GOOD FRAMES IN THE HART-SHELAH EXAMPLE

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Abstract. For a fixed natural number $n \geq 1$, the Hart-Shelah example is an abstract elementary class (AEC) with amalgamation that is categorical exactly in the infinite cardinals less than or equal to $\aleph_n$.

We investigate recently-isolated properties of AECs in the setting of this example. We isolate the exact amount of type-shortness holding in the example and show that it has a type-full good $\aleph_{n-1}$-frame which fails the existence property for uniqueness triples. This gives the first example of such a frame.

1. Introduction

In his milestone two-volume book on classification theory for abstract elementary classes (AECs) [She09a, She09b], Shelah introduces a central definition: good λ-frames. These are an axiomatic notion of forking for types of singletons over models of cardinality λ (see [She09a, Definition II.2.1] or Definition 2.7 here). One can think of the statement "an AEC $\mathbf{K}$ has a good λ-frame" as meaning that $\mathbf{K}$ is well-behaved in λ, where "well-behaved" in this context means something similar to superstability in the context of first-order model theory. With this in mind, a key question is:

Question 1.1 (The extension question). *Assume an AEC $\mathbf{K}$ has a good λ-frame. Under what conditions does it (or a subclass of saturated models) have a good λ^+-frame?*

Shelah's answer in [She09a, Chapter II] involves two dividing lines: the existence property for uniqueness triples, and smoothness of a certain ordering $\leq^{\text{NF}}_{\mathbf{K}_{\lambda^+}}$ (see Definitions 2.10, 2.13). Shelah calls a good frame satisfying the first property *weakly successful* and a good frame satisfying both properties is called *successful*. Assuming instances of the weak diamond, Shelah shows [She09a, II.5.9] that the failure of the first property implies many models in λ^{++}. In [She09a, II.8.7] (see

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also [JS13, Theorem 7.1.3]), Shelah shows that if the first property holds, then the failure of the second implies there exists $2^{\lambda^{++}}$ many models in λ^{++}.

However, Shelah does not give any examples showing that these two properties can fail (this is mentioned as part of the "empty half of the glass" in Shelah's introduction [She09a, N.4.(f)]). The present paper investigates these dividing lines in the specific setup of the Hart-Shelah example [HS90]. For a fixed[1] $n \in [3, \omega)$, the Hart-Shelah example is an AEC $\mathbf{K}^n$ that is categorical exactly in the interval $[\aleph_0, \aleph_{n-2}]$. It was investigated in details by Baldwin and Kolesnikov [BK09] who proved that $\mathbf{K}^n$ has (disjoint) amalgamation, is (Galois) stable exactly in the infinite cardinals less than or equal to $\aleph_{n-3}$, and is $(< \aleph_0, \leq \aleph_{n-3})$-tame (i.e. Galois types over models of size at most $\aleph_{n-3}$ are determined by their restrictions to finite sets, see Definition 2.1).

The Hart-Shelah example is a natural place to investigate good frames, since it has good behavior only below certain cardinals (around $\aleph_{n-3}$). The first author has shown [Bon14a, Theorem 10.2] that $\mathbf{K}^n$ has a good $\aleph_k$ frame for any $k \leq n-3$, but cannot have one above since stability is part of the definition of a good frame. Therefore at $\aleph_{n-3}$, the last cardinal when $\mathbf{K}^n$ has a good frame, the answer to the extension question must be negative, so one of the two dividing lines above must fail, i.e. the good frame is *not* successful. The next question is: *which* of these properties fails? We show that the first property must fail: the frame is not weakly successful. In fact, we give several proofs (Theorem 6.6, Corollary 7.4). On the other hand, we show that the frames strictly below $\aleph_{n-3}$ are successful[2]. This follows both from a concrete analysis of the Hart-Shelah example (Theorem 6.3) and from abstract results in the theory of good frames (Theorem 5.1).

Regarding the abstract theory, a focus of recent research has been the interaction of locality properties and frames. For example, the first author [Bon14a] (with slight improvements in [BVb, Corollary 6.9]) has shown that amalgamation and *tameness* (a locality property for types isolated by Grossberg and VanDieren [GV06]) implies a positive answer to the extension question (in particular, the Hart-Shelah example is *not* $(\aleph_{n-3}, \aleph_{n-2})$-tame[3]). A relative of tameness is type-shortness, introduced by the first author in [Bon14b, Definition 3.2]: roughly, it says that types of sequences are determined by their restriction to small subsequences. Sufficient amount of type-shortness implies (with a few additional technical conditions) that a good frame is weakly successful [Vasa, Section 11].

As already mentioned, Baldwin and Kolesnikov have shown that the Hart-Shelah example is $(< \aleph_0, \leq \aleph_{n-3})$-tame (see Fact 3.2); here (Theorem 4.1) we refine their argument to show that it is also $(< \aleph_0, < \aleph_{n-3})$-type short over models of size

[1]Note that our indexing follows [HS90] and [BK09] rather than [Bon14a].

[2]While there are no known examples, it is conceivable that there is a good frame that is not successful but can still be extended.

[3]This was already noticed by Baldwin and Kolesnikov using a different argument [BK09, Proposition 6.8].

less than or equal to $\aleph_{n-3}$ (i.e. types of sequences of length less than $\aleph_{n-3}$ are determined by their finite restrictions, see Definition 2.1). We prove that this is optimal: the result *cannot* be extended to types of length $\aleph_{n-3}$ (see Corollary 8.13).

We can also improve the aforementioned first author's construction of a good $\aleph_k$-frame (when $k \leq n-3$) in the Hart-Shelah example: the good frame built there is not type-full: forking is only defined for a certain (dense family) of basic types. We prove here that the good frame extends to a type-full one. This uses abstract constructions of good frames due to the second author [Vas16a] (as well as results of VanDieren on the symmetry property [Van]) when $k \geq 1$. When $k = 0$ we have to work more and develop new general tools to build good frames (see Section 8 and Appendix C).

The following summarizes our main results:

Theorem 1.2. *Let $n \in [3, \omega)$ and let $\mathbf{K}^n$ denote the AEC induced by the Hart-Shelah example. Then:*

1. $\mathbf{K}^n$ *is* $(< \aleph_0, < \aleph_{n-3})$*-type short over* $\leq \aleph_{n-3}$*-sized models and* $(< \aleph_0, \leq \aleph_{n-3})$*-tame for* $(< \aleph_{n-3})$*-length types.*
2. $\mathbf{K}^n$ *is* not $(< \aleph_{n-3}, \aleph_{n-3})$*-type short over* $\aleph_{n-3}$*-sized models.*
3. *For any $k \leq n-3$, there exists a unique type-full good $\aleph_k$-frame $\mathfrak{s}$ on $\mathbf{K}^n$. Morover:*
   (a) *If $k < n-3$, $\mathfrak{s}$ is successful good$^+$.*
   (b) *If $k = n-3$, $\mathfrak{s}$ is* not *weakly successful.*

Proof.

1. By Theorem 4.1.
2. By Corollary 8.13.
3. By Theorems 5.1 and Corollary 8.12. Note also that by canonicity (Fact 2.20), $\mathfrak{s}$ is unique, so extends $\mathfrak{s}^{k,n}$ (see Definition 3.3).
   (a) By Theorem 6.3, $\mathfrak{s}^{k,n}$ is successful. By Lemma 5.2, $\mathfrak{s}^{k,n}$ is good$^+$. Now apply Facts 2.20 and 2.17.
   (b) By Proposition 6.6, $\mathfrak{s}^{k,n}$ is not weakly successful and since $\mathfrak{s}$ extends $\mathfrak{s}^{k,n}$, $\mathfrak{s}$ is not weakly successful either.

$\square$

We discuss several open questions. First, one can ask whether the aforementioned second dividing line can fail:

Question 1.3 (See also Question 7.1 in [Jar16])**.** *Is there an example of a good λ-frame that is weakly successful but not successful?*

Second, one can ask whether there is any example at all of a good frame where the forking relation can be defined only for certain types:

Question 1.4. *Is there an example of a good λ-frame that does* not *extend to a type-full frame?*

We have not discussed good$^+$ in our introduction: it is a technical property of frames that allows one to extend frames without changing the order (see the background in Section 2). No negative examples are known.

Question 1.5. *Is there an example of a good λ-frame that is not good$^+$? Is there an example that is successful but not good$^+$?*

In a slightly different direction, we also do not know of an example of a good frame failing symmetry:

Question 1.6 (See also Question 4.14 in [VV])**.** *Is there an example of a triple $(\mathbf{K}, \perp, \mathrm{gS}^{bs})$ satisfying all the requirements from the definition of a good λ-frame except symmetry?*

In the various examples, the proofs of symmetry either uses disjoint amalgamation (as in [She09a, Example II.3.7]) or failure of the order property (see e.g. [BGKV16, Theorem 5.14]). Recently the second author [Vasc, Corollary 4.8] has shown that symmetry follows from (amalgamation, no maximal models, and) *solvability* in any $\mu > \lambda$ (see Appendix A; roughly it means that the union of a short chain of saturated model of cardinality μ is saturated, and there is a "constructible" witness). We do not know of an example of a good λ-frame where solvability in every $\mu > \lambda$ fails.

The background required to read this paper is a solid knowldge of AECs (including most of the material in [Bal09]). Familiarity with good frames and the Hart-Shelah example would be helpful, although we have tried to give a self-contained presentation and quote all the black boxes we need.

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2. Preliminaries: The abstract theory

Everywhere in this paper, $\mathbf{K}$ denotes a fixed AEC (that may or may not have structural properties such as amalgamation or arbitrarily large models). We assume the reader is familiar with concepts such as amalgamation, Galois types, tameness, type-shortness, stability, saturation, and splitting (see for example the first 12 chapters of [Bal09]). Our notation is standard and is described in the preliminaries of [Vas16b].

On tameness and type-shortness, we use the notation from [Bon14b, Definitions 3.1,3.2]:

Definition 2.1. *Let $\lambda \geq \mathrm{LS}(\mathbf{K})$ and let κ, μ be infinite cardinals*[4]

(1) $\mathbf{K}$ *is* $(<\kappa, \lambda)$-tame for μ-length types *if for any $M \in \mathbf{K}_\lambda$ and distinct $p, q \in \mathrm{gS}^\mu(M)$, there exists $A \subseteq |M|$ with $|A| < \kappa$ such that $p \upharpoonright A \neq q \upharpoonright A$. When $\mu = 1$ (i.e. we are only interested in types of length one), we omit it and just say that $\mathbf{K}$ is $(<\kappa, \lambda)$-tame.*
(2) $\mathbf{K}$ *is* $(<\kappa, \mu)$-type short over λ-sized models *if for any $M \in \mathbf{K}_\lambda$ and distinct $p, q \in \mathrm{gS}^\mu(M)$, there exists $I \subseteq \mu$ with $|I| < \kappa$ and $p^I \neq q^I$.*

We similary define variations such as "$\mathbf{K}$ is $(<\kappa, \leq \mu)$-type short over $\leq \lambda$-sized models.

2.1. **Superstability and symmetry.** We will rely on the following local version of superstability, already implicit in [SV99] and since then studied in many papers, e.g. [Van06, GVV, Vasa, BVa, GV, Van]. We quote the definition from [Vasa, Definition 10.1]:

Definition 2.2. $\mathbf{K}$ *is* μ-superstable *(or* superstable in μ*) if:*

(1) $\mu \geq \mathrm{LS}(\mathbf{K})$.
(2) $\mathbf{K}_\mu$ *is nonempty, has joint embedding, amalgamation, and no maximal models.*
(3) $\mathbf{K}$ *is stable in μ.*
(4) *There are no long splitting chains in μ:*
    For any limit ordinal $\delta < \mu^+$, for every sequence $\langle M_i \mid i < \delta \rangle$ of models of cardinality μ with M_{i+1} universal over M_i and for every $p \in \mathrm{gS}(\bigcup_{i<\delta} M_i)$, there exists $i < \delta$ such that p does not μ-split over M_i.

We will also use the concept of symmetry for splitting isolated in [Van]:

Definition 2.3. *For $\mu \geq \mathrm{LS}(\mathbf{K})$, we say that $\mathbf{K}$ has* μ-symmetry *(or* symmetry in μ*) if whenever models $M, M_0, N \in \mathbf{K}_\mu$ and elements a and b satisfy the conditions (1)-(4) below, then there exists M^b a limit model over M_0, containing b, so that $\mathrm{gtp}(a/M^b)$ does not μ-split over N.*

(1) *M is universal over M_0 and M_0 is a limit model over N.*
(2) $a \in |M| \backslash |M_0|$.
(3) $\mathrm{gtp}(a/M_0)$ *is non-algebraic and does not μ-split over N.*
(4) $\mathrm{gtp}(b/M)$ *is non-algebraic and does not μ-split over M_0.*

By an argument of Shelah and Villaveces [SV99, Theorem 2.2.1] (see also [GV, Corollary 6.3]), superstability holds below a categoricity cardinal.

Fact 2.4 (The Shelah-Villaveces Theorem)**.** *Let $\lambda > \mathrm{LS}(\mathbf{K})$. Assume that $\mathbf{K}_{<\lambda}$ has amalgamation and no maximal models. If $\mathbf{K}$ has arbitrarily large models and is categorical in λ, then $\mathbf{K}$ is superstable in any $\mu \in [\mathrm{LS}(\mathbf{K}), \lambda)$.*

[4] As opposed to the first author's original definition, we allow $\kappa \leq \mathrm{LS}(\mathbf{K})$ by making use of Galois types over sets, see the preliminaries of [Vas16b].

Remark 2.5. *We will only use the result when λ is a successor (in fact $\lambda = \mu^+$, where μ is the cardinal where we want to derive superstability). In this case there is an easier proof due to Shelah. See [She99, Lemma I.6.3] or [Bal09, Theorem 15.3].*

VanDieren [Van, Theorems 2,3] has shown that (in an AEC with amalgamation and no maximal models) symmetry in μ follows from categoricity in μ^+. This was improved in [VV, Corollary 7.3] and recently in [Vasc, Corollary 4.8], but we will only use VanDieren's original result:

Fact 2.6. *If $\mathbf{K}$ is μ-superstable and categorical in μ^+, then $\mathbf{K}$ has symmetry in μ.*

2.2. **Good frames.** Good λ-frames were introduced by Shelah in [She09a, Chapter II] as a bare-bone axiomatization of superstability. We give a simplified definition here:

Definition 2.7 (Definition II.2.1 in [She09a])**.** *A* good λ-frame *is a triple $\mathfrak{s} = (\mathbf{K}_\lambda, \perp, \mathrm{gS}^{bs})$ where:*

1. *$\mathbf{K}$ is an AEC such that:*
   (a) *$\lambda \geq \mathrm{LS}(\mathbf{K})$.*
   (b) *$\mathbf{K}_\lambda \neq \emptyset$.*
   (c) *$\mathbf{K}_\lambda$ has amalgamation, joint embedding, and no maximal models.*
   (d) *$\mathbf{K}$ is stable*[5] *in λ.*
2. *For each $M \in \mathbf{K}_\lambda$, $\mathrm{gS}^{bs}(M)$ (called the set of* basic types *over M) is a set of nonalgebraic Galois types over M satisfying the* density property*: if $M <_{\mathbf{K}} N$ are both in $\mathbf{K}_\lambda$, there exists $a \in |N| \backslash |M|$ such that $\mathrm{gtp}(a/M; N) \in \mathrm{gS}^{bs}(M)$.*
3. *$\perp$ is an (abstract) independence relation on the basic types satisfying invariance, monotonicity, extension existence, uniqueness, continuity, local character, and symmetry (see [She09a, Definition II.2.1] for the full definition of these properties).*

We say that $\mathfrak{s}$ is type-full *[She09a, Definition III.9.2.(1)] if for any $M \in \mathbf{K}_\lambda$, $\mathrm{gS}^{bs}(M)$ is the set of* all *nonalgebraic types over M. Rather than explicitly using the relation $\perp$, we will say that* $\mathrm{gtp}(a/M; N)$ does not fork *over M_0 if $a \overset{N}{\underset{M_0}{\perp}} M$ (this is well-defined by the invariance and monotonicity properties). We say that a good λ-frame $\mathfrak{s}$ is* on $\mathbf{K}$ *if the underlying AEC of $\mathfrak{s}$ is $\mathbf{K}_\lambda$, and similarly for other variations.*

Remark 2.8. *We will* not *use the axiom (B) [She09a, Definition II.2.1] requiring the existence of a superlimit model of size λ. In fact many papers (e.g. [JS13]) define good frames without this assumption.*

[5]In Shelah's original definition, only the set of basic types is required to be stable. However full stability follows, see [She09a, Claim II.4.2].

Remark 2.9. *We gave a shorter list of properties that in Shelah's original definition, but the other properties follow, see [She09a, Section II.2].*

The next technical property is of great importance in Chapter II and III of [She09a]. The definition below follows [JS13, Definition 4.1.5].

Definition 2.10. *Let $\lambda \geq \mathrm{LS}(\mathbf{K})$.*

(1) *For $M_0 \leq_{\mathbf{K}} M_\ell$ all in $\mathbf{K}_\lambda$, $\ell = 1, 2$, an* amalgam of M_1 and M_2 over M_0 *is a triple (f_1, f_2, N) such that $N \in \mathbf{K}_\lambda$ and $f_\ell : M_\ell \underset{M_0}{\longrightarrow} N$.*

(2) *Let (f_1^x, f_2^x, N^x), $x = a, b$ be amalgams of M_1 and M_2 over M_0. We say (f_1^a, f_2^a, N^a) and (f_1^b, f_2^b, N^b) are* equivalent over M_0 *if there exists $N_* \in \mathbf{K}_\lambda$ and $f^x : N^x \to N_*$ such that $f^b \circ f_1^b = f^a \circ f_1^a$ and $f^b \circ f_2^b = f^a \circ f_2^a$, namely, the following commutes:*

$$\begin{tikzcd}
 & N^a \arrow[r, dotted, "f^a"] & N_* \\
M_1 \arrow[ur, "f_1^a"] \arrow[rr, "f_1^b"] & & N^b \arrow[u, dotted, "f^b"'] \\
M_0 \arrow[u] \arrow[r] & M_2 \arrow[uu, "f_2^a"] \arrow[ur, "f_2^b"'] &
\end{tikzcd}$$

Note that being "equivalent over M_0" is an equivalence relation ([JS13, Proposition 4.3]).

(3) *Let $\mathfrak{s} = (\mathbf{K}_\lambda, \mathop{\smile}\limits^{\mid}, \mathrm{gS}^{bs})$ be a good λ-frame on $\mathbf{K}$.*

  (a) *A triple (a, M, N) is a* uniqueness triple *(for $\mathfrak{s}$) if $M \leq_{\mathbf{K}} N$ are both in $\mathbf{K}_\lambda$, $a \in |N| \backslash |M|$, $\mathrm{gtp}(a/M; N) \in \mathrm{gS}^{bs}(M)$, and for any $M_1 \geq_{\mathbf{K}} M$ in $\mathbf{K}_\lambda$, there exists a* unique *(up to equivalence over M) amalgam (f_1, f_2, N_1) of N and M_1 over M such that $\mathrm{gtp}(f_1(a)/f_2[M_1]; N_1)$ does not fork over M.*

  (b) *$\mathfrak{s}$ has the* existence property for uniqueness triples *(or is* weakly successful*) if for any $M \in \mathbf{K}_\lambda$ and any $p \in \mathrm{gS}^{bs}(M)$, one can write $p = \mathrm{gtp}(a/M; N)$ with (a, M, N) a uniqueness triple.*

The importance of the existence property for uniqueness triples is that it allows us to extend the nonforking relation to types of models (rather than just types of length one). This is done by Shelah in [She09a, Section II.6] but was subsequently simplified in [JS13], so we quote from the latter.

Definition 2.11. *Let $\mathfrak{s}$ be a weakly successful good λ-frame on $\mathbf{K}$, with $\mathbf{K}$ categorical in λ.*

(1) *[JS13, Definition 5.3.1] Define a 4-ary relation $\mathrm{NF}^* = \mathrm{NF}^*_{\mathfrak{s}}$ on $\mathbf{K}_\lambda$ by $\mathrm{NF}^*(N_0, N_1, N_2, N_3)$ if there is $\alpha^* < \lambda^+$ and for $\ell = 1, 2$ there are increasing continuous sequences $\langle N_{\ell,i} : i \leq \alpha^* \rangle$ and a sequence $\langle d_i : i < \alpha^* \rangle$ such that:*

(a) $\ell < 4$ *implies* $N_0 \leq_{\mathbf{K}} N_\ell \leq_{\mathbf{K}} N_3$.
(b) $N_{1,0} = N_0$, $N_{1,\alpha^*} = N_1$, $N_{2,0} = N_2$, $N_{2,\alpha^*} = N_3$.
(c) $i \leq \alpha^*$ *implies* $N_{1,i} \leq_{\mathbf{K}} N_{2,i}$.
(d) $d_i \in |N_{1,i+1}| \backslash |N_{1,i}|$.
(e) $(d_i, N_{1,i}, N_{1,i+1})$ *is a uniqueness triple.*
(f) $\text{gtp}(d_i/N_{2,i}; N_{2,i+1})$ *does not fork over* $N_{1,i}$.

(2) *[JS13, Definition 5.3.2] Define a 4-ary relation* $\text{NF} = \text{NF}_{\mathfrak{s}}$ *on* $\mathbf{K}_\lambda$ *by* $\text{NF}(M_0, M_1, M_2, M_3)$ *if there are models* N_0, N_1, N_2, N_3 *such that* $N_0 = M_0$, $\ell < 4$ *implies* $M_\ell \leq_{\mathbf{K}} N_\ell$ *and* $\text{NF}^*(N_0, N_1, N_2, N_3)$.

By [JS13, Theorem 5.5.4], NF satisfies several of the basic properties of forking:

Fact 2.12. *If* $\text{NF}(M_0, M_1, M_2, M_3)$, *then* $M_1 \cap M_2 = M_0$. *Moreover,* NF *respects* $\mathfrak{s}$ *and satisfies monotonicity, existence, weak uniqueness, symmetry, and long transitivity (see [JS13, Definition 5.2.1] for the definitions).*

Shelah [She09a, Definition III.1.1] says a weakly successful good frame is *successful* if an ordering $\leq^{\text{NF}}_{\mathbf{K}_{\lambda^+}}$ defined in term of the relation NF induces an AEC. We quote the full definition from [JS13].

Definition 2.13. *Let* $\mathfrak{s}$ *be a weakly successful good* λ*-frame on* $\mathbf{K}$, *with* $\mathbf{K}$ *categorical in* λ.

(1) *[JS13, Definition 6.1.2] Define a 4-ary relation* $\widehat{\text{NF}} = \widehat{\text{NF}}_{\mathfrak{s}}$ *on* $\mathbf{K}$ *by* $\widehat{\text{NF}}(N_0, N_1, M_0, M_1)$ *if:*
(a) $\ell < 2$ *implies that* $N_n \in \mathbf{K}_\lambda$, $M_n \in \mathbf{K}_{\lambda^+}$.
(b) *There is a pair of increasing continuous sequences* $\langle N_{0,\alpha} : \alpha < \lambda^+ \rangle$, $\langle N_{1,\alpha} : \alpha \leq \lambda^+ \rangle$ *such that for every* $\alpha < \lambda^+$, $\text{NF}(N_{0,\alpha}, N_{1,\alpha}, N_{0,\alpha+1}, N_{1,\alpha+1})$ *and for* $\ell < 2$, $N_{0,\ell} = N_\ell$, $M_\ell = N_{\ell,\lambda^+}$.

(2) *[JS13, Definition 6.1.4] For* $M_0 \leq_{\mathbf{K}} M_1$ *both in* $\mathbf{K}_{\lambda^+}$, $M_0 \leq^{\text{NF}}_{\mathbf{K}_{\lambda^+}} M_1$ *if there exists* $N_0, N_1 \in \mathbf{K}_\lambda$ *such that* $\widehat{\text{NF}}(N_0, N_1, M_0, M_1)$.

(3) *[JS13, Definition 10.1.1]* $\mathfrak{s}$ *is* successful *if* $\leq^{\text{NF}}_{\mathbf{K}_{\lambda^+}}$ *satisfies smoothness on the saturated models in* $\mathbf{K}_{\lambda^+}$: *whenever* $\delta < \lambda^{++}$ *is limit,* $\langle M_i : i \leq \delta \rangle$ *is a* $\leq^{\text{NF}}_{\mathbf{K}_{\lambda^+}}$*-increasing continuous sequence of saturated models of cardinality* λ^+, *and* $N \in \mathbf{K}_{\lambda^+}$ *is saturated such that* $i < \delta$ *implies* $M_i \leq^{\text{NF}}_{\mathbf{K}_{\lambda^+}} N$, *then* $M_\delta \leq^{\text{NF}}_{\mathbf{K}_{\lambda^+}} N$.

The point of successful is that they can be extended to a good λ^+-frame on the class of saturated model of cardinality λ^+ (see [JS13, Theorem 10.1.9]). The ordering of the class will be $\leq^{\text{NF}}_{\mathbf{K}_{\lambda^+}}$. Shelah also defines what it means for a frame to be good$^+$. If the frame is successful, this implies that $\leq^{\text{NF}}_{\mathbf{K}_{\lambda^+}}$ is just $\leq_{\mathbf{K}}$ and simplifies several arguments [She09a, Definition III.1.3, Conclusion III.1.8]:

Definition 2.14. *A good* λ*-frame* $\mathfrak{s}$ *on* $\mathbf{K}$ *is* good$^+$ *when the following is* impossible*:*

There exists an increasing continuous $\langle M_i : i < \lambda^+ \rangle$, $\langle N_i : i < \lambda^+ \rangle$, *a basic type* $p \in \text{gS}(M_0)$, *and* $\langle a_i : i < \lambda^+ \rangle$ *such that for any* $i < \lambda^+$*:*

(1) $i < \lambda^+$ *implies that* $M_i \leq_{\mathbf{K}} N_i$ *and both are in* $\mathbf{K}_\lambda$.
(2) $a_{i+1} \in |M_{i+2}|$ *and* $\text{gtp}(a_{i+1}/M_{i+1}; M_{i+2})$ *is a nonforking extension of* p, *but* $\text{gtp}(a_{i+1}/N_0; N_{i+2})$ *is not.*
(3) $\bigcup_{j<\lambda^+} M_j$ *is saturated.*

Fact 2.15. *Let* $\mathfrak{s}$ *be a successful good* λ*-frame on* $\mathbf{K}$. *The following are equivalent:*

(1) $\mathfrak{s}$ *is good*$^+$.
(2) *For* $M, N \in \mathbf{K}_{\lambda^+}$ *both saturated,* $M \leq^{\text{NF}}_{\mathbf{K}_{\lambda^+}} N$ *if and only if* $M \leq_{\mathbf{K}} N$.

Proof. (1) implies (2) is [She09a, Conclusion III.1.8]. Let us see that (2) implies (1): Suppose for a contradiction that $\langle M_i : i < \lambda^+ \rangle$, $\langle N_i : i < \lambda^+ \rangle$, p, $\langle a_i : i < \lambda^+ \rangle$ witness that $\mathfrak{s}$ is *not* good$^+$. Write $M_{\lambda^+} := \bigcup_{i<\lambda^+} M_i$, $N_{\lambda^+} := \bigcup_{i<\lambda^+} N_i$. Using [JS13, Proposition 6.1.6], we have that there exists a club $C \subseteq \lambda^+$ such that for any $i < j$ both in C, $\text{NF}(M_i, M_j, N_i, N_j)$. In particular (by monotonicity), $\text{NF}(M_i, M_{i+2}, N_i, N_{i+2})$. Pick any $i \in C$. Because NF respects $\mathfrak{s}$ (Fact 2.12), $\text{gtp}(a_{i+1}/N_i; N_{i+2})$ does not fork over M_i. By the properties of $\langle a_i : i < \lambda^+ \rangle$, $\text{gtp}(a_{i+1}/M_{i+1}; M_{i+2})$ is a nonforking extension of p. By transitivity, $\text{gtp}(a_{i+1}/N_i; N_{i+2})$ also is a nonforking extension of p, contradicting the definition of good$^+$. $\square$

Fact 2.16 (Conclusion III.1.8 in [She09a])**.** *Let* $\mathfrak{s}$ *be a successful good*$^+$ λ*-frame on* $\mathbf{K}$. *Then there exists a good* λ^+*-frame* $\mathfrak{s}^+$ *with underlying AEC the saturated models in* $\mathbf{K}$ *of size* λ^+ *(ordered with the strong substructure relation inherited from* $\mathbf{K}$*).*

We will also use that successful good$^+$ frame can be extended to be type-full.

Fact 2.17 (Claim III.9.6.(2B) in [She09a])**.** *If* $\mathfrak{s}$ *is a successful good*$^+$ λ*-frame on* $\mathbf{K}$ *and* $\mathbf{K}$ *is categorical in* λ, *then there exists a* type-full *successful good*$^+$ λ*-frame* $\mathfrak{t}$ *with underlying class* $\mathbf{K}_\lambda$.

The next result derives good frames from some tameness and categoricity. The statement is not optimal (e.g. categoricity in λ^+ can be replaced by categoricity in any $\mu > \lambda$) but suffices for our purpose.

Fact 2.18. *Assume that* $\mathbf{K}$ *has amalgamation and arbitrarily large models. Let* $\text{LS}(\mathbf{K}) < \lambda$ *be such that* $\mathbf{K}$ *is categorical in both* λ *and* λ^+. *Let* $\kappa \leq \lambda$ *be an infinite regular cardinal such that* $\lambda = \lambda^{<\kappa}$.

If $\mathbf{K}$ *is* $(\text{LS}(\mathbf{K}), \leq \lambda)$*-tame, then there is a type-full good* λ*-frame* $\mathfrak{s}$ *on* $\mathbf{K}$. *If in addition* $\mathbf{K}$ *is* $(<\kappa, \lambda)$*-type-short over* λ*-sized models, then* $\mathfrak{s}$ *is weakly successful.*

Proof. By Facts 2.4 and 2.6, $\mathbf{K}$ is superstable in any $\mu \in [\text{LS}(\mathbf{K}), \lambda]$, and has λ-symmetry. By [VV, Therem 6.4], there is a type-full good λ-frame $\mathfrak{s}$ on $\mathbf{K}_\lambda$. For the last sentence, combine [Vasb, Corollary 3.10] with the proof of [Vasd, Lemma A.14]. $\square$

Fact 2.18 gives a criteria for when a good frame is *weakly* successful, but when is it successful? This is answered by the next result, due to Adi Jarden [Jar16, Corollary 7.19] (note that the conjugation hypothesis there follows from [She09a, Claim III.1.21]).

Fact 2.19. *Let $\mathfrak{s}$ be a weakly successful good λ-frame on $\mathbf{K}$. If $\mathbf{K}$ is categorical in λ, has amalgamation in λ^+, and is (λ, λ^+)-tame, then $\mathfrak{s}$ is successful good$^+$.*

We will also make use of the following result, which tells us that if the AEC is categorical, there can be at most one good frame [Vasa, Theorem 9.7]:

Fact 2.20 (Canonicity of categorical good frames)**.** *Let $\mathfrak{s}$ and $\mathfrak{t}$ be good λ-frame on $\mathbf{K}$ with the same basic types. If $\mathbf{K}$ is categorical in λ, then $\mathfrak{s} = \mathfrak{t}$.*

3. Preliminaries: Hart-Shelah

Definition 3.1. *Fix $n \in [2, \omega)$. Let $\mathbf{K}^n$ be the AEC from the Hart-Shelah example. This class is $\mathbb{L}_{\omega_1,\omega}$-definable and a model in $\mathbf{K}^n$ consists of the following:*

- *I, some arbitrary index set*
- *$K = [I]^3$ with a membership relation for I*
- *H is a copy of $\mathbb{Z}_2$ with addition*
- *$G = \oplus_{u \in K} \mathbb{Z}_2$ with the evaluation map from $G \times K$ to $\mathbb{Z}_2$*
- *G^* is a set with a projection π_{G^*} onto K such that there is a 1-transitive action of G on each stalk $G^*_u = \pi^{-1}_{G^*}(u)$; we denote this action by $t_G(u, \gamma, x, y)$ for $u \in K$, $\gamma \in G$, and $x, y \in G^*_u$*
- *H^* is a set with a projection π_{H^*} onto K such that there is a 1-transitive action of $\mathbb{Z}_2$ on each stalk $H^*_u = \pi^{-1}_{H^*}(u)$ denoted by t_H*
- *Q is a $(n+1)$-ary relation on $(G^*)^n \times H^*$ satisfying the following:*
  - *We can permute the first n elements (the one from G^*) and preserve holding.*
  - *If $Q(x_1, \ldots, x_n, y)$ holds, then the indices of their stalks are* compatible, *which means the following: $x_\ell \in G^*_{u_\ell}$ and $y \in H^*_v$ such that $\{u_1, \ldots, u_n, v\}$ are all n element sets of some $n+1$ element subset of I.*
  - *Q is preserved by "even" actions in the following sense: suppose*
    - *$u_1, \ldots, u_n, v \in K$ are compatible*
    - *$x_\ell, x'_\ell \in G^*_{u_\ell}$ and $y, y' \in H^*_v$*
    - *$\gamma_\ell \in G$ and $\ell \in \mathbb{Z}_2$ are the unique elements that send x_ℓ or y to x'_ℓ or y'*

    then the following are equivalent
    - *$Q(x_1, \ldots, x_n, y)$ if and only if $Q(x'_1, \ldots, x'_n, y')$*
    - *$\gamma_1(v) + \cdots + \gamma_n(v) + \ell = 0 \mod 2$*

For $M, N \in \mathbf{K}^n$, $M \leq_{\mathbf{K}^n} N$ if and only if $M \prec_{\mathbb{L}_{\omega_1,\omega}} N$.

Fact 3.2 ([BK09])**.** *Let $n \in [2, \omega)$.*

(1) $\mathbf{K}^n$ *has disjoint amalgamation, joint embedding, and arbitrarily large models.*
(2) $\mathbf{K}^n$ *is model-complete: For* $M, N \in \mathbf{K}^n$, $M \leq_{\mathbf{K}^n} N$ *if and only if* $M \subseteq N$.
(3) *For any infinite cardinal* λ, $\mathbf{K}^n$ *is categorical in* λ *if and only if* $\lambda \leq \aleph_{n-2}$.
(4) $\mathbf{K}^n$ *is not stable in any* $\lambda \geq \aleph_{n-2}$.
(5) *If* $n \geq 3$, *then* $\mathbf{K}^n$ *is* $(< \aleph_0, \leq \aleph_{n-3})$*-tame, but it is* not $(\aleph_{n-3}, \aleph_{n-2})$*-tame.*

Note that the entire universe of a model of $\mathbf{K}^n$ is determined by the index I, so if $M \subsetneq N$, then $I(M) \subsetneq I(N)$. Thus it is natural to define a frame whose basic types are just the types of elements in I and nonforking is just nonalgebraicity. The following definition appears in the proof of [Bon14a, Theorem 10.2]:

Definition 3.3. *Let* $n \in [3, \omega)$. *For* $k \leq n-3$, *let* $\mathfrak{s}^{k,n} = (\mathbf{K}^n_{\aleph_k}, \perp, \mathrm{gS}^{bs})$ *be defined as follows:*

- $p \in \mathrm{gS}^{bs}(M)$ *if and only if* $p = \mathrm{gtp}(a/M; N)$ *for* $a \in I(N) \backslash I(M)$.
- $\mathrm{gtp}(a/M_1; M_2)$ *does not fork over* M_0 *if and only if* $a \in I(M_2) \backslash I(M_1)$.

Remark 3.4. *By [Bon14a, Theorem 10.2],* $\mathfrak{s}^{k,n}$ *is a good* $\aleph_k$*-frame.*

The notion of a solution is key to analyzing models of $\mathbf{K}^n$.

Definition 3.5 (Definitions 2.1 and 2.3 in [BK09])**.** *Let* $M \in \mathbf{K}^N$.

(1) $h = (f, g)$ *is a solution for* $W \subseteq K(M)$ *if and only if* $f \in \Pi_{u \in W} G^*_u(M)$ *and* $g \in \Pi_{u \in W} H^*_u(M)$ *such that, for all compatible* $u_1, \ldots, u_n, v \in W$, *we have*
$$M \vDash Q\left(f(u_1), \ldots . f(u_n), g(v)\right)$$
(2) $h = (f, g)$ *is a solution over* $A \subseteq I(M)$ *if and only if it is a solution for* $[A]^n$.
(3) $h = (f, g)$ *is a solution for* M *if and only if it is a solution for* $K(M)$.

Given $f : M \cong N$ *and solutions* h^M *for* M *and* h^N, *we say that* h^M *and* h^N *are conjugate by* f *if*
$$f^N = f \circ f^M \circ f^{-1} \text{ and } g^N = f \circ g^M \circ f^{-1}$$
We write this as $h^N = f \circ h^M \circ f^{-1}$.

A key notion is that of extending and amalgamating solutions.

Definition 3.6 (Definition 2.9 in [BK09])**.**

(1) *A solution* $h = (f, g)$ *extends another solution* $h' = (f', g')$ *if* $f' \subseteq f$ *and* $g' \subseteq g$.
(2) *We say that* $\mathbf{K}^n$ *has* k-amalgamation for solutions over sets of size λ *if given any* $M \in \mathbf{K}^n$, $A \subseteq I(M)$ *of size* λ, $\{b_1, \ldots, b_n\} \subseteq I(M)$, *and solutions* h_w *over* $A \cup \{b_i \mid i \in w\}$ *for every* $w \in [\{b_1, \ldots, b_b\}]^{n-1}$ *such that* $\bigcup_w h_w$ *is a function, there is a solution* h *for* $A \cup \{b_i \mid i \leq n\}$ *that extends all* h_w.

0-amalgamation is often referred to simply as the existence of solutions and 1-amalgamation is the extension of solutions.

Forgetting the Q predicate, $M \in \mathbf{K}^n$ is a bunch of affine copies of G^M, so an isomorphism is determined by a bijection between the copies and picking a 0 from each affine copy. However, adding Q complicates this picture. Solutions are the generalization of picking 0's to $\mathbf{K}^n$. Thus, amongst the models of $\mathbf{K}^n$ admitting solutions (which is at least $\mathbf{K}^n_{\aleph_{n-2}}$, see Fact 3.9), there is a strong, functorial correspondence between isomorphisms between M and N and pairs of solutions for M and N.

The following is implicit in [BK09], especially Lemma 2.6 there.

Fact 3.7. *We work in $\mathbf{K}^n$.*

(1) *Given $f : M \cong N$ and a solution h^M of M, there is a unique solution h^N of N that is conjugate to h^M by f. Moreover, if $f' : M' \cong N'$ extends[6] f and $h^{M'}$ is a solution of M' extending h^M, then the resulting $h^{N'}$ extends h^N.*
(2) *Given solutions h^M for M and h^N for N and a bijection $h_0 : I(M) \to I(N)$, there is a unique isomorphism $f : M \cong N$ extending h_0 such that h^M and h^N are conjugate by f. Moreover, if $h^{M'}$ and $h^{N'}$ are solutions for M' and N' that extend h^M and h^N, then the resulting f' extends f.*
(3) *These processes are inverses of each other: if we have [$f : M \cong N$ and a solution h^M of M]/[solutions h^M and h^N for M and N and a bijection $h_0 : I(M) \to I(N)$] and then apply [(1) and then (2)]/[(2) and then (1)], then [the resulting isomorphism is f]/[the resulting solutions for N is h^N],*

Lemma 3.8. *Suppose $M, N \in \mathbf{K}^n$ and $f_0 : I(M) \to I(N)$ is an injection. Then there is a unique extension to f_1 with domain $M - (G^*(M) \cup H^*(M))$ that must be extended by any strong embedding extending f_0.*

Proof. $M - (G^*(M) \cup H^*(M))$ is the definable closure of $I(M)$, so the value of f_0 on $I(M)$ determines the value on $M - (G^*(M) \cup H^*(M))$. $\square$

For the following, write $\aleph_{-1}$ for finite.

Fact 3.9. *Let $n \in [2, \omega)$, $k_0 < \omega$, and $k_1 \in \{-1\} \cup \omega$. The following are equivalent:*

(1) *$\mathbf{K}^n$ has k_0-amalgamation of solutions over $\aleph_{k_1}$-sized sets.*
(2) *$k_0 + k_1 \leq n - 2$.*

Proof. (1) implies (2) by the examples of [BK09, Section 6]. (2) implies (1) by combining [BK09, Lemmas 2.11, 2.14]. $\square$

We could do many more variations on the following, but I think this statement suffices for what we need to show.

[6] So $M \leq_{\mathbf{K}^n} M'$ and $N \leq_{\mathbf{K}^n} N'$.

Definition 3.10. *For $n \in [2, \omega)$ and I an index set, the* standard model for I *is the unique $M \in \mathbf{K}^n$ such that $G^*(M) = K \times G_K$, where $K := [I]^n$.*

Lemma 3.11. *Let $n \in [3, \omega)$. Given any $M \leq_{\mathbf{K}^n} N$ from $\mathbf{K}^n_{\leq \aleph_{n-3}}$, we may assume that they are standard. That is, if we write M^* for the standard model of $I(M)$ and N^* for the standard model on $I(N)$, then there is an isomorphism $f : N \cong_{I(N)} N^*$ that restricts to an isomorphism $M \cong_{I(M)} M^*$.*

Proof. Find a solution h^M for M and extend it to a solution h^N for N; this is possible by Fact 3.9 since $(n-3)+1 \leq n-2$. We have solutions h^{M^*} and h^{N^*} for M^* and N^* because they are the standard models and, thus, have solutions. Then Theorem 3.7 allows me to build an isomorphism between M and M^* and extend it to $f : N \cong N^*$, each of which extend the identity on I. □

4. Tameness and shortness

The following is a strengthening of [BK09, Theorem 5.1] to include type-shortness.

Theorem 4.1. *For $n \in [3, \omega)$, $\mathbf{K}^n$ is $(< \aleph_0, < \aleph_{n-3})$-type short over $\leq \aleph_{n-3}$-sized models and $(< \aleph_0, \leq \aleph_{n-3})$-tame for $(< \aleph_{n-3})$-length types. Moreover, these Galois types are equivalent to first-order existential (syntactic) types.*

Proof. For this proof, write $\text{tp}_\exists$ for the first-order existential type. We prove the type-shortness claim. The tameness result follows from [BK09, Theorem 5.1].

Let $M \in K^n_{\leq \aleph_{n-3}}$ and $M \leq_{\mathbf{K}^n} N^A, N^B$ with $A \subseteq |N^A|$, $B \subseteq |N^B|$ of size $\leq \aleph_{n-4}$ such that $\text{tp}_\exists(A/M; N^A) = \text{tp}_\exists(B/M; N^B)$. By [BK09, Lemma 4.2], we can find minimal, full substructures M^A and M^B. Additonally, for each finite $\mathbf{a} \in A$ and $\mathbf{b} \in B$, we can find minimal full substructures $M^{\mathbf{a}}$ and $M^{\mathbf{b}}$ in M^A and M^B. It's easy to see that M^A is the directed union of $\{M^{\mathbf{a}} \mid \mathbf{a} \in A\}$ and similarly for M^B; note that we don't necessarily have $M^{\mathbf{a}}, M^{\mathbf{a}'} \subseteq |M^{\mathbf{a} \cup \mathbf{a}'}|$, but they are in $M^{M^{\mathbf{a}} \cup M^{\mathbf{a}'}}$.

Set $M_0 = M^A \cap M$. We want to build $f_0 : M^A \to_{M_0} N^B$ such that $f_0(A) = B$. Similarly, construct M^B. Note that

$$M_0 = M^A \cap M = \cup_{\mathbf{a} \in M}(M^{\mathbf{a}} \cap M_0) = \cup_{\mathbf{b} \in M}(M^{\mathbf{b}} \cap M_0) = M^B \cap M_0$$

By assumption, we have $\text{tp}_\exists(A/M_0; M^A) = \text{tp}_\exists(B/M_0; M^A)$. Set $X = \{\pi^{M^A}(x) \mid x \in A \cap G^*(M^A)\}$ and $Y = \{\pi^{M^B}(x) \mid x \in B \cap G^*(M^B)\}$, indexed appropriately.

Claim: $\text{tp}_\exists(AX/M_0; M^A) = \text{tp}_\exists(BY/M_0; M^A)$

This is true because all of the added points are in the definable closure via an existential formula.

Thus, the induced partial map $f : AX \to BY$ is $\exists$-elementary. By Fact 3.9, we have extensions of solutions. Let h^{M^A} be a solution for M^A. Then we can restrict this to h^X which is a solution for X. Then we can define a solution h^Y for Y by conjugating it with f. Finally, we can extend h^Y to a solution h^{M^B} for

M^B. Since they satisfy the same existential type and the extensions are minimally constructed, we can define a bijection $h_0 : I(M^A) \to I(M^B)$ respecting the type. Given the two solutions and the bijection h_0, we can use Theorem 3.7 to find an isomorphism $f_0 : M^A \cong M^B$ extending h_0 and making these solutions conjugate. By construction, f_0 fixes M_0 and sends A to B.

Resolve M as $\langle M_i \mid i < \alpha \rangle$ starting with M_0 so $\|M_i\| \leq \aleph_{n-4}$. Then find increasing continuous $\langle M_i^A, M_i^B \mid i < \alpha \rangle$ by setting $M_0^A = M^A$ and M_{i+1}^A to be a disjoint amalgam[7] of M_{i+1} and M_i^A over M_i, and similarly for M_i^B.

Using extension of solutions, we can find an increasing chain of solutions $\langle h^{M_i} \mid i < \alpha \rangle$ for M_i. Using 2-amalgamation of solutions over $\leq \alpha_{n-4}$ sized sets[8], we can find increasing chains of solutions $\langle h^{M_i^A}, h^{M_i^B} \mid i < \alpha \rangle$ for M_i^A and M_i^B, respectively, such that $h^{M_i^A}$ also extends h^{M_i}.

By another application of Theorem 3.7.(2), this gives us an increasing sequence of isomorphism $\langle f_i : M_i^A \cong_{M_i} M_i^B \mid i < \alpha \rangle$; here we are using that $I(M_{i+1}^A) - I(M_i^A) = I(M_{i+1}^B) - I(M_i^B)$. At the top, we have that $f_\alpha : M^A \cong_M M^B$. This demonstrates that $\mathrm{gtp}(A/M; N^A) = \mathrm{gtp}(B/M; N^B)$. □

Baldwin and Kolesnikov [BK09] have shown that tameness fails at the next cardinal and we will see later (Corollary 8.13) that $\mathbf{K}^n$ is *not* $(< \aleph_{n-3}, \aleph_{n-3})$-type short over $\aleph_{n-3}$-sized models.

5. What the abstract theory tells us

We combine the abstract theory with the facts derived so far about the Hart-Shelah example.

We first give an abstract argument that in the Hart-Shelah example good frames below $\aleph_{n-3}$ are weakly successful (in fact successful):

Theorem 5.1. *Let $n \in [3, \omega)$. For any $k \in [1, n-3]$, there is a type-full good $\aleph_k$-frame $\mathfrak{s}$ on $\mathbf{K}^n$. Moreover, $\mathfrak{s}$ (and therefore $\mathfrak{s}^{k,n}$) is successful if $k < n-3$.*

Proof. Let $\lambda := \aleph_k$. First, assume that $k < n-3$. By Fact 3.2, $\mathbf{K}^n$ is categorical in λ, λ^+ and is $(< \aleph_0, \leq \lambda^+)$-tame. By Theorem 4.1, $\mathbf{K}$ is $(< \aleph_0, \lambda)$-type-short over λ-sized models. Thus one can apply Fact 2.18 (where κ there stands for $\aleph_0$ here) to get a weakly successful type-full good λ-frame $\mathfrak{s}$ on $\mathbf{K}^n$. By Fact 2.19, $\mathfrak{s}$ is actually successful. This implies that $\mathfrak{s}^{k,n}$ is successful by canonicity (Fact 2.20).

Second, assume $k = n-3$. We can still apply Fact 2.18 to get the existence of a type-full good λ-frame $\mathfrak{s}$, although we do not know it will be weakly successful (in fact this will fail, see Proposition 6.6). Then Fact 2.20 implies that $\mathfrak{s}^{k,n}$ is $\mathfrak{s}$ restricted to types in I. □

[7]Crucially, it is an amalgam such that $I(M_{i+1}^A) = I(M_i^A) \cup I(M_{i+1})$ with the union disjoint over $I(M_i)$; this is guaranteed by the second clause of the claim.

[8]Crucially, this holds here, but fails at the next cardinal. Thus, we couldn't use this argument to get $(< \aleph_0, \aleph_{n-3})$-type shortness or over $\aleph_{n-2}$ sized models.

Note that the case $k = 0$ is missing here, and will have to be treated differently (see Theorem 6.3 and Corollary 8.12; alternatively, see Appendix C). On the negative side, we show that $\mathfrak{s}^{n-3,n}$ cannot be successful. First, we show that it is good$^+$ (Definition 2.14).

Lemma 5.2. *For $n \in [3, \omega)$ and $k \leq n-3$, $\mathfrak{s}^{k,n}$ is good$^+$.*

Proof. Essentially this is because forking is trivial. In details, suppose that $\mathfrak{s}^{k,n}$ is not good$^+$ and fix $\langle M_i : i < \lambda^+ \rangle$, $\langle N_i : i < \lambda^+ \rangle$, $\langle a_i : i < \lambda^+ \rangle$ and p witnessing it. The set of $i < \lambda^+$ such that $M_{\lambda^+} \cap N_i = M_i$ is club, so pick such an i. Since $\mathrm{gtp}(a_{i+1}/M_{i+1}; M_{i+2})$ is a nonforking extension of p, we know that $a_{i+1} \in I(M_{i+2}) \backslash I(M_{i+1})$. Because $M_{\lambda^+} \cap N_i = M_i$, we have that $a_{i+1} \notin |N_i|$. Since $a_{i+1} \in I(M_{i+2})$, also $a_{i+1} \in I(N_{i+2})$. Therefore $\mathrm{gtp}(a_{i+1}/N_i; N_{i+2})$ does not fork over M_0, contradicting the defining assumption on $\langle N_i : i < \lambda^+ \rangle$. $\square$

Corollary 5.3. *For $n \in [3, \omega)$, $\mathfrak{s}^{n-3,n}$ is* not *successful.*

Proof. Suppose for a contradiction that $\mathfrak{s}^{n-3,n}$ is successful. Let $\lambda := \aleph_{n-3}$. By Fact 2.16, we can get a good λ^+-frame on the saturated models of $\mathbf{K}^n_{\lambda^+}$. Since $\mathbf{K}^n$ is categorical in λ^+, this gives a good λ^+-frame on $\mathbf{K}^n_{\lambda^+}$. In particular, $\mathbf{K}^n$ is stable in λ^+, contradicting Fact 3.2. $\square$

Notice that the proof gives no information as to which part of the definition of successful fails: i.e. whether $\mathfrak{s}^{n-3,n}$ has the existence property for uniqueness triples (and then smoothness for $\leq^{\mathrm{NF}}_{\mathbf{K}^n_{\lambda^+}}$ must fail) or not. To understand this, we take a closer look at uniqueness triples in the specific context of the Hart-Shelah example.

6. Uniqueness triples in Hart-Shelah

In this section, we show that the frame $\mathfrak{s}^{n-3,n}$ is *not* weakly successful. This follows from the fact that the existence of uniqueness triples corresponds exactly to amalgamation of solutions.

The following says that it is sufficient to check one point extensions when trying to build uniqueness triples.

Lemma 6.1. *Let $n \in [3, \omega)$ and let $k \leq n-3$. The good $\aleph_k$-frame $\mathfrak{s}^{k,n}$ (see Definition 3.3) is weakly successful if the following holds.*

- (∗) *Whenever $M, M_a, M_b, M_{ab} \in \mathbf{K}_{\aleph_k}$ are such that:*
  - (1) *$I(M_x) = I(M) \cup \{x\}$ for $x = a, b, ab$;*
  - (2) *$M \leq_{\mathbf{K}^n} M_a, M_b$ and $M_b \leq_{\mathbf{K}^n} M_{ab}$; and*
  - (3) *there is $f_\ell : M_a \to_M M_{ab}$ such that $f_\ell(a) = a$.*

  Then there is $f_* : M_{ab} \cong_{M_b} M_{ab}$ such that $f_* \circ f_1 = f_2$

Remark 6.2. *By an easy renaming exercise, we could have the range of f_ℓ be distinct one point extensions of M_b with $f_\ell(a)$ being that point.*

Proof of Lemma 6.1. Suppose that $(*)$ holds. Let $p = \mathrm{gtp}(a/M; N^+) \in \mathrm{gS}^{bs}(M)$ and find some[9] $M_a \leq_{\mathbf{K}^n} N^+$ so $I(M_a) = I(M) \cup \{a\}$. We want to show that this is a uniqueness triple. To this end, suppose that we have $N \succ M$, $N \leq_{\mathbf{K}^n} M_\ell$, and $f_\ell : M_a \to_M M_\ell$ with $f_\ell(a) \notin N$. Enumerate $I(N) - I(M) = \{a_i \mid i < \mu \leq \aleph_k\}$; Without loss of generality $I(M_1) \cap I(M_2) = I(N)$. Let $M_\ell^- \leq_{\mathbf{K}^n} M_\ell$ be such that $I(M_\ell^-) = \{f_\ell(a)\} \cup I(N)$.

Claim: We can find $f_-^* : M_1^- \cong_N M_2^-$ such that $(f_-^*)^{-1} \circ f_1 = f_2$.

This is enough: from the claim, we have $M_1^- \leq_{\mathbf{K}^n} M_1$ and $f_-^* : M_1^- \to M_2$. The class has disjoint amalgamation by Fact 3.2, so find a disjoint amalgam N^* with maps $g_\ell : M_\ell \to N^*$ such that $g_1 \restriction M_1^- = g_2 \circ f_-^*$. This is the witness required to have that (a, M, M_a) is a uniqueness triple.

Proof of the claim: We can find resolutions $\langle N_i : i < \mu \rangle$ and $\langle M_i^\ell \mid i < \mu \rangle$ such that

(1) $M \leq_{\mathbf{K}^n} N_i \leq_{\mathbf{K}^n} M_i^\ell \leq_{\mathbf{K}^n} M_\ell^-$ and $f_\ell(M_a) \leq_{\mathbf{K}^n} M_i^\ell$; and
(2) $I(N) = I(M) \cup \{a_j \mid j < i\}$ and $I(M_i^\ell) = I(N_i) \cup \{f_\ell(a)\}$.

The values of I for these models is specified, which determines K and G. Then G^* and H^* are just picked to be subsets of the larger models version that is closed under the relevant action. Since there are embeddings going everywhere, this can be done.

We build increasing, continuous $f_i^* : M_i^1 \cong_{N_i} M_i^2$ such that $f_i^* \circ f_1 = f_2$ by induction on $i \geq 1$.

- For $i = 1$, we use $(*)$ taking $b = a_0$ (and using the renamed formulation). This gives $f_1^* : M_1^1 \cong_{N_1} M_1^2$.
- For i limit, we take unions of everything.
- For $i = j + 1$, we have an instance of $(*)$:

$$\begin{tikzcd}
& M^1_{j+1} & \\
M^1_j \arrow[ur] \arrow[rr, "f_i^*"'] & & M^2_{j+1} \\
N_j \arrow[u] \arrow[r] & N_{j+1} \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

Then we can find $f_{i+1}^* : M_{j+1}^1 \cong M_{j+1}^2$ that works.

□

We can now give a direct proof of Theorem 5.1 that also treats the case $k = 0$.

Theorem 6.3. *Let $n \in [3, \omega)$. For any $k < n - 3$, $\mathfrak{s}^{k,n}$ is successful.*

[9] M_a is not unique, but there is such an M_a

Proof. By Fact 2.19 (as in the proof of Theorem 5.1), it is enough to show that $\mathfrak{s}^{k,n}$ is weakly successful. It suffices to show $(*)$ from Lemma 6.1. We start with a solution h on $I(M)$. Working inside M_{ab}, we can find extensions h_a^1, h_a^2, h_b of h that are solutions for $f_1(M_a), f_2(M_a), M_b$ by the extension property of solutions (which holds because 2-amalgamation does). Now, for $\ell = 1, 2$, amalgamate h_a^ℓ and h_b over h into h_{ab}^ℓ, which is a solution for M_{ab}. We use this to get a isomorphism f_*.

Set f_* to be the identity on $I(M_{ab}) = I(M) \cup \{a, b\}$. This determines its value on K, G, and $\mathbb{Z}_2$.

Let $x \in G_u^*(M_{ab})$ for $u \in K(M_{ab})$. There is a unique $\gamma \in G(M_{ab})$ such that $t_{G^*}^{M_{ab}}(u, f_{ab}^1(u), x, \gamma)$. Then, there is a unique $y \in G_u^*(M_{ab})$ such that $t_{G^*}^{M_{ab}}(u, f_{ab}^2(u), y, \gamma)$. Set $f_*(x) = y$.

Let $x \in H_u^*(M_{ab})$ for $u \in K(M_{ab})$. There is a unique $n \in H(M_{ab})$ such that $t_{H^*}^{M_{ab}}(u, f_{ab}^1(u), x, n)$. Then, there is a unique $y \in H_u^*(M_{ab})$ such that $t_{H^*}^{M_{ab}}(u, f_{ab}^2(u), y, n)$. Set $f_*(x) = y$.

This is a bijection on the universes, and clearly preserves all structure except maybe Q. So we show it preserves Q. It suffices to show one direction for positive instances of Q. So let $u_1, \dots, u_k, v$ be compatible from $K(M_{ab})$ and $x_j \in G_{u_j}^*(M_{ab}), y \in H_v^*(M_{ab})$ such that

$$M_{ab} \vDash Q(x_1, \dots, x_k, y)$$

Note, by definition of solutions, we have

$$M_{ab} \vDash Q\left(f_{ab}^1(u_1), \dots, f_{ab}^1(u_k), g_{ab}^1(v)\right)$$
$$M_{ab} \vDash Q\left(f_{ab}^2(u_1), \dots, f_{ab}^2(u_k), g_{ab}^2(v)\right)$$

By the properties of Q, we get $\gamma_j \in G(M_{ab})$ and $n \in H(M_{ab})$ such that

(1) $t_{G^*}^{M_{ab}}(u_j, f_{ab}^1(u_j), x_j, \gamma_j)$
(2) $t_{H^*}^{M_{ab}}(v, g_{ab}^1(v), y, n)$
(3) $\gamma_1(v) + \cdots + \gamma_k(v) + n \equiv 0 \mod 2$

Then, by definition of f_*, we have

(1) $t_{G^*}^{M_{ab}}(u_j, f_{ab}^2(u_j), f_*(x_j), \gamma_j)$
(2) $t_{H^*}^{M_{ab}}(v, g_{ab}^2(v), f_*(y), n)$

By the evenness of these shifts, we have that

$$M_{ab} \vDash Q\left(f_*(x_1), \dots, f_*(x_k), f_*(y)\right)$$

Perfect.

The commutativity condition is easy to check. □

The next two lemmas show that the uniqueness triples (if they exist) must be exactly the one point extensions. This can be seen from the abstract theory [She09a, Claim III.3.5] but we give a direct proof here.

Lemma 6.4. *Let $n \in [3, \omega)$ and let $k \le n-3$. If (a, M, M^+) is a uniqueness triple of $\mathfrak{s}^{k,n}$, then $I(M^+) = I(M) \cup \{a\}$.*

Recall (Definition 3.10) that the standard model is the one where G^* is literally equal to $K \times G$, so that we can easily recover 0's.

Proof. Deny. By Lemma 3.11, without loss of generality, we have that M is the standard model on $I(M) = X$ and M^+ is the standard model on $I(M^+) = X \cup X^+ \cup \{a\}$ (those unions are disjoint) with X^+ nonempty. Set N to be the standard model on $X \cup (2 \times X^+)$ and N_0, N_1 to be standard models on $X \cup 2 \times X^+ \cup \{a\}$. For $\ell = 0, 1$, define $f_\ell : M^+ \to_M N_\ell$ by

(1) f_ℓ is the identity on $X \cup \{a\}$ and sends $x \in X^+$ to (ℓ, x).
(2) The above determines the map on K, H, and G.
(3) $(u, x) \in G^*(M^+)$ goes to $(f_\ell(u), x) \in G^*(N_\ell)$.
(4) $(u, n) \in H^*(M^+)$ goes to $(f_\ell(u), n) \in H^*(N_\ell)$.

Then this is clearly a set-up for weak uniqueness. However, suppose there were a N^* with $g_\ell : N_\ell \to_N N^*$ such that $g_0 \circ f_0 = g_1 \circ f_1$. Let $x \in X^+$. Then

$$(0, x) = g_0(x) = f_0(g_0(x)) = f_1(g_1(x)) = f_1(1, x) = (1, x)$$

which is false. $\square$

Lemma 6.5. *Let $n \in [3, \omega)$ and let $k \leq n - 3$. Let $M \leq_{\mathbf{K}^n} N$ both be in $\mathbf{K}^n_{\aleph_k}$. If $\mathfrak{s}^{k,n}$ is weakly successful, then (a, M, N) is a uniqueness triple of $\mathfrak{s}^{k,n}$ if and only if $I(N) = I(M) \cup \{a\}$.*

Proof. Lemma 6.4 gives one direction. Conversely, let (a, M, N) with $I(N) = I(M) \cup \{a\}$. Since $\mathfrak{s}^{k,n}$ is weakly successful, there is some uniqueness triple (b, M', N') representing $\text{gtp}(a/M; N)$. By Lemma 6.4, we must have $I(N') = I(M') \cup \{b\}$. By Lemma 3.11, we have $(M, N) \cong (M', N')$ since they are both isomorphic to the standard model. This isomorphism must take a to b. Since $(a, M, N) \cong (b, M', N')$, the former is a uniqueness triple as well. $\square$

We deduce that $\mathfrak{s}^{n-3,n}$ is not even weakly successful.

Theorem 6.6. *For $n \in [3, \omega)$, $\mathfrak{s}^{n-3,n}$ is not weakly successful.*

Proof. Let $\lambda := \aleph_{k-3}$. At this cardinal, 2-amalgamation of solutions over sets of size λ fails. To witness this, we have:

- M of size λ with solution $h = (f, g)$
- M_a has a solution $h_a = (f_a, g_a)$
- M_b has a solution $h_b = (f_b, g_b)$
- M_{ab} has no solution that extends them both
- $I(M_x) = I(M) \cup \{x\}$ for $x = a, b, ab$

However, λ does have extension of solutions, so let $h_{ab} = (f_{ab}, g_{ab})$ be a solution for M_{ab} that extends h_b. h_{ab} is a solution for $I(M_a)$ in M_{ab}.[10] Set $f_1 : M_a \to_M M_{ab}$ to be the identity. Define $f_2 : M_a \to_M M_{ab}$ as follows:

[10]Note that it isn't a solution in M_a as $f_{ab}(u)$ might not be in M_a for $u \in M_a$.

- identity on $I(M) \cup \{a\}$, which determines it except on the affine stuff (in the sense of Lemma 3.8)
- Let $x \in G^*_u(M_a)$ for $u \in K(M_a)$. Set f_2 to send $f_a(u)$ to $f_{ab}(u)$ and the rest falls out by the G action
- Let $x \in H^*_u(M_a)$ for $u \in K(M_a)$. Set f_2 to send $g_a(u)$ to $g_{ab}(u)$ and the rest falls out by the G action.

This map commutes on M because if $u \in K(M)$, then $f_{ab}(u) = f_a(u) = f(u)$.

We claim that $\mathrm{gtp}(a/M; M_a)$ does not have a uniqueness triple. Suppose it does. By Lemma 6.5, (a, M, M_a) is one.

Suppose that we had N^* and $g_\ell : M_{ab} \to_{M_b} N^*$ such that $g_1 = g_2 f_2$ and $g_1(a) = g_2(f_2(a))$ (recalling that f_1 is the identity).

Claim: If $u \in K(M_a)$, then $g_1(G^*_u(M_{ab})) = g_2(G^*_u(M_{ab}))$.

There is $\gamma_u \in G(M_{ab})$ such that $f_{ab}(u) = f_a(u) + \gamma_u$. Given $x \in G^*_u(M_{ab})$,

$$g_1(x) = g_2(f_2(x)) = g_2(x + \gamma_u) = g_2(x) + \gamma_u$$

Thus $g_1(G^*_u(M_{ab}))$ and $g_2(G^*_u(M_{ab}))$ are both subsets of $G^*_u(N^*)$ that have a 1-transitive action of $G(M_{ab})$ and share points. $\dagger_{\text{Claim}}$

Now define $h^+ = (f^+, g^+)$ on M_{ab} by

$$\begin{aligned} f^+(u) &= g_1^{-1} \circ g_2 \circ f_{ab}(u) \\ g^+(u) &= g_1^{-1} \circ g_2 \circ g_{ab}(u) \end{aligned}$$

We claim h^+ extends both h_a and h_b. If $u \in K(M_b)$, then $f_{ab}(u) = f_b(u) \in M_b$, so

$$f^+(u) = g_1^{-1} \circ g_2 \circ f_{ab}(u) = f_{ab}(u) = f_b(u)$$

since the g_ℓ's fix M_b. Suppose $u \in K(M_a)$. First note that $g_1^{-1} \circ g_2 = f_2^{-1}$ by assumption. Also, since $f_2(f_a(u)) = f_{ab}(u)$ and f_2 respects the group action, $f_2(f_{ab}(u)) = f_a(u)$. Thus

$$f^+(u) = g_1^{-1} \circ g_2 \circ f_{ab}(u) = f_2^{-1} \circ f_{ab}(u) = f_a(u)$$

Similarly for g^+.

But this is our contradiction! h_a and h_b were not amalgamable, so there is no isomorphism. $\square$

7. Nonforking is disjoint amalgamation

Recall that if a good frame is weakly successful, one can define an independence relation NF for models (see Definition 2.11). We show here that NF in the Hart-Shelah example is just disjoint amalgamation, i.e. $\mathrm{NF}(M_0, M_1, M_2, M_3)$ holds if and only if $M_0 \leq_{\mathbf{K}^n} M_\ell \leq_{\mathbf{K}^n} M_3$ for $\ell < 4$ and $M_1 \cap M_2 = M_0$. We deduce another proof of Theorem 6.6.

We will use the following weakening of [BK09, Lemma 4.2]

Fact 7.1. *Let $n \in [2, \omega)$. If $M_0, M_1 \leq_{\mathbf{K}^n} N$, then there is $M_2 \leq_{\mathbf{K}^n} N$ such that $I(M_2) = I(M_0) \cup I(M_1)$ and $M_0, M_1 \leq_{\mathbf{K}^n} M_2$.*

Theorem 7.2. *Let $n \in [3, \omega)$ and let $k \leq n-3$. Let $\lambda := \aleph_k$ and let $M_0, M_1, M_2, M_3 \in \mathbf{K}^n_\lambda$ with $M_0 \leq_{\mathbf{K}^n} M_\ell \leq_{\mathbf{K}^n} M_3$ for $\ell < 4$. If $\mathfrak{s}^{k,n}$ is weakly successful, then $\mathrm{NF}_{\mathfrak{s}^{k,n}}(M_0, M_1, M_2, M_3)$ if and only if $M_1 \cap M_2 = M_0$.*

Proof. Write NF for $\mathrm{NF}_{\mathfrak{s}^{k,n}}$. The left to right direction follows from the properties of NF (Fact 2.12). Now assume that $M_1 \cap M_2 = M_0$.

Write $I(M_1) - I(M_0) = \{d_i \mid i < \alpha^*\}$. By induction, build increasing, continuous $M_{1,i} \leq_{\mathbf{K}^n} M_1$ for $i < \alpha^*$ so $I(M_{1,i}) = I(M_0) \cup \{d_j \mid j < i\}$. Again by induction, build increasing continuous $M_{2,i} \leq_{\mathbf{K}^n} M_3$ for $i \leq \alpha^*$ such that

- $I(M_{2,i}) = I(M_2) \cup \{d_j \mid j < i\}$
- $M_{1,i} \leq_{\mathbf{K}^n} M_{2,i}$

The successor stage of this construction is possible by Fact 7.1 and the limit is easy. Now it's easy to see that $\mathrm{gtp}(d_i / M_{2,i}; M_{2,i+1})$ does not fork over $M_{1,i}$. Furthermore by Lemma 6.5, $(d_i, M_{1,i}, M_{1,i+1})$ is a uniqueness triple. Thus letting $M_3' := M_{2,\alpha^*}$, we have that $\mathrm{NF}^*(M_0, M_1, M_2, M_3')$, so $\mathrm{NF}(M_0, M_1, M_2, M_3')$. By the monotonicity property of NF, $\mathrm{NF}(M_0, M_1, M_2, M_3)$ also holds. $\square$

We deduce another proof of Theorem 6.6. First we show that weakly successful implies successful in the context of Hart-Shelah:

Lemma 7.3. *Let $n \in [3, \omega)$ and let $k \leq n - 3$. If $\mathfrak{s}^{k,n}$ is weakly successful, then $\mathfrak{s}$ is successful (recall Definition 2.13). Moreover for $M_0, M_1 \in \mathbf{K}^n_{\lambda^+}$, $M_0 \leq^{\mathrm{NF}}_{\mathbf{K}^n_{\lambda^+}} M_1$ if and only if $M_0 \leq_{\mathbf{K}^n} M_1$.*

Proof. This is straightforward from Definition 2.13 and Theorem 7.2. $\square$

We deduce another proof of Theorem 6.6:

Corollary 7.4. *For $n \in [3, \omega)$, $\mathfrak{s}^{n-3,n}$ is not weakly successful.*

Proof. Assume for a contradiction that $\mathfrak{s}^{n-3,n}$ is weakly successful. By Lemma 7.3, $\mathfrak{s}^{n-3,n}$ is successful. This contradicts Corollary 5.3. $\square$

8. A type-full good frame at $\aleph_0$

We have seen that when $k < n - 3$, $\mathfrak{s}^{k,n}$ is successful good$^+$ and therefore by Fact 2.17 extends to a type-full frame. When $k = n - 3$, $\mathfrak{s}^{k,n}$ is not successful, but by Theorem 5.1, it still extends to a type-full frame if $k \geq 1$. In this section, we complete the picture by building a type-full frame when $k = 0$ and $n = 3$.

Recall that (when $n \geq 3$) $\mathbf{K}^n$ is a class of models of an $\mathbb{L}_{\omega_1,\omega}$ sentence, categorical in $\aleph_0$ and $\aleph_1$. Therefore by [She09a, Theorem II.3.4] (a generalization of earlier results in [She75, She83]), there will be a good $\aleph_0$-frame on $\mathbf{K}^n$ *provided* that $2^{\aleph_0} < 2^{\aleph_1}$. Therefore the result we want is at least consistent with ZFC, but we want to use the additional structure of the Hart-Shelah example to remove the cardinal arithmetic hypothesis.

So we take here a different approach than Shelah's, giving new cases on when an AEC has a good $\aleph_0$-frame (in Appendix C, we explore our argument further and also get new cases when good $\text{LS}(\mathbf{K})$-frames can be built, with $\text{LS}(\mathbf{K})$ potentially uncountable). As opposed to Shelah, we use Ehrenfeucht-Mostowski models (so assume that the AEC has arbitrarily large models), but do not need a very strong assumption on $\aleph_1$: although we use categoricity in $\aleph_1$ here (for simplicity), the argument generalize to just using solvability in $\aleph_1$, see Remark 8.8.

We start by studying what limit models look like in the Hart-Shelah example: Recall that we're working in a zone where we have extensions of solutions.

Theorem 8.1. *Let $n \in [3, \omega)$. Let $k \leq n - 3$ and let $M_0, M_1 \in \mathbf{K}^n_{\aleph_k}$. Then M_1 is universal over M_0 if and only if $|I(M_1) - I(M_0)| = \|M_1\|$. In particular, M_1 is universal over M_0 if and only if M_1 is limit over M_0.*

Proof. First suppose that M_1 is universal over M_0. We don't have maximal models, so let $M_0 \leq_{\mathbf{K}^n} N_*$ be such that $|I(N_*) - I(M_0)| = \|M_1\|$. We have that $\|N_*\| = \|M_1\|$, so there is an embedding $f : N_* \to_{M_0} M_1$. Then $f(I(N_*)) \subseteq I(M_1)$.

Now suppose that $|I(M_1) - I(M_0)| = \|M_1\|$ and let $M_0 \leq_{\mathbf{K}^n} N_*$ with $\|N_*\| = \|M_1\|$. Let $I^- \subseteq I(M_1) - I(M_0)$ be of size $|I(N_*) - I(M_0)|$ and let $M^- \leq_{\mathbf{K}^n} M_1$ have $I(M^-) = I(M_0) \cup I(M^-)$. Let (f, g) be a solution for M_0. Since we have extensions of solutions, we can extend this to solutions (f^-, g^-) on M^- and (f_*, g_*) on N_*. The whole point of solutions is that this allows us to build an isomorphism between M^- and N_* over M_0 by mapping the solutions to each other (see Theorem 3.7). $\square$

Shelah has defined a similar property [She09a, Definition 1.3.(2)][11]:

Definition 8.2. $\mathbf{K}$ *is* λ-saturative *(or* saturative in λ*) if for any $M_0 \leq_{\mathbf{K}} M_1 \leq_{\mathbf{K}} M_2$ all in $\mathbf{K}_\lambda$, if M_1 is limit over M_0, then M_2 is limit over M_0.*

So an immediate consequence of Theorem 8.1 is:

Corollary 8.3. *Let $n \in [3, \omega)$. For any $k \leq n - 3$, $\mathbf{K}^n$ is saturative in $\aleph_k$.*

We will use the following consequence of being saturative:

Lemma 8.4. *Assume that* $\text{LS}(\mathbf{K}) = \aleph_0$*, and $\mathbf{K}_{\aleph_0}$ has amalgamation, no maximal models, and is stable in $\aleph_0$. Let $\langle M_i : i \leq \omega \rangle$ be an increasing continuous chain in $\mathbf{K}_{\aleph_0}$. If $\mathbf{K}$ is categorical in $\aleph_0$ and saturative in $\aleph_0$, then there exists an increasing continuous chain $\langle N_i : i \leq \omega \rangle$ such that:*

(1) *For $i < \omega$, M_i is limit over N_i.*
(2) *For $i < \omega$, N_{i+1} is limit over N_i.*
(3) *$N_\omega = M_\omega$.*

[11]Shelah defines saturative as a property of frames, but it depends only on the class.

Proof. Let $\{a_n : n < \omega\}$ be an enumeration of $|M_\omega|$. We will build $\langle N_i : i \leq \omega \rangle$ satisfying (1) and (2) above and in addition that for each $i < \omega$, $\{a_n : n < i\} \cap |M_i| \subseteq |N_i|$. Clearly, this is enough.

This is possible. By categoricity in $\aleph_0$, any model of size $\aleph_0$ is limit, so pick any $N_0 \in \mathbf{K}_{\aleph_0}$ such that M_0 is limit over N_0. Now assume inductively that N_i has been defined for $i < \omega$. Since $\mathbf{K}$ is saturative in $\aleph_0$, M_{i+1} is limit over N_i. Since all limit models of the same cofinality are isomorphic, M_{i+1} is in particular $(\aleph_0, \omega \cdot \omega)$-limit over N_i. Fix an increasing continuous sequence $\langle M_{i+1,j} : j \leq \omega \cdot \omega \rangle$ witnessing it: $M_{i+1,0} = N_i$, $M_{i+1,\omega \cdot \omega} = M_{i+1}$, and $M_{i+1,j+1}$ is universal over $M_{i+1,j}$ for all $j < \omega \cdot \omega$. Now pick $j < \omega \cdot \omega$ big enough so that $\{a_n : n < i+1\} \cap |M_{i+1}| \subseteq |M_{i+1,j}|$. Let $N_{i+1} := M_{i+1,j+\omega}$. $\square$

Remark 8.5. *We do not know how to replace $\aleph_0$ by an uncountable cardinal in the argument above: it is not clear what to do at limit steps.*

To build the good frame, we will also use the transitivity property of splitting:

Definition 8.6. *We say that $\mathbf{K}$ satisfies* transitivity *in μ (or μ-*transitivity*) if whenever $M_0, M_1, M_2 \in \mathbf{K}_\mu$, M_1 is limit over M_0 and M_2 is limit over M_1, if $p \in \mathrm{gS}(M_2)$ does not μ-split over M_1 and $p \restriction M_1$ does not μ-split over M_0, we have that p does not μ-split over M_0.*

The following result of Shelah [She99, Claim 7.5] is key. For the convenience of the reader, we provide a proof in the appendix (see Fact B.7).

Fact 8.7. *Let $\mu \geq \mathrm{LS}(\mathbf{K})$. Assume that $\mathbf{K}_\mu$ has amalgamation and no maximal models. If $\mathbf{K}$ has arbitrarily large models and is categorical in μ^+, then $\mathbf{K}$ has transitivity in μ.*

Remark 8.8. *Although we will not use it, categoricity in μ^+ can be replaced by* solvability *in μ^+ (see Appendix A). By a recent result of the second author (Fact A.5), it follows from amalgamation, no maximal models, and categoricity in any $\lambda > \mu$.*

We will also use two lemmas on splitting isolated by VanDieren [Van06, Theorems I.4.10, I.4.12].

Fact 8.9. *Let $\mu \geq \mathrm{LS}(\mathbf{K})$. Assume that $\mathbf{K}_\mu$ has amalgamation, no maximal models, and is stable in μ. Let $M_0 \leq_{\mathbf{K}} M \leq_{\mathbf{K}} N$ all be in $\mathbf{K}_\mu$ such that M is universal over M_0.*

1. *Weak extension: If $p \in \mathrm{gS}(M)$ does not μ-split over M_0, then there exists $q \in \mathrm{gS}(N)$ extending p and not μ-splitting over M_0. Moreover q is algebraic if and only if p is algebraic.*
2. *Weak uniqueness: If $p, q \in \mathrm{gS}(N)$ do not μ-split over M_0 and $p \restriction M = q \restriction M$, then $p = q$.*

We are now ready to build the good frame:

Theorem 8.10. *If:*

(1) $\mathbf{K}$ *is superstable in* $\aleph_0$.
(2) $\mathbf{K}$ *has symmetry in* $\aleph_0$.
(3) $\mathbf{K}$ *has transitivity in* $\aleph_0$.
(4) $\mathbf{K}$ *is categorical in* $\aleph_0$.
(5) $\mathbf{K}$ *is saturative in* $\aleph_0$.

Then there exists a type-full good $\aleph_0$*-frame with underlying class* $\mathbf{K}_{\aleph_0}$.

Proof. By the superstability assumption, $\mathbf{K}_{\aleph_0}$ has amalgamation and no maximal models and is stable in $\aleph_0$. By the categoricity assumption, $\mathbf{K}_{\aleph_0}$ also has joint embedding. It remains to define an appropriate forking notion. For $M \leq_{\mathbf{K}} N$ both in $\mathbf{K}_{\aleph_0}$, let us say that $p \in \mathrm{gS}(N)$ *does not fork over* M if there exists $M_0 \in \mathbf{K}_{\aleph_0}$ such that M is universal over M_0 and p does not $\aleph_0$-split over M_0. We check that it has the required properties (see Definition 2.7):

(1) Invariance, monotonicity: Straightforward.
(2) Extension existence: By the weak extension property of splitting (Fact 8.9).
(3) Uniqueness: Let $M \leq_{\mathbf{K}} N$ both be in $\mathbf{K}_{\aleph_0}$ and let $p, q \in \mathrm{gS}(N)$ be nonforking over M such that $p \upharpoonright M = q \upharpoonright M$. Using the extension property, we can make N bigger if necessary to assume without loss of generality that N is limit over M. By categoricity, M is limit. Pick $\langle M_i : i \leq \omega \rangle$ increasing continuous witnessing it (so $M_\omega = M$ and M_{i+1} is universal over M_i for all $i < \omega$). By the superstability assumption, there exists $i < \omega$ such that $p \upharpoonright M$ does not $\aleph_0$-split over M_i and there exists $j < \omega$ such that $q \upharpoonright M$ does not $\aleph_0$-split over M_j. Let $i^* := i + j$. Then both $p \upharpoonright M$ and $q \upharpoonright M$ do not $\aleph_0$-split over M_{i^*}. By $\aleph_0$-transitivity, both p and q do not $\aleph_0$-split over M_{i^*}. Now use the weak uniqueness property of splitting (Fact 8.9).
(4) Continuity: In the type-full context, this follows from local character (see [She09a, Claim II.2.17.(3)]).
(5) Local character: Let $\delta < \omega_1$ be limit and let $\langle M_i : i \leq \delta \rangle$ be increasing continuous in $\mathbf{K}_{\aleph_0}$. Let $p \in \mathrm{gS}(M_\delta)$. We want to see that there exists $i < \delta$ such that p does not fork over M_i. We have that $\mathrm{cf}(\delta) = \omega$, so without loss of generality $\delta = \omega$. Let $\langle N_i : i \leq \omega \rangle$ be as given by Lemma 8.4 (we are using saturativity here). By superstability, there exists $i < \omega$ such that p does not $\aleph_0$-split over N_i. Because M_i is limit (hence universal) over M_i, this means that p does not fork over M_i, as desired.
(6) Symmetry: by $\aleph_0$-symmetry (see [VV, Theorem 4.13]).

□

Corollary 8.11. *Assume that* $\mathrm{LS}(\mathbf{K}) = \aleph_0$. *If:*

(1) $\mathbf{K}$ *has amalgamation in* $\aleph_0$.
(2) $\mathbf{K}$ *is categorical in* $\aleph_0$.

(3) $\mathbf{K}$ *is saturative in* $\aleph_0$.
(4) $\mathbf{K}$ *has arbitrarily large models and is categorical*[12] *in* $\aleph_1$.

Then there exists a type-full good $\aleph_0$*-frame with underlying class* $\mathbf{K}_{\aleph_0}$.

Proof. It is enough to check that the hypotheses of Theorem 8.10 are satisfied. First note that $\mathbf{K}$ has no maximal models in $\aleph_0$ because it has a model in $\aleph_1$ (by solvability) and is categorical in $\aleph_0$. Therefore by Fact 2.4, $\mathbf{K}$ is $\aleph_0$-superstable. By Fact 2.6, $\mathbf{K}$ has $\aleph_0$-symmetry. Finally by Fact 8.7, $\mathbf{K}$ has $\aleph_0$-transitivity. □

Corollary 8.12. *For* $n \in [3, \omega)$*, there exists a type-full good* $\aleph_0$*-frame on* $\mathbf{K}^n$.

Proof. By Fact 3.2 and Corollary 8.3, $\mathbf{K}^n$ satisfies the hypotheses of Corollary 8.11. □

The argument also allows us to prove that Theorem 4.1 is optimal, even when $n = 3$:

Corollary 8.13. *For* $n \in [3, \omega)$, $\mathbf{K}^n$ *is not* $(< \aleph_{n-3}, \aleph_{n-3})$*-type short over* $\aleph_{n-3}$*-sized models.*

Proof. Let $\lambda := \aleph_{n-3}$. By Theorem 5.1 (or Corollary 8.12 if $\lambda = \aleph_0$), there is a type-full good λ-frame $\mathfrak{s}$ on $\mathbf{K}_\lambda$. Assume for a contradiction that $\mathbf{K}^n$ is $(< \lambda, \lambda)$-type short over λ-sized models. We will prove that $\mathfrak{s}$ is weakly successful. This will imply (by Fact 2.20 and the definition of uniqueness triples) that $\mathfrak{s}^{n-3,n}$ is weakly successful, contradicting Theorem 6.6. First observe that by Theorem 4.1, $\mathbf{K}^n$ must be $(< \aleph_0, \lambda)$-type short over λ-sized models.

We now consider two cases.

- If $\lambda > \aleph_0$, then (recalling Facts 3.2 and 2.20) by Fact 2.18 (where κ there stands for $\aleph_0$ here), $\mathfrak{s}$ is weakly successful, which is the desired contradiction.
- If $\lambda = \aleph_0$, we proceed similarly. The details are in the appendix: by Theorem C.1, we get a weakly successful type-full good $\aleph_0$-frame on $\mathbf{K}^n$, which is the desired contradiction.

□

Appendix A. Solvability

We recall the definition of solvability and some consequences. This is used only in the appendix. Solvability was introduced by Shelah [She09a, Chapter IV] and is further studied in [GV, Vasc]). Shelah writes that solvability is perhaps the true analog of superstability in abstract elementary classes [She09a, N§4(B)] (in first-order, it is equivalent to the usual notion [GV, Corollary 6.4]).

When working with Ehrenfeucht-Mostowski (EM) models, we will use the notation from [She09a, Chapter IV]:

[12]This can be replaced by just assuming solvability, see Remark 8.8.

Definition A.1. *[She09a, Definition IV.0.8] For $\mu \geq \mathrm{LS}(\mathbf{K})$, let $\Upsilon_\mu[\mathbf{K}]$ be the set of Φ proper for linear orders (that is, Φ is a set $\{p_n : n < \omega\}$, where p_n is an n-variable quantifier-free type in a fixed vocabulary $\tau(\Phi)$ and the types in Φ can be used to generate a $\tau(\Phi)$-structure* $\mathrm{EM}(I, \Phi)$ *for each linear order I; that is,* $\mathrm{EM}(I, \Phi)$ *is the closure under the functions of $\tau(\Phi)$ of the universe of I and for any $i_0 < \ldots < i_{n-1}$ in I, $i_0 \ldots i_{n-1}$ realizes p_n) with:*

(1) $|\tau(\Phi)| \leq \mu$.
(2) *If I is a linear order of cardinality λ,* $\mathrm{EM}_{\tau(\mathbf{K})}(I, \Phi) \in \mathbf{K}_{\lambda+|\tau(\Phi)|+\mathrm{LS}(\mathbf{K})}$*, where $\tau(\mathbf{K})$ is the vocabulary of* $\mathbf{K}$ *and* $\mathrm{EM}_{\tau(\mathbf{K})}(I, \Phi)$ *denotes the reduct of* $\mathrm{EM}(I, \Phi)$ *to $\tau(\mathbf{K})$. Here we are implicitly also assuming that $\tau(\mathbf{K}) \subseteq \tau(\Phi)$.*
(3) *For $I \subseteq J$ linear orders,* $\mathrm{EM}_{\tau(\mathbf{K})}(I, \Phi) \leq_{\mathbf{K}} \mathrm{EM}_{\tau(\mathbf{K})}(J, \Phi)$.

We call Φ as above an EM blueprint.

The following follows from Shelah's presentation theorem. We will use it without explicit mention.

Fact A.2. *Let $\mu \geq \mathrm{LS}(\mathbf{K})$.* $\mathbf{K}$ *has arbitrarily large models if and only if $\Upsilon_\mu[\mathbf{K}] \neq \emptyset$.*

Definition A.3. *Let* $\mathrm{LS}(\mathbf{K}) \leq \mu \leq \lambda$.

(1) *$M \in \mathbf{K}$ is* universal in λ *if $M \in \mathbf{K}_\lambda$ and for any $N \in \mathbf{K}_\lambda$ there exists $f : N \to M$.*
(2) *[She09a, Definition IV.0.5] $M \in \mathbf{K}$ is* superlimit in λ *if:*
    (a) *M is universal in λ.*
    (b) *M has a proper extension.*
    (c) *For any limit ordinal $\delta < \lambda^+$ and any increasing continuous chain $\langle M_i : i \leq \delta \rangle$ in $\mathbf{K}_\lambda$, if $M \cong M_i$ for all $i < \delta$, then $M \cong M_\delta$.*
(3) *[She09a, Definition IV.1.4.(1)] We say that* Φ witnesses (λ, μ)-solvability *if:*
    (a) $\Phi \in \Upsilon_\mu[\mathbf{K}]$.
    (b) *If I is a linear order of size λ, then* $\mathrm{EM}_{\tau(\mathbf{K})}(I, \Phi)$ *is superlimit in λ.*
(4) $\mathbf{K}$ *is* (λ, μ)-solvable *if there exists Φ witnessing (λ, μ)-solvability.*
(5) $\mathbf{K}$ *is* λ-solvable *(or* solvable in λ*) if* $\mathbf{K}$ *is $(\lambda, \mathrm{LS}(\mathbf{K}))$-solvable.*

The following are straightforward:

Fact A.4.

(1) *Superlimits are unique: if M and N are superlimit models in λ, then $M \cong N$.*
(2) *Let $\lambda \geq \mathrm{LS}(\mathbf{K})$. If* $\mathbf{K}$ *has arbitrarily large models and is categorical in λ, then* $\mathbf{K}$ *is solvable in λ.*

Recently, the second author proved that solvability transfers down in AECs with amalgamation and no maximal models. In particular, categoricity implies solvability everywhere below.

Fact A.5 (Corollary 5.1 in [Vasc]). *Let $\lambda > \mu > \text{LS}(\mathbf{K})$ and assume that $\mathbf{K}_{<\lambda}$ has amalgamation and no maximal models. If $\mathbf{K}$ is solvable in λ, then $\mathbf{K}$ is solvable in μ.*

A consequence of Fact A.5 it that it is reasonable to assume solvability in a successor cardinal. In this context, Shelah has shown the following result [She99, Subfact 6.8] (a full proof appears in [She], the online version of [She99]).

Fact A.6. *Assume that $\mathbf{K}$ has amalgamation and no maximal models in $\text{LS}(\mathbf{K})$. If $\mathbf{K}$ is solvable in $\text{LS}(\mathbf{K})^+$, then there is an EM blueprint Φ witnessing it such that whenever I and J are linear orders of cardinality μ and $I \subsetneq J$, we have that $\text{EM}_{\tau(\mathbf{K})}(I, \Phi)$ is limit and $\text{EM}_{\tau(\mathbf{K})}(J, \Phi)$ is limit over $\text{EM}_{\tau(\mathbf{K})}(I, \Phi)$.*

For I a linear order, let I^* denote the reverse ordering. Given another linear ordering J, we order $I \times J$ with the inverse lexicographical ordering (i.e. $I \times J$ denotes copies of I ordered in type J). We do not use the regular lexicographical ordering to keep consistency with ordinal multiplication. We will use the following easy lemma (in [She87, Appendix A], Shelah imitates a construction of Laver [Lav71, Theorem 3.3] and claims much more):

Lemma A.7. *For any infinite cardinal μ, there exists a linear order I of cardinality μ such that for any non-zero countable ordinal α, $I \cong I \times \alpha \cong I \times \alpha^*$.*

Remark A.8. *When I is such an ordering and α and β are non-zero countable ordinals, $I \times (\alpha + \beta^*) \cong (I \times \alpha) + (I \times \beta^*) \cong I + I \cong I \times 2 \cong I$.*

Proof of Lemma A.7. Let $I := \mu \times \mathbb{Q}$ and use associativity of the product operation together with the fact that for any non-zero countable ordinal α, $\mathbb{Q} \cong \mathbb{Q} \times \alpha \cong \mathbb{Q} \times \alpha^*$, as they are all countable dense linear orders without endpoints. □

Appendix B. Transitivity of splitting

In this appendix, we prove Fact 8.7. This is based on Shelah's proof [She99, Claim 7.5], although we give an easier proof with simpler linear orderings. The reader can consult the proof of [Bal09, Theorem 15.3] for an exposition of a similar argument as the one used here. Throughout this appendix, we make the following *global* hypothesis:

Hypothesis B.1. *Setting $\mu := \text{LS}(\mathbf{K})$:*

(1) *$\mathbf{K}_\mu$ has amalgamation and no maximal models.*
(2) *$\mathbf{K}$ is solvable in μ^+.*
(3) *Φ is an EM blueprint satisfying the conclusion of Fact A.6.*
(4) *I is as given by Fact A.7.*

Fact B.2. *$\mathbf{K}$ is μ-superstable and has μ-symmetry (see Definitions 2.2 and 2.3).*

Proof. By (the proof of) Facts 2.4 and 2.6. □

Fact B.3 ([Van]). *If M_0, M_1, M_2 are all in $\mathbf{K}_\mu$ and M_1, M_2 are both limit over M_0, then $M_1 \cong_{M_0} M_2$.*

The following follows directly from the weak uniqueness and weak extension property of splitting (Fact 8.9).

Fact B.4 (Proposition 3.7 in [Vas16a]). *Let $M_0 \leq_{\mathbf{K}} M_1 \leq_{\mathbf{K}} M_1' \leq_{\mathbf{K}} M_2$ all be in $\mathbf{K}_\mu$, with M_1' universal over M_1. Let $p \in \mathrm{gS}(M_2)$. If $p \upharpoonright M_1'$ does not μ-split over M_0 and p does not μ-split over M_1, then p does not μ-split over M_0.*

The following notation will be used most often when J is an ordinal.

Notation B.5. *For J a linear order, let $I_J := I \times J$ and let $M_J := \mathrm{EM}_{\tau(\mathbf{K})}(I_J, \Phi)$.*

Lemma B.6. *For any $p \in \mathrm{gS}^{<\omega}(M_{\omega+\omega^*})$, there exists $n < \omega$ such that $p \upharpoonright M_{\omega+(\omega^* \setminus n^*)}$ does not μ-split over M_n.*

Proof. Let $J_1 := \omega$, $J_2 := J_1 + \omega^*$, $J_3 := J_2 + \omega$. For $\ell = 1, 2, 3$, write $N_\ell := M_{J_\ell}$. By the way Φ was chosen, N_3 is limit over N_2, hence p is realized in N_3, say $p = \mathrm{gtp}(\mathbf{a}/N_2; N_3)$. We have that $\mathbf{a} \in {}^{<\omega}\mathrm{EM}_{\tau(\Phi)}(B \cup C \cup D, \Phi)$, where $B \subseteq I_{J_1}$, $C \subseteq I_{J_2 \setminus J_1}$, $D \subseteq I_{J_3 \setminus J_2}$, and $B \cup C \cup D$ is finite. Pick $m < \omega$ big-enough such that $B \subseteq I_m$, and $C \subseteq I_{\omega+(\omega^* \setminus m^*)}$. We claim that there exists $n > m+1$ such that $p \upharpoonright M_{\omega+(\omega^* \setminus n^*)}$ does not μ-split over M_n. Suppose not and fix $n > m+1$.

Note that the interval $[n, n^*)$ (in J_2) is isomorphic to $\omega + \omega^*$. By Lemma A.7 and Remark A.8, there exists $f : I_{[n,n^*)} \cong I \cong I_{[n,n+1)}$. Let $g := f \cup \mathrm{id}_{J_3 \setminus I_{[n,n+1)}}$. Then g is an automorphism of J_3 that fixes I_n and $A \cup B \cup C$, and maps $I_{\omega+(\omega^* \setminus n^*)}$ onto I_{n+1}. Therefore using the induced automorphism of N_3 we get that $p \upharpoonright M_{n+1}$ μ-splits over M_n. Since n was arbitrary, this contradicts μ-superstability. $\square$

We can now prove Fact 8.7. For the convenience of the reader, we restate the essential part of Hypothesis B.1.

Fact B.7. *Set $\mu := \mathrm{LS}(\mathbf{K})$. Assume that $\mathbf{K}$ is solvable in μ^+ and $\mathbf{K}_\mu$ has amalgamation and no maximal models.*

Let $N_0 \leq_{\mathbf{K}} N_1 \leq_{\mathbf{K}} N_2$ all be in $\mathbf{K}_\mu$ such that N_1 is limit over N_0 and N_2 is limit over N_1. Let $p \in \mathrm{gS}^{<\omega}(N_2)$. If p does not μ-split over N_1 and $p \upharpoonright N_1$ does not μ-split over N_0, then p does not μ-split over N_0.

Proof. By Fact B.3, we can assume without loss of generality that $N_0 = M_0$, $N_1 = M_\omega$, and $N_2 = M_{\omega+\omega^*}$ (recall Notation B.5). By Lemma B.6, there exists $n < \omega$ such that $p \upharpoonright M_{\omega+(\omega^* \setminus n^*)}$ does not μ-split over M_n. Apply Fact B.4, where M_0, M_1, M_1', M_2' there stand for $N_0, M_n, N_1, M_{\omega+(\omega^* \setminus n^*)}$ here. We get that $p \upharpoonright M_{\omega+(\omega^* \setminus n^*)}$ does not μ-split over N_0. Now apply Fact B.4 a second time, where M_0, M_1, M_1', M_2' there stand for N_0, N_1, $M_{\omega+(\omega^* \setminus n^*)}$, N_2 here. $\square$

Appendix C. More on building good LS($\mathbf{K}$)-frames

We give more results based on the construction of a good frame in Section 8.

The goal here is to build a good frame in LS($\mathbf{K}$), assuming both locality and model-theoretic hypotheses (recall that Fact 2.18 gives a good λ-frame only when $\lambda > \text{LS}(\mathbf{K})$). The main result is:

Theorem C.1. *Set $\mu := \text{LS}(\mathbf{K})$. Assume that $\mathbf{K}$ is categorical in μ, has amalgamation in μ, and is solvable in μ^+. If:*

1. *$\mathbf{K}_\mu$ is ω-local.*
2. *$\mathbf{K}$ is $(< \aleph_0, \leq \mu)$-type short over μ-sized models.*

Then there is a weakly successful type-full good μ-frame on $\mathbf{K}$.

Recall [Bal09, Definition 11.4] that $\mathbf{K}_\mu$ is *ω-local* if whenever $\delta < \mu^+$ is a limit ordinal and $\langle M_i : i \leq \delta \rangle$ is an increasing continuous chain in $\mathbf{K}_\mu$, if $p, q \in \text{gS}(M_\delta)$ are such that $p \upharpoonright M_i = q \upharpoonright M_i$ for all $i < \delta$, then $p = q$. For example, if there is a type-full good μ-frame on $\mathbf{K}$, then $\mathbf{K}$ is ω-local (use the local character and uniqueness properties of forking). Also, any $(< \aleph_0)$-tame AEC is ω-local. Thus Theorem C.1 gives another proof that there is a weakly successful type-full good $\aleph_k$-frame on $\mathbf{K}^n$ when $k < n - 3$ (the case $k = 0$ being allowed).

We will use a weakening of the definition of a good frame introduced by Shelah [She09b, Definition VI.8.2.(2)]. The definition is further studied in [JS].

Definition C.2. $\mathfrak{s} = (\mathbf{K}, \mathop{\smile\!\!\!\!\!\!|}, \text{gS}^{bs})$ *is an* almost good λ-frame *if it satisfies all the axioms of a good λ-frame listed in Definition 2.7 except that the local character property is weakened to:*

> *For any limit ordinal $\delta < \lambda^+$ and any strictly increasing continuous chain $\langle M_i : i \leq \delta + 1 \rangle$ so that $i < \delta$ implies M_{i+1} is universal over M_i, there exists $i < \delta$ and $a \in |M_{\delta+1}| \backslash |M_\delta|$ such that $\text{gtp}(a/M_\delta; M_{\delta+1})$ does not fork over M_i.*

We say that an almost good λ-frame $\mathfrak{s}$ has the conjugation property *if whenever $M \leq_{\mathbf{K}} N$ are in $\mathbf{K}_\lambda$ and $p \in \text{gS}(N)$ does not fork over M, there exists an isomorphism $f : N \cong M$ such that $f(p) = p \upharpoonright M$.*

Remark C.3. *Even though local character is weakened, we still assume the continuity property.*

We define what it means for an almost good frame to be weakly successful as for good frames (see Definition 2.10). A key property is that weakly successful almost good frames extend to good frames. This was observed by Shelah in [She09b, Section VII.5] but we will use here the simpler argument from [JS, Theorem 4.3].

Fact C.4. *If $\mathfrak{s}$ is a weakly successful almost good λ-frame on $\mathbf{K}$ that satisfies the conjugation property, then there exists a type-full good λ-frame on $\mathbf{K}$.*

With the additional assumption of ω-locality, the proof of Theorem 8.10 gave an almost good λ-frame (saturativity was only used to get local character):

Theorem C.5. *Set $\mu := \text{LS}(\mathbf{K})$. Assume that $\mathbf{K}$ is categorical in μ, has amalgamation in μ, and is solvable in μ^+. If $\mathbf{K}_\mu$ is ω-local, then there exists an almost good μ-frame on $\mathbf{K}$ that satisfies the conjugation property.*

Proof. We follow the proof of Corollary 8.11 and define forking as there. The only property that we are missing is continuity: for any limit ordinal $\delta < \mu^+$ and any increasing continuous chain $\langle M_i : i \leq \delta \rangle$, if $p \in \text{gS}(M_\delta)$ is such that $p \upharpoonright M_i$ does not fork over M_0 for all $i < \delta$, then p does not fork over M_0.

We now use ω-locality to prove continuity. Fix p, δ, $\langle M_i : i \leq \delta \rangle$ as in the previous paragraph. By the extension property, let $q \in \text{gS}(M_\delta)$ be such that q extends $p \upharpoonright M_0$ and q does not fork over M_0. By uniqueness, $p \upharpoonright M_i = q \upharpoonright M_i$ for all $i < \delta$. By ω-locality, we must have that $p = q$, as desired.

Finally, the almost good frame satisfies the conjugation property by the proof of [She09a, Claim III.1.21]. □

Proof of Theorem C.1. By Theorem C.5, there is an almost good μ-frame $\mathfrak{s}$ on $\mathbf{K}$ that satisfies the conjugation property. We show that it is weakly successful.

For $M \leq_{\mathbf{K}} N$ both in $\mathbf{K}_\mu$ and $p \in \text{gS}^\alpha(N)$ with $\alpha \leq \mu^+$, let us say that *p does not fork over M* if for every finite $I \subseteq \alpha$ there exists $M_0 \leq_{\mathbf{K}} M$ with M universal over M_0 such that p^I does not μ-split over M_0. As in the proof of Theorem 8.10 (noting that in Fact B.7 transitivity holds for any type of finite length), this nonforking relation has the uniqueness property for types of finite length. By the shortness assumption, it has it for types of length at most μ too. It is easy to see that nonforking satisfies the local character property $(*)$ from [Vasb, Definition 3.7]. Therefore by [Vasb, Lemma 3.8] it also satisfies $(**)$ from [Vasb, Definition 3.7]. By [Vasb, Corollary 3.10], $\mathfrak{s}$ is weakly successful, as desired.

Now by Fact C.4, there exists a type-full good μ-frame $\mathfrak{t}$ on $\mathbf{K}$. By the same argument as before, $\mathfrak{t}$ is weakly successful, as needed. □

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