TOWARD A STABILITY THEORY OF TAME ABSTRACT ELEMENTARY CLASSES

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Abstract. We initiate a systematic investigation of the abstract elementary classes that have amalgamation, satisfy tameness (a locality property for orbital types), and are stable (in terms of the number of orbital types) in some cardinal. We define a cardinal parameter $\chi(\mathbf{K})$ which is the analog of $\kappa_r(T)$ from the first-order setup. In particular, a full characterization of the (high-enough) stability cardinals is possible assuming the singular cardinal hypothesis (SCH). Using tools of Boney and VanDieren, we show that limit models of length at least $\chi(\mathbf{K})$ are unique and deduce results on the saturation spectrum, including a full (eventual) characterization assuming SCH.

We also show (in ZFC) that if a class is stable on a tail of cardinals, then $\chi(\mathbf{K}) = \aleph_0$ (the converse is known). This indicates that there is a clear notion of superstability in this framework.

Contents

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1. Introduction

1.1. **Motivation and history.** Abstract elementary classes (AECs) are partially ordered classes $\mathbf{K} = (K, \leq_{\mathbf{K}})$ which satisfy several of the basic category-theoretic properties of classes of the form $(\text{Mod}(T), \preceq)$ for T a first-order theory. They were introduced by Saharon Shelah in the late seventies [She87] and encompass infinitary logics such as $\mathbb{L}_{\lambda^+,\omega}(Q)$ as well as several algebraic examples. One of Shelah's test questions is the eventual categoricity conjecture: an AEC categorical in *some* high-enough cardinal should be categorical in *all* high-enough cardinals.

Toward an approximation, work of Makkai and Shelah [MS90] studied classes of models of an $\mathbb{L}_{\kappa,\omega}$ theory categorical in a high-enough cardinal, when κ is a strongly compact cardinal. They proved [MS90, Proposition 1.13] that such a class has (eventual) amalgamation, joint embedding, and no maximal models. Thus one can work inside a monster model and look at the corresponding orbital types. Makkai and Shelah proved that the orbital types correspond to certain syntactic types, implying in particular that two orbital types are equal if all their restrictions of size less than κ are equal. They then went on to develop some theory of superstability and concluded that categoricity in some high-enough successor implies categoricity in all high-enough cardinals.

A common theme of recent work on AECs is to try to replace large cardinal hypotheses with their model-theoretic consequences. For example, regardless of whether there are large cardinals, many classes of interests have a monster model and satisfy a locality property for their orbital types (see the introduction to [GV06] or the list of examples in the recent survey [BVc]). Toward that end, Grossberg and VanDieren made the locality property isolated by Makkai and Shelah (and later also used by Shelah in another work [She99]) into a definition: Call an AEC *μ-tame* if its orbital types are determined by their μ-sized restrictions. Will Boney [Bon14b] has generalized the first steps in the work of Makkai and Shelah to AECs, showing that tameness follows from a large cardinal axiom (amalgamation also follows if one assumes categoricity). Earlier, Shelah had shown that Makkai and Shelah's downward part of the transfer holds assuming amalgamation (but not tameness) [She99] and Grossberg and VanDieren used Shelah's proof (their actual initial motivation for isolating tameness) to show that the upward part of the transfer holds in tame AECs with amalgamation.

Recently, the superstability theory of tame AECs with a monster model has seen a lot of development [Bon14a, Vas16b, BVb, VV, GV] and one can say that most of Makkai and Shelah's work has been generalized to the tame context (see also [Bal09, Problem D.9.(3)]). New concepts not featured in the Makkai and Shelah paper, such as good frames and limit models, have also seen extensive studies (e.g. in the previously-cited papers and in Shelah's book [She09a]). The theory of superstability for AECs has had several applications, including a full proof of Shelah's eventual categoricity conjecture in universal classes [Vasb].

While we showed with Grossberg [GV] that several possible definitions of superstability are all equivalent in the tame case, it was still open [GV, Question 1.7] whether stability on a tail of cardinals implied these possible definitions (e.g. locality of forking).

The present paper answers positively by developing the theory of *strictly stable* tame AECs with a monster model. We emphasize that this is *not* the first paper on strictly stable AECs. In their paper introducing tameness [GV06], Grossberg and VanDieren proved several fundamental results (see also [BKV06]). Shelah [She99, Sections 4,5] has made some important contributions without even assuming tameness; see also his work on universal classes [She09b, Chapter V.E]. Several recent works [BG, Vas16c, BVb, BVa] establish results on independence, the first stability cardinal, chains of saturated models, and limit models. The present paper aims to put these papers together and improve some of their results using either some of the superstability machinery mentioned above or (in the case of Shelah's tameness-free results) assuming tameness.

1.2. **Outline of the main results.** Fix an $\text{LS}(\mathbf{K})$-tame AEC $\mathbf{K}$ with a monster model. Assume that $\mathbf{K}$ is stable (defined by counting Galois types) in some cardinal. Let $\chi(\mathbf{K})$ be the minimal cardinal χ such that for all high-enough stability cardinals μ, any type over the union of a (μ, χ)-limit chain $\langle M_i : i < \chi \rangle$ does not μ-split over some M_i. Using results of Shelah [She99], we show (Theorem 8.6) that $\chi(\mathbf{K})$ is maximal such that for any stability cardinal μ, $\mu = \mu^{<\chi(\mathbf{K})}$ (thus $\chi(\mathbf{K})$ is the analog of the first-order $\kappa_r(T)$). This gives one direction of a stability spectrum theorem and proves that stability on a tail implies $\chi(\mathbf{K}) = \aleph_0$ (Corollary 8.16).

Using the singular cardinal hypothesis (SCH[1]) we establish the converse: for any high-enough μ, $\mu^{<\chi(\mathbf{K})} = \mu$ implies that $\mathbf{K}$ is stable in μ (this is implicit in the author's earlier work [Vas16b]). We then prove that $\chi(\mathbf{K})$ connects the stability spectrum with the behavior of limit and saturated models: any two limit models of length at least $\chi(\mathbf{K})$ (of the same big-enough size) are isomorphic[2] (Theorem 9.4) and unions of chains of λ-saturated models of length $\chi(\mathbf{K})$ are λ-saturated (with some mild restrictions on λ, weaker than those used in [BVb], see Theorem 9.9). In fact, each of these properties characterizes $\chi(\mathbf{K})$ (Theorem 11.3). We also prove a saturation spectrum theorem, showing assuming SCH that (when $\lambda < \lambda^{<\lambda}$ is high-enough), $\mathbf{K}$ is stable in λ if and only if $\mathbf{K}$ has a saturated model in λ (see Corollary 10.7). In ZFC, we show that having saturated models on a tail of cardinals implies superstability (Corollary 10.8), answering another question from [GV] (see after Remark 1.6 there).

The reader may ask how SCH is used in the above results. Roughly, it makes cardinal arithmetic well-behaved enough that for any big-enough cardinal λ, $\mathbf{K}$ will either be stable in λ or in unboundedly-many cardinals below λ. This is connected to defining the locality cardinal $\chi(\mathbf{K})$ using chains rather than as the least cardinal κ for which every type does not fork over a set of size less than κ (indeed, in AECs it is not even clear what exact form such a definition should take). Still several results of this paper hold (in ZFC) for "most" cardinals, and the role of SCH is only to deduce that "most" means "all".

By a result of Solovay [Sol74], SCH holds above a strongly compact. Thus our results which assume SCH hold also above a strongly compact. This shows that a stability theory (not just a superstability theory) can be developed in the context of the Makkai and Shelah paper, partially answering [She00, Problem 6.15].

1.3. **Future work.** We believe that an important test question is whether the aforementioned SCH hypothesis can be removed:

Question 1.1. Let $\mathbf{K}$ be an $\mathrm{LS}(\mathbf{K})$-tame AEC with a monster model. If $\mathbf{K}$ is stable in some cardinal, does there exists a cardinal λ_0 such that $\mathbf{K}$ is stable in any $\lambda = \lambda_0 + \lambda^{<\chi(\mathbf{K})}$?

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[1]That is, for every infinite singular cardinal λ, $\lambda^{\mathrm{cf}(\lambda)} = 2^{\mathrm{cf}(\lambda)} + \lambda^+$.

[2]Along the way, we improve work of Boney and VanDieren [BVa] by showing that the continuity hypothesis of their main result is redundant (Theorem 3.7).

By the present work, the answer to Question 1.1 is positive assuming the existence of large cardinals (take λ_0 to be the first strongly compact above $\text{LS}(\mathbf{K})$).

Apart from $\chi(\mathbf{K})$, several other cardinal parameters ($\lambda(\mathbf{K})$, $\lambda'(\mathbf{K})$, $\lambda''(\mathbf{K})$, and $\bar{\mu}(\mathbf{K})$) are defined in this paper. We give loose bounds on these cardinals (see e.g. Theorem 8.8, in particular they are all below $\beth_{\beth_{\left(2^{\text{LS}(\mathbf{K})}\right)^+}}$) but focus on eventual behavior. We believe it is a worthy endeavour (analog to the study of the behavior of the stability spectrum below $2^{|T|}$ in first-order) to try to say something more on these cardinals.

1.4. **Notes.** The background required to read this paper is a solid knowledge of tame AECs (as presented for example in [Bal09]). Familiarity with [Vas16b] would be very helpful. Results from the recent literature which we rely on can be used as black boxes.

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Note that at the beginning of several sections, we make global hypotheses assumed throughout the section. In the statement of the main results, these global hypotheses will be repeated.

2. Preliminaries

2.1. Basic notation.

Notation 2.1.

(1) We use the Hanf number notation from [Bal09, Notation 4.24]: for λ an infinite cardinal, write $h(\lambda) := \beth_{\left(2^{\lambda}\right)^+}$. When $\mathbf{K}$ is an AEC clear from context, $H_1 := h(\text{LS}(\mathbf{K}))$.
(2) For $\alpha < \mu$ both regular cardinals, let $S^{\alpha}_{\mu} := \{i < \mu \mid \text{cf}(i) = \alpha\}$.
(3) Let REG denote the class of regular cardinals.

2.2. **Monster model, Galois types, and tameness.** We say that an AEC $\mathbf{K}$ *has a monster model* if it has amalgamation, joint embedding, and arbitrarily large models. Equivalently, it has a (proper class sized) model-homogeneous universal model $\mathfrak{C}$. When $\mathbf{K}$ has a monster model, we fix such a $\mathfrak{C}$ and work inside it. Note that for our purpose amalgamation is the only essential property. Once we have it, we can partition $\mathbf{K}$ into disjoint pieces, each of which has joint embedding (see for example [Bal09, Lemma 16.14]). Further, for studying the eventual behavior of $\mathbf{K}$ assuming the existence of arbitrarily large models is natural.

We use the notation of [Vas16c] for Galois types. In particular, $\mathrm{gtp}(\bar{b}/A; N)$ denotes the Galois type of the sequence $\bar{b}$ over the set A, as computed in $N \in \mathbf{K}$. In case $\mathbf{K}$ has a monster model $\mathfrak{C}$, we write $\mathrm{gtp}(\bar{b}/A)$ instead of $\mathrm{gtp}(\bar{b}/A; \mathfrak{C})$. In this case, $\mathrm{gtp}(\bar{b}/A) = \mathrm{gtp}(\bar{c}/A)$ if and only if there exists an automorphism f of $\mathfrak{C}$ fixing A such that $f(\bar{b}) = \bar{c}$.

Observe that the definition of Galois types is completely semantic. Tameness is a locality property for types isolated by Grossberg and VanDieren [GV06] that, when it holds, allows us to recover some of the syntactic properties of first-order types. For a cardinal $\mu \geq \mathrm{LS}(\mathbf{K})$, we say that an AEC $\mathbf{K}$ with a monster model is *μ-tame* if whenever $\mathrm{gtp}(b/M) \neq \mathrm{gtp}(c/M)$, there exists $M_0 \in \mathbf{K}_{\leq \mu}$ such that $M_0 \leq_{\mathbf{K}} M$ and $\mathrm{gtp}(b/M_0) \neq \mathrm{gtp}(c/M_0)$. When assuming tameness in this paper, we will usually just assume that $\mathbf{K}$ is $\mathrm{LS}(\mathbf{K})$-tame. Indeed if $\mathbf{K}$ is μ-tame we can just replace $\mathbf{K}$ by $\mathbf{K}_{\geq \mu}$. Then $\mathrm{LS}(\mathbf{K}_{\geq \mu}) = \mu$, so $\mathbf{K}_{\geq \mu}$ will be $\mathrm{LS}(\mathbf{K}_{\geq \mu})$-tame.

Concepts such as stability and saturation are defined as in the first-order case but using Galois type. For example, an AEC $\mathbf{K}$ with a monster model is *stable in μ* if $|\mathrm{gS}(M)| \leq \mu$ for every $M \in \mathbf{K}_\mu$. For $\mu > \mathrm{LS}(\mathbf{K})$, a model $M \in \mathbf{K}$ is *μ-saturated* if every Galois type over a $\leq_{\mathbf{K}}$-substructure of M of size less than μ is realized. In the literature, these are often called "Galois stable" and "Galois saturated", but we omit the "Galois" prefix since there is no risk of confusion in this paper.

As in [She99, Definition 4.3]:

Definition 2.2. Let $\mathbf{K}$ be an AEC with a monster model and let α be a non-zero cardinal. We say that $\mathbf{K}$ *has the α-order property* if for every θ there exists $\langle \bar{a}_i : i < \theta \rangle$ such that for all $i < \theta$, $\ell(\bar{a}_i) = \alpha$ and for all $i_0 < i_1 < \theta$, $j_0 < j_1 < \theta$, $\mathrm{gtp}(\bar{a}_{i_0}\bar{a}_{i_1}/\emptyset) \neq \mathrm{gtp}(\bar{a}_{j_1}\bar{a}_{j_0}/\emptyset)$.

2.3. **Independence relations.** Recall [Gro] that an abstract class (AC) is a partial order $\mathbf{K} = (K, \leq_{\mathbf{K}})$ where K is a class of structures in a fixed vocabulary $\tau(\mathbf{K})$, $\mathbf{K}$ is closed under isomorphisms, and $M \leq_{\mathbf{K}} N$ implies $M \subseteq N$.

In this paper, an *independence relation* will be a pair $(\mathbf{K}, \mathop{\smile\hskip-0.9em\raise0.5ex\hbox{$\mid$}})$, where:

(1) $\mathbf{K}$ is a coherent[3] abstract class with amalgamation.
(2) $\mathop{\smile\hskip-0.9em\raise0.5ex\hbox{$\mid$}}$ is a 4-ary relation so that:

[3]that is, whenever $M_0 \subseteq M_1 \leq_{\mathbf{K}} M_2$ and $M_0 \leq_{\mathbf{K}} M_2$, we have that $M_0 \leq_{\mathbf{K}} M_1$.

(a) $\mathop{\smile}\limits^{|}(M, A, B, N)$ implies $M \leq_{\mathbf{K}} N$, $A, B \subseteq |N|$, $|A| \leq 1$. We write $A \overset{N}{\underset{M}{\mathop{\smile}\limits^{|}}} B$.

(b) $\mathop{\smile}\limits^{|}$ satisfies invariance, normality, and monotonicity (see [Vas16a, Definition 3.6] for the definitions).

(c) We also ask that $\mathop{\smile}\limits^{|}$ satisfies base monotonicity: if $A \overset{N}{\underset{M_0}{\mathop{\smile}\limits^{|}}} B$, $M_0 \leq_{\mathbf{K}} M \leq_{\mathbf{K}} N$, and $|M| \subseteq B$, then $A \overset{N}{\underset{M}{\mathop{\smile}\limits^{|}}} B$.

Note that this definition differs slightly from that in [Vas16a, Definition 3.6]: there additional parameters are added controlling the size of the left and right hand side, and base monotonicity is assumed. Here, the size of the left hand side is at most 1 and the size of the right hand side is not bounded. So in the terminology of [Vas16a], we are defining a $(\leq 1, [0, \infty))$-independence relation with base monotonicity.

When $\mathfrak{i} = (\mathbf{K}, \mathop{\smile}\limits^{|})$ is an independence relation and $p \in \text{gS}(B; N)$ (we make use of Galois types over sets, see [Vas16c, Definition 2.16]), we say that p *does not* $\mathfrak{i}$*-fork over* M if $a \overset{N}{\underset{M}{\mathop{\smile}\limits^{|}}} B$ for some (any) a realizing p in N. When $\mathfrak{i}$ is clear from context, we omit it and just say that p *does not fork over* M.

The following independence notion is central. It was introduced by Shelah in [She99, Definition 3.2].

Definition 2.3. Let $\mathbf{K}$ be a coherent abstract class with amalgamation, let $M \leq_{\mathbf{K}} N$, $p \in \text{gS}(N)$, and let $\mu \geq \|M\|$. We say that p *μ-splits over* M if there exists $N_1, N_2 \in \mathbf{K}_{\leq \mu}$ and f such that $M \leq_{\mathbf{K}} N_\ell \leq_{\mathbf{K}} N$ for $\ell = 1, 2$, $f : N_1 \cong_M N_2$, and $f(p \upharpoonright N_1) \neq p \upharpoonright N_2$.

For λ an infinite cardinal, we write $\mathfrak{i}_{\mu\text{-spl}}(\mathbf{K}_\lambda)$ for the independence relation with underlying class $\mathbf{K}_\lambda$ and underlying independence notion non μ-splitting.

2.4. **Universal orderings and limit models.** Work inside an abstract class $\mathbf{K}$. For $M <_{\mathbf{K}} N$, we say that N is *universal over* M (and write $M <_{\mathbf{K}}^{\text{univ}} N$) if for any $M' \in \mathbf{K}$ with $M \leq_{\mathbf{K}} M'$ and $\|M'\| = \|M\|$, there exists $f : M' \xrightarrow[M]{} N$. For a cardinal λ and a limit ordinal $\delta < \lambda^+$, we say that N is (λ, δ)*-limit over* M if there exists an increasing continuous chain $\langle N_i : i \leq \delta \rangle$ such that $N_0 = M$, $N_\delta = N$, and for any $i < \delta$, N_i is in $\mathbf{K}_\lambda$ and N_{i+1} is universal over N_i. We say that N is $(\lambda, \geq \delta)$*-limit over* M if there exists a regular cardinal $\gamma \in [\delta, \lambda^+)$ such that N is (λ, γ)-limit over M. Similarly define variations such as

$(\lambda, < \delta)$-limit. We will use without mention the basic facts about limit models in AECs: existence (assuming stability and a monster model) and uniqueness when they have the same cofinality. See [GVV16] for an introduction to the theory of limit models.

2.5. **Locality cardinals for independence.** One of the main object of study of this paper is $\chi(\mathbf{K})$ (see Definition 8.5), which roughly is the least length χ of an increasing continuous chain $\langle M_i : i \leq \chi \rangle$ where each model is universal over the previous one and so that for any $p \in \mathrm{gS}(M_\chi)$ there exists $i < \chi$ such that p does not $\|M_i\|$-split over M_i. We will first have to take a step back and argue about why there is such a least cardinal (e.g. show that there are no regular $\chi_0 < \chi_1 < \chi_2$ such that chains of χ_0 and χ_2 have the good property above but chains of length χ_1 do not). Further, several other related locality properties turn out to be useful.

We follow Shelah's approach from [She99] and define *classes* of locality cardinals, rather than directly taking a minimum (as in for example [GV06, Definition 4.3]). The cardinals $\underline{\kappa}^{\mathrm{wk}}$ are already in [She99, Definition 4.8], while $\underline{\kappa}^{\mathrm{cont}}$ is used in the proof of the Shelah-Villaveces theorem [SV99, Theorem 2.2.1], see also [BGVV].

Definition 2.4 (Locality cardinals)**.** Let $\mathfrak{i}$ be an independence relation. Let R be a partial order on $\mathbf{K}$ extending $\leq_{\mathbf{K}}$.

(1) $\underline{\kappa}(\mathfrak{i}, R)$ is the set of limit ordinals δ such that whenever $\langle M_i : i < \delta \rangle$ is an R-increasing chain[4], $N \in \mathbf{K}$ is such that $M_i \leq_{\mathbf{K}} N$ for all $i < \delta$, and $p \in \mathrm{gS}(\bigcup_{i<\delta} |M_i|; N)$, there exists $i < \delta$ such that p does not fork over M_i.
(2) $\underline{\kappa}^{\mathrm{wk}}(\mathfrak{i}, R)$ is the set of limit ordinals δ such that whenever $\langle M_i : i < \delta \rangle$ is an R-increasing chain, $N \in \mathbf{K}$ is such that $M_i \leq_{\mathbf{K}} N$ for all $i < \delta$, and $p \in \mathrm{gS}(\bigcup_{i<\delta} |M_i|; N)$, there exists $i < \delta$ such that $p \upharpoonright M_{i+1}$ does not fork over M_i.
(3) $\underline{\kappa}^{\mathrm{cont}}(\mathfrak{i}, R)$ is the set of limit ordinals δ such that whenever $\langle M_i : i < \delta \rangle$ is an R-increasing chain, $N \in \mathbf{K}$ is such that $M_i \leq_{\mathbf{K}} N$ for all $i < \delta$, and $p \in \mathrm{gS}(\bigcup_{i<\delta} |M_i|; N)$, if $p \upharpoonright M_i$ does not fork over M_0 for all $i < \delta$, then p does not fork over M_0.

When R is $\leq_{\mathbf{K}}$, we omit it. In this paper, R will mostly be $<_{\mathbf{K}}^{\mathrm{univ}}$ (see Section 2.4).

Note that for a limit ordinal δ, if $\mathrm{cf}(\delta) \in \underline{\kappa}(\mathfrak{i}, R)$, then $\delta \in \underline{\kappa}(\mathfrak{i}, R)$, and similarly for $\underline{\kappa}^{\mathrm{cont}}(\mathfrak{i}, R)$. An even stronger fact holds for $\underline{\kappa}^{\mathrm{wk}}(\mathfrak{i}, R)$:

[4] that is, $M_i R M_j$ for all $i < j < \delta$.

$\delta_0 < \delta_1$ and $\delta_0 \in \underline{\kappa}^{\text{wk}}(\mathfrak{i}, R)$ implies that $\delta_1 \in \underline{\kappa}^{\text{wk}}(\mathfrak{i}, R)$. Thus $\underline{\kappa}^{\text{wk}}(\mathfrak{i}, R)$ is an end segment of limit ordinals. In section 3 we will give conditions under which $\underline{\kappa}(\mathfrak{i}, R) \cap \text{REG}$ and $\underline{\kappa}^{\text{cont}}(\mathfrak{i}, R) \cap \text{REG}$ are also end segments. In this case, the following cardinals are especially interesting (note the absence of line under κ):

Definition 2.5. $\kappa(\mathfrak{i}, R) := \min(\underline{\kappa}(\mathfrak{i}, R))$. Similarly define $\kappa^{\text{wk}}(\mathfrak{i}, R)$ and $\kappa^{\text{cont}}(\mathfrak{i}, R)$.

The following lemma is [BGVV, Lemma 9.(2)] but we give the proof here for the convenience of the reader.

Lemma 2.6. Let $\mathfrak{i} = (\mathbf{K}, \downfree)$ be an independence relation. Let R be a partial order on $\mathbf{K}$ extending $\leq_{\mathbf{K}}$.

We have that $\underline{\kappa}^{\text{wk}}(\mathfrak{i}, R) \cap \underline{\kappa}^{\text{cont}}(\mathfrak{i}, R) \cap \text{REG} \subseteq \underline{\kappa}(\mathfrak{i}, R) \cap \text{REG}$.

Proof. We prove the following stronger claim. The result then follows from the definition of $\underline{\kappa}^{\text{cont}}$.

<u>Claim</u>: Let $\delta \in \underline{\kappa}^{\text{wk}}(\mathfrak{i}, R)$ be regular. For any R-increasing chain $\langle M_i : i < \delta \rangle$, any $N \in \mathbf{K}$ with with $i < \delta$ implies $M_i \leq_{\mathbf{K}} N$, and any $p \in \text{gS}(\bigcup_{i<\delta} |M_i|; N)$, there exists $i < \delta$ such that $p \upharpoonright M_j$ does not fork over M_i for all $j \in [i, \delta)$.

<u>Proof of Claim</u>: Suppose not. Then for every $i < \delta$ there exists $j \in [i, \delta)$ such that $p \upharpoonright M_j$ forks over M_i. Let j_i be the least such j. Now define an increasing sequence of ordinals $\langle \alpha_k : k < \delta \rangle$ by induction on k as follows: $\alpha_0 := 0$, $\alpha_{k+1} := j_{\alpha_k}$, and $\alpha_k := \sup_{k' < k} \alpha_{k'}$ for k limit. Note that by regularity of δ, $\alpha_k < \delta$ for any $k < \delta$, so the definition makes sense. Also note that $j_i > i$ (apply the definition of $\underline{\kappa}^{\text{wk}}$ to an appropriate constant sequence). Therefore $\langle \alpha_k : k < \delta \rangle$ is a strictly increasing sequence of ordinals, and if we let $\alpha_\delta := \sup_{k<\delta} \alpha_k$, we have that $\alpha_\delta = \delta$.

We can then apply the definition of $\underline{\kappa}^{\text{wk}}$ to the chain $\langle M_{\alpha_k} : k < \delta \rangle$. We get that there exists $k < \delta$ such that $p \upharpoonright M_{\alpha_{k+1}}$ does not fork over M_{α_k}. Since $\alpha_{k+1} = j_{\alpha_k}$, this contradicts the definition of j_{α_k}. $\dagger_{\text{Claim}}$ $\square$

Remark 2.7. The conclusion of Lemma 2.6 can be made into an equality assuming that $\mathfrak{i}$ satisfies a weak transitivity property (see the statement for splitting and $R = <^{\text{univ}}_{\mathbf{K}}$ in [Vas16b, Proposition 3.7]). This is not needed in this paper.

3. Continuity of forking

In this section, we aim to study the locality cardinals and give conditions under which $\underline{\kappa}^{\text{cont}}$ contains all limit ordinals. We work in an AEC with amalgamation and stability in a single cardinal μ:

Hypothesis 3.1.

(1) $\mathbf{K}$ is an AEC, $\mu \geq \text{LS}(\mathbf{K})$.
(2) $\mathbf{K}_\mu$ has amalgamation, joint embedding, and no maximal models in μ. Moreover $\mathbf{K}$ is stable in μ.
(3) $\mathfrak{i} = (\mathbf{K}_\mu, \perp)$ is an independence relation.

Remark 3.2. The results of this section generalize to AECs that may not have full amalgamation in μ, but only satisfy the properties from [SV99]: density of amalgamation bases, existence of universal extensions, and limit models being amalgamation bases.

We will use the following without comments:

Lemma 3.3. Let $\delta < \mu^+$ be a limit ordinal. Then $\delta \in \underline{\kappa}(\mathfrak{i}, <_{\mathbf{K}}^{\text{univ}})$ if and only if $\text{cf}(\delta) \in \underline{\kappa}(\mathfrak{i}, <_{\mathbf{K}}^{\text{univ}})$, and similarly for $\underline{\kappa}^{\text{cont}}(\mathfrak{i}, <_{\mathbf{K}}^{\text{univ}})$. In particular, $\kappa(\mathfrak{i}, <_{\mathbf{K}}^{\text{univ}})$ and $\kappa^{\text{cont}}(\mathfrak{i}, <_{\mathbf{K}}^{\text{univ}})$ are regular cardinals.

Proof. Use the uniqueness of limit models of the same cofinality. □

We will usually assume that $\mathfrak{i}$ has the weak uniqueness property:

Definition 3.4. $\mathfrak{i}$ has *weak uniqueness* if whenever $M_0 \leq_{\mathbf{K}} M \leq_{\mathbf{K}} N$ are all in $\mathbf{K}_\mu$ with M universal over M_0, $p, q \in \text{gS}(N)$ do not fork over M_0, and $p \upharpoonright M = q \upharpoonright M$, then $p = q$.

The reader can think of $\mathfrak{i}$ as non-μ-splitting (Definition 2.3), where such a property holds [Van06, Theorem I.4.12]. We state a more general version:

Fact 3.5 (Theorem 6.2 in [GV06])**.** $\mathfrak{i}_{\mu\text{-spl}}(\mathbf{K}_\mu)$ has weak uniqueness. More precisely, let $M_0 \leq_{\mathbf{K}} M \leq_{\mathbf{K}} N$ all be in $\mathbf{K}_{\geq \mu}$ with $M_0 \in \mathbf{K}_\mu$. Assume that M is universal over M_0 and $\mathbf{K}$ is $(\mu, \|N\|)$-tame (i.e. types over models of size $\|N\|$ are determined by their restrictions of size μ).

Let $p, q \in \text{gS}(N)$. If p, q both do not μ-split over M_0 and $p \upharpoonright M = q \upharpoonright M$, then $p = q$.

Weak uniqueness is used in the following standard pruning argument (recall from Notation 2.1 that $S^\alpha_{\mu^+}$ is the set of ordinals in μ^+ of cofinality α).

Lemma 3.6. Assume that $\mathfrak{i}$ has weak uniqueness and that $\alpha \in \underline{\kappa}(\mathfrak{i}, <^{\text{univ}}_{\mathbf{K}}) \cap \mu^+$. Let $\langle N_i : i \leq \mu^+ \rangle$ be increasing continuous with $N_{i+1} \in \mathbf{K}_\mu$ universal over N_i for all $i < \mu^+$. Let $\langle a_i : i < \mu^+ \rangle$ be in $|N_{\mu^+}|$. Then there exists a stationary set $S \subseteq S^{\text{cf}(\alpha)}_{\mu^+}$ such that for any $i < j$ both in S, $\text{gtp}(a_i / N_i; N_{\mu^+}) = \text{gtp}(a_j / N_i; N_{\mu^+})$.

Proof. Work inside N_{μ^+}. Without loss of generality, $\alpha = \text{cf}(\alpha)$. Because $\alpha \in \underline{\kappa}(\mathfrak{i}, <^{\text{univ}}_{\mathbf{K}})$, for each $i \in S^{\alpha}_{\mu^+}$, there exists $j_i < i$ such that $\text{gtp}(a_i / N_i)$ does not fork over N_{j_i}. Apply Fodor's lemma to get $i_0 < \mu^+$ and $S_1 \subseteq S^{\alpha}_{\mu^+}$ stationary on which the map $i \mapsto j_i$ is constantly i_0. Further pruning, we can get $S \subseteq S_1$ stationary on which the map $i \mapsto \text{gtp}(a_i / N_{i_0 + 1})$ is constantly $p \in \text{gS}(N_{i_0+1})$. Use weak uniqueness to see that such an S works. $\square$

The next two statements are the main results of this section, so we state Hypothesis 3.1 again at the beginning of each one.

Theorem 3.7 (The continuity theorem)**.** Let $\mathbf{K}$ be an AEC. Let $\mu \geq \text{LS}(\mathbf{K})$ be such that $\mathbf{K}_\mu$ has amalgamation, joint embedding, no maximal models, and is stable in μ. Let $\mathfrak{i} = (\mathbf{K}_\mu, \mathop{\smile\kern-0.9em\raise0.4ex\hbox{$\scriptstyle|$}})$ be an independence relation.

If $\mathfrak{i}$ has weak uniqueness and $\kappa(\mathfrak{i}, <^{\text{univ}}_{\mathbf{K}}) < \mu^+$, then $\underline{\kappa}^{\text{cont}}(\mathfrak{i}, <^{\text{univ}}_{\mathbf{K}})$ contains all the limit ordinals.

Proof. The idea is to follow the argument from the proof of case (a) in [SV99, Theorem 2.2.1], but to use Lemma 3.6 instead of using EM models to show that the two types are equal.

Let $\delta < \mu^+$ be a limit ordinal. Let $\langle M_i : i \leq \delta \rangle$ be increasing continuous in $\mathbf{K}_\mu$ such that M_{i+1} is universal over M_i for all $i < \delta$. Let $p \in \text{gS}(M_\delta)$ and assume that $p \restriction M_i$ does not fork over M_0 for all $i < \delta$. We want to see that p does not fork over M_0. Without loss of generality, $\delta = \text{cf}(\delta)$.

Suppose for a contradiction that p forks over M_0. Without loss of generality, for each $i < \delta$, M_{i+1} is (μ, ω)-limit over M_i (use universality). Build $\langle N_i : i \leq \mu^+ \rangle$ increasing continuous such that for each $i < \mu^+$, $N_{i+1} \in \mathbf{K}_\mu$ is (μ, ω)-limit over N_i. By uniqueness of limit models of the same cofinality, for each $i \in S^{\delta}_{\mu^+}$ there exists a club $c_i \subseteq i$ with enumeration $\langle c_{i,j} : j < \delta \rangle$ and $f_i : M_\delta \cong N_i$ such that $f_i[M_j] = N_{c_{i,j}}$ for each $j < \delta$. Let $a_i \in N_{\mu^+}$ realize $f_i(p)$.

By Lemma 3.6, there exists a stationary $S \subseteq \mu^+$ such that for any $i < j$ in S, $\text{gtp}(a_i / N_i; N_{\mu^+}) = \text{gtp}(a_j / N_i; N_{\mu^+})$. Fix any two such i and j. By further pruning, we can assume without loss of generality that

$c_{i,0} = c_{j,0} < i$. Pick $k < \delta$ such that $i < c_{j,k} < j$. Since $p \restriction M_k$ does not fork over M_0, we have that $f_j(p \restriction M_k) \in \mathrm{gS}(N_{c_{j,k}})$ does not fork over $f_j[M_0] = N_{c_{j,0}}$. By monotonicity, $f_j(p) \restriction N_i$ does not fork over $N_{c_{j,0}}$.

On the other hand, since p forks over M_0, we have that $f_i(p) \in \mathrm{gS}(N_i)$ forks over $f_i[M_0] = N_{c_{i,0}} = N_{c_{j,0}}$. Now, $f_i(p) = \mathrm{gtp}(a_i/N_i; N_{\mu^+})$ while $f_j(p) \restriction N_i = \mathrm{gtp}(a_j/N_i; N_{\mu^+})$. By assumption, these two types are equal, a contradiction to the fact that the first forks but the latter does not. $\square$

Remark 3.8. In Theorem 3.7 and Corollary 3.9 below, it is enough to assume the conclusion of Lemma 3.6 instead of weak uniqueness and $\kappa(\mathfrak{i}, <^{\mathrm{univ}}_{\mathbf{K}}) < \mu^+$.

Corollary 3.9. Let $\mathbf{K}$ be an AEC. Let $\mu \geq \mathrm{LS}(\mathbf{K})$ be such that $\mathbf{K}_\mu$ has amalgamation, joint embedding, no maximal models, and is stable in μ. Let $\mathfrak{i} = (\mathbf{K}_\mu, \mathop{\smile\!\!\!\!\!\!\;\,|}\,)$ be an independence relation.

If $\mathfrak{i}$ has weak uniqueness, then:

(1) If $\kappa(\mathfrak{i}, <^{\mathrm{univ}}_{\mathbf{K}}) < \mu^+$, then $\underline{\kappa}(\mathfrak{i}, <^{\mathrm{univ}}_{\mathbf{K}}) \cap \mathrm{REG} = \underline{\kappa}^{\mathrm{wk}}(\mathfrak{i}, <^{\mathrm{univ}}_{\mathbf{K}}) \cap \mathrm{REG}$.
(2) $\underline{\kappa}(\mathfrak{i}, <^{\mathrm{univ}}_{\mathbf{K}}) \cap \mathrm{REG}$ is an end segment of regular cardinals.

Proof.

(1) Clearly, $\underline{\kappa}(\mathfrak{i}, <^{\mathrm{univ}}_{\mathbf{K}}) \cap \mathrm{REG} \subseteq \underline{\kappa}^{\mathrm{wk}}(\mathfrak{i}, <^{\mathrm{univ}}_{\mathbf{K}}) \cap \mathrm{REG}$. For the converse, note that by Theorem 3.7 $\underline{\kappa}^{\mathrm{cont}}(\mathfrak{i}, <^{\mathrm{univ}}_{\mathbf{K}}) \cap \mathrm{REG} = \mathrm{REG}$ so by Lemma 2.6, $\underline{\kappa}^{\mathrm{wk}}(\mathfrak{i}, <^{\mathrm{univ}}_{\mathbf{K}}) \cap \mathrm{REG} = \underline{\kappa}^{\mathrm{wk}}(\mathfrak{i}, <^{\mathrm{univ}}_{\mathbf{K}}) \cap \underline{\kappa}^{\mathrm{cont}}(\mathfrak{i}, <^{\mathrm{univ}}_{\mathbf{K}}) \cap \mathrm{REG} \subseteq \underline{\kappa}(\mathfrak{i}, <^{\mathrm{univ}}_{\mathbf{K}}) \cap \mathrm{REG}$.
(2) We have that $\underline{\kappa}(\mathfrak{i}, <^{\mathrm{univ}}_{\mathbf{K}})$ vacuously contains all the limit ordinals greater than or equal to μ^+. Now let $\alpha \in \underline{\kappa}(\mathfrak{i}, <^{\mathrm{univ}}_{\mathbf{K}}) \cap \mu^+ \cap \mathrm{REG}$ and let $\beta \in [\alpha, \mu^+) \cap \mathrm{REG}$. By the first part, $\alpha \in \underline{\kappa}^{\mathrm{wk}}(\mathfrak{i}, <^{\mathrm{univ}}_{\mathbf{K}})$. By definition of $\underline{\kappa}^{\mathrm{wk}}$, $\beta \in \underline{\kappa}^{\mathrm{wk}}(\mathfrak{i}, <^{\mathrm{univ}}_{\mathbf{K}})$. By the first part again, $\beta \in \underline{\kappa}(\mathfrak{i}, <^{\mathrm{univ}}_{\mathbf{K}})$, as desired.

$\square$

3.1. **Applications.**

(1) Corollary 3.9 sheds light on Shelah's remark on [She99, p. 275]. Shelah explains that as opposed to the first-order case it is not clear that $\underline{\kappa}(\mathfrak{i}, \leq_{\mathbf{K}})$ is an initial segment. We have shown here that under reasonable assumptions $\underline{\kappa}(\mathfrak{i}, <^{\mathrm{univ}}_{\mathbf{K}})$ (note the universal ordering) *is* an initial segment.

(2) Just like in [Vas16b, Section 4], we can also transfer continuity upward to *any* increasing (i.e. not just $<_{\mathbf{K}}^{\text{univ}}$-increasing) chain of μ^+-saturated models.
(3) Corollary 3.9 says that $\underline{\kappa}$ and $\underline{\kappa}^{\text{wk}}$ are the same for our purpose. Thus a definition of superstability using a rank (as in [She99, Definition 5.1] or [BG, Section 7]) is (in reasonable cases) equivalent to a definition using $\underline{\kappa}$ (see e.g. Definition 8.15 here). This answers a question around [GV, Definition 7.2].
(4) Theorem 3.7 shows that the fourth hypothesis in [BVa, Theorem 1] (a uniqueness of limit models result for strictly stable classes) is redundant. Thus (as we will see in Section 9), Boney and VanDieren's result can be applied to any stable tame AEC with amalgamation.

3.2. **Canonicity.** As noted before (Fact 3.5), non-μ-splitting is an example of an independence relation which (in the context of Hypothesis 3.1) satisfies weak uniqueness. We show here that any other independence relation satisfying weak uniqueness must look like non-μ-splitting, in the sense that its locality cardinals $\underline{\kappa}$, $\underline{\kappa}^{\text{wk}}$, and $\underline{\kappa}^{\text{cont}}$ are the same. The result is implicit in [Vas16a, Section 9], but we give full details here.

Lemma 3.10 (Weak extension)**.** Let $M_0 \leq_{\mathbf{K}} M \leq_{\mathbf{K}} N$ all be in $\mathbf{K}_\mu$. Assume that M is universal over M_0. Let $p \in \text{gS}(M)$ and assume that p does not fork over M_0.

If $\mathfrak{i}$ has weak uniqueness, then there exists $q \in \text{gS}(N)$ extending p such that q does not fork over M_0.

Proof. We first prove the result when M is (μ, ω)-limit over M_0. In this case we can write $M = M_\omega$, where $\langle M_i : i \leq \omega \rangle$ is increasing continuous with M_{i+1} universal over M_i for each $i < \omega$.

Let $f : N \underset{M_1}{\longrightarrow} M$. Let $q := f^{-1}(p)$. Then $q \in \text{gS}(N)$ and by invariance q does not fork over M_0. It remains to show that q extends p. Let $q_M := q \upharpoonright M$. We want to see that $q_M = p$. By monotonicity, q_M does not fork over M_0. Moreover, $q_M \upharpoonright M_1 = p \upharpoonright M_1$. By weak uniqueness, this implies that $q_M = p$, as desired.

In the general case (when M is only universal over M_0), let $M' \in \mathbf{K}_\mu$ be (μ, ω)-limit over M_0. By universality, we can assume that $M_0 \leq_{\mathbf{K}} M' \leq_{\mathbf{K}} M$. By the special case we have just proven, there exists $q \in \text{gS}(N)$ extending $p \upharpoonright M'$ such that q does not fork over M_0. By weak uniqueness, we must have that also $q \upharpoonright M = p$, i.e. q extends p. $\square$

We repeat Hypothesis 3.1 here for the convenience of the reader.

Theorem 3.11. Let $\mathbf{K}$ be an AEC. Let $\mu \geq \text{LS}(\mathbf{K})$ be such that $\mathbf{K}_\mu$ has amalgamation, joint embedding, no maximal models, and is stable in μ. Let $\mathfrak{i} = (\mathbf{K}_\mu, \perp)$ be an independence relation with weak uniqueness.

Let $M_0 \leq_{\mathbf{K}} M_1 \leq_{\mathbf{K}} M \leq_{\mathbf{K}} N$ all be in $\mathbf{K}_\mu$ such that M_1 is universal over M_0 and M is universal over M_1. Let $p \in \text{gS}(N)$.

(1) If p does not fork over M_0, then p does not μ-split over M_1.
(2) If p does not μ-split over M_0, then p does not fork over M_0.

Proof.

(1) Let $N_1, N_2 \in \mathbf{K}_\mu$ and $f : N_1 \cong_{M_1} N_2$ be such that $M_1 \leq_{\mathbf{K}} N_\ell \leq_{\mathbf{K}} N$ for $\ell = 1, 2$. We want to see that $f(p \upharpoonright N_1) = p \upharpoonright N_2$. By monotonicity, $p \upharpoonright N_\ell$ does not fork over M_0 for $\ell = 1, 2$. Consequently, $f(p \upharpoonright N_1)$ does not fork over M_0. Furthermore, $f(p \upharpoonright N_1) \upharpoonright M_1 = p \upharpoonright M_1 = (p \upharpoonright N_2) \upharpoonright M_1$. Applying weak uniqueness, we get that $f(p \upharpoonright N_1) = p \upharpoonright N_2$.
(2) By weak extension (Lemma 3.10), find $q \in \text{gS}(N)$ such that q extends $p \upharpoonright M$ and q does not fork over M_0. By the previous part, q does not μ-split over M_1. By weak uniqueness, we must have that $p = q$. In particular, p does not fork over M_0.

$\square$

A straightforward consequence of Theorem 3.11 is:

Corollary 3.12. If $\mathfrak{i}$ has weak uniqueness, then:

$$\underline{\kappa}(\mathfrak{i}, <^{\text{univ}}_{\mathbf{K}}) = \underline{\kappa}(\mathfrak{i}_{\mu\text{-spl}}(\mathbf{K}_\mu), <^{\text{univ}}_{\mathbf{K}})$$

and similarly for $\underline{\kappa}^{\text{wk}}$ and $\underline{\kappa}^{\text{cont}}$.

This justifies the following definition:

Definition 3.13. Define $\underline{\kappa}(\mathbf{K}_\mu, <^{\text{univ}}_{\mathbf{K}}) := \underline{\kappa}(\mathfrak{i}_{\mu\text{-spl}}(\mathbf{K}_\mu), <^{\text{univ}}_{\mathbf{K}})$. Similarly define the other variations in terms of $\underline{\kappa}^{\text{wk}}$ and $\underline{\kappa}^{\text{cont}}$. Also define $\kappa(\mathbf{K}_\mu, <^{\text{univ}}_{\mathbf{K}})$ and its variations.

4. Indiscernibles and bounded equivalence relations

We review here the main tools for the study of strong splitting in the next section: indiscernibles and bounded equivalence relations. All throughout, we assume:

Hypothesis 4.1. $\mathbf{K}$ is an AEC with a monster model.

Remark 4.2. By working more locally, the results and definitions of this section could be adapted to the amalgamation-less setup (see for example [Vasa, Definition 2.3]).

Definition 4.3 (Indiscernibles, Definition 4.1 in [She99])**.** Let α be a non-zero cardinal, θ be an infinite cardinal, and let $\langle \bar{a}_i : i < \theta \rangle$ be a sequence of distinct elements each of length α. Let A be a set.

(1) We say that $\langle \bar{a}_i : i < \theta \rangle$ *is indiscernible over A in N* if for every $n < \omega$, every $i_0 < \ldots < i_{n-1} < \theta$, $j_0 < \ldots < j_{n-1} < \theta$, $\text{gtp}(\bar{a}_{i_0} \ldots \bar{a}_{i_n} / A) = \text{gtp}(\bar{a}_{j_0} \ldots \bar{a}_{j_n} / A)$. When $A = \emptyset$, we omit it and just say that $\langle \bar{a}_i : i < \theta \rangle$ is indiscernible.
(2) We say that $\langle \bar{a}_i : i < \theta \rangle$ is *strictly indiscernible* if there exists an EM blueprint Φ (whose vocabulary is allowed to have arbitrary size) an automorphism f of $\mathfrak{C}$ so that, letting $N' := \text{EM}_{\tau(\mathbf{K})}(\theta, \Phi)$:
   (a) For all $i < \theta$, $\bar{b}_i := f(\bar{a}_i) \in {}^{\alpha}|N'|$.
   (b) If for $i < \theta$, $\bar{b}_i = \langle b_{i,j} : j < \alpha \rangle$, then for all $j < \alpha$ there exists a unary $\tau(\Phi)$-function symbol ρ_j such that for all $i < \theta$, $b_{i,j} = \rho_j^{N'}(i)$.
(3) Let A be a set. We say that $\langle \bar{a}_i : i < \theta \rangle$ is *strictly indiscernible over A* if there exists an enumeration $\bar{a}$ of A such that $\langle \bar{a}_i \bar{a} : i < \theta \rangle$ is strictly indiscernible.

Any strict indiscernible sequence extends to arbitrary lengths: this follows from a use of first-order compactness in the EM language. The converse is also true. This follows from the more general extraction theorem, essentially due to Morley:

Fact 4.4. Let $\mathbf{I} := \langle \bar{a}_i : i < \theta \rangle$ be distinct such that $\ell(\bar{a}_i) = \alpha$ for all $i < \theta$. Let A be a set. If $\theta \geq h(\text{LS}(\mathbf{K}) + |\alpha| + |A|)$, then there exists $\mathbf{J} := \langle \bar{b}_i : i < \omega \rangle$ such that $\mathbf{J}$ is strictly indiscernible over A and for any $n < \omega$ there exists $i_0 < \ldots < i_{n-1} < \theta$ such that $\text{gtp}(\bar{b}_0 \ldots \bar{b}_{n-1} / A) = \text{gtp}(\bar{a}_{i_0} \ldots \bar{a}_{i_{n-1}} / A)$.

Fact 4.5. Let$\langle \bar{a}_i : i < \theta \rangle$ be indiscernible over A, with $\ell(\bar{a}_i) = \alpha$ for all $i < \theta$. The following are equivalent:

(1) For any infinite cardinal λ, there exists $\langle \bar{b}_i : i < \lambda \rangle$ that is indiscernible over A and such that $\bar{b}_i = \bar{a}_i$ for all $i < \theta$.
(2) For all infinite $\lambda < h(\theta + |A| + |\alpha| + \mathrm{LS}(\mathbf{K}))$ (recall Notation 2.1), there exists $\langle \bar{b}_i : i < \lambda \rangle$ as in (1).
(3) $\langle \bar{a}_i : i < \theta \rangle$ is strictly indiscernible over A.

We want to study bounded equivalence relations: they are the analog of Shelah's finite equivalence relations from the first-order setup but here the failure of compactness compels us to only ask for the number of classes to be bounded (i.e. a cardinal). The definition for homogeneous model theory appears in [HS00, Definition 1.4].

Definition 4.6. Let α be a non-zero cardinal and let A be a set. An *α-ary Galois equivalence relation on* A is an equivalence relation E on ${}^{\alpha}\mathfrak{C}$ such that for any automorphism f of $\mathfrak{C}$ fixing A, $\bar{b} E \bar{c}$ if and only if $f(\bar{b}) E f(\bar{c})$.

Definition 4.7. Let α be a non-zero cardinal, A be a set, and E be an α-ary Galois equivalence relation on A.

(1) Let $c(E)$ be the number of equivalence classes of E.
(2) We say that E is *bounded* if $c(E) < \infty$ (i.e. it is a cardinal).
(3) Let $\mathrm{SE}^{\alpha}(A)$ be the set of α-ary bounded Galois equivalence relations over A (S stands for strong).

Remark 4.8.

$$|\mathrm{SE}^{\alpha}(A)| \leq |2^{\mathrm{gS}^{\alpha+\alpha}(A)}| \leq 2^{2^{|A|+\mathrm{LS}(\mathbf{K})+\alpha}}$$

The next two results appear for homogeneous model theory in [HS00, Section 1]. The main difference here is that strictly indiscernible and indiscernibles need not coincide.

Lemma 4.9. Let $E \in \mathrm{SE}^{\alpha}(A)$. Let $\mathbf{I}$ be strictly indiscernible over A. For any $\bar{a}, \bar{b} \in \mathbf{I}$, we have that $\bar{a} E \bar{b}$.

Proof. Suppose not, say $\neg(\bar{a} E \bar{b})$. Fix any infinite cardinal $\lambda \geq |\mathbf{J}|$. By Theorem 4.5, $\mathbf{I}$ extends to a strictly indiscernible sequence $\mathbf{J}$ over A of cardinality λ. Thus $c(E) \geq \lambda$. Since λ was arbitrary, this contradicts the fact that E was bounded. $\square$

Lemma 4.10. Let A be a set and α be a non-zero cardinal. Let E be an α-ary Galois equivalence relation over A. The following are equivalent:

(1) E is bounded.
(2) $c(E) < h(|A| + \alpha + \mathrm{LS}(\mathbf{K}))$.

Proof. Let $\theta := h(|A| + \alpha + \mathrm{LS}(\mathbf{K}))$. If $c(E) < \theta$, E is bounded. Conversely if $c(E) \geq \theta$ then we can list θ non-equivalent elements as $\mathbf{I} := \langle \bar{a}_i : i < \theta \rangle$. By Fact 4.4, there exists a strictly indiscernible sequence over A $\langle \bar{b}_i : i < \omega \rangle$ reflecting some of the structure of $\mathbf{I}$. In particular, for $i < j < \omega$, $\neg(\bar{b}_i E \bar{b}_j)$. By Lemma 4.9, E cannot be bounded. $\square$

The following equivalence relation will play an important role (see [HS00, Corollary 4.7])

Definition 4.11. For all A and α, let $E_{\min,A,\alpha} := \bigcap \mathrm{SE}^\alpha(A)$.

By Remark 4.8 and a straightforward counting argument, we have that $E_{\min,A,\alpha} \in \mathrm{SE}^\alpha(A)$.

5. Strong splitting

We study the AEC analog of first-order strong splitting. It was introduced by Shelah in [She99, Definition 4.11]. In the next section, the analog of first-order dividing will be studied. Shelah also introduced it [She99, Definition 4.8] and showed how to connect it with strong splitting. After developing enough machinery, we will be able to connect Shelah's results on the locality cardinals for dividing [She99, Claim 5.5] to the locality cardinals for splitting.

All throughout this section, we assume:

Hypothesis 5.1. $\mathbf{K}$ is an AEC with a monster model.

Definition 5.2. Let μ be an infinite cardinal, $A \subseteq B$, $p \in \mathrm{gS}(B)$. We say that p $(< \mu)$*-strongly splits over* A if there exists a strictly indiscernible sequence $\langle \bar{a}_i : i < \omega \rangle$ over A with $\ell(\bar{a}_i) < \mu$ for all $i < \omega$ such that for any b realizing p, $\mathrm{gtp}(b\bar{a}_0/A) \neq \mathrm{gtp}(b\bar{a}_1/A)$. We say that p *explicitly* $(< \mu)$*-strongly splits over* A if the above holds with $\bar{a}_0\bar{a}_1 \in {}^{<\mu}B$.

μ-strongly splits means $(\leq \mu)$-strongly splits, which has the expected meaning.

Remark 5.3. For $\mu < \mu'$, if p [explicitly] $(< \mu)$-strongly splits over A, then p [explicitly] $(< \mu')$-strongly splits over A.

Lemma 5.4 (Base monotonicity of strong splitting)**.** Let $A \subseteq B \subseteq C$ and let $p \in \mathrm{gS}(C)$. Let $\mu > |B \backslash A|$ be infinite. If p $(< \mu)$-strongly splits over B, then p $(< \mu)$-strongly splits over A.

Proof. Let $\langle \bar{a}_i : i < \omega \rangle$ witness the strong splitting over B. Let $\bar{c}$ be an enumeration of $B \backslash A$. The sequence $\langle \bar{a}_i \bar{c} : i < \omega \rangle$ is strictly indiscernible over A. Moreover, for any b realizing p, $\text{gtp}(b\bar{c}\bar{a}_0/A) \neq \text{gtp}(b\bar{c}\bar{a}_1/A)$ if and only if $\text{gtp}(b\bar{a}_0/A\bar{c}) \neq \text{gtp}(b\bar{a}_1/A\bar{c})$ if and only if $\text{gtp}(b\bar{a}_0/B) \neq \text{gtp}(b\bar{a}_1/B)$, which holds by the strong splitting assumption. $\square$

Lemma 5.4 motivates the following definition:

Definition 5.5. For $\lambda \geq \text{LS}(\mathbf{K})$), we let $\mathbf{i}_{\lambda\text{-strong-spl}}(\mathbf{K}_\lambda)$ be the independence relation whose underlying class is $\mathbf{K}'$ and whose independence relation is non λ-strong-splitting.

Next, we state a key characterization lemma for strong splitting in terms of bounded equivalence relations. This is used in the proof of the next result, a kind of uniqueness of the non-strong-splitting extension. It appears already for homogeneous model theory in [HS00, Lemma 1.12]

Definition 5.6. Let $N \in \mathbf{K}$, $A \subseteq |N|$, and μ be an infinite cardinal. We say that N is *μ-saturated over A* if any type in $\text{gS}^{<\mu}(A)$ is realized in N.

Lemma 5.7. Let $N \in \mathbf{K}$ and let $A \subseteq |N|$. Assume that N is $(\aleph_1 + \mu)$-saturated over A. Let $p := \text{gtp}(b/N)$. The following are equivalent.

(1) p does not *explicitly* $(< \mu)$-strongly split over A.
(2) p does not $(< \mu)$-strongly split over A.
(3) For all $\alpha < \mu$, all $\bar{c}$, $\bar{d}$ in ${}^{\alpha}|N|$, $\bar{c} E_{\min,A,\alpha} \bar{d}$ implies $\text{gtp}(b\bar{c}/A) = \text{gtp}(b\bar{d}/A)$.

Proof. If p explicitly $(< \mu)$-strongly splits over A, then p $(< \mu)$-strongly splits over A. Thus (2) implies (1).

If p $(< \mu)$-splits strongly over A, let $\mathbf{I} = \langle \bar{a}_i : i < \omega \rangle$ witness it, with $\bar{a}_i \in {}^{\alpha}|\mathfrak{C}|$ for all $i < \omega$. By Lemma 4.9, $\bar{a}_0 E_{\min,A,\alpha} \bar{a}_1$. However by the strong splitting assumption $\text{gtp}(b\bar{a}_0/A) \neq \text{gtp}(b\bar{a}_1/A)$. This proves (3) implies (2).

It remains to show that (1) implies (3). Assume (1). Assume $\bar{c}$, $\bar{d}$ are in ${}^{\alpha}|N|$ such that $\bar{c} E_{\min,A,\alpha} \bar{d}$. Define an equivalence relation E on ${}^{\alpha}|\mathfrak{C}|$ as follows. $\bar{b}_0 E \bar{b}_1$ if and only if $\bar{b}_0 = \bar{b}_1$ or there exists $n < \omega$ and $\langle \mathbf{I}_i : i < n \rangle$ strictly indiscernible over A such that $\bar{b}_0 \in I_0$, $\bar{b}_1 \in I_{n-1}$ and for all $i < n - 1$, $\mathbf{I}_i \cap \mathbf{I}_{i+1} \neq \emptyset$. E is a Galois equivalence relation over A. Moreover if $\langle \bar{a}_i : i < \theta \rangle$ are in different equivalence classes and θ is sufficiently big, we can extract a strictly indiscernible sequence from it which will witness that all elements are actually in the same class. Therefore $E \in \mathrm{SE}^\alpha(A)$.

Since $\bar{c} E_{\min,A,\alpha} \bar{d}$, we have that $\bar{c} E \bar{d}$ and without loss of generality $\bar{c} \neq \bar{d}$. Let $\langle \mathbf{I}_i : i < n \rangle$ witness equivalence. By saturation, we can assume without loss of generality that $\mathbf{I}_i$ is in $|M|$ for all $i < n$. Now use the failure of explicit strong splitting to argue that $\mathrm{gtp}(b\bar{c}/A) = \mathrm{gtp}(b\bar{d}/A)$. $\square$

Lemma 5.8 (Toward uniqueness of non strong splitting)**.** Let $M \leq_{\mathbf{K}} N$ and let $A \subseteq |M|$. Assume that N is $(\aleph_1 + \mu)$-saturated over A and for every $\alpha < \mu$, $\bar{c} \in {}^\alpha|N|$, there is $\bar{d} \in {}^\alpha|M|$ such that $\bar{d} E_{\min,A,\alpha} \bar{c}$.

Let $p, q \in \mathrm{gS}(N)$ not $(< \mu)$-strongly split over A. If $p \restriction M = q \restriction M$, then $p \restriction B = q \restriction B$ for every $B \subseteq |N|$ with $|B| < \mu$.

Proof. Say $p = \mathrm{gtp}(a/N)$, $q = \mathrm{gtp}(b/N)$. Let $\bar{c} \in {}^{<\mu}|N|$. We want to see that $\mathrm{gtp}(a/\bar{c}) = \mathrm{gtp}(b/\bar{c})$. We will show that $\mathrm{gtp}(a\bar{c}/A) = \mathrm{gtp}(b\bar{c}/A)$. Pick $\bar{d}$ in M such that $\bar{c} E_{\min,A,\alpha} \bar{d}$. Then by Lemma 5.7, $\mathrm{gtp}(a\bar{c}/A) = \mathrm{gtp}(a\bar{d}/A)$. Since $p \restriction M = q \restriction M$, $\mathrm{gtp}(a\bar{d}/A) = \mathrm{gtp}(b\bar{d}/A)$. By Lemma 5.7 again, $\mathrm{gtp}(b\bar{d}/A) = \mathrm{gtp}(b\bar{c}/A)$. Combining these equalities, we get that $\mathrm{gtp}(a\bar{c}/A) = \mathrm{gtp}(b\bar{c}/A)$, as desired. $\square$

6. Dividing

Hypothesis 6.1. $\mathbf{K}$ is an AEC with a monster model.

The following notion generalizes first-order dividing and was introduced by Shelah [She99, Definition 4.8].

Definition 6.2. Let $A \subseteq B$, $p \in \mathrm{gS}(B)$. We say that *p divides over A* if there exists an infinite cardinal θ and a strictly indiscernible sequence $\langle \bar{b}_i : i < \theta \rangle$ over A as well as $\langle f_i : i < \theta \rangle$ automorphisms of $\mathfrak{C}$ fixing A such that $\bar{b}_0$ is an enumeration of B, $f_i(\bar{b}_0) = \bar{a}_i$ for all $i < \theta$, and $\langle f_i(p) : i < \theta \rangle$ is inconsistent.

It is clear from the definition that dividing induces an independence relation:

Definition 6.3. For $\lambda \geq \mathrm{LS}(\mathbf{K})$, we let $\mathfrak{i}_{\mathrm{div}}(\mathbf{K}_\lambda)$ be the independence relation whose underlying class is $\mathbf{K}'$ and whose independence relation is non-dividing.

The following fact about dividing was proven by Shelah in [She99, Claim 5.5.(2)]:

Fact 6.4. Let $\mu_1 \geq \mu_0 \geq \text{LS}(\mathbf{K})$. Let $\alpha < \mu_1^+$ be a regular cardinal. If $\mathbf{K}$ is stable in μ_1 and $\mu_1^\alpha > \mu_1$, then $\alpha \in \underline{\kappa}^{\text{wk}}(\mathbf{i}_{\text{div}}(\mathbf{K}_{\mu_0}))$ (recall Definition 2.4).

To see when strong splitting implies dividing, Shelah considered the following property:

Definition 6.5. $\mathbf{K}$ satisfies $(*)_{\mu,\theta,\sigma}$ if whenever $\langle \bar{a}_i : i < \delta \rangle$ is a strictly indiscernible sequence with $\ell(\bar{a}_i) < \mu$ for all $i < \delta$, then for any $\bar{b}$ with $\ell(\bar{b}) < \sigma$, there exists $u \subseteq \delta$ with $|u| < \theta$ such that for any $i, j \in \delta \backslash u$, $\text{gtp}(\bar{a}_i \bar{b}/\emptyset) = \text{gtp}(\bar{a}_j \bar{b}/\emptyset)$.

Fact 6.6 (Claim 4.12 in [She99])**.** Let $\mu^* := \text{LS}(\mathbf{K}) + \mu + \sigma$. If $\mathbf{K}$ does not have the μ^*-order property (recall Definition 2.2), then $(*)_{\mu^+,h(\mu^*),\sigma^+}$ holds.

Lemma 6.7. Let $A \subseteq B$. Let $p \in \text{gS}(B)$. Assume that $(*)_{|B|^+,\theta,\sigma}$ holds for some infinite cardinals θ and σ.

If p explicitly $|B|$-strongly splits over A, then p divides over A.

Proof. Let $\mu := |B|$. Let $\langle \bar{a}_i : i < \omega \rangle$ witness the explicit strong splitting (so $\ell(\bar{a}_i) = \mu$ for all $i < \omega$ and $\bar{a}_0 \in {}^\mu B$). Increase the indiscernible to assume without loss of generality that $\bar{a}_0$ enumerates B and increase further to get $\langle \bar{a}_i : i < \theta^+ \rangle$. Pick $\langle f_i : i < \theta^+ \rangle$ automorphisms of $\mathfrak{C}$ fixing A such that f_0 is the identity and $f_i(\bar{a}_0 \bar{a}_1) = \bar{a}_i \bar{a}_{i+1}$ for each $i < \theta^+$. We claim that $\langle \bar{a}_i : i < \theta^+ \rangle$, $\langle f_i : i < \theta^+ \rangle$ witness the dividing over A.

Indeed, suppose for a contradiction that b realizes $f_i(p)$ for each $i < \theta^+$. In particular, b realizes $f_0(p) = p$. By $(*)_{\mu^+,\theta,\sigma}$, there exists $i < \theta^+$ such that $\text{gtp}(b\bar{a}_i/A) = \text{gtp}(b\bar{a}_{i+1}/A)$. Applying f_i^{-1} to this equation, we get that $\text{gtp}(c\bar{a}_0/A) = \text{gtp}(c\bar{a}_1/A)$, where $c := f_i^{-1}(b)$. But since b realizes $f_i(p)$, c realizes p. This contradicts the strong splitting assumption. $\square$

We have arrived to the following result:

Lemma 6.8. Let $\mu_1 \geq \mu_0 \geq \text{LS}(\mathbf{K})$ be such that $\mathbf{K}$ is stable in both μ_0 and μ_1. Assume further that $\mathbf{K}$ does not have the μ_0-order property.

Let $\alpha < \mu_0^+$ be a regular cardinal. If $\mu_1^\alpha > \mu_1$, then:

$$\alpha \in \underline{\kappa}^{\text{wk}}(\mathbf{i}_{\mu_0\text{-strong-spl}}(\mathbf{K}_{\mu_0}), <^{\text{univ}}_{\mathbf{K}})$$

Proof. By Fact 6.6, $(*)_{\mu_0^+,h(\mu_0),\mu_0^+}$ holds.

Now let $\langle M_i : i < \alpha \rangle$ be $<_{\mathbf{K}}^{\text{univ}}$-increasing in $\mathbf{K}_\lambda$. Let $p \in \text{gS}(\bigcup_{i<\alpha} M_i)$. By Fact 6.4, there exists $i < \alpha$ such that $p \upharpoonright M_{i+1}$ does not divide over M_i. By Lemma 6.7, $p \upharpoonright M_{i+1}$ does not explicitly λ-strongly split over M_i. By Lemma 5.7 (recall that M_{i+1} is universal over M_i), $p \upharpoonright M_{i+1}$ does not λ-strongly split over M_i. $\square$

Our aim in the next section will be to show that non-strong splitting has weak uniqueness. This will allow us to apply the results of Section 3 and replace $\underline{\kappa}^{\text{wk}}$ by $\underline{\kappa}$.

7. Strong splitting in stable tame AECs

Hypothesis 7.1. $\mathbf{K}$ is an $\text{LS}(\mathbf{K})$-tame AEC with a monster model.

Why do we assume tameness? Because we would like to exploit the uniqueness of strong splitting (Lemma 5.8), but we want to be able to conclude $p = q$, and not just $p \upharpoonright B = q \upharpoonright B$ for every small B. This will allow us to use the tools of Section 3.

Definition 7.2. For $\mu \geq \text{LS}(\mathbf{K})$, let $\chi^*(\mu) \in [\mu^+, h(\mu)$ be the least cardinal χ^* such that whenever A has size at most μ and $\alpha < \mu^+$ then $c(E_{\min,A,\alpha}) < \chi^*$ (it exists by Lemma 4.10).

The following is technically different from the μ-forking defined in [Vas16b, Definition 4.2] (which uses μ-splitting), but it is patterned similarly.

Definition 7.3. For $p \in \text{gS}(B)$, we say that p *does not μ-fork over* (M_0, M) if:

(1) $M_0 \leq_{\mathbf{K}} M$, $|M| \subseteq B$.
(2) $M_0 \in \mathbf{K}_\mu$.
(3) M is $\chi^*(\mu)$-saturated over M_0.
(4) p does not μ-strongly split over M_0.

We say that p does not μ-fork over M if there exists M_0 such that p does not μ-fork over (M_0, M).

The basic properties are satisfied:

Lemma 7.4.

(1) (Invariance) For any automorphism f of $\mathfrak{C}$, $p \in \text{gS}(B)$ does not μ-fork over (M_0, M) if and only if $f(p)$ does not μ-fork over $(f[M_0], f[M])$.

(2) (Monotonicity) Let $M_0 \leq_{\mathbf{K}} M_0' \leq_{\mathbf{K}} M \leq_{\mathbf{K}} M'$, $|M'| \subseteq B$. Assume that $M_0, M_0' \in \mathbf{K}_\mu$ and M is $\chi^*(\mu)$-saturated over M_0'
Let $p \in \text{gS}(B)$ be such that p does not μ-fork over (M_0, M). Then:
  (a) p does not μ-fork over (M_0', M).
  (b) p does not μ-fork over (M_0, M').

Proof. Invariance is straightforward. We prove monotonicity. Assume that M_0, M_0', M, M', B, p are as in the statement. First we have to show that p does not μ-fork over (M_0', M). We know that p does not μ-strongly split over M_0. Since $M_0' \in \mathbf{K}_\mu$, Lemma 5.4 implies that p does not μ-strongly split over M_0', as desired.

Similarly, it follows directly from the definitions that p does not μ-fork over (M_0, M'). $\square$

This justfies the following definition:

Definition 7.5. For $\lambda \geq \text{LS}(\mathbf{K})$, we write $\mathbf{i}_{\mu\text{-forking}}(\mathbf{K}_\lambda)$ for the independence relation with class $\mathbf{K}_\lambda$ and independence relation induced by non-μ-forking.

We now want to show that under certain conditions $\mathbf{i}_{\mu\text{-forking}}(\mathbf{K}_\lambda)$ has weak uniqueness (see Definition 3.4). First, we show that when two types do not fork over the same sufficiently saturated model, then the "witness" M_0 to the μ-forking can be taken to be the same.

Lemma 7.6. Let M be $\chi^*(\mu)$-saturated. Let $|M| \subseteq B$. Let $p, q \in \text{gS}(B)$ and assume that both p and q do not μ-fork over M. Then there exists M_0 such that both p and q do not μ-fork over (M_0, M).

Proof. Say p does not fork over (M_p, M) and q does not fork over (M_q, M). Pick $M_0 \leq_{\mathbf{K}} M$ of size μ containing both M_p and M_q. This works since M is $\chi^*(\mu)$-saturated and $\chi^*(\mu) > \mu$. $\square$

Lemma 7.7. Let $\mu \geq \text{LS}(\mathbf{K})$. Let $M \in \mathbf{K}_{\geq \mu}$ and let B be a set with $|M| \subseteq B$. Let $p, q \in \text{gS}(B)$ and assume that $p \upharpoonright M = q \upharpoonright M$.

(1) (Uniqueness over χ^*-saturated models) If M is $\chi^*(\mu)$-saturated and p, q do not μ-fork over M, then $p = q$.
(2) (Weak uniqueness) Let $\lambda > \chi^*(\mu)$ be a stability cardinal. Let $M_0 \leq_{\mathbf{K}} M$ be such that $M_0, M \in \mathbf{K}_\lambda$ and M is universal over M_0. If p, q do not μ-fork over M_0, then $p = q$. In other words, $\mathbf{i}_{\mu\text{-forking}}(\mathbf{K}_\lambda)$ has weak uniqueness.

Proof.

(1) By Lemma 7.6, we can pick M_0 such that both p and q do not μ-fork over (M_0, M). By Lemma 5.8, $p = q$.
(2) Using stability, we can build $M' \in \mathbf{K}_\lambda$ that is $\chi^*(\mu)$-saturated with $M_0 \leq_{\mathbf{K}} M'$. Without loss of generality (using universality of M over M_0), $M' \leq_{\mathbf{K}} M$. By base monotonicity, both p and q do not μ-fork over M'. Since $p \upharpoonright M = q \upharpoonright M$, we also have that $p \upharpoonright M' = q \upharpoonright M'$. Now use the first part.

□

The following theorem is the main result of this section, so we repeat its global hypotheses here for convenience.

Theorem 7.8. Let $\mathbf{K}$ be an $\text{LS}(\mathbf{K})$-tame AEC with a monster model.

Let $\mu_0 \geq \text{LS}(\mathbf{K})$ be a stability cardinal. Let $\lambda > \chi^*(\mu_0)$ be another stability cardinal. For any $\mu_1 \geq \mu_0$, if $\mathbf{K}$ is stable in μ_1 then $\mu_1^{<\kappa(\mathbf{K}_\lambda, <^{\text{univ}}_{\mathbf{K}})} = \mu_1$ (recall Definition 3.13).

The proof uses several facts (recall from Hypothesis 7.1 that we are working inside the monster model of a tame AEC):

Fact 7.9 (Theorem 4.13 in [Vas16c])**.** The following are equivalent:

(1) $\mathbf{K}$ is stable in some $\mu \geq \text{LS}(\mathbf{K})$.
(2) $\mathbf{K}$ does not have the $\text{LS}(\mathbf{K})$-order property.

Fact 7.10 (Proposition 3.12 in [BGKV16])**.** For $M \leq_{\mathbf{K}} N$, $p \in \text{gS}(N)$, $\mu \in [\|M\|, \|N\|]$, p μ-splits over M if and only if p $\|M\|$-splits over M.

Fact 7.11 (Claim 3.3 in [She99])**.** Assume that $\mathbf{K}$ is stable in $\mu \geq \text{LS}(\mathbf{K})$. For any $M \in \mathbf{K}_{\geq \mu}$ and any $p \in \text{gS}(M)$, there exists $M_0 \leq_{\mathbf{K}} M$ with $M_0 \in \mathbf{K}_\mu$ such that p does not μ-split over M_0.

Proof of Theorem 7.8. We prove that for any regular cardinal $\alpha < \lambda^+$, if $\mu_1^\alpha > \mu_1$ then $\alpha \in \underline{\kappa}(\mathbf{K}_\lambda, <^{\text{univ}}_{\mathbf{K}})$. This suffices because the least cardinal α such that $\mu_1^\alpha > \mu_1$ is regular.

We will use without comments the fact proven in Section 3 that $\underline{\kappa}(\mathbf{K}_\lambda, <^{\text{univ}}_{\mathbf{K}}) \cap \text{REG}$ is an end segment. Note that by Lemma 7.7, $\mathfrak{i} := \mathfrak{i}_{\mu_0\text{-forking}}(\mathbf{K}_\lambda)$ has weak uniqueness and thus we can use the results from Section 3 also on $\mathfrak{i}$.

By Fact 7.10 and 7.11, $\mu_0^+ \in \underline{\kappa}(\mathbf{K}_\lambda, <^{\text{univ}}_{\mathbf{K}})$. Therefore we may assume that $\alpha < \mu_0^+$.

By Fact 7.9, $\mathbf{K}$ does not have the $\text{LS}(\mathbf{K})$-order property. By Lemma 6.8, $\alpha \in \underline{\kappa}^{\text{wk}}(\mathfrak{i}_{\mu_0\text{-strong-spl}}(\mathbf{K}_{\mu_0}), <^{\text{univ}}_{\mathbf{K}})$. As in [Vas16b, Section 4], this implies that $\alpha \in \underline{\kappa}^{\text{wk}}(\mathbf{i}_{\mu_0\text{-forking}}(\mathbf{K}_\lambda), <^{\text{univ}}_{\mathbf{K}})$. But by Corollary 3.12, this means that $\alpha \in \underline{\kappa}^{\text{wk}}(\mathbf{K}_\lambda, <^{\text{univ}}_{\mathbf{K}})$ which by Corollary 3.9 implies that $\alpha \in \underline{\kappa}(\mathbf{K}_\lambda, <^{\text{univ}}_{\mathbf{K}})$. $\square$

8. The stability spectrum of tame AECs

For an AEC $\mathbf{K}$ with a monster model, we define the *stability spectrum of* $\mathbf{K}$, $\text{Stab}(\mathbf{K})$ to be the class of cardinals $\mu \geq \text{LS}(\mathbf{K})$ such that $\mathbf{K}$ is stable in μ. We would like to study it assuming tameness. From earlier work, the following is known about $\text{Stab}(\mathbf{K})$ in tame AECs:

Fact 8.1. Let $\mathbf{K}$ be an $\text{LS}(\mathbf{K})$-tame AEC with a monster model.

(1) [Vas16c, Theorem 4.13] If $\text{Stab}(\mathbf{K}) \neq \emptyset$, then $\min(\text{Stab}(\mathbf{K})) < H_1$ (recall Notation 2.1).
(2) [GV06, Corollary 6.4][5] If $\mu \in \text{Stab}(\mathbf{K})$ and $\lambda = \lambda^\mu$, then $\lambda \in \text{Stab}(\mathbf{K})$.
(3) [BKV06, Theorem 1] If $\mu \in \text{Stab}(\mathbf{K})$, then $\mu^+ \in \text{Stab}(\mathbf{K})$.
(4) [Vas16b, Lemma 5.5] If $\langle \mu_i : i < \delta \rangle$ is strictly increasing in $\text{Stab}(\mathbf{K})$ and $\text{cf}(\delta) \in \underline{\kappa}(\mathbf{K}_{\mu_0}, <^{\text{univ}}_{\mathbf{K}})$, then $\sup_{i<\delta} \mu_i \in \text{Stab}(\mathbf{K})$.

Let us say that $\mathbf{K}$ is *stable* if $\text{Stab}(\mathbf{K}) \neq \emptyset$. In this case, it is natural to give a name to the first stability cardinal:

Definition 8.2. For $\mathbf{K}$ an AEC with a monster model, let $\lambda(\mathbf{K}) := \min(\text{Stab}(\mathbf{K}))$ (if $\text{Stab}(\mathbf{K}) = \emptyset$, let $\lambda(\mathbf{K}) := \infty$).

From Fact 8.1, if $\mathbf{K}$ is an $\text{LS}(\mathbf{K})$-tame AEC with a monster model, then $\lambda(\mathbf{K}) < \infty$ implies that $\lambda(\mathbf{K}) < H_1$.

It is natural to look at the sequence $\langle \kappa(\mathbf{K}_\mu, <^{\text{univ}}_{\mathbf{K}}) : \mu \in \text{Stab}(\mathbf{K}) \rangle$. From [Vas16b, Section 4], we have that:

Fact 8.3. Let $\mathbf{K}$ be an $\text{LS}(\mathbf{K})$-tame AEC with a monster model. If $\mu < \lambda$ are both in $\text{Stab}(\mathbf{K})$, then $\kappa(\mathbf{K}_\lambda, <^{\text{univ}}_{\mathbf{K}}) \leq \kappa(\mathbf{K}_\mu, <^{\text{univ}}_{\mathbf{K}})$.

Thus the sequence is decreasing and must stabilize somewhere. We let $\lambda'(\mathbf{K})$ be the first place where it stabilizes. One can think of it as the first "well-behaved" stability cardinal.

Definition 8.4. For $\mathbf{K}$ an $\text{LS}(\mathbf{K})$-tame AEC with a monster model, let $\lambda'(\mathbf{K})$ be the least stability cardinal λ such that $\kappa(\mathbf{K}_\lambda, <^{\text{univ}}_{\mathbf{K}}) \leq \kappa(\mathbf{K}_\mu, <^{\text{univ}}_{\mathbf{K}})$ for all $\mu \in \text{Stab}(\mathbf{K})$. When $\lambda(\mathbf{K}) = \infty$, we set $\lambda'(\mathbf{K}) = \infty$.

[5] Grossberg and VanDieren's proof shows that the assumption there that $\mu > H_1$ can be removed, see [Bal09, Theorem 12.10].

We do not know whether $\lambda'(\mathbf{K}) = \lambda(\mathbf{K})$, but we can show that $\lambda'(\mathbf{K}) < h(\lambda(\mathbf{K}))$ (see Theorem 8.8 below).

We also give a name to the value of the locality cardinal at $\lambda'(\mathbf{K})$.

Definition 8.5. For $\mathbf{K}$ an LS($\mathbf{K}$)-tame AEC with a monster model, let $\chi(\mathbf{K}) := \kappa(\mathbf{K}_{\lambda'(\mathbf{K})}, <^{\text{univ}}_{\mathbf{K}})$. Set $\chi(\mathbf{K}) = \infty$ if $\lambda'(\mathbf{K}) = \infty$.

We have the following characterization of $\chi(\mathbf{K})$.

Theorem 8.6. Let $\mathbf{K}$ be a stable LS($\mathbf{K}$)-tame AEC with a monster model.

$\chi(\mathbf{K})$ is the maximal cardinal χ such that for any $\mu \geq \text{LS}(\mathbf{K})$, if $\mathbf{K}$ is stable in μ then $\mu = \mu^{<\chi}$.

Proof. First, let $\mu \geq \text{LS}(\mathbf{K})$ be a stability cardinal. By Theorem 7.8, $\mu^{<\chi(\mathbf{K})} = \mu$.

Conversely, consider the cardinal $\mu := \beth_{\chi(\mathbf{K})}(\lambda'(\mathbf{K}))$. By Fact 8.1, $\mathbf{K}$ is stable in μ. However $\text{cf}(\mu) = \chi(\mathbf{K})$ so $\mu^{\chi(\mathbf{K})} > \mu$. In other words, there does not exist a cardinal $\chi > \chi(\mathbf{K})$ such that $\mu^{<\chi} = \mu$. $\square$

Remark 8.7. If $\mathbf{K}$ is the class of models of a first-order stable theory T ordered by $\preceq$, then $\chi(\mathbf{K}) = \kappa_r(T)$ (the least regular cardinal greater than or equal to $\kappa(T)$). Indeed, $\kappa(T) \leq \chi(\mathbf{K})$ by Theorem 8.6, so since $\chi(\mathbf{K})$ is regular, $\kappa_r(T) \leq \chi(\mathbf{K})$. Conversely, $\mu := \beth_{\kappa_r(T)}(|T|)$ is a stability cardinal such that $\mu^{\kappa_r(T)} > \mu$. Since we know that $\mu^{<\chi(\mathbf{K})} = \mu$, this shows that $\chi(\mathbf{K}) \leq \kappa_r(T)$.

We have the following inequalities:

Theorem 8.8. Let $\mathbf{K}$ be a stable LS($\mathbf{K}$)-tame AEC with a monster model.

(1) $\chi(\mathbf{K}) \leq \lambda(\mathbf{K}) < H_1$.
(2) $\lambda(\mathbf{K}) \leq \lambda'(\mathbf{K}) < h(\lambda(\mathbf{K})) < \beth_{H_1}$.

Proof.

(1) That $\lambda(\mathbf{K}) < H_1$ is Fact 8.1. Now by Theorem 8.6, $\lambda(\mathbf{K})^{<\chi(\mathbf{K})} = \lambda(\mathbf{K})$ and hence $\chi(\mathbf{K}) \leq \lambda(\mathbf{K})$.
(2) Let λ' be the least stability cardinal above $\chi^*(\lambda(\mathbf{K}))$ (see Definition 7.2). We have that $\lambda' < h(\lambda(\mathbf{K}))$. We claim that $\lambda'(\mathbf{K}) \leq \lambda'$. Indeed by Theorem 7.8, for any stability cardinal μ, we have that $\mu^{<\kappa(\mathbf{K}_{\lambda'}, <^{\text{univ}}_{\mathbf{K}})} = \mu$. We know that $\chi(\mathbf{K})$ is the maximal cardinal with that property, but on the other hand

we have that $\chi(\mathbf{K}) \le \kappa(\mathbf{K}_{\lambda'}, <^{\text{univ}}_{\mathbf{K}})$ by definition. We conclude that $\chi(\mathbf{K}) = \kappa(\mathbf{K}_{\lambda'}, <^{\text{univ}}_{\mathbf{K}})$, as desired.

$\square$

It is natural to ask whether a converse of Theorem 8.6 holds, i.e. is $\mathbf{K}$ stable in μ whenever $\mu = \mu^{<\chi}$? At present, the answer we can give is sensitive to cardinal arithmetic: Fact 8.1 does not give us enough tools to answer in ZFC. There is however a large class of cardinals on which there is no cardinal arithmetic problems. This is already implicit in [Vas16b, Section 5].

Definition 8.9. A cardinal μ is *θ-closed* if $\lambda^\theta < \mu$ for all $\lambda < \mu$. We say that μ is *almost θ-closed* if $\lambda^\theta \le \mu$ for all $\lambda < \mu$.

Lemma 8.10. Let $\mathbf{K}$ be an $\text{LS}(\mathbf{K})$-tame AEC with a monster model. If μ is almost $\lambda(\mathbf{K})$-closed, then either $\mathbf{K}$ is stable in μ or $\mathbf{K}$ is stable in unboundedly many cardinals below μ.

Proof. If $\mu^{\lambda(\mathbf{K})} = \mu$, then $\mathbf{K}$ is stable in μ by Fact 8.1.(2). Otherwise, μ is $\lambda(\mathbf{K})$-closed. Thus for any $\mu_0 < \mu$, $\mu_1 := \mu_0^{\lambda(\mathbf{K})}$ is such that $\mu_1 < \mu$ and $\mu_1^{\lambda(\mathbf{K})} = \mu_1$, hence $\mathbf{K}$ is stable in μ_1. Therefore $\mathbf{K}$ is stable in unboundedly many cardinals below μ. $\square$

Theorem 8.11. Let $\mathbf{K}$ be an $\text{LS}(\mathbf{K})$-tame AEC with a monster model. Let μ be almost $\lambda(\mathbf{K})$-closed. If $\mu = \mu^{<\chi(\mathbf{K})} + \lambda'(\mathbf{K})$, then $\mathbf{K}$ is stable in μ.

Proof. By Lemma 8.10, we may assume that $\mathbf{K}$ is stable in unboundedly many cardinals below μ. By König's theorem, $\text{cf}(\mu) \ge \chi(\mathbf{K})$. The result now follows from either Fact 8.1.(3) if μ is a successor or Fact 8.1.(4) if μ is limit. $\square$

Note that the class of almost $\lambda(\mathbf{K})$-closed cardinals forms a club, and on this class we have a complete characterization of stability: $\mathbf{K}$ is stable in $\mu \ge \lambda'(\mathbf{K})$ if and only if $\mu = \mu^{<\chi(\mathbf{K})}$. We do not know how to analyze the cardinals that are *not* almost $\lambda(\mathbf{K})$-closed in ZFC. Using hypotheses beyond ZFC, we can see that all big-enough cardinals are almost $\lambda(\mathbf{K})$-closed. For ease of notation, we define the following function:

Definition 8.12. For μ an infinite cardinal, $\theta(\mu)$ is the least cardinal θ such that any $\lambda \ge \theta$ is almost μ-closed. When such a θ does not exist, we write $\theta(\mu) = \infty$.

If λ is a strong limit cardinal, then $2^\lambda = \lambda^{\text{cf}(\lambda)}$ and so if $2^\lambda > \lambda^+$ we have that λ^+ is not almost $\text{cf}(\lambda)$-closed. Foreman and Woodin [FW91] have shown that it is consistent with ZFC and a large cardinal axiom that $2^\lambda > \lambda^+$ for all infinite cardinals λ. Therefore it is possible that $\theta(\aleph_0) = \infty$ (and hence $\theta(\mu) = \infty$ for any infinite cardinal μ). However, we have:

Fact 8.13. Let μ be an infinite cardinal.

(1) If SCH holds, then $\theta(\mu) = 2^\mu$.
(2) If $\kappa > \mu$ is strongly compact, then $\theta(\mu) \leq \kappa$.

Proof. The first fact follows from basic cardinal arithmetic (see [Jec03, Theorem 5.22]), and the third follows from a result of Solovay (see [Sol74] or [Jec03, Theorem 20.8]). $\square$

Corollary 8.14. Let $\mathbf{K}$ be an $\text{LS}(\mathbf{K})$-tame AEC with a monster model. For any $\mu \geq \lambda'(\mathbf{K}) + \theta(\lambda(\mathbf{K}))$, $\mathbf{K}$ is stable in μ if and only if $\mu = \mu^{<\chi(\mathbf{K})}$.

Proof. The left to right direction follows from Theorem 8.6 and the right to left direction is by Theorem 8.11 and the definition of $\theta(\lambda(\mathbf{K}))$. $\square$

We emphasize that for the right to left directions of Corollary 8.14 to be nontrivial, we need $\theta(\lambda(\mathbf{K})) < \infty$, which holds under various set-theoretic hypotheses by Fact 8.13. This is implicit in [Vas16b, Section 5]. The left to right direction is new and does not need the boundedness of $\theta(\lambda(\mathbf{K}))$ (Theorem 8.6).

A particular case of Theorem 8.6 derives superstability from stability in a tail of cardinals. The following concept is studied already in [She99, Lemma 6.3].

Definition 8.15. An AEC $\mathbf{K}$ is *μ-superstable* if:

(1) $\mu \geq \text{LS}(\mathbf{K})$.
(2) $\mathbf{K}_\mu$ is non-empty, has amalgamation, joint embedding, and no maximal models.
(3) $\mathbf{K}$ is stable in μ.
(4) $\underline{\kappa}(\mathbf{K}_\mu, <^{\text{univ}}_{\mathbf{K}})$ contains all the limit ordinals.

This definition has been well-studied and has numerous consequences in tame AECs, such as the existence of a well-behaved independence notion (a good frame), the union of a chain of λ-saturated being λ-saturated, or the uniqueness of limit models (see for example [GV] for a survey and history). Even though in tame AECs Definition 8.15

is (eventually) equivalent to all these consequences [GV], it was not known whether it followed from stability on a tail of cardinals (see [GV, Question 1.7]). We show here that it does (note that this is a ZFC result).

Corollary 8.16. Let $\mathbf{K}$ be an $\mathrm{LS}(\mathbf{K})$-tame AEC with a monster model. The following are equivalent.

(1) $\chi(\mathbf{K}) = \aleph_0$.
(2) There exists a stability cardinal $\mu \geq \mathrm{LS}(\mathbf{K})$ such that $\mu^{\aleph_0} > \mu$.
(3) $\mathbf{K}$ is $\lambda'(\mathbf{K})$-superstable.

The proof uses that μ-superstability implies stability in every $\mu' \geq \mu$ (this is a straightforward induction using Fact 8.1, see [Vas16b, Theorem 5.6]). We state a slightly stronger version:

Fact 8.17 (Proposition 10.10 in [Vas16a])**.** Let $\mathbf{K}$ be a μ-tame AEC with amalgamation. If $\mathbf{K}$ is μ-superstable, then $\mathbf{K}$ is μ'-superstable for every $\mu' \geq \mu$.

Proof of Corollary 8.16. If (1) holds, then (3) holds by definition of $\chi(\mathbf{K})$ and Corollary 3.9. By Fact 8.17, this implies stability in every $\mu \geq \lambda'(\mathbf{K})$, and in particular (2). Now if (2) holds then by Theorem 8.6 we must have that $\chi(\mathbf{K}) = \aleph_0$, so (1) holds. $\square$

Corollary 8.16 also partially answers [GV, Question 1.8], which asked whether the least μ such that $\mathbf{K}$ is μ-superstable must satisfy $\mu < H_1$. We know now that $\mu \leq \lambda'(\mathbf{K}) < \beth_{H_1}$, so there is a Hanf number for superstability but whether it is H_1 (rather than $\beth_{H_1}$) remains open.

Corollary 8.16 and the author's earlier work with Grossberg [GV] justifies the following definition for tame AECs:

Definition 8.18. Let $\mathbf{K}$ be an $\mathrm{LS}(\mathbf{K})$-tame AEC with a monster model. We say that $\mathbf{K}$ is *superstable* if $\chi(\mathbf{K}) = \aleph_0$.

9. Limit and saturated models

In this section, we aim to study the behavior of unions of chains of saturated models and the uniqueness of limit models. The following two results will be used[6]:

[6]The statement of the first result in [BVb] only gives that $\mu_0 < h(\mathrm{LS}(\mathbf{K})^+)$, but this stems from an imprecision in an early version of [Vas16c, Theorem 4.13] that has now been fixed.

Fact 9.1 (Theorem 6.10 in [BVb])**.** Let $\mathbf{K}$ be an $\mathrm{LS}(\mathbf{K})$-tame AEC with a monster model.

There exists a cardinal $\mu_0 < H_1$ such that for any $\lambda > \lambda_0 \geq \mu_0$, if:

(1) $\mathbf{K}$ is stable in unboundedly many cardinals below λ.
(2) $\mathbf{K}$ is stable in λ_0 and $\delta \in \underline{\kappa}(\mathbf{K}_{\lambda_0}, <^{\mathrm{univ}}_{\mathbf{K}})$

then whenever $\langle M_i : i < \delta \rangle$ is an increasing chain of λ-saturated models, we have that $\bigcup_{i<\delta} M_i$ is λ-saturated.

Fact 9.2 (Theorem 1 in [BVa])**.** Let $\mathbf{K}$ be an AEC and let $\mu \geq \mathrm{LS}(\mathbf{K})$. Assume that $\mathbf{K}_\mu$ has amalgamation, joint embedding, no maximal models, and is stable in μ. If:

(1) $\delta \in \underline{\kappa}(\mathbf{K}_\mu, <^{\mathrm{univ}}_{\mathbf{K}}) \cap \mu^+ \cap \mathrm{REG}$.
(2) All limit ordinals are in $\underline{\kappa}^{\mathrm{cont}}(\mathbf{K}_\mu, <^{\mathrm{univ}}_{\mathbf{K}})$.
(3) $\mathbf{K}$ has (μ, δ)-symmetry.

Then whenever $M_0, M_1, M_2 \in \mathbf{K}_\mu$ are such that both M_1 and M_2 are $(\mu, \geq \delta)$-limit over M_0 (recall Section 2.4), we have that $M_1 \cong_{M_0} M_2$.

We will not need to use the definition of (μ, δ)-symmetry, only the following fact, which combines [BVa, Theorem 18] and the proof of [Van, Theorem 2].

Fact 9.3. Let $\mathbf{K}$ be an AEC and let $\mu \geq \mathrm{LS}(\mathbf{K})$. Assume that $\mathbf{K}_\mu$ has amalgamation, joint embedding, no maximal models, and is stable in μ. Let $\delta < \mu^+$ be a regular cardinal. If whenever $\langle M_i : i < \delta \rangle$ is an increasing chain of saturated models in $\mathbf{K}_{\mu^+}$ we have that $\bigcup_{i<\delta} M_i$ is saturated, then $\mathbf{K}$ has (μ, δ)-symmetry.

We can conclude that in tame stable AECs, any two big-enough ($\geq \chi(\mathbf{K})$)-limits are isomorphic.

Theorem 9.4. Let $\mathbf{K}$ be an $\mathrm{LS}(\mathbf{K})$-tame AEC with a monster model.

There exists $\mu_0 < H_1$ such that for any stability cardinal $\mu \geq \lambda'(\mathbf{K}) + \mu_0$ and any $M_0, M_1, M_2 \in \mathbf{K}_\mu$, if both M_1 and M_2 are $(\mu, \geq \chi(\mathbf{K}))$-limit over M_0, then $M_1 \cong_{M_0} M_2$.

Proof. Let μ_0 be as given by Fact 9.1. Let μ be as in the statement of the theorem. By Fact 9.1 (where λ_0, λ there stand for μ, μ^+ here), we have that the union of an increasing chain of saturated models in $\mathbf{K}_{\mu^+}$ of length $\chi(\mathbf{K})$ is saturated. Therefore by Fact 9.3, $\mathbf{K}$ has $(\mu, \chi(\mathbf{K}))$-symmetry. Now apply Fact 9.2 (together with Theorem 3.7). $\square$

For ease of notation in later results, we define another invariant of $\mathbf{K}$:

Definition 9.5. For $\mathbf{K}$ an $\mathrm{LS}(\mathbf{K})$-tame AEC with a monster model, let $\lambda''(\mathbf{K})$ be the least stability cardinal $\lambda'' \geq \lambda'(\mathbf{K})$ such that for any stability cardinal $\mu \geq \lambda''$ and any $M_0, M_1, M_2 \in \mathbf{K}_\mu$, if both M_1 and M_2 are $(\mu, \geq \chi(\mathbf{K}))$-limit over M_0, then $M_1 \cong_{M_0} M_2$. If $\mathbf{K}$ is not stable, let $\lambda''(\mathbf{K}) := \infty$.

From Theorem 9.4, we have that there exists $\mu_0 < H_1$ such that $\lambda''(\mathbf{K}) = \lambda'(\mathbf{K}) + \mu_0$ but we do not know whether $\lambda'(\mathbf{K}) = \lambda''(\mathbf{K})$.

Corollary 9.6. Let $\mathbf{K}$ be an $\mathrm{LS}(\mathbf{K})$-tame AEC with a monster model. For any stability cardinal $\mu \geq \lambda''(\mathbf{K})$, there is a saturated model of cardinality μ.

Proof. There is a $(\mu, \chi(\mathbf{K}))$-limit model of cardinality μ, and it is saturated by Theorem 9.4. $\square$

In Fact 9.1, it is open whether Hypothesis (1) can be removed. We aim to show that it can, assuming SCH. We first revisit an argument of VanDieren [Van16] to show that one can assume stability in λ instead of stability in unboundedly many cardinals below λ.

Lemma 9.7. Let $\mathbf{K}$ be an $\mathrm{LS}(\mathbf{K})$-tame AEC with a monster model. Let $\mu > \mathrm{LS}(\mathbf{K})$. Assume that $\mathbf{K}$ is stable in both $\mathrm{LS}(\mathbf{K})$ and μ. Let $\langle M_i : i < \delta \rangle$ be an increasing chain of μ-saturated models. If $\mathrm{cf}(\delta) \in \underline{\kappa}(\mathbf{K}_{\mathrm{LS}(\mathbf{K})}, <^{\mathrm{univ}}_{\mathbf{K}})$ and any two $(\mu, \geq \mathrm{cf}(\delta))$-limit models are isomorphic, then $\bigcup_{i<\delta} M_i$ is μ-saturated.

Let us say a little bit about the argument. VanDieren [Van16] shows that superstability in λ and $\mu := \lambda^+$ combined with the uniqueness of limit models in λ^+ implies that unions of chains of λ^+-saturated models are λ^+-saturated. One can use VanDieren's argument to prove that superstability in unboundedly-many cardinals below μ implies that unions of chains of μ-saturated models are μ-saturated, and this generalizes to the stable case too. However the case that interests us here is when $\mathbf{K}$ is stable in μ and not necessarily in unboundedly many cardinals below (the reader can think of μ as being the successor of a singular cardinal of low cofinality). This is where tameness enters the picture: by assuming stability e.g. in $\mathrm{LS}(\mathbf{K})$ as well as $\mathrm{LS}(\mathbf{K})$-tameness, we can transfer the locality of splitting upward and the main idea of VanDieren's argument carries through. Still several details have to be provided, so a full proof is given here.

Proof of Lemma 9.7. For $M_0 \leq_{\mathbf{K}} M \leq_{\mathbf{K}} N$, let us say that $p \in \mathrm{gS}(N)$ *does not fork over* (M_0, M) if M is $\|M_0\|^+$-saturated over M_0 (recall Definition 5.6) and $M_0 \in \mathbf{K}_{\mathrm{LS}(\mathbf{K})}$. Say that p does not fork over M if there exists M_0 so that it does not fork over (M_0, M).

Without loss of generality, $\delta = \mathrm{cf}(\delta) < \mu$. Note that the uniqueness of limit models tells us that the $(\geq \delta, \mu)$-limit model is saturated. Let $M_\delta := \bigcup_{i<\delta} M_i$. Let $N \leq_{\mathbf{K}} M_\delta$ with $N \in \mathbf{K}_{<\mu}$. Let $p \in \mathrm{gS}(N)$. We want to see that p is realized in M_δ. We may assume without loss of generality that $M_i \in \mathbf{K}_\mu$ for all $i \leq \delta$. Let $q \in \mathrm{gS}(M_\delta)$ be an extension of p.

Since $\delta \in \underline{\kappa}(\mathbf{K}_{\mathrm{LS}(\mathbf{K})}, <^{\mathrm{univ}}_{\mathbf{K}})$, using [Vas16b, Section 4] there exists $i < \delta$ such that q does not fork over M_i. This means there exists $M_i^0 \leq_{\mathbf{K}} M_i$ such that $M_i^0 \in \mathbf{K}_{\mathrm{LS}(\mathbf{K})}$ and q does not fork over (M_i^0, M_i). Without loss of generality, $i = 0$. Let $\mu_0 := \mathrm{LS}(\mathbf{K}) + \delta$. Build $\langle N_i : i \leq \delta \rangle$ increasing continuous in $\mathbf{K}_{\mu_0}$ such that $M_0^0 \leq_{\mathbf{K}} N_0$, $N \leq_{\mathbf{K}} N_\delta$, and for all $i \leq \delta$, $N_i \leq_{\mathbf{K}} M_i$. Without loss of generality, $N = N_\delta$.

We build an increasing continuous directed system $\langle M_i^*, f_{i,j} : i \leq j < \delta \rangle$ such that for all $i < \delta$:

(1) $M_i^* \in \mathbf{K}_\mu$.
(2) $N_i \leq_{\mathbf{K}} M_i^* \leq_{\mathbf{K}} M_i$.
(3) $f_{i,i+1}$ fixes N_i.
(4) M_{i+1}^* is universal over M_i^*.

This is possible. Take $M_0^* := M_0$. At i limit, take M_i^{**} to be the a direct limit of the system fixing N_i and let $g : M_i^{**} \xrightarrow[N_i]{} M_i$ (remember that M_i is saturated). Let $M_i^* := g[M_i^{**}]$, and define the $f_{j,i}$'s accordingly. At successors, proceed similarly and define the $f_{i,j}$'s in the natural way.

This is enough. Let $(M_\delta^*, f_{i,\delta})_{i<\delta}$ be a direct limit of the system extending N_δ (note: we do *not* know that $M_\delta^* \leq_{\mathbf{K}} M_\delta$). We have that M_δ^* is a (μ, δ)-limit model, hence is saturated. Now find a saturated $\mathfrak{C} \in \mathbf{K}_\mu$ containing $M_\delta \cup M_\delta^*$ and such that for each $i < \delta$, there exists $f_{i,\delta}^*$ an automorphism of $\mathfrak{C}$ extending $f_{i,\delta}$ such that $f_{i,\delta}^*[N_\delta] \leq_{\mathbf{K}} M_\delta^*$. This is possible since M_δ^* is universal over M_i^* for each $i < \delta$. Let $N^* \leq_{\mathbf{K}} M_\delta^*$ be such that $N^* \in \mathbf{K}_{\mu_0}$ and $|N_\delta| \cup \bigcup_{i<\delta} |f_{i,\delta}^*[N_\delta]| \subseteq |N^*|$.

<u>Claim</u>: For any saturated $\widehat{M} \in \mathbf{K}_\mu$ with $M_\delta \leq_{\mathbf{K}} \widehat{M}$, there exists $\hat{q} \in \mathrm{gS}(\widehat{M})$ extending q and not forking over (M_0^0, N_0).

<u>Proof of Claim</u>: We know that M_0 is saturated. Thus there exists $f : M_0 \cong_{N_0} \widehat{M}$. Let $\hat{q} := f(q \restriction M_0)$. We have that $\hat{q} \in \mathrm{gS}(\widehat{M})$ and $\hat{q}$

does not fork over (M_0^0, N_0). Further, $\hat{q} \upharpoonright N_0 = q \upharpoonright N_0$. By uniqueness of nonforking (see [Vas16b, Lemma 5.3]), $\hat{q} \upharpoonright M = q$. $\dagger_{\text{Claim}}$

By the claim, there exists $\hat{q} \in \text{gS}(\mathfrak{C})$ extending q and not forking over (M_0^0, N_0). Because M_δ^* is (μ_0^+, μ)-limit, there exists $M^{**} \in \mathbf{K}_\mu$ saturated such that $N^* \leq_{\mathbf{K}} M^{**} \leq_{\mathbf{K}} M_\delta^*$ and M_δ^* is universal over M^{**}.

Since M_δ^* is universal over M^{**}, there is $b^* \in M_\delta^*$ realizing $\hat{q} \upharpoonright M^{**}$. Fix $i < \delta$ and $b \in M_i^*$ such that $f_{i,\delta}(b) = b^*$. We claim that b realizes p (this is enough since by construction $M_i^* \leq_{\mathbf{K}} M_i \leq_{\mathbf{K}} M_\delta$). We show a stronger statement: b realizes $\hat{q} \upharpoonright M'$, where $M' := (f_{i,\delta}^*)^{-1}[M^{**}]$. This is stronger because $N^* \leq_{\mathbf{K}} M^{**}$ so by definition of N^*, $N \leq_{\mathbf{K}} (f_{i,\delta}^*)^{-1}[N^*] \leq_{\mathbf{K}} M'$. Work inside $\mathfrak{C}$. Since $\hat{q}$ does not fork over (M_0^0, N_0), also $\hat{q} \upharpoonright M^{**} = \text{gtp}(b^*/M^{**})$ does not fork over (M_0^0, N_0). Therefore $\text{gtp}(b/M')$ does not fork over (M_0^0, N_0). Moreover, $\text{gtp}(b/N_0) = \text{gtp}(b^*/N_0) = \hat{q} \upharpoonright N_0$, since $f_{i,\delta}$ fixes N_0. By uniqueness, $\text{gtp}(b/M') = \hat{q} \upharpoonright M'$. In other words, b realizes $\hat{q} \upharpoonright M'$, as desired. $\square$

Remark 9.8. It is enough to assume that amalgamation and the other structural properties hold only in $\mathbf{K}_{[\text{LS}(\mathbf{K}),\mu]}$.

We have arrived to the second main result of this section. Note that the second case below is already known (Fact 9.1), but the others are new.

Theorem 9.9. Let $\mathbf{K}$ be an $\text{LS}(\mathbf{K})$-tame AEC with a monster model. Let $\lambda > \lambda''(\mathbf{K})$ and let $\langle M_i : i < \delta \rangle$ be an increasing chain of λ-saturated models. If $\text{cf}(\delta) \geq \chi(\mathbf{K})$, then $\bigcup_{i<\delta} M_i$ is λ-saturated provided that at least one of the following conditions hold:

(1) $\mathbf{K}$ is stable in λ.
(2) $\mathbf{K}$ is stable in unboundedly many cardinals below λ.
(3) $\lambda \geq \theta(\lambda(\mathbf{K}))$ (recall Definition 8.12).
(4) SCH holds and $\lambda \geq 2^{\lambda(\mathbf{K})}$.

Proof.

(1) We check that the hypotheses of Lemma 9.7 hold, with $\mathbf{K}, \mu$ there standing for $\mathbf{K}_{\geq \lambda''(\mathbf{K})}$, λ here. By definition and assumption, $\mathbf{K}$ is stable in both $\lambda''(\mathbf{K})$ and λ. Furthermore, $\text{cf}(\delta) \in \underline{\kappa}(\mathbf{K}_{\lambda''(\mathbf{K})}, <^{\text{univ}}_{\mathbf{K}})$ by definition of $\chi(\mathbf{K})$ and the fact that $\lambda''(\mathbf{K}) \geq \lambda'(\mathbf{K})$. Finally, any two $(\mu, \geq \text{cf}(\delta))$-limit models are isomorphic by definition of $\lambda''(\mathbf{K})$.
(2) If λ is a successor, then $\mathbf{K}$ is also stable in λ by Fact 8.1.(3) so we can use the first part. If λ is limit, then we can use the

first part with each stability cardinal $\mu \in (\lambda''(\mathbf{K}), \lambda)$ to see that the union of the chain is μ-saturated. As λ is limit, this implies that the union of the chain is λ-saturated.
(3) By definition of $\theta(\lambda(\mathbf{K}))$, λ is almost $\lambda(\mathbf{K})$-closed. By Lemma 8.10, either $\mathbf{K}$ is stable in λ or $\mathbf{K}$ is stable in unboundedly many cardinals below λ, so the result follows from the previous parts.
(4) This is a special case of the previous part, see Fact 8.13.

□

10. The saturation spectrum

Corollary 9.6 shows that there is a saturated model in all high-enough stability cardinals. It is natural to ask whether the converse is true, as in the first-order case. We show here that this holds assuming SCH, but prove several ZFC results along the way. Some of the proofs are inspired from the ones in homogeneous model theory (due to Shelah [She75], see also the exposition in [GL02]).

The following is standard and will be used without comments.

Fact 10.1. Let $\mathbf{K}$ be an AEC with a monster model. If $\mathrm{LS}(\mathbf{K}) < \mu \leq \lambda = \lambda^{<\mu}$, then $\mathbf{K}$ has a μ-saturated model of cardinality λ.

In particular, $\mathbf{K}$ has a saturated model in λ if $\lambda = \lambda^{<\lambda}$. We turn to studying what we can say about λ when $\mathbf{K}$ has a saturated model in λ.

Theorem 10.2. Let $\mathbf{K}$ be an $\mathrm{LS}(\mathbf{K})$-tame AEC with a monster model. Let $\mathrm{LS}(\mathbf{K}) < \lambda$. If $\mathbf{K}$ has a saturated model of cardinality λ and $\mathbf{K}$ is stable in unboundedly many cardinals below λ, then $\mathbf{K}$ is stable in λ.

Proof. By Fact 8.1.(3), we can assume without loss of generality that λ is a limit cardinal. Let $\delta := \mathrm{cf}(\lambda)$. Pick $\langle \lambda_i : i \leq \delta \rangle$ strictly increasing continuous such that $\lambda_\delta = \lambda$, $\lambda_0 \geq \mathrm{LS}(\mathbf{K})$, and $i < \delta$ implies that $\mathbf{K}$ is stable in λ_{i+1}. Let $M \in \mathbf{K}_\lambda$ be saturated and let $\langle M_i : i \leq \delta \rangle$ be an increasing continuous resolution of M such that for each $i < \delta$, $M_i \in \mathbf{K}_{\lambda_i}$ and M_{i+2} is universal over M_{i+1}.

<u>Claim</u>: For any $p \in \mathrm{gS}(M)$, there exists $i < \delta$ such that p does not λ-split over M_i.

<u>Proof of Claim</u>: If $\delta > \lambda_1$, then the result follows from Facts 7.10 and 7.11, so assume that $\delta \leq \lambda_1$. In particular, $\delta < \lambda$. Assume for a contradiction that $p \in \mathrm{gS}(M)$ is such that p λ-splits over M_i for every $i < \delta$. Then for every $i < \delta$ there exists N_1^i, N_2^i, f_i such that

$M_i \leq_{\mathbf{K}} N_\ell^i \leq_{\mathbf{K}} M$, $\ell = 1, 2$, $f_i : N_1^i \cong_{M_i} N_2^i$, and $f_i(p \upharpoonright N_1^i) \neq p \upharpoonright N_2^i$. By tameness, there exists $M_1^i \leq_{\mathbf{K}} N_1^i, M_2^i \leq_{\mathbf{K}} N_2^i$ both in $\mathbf{K}_{\text{LS}(\mathbf{K})}$ such that $f_i[M_1^i] = M_2^i$ and $f_i(p \upharpoonright M_1^i) \neq p \upharpoonright M_2^i$.

Let $N \leq_{\mathbf{K}} M$ have size $\mu := \text{LS}(\mathbf{K}) + \delta$ and be such that $M_\ell^i \leq_{\mathbf{K}} N$ for $\ell = 1, 2$ and $i < \delta$.

We have that $\mu < \lambda$ so since M is saturated, there exists $b \in |M|$ realizing $p \upharpoonright N$. Let $i < \delta$ be such that $b \in |M_i|$. By construction, we have that $f_i(p \upharpoonright M_1^i) \neq p \upharpoonright M_2^i$ but on the other hand $p \upharpoonright M_\ell^i = \text{gtp}(b/M_\ell^i; M)$ and $f_i(p \upharpoonright M_1^i) = \text{gtp}(b/M_2^i; M)$, since $f_i(b) = b$ (it fixes M_i). This is a contradiction. $\dagger_{\text{Claim}}$

Now assume for a contradiction that $\mathbf{K}$ is not stable in λ and let $\langle p_i : i < \lambda^+ \rangle$ be distinct members of $\text{gS}(M)$ (the saturated model must witness instability because it is universal). By the claim, for each $i < \lambda^+$ there exists $j_i < \delta$ such that p does not λ-split over M_{j_i}. By the pigeonhole principle, without loss of generality $j_i = j_0$ for each $i < \lambda^+$. Now $|\text{gS}(M_{j_0})| \leq |\text{gS}(M_{j_0+2})| = \|M_{j_0+2}\| < \lambda$, so by the pigeonhole principle again, without loss of generality $p_i \upharpoonright M_{j_0+2} = p_j \upharpoonright M_{j_0+2}$ for all $i < j < \lambda^+$. By weak uniqueness of non-λ-splitting and tameness, this implies that $p_i = p_j$, a contradiction. $\square$

We can also prove in ZFC that existence of a saturated model at a cardinal $\lambda < \lambda^{<\lambda}$ implies that the class is stable. We first recall the definition of another locality cardinal:

Definition 10.3 (Definition 4.4 in [GV06])**.** For $\mathbf{K}$ a $\text{LS}(\mathbf{K})$-tame AEC with a monster model, define $\bar{\mu}(\mathbf{K})$ to be the least cardinal $\mu > \text{LS}(\mathbf{K})$ such that for any $M \in \mathbf{K}$ and any $p \in \text{gS}(M)$, there exists $M_0 \in \mathbf{K}_{<\mu}$ with $M_0 \leq_{\mathbf{K}} M$ such that p does not $\|M_0\|$-split over M_0. Set $\bar{\mu}(\mathbf{K}) = \infty$ if there is no such cardinal.

We have that stability is equivalent to boundedness of $\bar{\mu}(\mathbf{K})$:

Theorem 10.4. Let $\mathbf{K}$ be an $\text{LS}(\mathbf{K})$-tame AEC with a monster model. The following are equivalent:

(1) $\mathbf{K}$ is stable.
(2) $\bar{\mu}(\mathbf{K}) < H_1$.
(3) $\bar{\mu}(\mathbf{K}) < \infty$.

Proof. (1) implies (2) is because by Fact 7.11, $\bar{\mu}(\mathbf{K}) \leq \lambda(\mathbf{K})^+$ and by Fact 8.1.(1), $\lambda(\mathbf{K})^+ < H_1$. (2) implies (3) is trivial. To see that (3) implies (1), let $\mu := \bar{\mu}(\mathbf{K})$. Pick any $\lambda_0 \geq \text{LS}(\mathbf{K})$ such that $\lambda_0 = \lambda_0^{<\mu}$ (e.g. $\lambda_0 = 2^\mu$), and pick any $\lambda > \lambda_0$ such that $\lambda^{\lambda_0} = \lambda$ (e.g. $\lambda =$

2^{λ_0}). We claim that $\mathbf{K}$ is stable in λ. Let $M \in \mathbf{K}_\lambda$, and extend it to $M' \in \mathbf{K}_\lambda$ that is μ-saturated. It is enough to see that $|\mathrm{gS}(M')| = \lambda$, so without loss of generality $M = M'$. Suppose that $|\mathrm{gS}(M)| > \lambda$ and let $\langle p_i : i < \lambda^+ \rangle$ be distinct members. By definition of μ, for each $i < \lambda^+$ there exists $M_i \in \mathbf{K}_{<\mu}$ such that $M_i \leq_{\mathbf{K}} M$ and p does not $\|M_i\|$-split over M_i. Since $\lambda = \lambda^{<\mu}$, we can assume without loss of generality that $M_i = M_0$ for all $i < \lambda^+$. Further, $|\mathrm{gS}(M_0)| \leq 2^{<\mu} \leq \lambda_0^{<\mu} = \lambda_0$, so we can pick $M_0' \leq_{\mathbf{K}} M$ with $M_0' \in \mathbf{K}_{\lambda_0}$ such that M_0' is universal over M_0. As $\lambda = \lambda^{\lambda_0}$, we can assume without loss of generality that $p_i \upharpoonright M_0' = p_j \upharpoonright M_0'$. By tameness and weak uniqueness of non-splitting, we conclude that $p_i = p_j$, a contradiction. $\square$

We will use that failure of local character of splitting allows us to build a tree of types, see the proof of [GV06, Fact 4.6].

Fact 10.5. Let $\mathbf{K}$ be an $\mathrm{LS}(\mathbf{K})$-tame AEC with a monster model. Let $\mathrm{LS}(\mathbf{K}) < \lambda$. If $\bar{\mu}(\mathbf{K}) > \lambda$, then there exists an increasing continuous tree $\langle M_\eta : \eta \in {}^{\leq\lambda}2 \rangle$, and tree of types $\langle p_\eta : \eta \in {}^{\leq\lambda}2 \rangle$, and sets $\langle A_\eta : \eta \in {}^{\leq\lambda}2 \rangle$ such that for all $\eta \in {}^{<\lambda}2$:

(1) $M_\eta \in \mathbf{K}_{<\lambda}$.
(2) $p_\eta \in \mathrm{gS}(M_\eta)$.
(3) $A_\eta \subseteq |M_{\eta\frown 0}| \cap |M_{\eta\frown 1}|$.
(4) $|A_\eta| \leq \mathrm{LS}(\mathbf{K})$.
(5) $p_{\eta\frown 0} \upharpoonright A_\eta \neq p_{\eta\frown 1} \upharpoonright A_\eta$.

Theorem 10.6. Let $\mathbf{K}$ be an $\mathrm{LS}(\mathbf{K})$-tame AEC with a monster model. Let $\mathrm{LS}(\mathbf{K}) < \lambda < \lambda^{<\lambda}$. If $\mathbf{K}$ has a saturated model of cardinality λ, then $\bar{\mu}(\mathbf{K}) \leq \lambda$. In particular, $\mathbf{K}$ is stable.

Proof. The last sentence is Theorem 10.4. Now suppose for a contradiction that $\bar{\mu}(\mathbf{K}) > \lambda$. Let $\langle M_\eta : \eta \in {}^{\leq\lambda}2 \rangle$, $\langle p_\eta : \eta \in {}^{\leq\lambda}2 \rangle$, and $\langle A_\eta : \eta \in {}^{\leq\lambda}2 \rangle$ be as given by Fact 10.5. Without loss of generality, $M_\eta \leq_{\mathbf{K}} M$ for each $\eta \in {}^{<\lambda}2$.

Since M is saturated, it realizes all types over M_η, for each $\eta \in {}^{<\lambda}2$. In particular, $2^{<\lambda} = \lambda$. If there exists $\mu < \lambda$ such that $2^\mu = \lambda$, then $2^{\mu'} = \lambda$ for all $\mu' \in [\mu, \lambda)$, hence $\lambda = \lambda^{<\lambda}$. Therefore λ is strong limit. Since $\lambda < \lambda^{<\lambda}$, this implies that λ is singular. Let $\delta := \mathrm{cf}(\lambda)$ and fix $\langle M_i : i \leq \delta \rangle$ an increasing continuous resolution of M such that $M_i \in \mathbf{K}_{<\lambda}$ for all $i < \delta$.

We build $\langle C_i : i < \delta \rangle$, $\langle \eta_i : i < \delta \rangle$ such that for all $i < \delta$:

(1) $|C_i| \leq \mathrm{LS}(\mathbf{K})$.

(2) $i < j < \delta$ implies η_j extends η_i.
(3) $p_{\eta_{i+1}} \upharpoonright C_i$ is omitted in M_i.

This is enough: Let $\eta := \bigcup_{i<\delta} \eta_i$ and let $C := \bigcup_{i<\delta} C_i$. We have that $p_\eta \upharpoonright C$ is not realized in M_i for any $i < \delta$, hence it is omitted in M. This contradicts saturation of M.

This is possible: Let η_0 be the empty sequence. At limits, take unions. Now assume that M_{η_i} has been built. Let $\lambda_i := \|M_i\|$ and let $\theta := \left(2^{\lambda_i}\right)^+$. Since λ is strong limit, $\theta < \lambda$. For $\alpha < \theta$, let $\tau_\alpha := \eta_i \frown 0_\alpha$, where 0_α denotes a sequence of zeroes of order type α. By the Erdős-Rado theorem, there exists $\alpha < \beta < \theta$ such that for any $c \in |M_i|$, c realizes $p_{\tau_\alpha} \upharpoonright A_{\tau_\alpha}$ if and only if c realizes $p_{\tau_\beta} \upharpoonright A_{\tau_\beta}$. So let $C_i := A_{\tau_\alpha} \cup A_{\tau_\beta}$ and $\eta_{i+1} := \tau_\alpha \frown 1$. Then by construction $p_{\eta_{i+1}} \upharpoonright C_i$ is omitted in M_i. $\square$

We have arrived to the following. Note that we need some set-theoretic hypotheses (e.g. assuming SCH, $\theta(H_1) = 2^{H_1}$, see Fact 8.13) to get that $\theta(H_1) < \infty$ otherwise the result holds vacuously.

Corollary 10.7. Let $\mathbf{K}$ be an LS($\mathbf{K}$)-tame AEC with a monster model. Let $\lambda \geq \beth_{H_1} + \theta(H_1)$ (recall Definition 8.12). The following are equivalent:

(1) $\mathbf{K}$ has a saturated model of cardinality λ.
(2) $\lambda = \lambda^{<\lambda}$ or $\mathbf{K}$ is stable in λ.

Proof. (2) implies (1) is Fact 10.1 and Corollary 9.6. Now assume (1) and $\lambda < \lambda^{<\lambda}$. By Theorem 10.6, $\mathbf{K}$ is stable. As $\lambda \geq \theta(H_1)$, we have that $\lambda \geq \theta(\lambda(\mathbf{K}))$. By Lemma 8.10, either $\mathbf{K}$ is stable in λ, or there are unboundedly many stability cardinals below λ. In the former case we are done and in the latter case, we can use Theorem 10.2. $\square$

When $\mathbf{K}$ is superstable (i.e. $\chi(\mathbf{K}) = \aleph_0$, see Definition 8.18), we obtain a characterization in ZFC. This answers a question around [GV, Remark 1.6].

Corollary 10.8. Let $\mathbf{K}$ be an LS($\mathbf{K}$)-tame AEC with a monster model. The following are equivalent:

(1) $\mathbf{K}$ is superstable.
(2) $\mathbf{K}$ has a saturated model of size λ for every $\lambda \geq \beth_{H_1}$.

Proof. (1) implies (2) is known [BVb] (or use Corollary 9.6 with Corollary 8.16 and Fact 8.17). Now assume (2). By Theorem 10.6, $\mathbf{K}$ is stable. We prove by induction on $\lambda \geq \beth_{H_1}^{\lambda(\mathbf{K})}$ that $\mathbf{K}$ is stable in λ. This implies superstability by Corollary 8.16.

If $\lambda = \beth_{H_1}^{\lambda(\mathbf{K})}$, then $\lambda^{\lambda(\mathbf{K})} = \lambda$ so $\mathbf{K}$ is stable in λ (see Fact 8.1). Now if $\lambda > \beth_{H_1}^{\lambda(\mathbf{K})}$, then by the induction hypothesis $\mathbf{K}$ is stable in unboundedly many cardinals below λ, hence the result follows from Theorem 10.2. $\square$

11. Characterizations of stability

In [GV], Grossberg and the author characterize superstability in terms of the behavior of saturated, limit, and superlimit models. We show that stability can be characterized analogously. In fact, we are able to give a precise characterization of $\chi(\mathbf{K})$.

Remark 11.1. Another important characterization of superstability in [GV] was solvability: roughly, the existence of an EM blueprint generating superlimit models. We do not know if there is a generalization of solvability to stability. Indeed it follows from the proof of [SV99, Theorem 2.2.1] that even an EM blueprint generating just universal (not superlimit) models would imply superstability (see also [BGVV]).

We see the next definition as the "stable" version of a superlimit model. Very similar notions appear already in [She87].

Definition 11.2. Let $\mathbf{K}$ be an AEC. For χ a regular cardinal, $M \in \mathbf{K}_{\geq \chi}$ is *χ-superlimit* if:

(1) M has a proper extension.
(2) M is universal in $\mathbf{K}_{\|M\|}$.
(3) For any increasing chain $\langle M_i : i < \chi \rangle$, if $i < \chi$ implies $M \cong M_i$, then $M \cong \bigcup_{i<\chi} M_i$.

We say that M is *$(\geq \chi)$-superlimit* if $\chi \leq \|M\|$ and for any regular $\chi' \in [\chi, \|M\|]$, M is χ'-superlimit.

In [GV], it was shown that the cardinals $\chi_0, \chi_2, \chi_3, \chi_4$ below are all equal to $\aleph_0$ if one of them is. We see the following characterization as a generalization to strictly stable AECs.

Theorem 11.3. Let $\mathbf{K}$ be a (not necessarily stable) LS($\mathbf{K}$)-tame AEC with a monster model. The following cardinals are all equal to $\chi(\mathbf{K})$.

(0) The minimal cardinal χ_0 such that for unboundedly many stability cardinals μ, $\chi_0 \in \underline{\kappa}(\mathbf{K}_\mu, <^{\text{univ}}_{\mathbf{K}})$.

(1) The minimal cardinal χ_1 such that for unboundedly many stability cardinals μ we have that $\mu < \mu^{\chi_1}$.
(2) The minimal regular cardinal χ_2 such that for unboundedly many cardinals μ, there exists a unique $(\mu, \geq \chi_2)$-limit model.
(3) The minimal regular cardinal χ_3 such that for unboundedly many μ, the union of any increasing chain of μ-saturated models of cofinality at least χ_3 is μ-saturated.
(4) The minimal regular cardinal χ_4 such that at unboundedly many stability cardinals there is a $(\geq \chi_4)$-superlimit model.
(5) The minimal regular cardinal χ_5 such that at unboundedly many H_1-closed cardinals $\mu < \mu^{\chi_5}$, there is a saturated model of cardinality μ.

Proof. Note that χ_0 is equal to $\chi(\mathbf{K})$ from the definition and the discussion around Definition 8.4. Also, χ_1 is equal to $\chi(\mathbf{K})$ by Theorem 8.6.

Next we show that for $\ell \in \{2, 3, 4, 5\}$, $\chi_\ell < \infty$ implies that $\mathbf{K}$ is stable.

If $\chi_2 < \infty$, then there exists in particular limit models and this implies stability. Also $\chi_4 < \infty$ implies stability by definition. If $\chi_5 < \infty$, then we have stability by Theorem 10.6. Finally, assume that $\chi_3 < \infty$. Build an increasing continuous chain of cardinals $\langle \mu_i : i \leq \chi_3 \rangle$ such that $\chi_3 + \text{LS}(\mathbf{K}) < \mu_0$, for each $i \leq \chi_3$ any increasing chain of μ_i-saturated models of length at least χ_3 is μ_i-saturated, and $2^{\mu_i} < \mu_{i+1}$ for all $i < \chi_3$. Let $\mu := \mu_{\chi_3}$. Build an increasing chain $\langle M_i : i < \chi_3 \rangle$ such that $M_{i+1} \in \mathbf{K}_{2^{\mu_i}}$ and M_{i+1} is μ_i-saturated. Now by construction $M := \bigcup_{i<\chi_3} M_i$ is in $\mathbf{K}_\mu$ and is saturated. Since $\text{cf}(\mu) = \chi_3$, we have that $\mu < \mu^{<\mu}$. By Theorem 10.6, $\mathbf{K}$ is stable. We have shown that we can assume without loss of generality that $\mathbf{K}$ is stable.

Now if μ is H_1-closed such that $\mu < \mu^{\chi_5}$ and there is a saturated model of cardinality μ, then by Lemma 8.10, either $\mathbf{K}$ is stable in μ or stable in unboundedly many cardinals below μ. In the latter case, Theorem 10.2 implies that $\mathbf{K}$ is stable in μ. By Theorem 8.6, $\mu = \mu^{<\chi(\mathbf{K})}$. Since $\mu < \mu^{\chi_5}$, this implies that $\chi(\mathbf{K}) \leq \chi_5$. We also have that $\chi_5 \leq \chi(\mathbf{K})$ (Corollary 9.6 and Theorem 8.6) hence $\chi_5 = \chi(\mathbf{K})$. It remains to see that $\chi(\mathbf{K}) = \chi_\ell$ for $\ell \in \{2, 3, 4\}$.

We first claim that $\chi_3 = \chi_4$. Indeed, if we have a $(\geq \chi_4)$-superlimit at a stability cardinal μ, then it must be saturated and witnesses that the union of an increasing chain of μ-saturated models of length at least χ_4 is μ-saturated. Conversely, We have shown above how to build a saturated model in a cardinal μ such that the union of an increasing chain of μ-saturated of length at least χ_3 is μ-saturated. Such a saturated model must be a $(\geq \chi_3)$-superlimit.

We also have that $\chi_2 \leq \chi_4 = \chi_3$, as it is easy to see that a $(\geq \chi_4)$-superlimit model in a stability cardinal μ must be unique and also a $(\mu, \geq \chi_4)$-limit model. Also, $\chi_3 \leq \chi(\mathbf{K})$ (Theorem 9.9). It remains to show that $\chi(\mathbf{K}) \leq \chi_2$. This follows from the proof of [GV, Theorem 4.30]. □

References

[Bal09] John T. Baldwin, *Categoricity*, University Lecture Series, vol. 50, American Mathematical Society, 2009.

[BG] Will Boney and Rami Grossberg, *Forking in short and tame AECs*, Preprint. URL: `http://arxiv.org/abs/1306.6562v10`.

[BGKV16] Will Boney, Rami Grossberg, Alexei Kolesnikov, and Sebastien Vasey, *Canonical forking in AECs*, Annals of Pure and Applied Logic **167** (2016), no. 7, 590–613.

[BGVV] Will Boney, Rami Grossberg, Monica VanDieren, and Sebastien Vasey, *Superstability from categoricity in abstract elementary classes*, Preprint. URL: `http://arxiv.org/abs/1609.07101v1`.

[BKV06] John T. Baldwin, David Kueker, and Monica VanDieren, *Upward stability transfer for tame abstract elementary classes*, Notre Dame Journal of Formal Logic **47** (2006), no. 2, 291–298.

[Bon14a] Will Boney, *Tameness and extending frames*, Journal of Mathematical Logic **14** (2014), no. 2, 1450007.

[Bon14b] ———, *Tameness from large cardinal axioms*, The Journal of Symbolic Logic **79** (2014), no. 4, 1092–1119.

[BVa] Will Boney and Monica VanDieren, *Limit models in strictly stable abstract elementary classes*, Preprint. URL: `http://arxiv.org/abs/1508.04717v3`.

[BVb] Will Boney and Sebastien Vasey, *Chains of saturated models in AECs*, Preprint. URL: `http://arxiv.org/abs/1503.08781v3`.

[BVc] ———, *A survey on tame abstract elementary classes*, To appear in Beyond first order model theory. URL: `http://arxiv.org/abs/1512.00060v4`.

[FW91] Matthew Foreman and Hugh Woodin, *The generalized continuum hypothesis can fail everywhere*, Annals of Mathematics **133** (1991), no. 1, 1–35.

[GL02] Rami Grossberg and Olivier Lessmann, *Shelah's stability spectrum and homogeneity spectrum in finite diagrams*, Archive for Mathematical Logic **41** (2002), no. 1, 1–31.

[Gro] Rami Grossberg, *A course in model theory I*, A book in preparation.

[GV] Rami Grossberg and Sebastien Vasey, *Superstability in abstract elementary classes*, Preprint. URL: `http://arxiv.org/abs/1507.04223v3`.

[GV06] Rami Grossberg and Monica VanDieren, *Galois-stability for tame abstract elementary classes*, Journal of Mathematical Logic **6** (2006), no. 1, 25–49.

[GVV16] Rami Grossberg, Monica VanDieren, and Andrés Villaveces, *Uniqueness of limit models in classes with amalgamation*, Mathematical Logic Quarterly **62** (2016), 367–382.

[HS00] Tapani Hyttinen and Saharon Shelah, *Strong splitting in stable homogeneous models*, Annals of Pure and Applied Logic **103** (2000), 201–228.

[Jec03] Thomas Jech, *Set theory*, 3rd ed., Springer-Verlag, 2003.

[MS90] Michael Makkai and Saharon Shelah, *Categoricity of theories in $L_{\kappa,\omega}$, with κ a compact cardinal*, Annals of Pure and Applied Logic **47** (1990), 41–97.

[She75] Saharon Shelah, *The lazy model theoretician's guide to stability*, Logique et Analyse **18** (1975), 241–308.

[She87] ———, *Classification of non elementary classes II. Abstract elementary classes*, Classification Theory (Chicago, IL, 1985) (John T. Baldwin, ed.), Lecture Notes in Mathematics, vol. 1292, Springer-Verlag, 1987, pp. 419–497.

[She99] ———, *Categoricity for abstract classes with amalgamation*, Annals of Pure and Applied Logic **98** (1999), no. 1, 261–294.

[She00] ———, *On what I do not understand (and have something to say), model theory*, Math. Japonica **51** (2000), 329–377.

[She09a] ———, *Classification theory for abstract elementary classes*, Studies in Logic: Mathematical logic and foundations, vol. 18, College Publications, 2009.

[She09b] ———, *Classification theory for abstract elementary classes 2*, Studies in Logic: Mathematical logic and foundations, vol. 20, College Publications, 2009.

[Sol74] Robert M. Solovay, *Strongly compact cardinals and the GCH*, Proceedings of the Tarski Symposium, Proc. Sympos. Pure Math., vol. 25, American Mathematical Society, 1974, pp. 365–372.

[SV99] Saharon Shelah and Andrés Villaveces, *Toward categoricity for classes with no maximal models*, Annals of Pure and Applied Logic **97** (1999), 1–25.

[Van] Monica VanDieren, *Superstability and symmetry*, Annals of Pure and Applied Logic, To appear. URL: `http://arxiv.org/abs/1507.01990v4`.

[Van06] ———, *Categoricity in abstract elementary classes with no maximal models*, Annals of Pure and Applied Logic **141** (2006), 108–147.

[Van16] ———, *Symmetry and the union of saturated models in superstable abstract elementary classes*, Annals of Pure and Applied Logic **167** (2016), no. 4, 395–407.

[Vasa] Sebastien Vasey, *Saturation and solvability in abstract elementary classes with amalgamation*, Preprint. URL: `http://arxiv.org/abs/1604.07743v2`.

[Vasb] ———, *Shelah's eventual categoricity conjecture in universal classes. Part II*, Preprint. URL: `http://arxiv.org/abs/1602.02633v2`.

[Vas16a] ———, *Building independence relations in abstract elementary classes*, Annals of Pure and Applied Logic **167** (2016), no. 11, 1029–1092.

[Vas16b] ———, *Forking and superstability in tame AECs*, The Journal of Symbolic Logic **81** (2016), no. 1, 357–383.

[Vas16c] ______, *Infinitary stability theory*, Archive for Mathematical Logic **55** (2016), 567–592.

[VV] Monica VanDieren and Sebastien Vasey, *Symmetry in abstract elementary classes with amalgamation*, Preprint. URL: http://arxiv.org/abs/1508.03252v3.

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