primes. Thus we can apply Corollary 10.9. $\square$

11. Categoricity in a limit without primes

In this section, we prove Corollary 0.3 from the abstract and more generally give an exposition of Shelah's proof of the eventual categoricity conjecture in AECs with amalgamation [She09a, Theorem IV.7.12] (assuming the weak generalized continnum hypothesis). Note that while the eventual version of Corollary 0.3 is due to Shelah, here we improve the Hanf number to H_1, and moreover give alternate proofs for several of the hard steps in Shelah's argument.

Most of the results of this section will use the weak generalized continuum hypothesis. We adopt the following notation:

Notation 11.1. For a cardinal λ, $\mathrm{WGCH}(\lambda)$ is the statement "$2^\lambda < 2^{\lambda^+}$". More generally, for S a class of cardinals, $\mathrm{WGCH}(S)$ is the statement "$\mathrm{WGCH}(\lambda)$ for all $\lambda \in S$". WGCH will stand for $\mathrm{WGCH}(\mathrm{Card})$, where Card is the class of all cardinals.

We assume familiarity with the definitions of a *weakly successful, successful,* and *ω-successful* good λ-frame (see [She09a, Definition III.1.1], and on weakly successful see Definition E.6). We use the notation from [JS13]. We will say a good λ-frame is *successful$^+$* if it is successful and

$\leq^{\mathrm{NF}}_{\mathbf{K}_{\lambda^+}}$ is just $\leq_{\mathbf{K}_{\lambda^+}}$ on the saturated models in $\mathbf{K}_{\lambda^+}$, see [JS13, Definition 6.1.4]. We say a good λ-frame is ω-*successful*[+] if it is ω-successful and successful[+].

We first state the unpublished Claim of Shelah mentioned in the introduction. This stems from [She09a, Discussion III.12.40]. A proof should appear in [She].

Claim 11.2. Let $\mathfrak{s}$ be an ω-successful[+] good λ-frame on $\mathbf{K}$. Assume WGCH($[\lambda, \lambda^{+\omega})$). If $\mathbf{K}^{\lambda^{+\omega}\text{-sat}}$ is categorical in some $\mu > \lambda^{+\omega}$, then $\mathbf{K}^{\lambda^{+\omega}\text{-sat}}$ is categorical in all $\mu' > \lambda^{+\omega}$.

Next, we discuss how to obtain an ω-successful[+] good frame from a good frame. The proof of the following fact is contained in the proof of [She09a, Theorem IV.7.12] (see $\odot_4$ there). We give a full proof in Appendix E. Note that as opposed to the results in [She09a, Section II.5], we do *not* assume that $\mathbf{K}$ has few models in λ^{+2}.

Fact 11.3. Assume WGCH(λ). If $\mathfrak{s}$ is a good λ-frame on $\mathbf{K}$, $\mathbf{K}$ is categorical in λ, has amalgamation in λ^+ and is stable in λ^+, then $\mathfrak{s}$ is weakly successful.

We can obtain the stability hypothesis and successfulness using weak tameness:

Fact 11.4 (Theorem 4.5 in [BKV06])**.** Let $\lambda \geq \mathrm{LS}(\mathbf{K})$, be such that $\mathbf{K}$ has amalgamation in λ and λ^+, and $\mathbf{K}$ is stable in λ. If $\mathbf{K}$ is (λ, λ^+)-weakly tame, then $\mathbf{K}$ is stable in λ^+.

Fact 11.5 (Corollary 7.19 in [Jar16])**.** If $\mathfrak{s}$ is a weakly successful good λ-frame on $\mathbf{K}$, $\mathbf{K}$ is categorical[6] in λ, $\mathbf{K}$ has amalgamation in λ^+, and $\mathbf{K}$ is (λ, λ^+)-weakly tame, then $\mathfrak{s}$ is successful[+].

Remark 11.6. Although we will not need it, the converse (i.e. obtaining weak tameness from being successful[+]) is also true, see [JS13, Theorem 7.1.13.(b)].

Corollary 11.7. Assume WGCH(λ). If $\mathfrak{s}$ is a good λ-frame on $\mathbf{K}$, $\mathbf{K}$ is categorical in λ, has amalgamation in λ^+, and is (λ, λ^+)-weakly tame, then $\mathfrak{s}$ is successful[+].

Proof. By Fact 11.4, $\mathbf{K}$ is stable in λ^+. By Fact 11.3, $\mathfrak{s}$ is weakly successful. By Fact 11.5, $\mathfrak{s}$ is successful[+]. □

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[6]Note that by [She09a, Claim III.1.21], this implies the conjugation property, so Hypothesis 6.5 in [Jar16] is satisfied.

Using Fact 7.13 to build the good frame, we obtain:

Lemma 11.8. Assume that $\mathbf{K}$ has amalgamation and no maximal models. Assume $\mathrm{WGCH}([\mathrm{LS}(\mathbf{K}), \mathrm{LS}(\mathbf{K})^{+\omega}))$ and Claim 11.2. If $\mathbf{K}$ is categorical in a $\lambda > \mathrm{LS}(\mathbf{K})^{+\omega}$ and:

  (1) The model of size λ is $\mathrm{LS}(\mathbf{K})^{++}$-saturated.
  (2) $\mathbf{K}$ is $(\mathrm{LS}(\mathbf{K}), < \mathrm{LS}(\mathbf{K})^{+\omega})$-weakly tame.

Then $\mathbf{K}^{\mathrm{LS}(\mathbf{K})^{+\omega}\text{-sat}}$ is categorical in all $\lambda' > \mathrm{LS}(\mathbf{K})^{+\omega}$. In particular, $\mathbf{K}$ is categorical in all $\lambda' \geq \min(\lambda, \sup_{n<\omega} h(\mathrm{LS}(\mathbf{K})^{+n}))$.

Proof. By Fact 7.8, $\mathbf{K}$ has symmetry in $\mathrm{LS}(\mathbf{K})$ and $\mathrm{LS}(\mathbf{K})^{+}$. By Fact 7.3, $\mathbf{K}$ is superstable in every $\chi \in [\mathrm{LS}(\mathbf{K}), \lambda)$. By Fact 7.13, there is a good $\mathrm{LS}(\mathbf{K})^{+}$-frame $\mathfrak{s}$ on $\mathbf{K}^{\mathrm{LS}(\mathbf{K})^{+}\text{-sat}}$. By repeated applications of Corollary 11.7, $\mathfrak{s}$ is ω-successful^{+}. By Claim 11.2, $\mathbf{K}^{\mathrm{LS}(\mathbf{K})^{+\omega}\text{-sat}}$ is categorical in all $\lambda' > \mathrm{LS}(\mathbf{K})^{+\omega}$, and hence by Fact 9.2, $\mathbf{K}$ is categorical in every $\lambda' \geq \sup_{n<\omega} h(\mathrm{LS}(\mathbf{K})^{+\omega})$. In particular by Fact 7.3, $\mathbf{K}$ is stable in λ, so the model of size λ is saturated (hence $\mathrm{LS}(\mathbf{K})^{+\omega}$-saturated), and so $\mathbf{K}$ must be categorical in all $\lambda' \geq \min(\lambda, \sup_{n<\omega} h(\mathrm{LS}(\mathbf{K})^{+\omega}))$. □

As a corollary, we obtain Corollary 0.3 from the abstract. We restate it here in a slightly stronger form:

Corollary 11.9. Assume $\mathrm{WGCH}([\mathrm{LS}(\mathbf{K}), \mathrm{LS}(\mathbf{K})^{+\omega}))$ and Claim 11.2. Assume that $\mathbf{K}$ has amalgamation, arbitrarily large models, and is $\mathrm{LS}(\mathbf{K})$-tame. If $\mathbf{K}$ is categorical in a $\lambda > \mathrm{LS}(\mathbf{K})$, then there exists $\chi < H_1$ such that $\mathbf{K}$ is categorical in all $\lambda' \geq \min(\lambda, \chi)$.

Proof. By Fact 7.10, without loss of generality $\mathbf{K}$ has no maximal models. By Theorem 9.8 we can assume without loss of generality that $\lambda \geq H_1$ (then we can use Corollary 10.3 to transfer categoricity downward). By Fact 7.3, $\mathbf{K}$ is $\mathrm{LS}(\mathbf{K})$-superstable. By[7] Fact 7.5, $\mathbf{K}$ is stable in λ, hence the model of size λ is saturated. By Lemma 11.8, $\mathbf{K}$ is categorical in all $\lambda' \geq h(\mathrm{LS}(\mathbf{K})^{+\omega})$. In particular, $\mathbf{K}$ is categorical in $(h(\mathrm{LS}(\mathbf{K})^{+\omega}))^{+}$. By Corollary 10.6, there exists $\chi < H_1$ such that $\mathbf{K}$ is categorical in all $\lambda' \geq \chi$. □

Without tameness, we can obtain the hypotheses of Lemma 11.8 from categoricity in a high-enough cardinal. More precisely, to obtain enough weak tameness, we use Theorem 8.4. To obtain the first condition in

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[7]In our case, we only use stability in λ, which follows by [Vas16, Theorem 5.6].

Lemma 11.8 (i.e. that the model in the categoricity cardinal is suffi-
ciently saturated), we will use[8] Fact 7.8.

This allows us to give a proof of [She09a, Theorem IV.7.12]. Note that,
while we give a slightly different proof to attempt to convince doubters,
the result is due to Shelah. In fact, Shelah assumes amalgamation more
locally but we haven't fully checked his general proof. As explained
in the introduction, we avoid relying on PCF theory or on Shelah's
construction of certain linear orders in [She09a, Sections IV.5, IV.6].

Fact 11.10. Assume Claim 11.2. Assume that $\mathbf{K}$ has amalgamation.
Let λ and μ be cardinals such that:

  (1) $\mathbf{K}$ is categorical in λ.
  (2) μ is a limit cardinal with $\mathrm{cf}(\mu) > \mathrm{LS}(\mathbf{K})$.
  (3) For all $\chi < \mu$, $h(\chi) < \lambda$.
  (4) For unboundedly many $\chi < \mu$, $\mathrm{WGCH}([\chi, \chi^{+\omega}))$.

Then there exists $\mu_* < \mu$ such that $\mathbf{K}$ is categorical in all $\lambda' \geq h(\mu_*)$.

Proof. By Fact 7.10, without loss of generality $\mathbf{K}$ has no maximal mod-
els. By Fact 7.8 (used with $\mathbf{K}$ there standing for $\mathbf{K}_{\geq \chi^+}$ here, for each
$\chi < \mu$), the model of size λ is μ-saturated. By Theorem 8.4, there exists
$\chi < \mu$ so that $\mathbf{K}$ is $(\chi, < \mu)$-weakly tame. Increasing χ if necessary,
assume without loss of generality that $\mathrm{WGCH}((\chi, \chi^{+\omega}])$ holds. Now
apply Lemma 11.8 with $\mathbf{K}$ there standing for $\mathbf{K}_{\geq \chi}$ here. We get that
$\mathbf{K}$ is categorical in all $\lambda' \geq h(\chi^{+\omega})$, so we obtain the desired conclusion
with $\chi_* := \chi^{+\omega}$. □

Remark 11.11. The proof of Fact 11.10 given above goes through as-
suming only that $\mathbf{K}$ has amalgamation below the categoricity cardinal
λ.

Corollary 11.12. Assume Claim 11.2 and WGCH. If $\mathbf{K}$ has amalga-
mation and is categorical in *some* $\lambda \geq h(\aleph_{\mathrm{LS}(\mathbf{K})^+})$, then $\mathbf{K}$ is categorical
in *all* $\lambda' \geq h(\aleph_{\mathrm{LS}(\mathbf{K})^+})$.

Proof. Set $\mu := \aleph_{\mathrm{LS}(\mathbf{K})^+}$ in Fact 11.10. □

We can also state a version using large cardinals instead of amalgama-
tion. This is implicit in Shelah's work (see the remark after [She09a,

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[8]Shelah claims [She09a, Claim IV.7.8] a slightly different result using PCF theory
and the existence of certain linear orders: under amalgamation and no maximal
models, for $\mu > \mathrm{LS}(\mathbf{K})$, categoricity in some λ so that $\lambda \geq \aleph_{\mu+4} + 2^{2^\mu}$ implies that
the model of size λ is μ-saturated.

Theorem IV.7.12]), but to the best of our knowledge, the details have not appeared in print before. We will use the following fact, which follows from [SK96, She01b]. Note that while the results there are stated when $\mathbf{K}$ is the class of models of an $\mathbb{L}_{\kappa,\omega}$-theory, Boney observed that the proofs go through just as well in an AEC $\mathbf{K}$ with $\kappa > \mathrm{LS}(\mathbf{K})$, see the discussion around [Bon14b, Theorem 7.6].

Fact 11.13. Let $\mathbf{K}$ be an AEC and let $\kappa > \mathrm{LS}(\mathbf{K})$ be a measurable cardinal. Let $\lambda \geq h(\kappa)$ be such that $\mathbf{K}$ is categorical in λ. Then:

(1) [SK96] $\mathbf{K}_{[\kappa,\lambda)}$ has amalgamation and no maximal models.
(2) [She01b, Claim 1.16] The model of size λ is saturated.
(3) [She01b, Corollary 3.7] $\mathbf{K}$ is $(\kappa, < \lambda)$-tame.
(4) [She01b, Theorem 3.16] If λ is a successor cardinal, then $\mathbf{K}$ is categorical in all $\lambda' \geq h(\kappa)$.

Corollary 11.14. Assume Claim 11.2 and WGCH. Let $\kappa > \mathrm{LS}(\mathbf{K})$ be a measurable cardinal. If $\mathbf{K}$ is categorical in *some* $\lambda \geq h(\kappa)$, then $\mathbf{K}$ is categorical in *all*[9] $\lambda' \geq h(\kappa)$

Proof. By Fact 11.13.(1), $\mathbf{K}_{[\kappa,\lambda)}$ has amalgamation (and no maximal models, by taking ultrapowers). Note that by Remark 11.11 we do not need amalgamation in $\mathbf{K}_{\geq \lambda}$. By Fact 11.13.(2, the model of size λ is saturated. Let $\mu := \aleph_{\kappa^+}$. By Theorem 8.4, there exists $\chi < \mu$ so that $\mathbf{K}$ is $(\chi, < \mu)$-weakly tame. By Lemma 11.8 (with $\mathbf{K}$ there standing for $\mathbf{K}_{\geq \chi}$ here), $\mathbf{K}$ is categorical in all $\lambda' \geq h(\mu)$. In particular, $\mathbf{K}$ is categorical in $(h(\mu))^+$. By Fact 11.13.(4) (or by Corollary 10.6, since by Fact 11.13.(3) $\mathbf{K}$ has enough tameness), $\mathbf{K}$ is also categorical in all $\lambda' \in [h(\kappa), h(\mu))$. $\qquad\square$

The same proof gives:

Corollary 11.15. Assume Claim 11.2 and WGCH. Let κ be a measurable cardinal and let T be a theory in $\mathbb{L}_{\kappa,\omega}$. If T is categorical in *some* $\lambda \geq h(|T| + \kappa)$, then T is categorical in *all* $\lambda' \geq h(|T| + \kappa)$.

12. SUMMARY

Table 1 summarizes several known approximations of Shelah's eventual categoricity conjecture, for a fixed AEC $\mathbf{K}$. The topmost line and leftmost column contain properties that are either model-theoretic, set-theoretic, or about the categoricity cardinal. The intersection of a line

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[9]The proof gives that there exists $\chi < h(\kappa)$ such that $\mathbf{K}$ is categorical in all $\lambda' \geq \chi$.

and a column gives a known categoricity transfer for a class having these properties. "AP" stands for "$\mathbf{K}$ has the amalgamation property", "Primes" is short for "$\mathbf{K}$ has primes" (Definition 1.3), "s.c.$'$ is short for "strongly compact", and $(*)_{\mathbf{K}}$ is the statement "LS$(\mathbf{K}) = \kappa$, $\mathbf{K}$ has amalgamation, and $\mathbf{K}$ is LS$(\mathbf{K})$-tame".

Each transfer is described by its type, a comma, and a threshold μ. A "Full" type means that categoricity in *some* $\lambda \geq \mu$ implies categoricity in *all* $\lambda' \geq \mu$. A "Down" type means that we only know a downward transfer: categoricity in some $\lambda \geq \mu$ implies categoricity in all $\lambda' \in [\mu, \lambda]$ (in this case, we can still do an argument similar to the existence of Hanf numbers [Han60] to deduce Shelah's *eventual* categoricity conjecture, see [Bal09, Conclusion 15.13]). A "Partial" type means that we only know that categoricity in some $\lambda \geq \mu$ implies categoricity in *some* λ' with $\lambda' \neq \lambda$ (we do *not* require that $\lambda' \geq \mu$). When reading the line beginning with "categ. in a successor", one should assume that the starting categoricity cardinal λ is a successor.

For example, the first entry says that if an AEC $\mathbf{K}$ satisfies the amalgamation property and WGCH together with Claim 11.2 hold, then categoricity in some $\lambda' \geq h(\aleph_{\mathrm{LS}(\mathbf{K})^+})$ implies categoricity in all $\lambda' \geq h(\aleph_{\mathrm{LS}(\mathbf{K})^+})$.

Note that in the first column, it is enough to assume amalgamation below the categoricity cardinal (see Remark 11.11). So by Fact 11.13.(1) and because strongly compact cardinals are measurable, we can see the properties in the topmost row as being arranged in increasing order of strength. Moreover, the existence of a strongly compact cardinal implies $(*)$: amalgamation follows from the methods of [MS90, Proposition 1.13] and tameness from the main theorem of [Bon14b].

Fact 12.1. Let $\mathbf{K}$ be an AEC and let $\kappa > \mathrm{LS}(\mathbf{K})$ be a strongly compact cardinal. Let $\lambda \geq h(\kappa)$. If $\mathbf{K}$ is categorical in λ, then $(*)_{\mathbf{K}_{\geq \kappa}}$ holds.

Remark 12.2. An analog of $(*)_{\mathbf{K}}$ in the case κ is measurable would be given by conclusions (1)-(3) in Fact 11.13.

The results in the first row are Corollary 11.12 and Corollary 11.14 (for the strongly compact case, recall that the properties in the topmost row are in increasing order of strength). The first result in the second row is the downward transfer of [She99] (see also Corollary B.6 for an alternate proof). The second is Fact 11.13.(4). The third is given by Corollary 10.3 (recalling Fact 12.1).

The last two results in the first column are given by Corollary 9.7 (categoricity above $\beth_{\beth_{H_1}}$ implies categoricity in $\beth_{H_1}$). As for the last

	AP	$\kappa > \mathrm{LS}(\mathbf{K})$ measurable	$\kappa > \mathrm{LS}(\mathbf{K})$ s.c. or $(*)_{\mathbf{K}}$
WGCH and 11.2	Full, $h(\aleph_{\mathrm{LS}(\mathbf{K})^+})$	Full, $h(\kappa)$	Full, $h(\kappa)$
Categ. in a successor	Down, $\beth_{H_1}$	Down, $h(\kappa)$	Full, $h(\kappa)$
Primes	Partial, $\beth_{\beth_{H_1}}$	Down, $h(\kappa)$	Full, $h(\kappa)$
No extra hypothesis	Partial, $\beth_{\beth_{H_1}}$	Partial, $h(\kappa)^+$	Partial, $h(\kappa)$

TABLE 1. Some approximations to Shelah's categoricity conjecture.

column, the third result is by Corollary 10.9, and the fourth is by Theorem 9.8. Very similar proofs (using Fact 11.13 to deduce the needed amount of amalgamation and tameness) give the corresponding results in the second column.

APPENDIX A. SHRINKING GOOD FRAMES

We state a generalization of Theorem 6.14 to frames that are only defined over classes of saturated models (Shelah studies these frames in more details in [She]). This allows us to replace the assumption of tameness by only weak tameness in several results (see Appendix B).

We start by giving a precise definition of these frames (we call them shrinking frames for reasons that will soon become apparent).

Definition A.1 (Shrinking frame). Let λ be an infinite cardinal and let $\theta > \lambda$ be a cardinal or ∞. Let $\mathcal{F} := [\lambda, \theta)$ and let $\mathbf{K}$ be an AEC.

We say that $\langle \mathfrak{s}_\mu : \mu \in \mathcal{F} \rangle$ is a *shrinking type-full good $\mathcal{F}$-frame on $\mathbf{K}_{\mathcal{F}}$* (or on $\mathbf{K}$) if:

(1) $\mathbf{K}$ is $(\lambda, < \theta)$-weakly tame.
(2) $\mathfrak{s}_\lambda$ has underlying class $\mathbf{K}_\lambda$.
(3) For each $\mu \in \mathcal{F}$, $\mathfrak{s}_\mu$ is a type-full good μ-frame with $\mathbf{K}_{\mathfrak{s}_\mu} = \mathbf{K}_\mu^{\mu\text{-sat}}$. In particular, $\mathbf{K}$ is categorical in λ.

The reason for the name *shrinking* is that if $\mu < \mu'$ are in $\mathcal{F}$, then the AEC generated by $\mathbf{K}_{\mathfrak{s}_{\mu'}}$ is $\mathbf{K}^{\mu'\text{-sat}}$, but the underlying class $\mathbf{K}_{\mathfrak{s}_\mu}$ is only $\mathbf{K}^{\mu\text{-sat}}$ which could be a proper subclass of $\mathbf{K}^{\mu'\text{-sat}}$ (if $\mathbf{K}$ is not categorical in μ'). Note that that a type-full good $[\lambda, \theta)$-frame (which is categorical in λ) induces a shrinking frame in a natural way.

Proposition A.2. If $\mathfrak{s}$ is a type-full good $[\lambda, \theta)$-frame and $\mathbf{K}_{\mathfrak{s}}$ is categorical in λ, then $\langle \mathfrak{s} \restriction \mathbf{K}^{\mu\text{-sat}} : \mu \in [\lambda, \theta) \rangle$ is a shrinking type-full good $[\lambda, \theta)$-frame.

Proof. Straightforward, recalling Fact 2.6. □

Shrinking frames can be built using Fact 7.13:

Theorem A.3. Let $\mathbf{K}$ be an AEC. Let $\lambda > \mathrm{LS}(\mathbf{K})$. Assume that for every $\mu \in [\mathrm{LS}(\mathbf{K}), \lambda)$, $\mathbf{K}$ is μ-superstable and has μ-symmetry.

If $\mathbf{K}$ is $(\mathrm{LS}(\mathbf{K}), < \lambda)$-weakly tame, then there exists a shrinking type-full good $[\mathrm{LS}(\mathbf{K})^+, \lambda)$-frame on $\mathbf{K}$.

Proof. Let $\mathcal{F} := [\mathrm{LS}(\mathbf{K})^+, \lambda)$. By Fact 7.13, for each $\mu \in \mathcal{F}$, there exists a type-full good μ-frame on $\mathbf{K}_\mu^{\mu\text{-sat}}$. The result follows. □

We now study how forking in two different cardinals interact in a shrinking frame. The following notion is key:

Definition A.4. Let $\mathbf{K}$ be an AEC. Let $\mathrm{LS}(\mathbf{K}) \leq \lambda < \mu$. Let $\mathfrak{s}_\lambda$ be a type-full good λ-frame on $\mathbf{K}_\lambda^{\lambda\text{-sat}}$ and $\mathfrak{s}_\mu$ be a type-full good μ-frame on $\mathbf{K}_\mu^{\mu\text{-sat}}$. We say that $\mathfrak{s}_\lambda$ and $\mathfrak{s}_\mu$ are *compatible* if for any $M \leq_{\mathbf{K}} N$ in $\mathbf{K}_{\mathfrak{s}_\mu}$, and $p \in \mathrm{gS}(N)$, p does not $\mathfrak{s}_\mu$-fork over M if and only if there exists $M_0 \leq_{\mathbf{K}} M$ with $M_0 \in \mathbf{K}_{\mathfrak{s}_\lambda}$ so that $p \restriction N_0$ does not $\mathfrak{s}_\lambda$-fork over M_0 for every $N_0 \in \mathbf{K}_{\mathfrak{s}_\lambda}$ with $M_0 \leq_{\mathbf{K}} N_0 \leq_{\mathbf{K}} N$.

Intuitively, compatibility says that that forking in $\mathfrak{s}_\mu$ can be computed using forking in $\mathfrak{s}_\lambda$. In fact, it can be described in a canonical way (i.e. using Shelah's description of the extended frame, see [She09a, Section II.2]). The following result is key:

Theorem A.5. Let $\langle \mathfrak{s}_\mu : \mu \in \mathcal{F} \rangle$ be a shrinking type-full good $\mathcal{F}$-frame on the AEC $\mathbf{K}$. Let $\lambda < \mu$ be in $\mathcal{F}$. Then $\mathfrak{s}_\lambda$ and $\mathfrak{s}_\mu$ are compatible.

For the proof, we will use the following result which gives an explicit description of forking in any categorical good frame:

Fact A.6 (The canonicity theorem, 9.6 in [Vasa]). Let $\mathfrak{s}$ be a type-full good λ-frame with underlying class $\mathbf{K}_\lambda$. If $M \leq_{\mathbf{K}} N$ are limit models in $\mathbf{K}_\lambda$, then for any $p \in \mathrm{gS}(N)$, p does not $\mathfrak{s}$-fork over M if and only if there exists $M' \in \mathbf{K}_\lambda$ such that M is limit over M' and p does not λ-split over M'.

Proof of Theorem A.5. Note that by uniqueness of limit models, every model in $\mathbf{K}_{\mathfrak{s}_\mu}$ is limit.

For $M, N \in \mathbf{K}_{\mathfrak{s}_\mu}$ with $M \leq_{\mathbf{K}} N$, let us say that $p \in \mathrm{gS}(N)$ *does not* $(\geq \mathfrak{s}_\lambda)$-*fork over* M if it satisfies the condition in Definition A.4, namely there exists $M_0 \leq_{\mathbf{K}} M$ with $M_0 \in \mathbf{K}_{\mathfrak{s}_\lambda}$ so that $p \restriction N_0$ does not $\mathfrak{s}_\lambda$-fork

over M_0 for every $N_0 \in \mathbf{K}_{\mathfrak{s}_\lambda}$ with $M_0 \leq_{\mathbf{K}} N_0 \leq_{\mathbf{K}} N$. Let us say that p *does not μ-fork over M* if it satisfies the description of the canonicity theorem, namely there exists $M' \in \mathbf{K}_{\mathfrak{s}_\mu}$ such that M is limit over M' and p does not μ-split over M'. Notice that by the canonicity theorem (Fact A.6), p does not $\mathfrak{s}_\mu$-fork over M if and only if p does not μ-fork over M. Thus it is enough to show that p does not $(\geq \mathfrak{s}_\lambda)$-fork over M if and only if p does not μ-fork over M. We first show one direction:

<u>Claim</u>. Let $M \leq_{\mathbf{K}} N$ both be in $\mathbf{K}_{\mathfrak{s}_\mu}$ and let $p \in \mathrm{gS}(N)$. If p does not $(\geq \mathfrak{s}_\lambda)$-fork over M, then p does not μ-fork over M.

<u>Proof of Claim</u>. We know that M is limit, so let $\langle M_i : i < \delta \rangle$ witness it, i.e. δ is limit, for all $i < \delta$, $M_i \in \mathbf{K}_{\mathfrak{s}_\mu}$, M_{i+1} is universal over M_i, and $\bigcup_{i < \delta} M_i = M$. By [She09a, Claim II.2.11.(5)], there exists $i < \delta$ such that $p \restriction M$ does not $(\geq \mathfrak{s}_\lambda)$-fork over M_i. By [She09a, Claim II.2.11.(4)], p does not $(\geq \mathfrak{s}_\lambda)$-fork over M_i. By weak tameness and the uniqueness property of $\mathfrak{s}$, $(\geq \mathfrak{s}_\lambda)$-forking has the uniqueness property (see the proof of [Bon14a, Theorem 3.2]). By [BGKV, Lemma 4.2], $(\geq \mathfrak{s}_\lambda)$-nonforking must be extended by μ-nonsplitting, so p does not μ-split over M_i. Therefore p does not μ-fork over M, as desired. $\dagger_{\text{Claim}}$.

Now as observed above, $(\geq \mathfrak{s}_\lambda)$-forking has the uniqueness property. Also, μ-forking has the extension property (as $\mathfrak{s}_\mu$-forking has it). The claim tells us that μ-nonforking extends $(\geq \mathfrak{s}_\lambda)$-forking and hence by [BGKV, Lemma 4.1], they are the same. $\qquad\square$

Thus we can define a global notion of forking inside the frame:

Definition A.7. Assume that $\langle \mathfrak{s}_\mu : \mu \in \mathcal{F} \rangle$ is a shrinking type-full good $[\lambda, \theta)$-frame. Let $\mu \leq \mu'$ be in $\mathcal{F}$ and let $M \leq_{\mathbf{K}} N$ be such that $M \in \mathbf{K}_{\mathfrak{s}_\mu}$ and $M' \in \mathbf{K}_{\mathfrak{s}_{\mu'}}$. Let $p \in \mathrm{gS}(N)$. We say that p *does not fork over M* if there exists $M_0 \leq_{\mathbf{K}} M$ so that $M_0 \in \mathbf{K}_{\mathfrak{s}_\lambda}$ and for every $N_0 \in \mathbf{K}_{\mathfrak{s}_\lambda}$ with $M_0 \leq_{\mathbf{K}} N_0 \leq_{\mathbf{K}} N$, $p \restriction N_0$ does not $\mathfrak{s}_\lambda$-fork over M_0.

Theorem A.8. Assume that $\langle \mathfrak{s}_\mu : \mu \in \mathcal{F} \rangle$ is a shrinking type-full good $[\lambda, \theta)$-frame. Then forking (as defined in Definition A.7) has the usual properties: invariance, monotonicity, extension, uniqueness, transitivity, local character, and symmetry.

Proof sketch. Invariance, monotonicity, transitivity, and local character are straightforward. Symmetry is also straightforward (once we have it when the domain and the base have the same size, it is a simple use of monotonicity). Uniqueness is by weak tameness, and extension is as in [VV, Proposition 5.1]. $\qquad\square$

We can now state a generalization of Theorem 6.14 and sketch a proof:

Theorem A.9. Let $\langle \mathfrak{s}_\mu : \mu \in [\lambda, \theta) \rangle$ be a shrinking type-full good $[\lambda, \theta)$-frame on $\mathbf{K}$. Let $\mu \in [\lambda, \theta)$. If $\mathbf{K}^{\mu\text{-sat}}$ is categorical in μ^+, then $\mathbf{K}$ is categorical in every $\mu \in [\lambda, \theta]$.

Proof sketch. First note that in the upward transfer of Grossberg and VanDieren (Fact 6.12), it is implicit that tameness can be weakened to weak tameness (Remark 8.7). The rest of the proof of Theorem 6.14 (the downward part) is as before, we use Theorem A.8 and make sure that anytime a resolution is taken, all the components are saturated. $\square$

Appendix B. More on weak tameness

We use Theorem A.9 to replace tameness by weak tameness in some of the results of the second part of this paper. Everywhere in this section, we assume:

Hypothesis B.1. $\mathbf{K}$ is an AEC with amalgamation.

First, we state a stronger version of the main lemma (Lemma 10.2):

Lemma B.2. Assume that $\mathbf{K}$ has no maximal models. Let $\theta \geq \lambda > \mathrm{LS}(\mathbf{K})^+$ be such that $\mathbf{K}$ is $(\mathrm{LS}(\mathbf{K}), < \theta)$-weakly tame. Assume that λ is a successor cardinal. If $\mathbf{K}$ is categorical in λ, then $\mathbf{K}^{\mathrm{LS}(\mathbf{K})^+\text{-sat}}$ is categorical in all $\mu \in [\mathrm{LS}(\mathbf{K})^+, \theta]$.

Proof. By the upward transfer of Grossberg and VanDieren (Fact 1.6), $\mathbf{K}$ (and therefore $\mathbf{K}^{\mathrm{LS}(\mathbf{K})^+\text{-sat}}$) is categorical in every $\lambda' \in [\lambda, \theta]$. It remains to show the downward part. By Fact 7.3, $\mathbf{K}$ is superstable in every $\mu \in [\mathrm{LS}(\mathbf{K}), \lambda)$. Since λ is a successor, the model of size λ is saturated. By Fact 7.8, $\mathbf{K}$ has symmetry in every $\mu \in [\mathrm{LS}(\mathbf{K}), \lambda)$. By Theorem A.3, there is a shrinking type-full good $[\mathrm{LS}(\mathbf{K})^+, \lambda)$-frame on $\mathbf{K}$. By Theorem A.9, $\mathbf{K}^{\mathrm{LS}(\mathbf{K})^+\text{-sat}}$ is categorical in all $\lambda' \in [\mathrm{LS}(\mathbf{K})^+, \lambda]$. $\square$

We can improve on Corollary 10.3:

Corollary B.3. Assume that $\mathbf{K}$ has arbitrarily large models. Let $\mathrm{LS}(\mathbf{K}) < \lambda_0 < \lambda$. Assume that $\mathbf{K}$ is $(\mathrm{LS}(\mathbf{K}), < \lambda)$-weakly tame. If λ is a successor cardinal and $\mathbf{K}$ is categorical in λ_0 and λ, then $\mathbf{K}$ is categorical in all $\lambda' \in (\lambda_0, \lambda)$.

Proof. As in the proof of Corollary 10.3, using Lemma B.2 (with λ, θ there standing for λ, λ here). □

Corollary 10.6 can similarly be generalized:

Corollary B.4. Let $\lambda \geq H_1$ be a successor cardinal and assume that **K** is $(LS(\mathbf{K}), < \lambda)$-weakly tame. If **K** is categorical in λ, then there exists $\chi < H_1$ such that **K** is categorical in all $\lambda' \in [\chi, \lambda)$.

Proof. As in the proof of Corollary 10.6, using Lemma B.2 (with λ, θ there standing for λ, λ here). □

Remark B.5. It is unclear how to generalize the results using primes: the proof of Fact 10.8 uses tameness (for all models) heavily, and we do not know how to generalize it to weakly tame AECs.

We can use Corollary B.4 to give an alternate proof to the main theorem of [She99].

Corollary B.6. If **K** is categorical in *some successor* $\lambda \geq \beth_{H_1}$, then there exists $\mu < \beth_{H_1}$ such that **K** is categorical in *all* $\lambda' \in [\mu, \lambda)$.

Proof. Without loss of generality (Fact 7.10), **K** has no maximal models. By Fact 7.3, **K** is stable below λ, so the model of size λ is saturated. By Theorem 8.4, there exists $\chi < H_1$ such that **K** is $(\chi, < \lambda)$-weakly tame. By Corollary B.4 (applied to $\mathbf{K}_{\geq \chi}$), there exists $\mu < h(\chi) < \beth_{H_1}$ such that **K** is categorical in all $\lambda' \in [\mu, \lambda)$. □

Generalizing Corollary 11.9 is harder. The problem is how to ensure that the model in the categoricity cardinal has enough saturation. We give a consistency result in case $\lambda \geq H_1$.

Corollary B.7. Assume that $2^{LS(\mathbf{K})} = 2^{LS(\mathbf{K})^+}$, WGCH($[LS(\mathbf{K})^+, LS(\mathbf{K})^{+\omega})$), and Claim 11.2 holds. Assume that **K** is $(LS(\mathbf{K}), < H_1)$-weakly tame. If **K** is categorical in *some* $\lambda \geq H_1$, then there exists $\chi < H_1$ such that **K** is categorical in *all* $\lambda' \geq \chi$.

Proof. Without loss of generality (Fact 7.10), **K** has no maximal models. By Fact 7.3, **K** is superstable in every $\mu \in [LS(\mathbf{K}), \lambda)$. Since $2^{LS(\mathbf{K})} = 2^{LS(\mathbf{K})^+}$, $H_1 = h(LS(\mathbf{K})^+)$, so by Fact 7.8, **K** has symmetry in $LS(\mathbf{K})^+$. By Fact 7.6, $\mathbf{K}^{LS(\mathbf{K})^+\text{-sat}}$ is an AEC with $LS(\mathbf{K}^{LS(\mathbf{K})^+\text{-sat}}) = LS(\mathbf{K})^+$. In particular, the model of size λ is $LS(\mathbf{K})^+$-saturated. By Fact 7.13, there exists a type-full good $LS(\mathbf{K})^+$-frame $\mathfrak{s}$ on $\mathbf{K}^{LS(\mathbf{K})^+\text{-sat}}_{LS(\mathbf{K})^+}$. By iterating Corollary 11.7, $\mathfrak{s}$ is ω-successful$^+$. As in the proof of

Corollary 11.9, we get that $\mathbf{K}$ is categorical on a tail of cardinals. By Theorem 8.4, $\mathbf{K}$ is χ-weakly tame, so combining this with the hypothesis of $(\mathrm{LS}(\mathbf{K}), < H_1)$-tameness, $\mathbf{K}$ is $\mathrm{LS}(\mathbf{K})$-weakly tame. Now apply Corollary B.4. $\qquad\square$

Appendix C. Superstability for long types

We generalize Definition 7.1 to types of more than one element and use it to prove an extension property for 1-forking (recall Definition 4.2). This is used to give a converse to Lemma 6.7 in the next appendix (but is not needed for the main body of this paper). Everywhere below, $\mathbf{K}$ is an AEC.

Definition C.1. Let $\alpha \leq \omega$ be a cardinal. $\mathbf{K}$ is $(< \alpha, \mu)$-*superstable* (or $(< \alpha)$-*superstable in* μ) if it satisfies Definition C.1 except that in addition in condition (4) there we allow $p \in \mathrm{gS}^{<\alpha}(M_\delta)$ rather than just $p \in \mathrm{gS}(M_\delta)$ (that is, p need not have length one). $(\leq \alpha, \mu)$-*superstable* means $(< \alpha^+, \mu)$-superstable. When $\alpha = 2$, we omit it (that is, μ-superstable means $(< 2, \mu)$-superstable which is the same as $(\leq 1, \mu)$-superstable).

While not formally equivalent (although we do not know of any examples separating the two), μ-superstability and $(< \omega, \mu)$-superstability are very close. For example, the proof of Fact 7.3 also gives:

Fact C.2. Let $\mu \geq \mathrm{LS}(\mathbf{K})$. If $\mathbf{K}$ has amalgamation, no maximal models, and is categorical in a $\lambda > \mu$, then $\mathbf{K}$ is $(< \omega, \mu)$-superstable.

Even without categoricity, we can obtain eventual $(< \omega)$-superstability from just (≤ 1)-superstability and tameness. This uses another equivalent definition of superstability: solvability:

Theorem C.3. Assume $\mathbf{K}$ has amalgamation, no maximal models, and is $\mathrm{LS}(\mathbf{K})$-tame. If $\mathbf{K}$ is $\mathrm{LS}(\mathbf{K})$-superstable, then there exists $\mu_0 < H_1$ such that $\mathbf{K}$ is $(< \omega)$-superstable in every $\mu \geq \mu_0$.

Proof sketch. By [GV, Theorem 5.43][10], there exists $\mu_0 < H_1$ such that $\mathbf{K}$ is (μ_0, μ)-solvable for every $\mu \geq \mu_0$. This means [She09a, Definition IV.1.4.(1)] that for every $\mu \geq \mu_0$, there exists an EM Blueprint Φ so that $\mathrm{EM}_{\tau(\mathbf{K})}(I, \Phi)$ is a superlimit in $\mathbf{K}$ for every linear order I of size μ. Intuitively, it gives a weak version of categoricity in μ. As

---

[10]There, $(< \mathrm{LS}(\mathbf{K}))$-tameness was assumed, but it can be improved to $\mathrm{LS}(\mathbf{K})$-tameness using the updated version of [Vasb] to get the corresponding version of Fact 5.15 in [GV].

observed in [GV, Section 6], this weak version is enough for the proof of the Shelah-Villaveces theorem to go through, hence by Fact C.2, $\mathbf{K}$ is $(<\omega)$-superstable in μ for every $\mu \geq \mu_0$. $\square$

Remark C.4. If $\mathbf{K}$ has amalgamation, is $\mathrm{LS}(\mathbf{K})$-tame for types of length less than ω, and is $(<\omega, \mathrm{LS}(\mathbf{K}))$-superstable, then (by the proof of [Vasa, Proposition 10.10]) $\mathbf{K}$ is $(<\omega)$-superstable in every $\mu \geq \mathrm{LS}(\mathbf{K})$. However here we want to stick to using regular tameness (i.e. tameness for types of length one).

To prove the extension property for 1-forking, we will use:

Fact C.5 (Extension property for splitting, Proposition 5.1 in [VV])**.** Let $\mathbf{K}$ be an AEC, $\theta > \mathrm{LS}(\mathbf{K})$. Let $\alpha \leq \omega$ be a cardinal and assume that $\mathbf{K}$ is $(<\alpha)$-superstable in every $\mu \in [\mathrm{LS}(\mathbf{K}), \theta)$. Let $M_0 \leq_{\mathbf{K}} M \leq_{\mathbf{K}} N$ be in $\mathbf{K}_{[\mathrm{LS}(\mathbf{K}),\theta)}$, with M limit over M_0. Let $p \in \mathrm{gS}^{<\alpha}(M)$ be such that p does not $\mathrm{LS}(\mathbf{K})$-split over M_0. Then there exists an extension $q \in \mathrm{gS}^{<\alpha}(N)$ of p which does not $\mathrm{LS}(\mathbf{K})$-split over M_0.

Theorem C.6. Let $\theta > \mathrm{LS}(\mathbf{K})$. Write $\mathcal{F} := [\mathrm{LS}(\mathbf{K}), \theta)$. Let $\mathfrak{s}$ be a type-full good $\mathcal{F}$-frame with underlying class $\mathbf{K}_{\mathcal{F}}$. Let $\alpha \leq \omega$ be a cardinal and assume that $\mathbf{K}$ is $(<\alpha, \mu)$-superstable for every $\mu \in \mathcal{F}$. Let $M \leq_{\mathbf{K}} N$ be in $\mathbf{K}_{\mathcal{F}}$ with M a limit model. Let $p \in \mathrm{gS}^{<\alpha}(M)$. Then there exists $q \in \mathrm{gS}^{<\alpha}(N)$ that extends p so that q does not 1-$\mathfrak{s}$-fork over M (recall Definition 4.2).

Proof. Without loss of generality, N is a limit model. Let $\mu := \|M\|$. By $(<\alpha)$-superstability, there exists $M_0 \in \mathbf{K}_\mu$ such that M is limit over M_0 and p does not μ-split over M_0. By Fact C.5, there exists $q \in \mathrm{gS}(N)$ extending p so that q does not μ-split over M_0. We claim that q does not 1-$\mathfrak{s}$-fork over M. Let $I \subseteq \ell(p)$ have size one. By monotonicity of splitting, q^I does not μ-split over M_0. By local character, let $N_0 \leq_{\mathbf{K}} N$ be such that $M \leq_{\mathbf{K}} N_0$, $N_0 \in \mathbf{K}_\mu$, N_0 is limit, and q^I does not $\mathfrak{s}$-fork over N_0. By monotonicity of splitting again, $q^I \upharpoonright N_0$ does not μ-split over M_0. By the canonicity theorem (Fact A.6) applied to the frame $\mathfrak{s} \upharpoonright \mathbf{K}_\mu$, $q^I \upharpoonright N_0$ does not $\mathfrak{s}$-fork over M. By transitivity, q^I does not $\mathfrak{s}$-fork over M, as desired. $\square$

APPENDIX D. MORE ON GLOBAL ORTHOGONALITY

Assuming superstability for types of length two, we prove a converse to Lemma 6.7, partially answering Question 6.8. We then prove a few more facts about global orthogonality and derive an alternative proof

of the upward categoricity transfer of Grossberg and VanDieren (Fact 1.6). This material is not needed for the main body of this paper.

Hypothesis D.1.

(1) $\mathbf{K}$ is an AEC.
(2) $\theta > \mathrm{LS}(\mathbf{K})$ is a cardinal or ∞. We set $\mathcal{F} := [\mathrm{LS}(\mathbf{K}), \theta)$.
(3) $\mathfrak{s} = (\mathbf{K}_{\mathcal{F}}, \downarrow)$ is a type-full good $\mathcal{F}$-frame.
(4) $\mathbf{K}$ is (≤ 2)-superstable in every $\mu \in \mathcal{F}$.

Remark D.2. Compared to Hypothesis 6.1, we have added (≤ 2)-superstability. Note that this would follow automatically if $\mathfrak{s}$ was a type-full good frame for types of length two, hence it is a minor addition. It also holds if $\mathbf{K}$ is categorical above $\mathcal{F}$ (Fact C.2) or even if it is just tame (Theorem C.3).

Lemma D.3. Let $M_0 \leq_{\mathbf{K}} M$ be both in $\mathbf{K}_{\mathcal{F}}$ with $M_0 \in \mathbf{K}_{\mathrm{LS}(\mathbf{K})}$ limit. Let $p, q \in \mathrm{gS}(M)$ be nonalgebraic so that both do not fork over M_0. If $p \underset{\mathrm{wk}}{\perp} q$, then $p \upharpoonright M_0 \underset{\mathrm{wk}}{\perp} q \upharpoonright M_0$.

Proof. Assume that $p \upharpoonright M_0 \underset{\mathrm{wk}}{\not\perp} q \upharpoonright M_0$. We show that $p \underset{\mathrm{wk}}{\not\perp} q$. Let $N \geq_{\mathbf{K}} M_0$, $a, b \in |N|$ realize in N $p \upharpoonright M_0$ and $q \upharpoonright M_0$ respectively and such that $\langle ab \rangle$ is not independent in (M_0, N). Let $r := \mathrm{gtp}(ab/M_0; N)$. By Theorem C.6, there exists $r' \in \mathrm{gS}^2(M)$ that extends r and so that r' does not 1-$\mathfrak{s}$-fork over M_0 (recall Definition 4.2). Let $N' \geq_{\mathbf{K}} M$ and let $\langle a'b' \rangle$ realize r' in N'. Then $\mathrm{gtp}(a'/M; N')$ does not fork over M_0 and extends $p \upharpoonright M_0$, hence a' must realize p in N'. Similarly, b' realizes q. We claim that $\langle a'b' \rangle$ is *not* independent in (M, N'), hence $p \underset{\mathrm{wk}}{\not\perp} q$. If $\langle a'b' \rangle$ were independent in (M, N'), there would exist $N'' \geq_{\mathbf{K}} N'$ and $M' \leq_{\mathbf{K}} N''$ so that $M \leq_{\mathbf{K}} M'$, $b \in |M'|$, and $\mathrm{gtp}(a'/M'; N'')$ does not fork over M. By transitivity, $\mathrm{gtp}(a'/M'; N'')$ does not fork over M_0. This shows that $\langle a'b' \rangle$ is independent in (M_0, N''), so since $\mathrm{gtp}(a'b'/M_0; N'') = \mathrm{gtp}(ab/M_0; N)$, we must have that $\langle ab \rangle$ is independent in (M_0, N), a contradiction. $\square$

We obtain:

Theorem D.4. Let $M \in \mathbf{K}_{\mathcal{F}}$ and $p, q \in \mathrm{gS}(M)$. Then:

(1) If $M \in \mathbf{K}_{\mathcal{F}}^{\mathrm{LS}(\mathbf{K})\text{-sat}}$, then $p \perp q$ if and only if $p \underset{\mathrm{wk}}{\perp} q$.

(2) If $M \in \mathbf{K}_{\mathcal{F}}^{\mathrm{LS}(\mathbf{K})\text{-sat}}$, then $p \perp q$ if and only if $q \perp p$.

(3) If $M_0 \in \mathbf{K}_{\mathcal{F}}^{\mathrm{LS}(\mathbf{K})\text{-sat}}$ is such that $M_0 \leq_{\mathbf{K}} M$ and both p and q do not fork over M_0, then $p \perp q$ if and only if $p \upharpoonright M_0 \perp q \upharpoonright M_0$.

Proof.

(1) If $p \perp q$, then $p \underset{\text{wk}}{\perp} q$ by definition. Conversely, assume that $p \underset{\text{wk}}{\perp} q$. Fix a limit $M_0 \in \mathbf{K}_{\text{LS}(\mathbf{K})}$ such that $M_0 \leq_{\mathbf{K}} M$ and both p and q do not fork over M_0. By Lemma D.3, $p \upharpoonright M_0 \underset{\text{wk}}{\perp} q \upharpoonright M_0$. By Lemma 4.13.(1), $p \upharpoonright M_0 \perp q \upharpoonright M_0$. By Lemma 6.7, $p \perp q$.

(2) A similar proof, using (2) instead of (1) in Lemma 4.13.

(3) By local character and transitivity, we can fix a limit $M_0' \in \mathbf{K}_{\text{LS}(\mathbf{K})}$ such that $M_0' \leq_{\mathbf{K}} M_0$ and both p and q do not fork over M_0'. Now by what has been proven above and Lemmas 6.7 and D.3, $p \perp q$ if and only if $p \upharpoonright M_0' \perp q \upharpoonright M_0'$ if and only if $p \upharpoonright M_0 \perp q \upharpoonright M_0$.

$\square$

We can now give another proof of the upward transfer of unidimensionality (the second part of the proof of Theorem 6.13). This does not use Fact 6.12.

Lemma D.5. Let $\mu < \lambda$ be in $\mathcal{F}$. If $\mathfrak{s}$ is μ-unidimensional, then $\mathfrak{s}$ is λ-unidimensional.

Proof. Assume that $\mathfrak{s}$ is *not* λ-unidimensional. Let $M_0 \in \mathbf{K}_\mu$ be limit and let $p_0 \in \text{gS}(M_0)$ be minimal. We show that there exists a limit $M_0' \in \mathbf{K}_\mu$, $p_0', q_0' \in \text{gS}(M_0')$ such that p_0' extends p_0 and $p_0' \perp q_0'$. This will show that $\mathbf{K}$ is not μ-unidimensional by Lemma 5.6. Let $M \in \mathbf{K}_\lambda$ be saturated such that $M_0 \leq_{\mathbf{K}} M$ and let $p \in \text{gS}(M)$ be the nonforking extension of p_0. By non-λ-unidimensionality (and Lemma 5.6), there exists $q \in \text{gS}(M)$ so that $p \perp q$. Let $M_0' \in \mathbf{K}_\mu$ be limit such that $M_0 \leq_{\mathbf{K}} M_0' \leq_{\mathbf{K}} M$ and q does not fork over M_0'. Let $p_0' := p \upharpoonright M_0'$, $q_0' := q \upharpoonright M_0'$. By Theorem D.4, $p_0' \perp q_0'$, as desired. $\square$

We obtain the promised alternate proof to Grossberg-VanDieren. For this corollary, we drop Hypothesis D.1.

Corollary D.6. Let $\mathbf{K}$ be an AEC with amalgamation and arbitrarily large models. If $\mathbf{K}$ is $\text{LS}(\mathbf{K})$-tame and categorical in a successor $\lambda > \text{LS}(\mathbf{K})^+$, then $\mathbf{K}$ is categorical in all $\mu \geq \lambda$.

Proof. By Fact 7.10, we can assume without loss of generality that $\mathbf{K}$ has no maximal models. By Fact 7.3, $\mathbf{K}$ is $\text{LS}(\mathbf{K})$-superstable. By Fact 7.5, $\mathbf{K}$ is superstable and has symmetry in every $\mu \geq \text{LS}(\mathbf{K})$. By Theorem 9.8, $\mathbf{K}$ is categorical in a proper class of cardinals. By Fact

C.2, $\mathbf{K}$ is $(< \omega)$-superstable in every $\mu \geq \mathrm{LS}(\mathbf{K})$. Now say $\lambda = \lambda_0^+$. By Facts 7.13 and 7.14, there exists a type-full good $(\geq \lambda_0)$-frame $\mathfrak{s}$ with underlying class $\mathbf{K}_{\geq \mathrm{LS}(\mathbf{K})^+}^{\mathrm{LS}(\mathbf{K})^+\text{-sat}}$. By Fact 2.6, we can restrict the frame further to have underlying class $\mathbf{K}_{\geq \lambda_0}^{\lambda_0\text{-sat}}$. By Corollary 6.14 (using Lemma D.5 to transfer unidimensionality up), $\mathbf{K}^{\lambda_0\text{-sat}}$ is categorical in every $\mu \geq \lambda_0$. Now $\mathbf{K}_{\geq \lambda}^{\lambda_0\text{-sat}} = \mathbf{K}_{\geq \lambda}$ (by categoricity in λ), so the result follows. $\square$

Remark D.7. Similarly to what was said in Remark 10.4, it is also possible to use this argument to prove that a $\mathrm{LS}(\mathbf{K})$-tame AEC with amalgamation and arbitrarily large models categorical in $\mathrm{LS}(\mathbf{K})$ and $\mathrm{LS}(\mathbf{K})^+$ is categorical everywhere [GV06a, Theorem 6.3]. However we do not know that we have a good frame in $\mathrm{LS}(\mathbf{K})$, so the proof is more complicated.

Remark D.8. As opposed to Grossberg and VanDieren's proof, our proof of Corollary D.6 is global: it couldn't be turned into an argument for Fact 6.12 (e.g. if θ there is $\mathrm{LS}(\mathbf{K})^{+\omega}$ we cannot use Theorem 9.8). It is also not clear how to replace tameness with weak tameness in the proof.

Appendix E. A proof of Fact 11.3

Hypothesis E.1. $\mathbf{K}$ is an AEC, $\lambda \geq \mathrm{LS}(\mathbf{K})$.

We give a full proof of Fact 11.3. Shelah's proof in $\odot_4$ of the proof of [She09a, Theorem IV.7.12] skips some steps (for example it is not clear how we can make sure there that $f_{\eta \frown 0}[M_{\ell(\eta)+1}] = f_{\eta \frown 1}[M_{\ell(\eta)+1}]$). The proof we give here is similar in spirit to Shelah's, but we use a stronger blackbox (relying on weak diamond) which avoids having to deal with all of Shelah's renaming steps.

We will use the combinatorial principle Θ_{λ^+} is introduced (for $\lambda = \aleph_0$) in [DS78].

Definition E.2. Θ_{λ^+} holds if for every $\langle f_\eta \in {}^{\lambda^+}\lambda^+ : \eta \in {}^{\lambda^+}2 \rangle$, there exists $\eta \in {}^{\lambda^+}2$ such that the set

$$S_\eta := \{\delta < \lambda^+ \mid \exists \nu \in {}^{\lambda^+}2 : f_\eta \restriction \delta = f_\nu \restriction \delta \wedge \eta \restriction \delta = \nu \restriction \delta \wedge \eta(\delta) \neq \nu(\delta)\}$$

is stationary.

Remark E.3. Instead of requiring that $f_\eta \in {}^{\lambda^+}\lambda^+$, we can assume only that f_η is a partial function from λ^+ to λ^+ (we can always extend f_η arbitrarily to an actual function).

Fact E.4 (6.1 in [DS78]). If $2^\lambda < 2^{\lambda^+}$, then Θ_{λ^+} holds.

In [She01a, Claim 1.4.(2)], Shelah shows assuming the weak diamond that given a tree witnessing failure of amalgamation in λ, there cannot be a universal model of cardinality λ^+. A similar proof gives a two-dimensional version:

Lemma E.5. Assume $2^\lambda < 2^{\lambda^+}$. Let $\langle M_\eta : \eta \in {}^{\leq\lambda^+}2 \rangle$ be a strictly increasing continuous tree with $M_\eta \in \mathbf{K}_\lambda$ for all $\eta \in {}^{<\lambda^+}2$. Let $\langle M_\alpha : \alpha \leq \lambda^+ \rangle$ be an increasing continuous chain such that for every $\eta \in {}^{\leq\lambda^+}2$ there exists $f_\eta : M_{\ell(\eta)} \to M_\eta$ so that for any $\alpha < \ell(\eta)$, $f_\eta \upharpoonright M_\alpha = f_{\eta\upharpoonright\alpha}$.

Assume that there exists $N \in \mathbf{K}_{\lambda^+}$ with $M_{\lambda^+} \leq_{\mathbf{K}} N$ such that for all $\eta \in {}^{\lambda^+}2$, there exists $g_\eta : M_\eta \to N$ such that the following diagram commutes:

$$
\begin{array}{ccc}
N & & \\
\uparrow & \nwarrow^{g_\eta} & \\
M_{\lambda^+} & \xrightarrow{\ f_\eta\ } & M_\eta
\end{array}
$$

Then there exists $\rho \in {}^{<\lambda^+}2$, $\eta, \nu \in {}^{\lambda^+}2$, $\delta < \lambda^+$ such that $\rho = \eta \upharpoonright \delta = \nu \upharpoonright \delta$, $0 = \eta(\delta) \neq \nu(\delta) = 1$, and the following diagram commutes:

$$
\begin{array}{ccccc}
& & M_{\rho\frown 0} & \xrightarrow{g_\eta \upharpoonright M_{\rho\frown 0}} & N \\
& \nearrow^{f_{\rho\frown 0}} & \uparrow & & \uparrow^{g_\nu \upharpoonright M_{\rho\frown 1}} \\
M_{\delta+1} & & \xrightarrow{\ f_{\rho\frown 1}\ } & M_{\rho\frown 1} & \\
\uparrow & & & \nearrow & \\
M_\delta & \xrightarrow{\ f_\rho\ } & M_\rho & &
\end{array}
$$

Proof. Fix $N \in \mathbf{K}_{\lambda^+}$, f_η, g_η as in the statement of the lemma. By renaming everything, we can assume without loss of generality that $|N| \subseteq \lambda^+$ and $|M_\eta| \subseteq \lambda^+$ for all $\eta \in {}^{\leq\lambda^+}2$.

By Fact E.4, Θ_{λ^+} holds. We use it with the sequence $\langle g_\eta : \eta \in {}^{\lambda^+}2 \rangle$ (see Remark E.3). We obtain $\eta \in {}^{\lambda^+}2$ such that the set S_η of Definition

E.2 is stationary. Let $C := \{\delta < \lambda^+ \mid |M_{\eta \restriction \delta}| \subseteq \delta\}$. Clearly, C is club so let $\delta \in S_\eta \cap C$ and let $\nu \in {}^{\lambda^+}2$ be as given by the definition of S_η (i.e. $g_\eta \restriction \delta = f_\nu \restriction \delta$, $\eta \restriction \delta = \nu \restriction \delta$, and without loss of generality $0 = \eta(\delta) \neq \nu(\delta) = 1$). Let $\rho := \eta \restriction \delta = \nu \restriction \delta$. Then ρ, η, ν, δ are as desired. The main point is that since $|M_\rho| \subseteq \delta$ (by definition of δ), $g_\eta \restriction M_\rho = g_\nu \restriction M_\rho$.

$\square$

The next technical property is of great importance in Chapter II and III of [She09a]. The definition below follows [JS13, Definition 4.1.5] (but as usual, we work only with type-full frames).

Definition E.6.

(1) For $M_0 \leq_{\mathbf{K}} M_\ell$ all in $\mathbf{K}_\lambda$, $\ell = 1, 2$, an *amalgam of M_1 and M_2 over M_0* is a triple (f_1, f_2, N) such that $N \in \mathbf{K}_\lambda$ and $f_\ell : M_\ell \xrightarrow[M_0]{} N$.

(2) Let (f_1^x, f_2^x, N^x), $x = a, b$ be amalgams of M_1 and M_2 over M_0. We say (f_1^a, f_2^a, N^a) and (f_1^b, f_2^b, N^b) are *equivalent over M_0* if there exists $N_* \in \mathbf{K}_\lambda$ and $f^x : N^x \to N_*$ such that $f^b \circ f_1^b = f^a \circ f_1^a$ and $f^b \circ f_2^b = f^a \circ f_2^a$, namely, the following commutes:

Note that being "equivalent over M_0" is an equivalence relation ([JS13, Proposition 4.3]).

(3) Let $\mathfrak{s}$ be a type-full good λ-frame on $\mathbf{K}$.

(a) Let $\mathbf{K}_{\mathfrak{s}}^{3, \mathrm{uq}}$ denote the set of triples (a, M, N) such that $M \leq_{\mathbf{K}} N$ are in $\mathbf{K}_\lambda$, $a \in |N| \backslash |M|$ and for any $M_1 \geq_{\mathbf{K}} M$ in $\mathbf{K}_\lambda$, there exists a *unique* (up to equivalence over M) amalgam (f_1, f_2, N_1) of N and M_1 over M such that $\mathrm{gtp}(f_1(a)/f_2[M_1]; N_1)$ does not fork over M. We call the elements of $\mathbf{K}_{\mathfrak{s}}^{3, \mathrm{uq}}$ *uniqueness triples.* When $\mathfrak{s}$ is clear from context, we just write $\mathbf{K}^{3, \mathrm{uq}}$.

(b) $\mathbf{K}_{\mathfrak{s}}^{3,\mathrm{uq}}$ has the *existence property* if for any $M \in \mathbf{K}_\lambda$ and any nonalgebraic $p \in \mathrm{gS}(M)$, one can write $p = \mathrm{gtp}(a/M; N)$ with $(a, M, N) \in \mathbf{K}_{\mathfrak{s}}^{3,\mathrm{uq}}$.

(c) We say that $\mathfrak{s}$ *has the existence property for uniqueness triples* or $\mathfrak{s}$ *is weakly successful* if $\mathbf{K}_{\mathfrak{s}}^{3,\mathrm{uq}}$ has the existence property.

Remark E.7. Let $M_0 \leq_{\mathbf{K}} M_1$, $M_0 \leq_{\mathbf{K}} M_2$ all be in $\mathbf{K}_\lambda$.

(1) If (f_1, f_2, N) is an amalgam of M_1 and M_2 over M_0, there exists an equivalent amalgam (g_1, g_2, N') of M_1 and M_2 over M_0 with $g_2 = \mathrm{id}_{M_2}$.

(2) For $x = a, b$, assume that (f_1^x, f_2^x, N^x) are *non-equivalent* amalgams of M_1 and M_2 over M_0. We have the following monotonicity properties:

  (a) For $x = a, b$, if $N^x \leq_{\mathbf{K}} N_*^x$, then (f_1^x, f_2^x, N_*^x) are non-equivalent amalgams of M_1 and M_2 over M_0.

  (b) If $M_1 \leq_{\mathbf{K}} M_1'$, $M_2 \leq_{\mathbf{K}} M_2'$ (with $M_1', M_2' \in \mathbf{K}_\lambda$), and for $x = a, b$, there exists $g_1^x \supseteq f_1^x, g_2^x \supseteq f_2^x$ such that (g_1^x, g_2^x, N^x) is an amalgam of M_1' and M_2' over M_0, then (g_1^a, g_2^a, N^a) and (g_1^b, g_2^b, N^b) are not equivalent over M_0.

We are now ready to prove the desired result.

Theorem E.8 (Shelah). Assume $2^\lambda < 2^{\lambda^+}$. Let $\mathfrak{s}$ be a good λ-frame on $\mathbf{K}$. If $\mathbf{K}$ is categorical in λ and for any saturated $M \in \mathbf{K}_{\lambda^+}$ there exists $N \in \mathbf{K}_{\lambda^+}$ universal over M, then $\mathfrak{s}$ is weakly successful.

Proof. Below, we assume to simplify the notation that $\mathfrak{s}$ is type-full but the same proof goes through in the general case. So suppose the conclusion of the theorem fails. Fix $\langle M_\alpha : \alpha < \lambda^+ \rangle$ increasing continuous in $\mathbf{K}_\lambda$ such that $M_{\alpha+1}$ is limit over M_α for all $\alpha < \lambda^+$. Let $M_{\lambda^+} := \bigcup_{\alpha<\lambda^+} M_\alpha$. Since $\mathfrak{s}$ is not weakly successful, there exists a nonalgebraic type $p \in \mathrm{gS}(M_0)$ which cannot be represented by a uniqueness triple. Say $p = \mathrm{gtp}(a/M_0; M)$.

We build a strictly increasing continuous tree $\langle M_\eta : \eta \in {}^{\leq\lambda^+}2 \rangle$ with $M_\eta \in \mathbf{K}_\lambda$ for $\eta \in {}^{<\lambda^+}2$, as well as a strictly increasing continuous tree of embeddings $\langle f_\eta : \eta \in {}^{\leq\lambda^+}2 \rangle$ such that for any $\eta \in {}^{<\lambda^+}2$:

(1) $f_\eta : M_{\ell(\eta)} \to M_\eta$.
(2) $M_{<>} = M$ and $f_{<>} = \mathrm{id}_{M_0}$.
(3) $\mathrm{gtp}(a/f_\eta[M_{\ell(\eta)}]; M_\eta)$ does not fork over M_0.
(4) There is *no* $N \in \mathbf{K}_\lambda$ and $g_\ell : M_{\eta^\frown l} \to N$, $\ell = 0, 1$, such that the following diagram commutes:
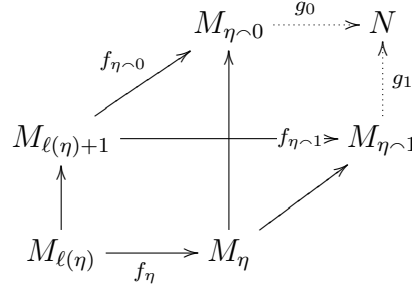
This is enough: We have that M_{λ^+} is saturated and we know by assumption that there is a universal model N over M_{λ^+} in $\mathbf{K}_{\lambda^+}$. In particular, for every $\eta \in {}^{\lambda^+}2$, there exists $g_\eta : M_\eta \to N$ such that $f_\eta \subseteq g_\eta$. By Lemma E.5, requirement (4) must fail somewhere in the construction, contradiction.

This is possible: The construction is by induction on the length of $\eta \in {}^{\leq\lambda^+}2$. If $\ell(\eta) = 0$, then (2) specifies what to do and if the length is limit, we take unions (and use the local character and transitivity properties of forking to see that (3) is preserved). Assume now that $\alpha < \lambda^+$ and that $\eta \in {}^\alpha 2$ is such that M_η, f_η have been defined. We want to build $M_{\eta^\frown\ell}$, $f_{\eta^\frown\ell}$ for $\ell = 0, 1$. Let $q := \mathrm{gtp}(a/f_\eta[M_\alpha]; M_\eta)$. We know that q is the nonforking extension of p (by (3) and the definition of p), so by the conjugation property (Fact 3.3, note that by assumption $\mathbf{K}$ is categorical in λ) p and q are conjugates, hence q cannot be represented by a uniqueness triple. Therefore $(a, f_\eta[M_\alpha], M_\eta)$ is not a uniqueness triple. This means that there exists $M'_\alpha \in \mathbf{K}_\lambda$ with $f_\eta[M_\alpha] \leq M'_\alpha$ and two non-equivalent amalgams $(f_1^\ell, f_2^\ell, M_{\eta^\frown\ell})$ such that $\mathrm{gtp}(f_2^\ell(a)/f_1^\ell[M'_\alpha]; M_{\eta^\frown\ell})$ does not fork over M_η for $\ell = 0, 1$. Without loss of generality (Remark E.7) f_2^ℓ is the identity for $\ell = 0, 1$:
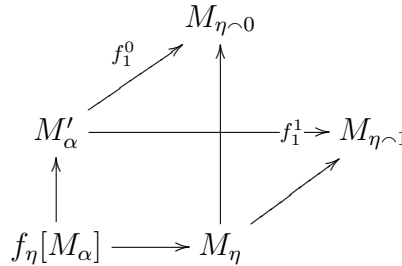
By the monotonicity property of being non-equivalent amalgams (Remark E.7) and the extension property of forking, we can increase M'_α, M_η^0, and M_η^1 to assume without loss of generality that M'_α is limit over

$f_\eta[M_\alpha]$. In particular, there exists $g : M_{\alpha+1} \cong f_\eta[M_\alpha]$ with $f_\eta \subseteq g$. For $\ell = 0, 1$, let $f_{\eta \frown \ell} := f_1^\ell \circ g$. $\qquad\square$

Proof of Fact 11.3. Note that amalgamation and stability in λ^+ imply that over every $M \in \mathbf{K}_{\lambda^+}$ there exists $N \in \mathbf{K}_{\lambda^+}$ universal over M. Thus the hypotheses of Theorem E.8 hold. $\qquad\square$

References

[Bal09]   John T. Baldwin, *Categoricity*, University Lecture Series, vol. 50, American Mathematical Society, 2009.

[BG]   Will Boney and Rami Grossberg, *Forking in short and tame AECs*, Preprint. URL: `http://arxiv.org/abs/1306.6562v9`.

[BGKV]   Will Boney, Rami Grossberg, Alexei Kolesnikov, and Sebastien Vasey, *Canonical forking in AECs*, Preprint. URL: `http://arxiv.org/abs/1404.1494v2`.

[BK09]   John T. Baldwin and Alexei Kolesnikov, *Categoricity, amalgamation, and tameness*, Israel Journal of Mathematics **170** (2009), 411–443.

[BKV06]   John T. Baldwin, David Kueker, and Monica VanDieren, *Upward stability transfer for tame abstract elementary classes*, Notre Dame Journal of Formal Logic **47** (2006), no. 2, 291–298.

[Bona]   Will Boney, *Shelah's omitting types theorem in excruciating detail*, Unpublished note. URL: `http://math.harvard.edu/~wboney/notes/SOTTNotes.pdf`.

[Bonb]   _____, *Tameness in groups and similar AECs*, In preparation.

[Bon14a]   _____, *Tameness and extending frames*, Journal of Mathematical Logic **14** (2014), no. 2, 1450007.

[Bon14b]   _____, *Tameness from large cardinal axioms*, The Journal of Symbolic Logic **79** (2014), no. 4, 1092–1119.

[BVa]   Will Boney and Sebastien Vasey, *Chains of saturated models in AECs*, Preprint. URL: `http://arxiv.org/abs/1503.08781v3`.

[BVb]   _____, *Tameness and frames revisited*, Preprint. URL: `http://arxiv.org/abs/1406.5980v5`.

[DS78]   Keith J. Devlin and Saharon Shelah, *A weak version of $\Diamond$ which follows from $2^{\aleph_0} < 2^{\aleph_1}$*, Israel Journal of Mathematics **29** (1978), no. 2, 239–247.

[GV]   Rami Grossberg and Sebastien Vasey, *Superstability in abstract elementary classes*, Preprint. URL: `http://arxiv.org/abs/1507.04223v2`.

[GV06a]   Rami Grossberg and Monica VanDieren, *Categoricity from one successor cardinal in tame abstract elementary classes*, Journal of Mathematical Logic **6** (2006), no. 2, 181–201.

[GV06b]   _____, *Galois-stability for tame abstract elementary classes*, Journal of Mathematical Logic **6** (2006), no. 1, 25–49.

[GV06c]   _____, *Shelah's categoricity conjecture from a successor for tame abstract elementary classes*, The Journal of Symbolic Logic **71** (2006), no. 2, 553–568.

[GVV]   Rami Grossberg, Monica VanDieren, and Andrés Villaveces, *Uniqueness of limit models in classes with amalgamation*, Mathematical Logic Quarterly, To appear. URL: `http://arxiv.org/abs/1507.02118v1`.

[Han60] William P. Hanf, *Models of languages with infinitely long expressions*, Abstracts of contributed papers, First International Congress for Logic, Methodology, and Philosophy of Science, Stanford University Press, 1960, p. 24.

[HS90] Bradd Hart and Saharon Shelah, *Categoricity over P for first order T or categoricity for $\phi \in \mathbb{L}_{\omega_1, \omega}$ can stop at $\aleph_k$ while holding for $\aleph_0, \ldots, \aleph_{k-1}$*, Israel Journal of Mathematics **70** (1990), 219–235.

[Jar16] Adi Jarden, *Tameness, uniqueness triples, and amalgamation*, Annals of Pure and Applied Logic **167** (2016), no. 2, 155–188.

[JS12] Adi Jarden and Alon Sitton, *Independence, dimension and continuity in non-forking frames*, The Journal of Symbolic Logic **78** (2012), no. 2, 602–632.

[JS13] Adi Jarden and Saharon Shelah, *Non-forking frames in abstract elementary classes*, Annals of Pure and Applied Logic **164** (2013), 135–191.

[MS90] Michael Makkai and Saharon Shelah, *Categoricity of theories in $\mathbb{L}_{\kappa, \omega}$, with κ a compact cardinal*, Annals of Pure and Applied Logic **47** (1990), 41–97.

[She] Saharon Shelah, *Eventual categoricity spectrum and frames*, Paper number 842 on Shelah's publication list. Preliminary draft from Oct. 3, 2014 (obtained from the author).

[She90] ———, *Classification theory and the number of non-isomorphic models*, 2nd ed., Studies in logic and the foundations of mathematics, vol. 92, North-Holland, 1990.

[She99] ———, *Categoricity for abstract classes with amalgamation*, Annals of Pure and Applied Logic **98** (1999), no. 1, 261–294.

[She00] ———, *On what I do not understand (and have something to say), model theory*, Math. Japonica **51** (2000), 329–377.

[She01a] ———, *Categoricity of an abstract elementary class in two successive cardinals*, Israel Journal of Mathematics **126** (2001), 29–128.

[She01b] ———, *Categoricity of theories in $\mathbb{L}_{\kappa, \omega}$, when κ is a measurable cardinal. Part II*, Fundamenta Mathematica **170** (2001), 165–196.

[She09a] ———, *Classification theory for abstract elementary classes*, Studies in Logic: Mathematical logic and foundations, vol. 18, College Publications, 2009.

[She09b] ———, *Classification theory for abstract elementary classes 2*, Studies in Logic: Mathematical logic and foundations, vol. 20, College Publications, 2009.

[SK96] Saharon Shelah and Oren Kolman, *Categoricity of theories in $\mathbb{L}_{\kappa, \omega}$, when κ is a measurable cardinal. Part I*, Fundamentae Mathematica **151** (1996), 209–240.

[SV99] Saharon Shelah and Andrés Villaveces, *Toward categoricity for classes with no maximal models*, Annals of Pure and Applied Logic **97** (1999), 1–25.

[Van] Monica VanDieren, *Superstability and symmetry*, Preprint. URL: `http://arxiv.org/abs/1507.01990v2`.

[Van06] ———, *Categoricity in abstract elementary classes with no maximal models*, Annals of Pure and Applied Logic **141** (2006), 108–147.

[Van16]   _____, *Symmetry and the union of saturated models in superstable abstract elementary classes*, Annals of Pure and Applied Logic **167** (2016), no. 4, 395–407.

[Vasa]   Sebastien Vasey, *Building independence relations in abstract elementary classes*, Annals of Pure and Applied Logic, To appear. URL: `http://arxiv.org/abs/1503.01366v6`.

[Vasb]   _____, *Infinitary stability theory*, Archive for Mathematical Logic, To appear. URL: `http://arxiv.org/abs/1412.3313v6`.

[Vasc]   _____, *Shelah's eventual categoricity conjecture in tame AECs with primes*, Preprint. URL: `http://arxiv.org/abs/1509.04102v3`.

[Vasd]   _____, *Shelah's eventual categoricity conjecture in universal classes. Part I*, Preprint. URL: `http://arxiv.org/abs/1506.07024v9`.

[Vase]   _____, *Shelah's eventual categoricity conjecture in universal classes. Part II*, Preprint. URL: `http://arxiv.org/abs/1602.02633v1`.

[Vas16]   _____, *Forking and superstability in tame AECs*, The Journal of Symbolic Logic **81** (2016), no. 1, 357–383.

[VV]   Monica VanDieren and Sebastien Vasey, *Symmetry in abstract elementary classes with amalgamation*, Preprint. URL: `http://arxiv.org/abs/1508.03252v3`.

[VZ14]   Andrés Villaveces and Pedro H. Zambrano, *Around independence and domination in metric abstract elementary classes: assuming uniqueness of limit models*, Mathematical Logic Quarterly **60** (2014), no. 3, 211–227.

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