ON CATEGORICITY IN SUCCESSIVE CARDINALS

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Abstract. We investigate, in ZFC, the behavior of abstract elementary classes (AECs) categorical in many successive small cardinals. We prove for example that a universal $\mathbb{L}_{\omega_1,\omega}$ sentence categorical on an end segment of cardinals below $\beth_\omega$ must be categorical also everywhere above $\beth_\omega$. This is done without any additional model-theoretic hypotheses (such as amalgamation or arbitrarily large models) and generalizes to the much broader framework of tame AECs with weak amalgamation and coherent sequences.

Contents

1. Introduction

The upward Löwenheim-Skolem theorem says that any first-order theory with an infinite model has models of arbitrarily large cardinalities. This result is no longer true outside of first-order logics, for example for theories in $\mathbb{L}_{\omega_1,\omega}$. For this more general case, it is reasonable to ask which properties of small models suffice to guarantee existence of bigger models. In that light, the following early result of Shelah [She87a] is remarkable:

Fact 1.1. An $\mathbb{L}_{\omega_1,\omega}$ sentence which is categorical in both $\aleph_0$ and $\aleph_1$ has a model of size $\aleph_2$.

Shelah proved in fact a more general theorem, valid for any "reasonably definable" abstract elementary class (AEC) with countable Löwenheim-Skolem number (see [She09a, I.3.11] for the details). In particular, he answered in the negative Baldwin's

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question: can a sentence in $L(Q)$ have exactly one uncountable model? See for example [Gro02, §4] for a more detailed history.

It is natural to ask whether Fact 1.1 generalizes to *any* AEC. More specifically:

Question 1.2. Assume $\mathbf{K}$ is an AEC with Löwenheim-Skolem-Tarski number λ. If $\mathbf{K}$ is categorical in both λ and λ^+, must it have a model of cardinality λ^{++}?

Question 1.2 is still open. Partial approximations immensely stimulated the field: Shelah [She01] has shown assuming some set-theoretic hypotheses that categoricity in *three* successive cardinals suffices:

Fact 1.3. Assume[1] $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$. Let $\mathbf{K}$ be an AEC with Löwenheim-Skolem-Tarski number λ. If $\mathbf{K}$ is categorical in λ, λ^+, and λ^{++}, then it has a model of size λ^{+3}.

Remark 1.4. One of the byproduct of Shelah's proof is the machinery of *good frames*, developed in Shelah's two volume book [She09a]. Essentially, an AEC has a *good λ-frame* if $\mathbf{K}_\lambda$, its class of models of cardinality λ, behaves "well" in the sense that it has several structural properties, including a superstable-like forking notion. Good frames have subsequently been used in many results, for example in the author's proof of the eventual categoricity conjecture for universal classes [Vas17b, Vas17c], in the recent proof of the eventual categoricity conjecture from large cardinals [SV], and in the full analysis of the categoricity spectrum of AECs with amalgamation [Vasa].

More ambitiously, one can ask when it is possible not only to prove the existence of a model from successive categoricity, but to prove the existence of *arbitrarily large models*, or even the existence of a *unique* model in all cardinalities. Another milestone result of Shelah in that direction is [She83a, She83b]:

Fact 1.5. Assume $2^{\aleph_n} < 2^{\aleph_{n+1}}$ for all $n < \omega$. If an $\mathbb{L}_{\omega_1,\omega}$ sentence is categorical in every $\aleph_n$, then it is categorical in every infinite cardinal.

This has recently been generalized to AECs by Shelah and the author [SV]. An example of Hart and Shelah [HS90, BK09] shows that one needs in general to assume categoricity at all $\aleph_n$'s to deduce categoricity further up. However, Mazari-Armida and the author showed that this restriction does not apply to certain simpler sentences:

Fact 1.6 ([MAV])**.** Assume $2^{\aleph_0} < 2^{\aleph_1}$. If a *universal* $\mathbb{L}_{\omega_1,\omega}$ sentence is categorical in $\aleph_1$, then it is categorical in all uncountable cardinals.

Here, an $\mathbb{L}_{\omega_1,\omega}$ sentence is *universal* if it is of the form $\forall \bar{x}\psi$, with $\psi \in \mathbb{L}_{\omega_1,\omega}$ quantifier free. Classes of models of such sentences are *universal classes* (classes of structures closed under isomorphisms, unions of chains, and substructures). See for example [Vas17b].

In the present paper, we aim to prove results along the ones above but *in ZFC*. One challenge is that it is harder to obtain amalgamation in this case: Shelah [She09a,

[1] Shelah originally proved this assuming in addition a saturation condition on the weak diamond ideal, but this was subsequently removed [She09a, VI.8.1(3)].

I.3.8] has shown that categoricity in λ and λ^+ implies amalgamation in λ *assuming* that $2^\lambda < 2^{\lambda^+}$. However, the set-theoretic hypothesis cannot in general be removed:

Example 1.7 ([She09a, §I.6])**.** Assuming Martin's axiom, there is an AEC (axiomatizable in $\mathbb{L}(Q)$) that is categorical in every cardinal in $[\aleph_0, 2^{\aleph_0})$, fails amalgamation everywhere below $2^{\aleph_0}$, and has no model of cardinality $\left(2^{\aleph_0}\right)^+$.

This example leads to the following weakening of Question 1.2, which is also open:

Question 1.8. Assume $\mathbf{K}$ is an AEC with Löwenheim-Skolem-Tarski number λ, categorical in every cardinal in $[\lambda, 2^\lambda]$. Must $\mathbf{K}$ have a model of cardinality $\left(2^\lambda\right)^+$?

Similarly, looking at Fact 1.5 suggests that to replace $\aleph_\omega$ by $\beth_\omega$ may be interesting:

Question 1.9. Assume $\mathbf{K}$ is an AEC with Löwenheim-Skolem-Tarski number λ, categorical in every cardinal in $[\lambda, \beth_\omega(\lambda))$. Must $\mathbf{K}$ be categorical everywhere above λ?

The present paper makes the following contributions:

(1) By a very short proof, putting together several results of Shelah, we observe (Corollary 3.7) that if $\mu > \text{LS}(\mathbf{K})$ is *limit* and $\mathbf{K}$ is categorical everywhere in $[\text{LS}(\mathbf{K}), \mu]$, then $\mathbf{K}$ has a model of cardinality μ^+. This is a (very partial) approximation to Questions 1.8 and 1.9, but this is essentially all the author is currently able to say for general AECs.

(2) We give a positive answer to Question 1.9 in the special case that $\mathbf{K}$ is a universal class (Corollary 5.10). This gives a ZFC version of the Facts presented above, and partially answers Question 4.3 in [MAV] (where we worked near $\aleph_1$ and assumed the weak continuum hypothesis, see Fact 1.6). We more generally answer Question 1.9 for tame AECs with weak amalgamation and coherent sequences (see Section 2 for the definitions):

Corollary 5.9. Assume $\mathbf{K}$ is a $\text{LS}(\mathbf{K})$-tame AEC with weak amalgamation and coherent sequences. If $\mathbf{K}$ is categorical everywhere in $[\text{LS}(\mathbf{K}), \beth_\omega(\text{LS}(\mathbf{K})))$, then $\mathbf{K}$ is categorical everywhere above $\text{LS}(\mathbf{K})$.

Tame AECs with weak amalgamation and coherent sequences encompass for example also multiuniversal classes [ABV], or Zilber's quasiminimal classes [Zil05]. Interestingly, even in the case of tame AECs with full amalgamation, the result is not trivial: it is only known [GV06c, GV06a] in case the AEC also has arbitrarily large models.

Essentially, the proof of Corollary 5.9 proceeds as follows: we use tameness and the weak amount of amalgamation to build a good $\beth_\omega(\lambda)$-frame (we have set $\lambda := \text{LS}(\mathbf{K})$). There are several technical difficulties here because of the lack of amalgamation, but otherwise the construction follows that in [Vas16a]. One key is a local character theorem proven in [Vasc, 4.12] (Lemma 4.6 here). Another key is that we can show the model of cardinality $\beth_{n+1}(\lambda)$, for $n < \omega$, is locally $\beth_n(\lambda)$-model-homogeneous (Definition 4.1) and thus that every model of cardinality $\beth_\omega(\lambda)$ is locally model-homogeneous. It is also crucial that $\beth_\omega(\lambda)$ has cofinality ω, as otherwise the construction of certain orbital types would have a problematic limit stage. Once the good frame is built, a known upward transfer of good frames in tame AECs with weak amalgamation (see [Bon14] and [Vas17b, 4.16]) is used to prove that the AEC has arbitrarily large models and eventual amalgamation. After some more work, it then becomes possible to use the result of Grossberg and VanDieren [GV06a] showing that in tame AECs with amalgamation and arbitrarily large models, categoricity in two successive cardinals implies categoricity above those cardinals.

We assume the reader has some basic knowledge of AECs (see e.g. [She09a, Bal09, Gro02]), although we briefly repeat some definitions in the preliminaries. In the last section, we will also assume some familiarity with the material in [Vas16a] regarding good frames. Other results we use can be regarded as black boxes.

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2. Preliminaries

Given a structure M, write $|M|$ for its universe and $\|M\|$ for the cardinality of its universe. We often do not distinguish between M and $|M|$, writing e.g. $a \in M$ instead of $a \in |M|$. We write $M \subseteq N$ to mean that M is a substructure of N.

2.1. **Abstract elementary classes.**

An *abstract class* is a pair $\mathbf{K} = (K, \leq_{\mathbf{K}})$, where K is a class of structures in a fixed vocabulary $\tau = \tau(\mathbf{K})$ and $\leq_{\mathbf{K}}$ is a partial order, $M \leq_{\mathbf{K}} N$ implies $M \subseteq N$, and both K and $\leq_{\mathbf{K}}$ respect isomorphisms (the definition is due to Grossberg). We often do not distinguish between K (the class of structures) and $\mathbf{K}$ (the *ordered* class of structures). Any abstract class admits a notion of $\mathbf{K}$-*embedding*: these are functions $f : M \to N$ such that $f : M \cong f[M]$ and $f[M] \leq_{\mathbf{K}} N$. Thus one can naturally see $\mathbf{K}$ as a category. Unless explicitly stated, any map $f : M \to N$ in this paper will be a $\mathbf{K}$-embedding. We write $f : M \xrightarrow[A]{} N$ to mean that f is a $\mathbf{K}$-embedding from M into N which fixes the set A pointwise (so $A \subseteq |M|$). We similarly write $f : M \cong_A N$ for isomorphisms from M onto N fixing A.

For λ a cardinal, we will write $\mathbf{K}_\lambda$ for the restriction of $\mathbf{K}$ to models of cardinality λ. Similarly define $\mathbf{K}_{\geq\lambda}$, $\mathbf{K}_{<\lambda}$, or more generally $\mathbf{K}_\Theta$, where Θ is a class of cardinals.

For an abstract class $\mathbf{K}$, we denote by $\mathbb{I}(\mathbf{K})$ the number of models in $\mathbf{K}$ up to isomorphism (i.e. the cardinality of $\mathbf{K}/_{\cong}$). We write $\mathbb{I}(\mathbf{K}, \lambda)$ instead of $\mathbb{I}(\mathbf{K}_\lambda)$. When $\mathbb{I}(\mathbf{K}) = 1$, we say that $\mathbf{K}$ is *categorical*. We say that $\mathbf{K}$ is *categorical in* λ if $\mathbf{K}_\lambda$ is categorical, i.e. $\mathbb{I}(\mathbf{K}, \lambda) = 1$.

We say that $\mathbf{K}$ has *amalgamation* if for any $M_0 \leq_{\mathbf{K}} M_\ell$, $\ell = 1, 2$, there is $M_3 \in \mathbf{K}$ and $\mathbf{K}$-embeddings $f_\ell : M_\ell \xrightarrow[M_0]{} M_3$, $\ell = 1, 2$. $\mathbf{K}$ has *joint embedding* if any two models can be $\mathbf{K}$-embedded in a common model. $\mathbf{K}$ has *no maximal models* if for any $M \in \mathbf{K}$ there exists $N \in \mathbf{K}$ with $M \leq_{\mathbf{K}} N$ and $M \neq N$ (we write $M <_{\mathbf{K}} N$). Localized concepts such as *amalgamation in* λ mean that $\mathbf{K}_\lambda$ has amalgamation.

The definition of an abstract elementary class is due to Shelah [She87a]:

Definition 2.1. An *abstract elementary class (AEC)* is an abstract class $\mathbf{K}$ in a finitary vocabulary satisfying:

(1) Coherence: if $M_0, M_1, M_2 \in \mathbf{K}$, $M_0 \subseteq M_1 \leq_{\mathbf{K}} M_2$ and $M_0 \leq_{\mathbf{K}} M_2$, then $M_0 \leq_{\mathbf{K}} M_1$.
(2) Tarski-Vaught chain axioms: if $\langle M_i : i \in I \rangle$ is a $\leq_{\mathbf{K}}$-directed system and $M := \bigcup_{i \in I} M_i$, then:
  (a) $M \in \mathbf{K}$.
  (b) $M_i \leq_{\mathbf{K}} M$ for all $i \in I$.
  (c) If $N \in \mathbf{K}$ is such that $M_i \leq_{\mathbf{K}} N$ for all $i \in I$, then $M \leq_{\mathbf{K}} N$.
(3) Löwenheim-Skolem-Tarski axiom: there exists a cardinal $\lambda \geq |\tau(\mathbf{K})| + \aleph_0$ such that for any $N \in \mathbf{K}$ and any $A \subseteq |N|$, there exists $M \in \mathbf{K}$ with $M \leq_{\mathbf{K}} N$, $A \subseteq |M|$, and $\|M\| \leq |A| + \lambda$. We write $\text{LS}(\mathbf{K})$ for the least such λ.

2.2. **Types.** In any abstract class $\mathbf{K}$, we can define a semantic notion of type, called Galois or orbital types in the literature (such types were introduced by Shelah in [She87b]). For $M \in \mathbf{K}$, $A \subseteq |M|$, and $b \in M$, we write $\mathbf{tp}_{\mathbf{K}}(b/A; M)$ for the orbital type of b over A as computed in M (usually $\mathbf{K}$ will be clear from context and we will omit it from the notation). It is the finest notion of type respecting $\mathbf{K}$-embeddings, see [Vas16b, 2.16] for a formal definition. For $M \in \mathbf{K}$, we write $\mathbf{S}_{\mathbf{K}}(M) = \mathbf{S}(M)$ for $\{\mathbf{tp}(b/M; N) \mid M \leq_{\mathbf{K}} N\}$, the class[2] of all types over M. We define naturally what it means for a type to be realized inside a model, to extend another type, and to take the image of a type by a $\mathbf{K}$-embedding. We call a type p *algebraic* if it can be written as $p = \mathbf{tp}(a/M; N)$, with $a \in M$.

When $\mathbf{K}$ is an elementary class, $\mathbf{tp}(b/A; M)$ contains the same information as the usual notion of $\mathbb{L}_{\omega,\omega}$-syntactic type. In particular, types in an elementary class are determined by their restrictions to finite sets. This idea was abstracted in [GV06b] and made into the following definition: for χ an infinite cardinal, an abstract class $\mathbf{K}$ is $(< \chi)$*-tame* if for any $M \in \mathbf{K}$ and any distinct $p, q \in \mathbf{S}(M)$, there exists $A \subseteq |M|$ such that $|A| < \chi$ and $p \upharpoonright A \neq q \upharpoonright A$. We say that $\mathbf{K}$ is χ*-tame* if it is $(< \chi^+)$-tame. Thus elementary classes are $(< \aleph_0)$-tame, but there are examples of non-tame AECs, see e.g. [BV17a, 3.2.2].

2.3. **Weak amalgamation, intersections, and coherent sequences of types.** Weak amalgamation was first introduced in [Vas17b, 4.11]. It can be seen as a common weakening of amalgamation and having certain kinds of prime models.

Definition 2.2. Let $\mathbf{K}$ be an abstract class and let $M \in \mathbf{K}$. We say that M is a *weak amalgamation base* (in $\mathbf{K}$) if for any $N_1, N_2 \in \mathbf{K}$ with $M \leq_{\mathbf{K}} N_1$, $M \leq_{\mathbf{K}} N_2$, and any $a_1 \in N_1$, $a_2 \in N_2$, *if* $\mathbf{tp}(a_1/M; N_1) = \mathbf{tp}(a_2/M; N_2)$, *then* there exists N_1^0, N_2', f such that:

(1) $M \leq_{\mathbf{K}} N_1^0 \leq_{\mathbf{K}} N_1$.
(2) $a_1 \in N_1^0$.
(3) $N_2 \leq_{\mathbf{K}} N_2'$.
(4) $f : N_1^0 \xrightarrow[M]{} N_2'$.
(5) $f(a_1) = a_2$.

[2] If $\mathbf{K}$ is an AEC, $\mathbf{S}(M)$ will of course be a set.

We say that $\mathbf{K}$ has *weak amalgamation* if every object of $\mathbf{K}$ is a weak amalgamation base.

Note that amalgamation implies weak amalgamation (see below). Another example of abstract classes with weak amalgamation are those that have intersections:

Definition 2.3. An abstract class $\mathbf{K}$ *has intersections* if for any $N \in \mathbf{K}$ and any $A \subseteq |N|$, the set $\bigcap\{M \in \mathbf{K} \mid M \leq_{\mathbf{K}} N, A \subseteq |N|\}$ induces a $\mathbf{K}$-substructure of N. We write $\text{cl}^N(A)$ for this substructure.

Remark 2.4. By [BS08, 1.3] or [Vas17b, 2.18], in an abstract class with intersections, $\mathbf{tp}(a_1/M; N_1) = \mathbf{tp}(a_2/M; N_2)$ if and only if there exists an isomorphism $f : \text{cl}^{N_1}(a_1 M) \cong_M \text{cl}^{N_2}(a_2 M)$ such that $f(a_1) = a_2$. In particular, abstract classes with intersections have weak amalgamation.

The following characterizes when weak amalgamation implies amalgamation.

Fact 2.5 ([Vas17b, 4.14])**.** Let $\mathbf{K}$ be an AEC and let $\lambda \geq \text{LS}(\mathbf{K})$. The following are equivalent:

(1) $\mathbf{K}_\lambda$ has weak amalgamation and for any $M \leq_{\mathbf{K}} N$ both in $\mathbf{K}_\lambda$, any $p \in \mathbf{S}(M)$ can be extended to a type in $\mathbf{S}(N)$.
(2) $\mathbf{K}_\lambda$ has amalgamation.

The definitions below are well known in AECs with amalgamation, see [Bal09, §11].

Definition 2.6. Let $\mathbf{K}$ be an abstract class, $\bar{M} = \langle M_i : i < \alpha \rangle$ be increasing continuous, and let $\bar{p} = \langle p_i : i < \alpha \rangle$ be an increasing chain of types with $p_i \in \mathbf{S}(M_i)$ for all $i < \alpha$.

(1) We say that $\bar{p}$ is *local* if for any limit $i < \alpha$ and any $q \in \mathbf{S}(M_i)$, if $q \upharpoonright M_j = p_j$ for all $j < i$, then $q = p_i$. We say that $\bar{M}$ is *local* if any increasing chain of types $\bar{p}$ as above is local.
(2) We say that $\bar{p}$ is *coherent* if there exists $\langle N_i : i < \alpha \rangle$ increasing continuous, $\langle a_i : i < \alpha \rangle$, and $\langle f_i : i + 1 < \alpha \rangle$ such that for all $i < \alpha$:
  (a) $M_i \leq_{\mathbf{K}} N_i$.
  (b) $a_i \in |N_i|$.
  (c) $p_i = \mathbf{tp}(a_i/M_i; N_i)$.
  (d) $f_i : N_i \xrightarrow[M_i]{} N_{i+1}$ if $i + 1 < \alpha$.
  (e) $f_i(a_i) = a_{i+1}$ if $i + 1 < \alpha$.

  We say that $\bar{M}$ is *coherent* if any $\bar{p}$ as above is coherent. Finally, we say that $\mathbf{K}$ *has coherent sequences* if any $\bar{M}$ as above is coherent.

The following is immediate from the definitions (take a directed colimit). See [Bal09, 11.5].

Fact 2.7. Let $\mathbf{K}$ be an AEC. Let δ be a limit ordinal, let $\bar{M} = \langle M_i : i < \delta \rangle$ be increasing continuous in $\mathbf{K}$, and let $\bar{p} = \langle p_i : i < \delta \rangle$ be an increasing chain of types with $p_i \in \mathbf{S}(M_i)$ for all $i < \delta$. If $\bar{p}$ is local and coherent, then there exists $q \in \mathbf{S}(\bigcup_{i<\delta} M_i)$ so that $q \upharpoonright M_i = q_i$ for all $i < \delta$.

We finish by proving some easy results about building coherent sequences of types: it can be done assuming amalgamation or assuming the AEC has intersections[3].

Lemma 2.8. Let $\mathbf{K}$ be an AEC. Let δ be a limit ordinal, let $\bar{M} = \langle M_i : i < \delta \rangle$ be increasing continuous in $\mathbf{K}$, and let $\bar{p} = \langle p_i : i < \delta \rangle$ be an increasing chain of types with $p_i \in \mathbf{S}(M_i)$ for all $i < \delta$. Assume that $\bar{p}$ is local.

(1) If $\mathbf{K}_{<\sup_{i<\delta}(\|M_i\|+\text{LS}(\mathbf{K}))^+}$ has amalgamation, then $\bar{p}$ is coherent.
(2) If $\mathbf{K}$ has intersections, then $\bar{p}$ is coherent.

Proof.

(1) This is well known. See the proof of [Bal09, 11.5].
(2) We build the coherence witnesses $\langle N_i : i < \alpha \rangle$, $\langle a_i : i < \alpha \rangle$, $\langle f_i : i + 1 < \alpha \rangle$ inductively such that for all $i < \alpha$, $N_i = \text{cl}^{N_i}(M_i a_i)$. The base case is trivial, and at limits we can take directed colimits (and use locality to see type equality is preserved; the closure condition will be preserved by [Vas17b, 2.14(6)]). At successors, we are given N_i and a_i and want to build N_{i+1}, a_{i+1}, and f_i. Pick N'_{i+1} and a_{i+1} such that $p_{i+1} = \mathbf{tp}(a_{i+1}/M_{i+1}; N'_{i+1})$. Then $p_{i+1} \upharpoonright M_i = p_i$, so there exists an isomorphism $f_i : N_i \cong \text{cl}^{N'_{i+1}}(M_i a_{i+1})$ such that $f(a_i) = a_{i+1}$. Let $N_{i+1} := \text{cl}^{N'_{i+1}}(M_{i+1} a_{i+1})$. By the coherence axiom of AECs, f_i is a $\mathbf{K}$-embedding of N_i into N_{i+1}, as desired.

$\square$

3. Existence from successive categoricity below a limit

In this section, we work in an arbitrary AEC (i.e. we do not assume tameness or weak amalgamation) and prove some relatively easy results about deriving no maximal models from categoricity below a limit.

The following easy observation will be used in the later sections:

Lemma 3.1. Let $\mathbf{K}$ be an AEC and let $\mu > \text{LS}(\mathbf{K})$ be a limit cardinal of countable cofinality. If $\mathbf{K}$ is categorical in unboundedly-many cardinals below μ, then $\mathbf{K}_\mu \neq \emptyset$.

Proof. Since μ has countable cofinality, we can pick $\langle \mu_i : i < \omega \rangle$ increasing cofinal in μ such that $\mu_0 \geq \text{LS}(\mathbf{K})$ and μ_i is a categoricity cardinal for each $i < \omega$. Now we build an increasing chain $\langle M_i : i < \omega \rangle$ such that $M_i \in \mathbf{K}_{\mu_i}$ for all $i < \omega$. Then the union of the chain will be in $\mathbf{K}_\mu$. For $i = 0$, take any $M_0 \in \mathbf{K}_{\mu_0}$. For $i = j+1$, given M_j, first pick any $N_i \in \mathbf{K}_{\mu_i}$. Pick $N_i^0 \leq_{\mathbf{K}} N_i$ with $N_i^0 \in \mathbf{K}_{\mu_j}$. By categoricity, there is an isomorphism $f : N_i^0 \cong M_j$. With some renaming, we can extend this isomorphism to $g : N_i \cong M_i$, for some M_i with $M_j \leq_{\mathbf{K}} M_i$. Then $M_i \in \mathbf{K}_{\mu_i}$, as desired. $\square$

The next two definitions are due to Shelah [She01].

[3]For the proof below, we assume familiarity with the basic properties of AECs admitting intersections, see [Vas17b, §2].

Definition 3.2. An *existence triple* in an abstract class $\mathbf{K}$ is a triple (a, M, N) with $M \leq_{\mathbf{K}} N$ both in $\mathbf{K}$ and $a \in N \backslash M$.

Definition 3.3. We say an abstract class $\mathbf{K}$ has *weak extension* if for any existence triple in (a, M, N) in $\mathbf{K}$, there exists an existence triple (b, M', N') in $\mathbf{K}$ with $a = b$, $M <_{\mathbf{K}} M'$, and $N <_{\mathbf{K}} N'$. We say that (b, M', N') is a *strict extension* (a, M, N).

The next two results are essentially due to Shelah, see [She09a, §VI.1]. We give full proofs here because they are short and (for the second one) slightly simpler than Shelah's.

Lemma 3.4. Let $\mathbf{K}$ be an AEC and let $\mu > \mathrm{LS}(\mathbf{K})$. If:

(1) There exists an existence triple in $\mathbf{K}_{\mathrm{LS}(\mathbf{K})}$.
(2) For every $\lambda \in [\mathrm{LS}(\mathbf{K}), \mu)$, $\mathbf{K}_\lambda$ has weak extension.

Then not every element of $\mathbf{K}_\mu$ is maximal.

Proof. Pick an existence triple (a, M, N) in $\mathbf{K}_{\mathrm{LS}(\mathbf{K})}$. Now build $\langle M_i : i \leq \mu \rangle$, $\langle N_i : i \leq \mu \rangle$ both increasing continuous such that for every $i < \mu$:

(1) $M_0 = M$, $N_0 = N$.
(2) $M_i, N_i \in \mathbf{K}_{|i|+\mathrm{LS}(\mathbf{K})}$.
(3) $M_i \leq_{\mathbf{K}} N_i$.
(4) $a \in N_i \backslash M_i$.
(5) $M_i <_{\mathbf{K}} M_i$.

This is possible using the weak extension property at successor stages and taking unions at limit stages. In the end, $M_\mu <_{\mathbf{K}} N_\mu$, since $a \in N_\mu \backslash M_\mu$. Thus M_μ is not maximal. Since $\langle M_i : i \leq \mu \rangle$ was strictly increasing, $M_\mu \in \mathbf{K}_\mu$. $\square$

Lemma 3.5. Let $\mathbf{K}$ be an AEC. If $\mathbf{K}$ is categorical in $\mathrm{LS}(\mathbf{K})$ and $\mathbf{K}_{\mathrm{LS}(\mathbf{K})^+}$ has no maximal models, then $\mathbf{K}_{\mathrm{LS}(\mathbf{K})}$ has weak extension.

Proof. Let $\lambda := \mathrm{LS}(\mathbf{K})$. Let (a, M, N) be an existence triple in $\mathbf{K}_\lambda$. We want to find a strict extension of (a, M, N). We build $\langle M_i : i \leq \lambda^+ \rangle$ increasing continuous and $\langle a_i : i < \lambda^+ \rangle$ such that for all $i < \lambda^+$:

(1) (a_i, M_i, M_{i+1}) is an existence triple in $\mathbf{K}_\lambda$.
(2) There exists an isomorphism $f_i : N \cong M_{i+1}$ so that $f_i[M] = M_i$ and $f_i(a) = a_i$.

This is possible by categoricity. This is enough: since $\mathbf{K}_{\lambda^+}$ has no maximal models, there exists $M' \in \mathbf{K}_{\lambda^+}$ with $M_{\lambda^+} <_{\mathbf{K}} M'$. Let $\langle M_i' : i \leq \lambda^+ \rangle$ be an increasing continuous resolution of M'. Let $C \subseteq \lambda^+$ be a club such that $i \in C$ implies $M_i \leq_{\mathbf{K}} M_i'$ and moreover $M_{\lambda^+} \cap M_i' = M_i$. Pick $b \in M' \backslash M_{\lambda^+}$ and pick $i \in C$ so that $b \in M_i'$. Then $M_i <_{\mathbf{K}} M_i'$ and one can pick $j < \lambda^+$ so that $M_{i+1} <_{\mathbf{K}} M_j'$. Then (a_i, M_i', M_j') is a strict extension of (a_i, M_i, M_{i+1}). Taking an isomorphic copy, we obtain a strict extension of the original triple (a, M, N). $\square$

Putting the two lemmas together, we obtain:

Theorem 3.6. Let $\mathbf{K}$ be an AEC and let $\mu > \mathrm{LS}(\mathbf{K})$ be a limit cardinal. If $\mathbf{K}$ is categorical in every cardinal in $[\mathrm{LS}(\mathbf{K}), \mu)$, then not every element of $\mathbf{K}_\mu$ is maximal.

Proof. Note that for every $\lambda \in [\mathrm{LS}(\mathbf{K}), \mu)$, both $\mathbf{K}_\lambda$ and $\mathbf{K}_{\lambda^+}$ are not empty and have no maximal models (we are using that μ is limit to deduce this for $\mathbf{K}_{\lambda^+}$). In particular, there is an existence triple in $\mathbf{K}_{\mathrm{LS}(\mathbf{K})}$. Moreover by Lemma 3.5 (applied to $\mathbf{K}_{\geq\lambda}$), for any $\lambda \in [\mathrm{LS}(\mathbf{K}), \mu)$, $\mathbf{K}_\lambda$ has the weak extension property. By Lemma 3.4, we get the result. $\square$

Corollary 3.7. Let $\mathbf{K}$ be an AEC and let $\mu > \mathrm{LS}(\mathbf{K})$ be a limit cardinal. If $\mathbf{K}$ is categorical in every cardinal in $[\mathrm{LS}(\mathbf{K}), \mu]$, then $\mathbf{K}_{\mu^+} \neq \emptyset$.

Proof. By Theorem 3.6, not every model in $\mathbf{K}_\mu$ is maximal. By categoricity in μ, this implies that $\mathbf{K}_\mu$ has no maximal models, hence that $\mathbf{K}_{\mu^+} \neq \emptyset$. $\square$

4. Local saturation and splitting

The present section is the core of the paper. We adapt known results about saturated models and splitting to setups without amalgamation (but often with weak amalgamation and/or tameness). For several results, categoricity is not needed, but in the end we will use it to put everything together.

The following are localized definitions of the well known variations of saturation:

Definition 4.1. Let $\mathbf{K}$ be an AEC. For θ an infinite cardinal, a model M is *locally θ-saturated* if for any $N \geq_{\mathbf{K}} M$ and any $A \subseteq |M|$ with $|A| < \theta$, we have that $\mathbf{S}(A; N) = \mathbf{S}(A; M)$. For $M_0 \leq_{\mathbf{K}} M$, we say that M is *locally θ-universal over M_0* if for any $M_0 \leq_{\mathbf{K}} N_0 \leq_{\mathbf{K}} N$ with $M \leq_{\mathbf{K}} N$ and $\|N_0\| < \theta$, N_0 embeds into M over M_0. When $\theta = \|M_0\|^+$, we omit it. M is *locally θ-model-homogeneous* if it is θ-universal over M_0 for any $M_0 \leq_{\mathbf{K}} M$ with $\|M_0\| < \theta$. We say that M is *locally saturated* when it is locally $\|M\|$-saturated, and similarly for locally model-homogeneous.

The usual exhaustion argument shows that locally model-homogeneous model can be built assuming some cardinal arithmetic. See for example the proof of Theorem 1 in [Ros97].

Fact 4.2. Let $\mathbf{K}$ be an AEC. For any $M \in \mathbf{K}$ and any regular cardinal $\theta > \mathrm{LS}(\mathbf{K})$, there exists $N \in \mathbf{K}$ such that $M \leq_{\mathbf{K}} N$, N is locally θ-model-homogeneous, and $\|N\| \leq \|M\|^{<\theta}$.

Assuming categoricity in a suitable unbounded set, we can build locally model-homogeneous models.

Lemma 4.3. Let $\mathbf{K}$ be an AEC and let $\mu > \mathrm{LS}(\mathbf{K})$ be a strong limit cardinal. If for every $\theta < \mu$ there exists $\lambda < \mu$ such that $\lambda = \lambda^\theta$ and $\mathbf{K}$ is categorical in λ, then every object of $\mathbf{K}_\mu$ is locally model-homogeneous.

Proof. Let $M \in \mathbf{K}_\mu$. Fix $\theta < \mu$, $N \in \mathbf{K}_\mu$ with $M \leq_{\mathbf{K}} N$, and $M_0, N_0 \in \mathbf{K}_{\leq\theta}$ with $M_0 \leq_{\mathbf{K}} M$, $M_0 \leq_{\mathbf{K}} N_0 \leq_{\mathbf{K}} N$. We have to see that N_0 embeds inside M over M_0. Without loss of generality, $\theta \geq \mathrm{LS}(\mathbf{K})$. By assumption, we can pick a categoricity cardinal $\lambda < \mu$ such that $\lambda = \lambda^\theta$. Let $M_0' \leq_{\mathbf{K}} M$ be such that $M_0 \leq_{\mathbf{K}} M_0'$ and $M_0' \in \mathbf{K}_\lambda$. By categoricity in λ and Fact 4.2, M_0' is locally θ^+-model-homogeneous. In particular, N_0 embeds into M_0' over M_0. Thus N_0 embeds into M over M_0, as desired. □

The following notion was introduced by Shelah [She99, 3.2] for AECs with amalgamation.

Definition 4.4 (Splitting)**.** Let $\mathbf{K}$ be an AEC, let $N \in \mathbf{K}$, $A \subseteq |N|$, let $p \in \mathbf{S}(N)$, and let θ be an infinite cardinal with $|A| < \theta$. We say that p $(< \theta)$*-splits over* A if there exists $M_1, M_2 \in \mathbf{K}_{<\theta}$ such that $A \subseteq M_\ell \leq_{\mathbf{K}} N$ for $\ell = 1, 2$ and $f : M_1 \cong_A M_2$ so that $f(p \upharpoonright M_1) \neq p \upharpoonright M_2$. We say that p θ*-splits over* A if it $(< \theta^+)$-splits over A. We say that p *splits over* A if it $(|A| + \aleph_0)$-splits over A.

Remark 4.5. Let $N \in \mathbf{K}$, $A \subseteq |N|$, let $p \in \mathbf{S}(N)$ and let θ be an infinite cardinal with $|A| < \theta$. If p $(< \theta)$-splits over A, then there exists $N_0 \leq_{\mathbf{K}} N$ with $A \subseteq |N_0|$ and $\|N_0\| < \theta$ such that $p \upharpoonright N_0$ $(< \theta)$-splits over A.

The following is a generalization of [Vasc, 4.12].

Lemma 4.6 (Local character)**.** Let $\mathbf{K}$ be an AEC, let $M \in \mathbf{K}_{\geq \aleph_0}$, let δ be a regular cardinal, and let $\langle A_i : i \leq \delta \rangle$ be an increasing continuous chain of sets with $A_\delta = |M|$. Let θ be either $\|M\|^+$ or $\sup_{i<\delta} |A_i|^+$. Let $p \in \mathbf{S}(M)$. If there exists a regular $\chi \leq \|M\|$ such that:

(1) $\mathbf{K}_{<\theta}$ is $(< \chi)$-tame.
(2) M is locally $(\chi + \delta^+)$-saturated.

Then there exists $i < \delta$ such that p does not $(< \theta)$-split over A_i.

Proof. Suppose not. Then for each $i < \delta$, there exists $M_i^1, M_i^2 \in \mathbf{K}_{<\theta}$ and $f_i : M_i^1 \cong_{A_i} M_i^2$ so that $A_i \subseteq M_i^\ell \leq_{\mathbf{K}} M$, $\ell = 1, 2$ and $f_i(p \upharpoonright M_i^1) \neq p \upharpoonright M_i^2$.

By $(< \chi)$-tameness, we can find $A_i^1 \subseteq |M_i^1|$ of cardinality strictly less than χ so that $f_i(p \upharpoonright A_i^1) \neq p \upharpoonright A_i^2$ (we have set $A_i^2 := f_i[A_i^1]$).

Let $A := \bigcup_{i<\delta}(A_i^1 \cup A_i^2)$. Recall that p is realized in an extension of M. Moreover, if $\delta < \chi$ then $|A| < \chi$ and if $\delta \geq \chi$ then $|A| \leq \delta$. In either case, M is locally $|A|^+$-saturated, so $p \upharpoonright A$ is realized in M, say by a. Since δ is a limit ordinal, there exists $i < \delta$ such that $a \in A_i$. Now, fix an extension $g_i : M \cong M'$ of f_i (so $M_i^2 \leq_{\mathbf{K}} M'$). We have:

$$f_i(p \upharpoonright A_i^1) = g_i(\mathbf{tp}(a/A_i^1; M)) = \mathbf{tp}(g_i(a)/A_i^2; M') = \mathbf{tp}(a/A_i^2; M')$$

Where we have used that g_i fixes A_i, so $g_i(a) = a$. On the other hand, since $A_i^2 \subseteq A$,

$$p \upharpoonright A_i^2 = \mathbf{tp}(a/A_i^2; M)$$

Finally, observe that $M_i^2 \leq_{\mathbf{K}} M$ and $M_i^2 \leq_{\mathbf{K}} M'$ by construction. Therefore since $a \in A_i \subseteq M_i^2$, $\mathbf{tp}(a/A_i^2; M') = \mathbf{tp}(a/A_i^2; M_i^2) = \mathbf{tp}(a/A_i^2; M)$. We have shown that $f_i(p \upharpoonright A_i^1) = p \upharpoonright A_i^2$, contradicting the definition of A_i^1, A_i^2, and f_i. □

We now generalize the weak uniqueness and extension properties of splitting first isolated by VanDieren [Van06, I.4.10].

Lemma 4.7 (Weak uniqueness)**.** Let $M_0 \leq_{\mathbf{K}} M_1 \leq_{\mathbf{K}} N$ all be in $\mathbf{K}_{\geq \mathrm{LS}(\mathbf{K})}$. Let $p, q \in \mathbf{S}(N)$. If M_1 is locally universal over M_0, p and q both do not split over M_0, and $p \upharpoonright M_1 = q \upharpoonright M_1$, then $p \upharpoonright A = q \upharpoonright A$ for any $A \subseteq |N|$ with $|A| \leq \|M_0\|$.

Proof. Fix $A \subseteq |N|$ with $|A| \leq \|M_0\|$. Pick $N_0 \leq_{\mathbf{K}} N$ with $A \subseteq |N_0|$ and $\|N_0\| \leq \|M_0\|$. Since M_1 is locally universal over M_0, there exists $f : N_0 \xrightarrow[M_0]{} M_1$. By the definition of nonsplitting (where M_1, M_2 there stand for N_0, $f[N_0]$ here), we must have that $f(p \upharpoonright N_0) = p \upharpoonright f[N_0]$ and $f(q \upharpoonright N_0) = q \upharpoonright f[N_0]$. Since $p \upharpoonright M_1 = q \upharpoonright M_1$, $f \upharpoonright f[N_0] = q \upharpoonright f[N_0]$. Therefore $f(p \upharpoonright N_0) = f(q \upharpoonright N_0)$, hence $p \upharpoonright N_0 = q \upharpoonright N_0$. □

Lemma 4.8 (Weak extension)**.** Let $M_0 \leq_{\mathbf{K}} M_1 \leq_{\mathbf{K}} M$ all be in $\mathbf{K}_{\geq \mathrm{LS}(\mathbf{K})}$ and let $p \in \mathbf{S}(M)$. Let $N_0 \in \mathbf{K}$ be such that $M_1 \leq_{\mathbf{K}} N_0$. If:

(1) p does not split over M_0.
(2) N_0 embeds into M over M_1.

Then there exists $q_0 \in \mathbf{S}(N_0)$ such that q_0 extends $p \upharpoonright M_1$ and q_0 does not split over M_0. Moreover, if p is not algebraic then q_0 can be taken to be nonalgebraic.

Proof. Use the hypothesis to fix $f : N_0 \xrightarrow[M_1]{} M$. Let $q_0 := f^{-1}(p \upharpoonright f[N_0])$. By invariance, q_0 does not split over M_0 and $q_0 \upharpoonright M_1 = p \upharpoonright M_1$. To see the moreover part, assume that q_0 is algebraic. Then $p \upharpoonright f[N_0]$ is algebraic, hence p is algebraic. □

We now work toward improving the statement of weak extension. Roughly, we would like to prove that if M is model-homogeneous and N is an extension of M of the same cardinality, then types over M have nonsplitting extensions over N. This is not immediate from Lemma 4.8 because without amalgamation we do not know whether N embeds into M. Instead, we will first build small approximations of the type we want to build, and then use tameness and existence of coherent sequences of types to take the limit of these approximations. The following technical definition will be key (it can be seen as a replacement for the notion of a limit model in this context):

Definition 4.9. Let $\mathbf{K}$ be an AEC. We call $N \in \mathbf{K}$ *nicely resolvable over* M if there exists a limit ordinal δ and an increasing continuous chain $\bar{N} = \langle N_i : i < \delta \rangle$ such that:

(1) $N = \bigcup_{i<\delta} N_i$, $N_0 = M$.
(2) $\|N_i\| < \|N\|$ for all $i < \delta$.
(3) N_{i+1} is locally universal over N_i for all $i < \delta$.
(4) $\bar{N}$ is local and coherent (Definition 2.6).
(5) For any $p \in \mathbf{S}(\bigcup_{i<\delta} N_i)$, there exists $i < \delta$ so that p does not split over N_i.

We call $\langle N_i : i < \delta \rangle$ a *nice resolution of* N *(over* M*)*.

From categoricity in a suitable unbounded set of cardinals below a strong limit of countable cofinality μ, we can get nice resolutions. Note that Lemma 3.1 tells us that from the hypotheses of Lemma 4.10 there will be a model in $\mathbf{K}_\mu$.

Lemma 4.10. Let $\mathbf{K}$ be an AEC and let $\mu > \mathrm{LS}(\mathbf{K})$ be a strong limit cardinal of countable cofinality. If:

(1) For any $\theta < \mu$ there exists $\lambda < \mu$ such that $\lambda = \lambda^\theta$ and $\mathbf{K}$ is categorical in λ.
(2) $\mathbf{K}_{<\mu}$ is $\mathrm{LS}(\mathbf{K})$-tame.
(3) $\mathbf{K}_{<\mu}$ has coherent sequences.

Then for any $N \in \mathbf{K}_\mu$ and any $M \in \mathbf{K}_{[\mathrm{LS}(\mathbf{K}),\mu)}$ with $M \leq_{\mathbf{K}} N$, N is nicely resolvable over M.

Proof. Let $N \in \mathbf{K}_\mu$ and let $M \in \mathbf{K}_{[\mathrm{LS}(\mathbf{K}),\mu)}$ with $M \leq_{\mathbf{K}} N$. Pick an increasing sequence $\langle \mu_i : i < \omega \rangle$ cofinal in μ with $\|M\| \leq \mu_0$ and so that for any $i < \omega$, $\mu_{i+1}^{\mu_i} = \mu_{i+1}$ and $\mathbf{K}$ is categorical in μ_i. This is possible by assumption. By increasing M if needed, we can assume without loss of generality that $M \in \mathbf{K}_{\mu_0}$. Now pick any increasing sequence $\langle N_i : i < \omega \rangle$ such that $N_0 = M$, $\bigcup_{i<\omega} N_i = N$, and $N_i \in \mathbf{K}_{\mu_i}$ for all $i < \omega$. We claim this is a nice resolution of N over M.

As in the proof of Lemma 4.3, for any $i < \omega$, N_{i+1} is locally universal over N_i. Also, $\bar{N} = \langle N_i : i < \omega \rangle$ is trivially local since there are no limit ordinals below ω. Furthermore, $\bar{N}$ is coherent because by assumption $\mathbf{K}_{<\mu}$ is coherent. Finally, pick $p \in \mathbf{S}(N)$. Note that by Lemma 4.3, N is locally model-homogeneous, hence locally saturated. Thus Lemma 4.6 (where δ, χ there stands for $\omega, \mathrm{LS}(\mathbf{K})^+$ here) implies there exists $i < \omega$ so that p does not split over N_i. $\square$

From the existence of nice resolutions, we can now prove the desired extension property:

Lemma 4.11 (Extension)**.** Let $M_0 \leq_{\mathbf{K}} M_1 \leq_{\mathbf{K}} M \leq_{\mathbf{K}} N$ all be in $\mathbf{K}_{\geq \mathrm{LS}(\mathbf{K})}$. Let $p \in \mathbf{S}(M)$. If:

(1) p does not split over M_0.
(2) M_1 is locally universal over M_0.
(3) M is $(\|M\|)$-locally universal over M_1.
(4) $\|N\| = \|M\|$.
(5) N is nicely resolvable over M_1.
(6) $\mathbf{K}_{\leq \|N\|}$ is $\|M_0\|$-tame.

Then there exists $q \in \mathbf{S}(N)$ which extends p and does not split over M_0. Moreover, q is not algebraic if p is not algebraic.

Proof. Fix $\bar{N} = \langle N_i : i < \delta \rangle$ a nice resolution of N over M_1. For $i < \delta$, let $q_i \in \mathbf{S}(N_i)$ be as given by weak extension: it extends $p \upharpoonright M_1$ and does not split over M_0. Moreover it is nonalgebraic if p is nonalgebraic. By weak uniqueness (and tameness), $q_j \upharpoonright N_i = q_i$ for $i < j < \delta$. By Fact 2.7, there exists $q \in \mathbf{S}(N)$ such that $q \upharpoonright N_i = q_i$ for all $i < \delta$. Note that q cannot be algebraic if all the q_i's are nonalgebraic. By the properties of $\bar{N}$, there exists $i < \delta$ so that q does not split over N_i.

<u>Claim</u>: q does not split over M_0.

<u>Proof of Claim:</u> By Remark 4.5, it suffices to see that $q \upharpoonright N_0'$ does not split over M_0 for any $N_0' \in \mathbf{K}_{<\|N\|}$ with $N_{i+1} \leq_{\mathbf{K}} N_0' \leq_{\mathbf{K}} N$. So let $q_0' \in \mathbf{S}(N_0')$ be as given by

weak extension (extending $p \upharpoonright M_1$ and not splitting over M_0). By weak uniqueness, $q_0' \upharpoonright N_{i+1} = q_{i+1}$. Since q does not split over N_i, $q \upharpoonright N_0'$ does not split over N_i. By monotonicity, also q_0' does not split over N_i. By weak uniqueness again (recalling that by definition N_{i+1} is locally universal over N_i), $q_0' = q \upharpoonright N_0'$. In particular, $q \upharpoonright N_0'$ does not split over M_0. $\dagger_{\text{Claim}}$

Now since $q \upharpoonright M_1 = p \upharpoonright M_1$, weak uniqueness and tameness imply that $q \upharpoonright M = p$, as desired. $\square$

5. Good frames and the main theorem

We use the tools of the previous section to build a good frame, a local forking-like notion. We assume some familiarity with good frames (see [She09a, Chapter 2]) here. We will use the definition and notation from [Vas16a, §2.4] (which we do not repeat here). Recall that a *good*$^{-S}$ λ-frame is a good frame, except that it may fail the symmetry property. The following key result tells us that good frame can be transferred up in tame AECs with weak amalgamation:

Fact 5.1. Let $\mathfrak{s}$ be a good^{-S} λ-frame on an AEC $\mathbf{K}$. If $\mathbf{K}$ is λ-tame and has weak amalgamation, then $\mathfrak{s}$ is a good λ-frame and extends to a good $[\lambda, \infty)$-frame on all of $\mathbf{K}_{\geq \lambda}$. In particular, $\mathbf{K}_{\geq \lambda}$ has amalgamation and arbitrarily large models.

Proof. By [Vas17b, 4.16], $\mathbf{K}_{\geq \lambda}$ has amalgamation. By [Bon14], $\mathfrak{s}$ extends to a good^{-S} $[\lambda, \infty)$-frame $\mathfrak{t}$ on $\mathbf{K}$. By [Vas16a, 6.14], $\mathfrak{t}$, and hence $\mathfrak{s}$, also has symmetry[4]. The "in particular" part follows from the definition of a good $[\lambda, \infty)$-frame. $\square$

For $\mathbf{K}$ an AEC and $M \in \mathbf{K}$, we let $\mathbf{K}_M$ denote the AEC obtained by adding constant symbols for M (see [Vas17b, 2.20] for the precise definition). Roughly speaking, it is the AEC of models above M.

Lemma 5.2 (Main lemma)**.** Let $\mathbf{K}$ be an AEC and let $\mu > \text{LS}(\mathbf{K})$ be a strong limit cardinal of countable cofinality. *If*:

(1) $\mathbf{K}_\mu$ has weak amalgamation.
(2) $\mathbf{K}_{<\mu}$ has coherent sequences.
(3) $\mathbf{K}_{\leq \mu}$ is $\text{LS}(\mathbf{K})$-tame.
(4) For every $\theta < \mu$, there exists $\lambda < \mu$ such that $\lambda = \lambda^\theta$ and $\mathbf{K}$ is categorical in λ.

Then for any non-maximal $M \in \mathbf{K}_\mu$, $\mathbf{K}_M$ has a type-full good^{-S} μ-frame.

Proof. For $N_1 \leq_{\mathbf{K}} N_2$ both in $\mathbf{K}_\mu$ and $p \in \mathbf{S}(N_2)$, we say that *p does not fork over N_1* if there exists $N_1^0 \in \mathbf{K}_{<\mu}$ with $N_1^0 \leq_{\mathbf{K}} N_1$ such that p does not split over N_1^0. We prove several claims:

- Nonforking is invariant under isomorphisms, and if $N_1 \leq_{\mathbf{K}} N_2 \leq_{\mathbf{K}} N_3$ are all in $\mathbf{K}_\mu$ and $p \in \mathbf{S}(N_3)$ does not fork over N_1, then $p \upharpoonright N_2$ does not fork over N_1 and p does not fork over N_2. This is immediate from the definition and the basic properties of splitting.

[4] This will not be used in the present paper.

- If $N_1 \leq_{\mathbf{K}} N_2$ are both in $\mathbf{K}_\mu$, $p, q \in \mathbf{S}(N_2)$ both do not fork over N_1 and $p \upharpoonright N_1 = q \upharpoonright N_1$, then $p = q$. To see this use the Löwenheim-Skolem axiom, fix $N_1^0 \leq_{\mathbf{K}} N_1$ such that $N_1^0 \in \mathbf{K}_{<\mu}$ and both p and q do not split over N_1^0. By Lemma 4.3, all the models in $\mathbf{K}_\mu$ are locally model-homogeneous, so in particular N_1 is locally universal over N_1^0, so by tameness and weak uniqueness (Lemma 4.7), $p = q$.
- If $N \in \mathbf{K}_\mu$ and $p \in \mathbf{S}(N)$, then p does not fork over N. This follows directly from the definition of nice resolvability and Lemma 4.10.
- If $N_1 \leq_{\mathbf{K}} N_2$ are both in $\mathbf{K}_\mu$ and $p \in \mathbf{S}(N_1)$, then there exists $q \in \mathbf{S}(N_2)$ such that q extends p and q does not fork over N_1. Moreover, q can be taken to be nonalgebraic if p is nonalgebraic. To see this, first note that we have observed previously that p does not fork over N_1, hence there is $M_0 \in \mathbf{K}_{[\mathrm{LS}(\mathbf{K}), \mu)}$ such that $M_0 \leq_{\mathbf{K}} N_1$ and p does not split over M_0. Now we can pick $M_1 \leq_{\mathbf{K}} N_1$ with $M_0 \leq_{\mathbf{K}} M_1$, $\|M_0\| < \|M_1\| < \|N_1\|$, and $\|M_1\|^{\|M_0\|} = \|M_1\|$, and $\mathbf{K}$ is categorical in $\|M_1\|$. Then it follows that M_1 is locally universal over M_0 (see Fact 4.2). Also, Lemma 4.10 implies that N_1 is nicely resolvable over M_1. Now apply Lemma 4.11, where M, N there stand for N_1, N_2 here.
- If $\delta < \mu^+$ is a limit ordinal, $\langle N_i : i \leq \delta \rangle$ is increasing continuous in $\mathbf{K}_\mu$, and $p \in \mathbf{S}(N_\delta)$, then there exists $i < \delta$ such that p does not fork over N_i. Indeed, suppose not. Assume without loss of generality that δ is regular. In particular, $\delta < \mu$ and so $\delta^+ < \mu$. Let $\theta := \mathrm{LS}(\mathbf{K})^+ + \delta^+$. Pick $\lambda < \mu$ such that $\mathbf{K}$ is categorical in λ and $\lambda^\theta = \lambda$. By Fact 4.2, the model in $\mathbf{K}_\lambda$ is locally θ^+-model-homogeneous. Build $\langle N_i^0 : i < \delta \rangle$ and $\langle N_i^1 : i < \delta \rangle$ increasing in $\mathbf{K}_\lambda$ such that for all $i < \delta$, $N_i^0 \leq_{\mathbf{K}} N_i$, $N_i^0 \leq_{\mathbf{K}} N_i^1 \leq_{\mathbf{K}} N_\delta$, $N_i^1 \cap N_i \subseteq N_{i+1}^0$, $p \upharpoonright N_i^1$ splits over N_i^0. This is possible (see [Vas16a, 4.11] for a very similar construction). At the end, let $N_\delta^\ell := \bigcup_{i<\delta} N_i^\ell$. Observe that $N_\delta^1 = N_\delta^0$ so by Lemma 4.6, there exists $i < \delta$ so that $p \upharpoonright N_\delta^1$ does not split over N_i^0. This is a contradiction, since we assumed that $p \upharpoonright N_i^1$ splits over N_i^0.
- For any $N \in \mathbf{K}_\mu$, $|\mathbf{S}(N)| \leq \mu$. Indeed, if $\langle p_i : i < \mu^+ \rangle$ are types in $\mathbf{S}(N)$, we can first use Lemma 4.10 to find a nice resolution $\langle N_j : j < \omega \rangle$ of N. In particular, for all $i < \mu^+$ there exists $j_i < \omega$ with p not splitting over N_{j_i}. By the usual pruning argument (using that μ is strong limit to see that $|\mathbf{S}(N_j)| < \mu$ for each $j < \mu$), there exists $j < \omega$ and an unbounded subset $X \subseteq \mu^+$ such that for $i_1, i_2 \in X$, p_{i_1}, p_{i_2} both do not split over N_j and have the same restriction to N_{j+1}. By weak uniqueness, $p_{i_1} = p_{i_2}$. This shows that $|\mathbf{S}(N)| \leq \mu$.

Now fix a non-maximal $M \in \mathbf{K}_\mu$. We identify any $N \in \mathbf{K}_\mu$ such that $M \leq_{\mathbf{K}} N$ with the corresponding member of $\mathbf{K}_M$ (this will not yield to confusion). We define nonforking in the frame using the nonforking relation defined above. The basic types are the nonalgebraic types. We have just established that invariance, monotonicity, uniqueness, extension, and local character hold in the frame. Therefore transitivity and continuity automatically follow (see [She09a, II.2.17,II.2.18]). Also note that $\mathbf{K}_M$ is not empty (it contains a copy of M) and has no maximal models of cardinality μ: given $M \leq_{\mathbf{K}} N$ in $\mathbf{K}_\mu$, we can fix a nonalgebraic $p \in \mathbf{S}(M)$ (which exists as M is not maximal) and take its nonforking extension to $\mathbf{S}(N)$. We have observed this extension is not algebraic, hence N cannot be maximal either. We have also seen that $\mathbf{K}$ (hence $\mathbf{K}_M$) is stable in μ. Finally, $\mathbf{K}$ has amalgamation in μ. This follows from Fact 2.5 and the extension property of nonforking. We deduce that $\mathbf{K}_M$ has amalgamation in μ and also joint embedding in μ. $\square$

Remark 5.3. By Lemma 3.1, any AEC satisfying hypotheses of Lemma 5.2 will have a model in $\mathbf{K}_\mu$, but we do not in general know how to find a non-maximal one. Theorem 3.6 gives a way by assuming categoricity in more cardinals.

We deduce arbitrarily large models from categoricity in enough small cardinals:

Theorem 5.4. Let $\mathbf{K}$ be an $\mathrm{LS}(\mathbf{K})$-tame AEC with weak amalgamation and coherent sequences, and let $\mu > \mathrm{LS}(\mathbf{K})$ be a strong limit cardinal. *If*:

(1) Not every element of $\mathbf{K}_\mu$ is maximal.
(2) For every $\theta < \mu$ there exists $\lambda < \mu$ such that $\lambda = \lambda^\theta$ and $\mathbf{K}$ is categorical in λ.

Then $\mathbf{K}_{\geq \mu}$ has amalgamation and arbitrarily large models.

Proof. Assume without loss of generality that μ has countable cofinality (otherwise, it is easy to find a smaller cardinal which has countable cofinality and still satisfies the hypotheses). Fix a non-maximal $M \in \mathbf{K}_\mu$. By Lemma 5.2, there is a good^{-S} μ-frame on $\mathbf{K}_M$. It is easy to check that $\mathbf{K}_M$ has weak amalgamation and is μ-tame. Therefore by Fact 5.1, $\mathbf{K}_M$ has amalgamation and arbitrarily large model. In particular, $\mathbf{K}$ has arbitrarily large models and M is an amalgamation base in $\mathbf{K}$. Since any maximal model is an amalgamation base, we deduce that $\mathbf{K}_{\geq \mu}$ has amalgamation. $\square$

We now work toward transferring categoricity assuming arbitrarily large models. We will use the following upward categoricity transfer of Grossberg and VanDieren for tame AECs with both amalgamation and arbitrarily large models.

Fact 5.5 ([GV06a, 6.3])**.** Let $\mathbf{K}$ be an $\mathrm{LS}(\mathbf{K})$-tame AEC with amalgamation and arbitrarily large models. If $\mathbf{K}$ is categorical in $\mathrm{LS}(\mathbf{K})$ and in $\mathrm{LS}(\mathbf{K})^+$, then $\mathbf{K}$ is categorical everywhere above $\mathrm{LS}(\mathbf{K})$.

Toward deriving global amalgamation, we show how to build a good frame assuming arbitrarily large models and enough categoricity, tameness, and amalgamation in low cardinals. We will use recent results from [BGVV17] and [Vas17a], but the reader can regard them as black boxes.

Lemma 5.6. Let $\mathbf{K}$ be an AEC. *If*:

(1) $\mathbf{K}$ has arbitrarily large models.
(2) $\mathbf{K}$ is categorical in $\mathrm{LS}(\mathbf{K})$.
(3) $\mathbf{K}$ is categorical in $\mathrm{LS}(\mathbf{K})^+$.
(4) $\mathbf{K}_{\mathrm{LS}(\mathbf{K})}$ has amalgamation.
(5) $\mathbf{K}_{\mathrm{LS}(\mathbf{K})^+}$ has weak amalgamation.
(6) $\mathbf{K}_{\leq \mathrm{LS}(\mathbf{K})^+}$ is $\mathrm{LS}(\mathbf{K})$-tame.

Then $\mathbf{K}$ has a good $\mathrm{LS}(\mathbf{K})^+$-frame.

Proof. Let $\lambda := \text{LS}(\mathbf{K})$. By [BGVV17], $\mathbf{K}$ is λ-superstable (essentially, this means that splitting has what is called the $\aleph_0$-local character for universal chains in [Vas16a]) and by [Vas17a, 5.7(1)], $\mathbf{K}$ has λ-symmetry. We now attempt to build a good^{-S}-frame as in [Vas16a]. We can prove extension without using amalgamation in λ^+, hence by Fact 2.5 we obtain amalgamation in λ^+, and hence obtain a good^{-S} λ^+-frame. Symmetry is then proven as in [BV17b, 6.8] (and is not needed for the present paper). $\square$

We can now prove a generalization of Fact 5.5 for tame AECs with only weak amalgamation (but still with arbitrarily large models).

Theorem 5.7. Let $\mathbf{K}$ be an $\text{LS}(\mathbf{K})$-tame AEC with weak amalgamation and arbitrarily large models. If $\mathbf{K}_{\text{LS}(\mathbf{K})}$ has amalgamation, and $\mathbf{K}$ is categorical in both $\text{LS}(\mathbf{K})$ and $\text{LS}(\mathbf{K})^+$, then $\mathbf{K}_{\geq \text{LS}(\mathbf{K})}$ has amalgamation and $\mathbf{K}$ is categorical everywhere above $\text{LS}(\mathbf{K})$.

Proof. By Lemma 5.6, $\mathbf{K}$ has a good $\text{LS}(\mathbf{K})^+$-frame. By Fact 5.1, it follows that $\mathbf{K}_{\geq \text{LS}(\mathbf{K})^+}$ (and hence $\mathbf{K}_{\geq \text{LS}(\mathbf{K})}$) has amalgamation. Now apply Fact 5.5. $\square$

Lemma 5.6 still asked for full amalgamation in one cardinal. However we can use the weak diamond to derive it from successive categoricity:

Corollary 5.8. Let $\mathbf{K}$ be an $\text{LS}(\mathbf{K})$-tame AEC with weak amalgamation and arbitrarily large models. Let $\theta > \text{LS}(\mathbf{K})$ be least such that $2^{\text{LS}(\mathbf{K})} < 2^\theta$. If $\mathbf{K}$ is categorical in every cardinal in $[\text{LS}(\mathbf{K}), \theta]$, then $\mathbf{K}$ is categorical everywhere above $\text{LS}(\mathbf{K})$. Moreover, there exists $\lambda \in [\text{LS}(\mathbf{K}), \theta)$ such that $\mathbf{K}_{\geq \lambda}$ has amalgamation

Proof. Generalizing [She09a, I.3.8] (using the corresponding generalization of the Devlin-Shelah weak diamond [DS78] as in [SV99, 1.2.4]), we can find $\lambda \in [\text{LS}(\mathbf{K}), \theta)$ such that $\mathbf{K}_\lambda$ has amalgamation. Now apply Theorem 5.7 to $\mathbf{K}_{\geq \lambda}$. $\square$

Putting together all the results proven so far, we obtain the main result of the paper:

Corollary 5.9. Let $\mathbf{K}$ be an $\text{LS}(\mathbf{K})$-tame AEC with weak amalgamation and coherent sequences. If $\mathbf{K}$ is categorical in every cardinal in $[\text{LS}(\mathbf{K}), \beth_\omega(\text{LS}(\mathbf{K})))$, then $\mathbf{K}$ is categorical everywhere above $\text{LS}(\mathbf{K})$.

Proof. Let $\mu := \beth_\omega(\text{LS}(\mathbf{K}))$. By Theorem 3.6, not every element of $\mathbf{K}_\mu$ is maximal. By Theorem 5.4, $\mathbf{K}$ has arbitrarily large models. Now apply Corollary 5.8. $\square$

Specializing to universal classes, we get:

Corollary 5.10. If a universal $\mathbb{L}_{\omega_1,\omega}$ sentence is categorical in an end segment of cardinals strictly below $\beth_\omega$, then it is also categorical everywhere above $\beth_\omega$.

Proof. Let ϕ be a universal $\mathbb{L}_{\omega_1,\omega}$ sentence and let $\mathbf{K}$ be its the class of models (ordered with substructure). Note that $\mathbf{K}$ has intersections (the closure operator computed inside N is the closure under the functions of N). By Lemma 2.8, $\mathbf{K}$ has coherent sequences. By Remark 2.4, $\mathbf{K}$ has weak amalgamation. By [Vas17b, 3.7],

$\mathbf{K}$ is $\text{LS}(\mathbf{K})$-tame. Let $\chi \in [\aleph_0, \beth_\omega)$ be such that $\mathbf{K}$ is categorical in every cardinal in $[\chi, \beth_\omega)$. Now apply Corollary 5.9 to $\mathbf{K}_{\geq \chi}$. $\square$

For completeness, we also point out that the last two results can be drastically improved assuming the weak GCH (see also [MAV] for how to improve even more when the Löwenheim-Skolem-Tarski number is $\aleph_0$). On μ_{unif}, see [She09b, VII.0.4] for a definition and [She09b, VII.9.4] for what is known. It seems that for all practical purposes the reader can take $\mu_{\text{unif}}(\lambda^{++}, 2^{\lambda^+})$ to mean $2^{\lambda^{++}}$.

Theorem 5.11. Let $\mathbf{K}$ be an AEC with Löwenheim-Skolem-Tarski number λ. Assume that $\mathbf{K}$ is λ^+-tame and has weak amalgamation. Assume further that $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$. If $\mathbf{K}$ is categorical in λ, $\mathbf{K}$ is categorical in λ^+, and $1 \leq \mathbb{I}(\mathbf{K}, \lambda^{++}) < \mu_{\text{unif}}(\lambda^{++}, 2^{\lambda^+})$, then $\mathbf{K}$ is categorical everywhere above λ.

Proof. By results of Shelah on building good frames, there is a good λ-frame on $\mathbf{K}$ and $\mathbf{K}_{\leq \lambda^+}$ is λ-tame (see [Vasb, 7.1] for an outline of the proof). By Fact 5.1, $\mathbf{K}$ has arbitrarily large models. Now apply Corollary 5.8. $\square$

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