QUASIMINIMAL ABSTRACT ELEMENTARY CLASSES

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Abstract. We propose the notion of a quasiminimal abstract elementary class (AEC). This is an AEC satisfying four semantic conditions: countable Löwenheim-Skolem-Tarski number, existence of a prime model, closure under intersections, and uniqueness of the generic orbital type over every countable model. We exhibit a correspondence between Zilber's quasimimimal pregeometry classes and quasiminimal AECs: any quasimimimal pregeometry class induces a quasiminimal AEC (this was known), and for any quasiminimal AEC there is a natural functorial expansion that induces a quasiminimal pregeometry class.

We show in particular that the exchange axiom is redundant in Zilber's definition of a quasiminimal pregeometry class. We also study a (non-quasiminimal) example of Shelah where exchange fails, and show that it has a good frame that cannot be extended to be type-full.

Contents

1. Introduction

Quasiminimal pregeometry classes were introduced by Zilber [Zil05a] in order to prove a categoricity theorem for the so-called pseudo-exponential fields. Quasiminimal pregeometry classes are a class of structures carrying a pregeometry satisfying several axioms. Roughly (see Definition 4.5) the axioms specify that the countable structures are quite homogeneous and that the generic type over them is unique (where types here are syntactic quantifier-free types). The original axioms included an "excellence" condition, but it has since been shown [BHH$^+$14] that this follows

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from the rest. Zilber showed that a quasiminimal pregeometry class has at most one model in every uncountable cardinal, and in fact the structures are determined by their dimension. Note that quasiminimal pregeometry classes are typically non-elementary (see [Kir10, §5]): they are axiomatizable in $\mathbb{L}_{\omega_1,\omega}(Q)$ (where Q is the quantifier "there exists uncountably many") but not even in $\mathbb{L}_{\omega_1,\omega}$.

The framework of abstract elementary classes (AECs) was introduced by Saharon Shelah [She87a] and encompasses for example classes of models of an $\mathbb{L}_{\omega_1,\omega}(Q)$ theory. Therefore quasiminimal pregeometry classes can be naturally seen as AECs (see Fact 4.8). In this paper, we show that a converse holds: there is a natural class of AECs, which we call the *quasiminimal AECs*, that corresponds to quasiminimal pregeometry classes. Quasiminimal AECs are required to satisfy four purely semantic properties (see Definition 4.1), the most important of which are that the AEC must, in a technical sense, be closed under intersections (this is called "admitting intersections", see Definition 3.1) and over each countable model M there must be a *unique* orbital (Galois) type that is not realized inside M.

It is straightforward (and implicit e.g. in [Kir10, §4], see also [HK16, Lemma 2.87]) to see that any quasiminimal pregeometry class is a quasiminimal AEC, but here we prove a converse (Theorem 4.21). We have to solve two difficulties:

(1) The axioms of quasiminimal pregeometry classes are very syntactic because they are phrased in terms of quantifier-free types. For example, one of the axioms (II.(2) in Definition 4.5) specifies that the models must have some syntactic homogeneity.
(2) Nothing in the definition of quasiminimal AECs says that the models must carry a pregeometry. It is not clear that the natural closure $\mathrm{cl}^M(A)$ given by the intersections of all the $\mathbf{K}$-substructures of M containing A satisfies exchange.

To get around the first difficulty, we use a recent joint work with Shelah [SV] together with the technique of adding relation symbols for small Galois types to the vocabulary (called the Galois Morleyization in [Vas16b]). To get around the second difficulty, we develop new tools to prove the exchange axiom of pregeometries in any setup where we know that the other axioms of pregeometries hold. We show (Corollary 2.12) that any *homogeneous* closure space satisfying the finite character axiom of pregeometries also satisfies the exchange axiom (to the best of our knowledge, this is new[1]). As a consequence, the exchange axiom is redundant in the definition of a quasiminimal pregeometry class (Corollary 4.12)[2].

An immediate corollary of the correspondence between quasiminimal AECs and quasiminimal pregeometry classes is that a quasiminimal AEC has at most one

[1] Although related to the study of quasiminimal structures in Itai-Tsuboi-Wakai [ITW04] and later Pillay-Tanović [PT11], Corollary 2.12 is different. It gives a stronger conclusion using stronger hypotheses, see Remarks 2.3, 2.14.

[2] Interestingly, exchange was initially not part of Zilber's definition of quasiminimal pregeometry classes (see [Zil05b, §5]) but was added later. Some sources claim that the axiom is necessary, see [Bal09, Remark 2.24] or [Kir10, p. 554], but this seems to be due to a related counterexample that does not fit in the framework of quasiminimal pregeometry classes (see the discussion in Remark 2.14).

model in every uncountable cardinal (Corollary 4.22). This can be seen as a generalization of the fact that algebraically closed fields of a fixed characteristic are uncountably categorical (indeed, algebraically closed fields are closed under intersections and if F is a field, a, b are transcendental over F, then a and b satisfy the same type over F).

In the last section of this paper, we study a (non-quasiminimal) example of Shelah that is quite well-behaved but where exchange fails. We point out that in this setup there is a good frame that cannot be extended to be type-full (Theorem 5.8). This answers a question of Boney and the author [BV, Question 1.4].

Throughout this paper, we assume basic familiarity with AECs (see [Bal09]). We use the notation from [Vas16b]. In particular, $\operatorname{gtp}(\bar{b}/A; N)$ denotes the Galois type of $\bar{b}$ over A as computed in N.

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2. Exchange in homogeneous closure spaces

In this section, we study closure spaces, which are objects satisfying the monotonicity and transitivity axioms of pregeometries. We want to know whether they satisfy the exchange axiom when they are homogeneous. We give criterias for when this is the case (Corollary 2.12). To the best of our knowledge, this is new (but see Remarks 2.3, 2.14).

The following definition is standard, see e.g. [CR70].

Definition 2.1. A closure space is a pair $W = (X, \mathrm{cl})$, where:

(1) X is a set.
(2) $\mathrm{cl} : \mathcal{P}(X) \to \mathcal{P}(X)$ satisfies:
  (a) Monotonicity: For any $A \subseteq X$, $A \subseteq \mathrm{cl}(A)$.
  (b) Transitivity: For any $A, B \subseteq X$, $A \subseteq \mathrm{cl}(B)$ implies $\mathrm{cl}(A) \subseteq \mathrm{cl}(B)$.

We write $|W|$ for X and cl^W for cl (but when W is clear from context we might forget it). For $a \in A$, we will often write $\mathrm{cl}(a)$ instead of $\mathrm{cl}(\{a\})$. Similarly, for sets $A, B \subseteq |W|$ and $a \in |W|$, we will write $\mathrm{cl}(Aa)$ instead of $\mathrm{cl}(A \cup \{a\})$ and $\mathrm{cl}(AB)$ instead of $\mathrm{cl}(A \cup B)$.

Definition 2.2. Let W be a closure space.

(1) For closure spaces W_1, W_2, we say that a function $f : |W_1| \to |W_2|$ is an *isomorphism* if it is a bijection and for any $A \subseteq |W_1|$, $f[\mathrm{cl}^{W_1}(A)] = \mathrm{cl}^{W_2}(f[A])$. When $W_1 = W_2 = W$, we say that f is an *automorphism* of W.
(2) We say that $A \subseteq |W|$ is *closed* if $\mathrm{cl}^W(A) = A$.
(3) For μ an infinite cardinal, we say that W is *μ-homogeneous* if for any closed set A with $|A| < \mu$ and any $a, b \in |W| \backslash A$, there exists an automorphism of W that fixes A pointwise and sends a to b.

(4) Let $\mathrm{LS}(W)$ be the least infinite cardinal μ such that for any $A \subseteq |W|$, $|\operatorname{cl}^W(A)| \leq |A| + \mu$.
(5) Let $\kappa(W)$ be the least infinite cardinal κ such that for any $A \subseteq |W|$, $a \in \operatorname{cl}^W(A)$ implies that there exists $A_0 \subseteq A$ with $|A_0| < \kappa$ and $a \in \operatorname{cl}^W(A_0)$. We say that W has *finite character* if $\kappa(W) = \aleph_0$.
(6) We say that W has *exchange over* A if $A \subseteq |W|$ and for any a, b, if $a \in \operatorname{cl}^W(Ab) \backslash \operatorname{cl}^W(A)$, then $b \in \operatorname{cl}^W(Aa)$. We say that W has *exchange* if it has exchange over every $A \subseteq |W|$.
(7) We say that W is a *pregeometry* if it has finite character and exchange.

Remark 2.3. The notion of homogeneity considered here is *not* the same as that considered in [PT11, §4]. There the notion is defined syntactically using first-order types and here we use automorphisms. The notion used here is stronger: two elements could satisfy the same first-order type but not the same type e.g. in an infinitary logic. This is used in the proof of Theorem 2.11.(3): if $(I, <)$ is a dense linear order and $b < c$, then b and c will satisfy the same first-order type over $(-\infty, b)$, but there cannot be an automorphism sending b to c fixing $(-\infty, b)$. Thus $I = \mathbb{Q} \times \omega_1$ cannot be a counterexample to Theorem 2.11.(3). In the proof of Theorem 4.11, we will build a (Galois) saturated model M and work with the pregeometry generated by a certain closure operator inside it. The (orbital) homogeneity of M will give homogeneity of the pregeometry in the strong sense given here.

Remark 2.4.

(1) $\mathrm{LS}(W) \leq \|W\| + \aleph_0$ and $\kappa(W) \leq \|W\|^+ + \aleph_0$.
(2) $\mathrm{LS}(W) \leq \kappa(W) \cdot \sup_{A \subseteq |W|, |A| < \kappa(W)} |\operatorname{cl}^W(A)|$.

Definition 2.5. For $A \subseteq |W|$, let W_A be the following closure space: $|W_A| := |W| \backslash A$, and $\operatorname{cl}^{W_A}(B) := \operatorname{cl}^W(AB) \cap |W_A|$.

Remark 2.6. Let W be a closure space.

(1) For μ an infinite cardinal, if W is μ-homogeneous, $A \subseteq |W|$ and $|A| < \mu$, then W_A is μ-homogeneous.
(2) W has exchange over A if and only if W_A has exchange over $\emptyset$.
(3) W has exchange if and only if W has exchange over every A with $|A| < \kappa(W)$.

Closure spaces where exchange always fails are studied in the literature under the names "antimatroid" or "convex geometry" [EJ85]. One of the first observation one can make is that there is a natural ordering in this context:

Definition 2.7. Let W be a closure space. For $a, b \in |W|$, say $a \leq b$ if $a \in \operatorname{cl}(b)$. We say $a < b$ if $a \leq b$ but $b \not\leq a$.

Remark 2.8. By the transitivity axiom, $(|W|, \leq)$ is a pre-order. We will often think of it as a partial order, i.e. identify a, b such that $a \leq b$ and $b \leq a$.

Remark 2.9. Let W be a closure space where $\emptyset$ is closed. Then W fails exchange over $\emptyset$ if and only if there exists $a, b \in |W|$ such that $a < b$.

To give conditions under which exchange follows from homogeneity, we will study the ordering $(|W|, \leq)$. If exchange fails, it must be linear:

Lemma 2.10. If W is $\mathrm{LS}(W)^+$-homogeneous, $\emptyset$ is closed, and W fails exchange over $\emptyset$, then $(|W|, \leq)$ is (if we identify a, b with $a \leq b$ and $b \leq a$) a dense linear order without endpoints.

Proof. Using failure of exchange, fix a, b such that $a \in \mathrm{cl}(b)$ but $b \notin \mathrm{cl}(a)$. Let c, d be given such that $d \not\leq c$. Then $d \notin \mathrm{cl}(c)$. We show that $c \leq d$, i.e. $c \in \mathrm{cl}(d)$. By homogeneity, there exists an automorphism f of W taking c to a. Let $d' := f(d)$. Then $d' \notin \mathrm{cl}(a)$. Note that $|\mathrm{cl}(a)| \leq \mathrm{LS}(W)$ so by $\mathrm{LS}(W)^+$-homogeneity, there exists an automorphism g of W fixing $\mathrm{cl}(a)$ and sending d' to b. Applying $f^{-1} \circ g^{-1}$ to the formula $a \in \mathrm{cl}(b)$, we obtain $c \in \mathrm{cl}(d)$, as desired.

We have shown that $(|W|, \leq)$ is a linear order. That it is dense and without endpoints similarly follow from homogeneity. $\square$

Theorem 2.11. Let W be a $\mathrm{LS}(W)^+$-homogeneous closure space where $\emptyset$ is closed. Then W has exchange over $\emptyset$ if at least one of the following conditions hold:

(1) $\|W\| < \aleph_0$.
(2) $\|W\| \geq \mathrm{LS}(W)^{++}$.
(3) $\kappa(W) = \aleph_0$.

Proof. Suppose for a contradiction that exchange over $\emptyset$ fails.

For $b \in |W|$, write $(-\infty, b) := \{a \in |W| : a < b\}$, and similarly for $(-\infty, b]$. Note that if $A \subseteq |W|$ is closed and $a \in A$, then by definition of $\leq$ and the transitivity axiom, $(-\infty, a] \subseteq A$. Similarly, if $b \notin A$ then by Lemma 2.10 $A \subseteq (-\infty, b)$.

(1) If $\|W\| < \aleph_0$, then Lemma 2.10 directly gives a contradiction.
(2) Let $A \subseteq |W|$ be closed such that $|A| \leq \mathrm{LS}(W)$ and let $B \subseteq |W|$ be closed with $A \subseteq B$ and $|B| = \mathrm{LS}(W)^+$. Let $a \in A$ and let $b \notin B$. Then $(-\infty, a) \subseteq A$ and $B \subseteq (-\infty, b)$. Therefore $|(-\infty, a)| \leq \mathrm{LS}(W)$ and $|(-\infty, b)| \geq \mathrm{LS}(W)^+$. However by homogeneity there exists an automorphism of W sending a to b, a contradiction.
(3) We first prove two claims.
Claim 1: If $b \in |W|$, then $\mathrm{cl}((-\infty, b)) = (-\infty, b]$.
Proof of Claim 1: Let $B := \mathrm{cl}(-\infty, b)$. First note that $B \subseteq \mathrm{cl}(b)$, hence $|B| \leq \mathrm{LS}(W)$ (so we can apply homogeneity to it) and $B \subseteq (-\infty, b]$. By monotonicity, $(-\infty, b) \subseteq B$. Also, if $B \neq (-\infty, b)$, then $b \in B$ (say $c \in B \backslash (-\infty, b)$. Then $c \not< b$, so by Lemma 2.10, $b \leq c$, so since B is closed $b \in B$). Thus if $b \notin B$, then $B = (-\infty, b)$. This is impossible: take $c \in |W|$ such that $b < c$ (exists by Lemma 2.10). Then there is an automorphism of W taking b to c fixing B, which is impossible as b is a least upper bound of B but c is not. Therefore $(-\infty, b] \subseteq B$. $\dagger_{\text{Claim 1}}$
Claim 2: If $\langle A_i : i \in I \rangle$ is a non-empty collection of subsets of $|W|$, then $\mathrm{cl}(\bigcup_{i \in I} A_i) = \bigcup_{i \in I} \mathrm{cl}(A_i)$.
Proof of Claim 2: Clearly, the right hand side is contained in the left hand side. We show the other inclusion. Let $A := \bigcup_{i \in I} A_i$. Let $a \in \mathrm{cl}(A)$. By finite character, there exists a finite $A' \subseteq A$ such that $a \in \mathrm{cl}(A')$. Since $\emptyset$ is closed, A' cannot be empty. Say $A' = \{a_0, \ldots, a_{n-1}\}$, with $a_0 \leq a_1 \leq \ldots \leq a_{n-1}$ (we are implicitly using Lemma 2.10). Then $a \in \mathrm{cl}(a_{n-1})$. Pick $i \in I$ such that $a_{n-1} \in A_i$. Then $a \in \mathrm{cl}(A_i)$, as desired. $\dagger_{\text{Claim 2}}$

Now pick any $b \in |W|$. Note that (using Lemma 2.10) $(-\infty, b) = \bigcup_{a<b}(-\infty, a) = \bigcup_{a<b}(-\infty, a]$. However on the one hand, by Claim 1, $\mathrm{cl}(\bigcup_{a<b}(-\infty, a)) = \mathrm{cl}((-\infty, b)) = (-\infty, b]$ but on the other hand, by Claim 2, $\mathrm{cl}(\bigcup_{a<b}(-\infty, a)) = \bigcup_{a<b}\mathrm{cl}((-\infty, a)) = \bigcup_{a<b}(-\infty, a] = (-\infty, b)$, a contradiction.

$\square$

Corollary 2.12. Let W be a $(\kappa(W) + \mathrm{LS}(W)^+)$-homogeneous closure space. If either $\kappa(W) = \aleph_0$ or $\|W\| \notin [\aleph_0, \mathrm{LS}(W)^+]$, then W has exchange.

Proof. Let $\mu := \kappa(W) + \mathrm{LS}(W)^+$. By Remark 2.6, it is enough to see that W has exchange over every set A with $|A| < \kappa(W)$. Fix such an A. By Remark 2.6, it is enough to see that $W' := W_A$ has exchange over $\emptyset$. Note that W' is μ-homogeneous and $\mathrm{LS}(W') \leq \mathrm{LS}(W)$. Let $B := \mathrm{cl}^{W'}(\emptyset)$. We have that $|B| \leq \mathrm{LS}(W)$. Let $W'' := W'_B$. We show that W'' has exchange over $\emptyset$. Note that W'' is still μ-homogeneous. Moreover $\emptyset$ is closed in W''. Observe that $\kappa(W) = \aleph_0$ implies that $\kappa(W'') = \aleph_0$, $\|W''\| \leq \|W\|$, but $\|W\| \geq \mathrm{LS}(W)^{++}$ implies that $\|W''\| \geq \mathrm{LS}(W)^{++}$. Therefore by Theorem 2.11, W'' has exchange over $\emptyset$, as desired. $\square$

We give a few examples showing that the hypotheses of Corollary 2.12 are near optimal:

Example 2.13.

(1) On any partial order $\mathbb{P}$, one can define a closure operator cl_1 by $\mathrm{cl}_1(A) := \{b \in \mathbb{P} \mid \exists a \in A : b \leq a\}$. The resulting closure space W_1 has exchange over $\emptyset$ if and only if there are no $a, b \in \mathbb{P}$ with $a < b$. Note that if $\mathbb{P}$ is e.g. a dense linear order, then W_1 is not $\aleph_1$-homogeneous.
(2) On the other hand, one can define $\mathrm{cl}_2(A) := \mathrm{cl}_1(A) \cup \{b \in \mathbb{P} \mid \forall c(c < b \to c \in \mathrm{cl}_1(A))\}$. This gives a closure space W_2. If $\mathbb{P} = \mathbb{Q}$, then W_2 is $\aleph_1$-homogeneous and does not have exchange over $\emptyset$ but note that $\kappa(W) = \aleph_1$, as witnessed by the fact that the statement "$0 \in \mathrm{cl}((-\infty, 0))$" is not witnessed by a finite subset of $(-\infty, 0)$.
(3) The closure space W_3 induced by $\mathbb{P} = \mathbb{Q} \times \omega_1$ and the closure operator cl_2 is also $\aleph_1$-homogeneous, satisfies $\mathrm{LS}(W_3) = \aleph_0$, $\kappa(W_3) = \aleph_1$, and does not have exchange over $\emptyset$.

Remark 2.14. In [PT11, §5], Pillay and Tanović (generalizing an earlier result of Itai, Tsuboi, and Wakai [ITW04, Proposition 2.8]) prove (roughly) that any quasiminimal structure (i.e. every definable set is either countable or co-countable) of size at least $\aleph_2$ induces a pregeometry. This is a (more general) version of Corollary 2.12 for the case $\kappa(W) = \aleph_0$, $\|W\| \geq \aleph_2$, and $\mathrm{LS}(W) = \aleph_0$ (note that one can see any such W as a structure by adding an n-ary function for the closure of each set of size n).

Note that in the Pillay-Tanović context the hypothesis that the size should be at least $\aleph_2$ is needed: consider [ITW04, Example 2.2.(3a)] the structure $M := (\mathbb{Q} \times \omega_1, <)$. The closure space induced by M is as in W_1 from Example 2.13, so it does not have exchange. Note that M is homogeneous in the model-theoretic sense that every countable partial elementary mapping from M into M can be extended (and also in the syntactic sense of [PT11, §4], see Remark 2.3), but this does *not* make the corresponding closure space homogeneous in the sense of Definition 2.2.(3).

3. On AECs admitting intersections

We recall the definition of an AEC admitting intersections, first appearing in Baldwin and Shelah [BS08, Definition 1.2]. We give a few known facts and show (Theorem 3.5) that admitting intersections transfers up in AECs: if $\mathbf{K}_\lambda$ admits intersections, then $\mathbf{K}_{\geq\lambda}$ admits intersections.

As in [Gro], we call an *abstract class* a pair $\mathbf{K} = (K, \leq_{\mathbf{K}})$ where K is a class of structures in a fixed vocabulary $\tau = \tau(\mathbf{K})$, $\leq_{\mathbf{K}}$ is a partial order on K, both K and $\leq_{\mathbf{K}}$ are closed under isomorphisms, and for $M, N \in K$, $M \leq_{\mathbf{K}} N$ implies $M \subseteq N$. We say that an abstract class is *coherent* if for $M_0, M_1, M_2 \in \mathbf{K}$, $M_0 \subseteq M_1 \leq_{\mathbf{K}} M_2$ and $M_0 \leq_{\mathbf{K}} M_2$ imply $M_0 \leq_{\mathbf{K}} M_1$.

Note that any AEC is a coherent abstract class, and if $\mathbf{K}$ is an AEC and λ is a cardinal, then $\mathbf{K}_\lambda$ is also a coherent abstract class.

Definition 3.1. Let $\mathbf{K}$ be a coherent abstract class. Let $N \in \mathbf{K}$ and let $A \subseteq |N|$.

(1) Let $\mathrm{cl}^N(A)$ be the set $\bigcap\{|M| : M \leq_{\mathbf{K}} N \wedge A \subseteq |M|\}$. Note that $\mathrm{cl}^N(A)$ induces a $\tau(\mathbf{K})$-substructure of N, so we will abuse notation and also write $\mathrm{cl}^N(A)$ for this substructure.
(2) We say that N *admits intersections over* A if $\mathrm{cl}^N(A) \leq_{\mathbf{K}} N$ (more formally, there exists $M \leq_{\mathbf{K}} N$ such that $|M| = \mathrm{cl}^N(A)$).
(3) We say that N *admits intersections* if it admits intersections over all $A \subseteq |N|$.
(4) We say that $\mathbf{K}$ *admits intersections* if every $N \in \mathbf{K}$ admits intersections.

Remark 3.2. Let $\mathbf{K}$ be a coherent abstract class and let $N \in \mathbf{K}$. Then $(|N|, \mathrm{cl}^N)$ is a closure space and any $M \leq_{\mathbf{K}} N$ is closed.

The following characterization of admitting intersections in terms of the existence of a certain closure operator will be used often in this paper. Part of it appears already (for AECs) in [Vas, Theorem 2.11].

Fact 3.3. Let $\mathbf{K}$ be a coherent abstract class and let $N \in \mathbf{K}$. The following are equivalent:

(1) N admits intersections.
(2) For every non-empty collection S of $\mathbf{K}$-substructures of N, we have that $\bigcap S \leq_{\mathbf{K}} N$.
(3) There is a closure space W such that:
    (a) $|W| = |N|$.
    (b) The closed sets in W are exactly the sets of the form $|M|$ for $M \leq_{\mathbf{K}} N$.

Proof. That (1) is equivalent to (2) is an exercise in the definition, left to the reader (and not used in this paper). Also, (1) implies (3) is clear: take $W := (|N|, \mathrm{cl}^N)$. We prove that (3) implies (1). Let (W, cl^W) be a closure space on $|N|$ such that the closed sets are exactly the $\mathbf{K}$-substructures of N. Let $A \subseteq |N|$ and let $M := \mathrm{cl}^W(A)$.

By assumption, $M \leq_{\mathbf{K}} N$ so it suffices to see that $M = \text{cl}^N(A)$. Let $M' \leq_{\mathbf{K}} N$ be such that $A \subseteq |M'|$. By assumption, M' is closed in W, so $M = \text{cl}^W(A) \subseteq |M'|$. By coherence, $M \leq_{\mathbf{K}} M'$. Since M' was arbitrary, it follows from the definition of cl^N that $\text{cl}^N(A) = M$, as desired. $\square$

The next result is observed (for AECs) in [Vas, Proposition 2.14.(4)]. The proof generalizes to coherent abstract classes.

Fact 3.4. Let $\mathbf{K}$ be a coherent abstract class and let $M \leq_{\mathbf{K}} N$ both be in $\mathbf{K}$. Let $A \subseteq |M|$ If N admits intersections over A, then M admits intersections over A and $\text{cl}^M(A) = \text{cl}^N(A)$.

We now show that admitting intersections transfer up. This is quite routine using the characterization of Fact 3.3 but we give a full proof.

Theorem 3.5. Let $\mathbf{K}$ be an AEC and let $\lambda \geq \text{LS}(\mathbf{K})$. Let $N \in \mathbf{K}_{\geq \lambda}$. If M admits intersections for all $M \in \mathbf{K}_\lambda$ with $M \leq_{\mathbf{K}} N$, then N admits intersections. In particular if $\mathbf{K}_\lambda$ admits intersections, then $\mathbf{K}_{\geq \lambda}$ admits intersections.

Proof. First note that if $M \in \mathbf{K}_{<\lambda}$ and $M \leq_{\mathbf{K}} N$, then M also admits intersections by Fact 3.4. Now on to the proof. By Fact 3.3, it is enough to find a closure space W with universe $|N|$ such that the closed sets of W are exactly the $\mathbf{K}$-substructures of N. Define $\text{cl}^W : \mathcal{P}(|N|) \to \mathcal{P}(|N|)$ as follows:

$$\text{cl}^W(A) := \bigcup \{\text{cl}^M(A \cap |M|) : M \leq_{\mathbf{K}} N \wedge M \in \mathbf{K}_\lambda\}$$

We claim that $W := (|N|, \text{cl}^W)$ is the required closure space. We prove this via a chain of claims:

<u>Claim 1</u>: For any $M \in \mathbf{K}_{\leq \lambda}$ with $M \leq_{\mathbf{K}} N$, $\text{cl}^M = \text{cl}^W \upharpoonright \mathcal{P}(|M|)$.

<u>Proof of Claim 1</u>: Let $A \subseteq |M|$. By definition, $\text{cl}^M(A) \subseteq \text{cl}^W(A)$. Conversely, let $a \in \text{cl}^W(A)$. Pick $M' \in \mathbf{K}_\lambda$ such that $M' \leq_{\mathbf{K}} N$ and $a \in \text{cl}^{M'}(A \cap |M'|)$. Pick $M'' \in \mathbf{K}_\lambda$ with $M' \leq_{\mathbf{K}} M''$ and $M \leq_{\mathbf{K}} M''$. By monotonicity and Fact 3.4, $a \in \text{cl}^{M''}(A) = \text{cl}^M(A)$, as desired. $\dagger_{\text{Claim 1}}$

<u>Claim 2</u>: If $A' \subseteq \text{cl}^W(A)$ is such that $|A'| \leq \lambda$, then there exists $A_0 \subseteq A$ such that $|A_0| \leq \lambda$ and $A' \subseteq \text{cl}^W(A_0)$.

<u>Proof of Claim 2</u>: Pick $M \in \mathbf{K}_\lambda$ witnessing that $a \in \text{cl}^W(A)$ and let $A_0 := A \cap |M|$. $\dagger_{\text{Claim 2}}$

<u>Claim 3</u>: W is a closure space.

<u>Proof of Claim 3</u>:

- Monotonicity: Let $A \subseteq |N|$. We want to see that $A \subseteq \text{cl}^W(A)$. Let $a \in A$. Using the Löwenheim-Skolem-Tarski axiom, fix $M \in \mathbf{K}_\lambda$ with $M \leq_{\mathbf{K}} N$ and $a \in |M|$. Now since $(|M|, \text{cl}^M)$ is a closure space (Remark 3.2), $a \in \text{cl}^M(a) \subseteq \text{cl}^W(A)$, as needed.
- Transitivity: Let $A, B \subseteq |N|$. First note that $A \subseteq B$ implies $\text{cl}^W(A) \subseteq \text{cl}^W(B)$. Now assume that $A \subseteq \text{cl}^W(B)$. We show that $\text{cl}^W(A) \subseteq \text{cl}^W(B)$.

Let $a \in \mathrm{cl}^W(A)$. By Claim 2, there exists $A_0 \subseteq A$ with $|A_0| \leq \lambda$ such that $a \in \mathrm{cl}^W(A_0)$. Since by assumption $A_0 \subseteq \mathrm{cl}^W(B)$, there exists by Claim 2 again $B_0 \subseteq B$ such that $|B_0| \leq \lambda$ and $A_0 \subseteq \mathrm{cl}^W(B_0)$. Pick $M \in \mathbf{K}_\lambda$ such that $A_0 \cup B_0 \subseteq |M|$ and $M \leq_{\mathbf{K}} N$. By Claim 1, $\mathrm{cl}^W(A_0) = \mathrm{cl}^M(A_0)$. By transitivity in the closure space $(|M|, \mathrm{cl}^M)$, $\mathrm{cl}^M(A_0) \subseteq \mathrm{cl}^M(B_0)$. By Claim 1 again, $\mathrm{cl}^M(B_0) = \mathrm{cl}^W(B_0)$. By what has been said earlier, $\mathrm{cl}^W(B_0) \subseteq \mathrm{cl}^W(B)$. It follows that $\mathrm{cl}^W(A_0) \subseteq \mathrm{cl}^W(B)$, hence $a \in \mathrm{cl}^W(B)$ as desired.

$\dagger_{\text{Claim 3}}$

Claim 4: If $M \leq_{\mathbf{K}} N$, then $\mathrm{cl}^W(M) = M$.

Proof of Claim 4: Let $a \in \mathrm{cl}^W(M)$. By Claim 2 and monotonicity we can pick $M_0 \in \mathbf{K}_{\leq\lambda}$ with $M_0 \leq_{\mathbf{K}} M$ such that $a \in \mathrm{cl}^W(M_0)$. By Claim 1, $a \in \mathrm{cl}^{M_0}(M_0) = M_0 \leq_{\mathbf{K}} M$. Therefore $a \in |M|$, as desired. $\dagger_{\text{Claim 4}}$

Claim 5: For any $A \subseteq |N|$, $\mathrm{cl}^W(A) \leq_{\mathbf{K}} N$.

Proof of Claim 5: From Claim 2, it is easy to see that $\mathrm{cl}^W(A) = \bigcup\{\mathrm{cl}^W(A_0) : A_0 \subseteq A \wedge |A_0| \leq \lambda\}$. Therefore by the chain and coherence axioms of AECs, it is enough to show the claim when $|A| \leq \lambda$. Pick $M \in \mathbf{K}_\lambda$ with $M \leq_{\mathbf{K}} N$ and $A \subseteq |M|$. By Claim 1, $\mathrm{cl}^W(A) = \mathrm{cl}^M(A)$. By Fact 3.4, $\mathrm{cl}^M(A) \leq_{\mathbf{K}} M$. Since $M \leq_{\mathbf{K}} N$, the result follows. $\dagger_{\text{Claim 5}}$

Putting together Claim 3, 4, and 5, we have the desired result. $\square$

We will use two facts about AECs admitting intersections in the next section. First, the closure operator has finite character [Vas, Proposition 2.14.(6)].

Fact 3.6. Let $\mathbf{K}$ be an AEC and let $N \in \mathbf{K}$. If N admits intersections, then $\kappa((|N|, \mathrm{cl}^N)) = \aleph_0$.

Second, Galois types can be characterized nicely (see [BS08, Lemma 1.3.(1)] or [Vas, Proposition 2.18]).

Fact 3.7. Let $\mathbf{K}$ be an AEC admitting intersections. Then $\mathrm{gtp}(\bar{b}_1/A; N_1) = \mathrm{gtp}(\bar{b}_2/A; N_2)$ if and only if there exists $f : \mathrm{cl}^{N_1}(A\bar{b}_1) \cong_A \mathrm{cl}^{N_2}(A\bar{b}_2)$ such that $f(\bar{b}_1) = \bar{b}_2$.

4. Quasiminimal AECs

In this section, we define quasiminimal AECs and show that they are essentially the same as quasiminimal pregeometry classes.

Following Shelah [She09a, II.1.9.(1A)], we will write $\mathrm{gS}^{\mathrm{na}}(M)$ for the set of *nonalgebraic* types over M: that is, the set of $p \in \mathrm{gS}(M)$ such that $p = \mathrm{gtp}(a/M; N)$ with $a \notin |M|$ (in the context of this paper, there will be a unique nonalgebraic type which we will call *the generic type*). We say that $M \in \mathbf{K}$ is *prime* if for any $N \in \mathbf{K}$, there exists $f : M \to N$.

Definition 4.1. An AEC $\mathbf{K}$ is *quasiminimal* if:

(1) $\mathrm{LS}(\mathbf{K}) = \aleph_0$.
(2) There is a prime model in $\mathbf{K}$.

(3) $\mathbf{K}_{\leq \aleph_0}$ admits intersections.
(4) (Uniqueness of the generic type) For any $M \in \mathbf{K}_{\leq \aleph_0}$, $|\operatorname{gS}^{\mathrm{na}}(M)| \leq 1$.

We say that $\mathbf{K}$ is *unbounded* if it satisfies in addition:

(5) There exists $\langle M_i : i < \omega \rangle$ strictly increasing in $\mathbf{K}$.

Remark 4.2. It is possible for a quasiminimal AEC to have maximal countable models. However if it is unbounded it turns out it will not have any maximal models (this is a consequence of the equivalence with quasiminimal pregeometry classes, see Corollary 4.22).

Remark 4.3. We obtain an equivalent definition if we replace axiom (2) by:

(2)' $\mathbf{K} \neq \emptyset$ and $\mathbf{K}_{\leq \aleph_0}$ has joint embedding.

Why? That (2) implies (2)' (modulo the other axioms) is given by Lemma 4.9. For the other direction, one can use joint embedding to see that $\operatorname{cl}^M(\emptyset)$ is a prime model for any $M \in \mathbf{K}_{\leq \aleph_0}$.

Remark 4.4. What happens if in Definition 4.1 one replaces $\aleph_0$ with an uncountable cardinal λ? Then the natural generalizations of Lemmas 4.9, 4.10, and Theorem 4.11 hold but we do not know whether Lemma 4.14 generalizes.

For the convenience of the reader, we repeat here the definition of a quasiminimal pregeometry class. We use the numbering and presentation from Kirby [Kir10], see there for more details on the terminology. We omit axiom III (excellence), since it has been shown [BHH+14] that it follows from the rest. We have added axiom 0.(3) that also appears in Haykazyan [Hay16, Definition 2.2] and corresponds to (2) in the definition of a quasiminimal AEC, as well as axiom 0.(1) which requires that the class be non-empty and that the vocabulary be countable (this can be assumed without loss of generality, see [Kir10, Proposition 5.2]).

As in Definition 4.1, we call the class *unbounded* if it has an infinite dimensional model (this is the nontrivial case that interests us here).

Definition 4.5. A *quasiminimal pregeometry class* is a class $\mathcal{C}$ of pairs (H, cl_H), where H is a τ-structure (for a fixed vocabulary $\tau = \tau(\mathcal{C})$) and $\operatorname{cl}_H : \mathcal{P}(|H|) \to \mathcal{P}(|H|)$ is a function satisfying the following axioms.

0: (1) $|\tau(\mathcal{C})| \leq \aleph_0$ and $\mathcal{C} \neq \emptyset$.
(2) If $(H, \operatorname{cl}_H), (H', \operatorname{cl}_{H'})$ are both τ-structures with functions on their powersets and $f : H \cong H'$ is also an isomorphism from $(|H|, \operatorname{cl}_H)$ onto $(|H'|, \operatorname{cl}_{H'})$, then $(H', \operatorname{cl}_{H'}) \in \mathcal{C}$.
(3) If $(H, \operatorname{cl}_H), (H', \operatorname{cl}_{H'}) \in \mathcal{C}$, then H and H' satisfy the same quantifier-free sentences.

I: (1) For each $(H, \operatorname{cl}_H) \in \mathcal{C}$, $(|H|, \operatorname{cl}_H)$ is a pregeometry such that the closure of any finite set is countable.
(2) If $(H, \operatorname{cl}_H) \in \mathcal{C}$ and $X \subseteq |H|$, then the $\tau(\mathcal{C})$-structure induced by $\operatorname{cl}_H(X)$ together with the appropriate restriction of cl_H is in $\mathcal{C}$.
(3) If $(H, \operatorname{cl}_H), (H', \operatorname{cl}_{H'}) \in \mathcal{C}$, $X \subseteq |H|$, $y \in \operatorname{cl}_H(X)$, and $f : H \rightharpoonup H'$ is a partial embedding with $X \cup \{y\} \subseteq \operatorname{preim}(f)$, then $f(y) \in \operatorname{cl}_{H'}(f[X])$.

II: Let $(H, \operatorname{cl}_H), (H', \operatorname{cl}_{H'}) \in \mathcal{C}$. Let $G \subseteq H$ and $G' \subseteq H'$ be countable closed subsets or empty and let $g : G \to G'$ be an isomorphism.

(1) If $x \in |H|$ and $x' \in |H'|$ are independent from G and G' respectively, then $g \cup \{(x, x')\}$ is a partial embedding.
(2) If $g \cup f : H \rightharpoonup H'$ is a partial embedding, f has finite preimage X, and $y \in \mathrm{cl}_H(X \cup G)$, then there is $y' \in H'$ such that $g \cup f \cup \{(y, y')\}$ is a partial embedding.

We say that $\mathcal{C}$ is *unbounded* if it satisfies in addition:

IV: (1) $\mathcal{C}$ is closed under unions of increasing chains: If δ is a limit ordinal and $\langle (H_i, \mathrm{cl}_{H_i}) : i < \delta \rangle$ is increasing with respect to being a closed substructure (i.e. for each $i < \delta$, $H_i \subseteq H_{i+1}$ and $\mathrm{cl}_{H_{i+1}} \upharpoonright \mathcal{P}(|H_i|) = \mathrm{cl}_{H_i}$), then $(H_\delta, \mathrm{cl}_{H_\delta}) \in \mathcal{C}$, where $H_\delta = \bigcup_{i<\delta} H_i$ and $\mathrm{cl}_{H_\delta}(X) = \bigcup_{i<\delta} \mathrm{cl}_{H_i}(X \cap |H_i|)$.
(2) $\mathcal{C}$ contains an infinite dimensional model (i.e. there exists $(H, \mathrm{cl}_H) \in \mathcal{C}$ with $\langle a_i : i < \omega \rangle$ in H such that $a_i \notin \mathrm{cl}_H(\{a_j : j < i\})$ for all $i < \omega$).

Remark 4.6. If $\mathcal{C}$ is a quasimiminal pregeometry class, $(H, \mathrm{cl}_1), (H, \mathrm{cl}_2) \in \mathcal{C}$, then by axiom I.(3) used with the identity embedding, $\mathrm{cl}_1 = \mathrm{cl}_2$. In other words, once $\mathcal{C}$ is fixed the pregeometry is determined by the structure (see also the discussion after [Kir10, Example 1.2]).

It is straightforward to show that quasiminimal pregeometry classes are (after forgetting the pregeometry and ordering them with "being a closed substructure") quasiminimal AECs. That they are AECs is noted in [Kir10, §4]. In fact, the exchange axiom is not necessary for this. We sketch a proof here for completeness.

Definition 4.7. Let $\mathcal{C}$ be a quasiminimal pregeometry class.

(1) For $(H, \mathrm{cl}_H), (H', \mathrm{cl}_{H'}) \in \mathcal{C}$, we write $(H, \mathrm{cl}_H) \leq_{\mathcal{C}} (H', \mathrm{cl}_{H'})$ if $H \subseteq H'$ and $\mathrm{cl}_{H'} \upharpoonright \mathcal{P}(|H|) = \mathrm{cl}_H$.
(2) Let $\mathbf{K} = \mathbf{K}(\mathcal{C}) := (K(\mathcal{C}), \leq_{\mathbf{K}})$ be defined as follows:
    (a) $K(\mathcal{C}) := \{M \mid \exists\, \mathrm{cl} : (M, \mathrm{cl}) \in \mathcal{C}\}$.
    (b) $M \leq_{\mathbf{K}} N$ if for some $\mathrm{cl}_M, \mathrm{cl}_N$, $(M, \mathrm{cl}_M), (N, \mathrm{cl}_N) \in \mathcal{C}$ and $(M, \mathrm{cl}_M) \leq_{\mathcal{C}} (N, \mathrm{cl}_N)$.

Fact 4.8. If $\mathcal{C}$ satisfies all the axioms of a quasiminimal pregeometry class except that in I.(1) cl_H may not have exchange, then $\mathbf{K}(\mathcal{C})$ is a quasiminimal AEC. Moreover $\mathcal{C}$ is unbounded if and only if $\mathbf{K}(\mathcal{C})$ is unbounded.

Proof. We will use Remark 4.6 without explicit mention. Let $\mathbf{K} := \mathbf{K}(\mathcal{C})$. By axioms 0, I, the finite character axiom of pregeometries, and (if $\mathcal{C}$ is unbounded; note that if $\mathcal{C}$ is bounded there are no infinite increasing chains) IV, $\mathbf{K}$ is an AEC. Since the closure of any finite set is countable (axiom I.(1)) and $|\tau(\mathcal{C})| \leq \aleph_0$ (axiom 0.(1)), $\mathrm{LS}(\mathbf{K}) = \aleph_0$. This proves that (1) in Definition 4.1 holds.

As for axiom (2), by axiom 0.(1), $\mathcal{C} \neq \emptyset$. Let $(M, \mathrm{cl}_M) \in \mathcal{C}$ and let $M_0 := \mathrm{cl}_M(\emptyset)$. By axiom I.(2), $(M_0, \mathrm{cl}_M \upharpoonright \mathcal{P}(|M_0|)) \in \mathcal{C}$. We show that M_0 is the desired prime model. Let $N \in \mathbf{K}$. This means that $(N, \mathrm{cl}_N) \in \mathcal{C}$. By axiom 0.(3), the empty map is a partial embedding from M into N. Using axiom II.(2) ω-many times (see the proof of [Kir10, Theorem 2.1]), we can extend it to a map $f_0 : M_0 = \mathrm{cl}_M(\emptyset) \cong \mathrm{cl}_N(\emptyset)$. Since $\mathrm{cl}_N(\emptyset) \leq_{\mathbf{K}} N$ (for the same reason that $M_0 \leq_{\mathbf{K}} M$), f_0 witnesses that M_0 embeds into N, as desired.

Why does $\mathbf{K}_{\leq \aleph_0}$ admit intersections (axiom (3) in Definition 4.1)? This is by the characterization in Fact 3.3 (use the definition of $\leq_{\mathbf{K}}$ and axiom I.(2).

Let us check axiom (4) in Definition 4.1. Let $M \in \mathbf{K}_{\leq \aleph_0}$. We want to show that $|\operatorname{gS}^{\text{na}}(M)| \leq 1$. Let $p_1, p_2 \in \operatorname{gS}^{\text{na}}(M)$. Say $p_\ell = \operatorname{gtp}(a_\ell / M; N_\ell)$, $\ell = 1, 2$. We want to see that $p_1 = p_2$. Without loss of generality (since we have just seen that $\mathbf{K}_{\leq \aleph_0}$ admits intersections), $N_\ell = \operatorname{cl}^{N_\ell}(M a_\ell)$. We show that there exists $f : N_1 \cong_M N_2$ with $f(a_1) = a_2$. We use axiom II.(1), where G, G', H, H', g, x, x' there stand for $M, M, N_1, N_2, \operatorname{id}_M, a_1, a_2$ here. We get that $\operatorname{id}_M \cup \{a_1, a_2\}$ is a partial embedding from N_1 to N_2. Now use axiom II.(2) ω-many times (as in the second paragraph of this proof) to extend this partial embedding to an isomorphism $f : N_1 \cong_M N_2$. By construction, we will have that $f(a_1) = a_2$, as desired.

Finally, it is straightforward to see that (5) holds if and only if $\mathcal{C}$ is unbounded, as desired. $\square$

We now start going toward the other direction. For this we first prove a couple of lemmas about quasiminimal AECs. In particular, we want to show that they have amalgamation, joint embedding, are stable, tame, and that the closure operator induces a pregeometry on them.

Lemma 4.9. If $\mathbf{K}$ is a quasiminimal AEC, then $\mathbf{K}_{\leq \aleph_0}$ has amalgamation and joint embedding.

Proof. We prove amalgamation, and joint embedding can then be obtained from the existence of the prime model and some renaming. By the "in particular" part of [Vas, Theorem 4.14], it is enough to prove the so-called type extension property in $\mathbf{K}_{\leq \aleph_0}$. This is given by the following claim:

Claim: If $M \leq_{\mathbf{K}} N$ are both in $\mathbf{K}_{\leq \aleph_0}$ and $p \in \operatorname{gS}(M)$, then there exists $q \in \operatorname{gS}(N)$ extending p.

Proof of Claim: Say $p = \operatorname{gtp}(a/M; N')$. If $a \in |M|$ (i.e. p is algebraic), let $q := \operatorname{gtp}(a/N; N)$. Assume now that $a \notin |M|$. If $M = N$, take $q = p$, so assume also that $M <_{\mathbf{K}} N$. Let $b \in |N| \backslash |M|$ and let $p' := \operatorname{gtp}(b/M; N)$. By uniqueness of the generic type, $p' = p$. Therefore $q := \operatorname{gtp}(b/N; N)$ is as desired. $\dagger_{\text{Claim}}$ $\square$

Lemma 4.10. If $\mathbf{K}$ is a quasiminimal AEC, then $\mathbf{K}$ is (Galois) stable in $\aleph_0$.

Proof. By uniqueness of the generic type. $\square$

Theorem 4.11. If $\mathbf{K}$ is a quasiminimal AEC, then $\mathbf{K}$ admits intersections and for any $N \in \mathbf{K}$, $(|N|, \operatorname{cl}^N)$ is a pregeometry whose closed sets are exactly the $\mathbf{K}$-substructures of N.

Proof. That $\mathbf{K}$ admits intersection is Theorem 3.5. Now let $N \in \mathbf{K}$ and let $W := (|N|, \operatorname{cl}^N)$. By Remark 3.2, W is a closure space and by Fact 3.3, its closed sets are exactly the $\mathbf{K}$-substructures of N. By Fact 3.6, $\kappa(W) = \aleph_0$, i.e. W has finite character. It remains to see that W has exchange. Let $a, b \in |N|$ and let $A \subseteq |N|$. Assume that $a \in \operatorname{cl}^N(Ab) \backslash \operatorname{cl}^N(A)$. We want to see that $b \in \operatorname{cl}^N(Aa)$. By finite character we can assume without loss of generality that $|A| \leq \aleph_0$. Using the Löwenheim-Skolem-Tarski axiom, we may also assume that $N \in \mathbf{K}_{\leq \aleph_0}$.

Using stability, let $N' \in \mathbf{K}_{\leq \aleph_1}$ be such that $N \leq_{\mathbf{K}} N'$ and N' is $\aleph_1$-saturated (this can be done even if there is a countable maximal model above N. In this case this will be the desired N'). Then $W' := (|N'|, \mathrm{cl}^{N'})$ is a closure space with $\kappa(W') = \aleph_0$ which (using uniqueness of the generic type) is $\aleph_1$-homogeneous. Therefore by Corollary 2.12, W' satisfies exchange. It follows immediately (see Fact 3.4) that W also satisfies exchange. $\square$

Corollary 4.12. If $\mathcal{C}$ satisfies all the axioms of a quasiminimal pregeometry class except that in I.(1) cl_H may not have exchange, then $\mathcal{C}$ is a quasiminimal pregeometry class.

Proof. By Fact 4.8, $\mathbf{K}(\mathcal{C})$ is a quasiminimal AEC. By Theorem 4.11, $(|M|, \mathrm{cl}^M)$ is a pregeometry for every $M \in \mathbf{K}$. The result now follows from Remark 4.6. $\square$

To prove tameness, we will use:

Fact 4.13 ([SV])**.** If an AEC $\mathbf{K}$ is stable in $\aleph_0$, has amalgamation in $\aleph_0$, and has joint embedding in $\aleph_0$, then $\mathbf{K}$ is $(< \aleph_0, \aleph_0)$-tame for types of finite length. That is, if $M \in \mathbf{K}_{\aleph_0}$ and $p \neq q$ are both in $\mathrm{gS}^{<\omega}(M)$, then there exists a finite $A \subseteq |M|$ such that $p \upharpoonright A \neq q \upharpoonright A$.

Lemma 4.14. If $\mathbf{K}$ is a quasiminimal AEC, then $\mathbf{K}$ is $(< \aleph_0, \aleph_0)$-tame for types of finite length.

Proof. This is a consequence of Fact 4.13 together with Lemmas 4.9 and 4.10. $\square$

We are now ready to state a correspondence between quasiminimal AECs and quasiminimal pregeometry classes. We will use the concept of a functorial expansion, which basically is an expansion of the vocabulary of the class that does not change anything about how the class behaves. Functorial expansions are defined in [Vas16b, Definition 3.1].

Definition 4.15. Let $\mathbf{K} = (K, \leq_{\mathbf{K}})$ be an abstract class. A *functorial expansion* of $\mathbf{K}$ is a class $\widehat{K}$ satisfying the following properties:

(1) $\widehat{K}$ is a class of $\hat{\tau}$-structures, where $\hat{\tau}$ is a fixed vocabulary extending $\tau(\mathbf{K})$.
(2) The map $\widehat{M} \mapsto \widehat{M} \upharpoonright \tau(\mathbf{K})$ is a bijection from $\widehat{K}$ onto K. For $M \in K$, we will write $\widehat{M}$ for the unique element of $\widehat{K}$ whose reduct is M. When we write "$\widehat{M} \in \widehat{K}$", it is understood that $M = \widehat{M} \upharpoonright \tau(K)$.
(3) Invariance: For $M, N \in K$, if $f : M \cong N$, then $f : \widehat{M} \cong \widehat{N}$.
(4) Monotonicity: If $M \leq_{\mathbf{K}} N$ are in K, then $\widehat{M} \subseteq \widehat{N}$.

Remark 4.16 (Proposition 3.8 in [Vas16b])**.** If $\widehat{K}$ is a functorial expansion of $\mathbf{K}$, then we can order $\widehat{K}$ by $\widehat{M} \leq_{\widehat{\mathbf{K}}} \widehat{N}$ if and only if $M \leq_{\mathbf{K}} N$. This gives an abstract class $\widehat{\mathbf{K}} := (\widehat{K}, \leq_{\widehat{\mathbf{K}}})$.

The specific functorial expansion we will use is what we call the $(< \aleph_0)$-Galois-Morleyization [Vas16b, Definition 3.3]. It consists in adding a relation for each Galois type over a "small" set (here small means finite, but in general for the $(< \kappa)$-Galois Morleyization small means "of size less than κ").

Definition 4.17. Let $\mathbf{K} = (K, \leq_{\mathbf{K}})$ be an AEC. Define an expansion $\hat{\tau}$ of $\tau(K)$ by adding a relation symbol R_p of arity $\ell(p)$ for each $p \in \mathrm{gS}^{<\omega}(\emptyset)$. Expand each $N \in K$ to a $\hat{\tau}$-structure $\widehat{N}$ by specifying that for each $\bar{a} \in {}^{<\omega}|\widehat{N}|$, $R_p^{\widehat{N}}(\bar{a})$ (where $R_p^{\widehat{N}}$ is the interpretation of R_p inside $\widehat{N}$) holds exactly when $\mathrm{gtp}(\bar{a}/\emptyset; N) = p$. Let $\widehat{\mathbf{K}}$ be the class of all such $\widehat{N}$, ordered as in Remark 4.16. We call $\widehat{\mathbf{K}}$ the $(< \aleph_0)$-*Galois Morleyization* of $\mathbf{K}$.

Remark 4.18. Let $\mathbf{K}$ be an AEC and let $\widehat{\mathbf{K}}$ be the $(< \aleph_0)$-Galois Morleyization of $\mathbf{K}$. Then $|\tau(\widehat{\mathbf{K}})| = |\mathrm{gS}^{<\omega}(\emptyset)| + |\tau(\mathbf{K})|$.

The basic facts about the Galois Morleyization that we will use are below. The most important says that if Galois types are determined by their finite restrictions, then they will become equivalent to quantifier-free types in the Galois Morleyization.

Fact 4.19. Let $\mathbf{K}$ be an AEC and let $\widehat{\mathbf{K}} = (\widehat{K}, \leq_{\widehat{\mathbf{K}}})$ be its $(< \aleph_0)$-Galois Morleyization.

(1) [Vas16b, Proposition 3.5] $\widehat{K}$ is a functorial expansion of $\mathbf{K}$.
(2) [Vas16b, Proposition 3.8] $\widehat{\mathbf{K}}$ is an AEC with $\mathrm{LS}(\widehat{\mathbf{K}}) = \mathrm{LS}(\mathbf{K}) + |\tau(\widehat{\mathbf{K}})|$.
(3) [Vas16b, Theorem 3.16] If $\mathbf{K}$ is $(< \aleph_0, \aleph_0)$-tame for types of finite length, then for any $M \in \widehat{\mathbf{K}}_{\leq \aleph_0}$, any two N, N' with $M \leq_{\widehat{\mathbf{K}}} N$, $M \leq_{\widehat{\mathbf{K}}} N'$ and any $\bar{a} \in {}^{<\omega}|N|$, $\bar{b} \in {}^{<\omega}|N'|$, if the $\tau(\widehat{\mathbf{K}})$-quantifier-free type of $\bar{a}$ over M inside N is the same as the $\tau(\widehat{\mathbf{K}})$-quantifier-free type of $\bar{b}$ over M inside N', then $\mathrm{gtp}(\bar{a}/M; N) = \mathrm{gtp}(\bar{b}/M; N')$. This also holds if M is empty.

We have arrived to the definition of the correspondence between quasiminimal AECs and quasiminimal pregeometry classes, and the proof that it works:

Definition 4.20. For $\mathbf{K}$, a quasimiminal AEC let $\mathcal{C}(\mathbf{K})$ be the class $\{(M, \mathrm{cl}^M) \mid M \in \widehat{\mathbf{K}}\}$, where $\widehat{\mathbf{K}}$ is the $(< \aleph_0)$-Galois Morleyization of $\mathbf{K}$.

Theorem 4.21. If $\mathbf{K}$ is a quasiminimal AEC, then $\mathcal{C}(\mathbf{K})$ is a quasiminimal pregeometry class, which is unbounded if and only if $\mathbf{K}$ is. Moreover $\mathbf{K}(\mathcal{C}(\mathbf{K}))$ is the $(< \aleph_0)$-Galois Morleyization of $\mathbf{K}$.

Proof. Let $\mathcal{C} := \mathcal{C}(\mathbf{K})$. It is clear that the elements of $\mathcal{C}$ are of the right form. The moreover part is clear from the definition of $\mathbf{K}(\mathcal{C}(\mathbf{K}))$. We check all the conditions of Definition 4.5. We will use without comments that $\mathbf{K}_{\leq \aleph_0}$ is has amalgamation and joint embedding (Lemma 4.9) and is stable in $\aleph_0$ (Lemma 4.10).

0: (1) Since $\mathbf{K}$ has a prime model (axiom (2) in Definition 4.1), $\mathcal{C} \neq \emptyset$. By Remark 4.18, $|\tau(\mathcal{C})| = |\tau(\widehat{\mathbf{K}})| \leq |\mathrm{gS}^{<\omega}(\emptyset)| + |\tau(\mathbf{K})|$. Since $\mathrm{LS}(\mathbf{K}) = \aleph_0$, we have that $|\tau(\mathbf{K})| \leq \aleph_0$. Using that $\mathbf{K} \neq \emptyset$, pick $M \in \mathbf{K}_{\leq \aleph_0}$. Since $\mathbf{K}_{\leq \aleph_0}$ has amalgamation and joint embedding, there is an injection from $\mathrm{gS}^{<\omega}(\emptyset)$ into $\mathrm{gS}^{<\omega}(M)$. By amalgamation and stability, there exists $M' \in \mathbf{K}_{\aleph_0}$ universal over M. Therefore $|\mathrm{gS}^{<\omega}(M)| \leq \aleph_0$. Thus $|\tau(\widehat{\mathbf{K}})| \leq \aleph_0$, as desired.
(2) This is clear. In fact, if $f : M \cong N$, then by definition of cl^M and cl^N, f is automatically an isomorphism from $(|M|, \mathrm{cl}^M)$ onto $(|N|, \mathrm{cl}^N)$.

(3) Let $(M, \mathrm{cl}^M), (N, \mathrm{cl}^N) \in \mathcal{C}$. If $M_0 \leq_{\widehat{\mathbf{K}}} M$, then $M_0 \subseteq M$ so M_0 and M satisfy the same quantifier-free sentences, so without loss of generality M and N are already countable. Now use that $\mathbf{K}_{\leq \aleph_0}$ has joint embedding (by Lemma 4.9).

I: (1) Let $(M, \mathrm{cl}^M) \in \mathcal{C}$. By Theorem 4.11, (M, cl^M) is a pregeometry. Moreover if $A \subseteq |M|$ is finite then $|\mathrm{cl}^M(A)| \leq \mathrm{LS}(\mathbf{K}) = \aleph_0$, as desired.

(2) Let $(M, \mathrm{cl}^M) \in \mathcal{C}$ and $X \subseteq |M|$. By definition of admitting intersections, $\mathrm{cl}^M(X) \leq_{\mathbf{K}} M$ and so the result follows.

(3) At that point it will be useful to prove a claim:
Claim: Let $(H, \mathrm{cl}_H), (H', \mathrm{cl}_{H'}) \in \mathcal{C}$. Let $G \subseteq H$ and $G' \subseteq H'$ be countable closed subsets or empty and let $g : G \to G'$ be an isomorphism. If $g \cup f : H \rightharpoonup H'$ is a partial embedding where f has finite preimage X, then for any enumeration $\bar{a}$ of X and $\bar{G}$ of G, $\mathrm{gtp}(\bar{a}\bar{G}/\emptyset; H) = \mathrm{gtp}(f(\bar{a})f(\bar{G})/\emptyset; H')$.
Proof of Claim: By renaming, we may assume that $G = G'$ so what we really have to prove is that $\mathrm{gtp}(\bar{a}/G; H) = \mathrm{gtp}(f(\bar{a})/G; H')$. Note that by assumption $\bar{a}$ and $f(\bar{a})$ satisfy the same $\tau(\widehat{\mathbf{K}})$-quantifier-free types over G. Therefore the result follows from Fact 4.19.(3) (recalling Lemma 4.14). $\dagger_{\text{Claim 1}}$
Now let $(M, \mathrm{cl}^M), (M', \mathrm{cl}^{M'}) \in \mathcal{C}$, $X \subseteq |M|$, $y \in \mathrm{cl}^M(X)$, and $f : M \rightharpoonup M'$ be a partial embedding with $X \cup \{y\} \subseteq \mathrm{preim}(f)$. We want to see that $f(y) \in \mathrm{cl}^{M'}(f[X])$. By finite character, we may assume without loss of generality that X is finite and therefore $\mathrm{preim}(f)$ is also finite. Let $\bar{a}$ be an enumeration of X. By the Claim, $\mathrm{gtp}(\bar{a}y/\emptyset; M) = \mathrm{gtp}(f(\bar{a})f(y)/\emptyset; M')$. The result now follows from the definition of the closure operator.

II: Let $(H, \mathrm{cl}_H), (H', \mathrm{cl}_{H'}) \in \mathcal{C}$. Let $G \subseteq H$ and $G' \subseteq H'$ be countable closed subsets or empty and let $g : G \to G'$ be an isomorphism.

(1) Let $x \in |H|$ and $x' \in |H'|$ be independent from G and G' respectively. We show that $g \cup \{(x, x')\}$ is a partial embedding. By renaming without loss of generality $G = G'$. By uniqueness of the generic type, $\mathrm{gtp}(x/G; H) = \mathrm{gtp}(x'/G; H')$. The result follows.

(2) Let $g \cup f : H \rightharpoonup H'$ be a partial embedding, where f has finite preimage X, and $y \in \mathrm{cl}_H(X \cup G)$. We want to find $y' \in H'$ such that $g \cup f \cup \{(y, y')\}$ is a partial embedding. Without loss of generality again, g is the identity. Let $\bar{a}$ be an enumeration of X. By the Claim, $\mathrm{gtp}(\bar{a}/G; H) = \mathrm{gtp}(f(\bar{a})/G; H')$. By Fact 3.7, there exists $h : \mathrm{cl}^H(G\bar{a}) \cong_G \mathrm{cl}^{H'}(G\bar{a}')$ such that $h(\bar{a}) = f(\bar{a})$. Let $y' := h(y)$.

IV: Assume that $\mathbf{K}$ is unbounded.

(1) Because $\mathbf{K}$ is an AEC and the closure operator has finite character.

(2) Since $\mathbf{K}$ is unbounded.

□

As a corollary, all the work on structural properties of quasiminimal pregeometry classes automatically applies also to quasiminimal AECs:

Corollary 4.22. Let $\mathbf{K}$ be a quasiminimal AEC.

(1) Let $M, N \in \mathbf{K}$ and let B_M, B_N be bases for $(|M|, \mathrm{cl}^M)$ and $(|N|, \mathrm{cl}^N)$ respectively. If f is a bijection from B_M onto B_N, then there exists an isomorphism $g : M \cong N$ with $f \subseteq g$.
(2) If $\mathbf{K}$ is unbounded, then:
  (a) $\mathbf{K}$ has no maximal models.
  (b) $\mathbf{K}$ has exactly $\aleph_0$ non-isomorphic countable models and $\mathbf{K}$ is categorical in every uncountable cardinal.

Proof. Let $\mathcal{C} := \mathcal{C}(\mathbf{K})$. By Theorem 4.21, $\mathcal{C}$ is a quasiminimal pregeometry class. By [BHH+14], it also satisfies the excellence axiom. By Zilber's main result on these classes [Zil05a] (or see [Kir10, Theorem 3.3] for an exposition), (1) holds for $\mathcal{C}$. Therefore it also holds for $\mathbf{K}(\mathcal{C})$, which is a just a functorial expansion of $\mathbf{K}$. Hence it also holds for $\mathbf{K}$. Similarly, (2) holds in unbounded quasimininal AECs (see [Kir10, §4]). $\square$

5. On a counterexample of Shelah

We give a (non-quasiminimal) example, due to Shelah, where the exchange property fails. We show that, in this example, there is a good frame which cannot be extended to be type-full, answering a question of Boney and the author [BV, Question 1.3]. We assume familiarity with good frames in this section, see [She09a, Chapter II]; we use the definition from [JS13, Definition 2.1.1].

The following definitions come from [She09b, Exercise VII.5.7]:

Definition 5.1.

(1) Let τ^* be the vocabulary consisting of only a single a unary function symbol F.
(2) Let ψ_0 be the following first-order τ^*-sentence:
$$\forall x : F(F(x)) = F(x)$$
(3) Let ψ be the following first-order τ^*-sentence:
$$\psi_0 \wedge \forall x \forall y : F(x) \neq x \wedge F(y) \neq y \to F(x) = F(y)$$
(4) Let $\phi(x)$ be the sentence $\exists y : F(y) = x \wedge y \neq x$.
(5) Let K^* be the class of τ^*-structures that satisfy ψ.
(6) Say $M \leq_{\mathbf{K}^*} N$ if $M, N \in K^*$ and $M \subseteq N$.
(7) Let $\mathbf{K}^* := (K^*, \leq_{\mathbf{K}^*})$.

Remark 5.2. If $M \in \mathbf{K}^*$, then $|\phi(M)| \leq 1$.

Recall (see [She87b]) that a class K of structures in a fixed vocabulary is a *universal class* if it is closed under isomorphisms, substructures, and unions of $\subseteq$-increasing chains. It is straightforward to check that K^* is a universal class, and it induces the AEC $\mathbf{K}^*$ with Löwenheim-Skolem-Tarski number $\aleph_0$. Thus it admits intersections. Moreover:

Fact 5.3 ($(*)_2$ in VII.5.7 of [She09b])**.** $\mathbf{K}^*$ has amalgamation.

Lemma 5.4. For any $M \in \mathbf{K}^*$, $2 \leq |\mathrm{gS}^{\mathrm{na}}(M)| \leq 3$.

Proof. For $M \leq_{\mathbf{K}^*} N$, there are three kinds of types realized in N (not necessarily all non-algebraic):

(1) The type of $a \in \phi(N)$.
(2) The type of $a \notin \phi(N)$ with $F^N(a) \neq a$.
(3) The type of $a \notin \phi(N)$ with $F^N(a) = a$.

Taking a suitable N, an instance of the last two can be found that is nonalgebraic. $\square$

Fact 5.5 ($(*)_8$ in VII.5.7 of [She09b])**.** $\mathbf{K}^*$ fails disjoint amalgamation (in any infinite cardinal).

Proof. Take $M_0 \leq_{\mathbf{K}^*} M_1 = M_2$ with $\phi(M_0) = \emptyset$, $\phi(M_\ell) = \{b\}$, $\ell = 1, 2$. Then it is clear that M_1 and M_2 cannot be disjointly amalgamated over M_0 (as any disjoint amalgam N would have to satisfy $|\phi(N)| \geq 2$). $\square$

Lemma 5.6. For any $N \in \mathbf{K}^*$ with $|\phi(N)| = 1$, $(|N|, \mathrm{cl}^N)$ does not have exchange over $\emptyset$.

Proof. First note that $\mathrm{cl}^N(\emptyset) = \emptyset$. Now pick $b \in \phi(N)$ and let $a \neq b$ be such that $F^N(a) = b$. Then $b \in \mathrm{cl}^N(a)$. However $F^N(b) = F^N(F^N(a)) = F^N(a) = b$, so $a \notin \mathrm{cl}^N(b)$. $\square$

For the rest of this section, we fix an infinite cardinal $\lambda \geq \aleph_0$. We define a good λ-frame $\mathfrak{s}$ (whose definition appears already in [She09b, VII.5.7]) and an object $\mathfrak{s}'$ satisfying all the axioms of good frames except existence. It turns out that $\mathfrak{s}'$ would be the only type-full extension of $\mathfrak{s}$, so $\mathfrak{s}$ cannot be extended to be type-full.

Definition 5.7. Define pre-λ-frames (see [She09a, Definition III.0.2]) $\mathfrak{s}, \mathfrak{s}'$ as follows:

(1) The underlying class of $\mathfrak{s}$ and $\mathfrak{s}'$ is $(\mathbf{K}^*)_\lambda$.
(2) The basic types of $\mathfrak{s}'$ are all the nonalgebraic types. The basic types of $\mathfrak{s}$ are all the nonalgebraic types of the form $\mathrm{gtp}(a/M; N)$ with $a \notin \phi(N)$.
(3) In both frames, $\mathrm{gtp}(a/M; N)$ does not fork over M_0 if and only if $F^N(a) \notin |N| \backslash |M|$.

Theorem 5.8.

(1) $\mathfrak{s}'$ is type-full but $\mathfrak{s}$ is not.
(2) The frames satisfy all the properties of good frames except perhaps existence.
(3) $\mathfrak{s}$ has existence. Therefore it is a good λ-frame.
(4) $\mathfrak{s}'$ fails existence.
(5) $\mathfrak{s}$ cannot be extended to be type-full.

Proof.

(1) Clear from the definitions.
(2) Straightforward. For example:

- Density of basic types in $\mathfrak{s}$: let $M <_{\mathbf{K}^*} N$. We may assume that $|\phi(M)| = 0$, $|\phi(N)| = 1$, so let $b \in \phi(N)$. Fix $a \neq b$ such that $F^N(a) = b$. Now since $b \notin |M|$, $a \notin |M|$, so $\text{gtp}(a/M; N)$ is basic.
- Uniqueness in $\mathfrak{s}'$: Use the description of the nonalgebraic types in the proof of Lemma 5.4.
- Symmetry: Suppose that (in one of the two frames) we have $a \underset{M_0}{\overset{N}{\perp}} M$. Let $b \in |M|$ be such that $\text{gtp}(b/M_0; M)$ is basic. Let $M_a := \text{cl}^N(M_0 a)$. Note that $|M_a| = |M_0| \cup \{a, F^N(a)\}$. We claim that $b \underset{M_0}{\overset{N}{\perp}} M_a$. Note that since $\text{gtp}(b/M_0; N)$ is basic, then also $\text{gtp}(b/M_a; N)$ is basic (check it for each frame). It remains to see that $F^N(b) \notin |M_a| \backslash |M_0|$. First note that as $a \notin |M|$, $F^N(b) = F^M(b) \neq a$. Further, if $F^N(a) \notin |M_0|$, then $F^N(a) \notin |M|$, so $F^N(a) \neq F^N(b) = F^M(b)$, as desired.

(3) Straightforward.
(4) As in the proof of Fact 5.5.
(5) Any type-full extension of $\mathfrak{s}$ would have to be $\mathfrak{s}'$ and we have shown that $\mathfrak{s}'$ fails existence.

$\square$

The following question seems much harder:

Question 5.9. Is there an example of a good-λ-frame which is categorical in λ and cannot be extended to be type-full?

Note that by [SV], if $\mathbf{K}_{\leq \aleph_0}$ is categorical, has amalgamation, joint embedding, no maximal models, and is stable in $\aleph_0$, then it has a type-full good $\aleph_0$-frame. Therefore by canonicity of categorical good frames [Vas16a, Theorem 9.7], the answer is negative when $\lambda = \aleph_0$.

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