TAMENESS FROM TWO SUCCESSIVE GOOD FRAMES

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Abstract. We show, assuming a mild set-theoretic hypothesis, that if an abstract elementary class (AEC) has a superstable-like forking notion for models of cardinality λ and a superstable-like forking notion for models of cardinality λ^+, then orbital types over saturated models of cardinality λ^+ are determined by their restrictions to submodels of cardinality λ. By a superstable-like forking notion, we mean here a good frame, a central concept of Shelah's book on AECs.

It is known that locality of orbital types together with the existence of a superstable-like notion for models of cardinality λ implies the existence of a superstable-like notion for models of cardinality λ^+, but here we prove the converse. An immediate consequence is that forking in λ^+ can be described in terms of forking in λ.

Contents

1. Introduction

Good frames are the central notion of Shelah's two volume book [She09a, She09b] on classification theory for abstract elementary classes (AECs). Roughly, an AEC $\mathbf{K}$ has a good λ-frame if its restriction to models of cardinality λ is reasonably well-behaved (e.g. has amalgamation, no maximal models, and is stable) and it admits an abstract notion of forking (for orbital types of elements over models of cardinality λ) that satisfies some of the basic properties of forking in a superstable elementary class: monotonicity, existence, uniqueness, symmetry, and local character. Here,

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local character is described not as "every type does not fork over a finite set" but as "every type over the union of an increasing continuous chain of models of cardinality λ does not fork over some member of the chain". The theory of good frames is used heavily in several recent results in the classification theory of AECs, including the author's proof of the categoricity conjecture in universal classes [Vas17b, Vas17c], see also [BVb] for a survey.

The reason for restricting oneself to models of cardinality λ is that the compactness theorem fails in general AECs, and so it is much easier in practice to exhibit a local notion of forking than it is to define forking globally for models of all sizes. Rather, Shelah's program is to start with a good λ-frame and then try to extend it to models of bigger sizes. For this, he describes a dividing line, being successful, and shows that if a good λ-frame is successful, then there is a good λ^+-frame on an appropriate subclass of $\mathbf{K}_{\lambda^+}$.

A related approach is to outright assume some weak amount of compactness. Tameness [GV06b] was proposed by Grossberg and VanDieren for that end: λ-tameness says that orbital types are determined by their restrictions of cardinality λ. This is a nontrivial assumption, since in AECs syntactic types are not as well-behaved as one might wish, so one defines types purely semantically (roughly, as the finest notion of type preserving isomorphisms and the $\mathbf{K}$-substructure relation). It is known that tameness follows from a large cardinal axiom (see Fact 2.17) and some amount of it can be derived from categoricity (see Fact 2.18). This paper gives another way to derive some tameness.

Grossberg conjectured in 2006 that, assuming amalgamation in λ^+, a good λ-frame extends to a good λ^+-frame if the class is λ-tame. The conjecture was almost proven by Boney [Bon14a], but a slightly stronger version of tameness was assumed there. Boney and the author later proved the full conjecture [BVc], see Fact 2.19. In this context, forking in the good λ^+-frame can be described in terms of forking in the good λ-frame. Let us call this result the *upward frame transfer theorem*.

This paper discusses the converse of the upward frame transfer theorem. Consider the following question: *if* there is a good λ-frame and a good λ^+-frame, can we say anything on how the two frames are related (i.e. can forking in λ^+ be described in terms of forking in λ?) and can we conclude some amount of tameness? We answer positively:

Corollary 5.19. Let $\mathbf{K}$ be an AEC and let $\lambda \geq \mathrm{LS}(\mathbf{K})$. Assume $2^\lambda < 2^{\lambda^+}$. If there is a categorical good λ-frame on $\mathbf{K}_\lambda$ and a good λ^+-frame on $\mathbf{K}_{\lambda^+}$, then $\mathbf{K}$ is (λ, λ^+)-weakly tame.

In the author's opinion, Corollary 5.19 is quite a surprising result since it shows that we cannot really study superstability in λ and λ^+ "independently": the two levels must in some sense be connected. Put another way, two successive local instances of superstability already give a nontrivial amount of compactness. In fact we anticipate that Corollary 5.19 could be used as a tool to *prove* some amount of tameness.

In Corollary 5.19 "categorical" simply means that $\mathbf{K}$ is assumed to be categorical in λ. We see it as a very mild assumption, since we can usually restrict to a subclass of saturated models if this is not the case, see the discussion after Definition 2.13.

As for (λ, λ^+)-weak tameness, it means that only types over *saturated* models of cardinality λ^+ are determined by their restrictions of cardinality λ. In fact, it is possible to obtain the same conclusion by starting with a good λ^+-frame on the class of *saturated* models in $\mathbf{K}$ of cardinality λ^+. In this case, we deduce that weak tameness is *equivalent* to the existence of a good λ^+-frame on the saturated models of cardinality λ^+, see Corollary 5.18.

An immediate consequence of Corollary 5.19 is that forking in λ^+ (at least over saturated models) can be described in terms of forking in λ. Indeed, the upward frame extension theorem gives a good λ^+-frame with such a property, and good frames on subclass of saturated models are canonical: there can be at most one, see Fact 2.12. In fact, assuming that forking in λ^+ is determined by forking in λ is equivalent to tameness (see [Bon14a, 3.2]) because of the uniqueness and local character properties of forking. In Corollary 5.19 we of course do *not* start with such an assumption: forking in λ^+ is *any* abstract notion satisfying some superstable-like properties for models of cardinality λ^+.

The proof of Corollary 5.19 goes as follows: we use $2^\lambda < 2^{\lambda^+}$ to derive that the good λ-frame is *weakly successful* (a dividing line introduced by Shelah in Chapter II of [She09a]). This is the only place where $2^\lambda < 2^{\lambda^+}$ is used. Being weakly successful imply that we can extend the good λ-frame to types of models of cardinality λ. We then have to show that the good λ-frame is also successful. This is equivalent to a certain reflecting down property of nonforking of models. Jarden [Jar16] has shown that successfulness follows from (λ, λ^+)-weak tameness and amalgamation in λ^+, and here we push Jarden's argument further by showing that having a good λ^+-frame suffices, see Theorem 5.15. A key issue that we constantly deal with is the question of whether a union of saturated models of cardinality λ^+ is saturated. In Section 3, we introduce a new property of forking, being decent, which characterizes a positive answer to this question and sheds further light on recent work of VanDieren [Van16a, Van16b]. The author believes it has independent interest.

Tameness has been used by Grossberg and VanDieren to prove an upward categoricity transfer from categoricity in two successive cardinals [GV06c, GV06a]. In Section 6, we revisit this result and show that tameness is in some sense needed to prove it. Although this could have been derived from the results of Shelah's books, this seems not to have been noticed before. Nevertheless, the results of this paper show that if an AEC is categorical λ and λ^+ and has a good frame in both λ and λ^+, then it is categorical in λ^{++}, see Corollary 6.1.

To read this paper, the reader should preferably have a solid knowledge of good frames, including knowing Chapter II of [She09a], [JS13], as well as [Jar16]. Still, we have tried to give all the definitions and relevant background facts in Section 2.

2. Preliminaries

2.1. **Notational conventions.** Given a structure M, write write $|M|$ for its universe and $\|M\|$ for the cardinality of its universe. We often do not distinguish between M and $|M|$, writing e.g. $a \in M$ or $a \in {}^{<\alpha}M$ instead of $a \in |M|$ and $a \in {}^{<\alpha}|M|$. We write $M \subseteq N$ to mean that M is a substructure of N.

2.2. **Abstract elementary classes.** An *abstract class* is a pair $\mathbf{K} = (K, \leq_{\mathbf{K}})$, where K is a class of structures in a fixed vocabulary $\tau = \tau(\mathbf{K})$ and $\leq_{\mathbf{K}}$ is a partial order, $M \leq_{\mathbf{K}} N$ implies $M \subseteq N$, and both K and $\leq_{\mathbf{K}}$ respect isomorphisms (the definition is due to Grossberg). Any abstract class admits a notion of $\mathbf{K}$-*embedding*: these are functions $f : M \to N$ such that $f : M \cong f[M]$ and $f[M] \leq_{\mathbf{K}} N$.

We often do not distinguish between K and $\mathbf{K}$. For λ a cardinal, we will write $\mathbf{K}_\lambda$ for the restriction of $\mathbf{K}$ to models of cardinality λ. Similarly define $\mathbf{K}_{\geq \lambda}$ or more generally $\mathbf{K}_S$, where S is a class of cardinals. We will also use the following notation:

Notation 2.1. For $\mathbf{K}$ an abstract class and $N \in \mathbf{K}$, write $\mathcal{P}_{\mathbf{K}}(N)$ for the set of $M \in \mathbf{K}$ with $M \leq_{\mathbf{K}} N$. Similarly define $\mathcal{P}_{\mathbf{K}_\lambda}(N)$, $\mathcal{P}_{\mathbf{K}_{<\lambda}}(N)$, etc.

For an abstract class $\mathbf{K}$, we denote by $\mathbb{I}(\mathbf{K})$ the number of models in $\mathbf{K}$ up to isomorphism (i.e. the cardinality of $\mathbf{K}/_{\cong}$). We write $\mathbb{I}(\mathbf{K}, \lambda)$ instead of $\mathbb{I}(\mathbf{K}_\lambda)$. When $\mathbb{I}(\mathbf{K}) = 1$, we say that $\mathbf{K}$ is *categorical*. We say that $\mathbf{K}$ is *categorical in* λ if $\mathbf{K}_\lambda$ is categorical, i.e. $\mathbb{I}(\mathbf{K}, \lambda) = 1$.

We say that $\mathbf{K}$ has *amalgamation* if for any $M_0 \leq_{\mathbf{K}} M_\ell$, $\ell = 1, 2$ there is $M_3 \in \mathbf{K}$ and $\mathbf{K}$-embeddings $f_\ell : M_\ell \to M_3$, $\ell = 1, 2$. $\mathbf{K}$ has *joint embedding* if any two models can be $\mathbf{K}$-embedded in a common model. $\mathbf{K}$ has *no maximal models* if for any $M \in \mathbf{K}$ there exists $N \in \mathbf{K}$ with $M \leq_{\mathbf{K}} N$ and $M \neq N$ (we write $M <_{\mathbf{K}} N$). Localized concepts such as *amalgamation in* λ mean that $\mathbf{K}_\lambda$ has amalgamation.

The definition of an abstract elementary class is due to Shelah [She87a]:

Definition 2.2. An *abstract elementary class (AEC)* is an abstract class $\mathbf{K}$ in a finitary vocabulary satisfying:

(1) Coherence: if $M_0, M_1, M_2 \in \mathbf{K}$, $M_0 \subseteq M_1 \leq_{\mathbf{K}} M_2$ and $M_0 \leq_{\mathbf{K}} M_2$, then $M_0 \leq_{\mathbf{K}} M_1$.
(2) Tarski-Vaught chain axioms: if $\langle M_i : i \in I \rangle$ is a $\leq_{\mathbf{K}}$-directed system and $M := \bigcup_{i \in I} M_i$, then:
   (a) $M \in \mathbf{K}$.
   (b) $M_i \leq_{\mathbf{K}} M$ for all $i \in I$.
   (c) Smoothness: if $N \in \mathbf{K}$ is such that $M_i \leq_{\mathbf{K}} N$ for all $i \in I$, then $M \leq_{\mathbf{K}} N$.
(3) Löwenheim-Skolem-Tarski (LST) axiom: there exists a cardinal $\lambda \geq |\tau(\mathbf{K})| + \aleph_0$ such that for any $N \in \mathbf{K}$ and any $A \subseteq |N|$, there exists $M \in \mathcal{P}_{\mathbf{K}_{\leq(|A|+\lambda)}}(N)$ with $A \subseteq |M|$. We write $\mathrm{LS}(\mathbf{K})$ for the least such λ.

2.3. **Types.** In any abstract class $\mathbf{K}$, we can define a semantic notion of type, called Galois or orbital types in the literature (such types were introduced by Shelah in [She87b]). For $M \in \mathbf{K}$, $A \subseteq |M|$, and $\bar{b} \in {}^{<\infty}M$, we write $\mathbf{tp}_{\mathbf{K}}(\bar{b}/A; M)$ for the orbital type of $\bar{b}$ over A as computed in M (usually $\mathbf{K}$ will be clear from context and we will omit it from the notation). It is the finest notion of type respecting $\mathbf{K}$-embeddings, see [Vas16b, 2.16] for a formal definition. When $\mathbf{K}$ is an elementary class, $\mathbf{tp}(\bar{b}/A; M)$ contains the same information as the usual notion of $\mathbb{L}_{\omega,\omega}$-syntactic type, but in general the two notions need not coincide [HS90].

The *length* of $\mathbf{tp}(\bar{b}/A; M)$ is the length of $\bar{b}$. For $M \in \mathbf{K}$ and α a cardinal, we write $\mathbf{S}^\alpha_{\mathbf{K}}(M) = \mathbf{S}^\alpha(M)$ for the set of types over M of length α. Similarly define $\mathbf{S}^{<\alpha}(M)$.

When $\alpha = 1$, we just write $\mathbf{S}(M)$. We define naturally what it means for a type to be realized inside a model, to extend another type, and to take the image of a type by a $\mathbf{K}$-embedding.

2.4. **Stability and saturation.** We say that an abstract class $\mathbf{K}$, is *stable in* λ (for λ an infinite cardinal) if $|\mathbf{S}(M)| \leq \lambda$ for any $M \in \mathbf{K}_\lambda$. If $\mathbf{K}$ is an AEC, $\lambda \geq \text{LS}(\mathbf{K})$, $\mathbf{K}$ is stable in λ and $\mathbf{K}$ has amalgamation in λ, then we will often use without comments the *existence of universal extension* [She09a, II.1.16]: for any $M \in \mathbf{K}_\lambda$, there exists $N \in \mathbf{K}_\lambda$ universal over M. This means that $M \leq_{\mathbf{K}} N$ and any extension of M of cardinality λ $\mathbf{K}$-embeds into N over M.

For $\mathbf{K}$ an AEC and $\lambda > \text{LS}(\mathbf{K})$, a model $N \in \mathbf{K}$ is called λ-*saturated* if for any $M \in \mathbf{K}_{<\lambda}$ with $M \leq_{\mathbf{K}} N$, any $p \in \mathbf{S}(M)$ is realized in N. N is called *saturated* if it is $\|N\|$-saturated.

We will often use without mention the *model-homogeneous = saturated lemma* [She09a, II.1.14]: it says that when $\mathbf{K}_{<\lambda}$ has amalgamation, then a model $N \in \mathbf{K}$ is λ-saturated if and only if it is λ-model-homogeneous. The latter means that for any $M \in \mathcal{P}_{\mathbf{K}_{<\lambda}}(N)$, any $M' \in \mathbf{K}_{<\lambda}$ $\leq_{\mathbf{K}}$-extending M can be $\mathbf{K}$-embedded into N over M. In particular, assuming amalgamation and joint embedding, there is at most one saturated model of a given cardinality. We write $\mathbf{K}^{\lambda\text{-sat}}$ for the abstract class of λ-saturated models in $\mathbf{K}$ (ordered by the appropriate restriction of $\leq_{\mathbf{K}}$).

2.5. **Frames.** Roughly, a *frame* consists of a class of models of the same cardinality together with an abstract notion of nonforking. The idea is that if the frame is "sufficiently nice", then it is possible to extend it to cover bigger models as well. This is the approach of Shelah's book [She09a], where he introduced *good frames*, where the abstract notion of nonforking is required to satisfy some of the basic properties of forking in a superstable $\mathbb{L}_{\omega,\omega}$-theory. We redefine here the much simpler concept of a frame, called pre-frame in [She09a, III.0.2], [Vas16a, 3.1]. We give a slightly different definition, as we do not include certain monotonicity axioms as part of the definition. Shelah assumes that nonforking is only defined for a certain class of types he calls the basic types. This complicates the notation and we have no use for basic types in this paper. In Shelah's terminology, our frames will always be *type-full*. In any case most of the known results about type-full frames carry over to the general ones with just basic types. This is certainly the case for the results of this paper.

Definition 2.3. Let λ be an infinite cardinal, and $\alpha \leq \lambda^+$ be a non-zero cardinal. A $(<\alpha, \lambda)$-*frame* consists of a pair $\mathfrak{s} = (\mathbf{K}, \perp)$, where:

(1) $\mathbf{K}$ is an abstract class with $\mathbf{K} = \mathbf{K}_\lambda$.
(2) $\perp$ is a 4-ary relation on pairs $(\bar{a}, M_0, M, N)$, where $M_0 \leq_{\mathbf{K}} M \leq_{\mathbf{K}} N$ and $\bar{a} \in {}^{<\alpha}N$. We write $\bar{a} \underset{M_0}{\overset{N}{\perp}} M$ instead of $\perp(\bar{a}, M_0, M, N)$.
(3) $\perp$ respects $\mathbf{K}$-embeddings: if $f : N \to N'$ is a $\mathbf{K}$-embedding and $M_0 \leq_{\mathbf{K}} M \leq_{\mathbf{K}} N$, $\bar{a} \in {}^{<\alpha}N$, then $\bar{a} \underset{M_0}{\overset{N}{\perp}} M$ if and only if $f(\bar{a}) \underset{f[M_0]}{\overset{N'}{\perp}} f[M]$.

We may write $\mathbf{K}_{\mathfrak{s}} = (K_{\mathfrak{s}}, \leq_{\mathfrak{s}})$ for $\mathbf{K}$ and $\underset{\mathfrak{s}}{\perp}$ for $\perp$. We say that $\mathfrak{s}$ is *on* $\mathbf{K}^*$ if $\mathbf{K}_{\mathfrak{s}} = \mathbf{K}^*$. A $(\leq \beta, \lambda)$-frame (for $\beta \leq \lambda$) is a $(< \beta^+, \lambda)$-frame. A λ-*frame* is a $(\leq 1, \lambda)$-frame.

Definition 2.4. We say a $(< \alpha, \lambda)$-frame $\mathfrak{t}$ *extends* a $(< \beta, \lambda)$-frame $\mathfrak{s}$ if $\beta \leq \alpha$, $\mathbf{K}_{\mathfrak{s}} = \mathbf{K}_{\mathfrak{t}}$, and for $M_0 \leq_{\mathfrak{s}} M \leq_{\mathfrak{s}} N$ and $\bar{a} \in {}^{<\beta}N$, $\underset{\mathfrak{t}}{\perp}(M_0, \bar{a}, M, N)$ if and only if $\underset{\mathfrak{s}}{\perp}(M_0, \bar{a}, M, N)$.

Since $\perp$ respects $\mathbf{K}$-embeddings, it respects types. Therefore we can define:

Definition 2.5. Let $\mathfrak{s}$ be a $(< \alpha, \lambda)$-frame. For $M_0 \leq_{\mathfrak{s}} M$ and $p \in \mathbf{S}^{<\alpha}(M)$, we say that p *does not* $\mathfrak{s}$*-fork over* M_0 if $\bar{a} \overset{N}{\underset{M_0}{\perp}} M$ whenever $p = \mathbf{tp}(\bar{a}/M; N)$. When $\mathfrak{s}$ is clear from context, we omit it and just say that p *does not fork over* M_0.

We will often consider frames whose underlying class is categorical:

Definition 2.6. We say that a $(< \alpha, \lambda)$-frame $\mathfrak{s}$ is *categorical* if $\mathbf{K}_{\mathfrak{s}}$ is categorical (i.e. it contains a single model up to isomorphism).

We will consider the following properties that forking may have in a frame:

Definition 2.7. Let $\mathfrak{s}$ be a $(< \alpha, \lambda)$-frame.

(1) $\mathfrak{s}$ has *non-order* if whenever $M_0 \leq_{\mathfrak{s}} M \leq_{\mathfrak{s}} N$, $\bar{a}, \bar{b} \in {}^{<\alpha}M$ and $A := \text{ran}(\bar{a}) = \text{ran}(\bar{b})$, then $\bar{a} \overset{N}{\underset{M_0}{\perp}} M$ if and only if $\bar{b} \overset{N}{\underset{M_0}{\perp}} M$. In this case we will write $A \overset{N}{\underset{M_0}{\perp}} M$.
(2) $\mathfrak{s}$ has *monotonicity* if whenever $M_0 \leq_{\mathfrak{s}} M_0' \leq_{\mathfrak{s}} M \leq_{\mathfrak{s}} N \leq_{\mathfrak{s}} N'$, $\bar{a} \in {}^{<\alpha}N'$, $I \subseteq \text{dom}(\bar{a})$, and $\bar{a} \overset{N'}{\underset{M_0}{\perp}} N$, we have that $\bar{a} \upharpoonright I \overset{N'}{\underset{M_0'}{\perp}} M$.
(3) $\mathfrak{s}$ has *existence* if whenever $M \leq_{\mathfrak{s}} N$, any $p \in \mathbf{S}^{<\alpha}(M)$ has a nonforking extension to $\mathbf{S}^{<\alpha}(N)$.
(4) $\mathfrak{s}$ has *uniqueness* if whenever $M \leq_{\mathfrak{s}} N$, if $p, q \in \mathbf{S}^{<\alpha}(N)$ both do not fork over M and $p \upharpoonright M = q \upharpoonright M$, then $p = q$.
(5) $\mathfrak{s}$ has *local character* if for any $\beta \leq \min(\alpha, \lambda)$, any limit ordinal $\delta < \lambda^+$ with $\text{cf}(\delta) \geq \beta$, any $\leq_{\mathfrak{s}}$-increasing continuous chain $\langle M_i : i \leq \delta \rangle$, and any $p \in \mathbf{S}^{<\beta}(M_\delta)$, there exists $i < \delta$ such that p does not fork over M_i.
(6) $\mathfrak{s}$ has *symmetry* if the following are equivalent for any $M \leq_{\mathfrak{s}} N$, $\bar{a}, \bar{b} \in {}^{<\alpha}N$.
   (a) There exists $M_{\bar{a}}, N_{\bar{a}}$ such that $N \leq_{\mathfrak{s}} N_{\bar{a}}$, $M \leq_{\mathfrak{s}} M_{\bar{a}} \leq_{\mathfrak{s}} N_{\bar{a}}$, $\bar{a} \in {}^{<\alpha}M_{\bar{a}}$, and $\mathbf{tp}(\bar{b}/M_{\bar{a}}; N_{\bar{a}})$ does not fork over M.
   (b) There exists $M_{\bar{b}}, N_{\bar{b}}$ such that $N \leq_{\mathfrak{s}} N_{\bar{b}}$, $M \leq_{\mathfrak{s}} M_{\bar{b}} \leq_{\mathfrak{s}} N_{\bar{b}}$, $\bar{b} \in {}^{<\alpha}M_{\bar{b}}$, and $\mathbf{tp}(\bar{a}/M_{\bar{b}}; N_{\bar{b}})$ does not fork over M.
(7) When $\alpha = \lambda^+$, $\mathfrak{s}$ has *long transitivity* if it has non-order and whenever $\gamma < \lambda^+$ is an ordinal (not necessarily limit), $\langle M_i : i \leq \gamma \rangle$, $\langle N_i : i \leq \gamma \rangle$

are $\leq_{\mathfrak{s}}$-increasing continuous, and $N_i \overset{N_{i+1}}{\underset{M_i}{\downarrow}} M_{i+1}$ for all $i < \gamma$, we have that $N_0 \overset{N_\gamma}{\underset{M_0}{\downarrow}} M_\gamma$.

Shelah's definition of a good frame (for types of length one) says that a frame must have all the properties above and its underlying class must be "reasonable" [She09a, II.2.1]. The prototypical example is the class of models of cardinality λ of a superstable $\mathbb{L}_{\omega,\omega}$-theory which is stable in λ. Taking this class with nonforking gives a good λ-frame (even a good $(\leq \lambda, \lambda)$-frame). We use the definition from [JS13, 2.1.1] (we omit the continuity property since it follows, see [JS13, 2.1.4]). We add the long transitivity property from [She09a, II.6.1] when the types have length λ.

Definition 2.8. We say a $(< \alpha, \lambda)$-frame $\mathfrak{s}$ is *good* if:

(1) There is an AEC $\mathbf{K}$ such that $\mathbf{K}_\lambda = \mathbf{K}_{\mathfrak{s}}$, $\text{LS}(\mathbf{K}) = \lambda$, and $\mathbf{K}_\lambda \neq \emptyset$. Moreover $\mathbf{K}$ is stable in λ and $\mathbf{K}_\lambda$ has amalgamation, joint embedding, and no maximal models.
(2) $\mathfrak{s}$ has non-order, monotonicity, existence, uniqueness, local character, symmetry, and (when $\alpha = \lambda^+$) long transitivity.

Remark 2.9. The AEC $\mathbf{K}$ in the definition is unique: any AEC is determined by its restriction to models of size $\text{LS}(\mathbf{K})$ [She09a, II.1.23]. We call $\mathbf{K}$ the AEC *generated by* $\mathfrak{s}$.

We will use the following conjugation property of good frames at a crucial point in the proof of Theorem 5.15:

Fact 2.10 (III.1.21 in [She09a])**.** Let $\mathfrak{s}$ be a categorical good λ-frame (see Definition 2.6). Let $M \leq_{\mathfrak{s}} N$ and let $p \in \mathbf{S}(N)$. If p does not fork over M, then p and $p \upharpoonright M$ are conjugate. That is, there exists an isomorphism $f : N \cong M$ such that $f(p) = p \upharpoonright M$.

Remark 2.11. The results of this paper also carry over (with essentially the same proofs) in the slightly weaker framework of semi-good λ-frames introduced in [JS13], where only "almost stability" (i.e. $|\mathbf{S}(M)| \leq \lambda^+$ for all $M \in \mathbf{K}_\lambda$) and the conjugation property are assumed. For example, in Corollary 5.18 we can assume only that there is a semi-good λ-frame on $\mathbf{K}_\lambda$ with conjugation (but we still should assume there is a good λ^+-frame on $\mathbf{K}_{\lambda^+}^{\lambda^+\text{-sat}}$).

Several sufficient conditions for the existence of a good frame are known. Assuming GCH, Shelah showed that the existence of a good λ-frame follows from categoricity in λ, λ^+, and a medium number of models in λ^{++} (see Fact 6.3). Good frames can also be built using a small amount of tameness that follows from amalgamation, no maximal models, and categoricity in a sufficiently high cardinal (see Section 2.7 and Fact 2.18).

Note that on a categorical good λ-frame, there is only one possible notion of nonforking with the required properties. In fact, nonforking can be given an explicit description, see [Vas16a, §9]. We will use this without comments:

Fact 2.12 (Canonicity of categorical good frames)**.** If $\mathfrak{s}$ and $\mathfrak{t}$ are categorical good λ-frames and $\mathbf{K}_{\mathfrak{s}} = \mathbf{K}_{\mathfrak{t}}$, then $\mathfrak{s} = \mathfrak{t}$.

2.6. **Superlimits.** As has been done in several recent papers (e.g. [JS13, Vas16a, Vas17a]), we have dropped the requirement that $\mathbf{K}$ has a superlimit in λ from Shelah's definition of a good frame:

Definition 2.13 (I.3.3 in [She09a])**.** Let $\mathbf{K}$ be an AEC and let $\lambda \geq \mathrm{LS}(\mathbf{K})$. A model N is *superlimit in* λ if:

(1) $N \in \mathbf{K}_\lambda$ and N has a proper extension.
(2) N is universal: any $M \in \mathbf{K}_\lambda$ $\mathbf{K}$-embeds into N.
(3) For any limit ordinal $\delta < \lambda^+$ and any increasing chain $\langle N_i : i < \delta \rangle$, if $N \cong N_i$ for all $i < \delta$, then $N \cong \bigcup_{i<\delta} N_i$.

We say that M is *superlimit* if it is superlimit in $\|M\|$.

Again, for a prototypical example consider a superstable elementary class which is stable in a cardinal λ and let M be a saturated model of cardinality λ. Then because unions of chains of λ-saturated are λ-saturated, M is superlimit in λ. In fact, an elementary class has a superlimit in some high-enough cardinal if and only if it is superstable [She12, 3.1].

There are no known examples of good λ-frames that do *not* have a superlimit in λ. In fact most constructions of good λ-frames give one, see for example [VV17, 6.4]. When a good λ-frame $\mathfrak{s}$ has a superlimit, we can restrict $\mathfrak{s}$ to the AEC generated by this superlimit (i.e the unique AEC $\mathbf{K}^*$ such that $\mathbf{K}_\lambda$ consists of isomorphic copies of the superlimit and is ordered by the appropriate restriction of $\leq_{\mathbf{K}}$) and obtain a new good frame that will be *categorical* in the sense of Definition 2.6. Thus in this paper we will often assume that the good frame is categorical to start with.

When one has a good λ-frame, it is natural to ask whether can be extended to a good λ^+-frame. It turns out that the behavior of the saturated model in λ^+ can be crucial for this purpose. Note that amalgamation and stability in λ indeed imply that there is a unique, universal, saturated model in λ^+. It is also known that there are no maximal models in λ^+ (see [She09a, II.4.13] or [JS13, 3.1.9]). A key property is whether union of chains of saturated models of cardinality λ^+ are saturated. In fact, it is easy to see that this is equivalent to the existence of a superlimit in λ^+. We will use it without comments and leave the proof to the reader:

Fact 2.14. Let $\mathfrak{s}$ be a good λ-frame generating an AEC $\mathbf{K}$. The following are equivalent:

(1) $\mathbf{K}$ has a superlimit in λ^+.
(2) The saturated model in $\mathbf{K}$ of cardinality λ^+ is superlimit.
(3) For any limit $\delta < \lambda^{++}$ and any increasing chain $\langle M_i : i < \delta \rangle$ of saturated model in $\mathbf{K}_{\lambda^+}$, $\bigcup_{i<\delta} M_i$ is saturated.

2.7. **Tameness.** In an elementary class, types coincides with sets of formulas so are in particular determined by their restrictions to small subsets of their domain. One may be interested in studying AECs where types have a similar behavior. Such a property is called tameness. Tameness was extracted from an argument of Shelah [She99] and made into a definition by Grossberg and VanDieren [GV06b] who used it to prove an upward categoricity transfer theorem [GV06c, GV06a]. Here we present the definitions we will use, and a few sufficient conditions for tameness (given as motivation but *not* used in this paper). We refer the reader to the survey of Boney and the author [BVb] for more on tame AECs.

Definition 2.15. For an abstract class $\mathbf{K}$, a class of types Γ, and an infinite cardinal χ, we say that $\mathbf{K}$ is $(<\chi)$-*tame* for Γ if for any $p, q \in \Gamma$ over the same set B, $p \upharpoonright A = q \upharpoonright A$ for all $A \subseteq B$ with $|A| < \chi$ implies that $p = q$. We say that $\mathbf{K}$ is χ-*tame for* Γ if it is $(<\chi^+)$-tame for Γ.

We will use the following variation from [Bal09, 11.6]:

Definition 2.16. An AEC $\mathbf{K}$ is $(<\chi, \lambda)$-*weakly tame* if $\mathbf{K}$ is $(<\chi)$-tame for the class of types of length one over *saturated* models of cardinality λ. When we omit the weakly, we mean that $\mathbf{K}$ is $(<\chi)$-tame for the class of types of length one over any model of cardinality λ. Define similarly variations such as $(\chi, <\lambda)$-*weakly tame*, or χ-*tame* (which means (χ, λ)-tame for all $\lambda \geq \chi$).

A consequence of the compactness theorem is that any elementary class is $(<\aleph_0)$-tame (for any class of types). Boney [Bon14b], building on work of Makkai and Shelah [MS90] showed that tameness follows from a large cardinal axiom:

Fact 2.17. If $\mathbf{K}$ is an AEC and $\chi > \mathrm{LS}(\mathbf{K})$ is strongly compact, then $\mathbf{K}$ is $(<\chi)$-tame for any class of types.

Recent work of Boney and Unger [BU] show that this can, in a sense, be reversed: the statement "For every AEC $\mathbf{K}$ there is χ such that $\mathbf{K}$ is χ-tame" is *equivalent* to a large cardinal axiom. It is however known that when the AEC is stability-theoretically well-behaved, some amount of tameness automatically holds. This was observed for categoricity in a cardinal of high-enough cardinal by Shelah [She99, II.2.3] and later improved to categoricity in any big-enough cardinal by the author [Vasb, 5.7(5)]:

Fact 2.18. Let $\mathbf{K}$ be an AEC with arbitrarily large models and let $\lambda \geq \beth_{\left(2^{\mathrm{LS}(\mathbf{K})}\right)^+}$. If $\mathbf{K}_{<\lambda}$ has amalgamation and no maximal models and $\mathbf{K}$ is categorical in λ, then there exists $\chi < \beth_{\left(2^{\mathrm{LS}(\mathbf{K})}\right)^+}$ such that $\mathbf{K}$ is $(\chi, <\lambda)$-weakly tame.

Very relevant to this paper is the fact that tameness allows one to transfer good frames up. This was conjectured by Grossberg in 2006. Boney [Bon14a] proved it with the additional assumption of tameness for types of length two, and this assumption is removed in recent joint work of the author with Boney [BVc, 6.9]:

Fact 2.19. Let $\mathfrak{s}$ be a good λ-frame generating an AEC $\mathbf{K}$. If $\mathbf{K}$ is (λ, λ^+)-tame and has amalgamation in λ^+, then there is a good λ^+-frame $\mathfrak{t}$ on $\mathbf{K}_{\lambda^+}$. Moreover $\mathfrak{t}$-nonforking can be described in terms of $\mathfrak{s}$-nonforking as follows: for $M \leq_{\mathfrak{t}} N$, $p \in \mathbf{S}(N)$ does not $\mathfrak{t}$-fork over M if and only if there exists $M_0 \in \mathcal{P}_{\mathbf{K}_\lambda} M$ so that for all $N_0 \in \mathcal{P}_{\mathbf{K}_\lambda} N$ with $M_0 \leq_{\mathbf{K}} N_0$, $p \upharpoonright N_0$ does not $\mathfrak{s}$-fork over M_0.

3. When is there a superlimit in λ^+?

Starting with a good λ-frame generating an AEC $\mathbf{K}$, it is natural to ask when $\mathbf{K}$ has a superlimit in λ^+, i.e. when the union of any increasing chains of λ^+-saturated models is λ^+-saturated. We should say that there are no known examples when this fails, but we are unable to prove it unconditionally. We give here the following condition on $\mathfrak{s}$-nonforking characterizing the existence of a superlimit in λ^+. The condition is extracted from the property $(**)_{M_1^*, M_2^*}$ in [She09a, II.8.5].

Definition 3.1. Let $\mathfrak{s}$ be a good λ-frame generating an AEC $\mathbf{K}$. We say that $\mathfrak{s}$ is *decent* if for any saturated model $N \in \mathbf{K}_{\lambda^+}$, any $M \in \mathcal{P}_{\mathbf{K}_\lambda}(N)$, and any $p \in \mathbf{S}(M)$, the nonforking extension of p to any model of cardinality λ is realized inside N.

Remark 3.2. Let $\mathfrak{s}$ be a good λ-frame generating an AEC $\mathbf{K}$. If $\mathfrak{s}$ is *not* decent, then by amalgamation there exists p, M, M', N, N' such that:

(1) $N \leq_{\mathbf{K}} N'$ are both in saturated in $\mathbf{K}_{\lambda^+}$.
(2) $M \leq_{\mathbf{K}} M'$ are both in $\mathbf{K}_\lambda$, $M \leq_{\mathbf{K}} N$, and $M' \leq_{\mathbf{K}} N'$.
(3) $p \in \mathbf{S}(M)$ and the nonforking extension of p to $\mathbf{S}(M')$ is not realized in N.

Theorem 3.3. Let $\mathfrak{s}$ be a good λ-frame generating an AEC $\mathbf{K}$. The following are equivalent:

(1) $\mathbf{K}$ has a superlimit model in λ^+.
(2) There exists a partial order $\leq^*$ on $K_{\lambda^+}^{\lambda^+\text{-sat}}$ such that:
  (a) Whenever $M_0 \leq_{\mathbf{K}} M_1$ are both in $\mathbf{K}_{\lambda^+}^{\lambda^+\text{-sat}}$, there exists $M_2 \in \mathbf{K}_{\lambda^+}^{\lambda^+\text{-sat}}$ such that $M_1 \leq_{\mathbf{K}} M_2$ and $M_0 \leq^* M_2$
  (b) For any increasing chain $\langle N_i : i < \omega \rangle$ in $\mathbf{K}_{\lambda^+}^{\lambda^+\text{-sat}}$ such that $N_i \leq^* N_{i+1}$ for all $i < \omega$, $\bigcup_{i<\omega} N_i$ is saturated.
(3) $\mathfrak{s}$ is decent.

Proof.

- (1) implies (2): Trivial (take $\leq^*$ to be $\leq_{\mathbf{K}}$).
- (3) implies (1): Assume (3). Let $\delta < \lambda^{++}$ be a limit ordinal and let $\langle N_i : i < \delta \rangle$ be an increasing chain of saturated models in $\mathbf{K}_{\lambda^+}$. We want to show that $N_\delta := \bigcup_{i<\delta} N_i$ is saturated. Without loss of generality, $\delta = \text{cf}(\delta) < \lambda^+$. Let $M \in \mathcal{P}_{\mathbf{K}_\lambda}(N_\delta)$. Build $\langle M_i : i < \delta \rangle$ increasing chain in $\mathbf{K}_\lambda$ such that for all $i < \delta$ $M_i \leq_{\mathbf{K}} N_i$ and $|M| \cap |N_i| \subseteq |M_i|$. This is possible using that each N_i is saturated. Let $M_\delta := \bigcup_{i<\delta} M_i$. Then $M \leq_{\mathbf{K}} M_\delta$. Let $p \in \mathbf{S}(M_\delta)$. By local character, there exists $i < \delta$ such that p does not fork over M_i. Since $\mathfrak{s}$ is decent, p is realized in N_i, hence in N_δ. This shows that N_δ is saturated, as desired.
- (2) implies (3): Assume (2) and suppose for a contradiction that $\mathfrak{s}$ is not decent. We build $\langle M_i : i \leq \omega \rangle$, $\langle N_i : i \leq \omega \rangle$ increasing continuous and a type $p \in \mathbf{S}(M_0)$ such that for all $i < \omega$: (writing p_M for the λ-nonforking extension of p to $\mathbf{S}(M)$):
  (1) $N_i \in \mathbf{K}_{\lambda^+}^{\lambda^+\text{-sat}}$.
  (2) $N_i \leq^* N_{i+1}$.
  (3) $M_i \in \mathcal{P}_{\mathbf{K}_\lambda}(N_i)$.
  (4) $p_{M_{i+1}}$ is not realized in N_i.

  This is enough: By assumption, N_ω is saturated. Therefore p_{M_ω} is realized inside N_ω, and therefore inside N_i for some $i < \omega$. This means in particular that $p_{M_{i+1}}$ is realized in N_i, a contradiction.

  This is possible: For $i = 0$, pick witnesses M_0, M_1, N_0, N_1', p as given by Remark 3.2, and then use the properties of $\leq^*$ to obtain $N_1 \in \mathbf{K}_{\lambda^+}^{\lambda^+\text{-sat}}$

such that $N_0 \leq^* N_1$ and $N_1' \leq_{\mathbf{K}} N_1$. Now given $\langle M_j : j \leq i+1 \rangle$, $\langle N_j : j \leq i+1 \rangle$, N_i and N_{i+1} are both saturated and so must be isomorphic over any common submodel of cardinality λ. Let $f : N_i \cong_{M_i} N_{i+1}$ and let $g : N_{i+1} \cong N_{i+2}'$ be an extension of f. Since g is an isomorphism, $g(p_{M_{i+1}})$ is not realized in N_{i+1}. Pick M_{i+2} in $\mathcal{P}_{\mathbf{K}_\lambda}(N_{i+2}')$ such that $M_{i+1} \leq_{\mathbf{K}} M_{i+2}$ and $g[M_{i+1}] \leq_{\mathbf{K}} M_{i+2}$. Finally, pick some $N_{i+2} \in \mathbf{K}_{\lambda^+}^{\lambda^+\text{-sat}}$ such that $N_{i+1} \leq^* N_{i+2}$ and $N_i' \leq_{\mathbf{K}} N_{i+2}$.

$\square$

Remark 3.4. Theorem 3.3 can be generalized to the weaker framework of a λ-superstable AEC (implicit for example [GVV16]; see [Vasa] for what forking is there and what properties it has). For this we ask in the definition of decent that M be limit, and we ask for example that M_{i+1} is limit over M_i in the proof of (3) implies (1) of Theorem 3.3. VanDieren [Van16a] has shown that λ-symmetry (a property akin to the symmetry property of good λ-frame, see [Vasa, §2.1]) is equivalent to the property that reduced towers are continuous, and if there is a superlimit in λ^+, then reduced towers are continuous. Thus decent is to "superlimit in λ^+" what λ-symmetry is to "reduced towers are continuous". In particular, being decent is a strengthening of the symmetry property. Note also that taking $\leq^*$ in (2) of Theorem 3.3 to be "being universal over", we obtain an alternate proof of the main theorem of [Van16b] which showed that λ and λ^+-superstability together with the uniqueness of limit models in λ^+ imply that the union of a chain of λ^+-saturated models is λ^+-saturated (see also the proof of Corollary 5.19 here).

In the rest of this section, we note that Shelah has introduced a related but more complicated property he calls good$^+$. We show that good$^+$ implies decent. We do not know whether the converse holds but it seems (see Fact 5.12) that wherever Shelah uses good$^+$, he only needs decent.

Definition 3.5 (III.1.3 in [She09a]). Let $\mathfrak{s}$ be a good λ-frame generating an AEC $\mathbf{K}$. We say that $\mathfrak{s}$ is *good*$^+$ if the following is *impossible*:

There exists increasing continuous chains in $\mathbf{K}_\lambda$ $\langle M_i : i < \lambda^+ \rangle$, $\langle N_i : i < \lambda^+ \rangle$, a type $p \in \mathbf{S}(M_0)$, and $\langle a_i : i < \lambda^+ \rangle$ such that for any $i < \lambda^+$:

(1) $M_i \leq_{\mathbf{K}} N_i$.
(2) $a_{i+1} \in |M_{i+2}|$ and $\mathbf{tp}(a_{i+1}, M_{i+1}, M_{i+2})$ is a nonforking extension of p, but $\mathbf{tp}(a_{i+1}, N_0, N_{i+2})$ forks over M_0.
(3) $\bigcup_{j<\lambda^+} M_j$ is saturated.

Lemma 3.6. Let $\mathfrak{s}$ be a good λ-frame. If $\mathfrak{s}$ is good$^+$, then $\mathfrak{s}$ is decent.

Proof. Let $\mathbf{K}$ be the AEC generated by $\mathfrak{s}$. Suppose that $\mathfrak{s}$ is not decent. Fix witnesses p, M, M', N, N' as given by Remark 3.2. We build increasing continuous chains in $\mathbf{K}_\lambda$ $\langle M_i : i < \lambda^+ \rangle$, $\langle N_i : i < \lambda^+ \rangle$ and $\langle a_i : i < \lambda^+ \rangle$ such that for all $i < \lambda^+$:

(1) $M_0 = M$, $N_0 = M'$.
(2) $M_i \leq_{\mathbf{K}} N$, $N_i \leq_{\mathbf{K}} N'$.
(3) $M_i \leq_{\mathbf{K}} N_i$.
(4) M_{i+1} is universal over M_i, N_{i+1} is universal over N_i.
(5) $a_i \in M_{i+1}$.

(6) $\mathbf{tp}(a_i/M_i; M_{i+1})$ is a nonforking extension of p.

This is possible since both M' and N' are saturated. This is enough: $M_{\lambda^+} := \bigcup_{i<\lambda^+} M_i$ is saturated and for any $i < \lambda^+$, $\text{tp}(a_{i+1}/N_0; N_{i+2})$ forks over M_0. If not, then by the uniqueness property of nonforking, we would have that a_{i+1} realizes p, which we assumed was impossible. □

4. From weak tameness to good frame

In this section, we briefly investigate how to generalize Fact 2.19 to AECs that are only (λ, λ^+)-weakly tame. The main problem is that the class $\mathbf{K}_{\lambda^+}^{\lambda^+\text{-sat}}$ may not be closed under unions of chains. Indeed, this is the only difficulty:

Theorem 4.1. Let $\mathfrak{s}$ be a good λ-frame generating an AEC $\mathbf{K}$. If $\mathbf{K}$ is (λ, λ^+)-weakly tame, the following are equivalent:

(1) $\mathfrak{s}$ is decent and $\mathbf{K}_{\lambda^+}^{\lambda^+\text{-sat}}$ has amalgamation.
(2) There is a good λ^+-frame on $\mathbf{K}_{\lambda^+}^{\lambda^+\text{-sat}}$.

Proof. (1) implies (2) is exactly as in the proof of Fact 2.19. (2) implies (1) is by Fact 2.14 and Theorem 3.3, since by definition the existence of a good λ^+-frame on $\mathbf{K}_{\lambda^+}^{\lambda^+\text{-sat}}$ implies that $\mathbf{K}_{\lambda^+}^{\lambda^+\text{-sat}}$ has amalgamation and $\mathbf{K}_{\lambda^+}^{\lambda^+\text{-sat}}$ generates an AEC, hence that $\mathbf{K}$ has a superlimit in λ^+. □

Remark 4.2. Just as in the statement of Fact 2.19, nonforking in the good λ^+-frame in Theorem 4.1 can be described in terms of the nonforking of $\mathfrak{s}$.

It is natural to ask whether the good λ^+-frame in Theorem 4.1 is also decent. We do not know the answer, but can answer positively assuming $2^\lambda < 2^{\lambda^+}$ (see Corollary 5.18) and in fact in this case $\mathfrak{s}$ is also good$^+$. We can show in ZFC that being good$^+$ transfers up. The proof adapts an argument of Shelah [She09a, III.1.6(2)].

Theorem 4.3. Let $\mathfrak{s}$ be a good$^+$ λ-frame generating an AEC $\mathbf{K}$. If $\mathbf{K}$ is (λ, λ^+)-weakly tame and $\mathbf{K}_{\lambda^+}^{\lambda^+\text{-sat}}$ has amalgamation, then there is a good$^+$ λ^+-frame on $\mathbf{K}_{\lambda^+}^{\lambda^+\text{-sat}}$.

Proof. By Lemma 3.6, $\mathfrak{s}$ is decent, so by Theorem 4.1 there is a good λ^+-frame $\mathfrak{t}$ on $\mathbf{K}_{\lambda^+}^{\lambda^+\text{-sat}}$. Moreover by Remark 4.2 $\mathfrak{t}$-nonforking is described in terms of $\mathfrak{s}$-nonforking as in the statement of Fact 2.19.

Suppose that $\mathfrak{t}$ is *not* good$^+$. Let $\langle M_i : i < \lambda^{++} \rangle$, $\langle N_i : i < \lambda^{++} \rangle$, p, $\langle a_i : i < \lambda^{++} \rangle$ be as in the definition of being good$^+$. Let $\langle M_j' : j < \lambda^+ \rangle$ be increasing continuous in $\mathbf{K}_\lambda$ such that $M_0 = \bigcup_{j<\lambda^+} M_j'$ and let $\langle N_j' : j < \lambda^+ \rangle$ be increasing continuous in $\mathbf{K}_\lambda$ such that $N_0 = \bigcup_{j<\lambda^+} N_j'$ and $M_j' \leq_{\mathbf{K}} N_j'$ for all $j < \lambda^+$.

By a standard pruning argument, there is $j^* < \lambda^+$ and an unbounded $S \subseteq \lambda^{++}$ of successor ordinals such that for all $i \in S$ and all $M' \in \mathcal{P}_{\mathbf{K}_\lambda}(M_i)$ with $M_{j^*}' \leq_{\mathbf{K}} M'$, $\mathbf{tp}(a_i/M'; M_{i+1})$ does not $\mathfrak{s}$-fork over M_{j^*}'. Now by assumption for all $i \in S$, $\mathbf{tp}(a_i/N_0; N_{i+1})$ $\mathfrak{t}$-forks over M_0, so by a pruning argument again, there is an unbounded $S' \subseteq S$ and $j^{**} \in [j^*, \lambda^+)$ such that for all $i \in S'$, $\mathbf{tp}(a_i/N_{j^{**}}'; N_{i+1})$

$\mathfrak{s}$-forks over M'_{j^*}. We build $\langle i_j : j < \lambda^+ \rangle$ and $\langle M_j^* : j < \lambda^+ \rangle$, $\langle N_j^* : j < \lambda^+ \rangle$ increasing continuous in $\mathbf{K}_\lambda$ such that for all $j < \lambda^+$:

(1) $i_j \in S'$.
(2) $M_j^* \leq_{\mathbf{K}} N_j^*$.
(3) M_{j+1}^* is universal over M_j^*.
(4) $M_0^* = M'_{j^*}$, $N_0^* = N'_{j^{**}}$.
(5) $M_j^* \leq_{\mathbf{K}} M_{i_j}$, $N_j^* \leq_{\mathbf{K}} N_{i_j}$.
(6) $a_{i_j} \in M_{j+1}^*$.

<u>This is enough</u>: Then by construction of S', j^*, and j^{**}, $\langle M_j^* : j < \lambda^+ \rangle$, $\langle N_j^* : j < \lambda^+ \rangle$, $\mathbf{tp}(a_{i_0}/M_0^*; M_1^*)$ and $\langle a_{i_j} : j < \lambda^+ \rangle$ witness that $\mathfrak{s}$ is not good$^+$.

<u>This is possible</u>: Let $M_{\lambda^{++}} := \bigcup_{i<\lambda^{++}} M_i$, $N_{\lambda^{++}} := \bigcup_{i<\lambda^{++}} N_i$. We are already given M_0^* and N_0^* and for j limit we take unions. Now assume inductively that $\langle M_k^* : k \leq j \rangle$, $\langle N_k^* : k \leq j \rangle$ and $\langle i_k : k < j \rangle$ are already given, with $M_j^* \leq_{\mathbf{K}} M_{\lambda^{++}}$ and $N_j^* \leq_{\mathbf{K}} N_{\lambda^{++}}$. Let $i_j \in S'$ be big-enough such that N_{i_j} contains N_j^*, M_{i_j} contains M_j^*, and $i_k < i_j$ for all $k < j$. Such an i_j exists since S' is unbounded. Now let $M^* \in \mathcal{P}_{\mathbf{K}_{\lambda^+}} M_{\lambda^{++}}$ contain M_j^* and a_{i_j} and be saturated. Such an M^* exists since $M_{\lambda^{++}}$ is saturated by assumption. Now pick $M_{j+1}^* \in \mathcal{P}_{\mathbf{K}_\lambda}(M^*)$ so that $a_{i_j} \in M_{j+1}^*$ and M_{j+1}^* is universal over M_j^*. This is possible since M^* is saturated. Finally, pick any $N_{j+1}^* \in \mathcal{P}_{\mathbf{K}_\lambda}(N_{\lambda^{++}})$ containing M_{j+1}^* and N_j^*. $\square$

We end this section by noting that in the context of Fact 2.19, $\mathfrak{s}$ is decent and hence by Theorem 4.1 there is a good λ^+-frame on $\mathbf{K}_{\lambda^+}^{\lambda^+\text{-sat}}$. In fact:

Lemma 4.4. Let $\mathfrak{s}$ be a good λ-frame generating an AEC $\mathbf{K}$. If there is a good λ^+-frame on $\mathbf{K}_{\lambda^+}$, then $\mathfrak{s}$ is decent.

The proof uses the uniqueness of limit models in good frames, due to Shelah [She09a, II.4.8] (see [Bon14a, 9.2] for a proof):

Fact 4.5. Let $\mathfrak{s}$ be a good λ-frame, $\delta_1, \delta_2 < \lambda^+$ be limit ordinals. Let $\langle M_i^\ell : i \leq \delta_\ell \rangle$, $\ell = 1, 2$, be increasing continuous. If for all $\ell = 1, 2$, $i < \delta_\ell$, M_{i+1}^ℓ is universal over M_i, then $M_{\delta_1}^1 \cong M_{\delta_2}^2$.

Proof of Lemma 4.4. By Theorem 3.3, it suffices to show that $\mathbf{K}$ has a superlimit in λ^+. We could apply two results of VanDieren [Van16a, Van16b] but we prefer to give a more explicit proof here.

Let $\langle N_i : i \leq \lambda^+ \rangle$ be increasing continuous in $\mathbf{K}_{\lambda^+}$ such that N_{i+1} is universal over N_i for all $i < \lambda^+$. This is possible since by definition of a good λ^+-frame, $\mathbf{K}$ is stable in λ^+ and has amalgamation in λ^+. Clearly, N_{λ^+} is saturated. Moreover by Fact 4.5, $N_{\lambda^+} \cong N_\omega$. Thus N_ω is also saturated. We chose $\langle N_i : i \leq \omega \rangle$ arbitrarily, therefore (2) of Theorem 3.3 holds with $\leq^*$ being "universal over or equal to". Thus (1) there holds: $\mathbf{K}$ has a superlimit in λ^+, as desired. $\square$

There are other variations on Lemma 4.4. For example, if $\mathfrak{s}$ is a good λ-frame generating an AEC $\mathbf{K}$, $\mathbf{K}$ has amalgamation in λ^+, and $\mathbf{K}$ is (λ, λ^+)-weakly tame, then $\mathfrak{s}$ is decent (to prove this, we transfer enough of the good λ-frame up, then apply results of VanDieren [Van16a, Van16b]).

5. From good frame to weak tameness

In this section, we look at a sufficient condition (due to Shelah) implying that a good λ-frame can be extended to a good λ^+-frame and prove its necessity.

It turns out it is convenient to first extend the good λ-frame to a good $(\leq \lambda, \lambda)$-frame. For this, the next technical property is of great importance, and it is key in Chapter II and III of [She09a]. The definition below follows [JS13, 4.1.5] (but as usual, we work only with type-full frames). Note that we will not use the exact content of the definition, only its consequence. We give the definition only for the benefit of the curious reader.

Definition 5.1. Let $\mathbf{K}$ be an abstract class and λ be a cardinal.

(1) For $M_0 \leq_{\mathbf{K}} M_\ell$ all in $\mathbf{K}_\lambda$, $\ell = 1, 2$, an *amalgam of* M_1 *and* M_2 *over* M_0 is a triple (f_1, f_2, N) such that $N \in \mathbf{K}_\lambda$ and $f_\ell : M_\ell \xrightarrow[M_0]{} N$.
(2) Let (f_1^x, f_2^x, N^x), $x = a, b$ be amalgams of M_1 and M_2 over M_0. We say (f_1^a, f_2^a, N^a) and (f_1^b, f_2^b, N^b) are *equivalent over* M_0 if there exists $N_* \in \mathbf{K}_\lambda$ and $f^x : N^x \to N_*$ such that $f^b \circ f_1^b = f^a \circ f_1^a$ and $f^b \circ f_2^b = f^a \circ f_2^a$, namely, the following commutes:

$$\begin{tikzcd}
 & N^a \arrow[r, dotted, "f^a"] & N_* \\
M_1 \arrow[ur, "f_1^a"] \arrow[rr, "f_1^b"] & & N^b \arrow[u, dotted, "f^b"'] \\
M_0 \arrow[u] \arrow[r] & M_2 \arrow[uu, "f_2^a"] \arrow[ur, "f_2^b"'] &
\end{tikzcd}$$

Note that being "equivalent over M_0" is an equivalence relation ([JS13, 4.3]).
(3) Let $\mathfrak{s}$ be a good $(< \alpha, \lambda)$-frame.
    (a) A *uniqueness triple* in $\mathfrak{s}$ is a triple $(\bar{a}, M, N)$ such that $M \leq_{\mathfrak{s}} N$, $\bar{a} \in {}^{<\alpha}N$ and for any $M_1 \geq_{\mathfrak{s}} M$, there exists a *unique* (up to equivalence over M) amalgam (f_1, f_2, N_1) of N and M_1 over M such that $\mathbf{tp}(f_1(a)/f_2[M_1]; N_1)$ does not fork over M.
    (b) $\mathfrak{s}$ has the *existence property for uniqueness triples* if for any $M \in \mathbf{K}_{\mathfrak{s}}$ and any $p \in \mathbf{S}^{<\alpha}(M)$, one can write $p = \mathbf{tp}(\bar{a}/M; N)$ with $(\bar{a}, M, N)$ a uniqueness triple.
    (c) We say that $\mathfrak{s}$ is *weakly successful* if its restriction to types of length one has the existence property for uniqueness triples.

As an additional motivation, we mention the closely related notion of a *domination triple*:

Definition 5.2. Let $\mathfrak{s}$ be a good $(\leq \lambda, \lambda)$-frame. A *domination triple* in $\mathfrak{s}$ is a triple $(\bar{a}, M, N)$ such that $M \leq_{\mathfrak{s}} N$, $\bar{a} \in {}^{\leq\lambda}N$, and whenever $M' \leq_{\mathfrak{s}} N'$ are such $M \leq_{\mathbf{K}} M'$, $N \leq_{\mathbf{K}} N'$, then $\bar{a} \overset{N'}{\underset{M}{\mathop{\smile\mkern-11mu|}}} M'$ implies $N \overset{N'}{\underset{M}{\mathop{\smile\mkern-11mu|}}} M'$.

The following fact shows that domination triples are the same as uniqueness triples once we have managed to get a good $(\leq \lambda, \lambda)$-frame. The advantage of uniqueness triples is that they can be defined already inside a good λ-frame.

Fact 5.3 (11.7, 11.8 in [Vas16a])**.** Let $\mathfrak{s}$ be a good $(\leq \lambda, \lambda)$-frame. Then in $\mathfrak{s}$ uniqueness triples and domination triples coincide.

The importance of weakly successful good frames is that they extend to longer types. This is due to Shelah:

Fact 5.4. Let $\mathfrak{s}$ be a categorical good λ-frame. If $\mathfrak{s}$ is weakly successful, then there is a unique good $(\leq \lambda, \lambda)$-frame extending $\mathfrak{s}$.

Proof. The uniqueness is [She09a, II.6.3]. Existence is the main result of [She09a, §II.6], although there Shelah only builds a nonforking relation for models satisfying the axioms of a good $(\leq \lambda, \lambda)$-frame. How to extend this to all types of length at most λ is done in [She09a, §III.9]. An outline of the full argument is in the proof of [Vas16a, 12.16(1)]. $\square$

It is not clear whether the converse of Fact 5.4 holds, but see Fact 5.12.

Shelah proved [She09a, II.5.11] that a categorical good λ-frame (generating an AEC $\mathbf{K}$) is weakly successful whenever $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$ and $\mathbf{K}$ has a "medium" number of models in λ^{++}. Shelah has also shown that being weakly successful follows from some stability in λ^+ and $2^\lambda < 2^{\lambda^+}$, see [Vas17a, E.8] for a proof:

Fact 5.5. Let $\mathfrak{s}$ be a categorical good λ-frame generating an AEC $\mathbf{K}$. Assume $2^\lambda < 2^{\lambda^+}$. If for every saturated $M \in \mathbf{K}_{\lambda^+}$ there is $N \in \mathbf{K}_{\lambda^+}$ universal over M, then $\mathfrak{s}$ is weakly successful.

Once we have a good $(\leq \lambda, \lambda)$-frame, we can define a candidate for a good λ^+-frame on the saturated models in $\mathbf{K}_{\lambda^+}$. Nonforking in λ^+ is defined in terms of nonforking in λ (in fact one can make sense of this definition even if we only start with a good λ-frame), but the problem is that we do not know that the class of saturated models in $\mathbf{K}_{\lambda^+}$ has amalgamation. To achieve this, the ordering $\leq_{\mathbf{K}}$ is changed to a new ordering $\leq_{\mathfrak{s}^+}$ so that nonforking "reflects down". The definition is due to Shelah [She09a, III.1.7] but we follow [JS13, 10.1.6]:

Definition 5.6. Let $\mathfrak{s} = (\mathbf{K}_{\mathfrak{s}}, \mathop{\smile}\limits^{\mid})$ be a good $(\leq \lambda, \lambda)$-frame generating an AEC $\mathbf{K}$. We define a pair $\mathfrak{s}^+ = (\mathbf{K}_{\mathfrak{s}^+}, \mathop{\smile}\limits^{\mid}_{\mathfrak{s}^+})$ as follows:

(1) $\mathbf{K}_{\mathfrak{s}^+} = (K_{\mathfrak{s}^+}, \leq_{\mathfrak{s}^+})$, where:
   (a) $K_{\mathfrak{s}^+}$ is the class of saturated models in $\mathbf{K}_{\lambda^+}$.
   (b) For $M, N \in K_{\mathfrak{s}^+}$, $M \leq_{\mathfrak{s}^+} N$ holds if and only if there exists increasing continuous chains $\langle M_i : i < \lambda^+ \rangle$, $\langle N_i : i < \lambda^+ \rangle$ in $\mathbf{K}_\lambda$ such that:
      (i) $M = \bigcup_{i<\lambda^+} M_i$.
      (ii) $N = \bigcup_{i<\lambda^+} N_i$.
      (iii) For all $i < j < \lambda^+$, $N_i \mathop{\smile}\limits^{N_j}_{M_i} M_j$.

(2) For $M_0 \leq_{\mathfrak{s}^+} M \leq_{\mathfrak{s}^+} N$ and $a \in N$, $\underset{\mathfrak{s}^+}{\downarrow}(M_0, a, M, N)$ holds if and only if there exists $M_0' \in \mathcal{P}_{\mathbf{K}_\lambda}(M_0)$ such that for all $M' \in \mathcal{P}_{\mathbf{K}_\lambda}(M)$ and all $N' \in \mathcal{P}_{\mathbf{K}_\lambda}(N)$, if $M_0' \leq_{\mathbf{K}} M' \leq_{\mathbf{K}} N'$ and $a \in N'$, then $\underset{\mathfrak{s}}{\downarrow}(M_0', a, M', N')$.

For a weakly successful categorical good λ-frame $\mathfrak{s}$, we define $\mathfrak{s}^+ := \mathfrak{t}^+$, where $\mathfrak{t}$ is the unique extension of $\mathfrak{s}$ to a good $(\leq \lambda, \lambda)$-frame (Fact 5.4).

As a motivation for the definition of $\leq_{\mathfrak{s}^+}$, observe that if $\langle M_i : i < \lambda^+ \rangle$, $\langle N_i i < \lambda^+ \rangle$ are increasing continuous in $\mathbf{K}_\lambda$ such that $M_i \leq_{\mathbf{K}} N_i$ for all $i < \lambda^+$, then it is known that there is a club $C \subseteq \lambda^+$ such that $M_j \cap N_i = M_i$ for $i < j$ both in C. We would like to conclude the stronger property that $N_i \overset{N_j}{\underset{M_i}{\downarrow}} M_j$ for $i < j$ both in C. If we are working with a superstable elementary class, this is true as nonforking has a strong finite character property, but in this more general setup this is not clear. Thus the property is built into the definition by changing the ordering. This creates a new problem: we do not know that this induces an AEC (smoothness is the problematic axiom). Note that there are no known examples of weakly successful good λ-frame $\mathfrak{s}$ where $\leq_{\mathfrak{s}^+}$ is not $\leq_{\mathbf{K}}$.

The following general properties of $\mathfrak{s}^+$ are known. They are all proven in [She09a, §II.7] but we cite from [JS13]:

Fact 5.7. Let $\mathfrak{s}$ be a weakly successful categorical good λ-frame generating an AEC $\mathbf{K}$.

(1) [JS13, 6.1.5] If $M, N \in \mathbf{K}_{\mathfrak{s}^+}$ and $M \leq_{\mathfrak{s}^+} N$, then $M \leq_{\mathbf{K}} N$.
(2) [JS13, 6.1.6(b),(d), 7.1.18(a)], $\mathbf{K}_{\mathfrak{s}^+}$ is an abstract class which is closed under unions of chains of length strictly less than λ^{++}.
(3) [JS13, 6.1.6(c)] $\mathbf{K}_{\mathfrak{s}^+}$ satisfies the following strengthening of the coherence axiom: if $M_0, M_1, M_2 \in \mathbf{K}_{\mathfrak{s}^+}$ are such that $M_0 \leq_{\mathfrak{s}^+} M_2$ and $M_0 \leq_{\mathbf{K}} M_1 \leq_{\mathbf{K}} M_2$, then $M_0 \leq_{\mathfrak{s}^+} M_1$.
(4) [JS13, 6.1.6(e), 7.1.18(c)] $\mathbf{K}_{\mathfrak{s}^+}$ has no maximal models and amalgamation.
(5) [JS13, 10.1.4] (the proof does not use that $\mathfrak{s}$ is successful) Let $M \leq_{\mathfrak{s}^+} N_\ell$ and let $a_\ell \in N_\ell$, $\ell = 1, 2$. Then $\mathbf{tp}_{\mathbf{K}}(a_1/M; N_1) = \mathbf{tp}_{\mathbf{K}}(a_2/M; N_2)$ if and only if $\mathbf{tp}_{\mathbf{K}_{\mathfrak{s}^+}}(a_1/M; N_1) = \mathbf{tp}_{\mathbf{K}_{\mathfrak{s}^+}}(a_2/M; N_2)$. In particular, $\mathfrak{s}^+$ is a λ^+-frame.

We will use the following terminology, taken from [Vas17a, 3.7(2)]:

Definition 5.8. Let $\mathfrak{s}$ be a good $(\leq \lambda, \lambda)$-frame generating an AEC $\mathbf{K}$. We say that $\mathfrak{s}$ *reflects down* if $M \leq_{\mathbf{K}} N$ implies $M \leq_{\mathfrak{s}^+} N$ for all $M, N \in \mathbf{K}_{\mathfrak{s}^+}$. We say that $\mathfrak{s}$ *almost reflects down* if $\mathbf{K}_{\mathfrak{s}^+}$ generates an AEC. We say that a weakly successful good λ-frame *[almost] reflects down* if its extension to a good $(\leq \lambda, \lambda)$-frame [almost] reflects down, see Fact 5.4.

Shelah uses the less descriptive "successful":

Definition 5.9 (III.1.1 in [She09a])**.** We say that a good λ-frame $\mathfrak{s}$ is *successful* if it is weakly successful and almost reflects down.

The point of this definition is that if it holds, then $\mathfrak{s}$ can be extended to a good λ^+-frame. Moreover $\mathfrak{s}$ is successful when there are few models in λ^{++}:

Fact 5.10 (III.1.9 in [She09a])**.** If $\mathfrak{s}$ is a successful categorical good λ-frame, then $\mathfrak{s}^+$ is a good$^+$ λ^+-frame.

Fact 5.11 (II.8.4, II.8.5 in [She09a], or see 7.1.3 in [JS13])**.** Let $\mathfrak{s}$ be a weakly successful categorical good λ-frame generating the AEC $\mathbf{K}$. If $\mathbb{I}(\mathbf{K}, \lambda^{++}) < 2^{\lambda^{++}}$, then $\mathfrak{s}$ is successful.

The downside of working only with a successful good λ-frame is that the ordering of $\mathbf{K}_{\mathfrak{s}^+}$ may not be $\leq_{\mathbf{K}}$ anymore. Thus the AEC generated by $\mathbf{K}_{\mathfrak{s}^+}$ may be different from the original one. For example it may fail to have arbitrarily large models even if the original one does. This is why we will focus on good frames that *reflect down*, i.e. $\leq_{\mathfrak{s}^+}$ is just $\leq_{\mathbf{K}}$. Several characterizations of this situation are known. (4) implies (1) below is due to Jarden and all the other implications are due to Shelah (but as usual we mostly quote from [JS13]):

Fact 5.12. Let $\mathfrak{s}$ be a categorical good λ-frame. The following are equivalent:

(1) $\mathfrak{s}$ extends to a good $(\leq \lambda, \lambda)$-frame which reflects down.
(2) $\mathfrak{s}$ is successful and good$^+$ (see Definition 3.5).
(3) $\mathfrak{s}$ is successful and decent (see Definition 3.1).
(4) $\mathfrak{s}$ is weakly successful and generates an AEC $\mathbf{K}$ which is (λ, λ^+)-weakly tame and so that $\mathbf{K}_{\lambda^+}^{\lambda^+\text{-sat}}$ has amalgamation.

Proof.

- <u>(1) implies (2)</u>: By [Vas17a, 3.11], $\mathfrak{s}$ is weakly successful. Since it reflects down, it is successful and $\leq_{\mathfrak{s}^+}$ is the restriction of $\leq_{\mathbf{K}}$ to $\mathbf{K}_{\lambda^+}^{\lambda^+\text{-sat}}$. By adapting the proof of [She09a, III.1.5(3)] (see [BVa, 2.14]), we get that $\mathfrak{s}$ is good$^+$.
- <u>(2) implies (3)</u>: By Lemma 3.6.
- <u>(3) implies (1)</u>: By definition of successful, $\mathfrak{s}$ is weakly successful, so by Fact 5.4 extends to a unique good $(\leq \lambda, \lambda)$-frame. Since $\mathfrak{s}$ is decent, we have that the order[1] $\preceq_{\lambda^+}^{\otimes}$ is the same as $\leq_{\mathbf{K}}$. Since $\mathfrak{s}$ is successful, all of the equivalent conditions of [JS13, 9.1.13] are false, so in particular for $M, N \in \mathbf{K}_{\mathfrak{s}^+}$, $M \leq_{\mathbf{K}} N$ implies $M \leq_{\mathfrak{s}^+} N$. Therefore $\mathfrak{s}$ reflects down.
- <u>(1) implies (4)</u>: we have already argued that $\mathfrak{s}$ is successful, and by definition $\leq_{\mathfrak{s}^+}$ is just the restriction of $\leq_{\mathbf{K}}$, i.e. $\mathbf{K}_{\mathfrak{s}^+} = \mathbf{K}_{\lambda^+}^{\lambda^+\text{-sat}}$. Now weak tameness follows from [She09a, III.1.10] and amalgamation is because by Fact 5.10 $\mathfrak{s}^+$ is a good λ^+-frame.
- <u>(4) implies (1)</u>: By [Jar16, 7.15].

$\square$

The aim of this section is to add another condition to Fact 5.12: the existence of a good λ^+-frame on $\mathbf{K}_{\lambda^+}^{\lambda^+\text{-sat}}$. Toward this, we will use the following ordering on pairs of saturated models, introduced by Jarden [Jar16, 7.5]:

[1]SV: cite?

Definition 5.13. Let $\mathfrak{s}$ be a weakly successful categorical good λ-frame generating an AEC $\mathbf{K}$. For pairs $(M, N), (M', N')$ in $\mathbf{K}_{\mathfrak{s}^+}$ with $M \leq_{\mathbf{K}} N$, $M' \leq_{\mathbf{K}} N'$, we write $(M, N) \lhd (M', N')$ if:

(1) $M \leq_{\mathfrak{s}^+} M'$.
(2) $N \leq_{\mathbf{K}} N'$.
(3) $M' \cap N \neq M$.

We write $(M, N) \unlhd (M', N')$ if $(M, N) \lhd (M', N')$ or $(M, N) = (M', N')$.

Fact 5.14. Let $\mathfrak{s}$ be a weakly successful categorical good λ-frame generating an AEC $\mathbf{K}$.

(1) (See the proof of [Jar16, 7.8]) For any $M \leq_{\mathbf{K}} N$ both in $\mathbf{K}_{\mathfrak{s}^+}$, there exists a pair (M', N') such that $(M, N) \unlhd (M', N')$ and (M', N') is $\lhd$-maximal.
(2) [JS13, 9.1.13] If $\mathfrak{s}$ is decent, $M \leq_{\mathbf{K}} N$ are both in $\mathbf{K}_{\mathfrak{s}^+}$, and $M \not\leq_{\mathfrak{s}^+} N$, then there exists $\langle M_i : i \leq \lambda^+ \rangle$ increasing continuous in $\mathbf{K}_{\mathfrak{s}^+}$ such that $M_{\lambda^+} = M$ and $M_i \leq_{\mathfrak{s}^+} N$ for all $i < \lambda^+$.

We have arrived to the main theorem of this section:

Theorem 5.15. Let $\mathfrak{s}$ be a weakly successful categorical good λ-frame generating an AEC $\mathbf{K}$. If there is a good λ^+-frame on $\mathbf{K}_{\lambda^+}^{\lambda^+\text{-sat}}$, then $\mathfrak{s}$ reflects down.

Proof. Let $\mathfrak{t}$ be the good λ^+-frame on $\mathbf{K}_{\lambda^+}^{\lambda^+\text{-sat}}$. Note that since $\mathfrak{t}$ is good, $\mathbf{K}_{\lambda^+}^{\lambda^+\text{-sat}}$ must generate an AEC, i.e. chains of λ^+-saturated models are λ^+-saturated, so by Fact 2.14 $\mathbf{K}$ has a superlimit in λ^+ which by Theorem 3.3 implies that $\mathfrak{s}$ is decent. Therefore we will later be able to apply Fact 5.14(2). We first show:

<u>Claim</u>: If $M \leq_{\mathbf{K}} N$ are both in $\mathbf{K}_{\mathfrak{s}^+}$, $a \in |N| \backslash |M|$ and $\mathbf{tp}_{\mathbf{K}}(a/M; N)$ is realized in a $\leq_{\mathfrak{s}^+}$-extension of M, then (M, N) is not $\lhd$-maximal.

<u>Proof of Claim</u>: Let $p := \mathbf{tp}_{\mathbf{K}}(a/M; N)$. Let N' be such that $N \leq_{\mathfrak{s}^+} N'$ and p is realized in N'. Say $p = \mathbf{tp}_{\mathbf{K}}(b/M; N')$. By amalgamation in $\mathbf{K}_{\lambda^+}^{\lambda^+\text{-sat}}$ (which holds since we are assuming there is a good frame on this class), and the fact that $\mathbf{tp}_{\mathbf{K}}(a/M; N) = \mathbf{tp}_{\mathbf{K}}(b/M; N')$, there exists $N'' \in \mathbf{K}_{\lambda^+}^{\lambda^+\text{-sat}}$ and $f : N' \xrightarrow[M]{} N''$ with $N \leq_{\mathbf{K}} N''$ such that $f(b) = a$. Consider the pair $(f[N'], N'')$. Since $\leq_{\mathfrak{s}^+}$ is invariant under isomorphisms and $M \leq_{\mathfrak{s}^+} N'$, $M \leq_{\mathfrak{s}^+} f[N']$. Moreover, $a \in |N| \backslash |M|$, so $b \in |N'| \backslash |M|$, and so $a = f(b) \in |f[N']| \backslash |M|$. Since we also have that $a \in |N|$, this implies that $f[N'] \cap N \neq M$, so $(M, N) \lhd (f[N'], N'')$, hence (M, N) is not $\lhd$-maximal. $\dagger_{\text{Claim}}$

Let $M, N \in \mathbf{K}_{\mathfrak{s}^+}$ be such that $M \leq_{\mathbf{K}} N$. We have to show that $M \leq_{\mathfrak{s}^+} N$. Suppose not. By Fact 5.14(1), there exists (M', N') such that $(M, N) \unlhd (M', N')$ and (M', N') is $\lhd$-maximal. Observe that $M' \not\leq_{\mathfrak{s}^+} N'$: if $M' \leq_{\mathfrak{s}^+} N'$, then since also $M \leq_{\mathfrak{s}^+} M'$ and $\mathbf{K}_{\mathfrak{s}^+}$ is an abstract class (Fact 5.7(2)), we would have that $M \leq_{\mathfrak{s}^+} N'$ so by Fact 5.7(3), $M \leq_{\mathfrak{s}^+} N$, a contradiction. By Fact 5.14(2), there exists $\langle M_i : i \leq \lambda^+ \rangle$ increasing continuous in $\mathbf{K}_{\mathfrak{s}^+}$ such that $M_{\lambda^+} = M'$ and $M_i \leq_{\mathfrak{s}^+} N'$ for all $i < \lambda^+$. Since $M' \not\leq_{\mathfrak{s}^+} N'$, we have in particular that $M' \neq N'$. Let $a \in N' \backslash M'$ and let $p := \mathbf{tp}_{\mathbf{K}}(a/M'; N')$. By local character in $\mathfrak{t}$ (the good λ^+-frame on $\mathbf{K}_{\lambda^+}^{\lambda^+\text{-sat}}$), there is $i < \lambda^+$ such that p does not $\mathfrak{t}$-fork over M_i. By conjugation in $\mathfrak{t}$ (Fact 2.10), this means that $p \upharpoonright M_i$ and p are isomorphic. However since $M_i \leq_{\mathfrak{s}^+} N'$, $p \restriction M_i$ is realized inside some $\leq_{\mathfrak{s}^+}$-extension of M_i, hence p is also realized inside some $\leq_{\mathfrak{s}^+}$-extension of M. Together with the claim, this contradicts the $\triangleleft$-maximality of (M', N'). $\square$

Remark 5.16. We are not using all the properties of the good λ^+-frame. In particular, it suffices that local character holds for chains of length λ^+.

We obtain a new characterization of when a weakly successful frame reflects down. We emphasize that only (3) implies (1) below is new. (1) implies (2) and (1) implies (3) are due to Shelah while (2) implies (1) is due to Jarden.

Corollary 5.17. Let $\mathfrak{s}$ be a weakly successful categorical good λ-frame generating the AEC $\mathbf{K}$. The following are equivalent:

(1) $\mathfrak{s}$ is successful and decent.
(2) $\mathbf{K}$ is (λ, λ^+)-weakly tame and $\mathbf{K}^{\lambda^+\text{-sat}}_{\lambda^+}$ has amalgamation.
(3) There is a good λ^+-frame on $\mathbf{K}^{\lambda^+\text{-sat}}_{\lambda^+}$.

Proof.

- <u>(1) is equivalent to (2)</u>: By Fact 5.12.
- <u>(1) implies (3)</u>: By Facts 5.10 and 5.12.
- <u>(3) implies (1)</u>: By Theorem 5.15 and Fact 5.12.

$\square$

We can combine our result with the weak GCH to obtain weak tameness from two successive good frames. This gives a converse to Theorem 4.1.

Corollary 5.18. Let $\mathbf{K}$ be an AEC, let $\lambda \geq \text{LS}(\mathbf{K})$ and assume that $2^\lambda < 2^{\lambda^+}$. Let $\mathfrak{s}$ be a categorical good λ-frame on $\mathbf{K}_\lambda$. The following are equivalent:

(1) There is a good$^+$ λ^+-frame on $\mathbf{K}^{\lambda^+\text{-sat}}_{\lambda^+}$.
(2) There is a good λ^+-frame on $\mathbf{K}^{\lambda^+\text{-sat}}_{\lambda^+}$.
(3) $\mathbf{K}$ is (λ, λ^+)-weakly tame, $\mathfrak{s}$ is decent, and $\mathbf{K}^{\lambda^+\text{-sat}}_{\lambda^+}$ has amalgamation.
(4) $\mathbf{K}$ is (λ, λ^+)-weakly tame and for every saturated $M \in \mathbf{K}_{\lambda^+}$ there is $N \in \mathbf{K}_{\lambda^+}$ universal over M.
(5) $\mathfrak{s}$ is successful and good$^+$.

Proof.

- <u>(1) implies (2)</u>: Trivial.
- <u>(2) implies (3)</u>: By Fact 5.5, $\mathfrak{s}$ is weakly successful. Now apply Corollary 5.17.
- <u>(3) implies (2)</u>: By Theorem 4.1.
- <u>(2) implies (4)</u>: Assume (2). Then by definition of a good λ^+-frame, for every saturated $M \in \mathbf{K}_{\lambda^+}$ there is $N \in \mathbf{K}_{\lambda^+}$ universal over M. Further we proved already that (3) holds. Therefore $\mathbf{K}$ is (λ, λ^+)-weakly tame.
- <u>(4) implies (5)</u>: By Fact 5.5, $\mathfrak{s}$ is weakly successful. Now apply (4) implies (2) in Fact 5.12.
- <u>(5) implies (1)</u>: By Fact 5.10 and since $\mathfrak{s}$ reflects down (Fact 5.12).

□

The observant reader may complain that assuming the existence of a good frame on $\mathbf{K}_{\lambda^+}^{\lambda^+\text{-sat}}$ implies by definition that there is a superlimit in λ^+. However we can also directly start with a good λ^+-frame on all of $\mathbf{K}_{\lambda^+}$:

Corollary 5.19. Let $\mathbf{K}$ be an AEC, let $\lambda \geq \text{LS}(\mathbf{K})$ and assume that $2^\lambda < 2^{\lambda^+}$. If there is a categorical good λ-frame on $\mathbf{K}_\lambda$ and a good λ^+-frame on $\mathbf{K}_{\lambda^+}$, then $\mathbf{K}$ is (λ, λ^+)-weakly tame.

Proof. Let $\mathfrak{s}$ be a categorical good λ-frame on $\mathbf{K}_\lambda$. By Lemma 4.4, $\mathfrak{s}$ is decent so by Theorem 3.3, $\mathbf{K}$ has a superlimit in λ^+. Thus by Fact 2.14 we can restrict the good λ^+-frame on $\mathbf{K}_{\lambda^+}$ to the class $\mathbf{K}_{\lambda^+}^{\lambda^+\text{-sat}}$, still obtain a good λ^+-frame, and apply Corollary 5.18. □

6. On categoricity in two successive cardinals

Grossberg and VanDieren have shown [GV06a, 6.3] that in a λ-tame AEC with amalgamation and no maximal models, categoricity in λ and λ^+ imply categoricity in all $\mu \geq \lambda$. In [Vas17a], the author gave a more local conclusion as well as a more abstract proof using good frames. Here, we give a converse: assuming GCH, if we can prove categoricity in λ^{++} from categoricity in λ and λ^+, then we must have some tameness. This follows from combining results of Shelah but seems not to have been noticed before. Only (2) implies (3) below really uses the results of this paper.

Corollary 6.1. Let $\mathbf{K}$ be an AEC and let $\lambda \geq \text{LS}(\mathbf{K})$. Assume $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$ and assume that $\mathbf{K}$ is categorical in λ and λ^+. The following are equivalent:

(1) There is a successful good λ-frame on $\mathbf{K}_\lambda$.
(2) There is a good λ-frame on $\mathbf{K}_\lambda$ and a good λ^+-frame on $\mathbf{K}_{\lambda^+}$.
(3) $\mathbf{K}$ is stable in λ, $\mathbf{K}$ is (λ, λ^+)-tame, $\mathbf{K}$ has amalgamation in λ^+, and $\mathbf{K}_{\lambda^{++}} \neq \emptyset$.
(4) $\mathbf{K}$ is categorical in λ^{++}.
(5) $1 \leq \mathbb{I}(\mathbf{K}, \lambda^{++}) < \mu_{\text{unif}}(\lambda^{++}, 2^{\lambda^+})$.
(6) $\mathbf{K}$ is stable in λ, $\mathbf{K}$ is stable in λ^+, and $1 \leq \mathbb{I}(\mathbf{K}, \lambda^{++}) < 2^{\lambda^{++}}$.

On μ_{unif}, see [She09b, VII.0.6] for a definition and [She09b, VII.9.4] for what is known. It seems that for all practical purpose the reader can take $\mu_{\text{unif}}(\lambda^{++}, 2^{\lambda^+})$ to mean $2^{\lambda^{++}}$.

Note that if the AEC $\mathbf{K}$ of Corollary 6.1 has arbitrarily large models, then stability in λ would follow from categoricity in λ^+ [She99, I.1.7(a)]. Thus we obtain that $\mathbf{K}$ is categorical in λ^{++} if and only if $\mathbf{K}$ is (λ, λ^+)-tame and has amalgamation in λ^+.

To prove Corollary 6.1, we will use several facts:

Fact 6.2 (I.3.8 in [She09a])**.** Let $\mathbf{K}$ be an AEC and let $\lambda \geq \text{LS}(\mathbf{K})$. Assume $2^\lambda < 2^{\lambda^+}$. If $\mathbf{K}$ is categorical in λ and $\mathbb{I}(\mathbf{K}, \lambda^+) < 2^{\lambda^+}$, then $\mathbf{K}$ has amalgamation in λ.

Fact 6.3. Let $\mathbf{K}$ be an AEC and let $\lambda \geq \mathrm{LS}(\mathbf{K})$. Assume $2^\lambda < 2^{\lambda^+}$ and assume that $\mathbf{K}$ is categorical in both λ and λ^+.

(1) If $\mathbf{K}_{\lambda^{++}} \neq \emptyset$ and $\mathbf{K}$ is stable in λ, then there is an almost good λ-frame (see [She09b, VI.8.2]) on $\mathbf{K}_\lambda$.
(2) If $\mathbf{K}_{\lambda^{++}} \neq \emptyset$, $\mathbf{K}$ has amalgamation in λ^+, $\mathbf{K}$ is stable in λ, and $\mathbf{K}$ is stable in λ^+, then there is a weakly successful good λ-frame on $\mathbf{K}_\lambda$.
(3) If $2^{\lambda^+} < 2^{\lambda^{++}}$, $1 \leq \mathbb{I}(\mathbf{K}, \lambda^{++}) < 2^{\lambda^{++}}$, $\mathbf{K}$ is stable in λ, and $\mathbf{K}$ is stable in λ^+, then there is a successful good λ-frame on $\mathbf{K}_\lambda$.
(4) If $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$ and $1 \leq \mathbb{I}(\mathbf{K}, \lambda^{++}) < \mu_{\mathrm{unif}}(\lambda^{++}, 2^{\lambda^+})$, then there is a successful good λ-frame on $\mathbf{K}_\lambda$.

Proof.

(1) By Fact 6.2, $\mathbf{K}$ has amalgamation in λ. We check that the hypotheses of [She09b, VI.8.1(2)] are satisfied. The only ones that we are not explicitly assuming are:
   (a) The extension property in $\mathbf{K}_\lambda$, i.e. for every $M \leq_{\mathbf{K}} N$ both in $\mathbf{K}_\lambda$ and every $p \in \mathbf{S}(M)$, if p is not algebraic (i.e. not realized inside M), then p has a *nonalgebraic* extension to $\mathbf{S}(N)$: holds by [She09b, VI.2.23(1)].
   (b) The existence of an inevitable type in $\mathbf{K}_\lambda$: holds by [She09b, VI.5.3(1)] and the density of minimal types (see the proof of $(*)_5$ in [She99, II.2.7]).
(2) By (1), there is an almost good λ-frame $\mathfrak{s}$ on $\mathbf{K}_\lambda$. The proof of [Vas17a, E.8] goes through even for almost good λ-frames and gives that $\mathfrak{s}$ is weakly successful (note that Shelah's proof of Fact 2.10 still goes through in almost good λ-frames). By [JS, 4.3], $\mathfrak{s}$ is in fact a good λ-frame.
(3) By Fact 6.2, $\mathbf{K}$ has amalgamation in λ and λ^+. By (2), there is a weakly successful good λ-frame $\mathfrak{s}$ on $\mathbf{K}_\lambda$. By Fact 5.11, $\mathfrak{s}$ is successful.
(4) By [She09b, VI.8.1], there is an almost good λ-frame $\mathfrak{s}$ on $\mathbf{K}_\lambda$. By [She09b, VII.6.17], $\mathfrak{s}$ has existence for a certain relative of uniqueness triples. Thus by [She09b, VII.7.19(2)], $\mathfrak{s}$ is actually a good λ-frame. By [She09a, II.5.11], $\mathfrak{s}$ is weakly successful. By Fact 5.11, $\mathfrak{s}$ is successful.

$\square$

Fact 6.4. Let $\mathbf{K}$ be an AEC and let $\lambda \geq \mathrm{LS}(\mathbf{K})$. If $\mathbf{K}$ is categorical in both λ and λ^+ and there is a successful good λ-frame on $\mathbf{K}_\lambda$, then $\mathbf{K}$ is categorical in λ^{++}.

Proof. Let $\mathfrak{s}$ be a successful good λ-frame on $\mathbf{K}_\lambda$. Since $\mathbf{K}$ is categorical in λ^+, there is a superlimit in λ^+, hence by Theorem 3.3 $\mathfrak{s}$ is decent. Now combine [She09a, III.2.10(2)] with [She09a, III.2.11(1)]. $\square$

Fact 6.5 ([BKV06])**.** Let $\mathbf{K}$ be an AEC and let $\lambda \geq \mathrm{LS}(\mathbf{K})$. If $\mathbf{K}$ has amalgamation in λ, is stable in λ, and is (λ, λ^+)-tame, then $\mathbf{K}$ is stable in λ^+.

Proof of Corollary 6.1. By categoricity in λ and λ^+ all good λ-frames on $\mathbf{K}_\lambda$ are categorical, $\mathbf{K}_{\lambda^+}^{\lambda^+\text{-sat}} = \mathbf{K}_{\lambda^+}$, and (λ, λ^+)-weak tameness is the same as (λ, λ^+)-tameness. By Theorem 3.3, any good λ-frame on $\mathbf{K}_\lambda$ is decent. We will use this without comments.

- (1) implies (2): By Fact 5.12, the definition of reflecting down (Definition 5.8) and the definition of $\mathfrak{s}^+$ (Definition 5.6), $\mathbf{K}_{\mathfrak{s}^+} = \mathbf{K}_{\lambda^+}^{\lambda^+\text{-sat}} = \mathbf{K}_{\lambda^+}$, so the result follows from Fact 5.10.
- (2) implies (3): By definition of a good λ-frame, $\mathbf{K}$ is stable in λ and by definition of a good λ^+-frame $\mathbf{K}$ has amalgamation in λ^+. Since a good λ^+-frame has no maximal models in λ^+, $\mathbf{K}_{\lambda^{++}} \neq \emptyset$. By Corollary 5.18, $\mathbf{K}$ is (λ, λ^+)-tame.
- (3) implies (1): By Fact 6.2, $\mathbf{K}$ has amalgamation in λ. By Fact 6.5, $\mathbf{K}$ is stable in λ^+. By Fact 6.3, there is a weakly successful good λ-frame $\mathfrak{s}$ on $\mathbf{K}_\lambda$. By Fact 5.12, $\mathfrak{s}$ is successful.
- (1) implies (4): By Fact 6.4.
- (4) implies (5): Trivial.
- (5) implies (1): By Fact 6.3.
- (5) implies (6): (5) trivially implies that $1 \leq \mathbb{I}(\mathbf{K}, \lambda^{++}) < 2^{\lambda^{++}}$. We have also seen that (5) implies (1) implies (2) which by definition implies stability in λ and λ^+.
- (6) implies (1): By Fact 6.3.

$\square$

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