SHELAH'S EVENTUAL CATEGORICITY CONJECTURE IN TAME AECS WITH PRIMES

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Abstract. A new case of Shelah's eventual categoricity conjecture is established:

Theorem 1. Let $\mathcal{K}$ be an AEC with amalgamation. Write $H_2 := \beth_{\left(2^{\beth_{\left(2^{\text{LS}(\mathcal{K})}\right)^+}}\right)^+}$. Assume that $\mathcal{K}$ is H_2-tame and $\mathcal{K}_{\ge H_2}$ has primes over sets of the form $M \cup \{a\}$. If $\mathcal{K}$ is categorical in some $\lambda > H_2$, then $\mathcal{K}$ is categorical in all $\lambda' \ge H_2$.

The result had previously been established when the stronger locality assumptions of full tameness and shortness are also required.

An application of the method of proof of Theorem 1 is that Shelah's categoricity conjecture holds in the context of homogeneous model theory (this was known, but our proof gives new cases):

Theorem 2. Let D be a homogeneous diagram in a first-order theory T. If D is categorical in a $\lambda > |T|$, then D is categorical in all $\lambda' \ge \min(\lambda, \beth_{(2^{|T|})^+})$.

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1. Introduction

Shelah's eventual categoricity conjecture is a major force in the development of classification theory for abstract elementary classes (AECs)[1].

Conjecture 1.1 (Shelah's eventual categoricity conjecture, N.4.2 in [She09])**.** An AEC categorical in a high-enough cardinal is categorical on a tail of cardinals.

In [Vasb], we established the conjecture for universal classes with the amalgamation property[2] (a universal class is a class of models closed under isomorphisms, substructures, and unions of $\subseteq$-increasing chains, see [She87]). The proof starts by noting that universal classes satisfy tameness: a locality property introduced in VanDieren's 2002 Ph.D. thesis (the relevant chapter appears in [GV06b]).

Fact 1.2 ([Bon])**.** Any universal class $\mathcal{K}$ is[3] $\mathrm{LS}(\mathcal{K})$-tame.

The proof generalizes to give a stronger locality property introduced in [Bon14]:

Definition 1.3. Let $\mathcal{K}$ be an AEC and let $\chi \geq \mathrm{LS}(\mathcal{K})$ be an infinite cardinal. $\mathcal{K}$ is *fully χ-tame and short* if for any $M \in \mathcal{K}$, any ordinal α, and any Galois types $p, q \in \mathrm{gS}^\alpha(M)$ of length α, $p = q$ if and only if $p^I \restriction M_0 = q^I \restriction M_0$ for any $M_0 \in \mathcal{K}_{\leq \chi}$ with $M_0 \leq M$ and any $I \subseteq \alpha$ with $|I| \leq \chi$.

Fact 1.4. Any universal class $\mathcal{K}$ is fully $\mathrm{LS}(\mathcal{K})$-tame and short.

Another important property of universal classes used in the proof of Shelah's eventual categoricity conjecture [Vasb, Corollary 5.20] is that they have primes. The definition is due to Shelah and appears in [She09, Section III.3]. For the convenience of the reader, we include it here:

Definition 1.5. Let $\mathcal{K}$ be an AEC.

(1) We say a triple (a, M, N) *represents* a Galois type p if $p = \mathrm{gtp}(a/M; N)$. In particular, $M \leq N$ and $a \in |N|$.

[1] For a history, see the introduction of [Vasb]. We assume here that the reader is familiar with the basics of AECs as presented in e.g. [Bal09].

[2] After the initial submission of this paper, we managed to remove the amalgamation hypothesis [Vasc].

[3] While the main idea of the proof is due to Will Boney, the fact that it applies to universal classes is due to the author. A full proof of Fact 1.2 appears as [Vasb, Theorem 3.7].

(2) A *prime triple* is a triple (a, M, N) representing a nonalgebraic Galois type p such that for every $N' \in \mathcal{K}$, $a' \in |N'|$, if $p = \mathrm{gtp}(a'/M; N')$ then there exists $f : N \underset{M}{\longrightarrow} N'$ so that $f(a) = a'$.
(3) We say that $\mathcal{K}$ *has primes* if for every $M \in \mathcal{K}$ and every nonalgebraic $p \in \mathrm{gS}(M)$, there exists a prime triple representing p.
(4) We define localizations such as "$\mathcal{K}_\lambda$ has primes" in the natural way.

By taking the closure of $|M| \cup \{a\}$ under the functions of N, we get:

Fact 1.6 (Remark 5.3 in [Vasb])**.** Any universal class has primes.

The proof of the eventual categoricity conjecture for universal classes with amalgamation in [Vasb] generalizes to give:

Fact 1.7 (Theorem 5.18 in [Vasb])**.** Fully tame and short AECs that have amalgamation and primes satisfy Shelah's eventual categoricity conjecture.

Many results only use the assumption of tameness (for example [GV06b, GV06c, GV06a, BKV06, Lie11, Vas16b, BVa]), while others use full tameness and shortness [BG, Vas16a] (but it is also unclear whether it is really needed there, see [Vas16a, Question 15.4]).

It is natural to ask whether shortness can be removed from Fact 1.7. We answer in the affirmative: Tame AECs with primes and amalgamation satisfy Shelah's eventual categoricity conjecture. To state this more precisely, we adopt notation from [Bal09, Chapter 14].

Notation 1.8. For λ an infinite cardinal, let $h(\lambda) := \beth_{(2^\lambda)^+}$. For $\mathcal{K}$ a fixed AEC, write $H_1 := h(\mathrm{LS}(\mathcal{K}))$ and $H_2 := h(H_1) = h(h(\mathrm{LS}(\mathcal{K})))$.

Main Theorem 3.8. Let $\mathcal{K}$ be an AEC with amalgamation. Assume that $\mathcal{K}$ is H_2-tame and $\mathcal{K}_{\geq H_2}$ has primes. If $\mathcal{K}$ is categorical in some $\lambda > H_2$, then $\mathcal{K}$ is categorical in all $\lambda' \geq H_2$.

This improves [Vasb, Theorem 5.18] which assumed full $\mathrm{LS}(\mathcal{K})$-tameness and shortness (so the improvement is on two counts: "full tameness and shortness" is replaced by "tameness" and "$\mathrm{LS}(\mathcal{K})$" is replaced by "H_2"). Compared to Grossberg and VanDieren's upward transfer [GV06a], we do *not* require categoricity in a successor cardinal, but we *do* require the categoricity cardinal to be at least H_2 and more importantly ask for the AEC to have primes.

Let us give a rough picture of the proof of both Theorem 3.8 and the earlier [Vasb, Theorem 5.18]. We will then explain where exactly the two proofs differ. The first step of the proof is to find a sub-AEC $\mathcal{K}'$ of $\mathcal{K}$ (typically a class of saturated models or just a tail: in the case of Theorem 3.8 we will have $\mathcal{K}' = \mathcal{K}_{\geq H_2}$) which is "well-behaved" in the sense of admitting a good-enough notion of independence. Typically, the first step does not use primes. The second step is to show that in $\mathcal{K}'$, categoricity in *some* $\lambda > \text{LS}(\mathcal{K}')$ implies categoricity in *all* $\lambda' > \text{LS}(\mathcal{K}')$. This uses orthogonality calculus and the existence of prime models. The third step pulls back this categoricity transfer to $\mathcal{K}$.

Now Shelah has developed orthogonality calculus in the context of what he calls successful good$^+$ λ-frames [She09, Section III.6]. It is known [Vas16a] that one can build such a frame using categoricity, amalgamation, and full tameness and shortness so this is how $\mathcal{K}'$ from the previous paragraph was chosen in [Vasb]. The orthogonality calculus part was just quoted from Shelah (although we did provide some proofs for the convenience of the reader). It is not known how to build a successful good$^+$ λ-frame using just categoricity, amalgamation, and *tameness*.

In this paper, we develop orthogonality calculus in the setup of good λ-frames with primes (i.e. we get rid of the successful good$^+$ hypothesis). Note that it is easier to build good frames than to build successful ones (see [Vas16b] and the recent [VV, Corollary 6.14]). In particular, this can be done with just amalgamation, categoricity, and tameness (the threshold cardinals are also lower than in the construction of a successful good frame).

To develop orthogonality calculus in good frames with primes, one has to change Shelah's definition of orthogonality (it uses the so-called uniqueness triples, which may not exist here) and check that the proofs needed for the categoricity transfer still go through. Lemma 2.2, saying that a definition of orthogonality in terms of "for all" is equivalent to one in terms of "there exists", has a different proof than Shelah's.

Let us justify the assumptions of Theorem 3.8. First of all, why do we ask for $\lambda > H_2$ and not e.g. $\lambda > H_1$ or even $\lambda > \text{LS}(\mathcal{K})$? The reason is that the argument uses categoricity in *two* cardinals, so we appeal to a downward categoricity transfer implicit in [She99, II.1.6] which proves (without using primes) that classes as in the hypothesis of Theorem 3.8 must be categorical in H_2. If we know that the class is categorical in two cardinals already, then we can work above $\text{LS}(\mathcal{K})$ (provided of course we adjust the levels at which tameness and primes occur). This is Theorem 3.4. Moreover if we know that for some $\chi < \lambda$, the class of

χ-saturated models of $\mathcal{K}$ has primes, then we can also lower the Hanf number from H_2 to H_1 (see Theorem 3.10).

Let us now discuss the structural assumptions on $\mathcal{K}$. Many classes occurring in practice have amalgamation. Grossberg conjectured [Gro02, Conjecture 2.3] that eventual amalgamation should follow from categoricity and, assuming that the class is *eventually syntactically characterizable* (see [Vasb, Section 4]), it does assuming the other assumptions: tameness and having primes. We now focus on these two assumptions.

A wide variety of AECs are tame (see e.g. the introduction to [GV06b] or the upcoming survey [BVb]), and many classes studied by algebraists have primes (one example are AECs which admit intersections, i.e. whenever $N \in \mathcal{K}$ and $A \subseteq |N|$, we have that $\bigcap\{M \leq N \mid A \subseteq |M|\} \leq N$. See [BS08] or [Vasb, Section 2]). Tameness is conjectured (see [GV06a, Conjecture 1.5]) to follow from categoricity and of course, the existence of prime models plays a key role in many categoricity transfer results including Morley's categoricity theorem and Shelah's generalization to excellent classes [She83a, She83b]. Currently, no general way[4] of building prime models in AECs is known except by going through the machinery of excellence [She09, Chapter III]. It is unknown whether excellence follows from categoricity.

In the special case of homogeneous model theory, it is easier to build prime models[5]. Let $\mathcal{K}$ be a class of models of a homogeneous diagram categorical in a $\lambda > H_2$. Clearly, $\mathcal{K}$ has amalgamation and is fully $\text{LS}(\mathcal{K})$-tame and short. By stability and [She70, Section 5], the class of H_2-saturated models of $\mathcal{K}$ has primes. The proof of Theorem 3.8 first argues without using primes that $\mathcal{K}$ is categorical in H_2. Hence the class of H_2-saturated models of $\mathcal{K}$ is just the class $\mathcal{K}_{\geq H_2}$, so it has primes. We apply Theorem 3.8 to obtain the eventual categoricity conjecture for homogeneous model theory. Actually Theorem 3.8 is not needed for that result: [Vasb, Theorem 5.18] suffices. However we can also improve on the Hanf number H_2 and obtain Theorem 2 from the abstract:

Theorem 4.22. Let D be a homogeneous diagram in a first-order theory T. If D is categorical in some $\lambda > |T|$, then D is categorical in all $\lambda' \geq \min(\lambda, h(|T|))$.

[4]We discuss homogeneous model theory and more generally finitary AECs later.

[5]We thank Rami Grossberg for asking us if the methods of [Vasb] could be adapted to this context.

When T is countable, a stronger result has been established by Lessmann [Les00]: categoricity in some uncountable cardinal implies categoricity in *all* uncountable cardinals. When T is uncountable, the eventual categoricity conjecture for homogeneous model theory is implicit in [She70, Section 7] and was also given a proof by Hyttinen [Hyt98]. More precisely, Hyttinen prove that categoricity in some $\lambda > |T|$ with $\lambda \neq \aleph_\omega(|T|)$ implies categoricity in all $\lambda' \geq \min(\lambda, h(|T|))$. Our proof of Theorem 4.22 is new and also covers the case $\lambda = \aleph_\omega(|T|)$. We do not know whether a similar result also holds in the framework of finitary AECs (there the categoricity conjecture has been solved for tame and *simple*[6] finitary AECs with countable Löwenheim-Skolem number [HK06][7]).

A continuation of the present paper is in [Vasa] (circulated after the initial submission of this paper), where orthogonality calculus is developed inside good frames that do not necessarily have primes. We establish there that the analog of Theorem 4.22 (i.e. the threshold is H_1) holds in any $\text{LS}(\mathcal{K})$-tame AEC with amalgamation and primes.

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2. Orthogonality with primes

In the proof of Fact 1.7, shortness was used only once: to build an independence notion sufficiently nice to allow the development of orthogonality calculus. More precisely, all we used was a technical statement on good frames being preserved when doing a certain change of AEC [She09, III.12.39]. We improve this statement to Theorem 2.18: while Shelah assumes that the good frame is successful good$^+$, here we do not make such assumption. Along the way, we develop orthogonality calculus in tame AECs with primes. Familiarity with Section 5 of [Vasb] would be helpful to understand the motivations of this section but all the relevant definitions will be given.

[6] In this context, stable does not imply simple.

[7] The argument is similar to the proof of Morley's categoricity theorem.

Recall [She09, Definition II.2.1][8] that a good λ-frame is a triple $\mathfrak{s} = (\mathcal{K}_\lambda, \mathop{\smile\!\!\!\!\!\mid}, \mathrm{gS}^{\mathrm{bs}})$ where:

(1) $\mathcal{K}$ is a nonempty AEC which has λ-amalgamation, λ-joint embedding, no maximal models, and is stable in λ.
(2) For each $M \in \mathcal{K}_\lambda$, $\mathrm{gS}^{\mathrm{bs}}(M)$ (called the set of *basic types* over M) is a set of nonalgebraic Galois types over M satisfying (among others) the *density property*: if $M < N$ are in $\mathcal{K}_\lambda$, there exists $a \in |N| \backslash |M|$ such that $\mathrm{gtp}(a/M; N) \in \mathrm{gS}^{\mathrm{bs}}(M)$.
(3) $\mathop{\smile\!\!\!\!\!\mid}$ is an (abstract) independence relation on types of elements over models in $\mathcal{K}_\lambda$ satisfying the basic properties of first-order forking in a superstable theory: invariance, monotonicity, extension, uniqueness, transitivity, local character, and symmetry (see [She09, Definition II.2.1]).

As in [She09, Definition II.6.35], we say that a good λ-frame $\mathfrak{s}$ is *type-full* if for each $M \in \mathcal{K}_\lambda$, $\mathrm{gS}^{\mathrm{bs}}(M)$ consists of *all* the nonalgebraic types over M. For simplicity, we focus on type-full good frames. Given a type-full good λ-frame $\mathfrak{s} = (\mathcal{K}_\lambda, \mathop{\smile\!\!\!\!\!\mid}, \mathrm{gS}^{\mathrm{bs}})$ and $M_0 \leq M$ both in $\mathcal{K}_\lambda$, we say that a nonalgebraic type $p \in \mathrm{gS}(M)$ *does not $\mathfrak{s}$-fork over* M_0 if it does not fork over M_0 according to the abstract independence relation $\mathop{\smile\!\!\!\!\!\mid}$ of $\mathfrak{s}$. When $\mathfrak{s}$ is clear from context, we just say that *p does not fork over M_0*. We say that a good λ-frame $\mathfrak{s}$ is *on* $\mathcal{K}_\lambda$ if its underlying class is $\mathcal{K}_\lambda$. We say that $\mathfrak{s}$ is *categorical* if $\mathcal{K}$ is categorical in λ and we say that it *has primes* if $\mathcal{K}_\lambda$ has primes (where we localize Definition 1.5 in the natural way).

In [She09, Section II.6], Shelah develops a theory of orthogonality for good frames. His assumptions include that the good frame is *weakly successful* [She09, Definition III.1.1], so in particular it expands to an independence relation NF for models in $\mathcal{K}_\lambda$. While weak successfulness follows from full tameness and shortness [Vas16a, Theorem 11.13], it is not clear if it follows from tameness only, so we do not adopt this assumption, instead we will assume that the good frame has primes.

Hypothesis 2.1. $\mathfrak{s} = (\mathcal{K}_\lambda, \mathop{\smile\!\!\!\!\!\mid}, \mathrm{gS}^{\mathrm{bs}})$ is a categorical type-full good λ-frame which has primes. We work inside $\mathfrak{s}$.

Hypothesis 2.1 is reasonable: By Fact 3.2, categorical good frames exist assuming categoricity, amalgamation, and tameness. As for assuming the existence of primes, this is an hypothesis of our main theorem

[8]The definition here is simpler and more general than the original: we will *not* use the axiom (B) requiring the existence of a superlimit model of size μ. Several papers (e.g. [JS13]) define good frames without this assumption.

(Theorem 3.8) and we have tried to justify it in the introduction. Se also Fact 4.6, which shows how to obtain the existence of primes in the setup of homogeneous model theory.

The definition of orthogonality we adopt is different from [She09, Definition III.6.2], though it is equivalent in Shelah's context (see [She09, Claim III.6.3]).

Definition 2.2. Let $M \in \mathcal{K}_\lambda$ and let $p, q \in \operatorname{gS}(M)$ be nonalgebraic. We say that p is *weakly orthogonal to* q and write $p \underset{\text{wk}}{\perp} q$ if for all prime triples (b, M, N) representing q (i.e. $q = \operatorname{gtp}(b/M; N)$, see Definition 1.5.(1)), we have that p has a unique extension to $\operatorname{gS}(N)$.

We say that p *is orthogonal to* q (written $p \perp q$) if for every $N \in \mathcal{K}_\lambda$ with $N \geq M$, $p' \underset{\text{wk}}{\perp} q'$, where p', q' are the nonforking extensions to N of p and q respectively.

For $p_\ell \in \operatorname{gS}(M_\ell)$ nonalgebraic, $\ell = 1, 2$, $p_1 \perp p_2$ if and only if there exists $N \geq M_\ell$, $\ell = 1, 2$ such that the nonforking extensions to N p_1' and p_2' of p_1 and p_2 respectively are orthogonal.

Remark 2.3. Formally, the definition of orthogonality depends on the frame but $\mathfrak{s}$ will always be fixed.

As noted before, in Shelah's framework for the development of orthogonality calculus, the good frame expands to a independence relation NF for *models* in $\mathcal{K}_\lambda$. Shelah uses many times the symmetry property of NF[9] : $\operatorname{NF}(M_0, M_1, M_2, M_3)$ if and only if $\operatorname{NF}(M_0, M_2, M_1, M_3)$. For example it is used to prove that orthogonality is symmetric: $p \perp q$ if and only if $q \perp p$. Here we do not know that NF exists (the good frame only gives us an independence relation for types of elements, not types of models) so cannot prove that orthogonality is symmetric. It turns out that this does not cause any problems for the result we are interested in. The next basic lemma says that we can replace the "for all" in Definition 2.2 by "there exists". This corresponds to [She09, Claim III.6.3], but the proof is different.

Lemma 2.4. Let $M \in \mathcal{K}_\lambda$ and $p, q \in \operatorname{gS}(M)$ be nonalgebraic. Then $p \underset{\text{wk}}{\perp} q$ if and only if there exists a prime triple (b, M, N) representing q such that p has a unique extension to $\operatorname{gS}(N)$.

Proof. The left to right direction is straightforward. Now assume (b, M, N) is a prime triple representing q such that p has a unique extension to

[9]For $M_0 \leq M_\ell \leq M_3$, $\ell = 1, 2$, $\operatorname{NF}(M_0, M_1, M_2, M_3)$ stands for "M_1 is independent of M_2 over M_0 in M_3".

$\mathrm{gS}(N)$. Let (b_2, M, N_2) be a prime triple representing q. We want to see that p has a unique extension to $\mathrm{gS}(N_2)$. Let $p_2 \in \mathrm{gS}(N_2)$ be an extension of p. By primeness of (b, M, N), there exists $f : N_2 \xrightarrow[M]{} N$ such that $f(b_2) = b$.

We have that $f(p_2)$ is an element of $\mathrm{gS}(N_2)$ and $N_2 \leq N$, so using amalgamation pick $p_2' \in \mathrm{gS}(N)$ extending $f(p_2)$. Now as f fixes M, $f(p_2)$ extends p, so p_2' extends p. Since by assumption p has a unique extension to $\mathrm{gS}(N)$, p_2' must be this unique extension, and in particular p_2' does not fork over M. By monotonicity, $f(p_2)$ does not fork over M. By invariance, p_2 does not fork over M. This shows that p_2 must be the unique extension of p to $\mathrm{gS}(N_2)$, as desired. □

We now show that weak orthogonality is the same as orthogonality. Recall that we are assuming categoricity in λ, so *all the models of size λ are limit (even superlimit)*. Thus we can use the following property, which Shelah proves for superlimit models $M, N \in \mathcal{K}_\lambda$:

Fact 2.5 (The conjugation property, Claim III.1.21 in [She09])**.** Let $M \leq N$ be in $\mathcal{K}_\lambda$, $\alpha < \lambda$, and let $(p_i)_{i<\alpha}$ be types in $\mathrm{gS}(N)$ that do not fork over M. Then there exists $f : N \cong M$ such that $f(p_i) = p_i \upharpoonright M$ for all $i < \alpha$.

Lemma 2.6 (Claim III.6.8.(4) in [She09])**.** For $M \in \mathcal{K}_\lambda$, $p, q \in \mathrm{gS}(M)$ nonalgebraic, $p \underset{\mathrm{wk}}{\perp} q$ if and only if $p \perp q$.

Proof. Clearly if $p \perp q$ then $p \underset{\mathrm{wk}}{\perp} q$. Conversely assume $p \underset{\mathrm{wk}}{\perp} q$ and let $N \geq M$. Let p', q' be the nonforking extensions to N of p, q respectively. We want to show that $p' \underset{\mathrm{wk}}{\perp} q'$. By the conjugation property, there exists $f : N \cong M$ such that $f(p') = p$ and $f(q) = q'$. Since weak orthogonality is invariant under isomorphism, $p' \underset{\mathrm{wk}}{\perp} q'$. □

We use orthogonality to study the following class of models, see [Vasb, Section 5] for motivation:

Definition 2.7. For $\mathcal{K}$ an AEC and $M \in \mathcal{K}$, let $\mathcal{K}_M$ be the AEC defined by adding constant symbols for the elements of M and requiring that M embeds inside every model of $\mathcal{K}_M$. That is, $L(\mathcal{K}_M) = L(\mathcal{K}) \cup \{c_a \mid a \in |M|\}$, where the c_a's are new constant symbols, and

$$\mathcal{K}_M := \{(N, c_a^N)_{a \in |M|} \mid N \in \mathcal{K} \text{ and } a \mapsto c_a^N \text{ is a } \mathcal{K}\text{-embedding from } M \text{ into } N\}$$

We order $\mathcal{K}_M$ by $(N_1, c_a^{N_1})_{a \in |M|} \leq (N_2, c_a^{N_2})$ if and only if $N_1 \leq N_2$ and $c_a^{N_1} = c_a^{N_2}$ for all $a \in |M|$.

Definition 2.8 (III.12.39 in [She09])**.** Let $M \in \mathcal{K}$ and let $p \in \mathrm{gS}(M)$. We define $\mathcal{K}_{\neg^* p}$ to be the class of $N \in \mathcal{K}_M$ such that $f(p)$ has a unique extension to $\mathrm{gS}(N \upharpoonright L(\mathcal{K}))$. Here $f : M \to N$ is given by $f(a) := c_a^N$. We order $\mathcal{K}_{\neg^* p}$ with the ordering induced from $\mathcal{K}_M$.

Remark 2.9. Let $p \in \mathrm{gS}(M)$ be nonalgebraic and suppose $M \leq N$ both are in $\mathcal{K}_\lambda$. If p has a unique extension to $\mathrm{gS}(N)$, then it must be the nonforking extension. Thus p is omitted in N. However even if p is omitted in N, p might have two nonalgebraic extensions to $\mathrm{gS}(N)$, so $\mathcal{K}_{\neg^* p}$ need not be the same as the class of models omitting p.

We will have to go back and forth between models in $\mathcal{K}_M$ and $\mathcal{K}$. We will use the following notation:

Notation 2.10. For $N \in \mathcal{K}$ with $M \leq N$, let (N, id) denote the canonical expansion of N to $\mathcal{K}_M$. In full, $(N, \mathrm{id}) = (N, c_a)_{a \in |M|}$, where $c_a = a$. We will often write e.g. "$(N, \mathrm{id}) \in \mathcal{K}_M$" (or $(N, \mathrm{id}) \in \mathcal{K}_{\neg^* p}$) as a shortcut to say that we pick an element of $\mathcal{K}_M$ whose reduct is N and whose constants are interpreted in the standard way. In particular, $M \leq N$.

Using local character and uniqueness, we get (see the proof of [Vasb, Proposition 5.9]):

Fact 2.11. Let $M \in \mathcal{K}_\lambda$, $p \in \mathrm{gS}(M)$. Then $(\mathcal{K}_{\neg^* p})_\lambda$ is an AEC in λ (that is, its models of size λ behave like an AEC, see [She09, Definition II.1.18])

Types realized by models with a canonical expansion in $\mathcal{K}_{\neg^* p}$ are orthogonal to p (in the sense of $\mathfrak{s}$):

Lemma 2.12. Fix $M \in \mathcal{K}_\lambda$ and let $p \in \mathrm{gS}(M)$ be nonalgebraic. Let $(N, \mathrm{id}) \in \mathcal{K}_{\neg^* p}$ (see Notation 2.10) be of size λ. For any $N_0 \in \mathcal{K}_\lambda$ with $M \leq N_0 \leq N$ and any $q \in \mathrm{gS}(N_0; N)$, we have that $p \perp q$.

Proof. Let p' be the nonforking extension of p to N_0. By Lemma 2.6, it is enough to show that p' is weakly orthogonal to q. Let (b, N_0, N') be a prime triple such that $\mathrm{gtp}(b/N_0; N') = q$ and $N' \leq N$ (exists since we are assuming that $\mathcal{K}_\lambda$ has primes). As $(N, \mathrm{id}) \in \mathcal{K}_{\neg^* p}$, p has a unique extension to N, hence a unique extension to N', which must be the nonforking extension so p' also has a unique extension to N'. By Lemma 2.4, this suffices to conclude that p' and q are weakly orthogonal. $\square$

Lemma 2.13. Suppose $M \leq N$ both are in $\mathcal{K}_\lambda$. Let $p \in \mathrm{gS}(M)$ be nonalgebraic and assume that $(N, \mathrm{id}) \in \mathcal{K}_{\neg^* p}$. Let $r \in \mathrm{gS}(N)$ be such that $p \perp r$. If (a, N, N') is a prime triple representing r, then $(N', \mathrm{id}) \in \mathcal{K}_{\neg^* p}$.

Proof. Write $p_N, p_{N'}$ for the nonforking extension of p to $\mathrm{gS}(N)$, $\mathrm{gS}(N')$ respectively and similarly for r. We have that $p_N \perp r$ so $p_{N'}$ is the unique extension of p_N to N'. Now if p' is an extension of p to $\mathrm{gS}(N')$, then $p' \upharpoonright N = p_N$ as $(N, \mathrm{id}) \in \mathcal{K}_{\neg^* p}$, so $p' = p_{N'}$ by the previous sentence. This shows that $(N', \mathrm{id}) \in \mathcal{K}_{\neg^* p}$, as desired. □

We want to study $\mathcal{K}_{\neg^* p}$ when $\mathcal{K}$ is not weakly uni-dimensional, where:

Definition 2.14 (Definition III.2.2.6 in [She09])**.** $\mathcal{K}_\lambda$ is *weakly uni-dimensional* if for every $M < M_\ell$, $\ell = 1, 2$ all in $\mathcal{K}_\lambda$, there is $c \in |M_2| \backslash |M|$ such that $\mathrm{gtp}(c/M; M_2)$ has more than one extension to $\mathrm{gS}(M_1)$.

The next lemma justifies the "uni-dimensional" terminology: if the class is *not* uni-dimensional, then there are two orthogonal types.

Lemma 2.15. If $\mathcal{K}_\lambda$ is not weakly uni-dimensional, there are $M \in \mathcal{K}_\lambda$ and types $p, q \in \mathrm{gS}(M)$ such that $p \perp q$.

Proof. Assume $\mathcal{K}_\lambda$ is not weakly uni-dimensional. This means that there exists $M < M_\ell$, $\ell = 1, 2$, all in $\mathcal{K}_\lambda$ such that for any $c \in |M_2| \backslash |M|$, $\mathrm{gtp}(c/M; M_2)$ has a unique extension to $\mathrm{gS}(M_1)$. Pick any $c \in |M_2| \backslash |M|$ and let $p := \mathrm{gtp}(c/M; M_2)$. Then $(M_1, \mathrm{id}) \in \mathcal{K}_{\neg^* p}$. So pick any $d \in |M_1| \backslash |M|$ and let $q := \mathrm{gtp}(d/M; M_1)$. By Lemma 2.12, $p \perp q$, as desired. □

We will have to go back and forth between Galois types in $\mathcal{K}_{\neg^* p}$ and $\mathcal{K}$. We write for example $\mathrm{gtp}_{\mathcal{K}}(a/N; N')$, $\mathrm{gS}_{\mathcal{K}}(N)$ for Galois types in $\mathcal{K}$, and $\mathrm{gtp}_{\mathcal{K}_{\neg^* p}}(a/(N, \mathrm{id}); (N', \mathrm{id}))$, $\mathrm{gS}_{\mathcal{K}_{\neg^* p}}(N)$ for Galois types in $\mathcal{K}_{\neg^* p}$. The following lemma is crucial:

Lemma 2.16. Let $M \leq N$ all be in $\mathcal{K}_\lambda$. Let $p \in \mathrm{gS}(M)$. If $(N, \mathrm{id}) \in \mathcal{K}_{\neg^* p}$, then the natural map $\mathrm{gtp}_{\mathcal{K}_{\neg^* p}}(a/(N, \mathrm{id}); (N', \mathrm{id})) \mapsto \mathrm{gtp}_{\mathcal{K}}(a/N; N')$ is a well-defined injection from $\mathrm{gS}_{\mathcal{K}_{\neg^* p}}(N)$ to $\mathrm{gS}_{\mathcal{K}}((N, \mathrm{id}))$.

Proof. It is easy to see that the map is well-defined, since the witnesses to the equality of two types in $\mathcal{K}_{\neg^* p}$ translate to witnesses to the equality of the two corresponding types in $\mathcal{K}$. Let us see that the map is injective. Assume that $\mathrm{gtp}_{\mathcal{K}}(a_1/N; N') = \mathrm{gtp}_{\mathcal{K}}(a_2/N; N')$. We have to see that $\mathrm{gtp}_{\mathcal{K}_{\neg^* p}}(a_1/(N, \mathrm{id}); (N', \mathrm{id})) = \mathrm{gtp}_{\mathcal{K}_{\neg^* p}}(a_2/(N, \mathrm{id}); (N', \mathrm{id}))$. Since $\mathcal{K}_\lambda$ has primes (Hypothesis 2.1), we can find $N_0' \in \mathcal{K}_\lambda$ and $f : N_0' \xrightarrow[N]{} N'$ such that $N \leq N_0' \leq N'$, $a_1 \in |N_0'|$, and $f(a_1) = a_2$. The map f also witnesses the equality of types inside $\mathcal{K}_{\neg^* p}$. $\square$

Definition 2.17. For $M \leq N$, p as in Lemma 2.16, for $r \in \mathrm{gS}_{\mathcal{K}_{\neg^* p}}((N, \mathrm{id}))$, we write $r_{\mathcal{K}}$ for the corresponding type in $\mathrm{gS}_{\mathcal{K}}((N, \mathrm{id}))$.

We have arrived to the main theorem of this section. This generalizes [She09, Claim III.12.39] which assumes in addition that $\mathfrak{s}$ is successful and good$^+$. We repeat Hypothesis 2.1 here.

Theorem 2.18. Let $\mathfrak{s} = (\mathcal{K}_\lambda, \mathop{\smile\hskip-0.9em\raise0.9ex\hbox{$\mid$}}, \mathrm{gS}^{\mathrm{bs}})$ be a categorical good λ-frame which has primes. If $\mathcal{K}_\lambda$ is not weakly uni-dimensional, then there exists $M \in \mathcal{K}_\lambda$ and $p \in \mathrm{gS}(M)$ such that $\mathfrak{s} \upharpoonright \mathcal{K}_{\neg^* p}$ (the expansion of $\mathfrak{s}$ to $\mathcal{K}_M$ restricted to the models in $\mathcal{K}_{\neg^* p}$) is a type-full good λ-frame with primes.

Proof. Assume $\mathcal{K}_\lambda$ is not weakly uni-dimensional. By Lemma 2.15, there exists $M \in \mathcal{K}_\lambda$ and types $p, q \in \mathrm{gS}(M)$ such that $p \perp q$.

Let $\mathfrak{s}_{\neg^* p} := \mathfrak{s} \upharpoonright \mathcal{K}_{\neg^* p}$. We check that it is a type-full good λ-frame with primes. For $N \geq M$, we write p_N for the nonforking extension of p to $\mathrm{gS}(N)$, and similarly for q_N. We will use Lemma 2.16 freely.

- $\mathcal{K}_{\neg^* p}$ is not empty, since $(M, \mathrm{id}) \in \mathcal{K}_{\neg^* p}$.
- $(\mathcal{K}_{\neg^* p})_\lambda$ is an AEC in λ by Fact 2.11.
- Nonforking has many of the usual properties: monotonicity, invariance, disjointness, local character, continuity, and transitivity all trivially follow from the definition of $\mathcal{K}_{\neg^* p}$.
- Nonforking has the uniqueness property: Let $\widehat{N} \in \mathcal{K}_{\neg^* p}$ have size λ. Without loss of generality, $\widehat{N} = (N, \mathrm{id})$ (so in particular $M \leq N$). Let $(N', \mathrm{id}) \in (\mathcal{K}_{\neg^* p})_\lambda$ be such that $N \leq N'$ and let $r_1, r_2 \in \mathrm{gS}_{\mathcal{K}_{\neg^* p}}((N', \mathrm{id}))$ be nonforking over N and such that $r_1 \upharpoonright (N, \mathrm{id}) = r_2 \upharpoonright (N, \mathrm{id})$. This implies that $(r_1)_{\mathcal{K}} \upharpoonright N = (r_2)_{\mathcal{K}} \upharpoonright N$ (recall Notation 2.17). Say $r_\ell = \mathrm{gtp}(a_\ell/(N', \mathrm{id}); (N_\ell, \mathrm{id}))$. Now by uniqueness of forking in $\mathfrak{s}$, $(r_1)_{\mathcal{K}} = (r_2)_{\mathcal{K}}$, and so by Lemma 2.16, $r_1 = r_2$.
- $(\mathcal{K}_{\neg^* p})_\lambda$ has primes: if (a, N, N') is a prime triple in $\mathcal{K}$ and $(N', \mathrm{id}) \in \mathcal{K}_{\neg^* p}$, then it is straightforward to check that $(a, (N, \mathrm{id}), (N', \mathrm{id}))$ is a prime triple in $\mathcal{K}_{\neg^* p}$.
- Nonforking has the extension property. Let $\widehat{N} \in \mathcal{K}_{\neg^* p}$ have size λ. Without loss of generality, $\widehat{N} = (N, \mathrm{id})$, so $M \leq N$. Let

$r \in \mathrm{gS}_{\mathcal{K}_{\neg^* p}}((N, \mathrm{id}))$ be nonalgebraic. By Lemma 2.12, $p \perp r_{\mathcal{K}}$. Let $N' \geq N$ be such that $(N', \mathrm{id}) \in (\mathcal{K}_{\neg^* p})_\lambda$. Let $r' \in \mathrm{gS}_{\mathcal{K}}(N')$ be the nonforking extension (in $\mathcal{K}$) of $r_{\mathcal{K}}$ to N'. Let (a, N', N'') be a prime triple such that $\mathrm{gtp}(a/N'; N'') = r'$. Because $p \perp r_{\mathcal{K}}$, the definition of orthogonality implies that $p \perp r'$. By Lemma 2.13, $(N'', \mathrm{id}) \in \mathcal{K}_{\neg^* p}$. Thus r' can be identified with a Galois type in $\mathcal{K}_{\neg^* p}$, as desired.

- $\mathcal{K}_{\neg^* p}$ has λ-amalgamation: because $(\mathcal{K}_{\neg^* p})_\lambda$ has the type extension property (for any Galois type $q \in \mathrm{gS}(N)$ and any $N' \geq N$, q extends to $\mathrm{gS}(N')$) and has primes, one can apply [Vasb, Theorem 4.14].
- $\mathcal{K}_{\neg^* p}$ has λ-joint embedding: Let $\widehat{N}$, $\widehat{N}' \in (\mathcal{K}_{\neg^* p})_\lambda$. Taking isomorphic images, we may assume that $\widehat{N} = (N, \mathrm{id})$, $\widehat{N}' = (N', \mathrm{id})$. In particular $M \leq N$ and $M \leq N'$. By λ-amalgamation over (M, id) (in $\mathcal{K}_{\neg^* p}$), $\widehat{N}$ and $\widehat{N}'$ embed inside a common element of $\mathcal{K}_{\neg^* p}$, as desired.
- $\mathcal{K}_{\neg^* p}$ is stable in λ: because $\mathcal{K}_{\neg^* p}$ has "fewer" Galois types than $\mathcal{K}$, and $\mathcal{K}$ is stable in λ.
- $(\mathcal{K}_{\neg^* p})_\lambda$ has no maximal models: This is where we use the negation of weakly uni-dimensional. Let $(N, \mathrm{id}) \in \mathcal{K}_{\neg^* p}$ be of size λ. Recall from above that there is a nonalgebraic type $q \in \mathrm{gS}(M)$ such that $p \perp q$. Let q_N be the nonforking extension of q to N and let (a, N, N') be a prime triple such that $q = \mathrm{gtp}(a/N; N')$. By Lemma 2.13, $(N', \mathrm{id}) \in \mathcal{K}_{\neg^* p}$. Moreover as $a \in |N'| \backslash |N|$, $N < N'$, as needed.
- $\mathfrak{s}_{\neg^* p}$ is type-full: because $\mathfrak{s}$ is.
- $\mathfrak{s}_{\neg^* p}$ has symmetry: Assume $a \overset{(N,\mathrm{id})}{\underset{(N_0,\mathrm{id})}{\perp}} (N_1, \mathrm{id})$, for $(N_0, \mathrm{id}), (N_1, \mathrm{id}), (N, \mathrm{id}) \in \mathcal{K}_{\neg^* p}$, $M \leq N_0 \leq N_1 \leq N$, and $a \in |N|$. Let $b \in |N_1|$. Without loss of generality, $a \notin |N_1|$ (if $a \in |N_1|$, then $a \in |N_0|$ by disjointness and as $b \overset{N}{\underset{N_0}{\perp}} N_0$, N_0 and N witness the symmetry). By symmetry in $\mathfrak{s}$, there exists $N_0', N' \in \mathcal{K}$ such that $N \leq N'$, $N_0 \leq N_0' \leq N'$, and $b \overset{N'}{\underset{N_0}{\perp}} N_0'$ (note that the first use of $\perp$ was in $\mathfrak{s}_{\neg^* p}$ and the second in $\mathfrak{s}$, but since the first is just the restriction of the first to models in $\mathcal{K}_{\neg^* p}$, we do not make the difference). Now let N_0'' be such that $N_0 \leq N_0'' \leq N_0'$ and (a, N_0, N_0'') is a prime triple. Since $r = \mathrm{gtp}(a/N_0; N_0'') = \mathrm{gtp}(a/N_0; N)$ and p is orthogonal to r (by Lemma 2.12), we have that $(N_0'', \mathrm{id}) \in \mathcal{K}_{\neg^* p}$.

By monotonicity, $b \underset{N_0}{\overset{N'}{\downfree}} N_0''$. Now let (b, N_0'', N'') be a prime triple with $N'' \leq N'$. By Lemma 2.13, $(N'', \mathrm{id}) \in \mathcal{K}_{\neg^* p}$ and by monotonicity, $b \underset{N_0}{\overset{N''}{\downfree}} N_0''$. Since all the models are in $\mathcal{K}_{\neg^* p}$, this shows that the nonforking happens in $\mathfrak{s}_{\neg^* p}$, as needed.

We have checked all the properties and therefore $\mathfrak{s}_{\neg^* p}$ is a type-full good λ-frame with primes. □

Assuming tameness and existence of primes above λ, we can conclude an equivalence between uni-dimensionality and categoricity. Once again, we repeat Hypothesis 2.1.

Theorem 2.19. Assume that $\mathcal{K}$ is an AEC categorical in λ which has a (type-full) good λ-frame. If $\mathcal{K}_{\geq \lambda}$ has primes and is λ-tame, then the following are equivalent:

(1) $\mathcal{K}$ is weakly uni-dimensional.
(2) $\mathcal{K}$ is categorical in all $\mu > \lambda$.
(3) $\mathcal{K}$ is categorical in some $\mu > \lambda$.

Proof. Exactly as in the proof of [Vasb, Theorem 5.16], except that we use Theorem 2.18. □

Remark 2.20. For the proof of Theorem 2.19 (and the other categoricity transfer theorems of this paper), the symmetry property is not needed.

3. Categoricity transfers in AECs with primes

In this section, we prove Theorem 1 from the abstract. We first recall that the existence of good frames follow from categoricity, amalgamation, and tameness. We use the following notation:

Notation 3.1. For $\mathcal{K}$ an AEC with amalgamation and $\lambda > \mathrm{LS}(\mathcal{K})$, we write $\mathcal{K}^{\lambda\text{-sat}}$ for the class of λ-saturated models in $\mathcal{K}_{\geq \lambda}$.

Fact 3.2. Let $\mathcal{K}$ be a $\mathrm{LS}(\mathcal{K})$-tame AEC with amalgamation and no maximal models. Let λ and μ be cardinals such that both λ and μ are strictly bigger than $\mathrm{LS}(\mathcal{K})$. If $\mathcal{K}$ is categorical in μ, then:

(1) $\mathcal{K}$ is stable in every cardinal.
(2) $\mathcal{K}^{\lambda\text{-sat}}$ is an AEC with $\mathrm{LS}(\mathcal{K}^{\lambda\text{-sat}}) = \lambda$.
(3) There exists a categorical type-full good λ-frame with underlying class $\mathcal{K}_\lambda^{\lambda\text{-sat}}$.

Proof. By the Shelah-Villaveces theorem [SV99, Theorem 2.2.1] (see [GV, Corollary 6.3] for a statement of the version with full amalgamation and the recent [BGVV] for a detailed proof), $\mathcal{K}$ is $\text{LS}(\mathcal{K})$-superstable (see for example [Vas16a, Definition 10.1]), in particular it is stable in $\text{LS}(\mathcal{K})$. Now we start to use $\text{LS}(\mathcal{K})$-tameness. By [VV, Corollary 6.10], $\mathcal{K}^{\lambda\text{-sat}}$ is an AEC with $\text{LS}(\mathcal{K}^{\lambda\text{-sat}}) = \lambda$. By [Vas16a, Theorem 10.8], there is a type-full good λ-frame with underlying class $\mathcal{K}_\lambda^{\lambda\text{-sat}}$ (and in particular stable in λ) By uniqueness of saturated models, $\mathcal{K}^{\lambda\text{-sat}}$ is categorical in λ. $\square$

We obtain a categoricity transfer for tame AECs with primes categorical in two cardinals. First we prove a more general lemma:

Lemma 3.3. Let $\mathcal{K}$ be a $\text{LS}(\mathcal{K})$-tame AEC with amalgamation and arbitrarily large models. Let λ and μ be cardinals such that $\text{LS}(\mathcal{K}) < \lambda < \mu$.

If $\mathcal{K}$ is categorical in μ and $\mathcal{K}^{\lambda\text{-sat}}$ has primes, then $\mathcal{K}^{\lambda\text{-sat}}$ is categorical in all $\mu' \geq \lambda$.

Proof. By partitioning $\mathcal{K}$ into disjoint AECs, each of which has joint embedding (see for example [Bal09, Lemma 16.14]) and working inside the unique piece that is categorical in μ, we can assume without loss of generality that $\mathcal{K}$ has joint embedding. Because $\mathcal{K}$ has arbitrarily large models, $\mathcal{K}$ also has no maximal models.

By Fact 3.2, there is a categorical type-full good λ-frame $\mathfrak{s}$ with underlying class $\mathcal{K}_\lambda^{\lambda\text{-sat}}$. Now apply Theorem 2.19 to $\mathfrak{s}$ and $\mathcal{K}^{\lambda\text{-sat}}$. $\square$

Theorem 3.4. Let $\mathcal{K}$ be a $\text{LS}(\mathcal{K})$-tame AEC with amalgamation and arbitrarily large models. Let λ and μ be cardinals such that $\text{LS}(\mathcal{K}) < \lambda < \mu$. Assume that $\mathcal{K}_{\geq\lambda}$ has primes.

If $\mathcal{K}$ is categorical in both λ and μ, then $\mathcal{K}$ is categorical in all $\mu' \geq \lambda$.

Proof. By categoricity, $\mathcal{K}^{\lambda\text{-sat}} = \mathcal{K}_{\geq\lambda}$. Now apply Lemma 3.3. $\square$

Remark 3.5. What if $\lambda = \text{LS}(\mathcal{K})$? Then it is open whether $\mathcal{K}$ has a good $\text{LS}(\mathcal{K})$-frame (see the discussion in [Vas16b, Section 3]). If it does, then we can use Theorem 2.19.

We present two transfers from categoricity in a single cardinal. The first uses the following downward transfer which follows from the proof of [Bal09, Theorem 14.9] (an exposition of [She99, II.1.6]).

Fact 3.6. Let $\mathcal{K}$ be an AEC with amalgamation and no maximal models. If $\mathcal{K}$ is categorical in a $\lambda > H_2$ (recall Notation 1.8) and the model of size λ is H_2^+-saturated, then $\mathcal{K}$ is categorical in H_2.

To get the optimal tameness bound, we will use[10]:

Fact 3.7 (Corollary 7.9 in [VV])**.** Let $\mathcal{K}$ be an AEC with amalgamation and no maximal models. Let $\mu \geq H_1$ and assume that $\mathcal{K}$ is categorical in a $\lambda > \mu$ so that the model of size λ is μ^+-saturated. Then there exists a categorical type-full good μ-frame with underlying class $\mathcal{K}_\mu^{\mu\text{-sat}}$.

Theorem 3.8. Let $\mathcal{K}$ be an AEC with amalgamation. Assume that $\mathcal{K}$ is H_2-tame and $\mathcal{K}_{\geq H_2}$ has primes. If $\mathcal{K}$ is categorical in some $\lambda > H_2$, then $\mathcal{K}$ is categorical in all $\lambda' \geq H_2$.

Proof. As in the proof of Lemma 3.3, we can assume without loss of generality that $\mathcal{K}$ has no maximal models. By Fact 3.2 (applied to the AEC $\mathcal{K}_{\geq H_2}$), $\mathcal{K}$ is in particular stable in λ, hence the model of size λ is saturated. By Fact 3.6, $\mathcal{K}$ is categorical in H_2. By Fact 3.7, there is a categorical type-full good H_2-frame $\mathfrak{s}$ with underlying class $\mathcal{K}_{H_2}^{H_2\text{-sat}}$. By categoricity in H_2, $\mathcal{K}^{H_2\text{-sat}} = \mathcal{K}_{\geq H_2}$. Now apply Theorem 2.19 to $\mathfrak{s}$. $\square$

We give a variation on Theorem 3.8 which gives a lower Hanf number but assumes that classes of saturated models have primes. We will use the following consequence of the omitting type theorem for AECs [She99, II.1.10] (or see [Bal09, Corollary 14.3]):

Fact 3.9. Let $\mathcal{K}$ be an AEC with amalgamation. Let $\lambda \geq \chi > \text{LS}(\mathcal{K})$ be cardinals. Assume that all the models of size λ are χ-saturated. Then all the models of size at least $\min(\lambda, \sup_{\chi_0 < \chi} h(\chi_0))$ are χ-saturated.

Theorem 3.10. Let $\mathcal{K}$ be a $\text{LS}(\mathcal{K})$-tame AEC with amalgamation and arbitrarily large models. Let $\lambda > \text{LS}(\mathcal{K})^+$ be such that $\mathcal{K}$ is categorical in λ and let $\chi \in (\text{LS}(\mathcal{K}), \lambda)$ be such that $\mathcal{K}^{\chi\text{-sat}}$ has primes. Then $\mathcal{K}$ is categorical in all $\lambda' \geq \min(\lambda, \sup_{\chi_0 < \chi} h(\chi_0))$.

Proof. As in the proof of Lemma 3.3, we may assume that $\mathcal{K}$ has no maximal models. By Lemma 3.3, $\mathcal{K}^{\chi\text{-sat}}$ is categorical in all $\lambda' \geq \chi$. By Fact 3.2, $\mathcal{K}$ is stable in λ, so the model of size λ is saturated,

[10]For a simpler proof of Theorem 3.8 from slightly stronger assumptions, replace "H_2-tame" by "χ-tame for some $\chi < H_2$. Then in the proof one can use Fact 3.2 together with Theorem 3.4, both applied to the class $\mathcal{K}_{\geq \chi}$.

hence χ-saturated. By Fact 3.9, all the models of size at least $\lambda_0' := \min(\lambda, \sup_{\chi_0 < \chi} h(\chi_0))$ are χ-saturated. In other words, $\mathcal{K}_{\geq \lambda_0'} = \mathcal{K}^{\chi\text{-sat}}_{\geq \lambda_0'}$. Since $\mathcal{K}^{\chi\text{-sat}}$ is categorical in all $\lambda' \geq \chi$, $\mathcal{K}$ is categorical in all $\lambda' \geq \lambda_0'$. $\square$

Remark 3.11. Theorem 3.8 and Theorem 3.10 have different strengths. It could be that we know our AEC $\mathcal{K}$ has primes but it is unclear that $\mathcal{K}^{\chi\text{-sat}}$ has primes for any χ. For example, $\mathcal{K}$ could be a universal class (or more generally an AEC admitting intersections). In this case we can use Theorem 3.8. On the other hand we may not know that $\mathcal{K}$ has primes but we could know how to build primes in $\mathcal{K}^{\chi\text{-sat}}$ (for example $\mathcal{K}$ could be an elementary class or more generally a class of homogeneous models, see the next section). There Theorem 3.10 applies.

4. Categoricity in homogeneous model theory

We use the results of the previous section to obtain Shelah's categoricity conjecture for homogeneous model theory, a nonelementary framework extending classical first-order model theory. It was introduced in [She70]. The idea is to look at a class of models of a first-order theory omitting a set of types and assume that this class has a very nice (sequentially homogeneous) monster model. We quote from the presentation in [GL02] but all the results on homogeneous model theory that we use initially appeared in either [She70] or [HS00].

The following definitions appear in [GL02]. They differ from (but are equivalent to) Shelah's original definitions from [She70].

Definition 4.1. Fix a first-order theory T.

(1) A set of T-types D is a *diagram in* T if it has the form $\{\text{tp}(\bar{a}/\emptyset; M) \mid \bar{a} \in {}^{<\omega}A\}$ for a model M of T.
(2) A model M of T is a *D-model* if $D(M) := \{\text{tp}(\bar{a}/\emptyset; M) \mid \bar{a} \in {}^{<\omega}|M|\} \subseteq D$.
(3) For D a diagram of T, we let $\mathcal{K}_D$ be the class of D-models of T, ordered with elementary substructure.
(4) For M a model of T, we write $S_D^{<\omega}(A; M)$ for the set of types of finite tuples over A which are realized in some D-model N with $N \preceq M$.

Definition 4.2. Let T be a first-order theory and D a diagram in T. A model M of T is *(D, λ)-homogeneous* if it is a D-model and for every $N \succeq M$, every $A \subseteq |M|$ with $|A| < \lambda$, every $p \in S_D^{<\omega}(A; N)$ is realized in M.

Definition 4.3. We say a diagram D in T is *homogeneous* if for every λ there exists a (D, λ)-homogeneous model of T.

It is straightforward to check the following (they will be used without mention):

Proposition 4.4. For D a homogeneous diagram in T:

(1) $\mathcal{K}_D$ is an AEC with $\text{LS}(\mathcal{K}_D) = |T|$.
(2) $\mathcal{K}$ has amalgamation, no maximal models, and is fully $\text{LS}(\mathcal{K})$-tame and short (in fact syntactic and Galois types coincide).
(3) For $\lambda > |T|$, a D-model M is (D, λ)-homogeneous if and only if $M \in \mathcal{K}_D^{\lambda\text{-sat}}$.

Note that in this framework it also makes sense to talk about the $|T|$-saturated models, so we let:

Definition 4.5. Let $\mathcal{K}_D^{|T|\text{-sat}}$ be the class of $(D, |T|)$-homogeneous models, ordered by elementary substructure.

To apply the results of the previous section, we must give conditions under which $\mathcal{K}_D^{\chi\text{-sat}}$ has primes. This is implicit in [She70, Section 5]:

Fact 4.6. Let D be a homogeneous diagram in T. If $\mathcal{K}_D$ is stable in $\chi \geq \text{LS}(\mathcal{K})$ then $\mathcal{K}_D^{\chi\text{-sat}}$ has primes.

Proof. By [She70, Theorem 5.11.(1)] (with μ, λ there standing for χ, χ here; in particular $2^\mu > \lambda$), D satisfies a property Shelah calls $(P, \chi, 1)$ (a form of density of isolated types, see [She70, Definition 5.4]). By the proof of [She70, Claim 5.2.(1)] and [She70, Theorem 5.3.(1)] there, this implies that the class $\mathcal{K}_D^{\chi\text{-sat}}$ has primes. $\square$

We immediately obtain:

Theorem 4.7. If a homogeneous diagram D in a first-order theory T is categorical in a $\lambda > |T|^+$, then it is categorical in all $\lambda' \geq \min(\lambda, h(|T|))$.

Proof. Note that $\mathcal{K}_D$ is stable in all cardinals by Fact 3.2. So we can combine Fact 4.6 and Theorem 3.10. $\square$

This proves Theorem 2 in the abstract modulo a small wrinkle: the case $\lambda = |T|^+$. One would like to use the categoricity transfer of Grossberg and VanDieren [GV06a] but they assume that $\mathcal{K}$ is categorical in a successor $\lambda > \text{LS}(\mathcal{K})^+$ since otherwise it is in general unclear whether there is a superlimit (see [She09, Definition N.2.4.(4)]) in $\mathrm{LS}(\mathcal{K})$ (one can get around this difficulty if $\mathrm{LS}(\mathcal{K}) = \aleph_0$, see [Les05]). However in the case of homogeneous model theory we can show that there *is* a superlimit, completing the proof. The key is that under stability, $(D, |T|)$-homogeneous models are closed under unions of chains. This is claimed without proof by Shelah in [She75, Theorem 1.15]. We give a proof here which imitates the first-order proof of Harnik [Har75]. Still it seems that a fair amount of forking calculus has to be developed first. All throughout, we assume:

Hypothesis 4.8. D is a homogeneous diagram in a first-order theory T. We work inside a $(D, \bar{\kappa})$-homogeneous model $\mathfrak{C}$ for $\bar{\kappa}$ a very big cardinal. In particular, all sets are assumed to be D-sets.

The following can be seen as a first approximation for forking in the homogeneous context. It was used by Shelah to prove the stability spectrum theorem in this framework (see Fact 4.11). We will not use the exact definition, only its consequences.

Definition 4.9 (Definition 4.1 in [She70])**.** A type $p \in S_D^{<\omega}(A)$ *strongly splits* over $B \subseteq A$ if there exists an indiscernible sequence $\langle \bar{a}_i : i < \omega \rangle$ over B and a formula $\phi(\bar{x}, \bar{y})$ such that $\phi(\bar{x}, \bar{a}_0) \in p$ and $\neg\phi(\bar{x}, \bar{a}_1) \in p$.

Definition 4.10. $\kappa(D)$ is the minimal cardinal κ such that for all A and all $p \in S_D^{<\omega}(A)$, there exists $B \subseteq A$ with $|B| < \kappa$ so that p does not strongly split over B.

The following is due to Shelah [She70, Theorem 4.4]. See also [GL02, Theorems 4.11, 4.14, 4.15]:

Fact 4.11. If D is stable in $\lambda_0 \geq |T|$ and $\lambda \geq \lambda_0$, then D is stable in λ if and only if $\lambda = \lambda^{<\kappa(D)}$.

We can define forking using strong splitting:

Definition 4.12 (Definition 3.1 in [HS00])**.** For $A \subseteq B$, $p \in S_D^{<\omega}(B)$ *does not fork over* A if there exists $A_0 \subseteq A$ such that:

(1) $|A_0| < \kappa(D)$.
(2) For every set C, there exists $q \in S_D^{<\omega}(B \cup C)$ such that q extends p and q does not strongly split over A_0.

Assuming that the base has a certain degree of saturation, forking behaves well:

Fact 4.13. Assume that D is stable in $\lambda \geq |T|$. Let M be (D, λ)-homogeneous and let $A \subseteq B \subseteq C$ be sets.

(1) (Monotonicity) For $p \in S_D^{<\omega}(C)$, if p does not fork over A, then $p \upharpoonright B$ does not fork over A and p does not fork over B.
(2) (Extension-existence) For any $p \in S_D^{<\omega}(M)$, there exists $q \in S_D^{<\omega}(M \cup B)$ that extends p and does not fork over M. Also, q is algebraic if and only if p is. Moreover if $p \in S_D^{<\omega}(M)$ does not strongly split over $A_0 \subseteq |M|$, then p does not fork over A_0.
(3) (Uniqueness) If $p, q \in S_D^{<\omega}(M \cup B)$ both do not fork over M and are such that $p \upharpoonright M = q \upharpoonright M$, then $p = q$.
(4) (Transitivity) For any $p \in S_D^{<\omega}(M \cup B)$, if p does not fork over M and $p \upharpoonright M$ does not fork over $A_0 \subseteq |M|$, then p does not fork over A_0.
(5) (Symmetry) If $\text{tp}(\bar{b}/M\bar{a})$ does not fork over M, then $\text{tp}(\bar{a}/M\bar{b})$ does not fork over M.
(6) (Local character) For any $p \in S_D^{<\omega}(M)$, there exists $A_0 \subseteq |M|$ such that $|A_0| < \kappa(D)$ and p does not fork over A_0. Moreover, for any $\langle M_i : i < \delta \rangle$ increasing chain of (D, λ)-homogeneous models, if $p \in S_D^{<\omega}(\bigcup_{i<\delta} M_i)$ and $\text{cf}(\delta) \geq \kappa(D)$, then there exists $i < \delta$ and $A_0 \subseteq |M_i|$ such that $|A_0| < \kappa(D)$ and p does not fork over A_0.

Proof. We use freely that (by [HS00, Lemma 1.9.(iv)]) a (D, λ)-homogeneous model is an a-saturated model in the sense of [HS00, Definition 1.8.(ii)].

(1) By [HS00, Lemma 3.2.(i)].
(2) By [HS00, Lemma 3.2.(iii), (v), (vi)] and the definitions of $\kappa(D)$ and forking.
(3) By [HS00, Lemma 3.4].
(4) By [HS00, Corollary 3.5.(iv)].
(5) By [HS00, Lemma 3.6].
(6) We prove the moreover part and the first part follows by taking $M_i := M$ for all $i < \delta$. Let $M_\delta := \bigcup_{i<\delta} M_i$. Without loss of generality, $\delta = \text{cf}(\delta) \geq \kappa(D)$. By definition of $\kappa(D)$, there exists $A_0 \subseteq |M_\delta|$ such that $|A_0| < \kappa(D)$ and p does not strongly split over A_0. By cofinality consideration, there exists $i < \delta$ such that $A_0 \subseteq |M_i|$. By the moreover part of extension-existence, for all $j \in [i, \delta)$, $p \upharpoonright M_j$ does not fork over A_0. By [HS00, Corollary 3.5.(i)], it follows that p does not fork over M_i, and therefore by transitivity over A_0.

$\square$

We will use the machinery of indiscernibles and averages. Note that by [GL02, Remark 3.4, Corollary 3.12], indiscernible sequences are indiscernible sets under stability. We will use this freely. The following directly follows from the definition of strong splitting:

Fact 4.14 (Theorem 5.3 in [GL02])**.** Assume that D is stable. For all infinite indiscernible sequences I over a set A and all elements b, there exists $J \subseteq I$ with $|J| < \kappa(D)$ such that $I \backslash J$ is indiscernible over $A \cup \{b\}$.

Definition 4.15. For I an indiscernible sequence of cardinality at least $\kappa(D)$, let $\mathrm{Av}(I/A)$ be the set of formulas $\phi(\bar{x}, \bar{a})$ with $\bar{a} \in {}^{<\omega}A$ such that for at least $\kappa(D)$-many elements $\bar{b}$ of I, $\models \phi[\bar{b}, \bar{a}]$.

Fact 4.16 (Theorem 5.5 in [GL02])**.** If D is stable and I is an indiscernible sequence of cardinality at least $\kappa(D)$, then $\mathrm{Av}(I/A) \in S_D^{<\omega}(A)$.

Fact 4.17. Assume that D is stable.

Let $A \subseteq B$ and let $p \in S_D^{<\omega}(B)$. If p does not fork over A, $|A| < \kappa(D)$, and p is nonalgebraic, then there exists an indiscernible set I over A with $|I| \geq \kappa(D)$ such that $\mathrm{Av}(I/M) = p$.

Proof. This is [HS00, Lemma 3.9]. We have to check that $p \upharpoonright A$ has unboundedly-many realizations, but this is easy using the extension-existence property of forking (Fact 4.13) and the assumption that p is nonalgebraic. $\square$

We can conclude:

Theorem 4.18. Let $\lambda \geq |T|$. Assume that D is stable in some $\mu \leq \lambda$. Let δ be a limit ordinal with $\mathrm{cf}(\delta) \geq \kappa(D)$ and let $\langle M_i : i < \delta \rangle$ be an increasing sequence of (D, λ)-homogeneous models. Then $\bigcup_{i<\delta} M_i$ is (D, λ)-homogeneous.

Proof. By cofinality consideration, we can assume without loss of generality that $\delta = \mathrm{cf}(\delta)$ and $\lambda > \delta$. Also without loss of generality, λ is regular. Let $M_\delta := \bigcup_{i<\delta} M_i$. Let $A \subseteq |M_\delta|$ have size less than λ and let $p \in S_D^{<\omega}(A)$. Let $q \in S_D^{<\omega}(M_\delta)$ be an extension of p and assume for sake of contradiction that q is not realized in M_δ. By the moreover part of local character (Fact 4.13), there exists $i < \delta$ and $B \subseteq |M_i|$ such that $|B| < \kappa(D)$ and q does not fork over B. By making A slightly bigger we can assume without loss of generality that $B \subseteq A$.

Since q is not realized in M_δ, q is nonalgebraic. By Fact 4.17, there exists an indiscernible set I over B with $\mathrm{Av}(I/M) = q$. Enlarging

I if necessary, $|I| = \lambda$. Since M_{i+1} is (D, λ)-homogeneous, we can assume without loss of generality that $I \subseteq |M_{i+1}|$. By Fact 4.14 used $|A|$-many times (recall $|A| < \lambda$), there exists $I_0 \subseteq I$ with $|I_0| = \lambda$ and I_0 indiscernible over A. Then $\text{Av}(I_0/M) = \text{Av}(I/M) = q$ so $p = \text{Av}(I_0/A)$. By definition of average, if $\phi(\bar{x}, \bar{a}) \in p$, there exists $\bar{b} \in I_0$ such that $\models \phi[\bar{b}, \bar{a}]$. By indiscernibility over A, this is true for any $\bar{b} \in I_0$, hence any element of I_0 realizes p. $\square$

Remark 4.19. When $\lambda > |T|$ and $\kappa(D) = \aleph_0$, Theorem 4.18 generalizes to superstable tame AECs with amalgamation (see [BVa] and the more recent [VV, Corollary 6.10]). We do not know whether there is a generalization of Theorem 4.18 to AECs when $\lambda = \text{LS}(\mathcal{K})$ (see also [VV, Question 6.12]).

In homogeneous model theory, superstability follows from categoricity:

Lemma 4.20. If a homogeneous diagram D in a first-order theory T is categorical in a $\lambda > |T|$, then $\kappa(D) = \aleph_0$.

Proof. By Fact 3.2 (applied to $\mathcal{K} := \mathcal{K}_D$, recall Proposition 4.4), D is stable in all cardinals and in particular in $\mu := \aleph_\omega(|T|)$. Since $\mu^{\aleph_0} > \mu$, Fact 4.11 gives $\kappa(D) = \aleph_0$. $\square$

Note that Lemma 4.20 was known when $\lambda \neq \aleph_\omega(|T|)$ (see [Hyt98, Theorem 3]). The case $\lambda = \aleph_\omega(|T|)$ is new (in fact, once Lemma 4.20 is proven for $\lambda = \aleph_\omega(|T|)$, Hyttinen's argument for transfering categoricity [Hyt98, Corollary 14.(ii)] goes through).

The referee asked if Lemma 4.20 had an easier proof using tools specific to homogeneous model theory. An easy proof of Lemma 4.20 when $\lambda \neq \aleph_\omega(|T|)$ runs as follows: By a standard Ehrenfeucht-Mostowski (EM) model argument of Morley (see for example [Bal09, Theorem 8.21]), D is stable in every $\mu \in [|T|, \lambda)$. If $\lambda > \aleph_\omega(|T|)$, then D is stable in $\mu := \aleph_\omega(|T|)$ and $\mu^{\aleph_0} > \mu$ so by the stability spectrum theorem (Fact 4.11), we must have that $\kappa(D) = \aleph_0$. If $\lambda < \aleph_\omega(|T|)$, λ is a successor and we can use other EM model tricks. Only the case $\lambda = \aleph_\omega(|T|)$ remains but to deal with it, we are not aware of any tools specific to the homogeneous setup. The proof above is in effect an application of a result of Shelah and Villaveces (see [SV99, Theorem 2.2.1] and the recent exposition [BGVV]) and an upward stability transfer of the author [Vas16b, Theorem 5.6].

We can conclude with a proof of Theorem 2 from the abstract. When $\lambda = |T|^+$, we could appeal to [GV06a] but prefer to prove a more general statement using primes:

Theorem 4.21. If a homogeneous diagram D in a first-order theory T is categorical in a $\lambda > |T|$, then the class $\mathcal{K}_D^{|T|\text{-sat}}$ of its $(D, |T|)$-homogeneous models is categorical in all $\lambda' \geq |T|$. In particular, if D is also categorical in $|T|$, then D is categorical in all $\lambda' \geq |T|$

Proof. Let $\mathcal{K} := \mathcal{K}_D$ be the class of D-models of T. By Proposition 4.4, $\mathcal{K}$ is a $\text{LS}(\mathcal{K})$-tame AEC (where $\text{LS}(\mathcal{K}) = |T|$) with amalgamation and no maximal models. Furthermore $\mathcal{K}$ is categorical in λ. By Lemma 4.20, $\kappa(D) = \aleph_0$. By Theorem 4.18, the union of any increasing chain of $(D, |T|)$-homogeneous models is $(D, |T|)$-homogeneous. Moreover, there is a unique $(D, |T|)$-homogeneous model of cardinality $|T|$ (see e.g. [GL02, Theorem 5.9]). So we get that:

(1) $\mathcal{K}_D^{|T|\text{-sat}}$ is an AEC with $\text{LS}(\mathcal{K}_D^{|T|\text{-sat}}) = \text{LS}(\mathcal{K})$.
(2) $\mathcal{K}_D^{|T|\text{-sat}}$ has amalgamation, no maximal models, and is $\text{LS}(\mathcal{K})$-tame.
(3) $\mathcal{K}_D^{|T|\text{-sat}}$ is categorical in $\text{LS}(\mathcal{K})$ and λ.

Thus the last sentence in the statement of the theorem follows from uniqueness of homogeneous models. Let us prove the first. By Fact 4.13, nonforking induces a type-full good $|T|$-frame on the class $(\mathcal{K}_D^{|T|\text{-sat}})_{|T|}$. By Fact 4.6, $\mathcal{K}_D^{|T|\text{-sat}}$ has primes. Now apply Theorem 2.19. $\square$

Theorem 4.22. If a homogeneous diagram D in a first-order theory T is categorical in a $\lambda > |T|$, then it is categorical in all $\lambda' \geq \min(\lambda, h(|T|))$.

Proof. By Theorem 4.21, $\mathcal{K}_D^{|T|\text{-sat}}$ is categorical in all $\lambda' \geq |T|$. In particular by categoricity in λ, $\left(\mathcal{K}_D^{|T|\text{-sat}}\right)_{\geq\lambda} = (\mathcal{K}_D)_{\geq\lambda}$, so $\mathcal{K}_D$ is categorical in all $\lambda' \geq \lambda$. To see that $\mathcal{K}_D$ is categorical in all $\lambda' \geq h(|T|)$, use Theorem 4.7 (or just directly Fact 3.9). $\square$

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