CHAINS OF SATURATED MODELS IN AECS

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Abstract. We study when a union of saturated models is saturated in the framework of tame abstract elementary classes (AECs) with amalgamation. Under a natural superstability assumption (which follows from categoricity in a high-enough cardinal), we prove:

Theorem 0.1. If K is a tame superstable AEC with amalgamation, then for all high-enough λ:
(1) The union of an increasing chain of λ-saturated models is λ-saturated.
(2) There exists a type-full good λ-frame with underlying class the saturated models of size λ.
(3) There exists a unique limit model of size λ.

Our proofs use independence calculus and a generalization of averages to this non first-order context.

Contents

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1. Introduction

Determining when a union of λ-saturated model is λ-saturated is an important dividing line in first-order model theory. Recall that Harnik and Shelah have shown:

Fact 1.1 ([Har75], Theorem III.3.11 in [She90] for the case $\lambda \leq |T|$)**.** Let T be a first-order theory.

- If T is superstable, then any increasing union of λ-saturated models is λ-saturated.
- If T is stable, then any increasing union of λ-saturated models of cofinality at least $|T|^+$ is λ-saturated.

A converse was later proven by Albert and Grossberg [AG90, Theorem 13]. Fact 1.1 can be used to prove:

Fact 1.2 (The saturation spectrum theorem, VIII.4.7 in [She90])**.** Let T be a stable first-order theory. Then T has a saturated model of size λ if and only if [T is stable in λ or $\lambda = \lambda^{<\lambda} + |D(T)|$].

Although not immediately evident from the statement, the proof of Fact 1.1 relies on the heavy machinery of forking and averages.

While the saturation spectrum theorem has been generalized to homogeneous model theory (see [She78, Theorem 1.13] or [GL02, Theorem 5.9]), to the best of our knowledge no explicit generalization of Fact 1.1 has been published in this context. Grossberg [Gro91] has proven a version of Fact 1.1 in the framework of stability theory inside a model. The proof uses averages but relies on a strong negation of the order property. Makkai and Shelah [MS90, Proposition 4.18] have given a generalization in the class of models of an $L_{\kappa,\omega}$ sentence where κ is a strongly compact cardinal. The proof uses independence calculus.

One can ask whether Fact 1.1 can also be generalized to abstract elementary classes (AECs), a general framework for classification theory introduced in [She87] (see [Gro02] for an introduction to AECs). In [She09a, Theorem I.5.39], Shelah proves a generalization of the superstable case of Fact 1.1 to "definable-enough" AECs with countable Löwenheim-Skolem number, using the weak continuum hypothesis.

In chapter II of [She09a], Shelah starts with a (weakly successful) good λ-frame (a local notion of superstability) on an abstract elementary class (AEC) K and wants to show that a union of saturated models is saturated in K_{λ^+}. For this purpose, he introduces a restriction $\leq^*$ of the ordering that allows him to prove the result for $\leq^*$-increasing chains (Theorem II.7.7 there). [She09a, Theorem I.5.39] is another generalization to the case of AECs with countable Löwenheim-Skolem number. Restricting the ordering of the AEC is somewhat artificial and one can ask what happens in the general case, and also if λ^+ is replaced by an arbitrary cardinal. Moreover, Shelah's methods to obtain a weakly successful good λ-frame typically use categoricity in two successive cardinals and the weak continuum hypothesis[1].

In [She99], Shelah had previously proven that a union of λ-saturated models is λ-saturated, for K an AEC with amalgamation, joint embedding, and no maximal models categorical in a successor $\lambda' > \lambda$ (see [Bal09, Chapter 15] for a writeup), but left the case $\lambda \geq \lambda'$ (or λ' not a successor) unexamined.

In this paper, we replace the local model-theoretic assumptions of Shelah with *global* ones, including *tameness*, a locality notion for types introduced by Grossberg and VanDieren [GV06]. We take advantage of recent developments in forking in tame AECs (especially by Boney and Grossberg [BG] and Vasey [Vasc, Vasb]) to generalize Fact 1.1 to tame abstract elementary classes with amalgamation. Our main result is:

Theorem 1.3. Assume K is a $(< \kappa)$-tame AEC with amalgamation. If $\kappa = \beth_\kappa > \mathrm{LS}(K)$ and K is categorical in some cardinal above κ, then for all $\lambda > (2^\kappa)^+$, $K^{\lambda\text{-sat}}$ (the class of λ-saturated models of K) is an AEC with $\mathrm{LS}(K^{\lambda\text{-sat}}) = \lambda$.

Proof. By Fact 2.21 and Corollary 4.5, the union of any chain of λ-saturated models is λ-saturated. Imitate the proof of [She90, Theorem III.3.12] to see that $\mathrm{LS}(K^{\lambda\text{-sat}}) = \lambda$. $\square$

Notice that if $K^{\lambda\text{-sat}}$ is an AEC, then any increasing union of λ-saturated models is λ-saturated. Thus, in contrast to Shelah's [She99] result, we obtain a *global* theorem that holds for all high-enough λ and not just those under the categoricity cardinal. Furthermore categoricity at a

[1]See for example [She09a, Theorem II.3.7]. Shelah also shows how to build a good frame in ZFC from more model-theoretic hypotheses in [She09a, Theorem IV.4.10], but he has to change the class and it is not clear his frame is weakly successful.

successor is not assumed. We can also replace the categoricity by various notions of superstability defined in terms of the local character for independence notions such as coheir or splitting. In fact, we can combine this result with the construction of a good frame in [Vasb] to obtain the theorem in the abstract (see Theorem 7.1 for a proof):

Theorem 1.4. If K is a tame superstable AEC with amalgamation (see Definition 2.18), then for all high-enough λ, there exists a *unique* limit model of size λ.

This proves an eventual version of a statement appearing in previous versions of [GVV] (see the discussion in Section 7).

It is very convenient to have $K^{\lambda\text{-sat}}$ an AEC, as saturated models are typically better behaved than arbitrary ones. This is crucial for example in Shelah's upward transfer of frames in [She09a, Chapter II], and is also used in [Vasb] to build an ω-successful good frame (and later a global independence notion). We also prove a result for the strictly stable case (see Theorem 6.10 for a proof):

Theorem 1.5. Let K be a stable $(<\kappa)$-tame AEC with amalgamation, $\kappa \geq \text{LS}(K)$. Then there exists $\chi_0 \leq \lambda_0 < \beth_{(2^\kappa)^+}$ such that whenever $\lambda \geq \lambda_0$ is such that $\mu^{<\chi_0} < \lambda$ for all $\mu < \lambda$, the union of an increasing chain of λ-saturated models of cofinality at least χ_0 is λ-saturated.

One caveat here is the introduction of cardinal arithmetic. When dealing with compact classes (or even just $(<\omega)$-tame classes), the map $\lambda \mapsto \lambda^{<\omega}$ can be used freely. Even in the work of Makkai and Shelah [MS90], where κ is strongly compact and the class is $(<\kappa)$-tame, the map $\lambda \mapsto \lambda^{<\kappa}$ is constant on most cardinals (those with cofinality at least κ) by a result of Solovay. However, in our context of $(<\kappa)$-tameness for $\kappa > \omega$ but not strongly compact, this function can be much wilder. Thus, we need to introduce assumptions that this map is well-behaved. Using various tricks, we can bypass these assumptions in the superstable case but are unable to do so in the stable case. For example in the theorem above, the cardinal arithmetic assumption can be replaced by "K is stable in μ for unboundedly many $\mu < \lambda$", which is always true in case K is superstable.

We use two main methods: The first method is pure independence calculus, relying on a well-behaved independence relation (coheir), whose existence in our context is proven in [BG, Vasc]. This works well in the superstable case if we define superstability in terms of coheir (called strong superstability in [Vasb]) but we do not know how to make it work for weaker definitions of superstability (such as superstability defined in terms of splitting, a more classical definition implicit for example in [GVV]). The second method is the use of syntactic averages, developed by Shelah in [She09b, Chapter V]. We end up proving a result on chains of saturated models in the framework of stability theory inside a model and then translate to AECs using *Galois Morleyization*, introduced in [Vasc]. This method typically gives lower Hanf numbers and allows us to use superstability defined in terms of splitting.

The paper is organized as follows. Section 2 introduces the necessary preliminaries, but contains nothing new[2] to those familiar with [Vasb]. Section 3 gives the argument using independence calculus culminating in Theorems 3.14 and 4.4. Both of these arguments work just using forking relations, drawing inspiration from Makkai and Shelah, rather than the classical first-order argument using averages. Section 5 develops averages in our context based on earlier work of Shelah and culminates in the more local Theorem 5.23. Section 6 translates the local result to AECs and Section 7 proves consequences such as the uniqueness of limit models from superstability.

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2. Preliminaries

We review some of the basics of abstract elementary classes and fix some notation. The reader is advised to skim through this section quickly and go back to it as needed. We assume familiarity with a basic text on AECs such as [Bal09] or [Gro] and refer the reader to the preliminaries of [Vasc] for more details and motivations on the concepts used in this paper.

2.1. **Set theoretic terminology.** We will often use the following function:

Definition 2.1 (Hanf function)**.** For λ an infinite cardinal, define $h(\lambda) := \beth_{(2^\lambda)^+}$.

Note that for λ infinite, $\lambda = \beth_\lambda$ if and only if for all $\mu < \lambda$, $h(\mu) < \lambda$.

Definition 2.2. For κ an infinite cardinal, let κ_r be the least regular cardinal $\geq \kappa$. That is, κ_r is κ^+ if κ is singular and κ otherwise.

[2] Except for the second part of Proposition 2.15 which we do not use.

2.2. **Abstract classes.** Recall[3] [Vasc, Definition 2.7] that an abstract class (AC for short) is a pair $(K, \leq)$, where K is a class of structures of the same (possibly infinitary) language and $\leq$ is an ordering on K extending substructure and respecting isomorphisms. We will use the same notation as in [Vasc]; for example $M < N$ means $M \leq N$ and $M \neq N$, and $K_{\geq \lambda} := \{M \in K \mid \|M\| \geq \lambda\}$.

At one point in this paper, we will mention μ-AEC. μ-AECs will be introduced in details in [BGV]. A formal definition appears as [Vasc, Definition 2.13]. Here it suffices to say that μ-AECs are generalizations of AECs where we only require the language to be $(< \mu)$-ary and the class to be closed under chains of cofinality $\geq \mu$. For example, $\aleph_0$-AECs are exactly AECs.

Recall also from [Vasb, Definition 2.10] that a *coherent abstract class* is an abstract class satisfying the coherence axiom (that is, if $M_0 \subseteq M_1 \leq M_2$ and $M_0 \leq M_2$, then $M_0 \leq M_1$). We will use the following weakening of the existence of a Löwenheim-Skolem number:

Definition 2.3 (Definition 2.11 in [Vasb])**.** An abstract class K is $(< \lambda)$-*closed* if for any $M \in K$ and $A \subseteq |M|$ with $|A| < \lambda$, there exists $M_0 \leq M$ which contains A and has size less than λ. λ-closed means $(< \lambda^+)$-closed.

Remark 2.4. An AEC K is $(< \lambda)$-closed in every $\lambda > \mathrm{LS}(K)$.

As in the preliminaries of [Vasc], we can define a notion of embedding for abstract classes and go on to define amalgamation, joint embedding, no maximal models, Galois types, and tameness.

Using Galois types, a natural notion of saturation can be defined (see [Vasc, Definition 2.22] for more explanation on the definition):

Definition 2.5. Let K be an abstract class and μ be an infinite cardinal.

(1) For $N \in K$, $A \subseteq |N|$ is *μ-saturated for types of length α in N* if for any $A_0 \subseteq A$ of size less than μ, any $p \in \mathrm{gS}^{\alpha}(A_0; N)$ is realized inside A. Define *μ-saturated for types of length less than α in N* similarly. A is *μ-saturated in N* if it is μ-saturated for types of length less than μ in N.
(2) A model $M \in K$ is *μ-saturated for types of length α* if it is μ-saturated for types of length α in N for all $N \geq M$. When $\mu = \|M\|$, we omit it.

[3] The definition is due to Rami Grossberg and appears in [Gro].

(3) We write $K^{\mu\text{-sat}}$ for the class of μ-saturated models of $K_{\geq \mu}$ (ordered by the ordering of K).

Remark 2.6.

(1) This is a semantic (Galois) notion of saturation, different from the syntactic one defined at the beginning of Section 5. It will always be clear from context which we refer to.
(2) It is easy to check that if K is an AEC with amalgamation, $\mu > \mathrm{LS}(K)$, then $K^{\mu\text{-sat}}$ is a μ-AEC with $\mathrm{LS}(K^{\mu\text{-sat}}) = \mathrm{LS}(K)^{<\mu}$.
(3) By [She09a, Lemma II.1.14], if K is an AEC with amalgamation and $\mu > \mathrm{LS}(K)$, $M \in K$ is μ-saturated if and only if M is μ-saturated for types of length one[4]. We will use this fact freely.

We recall there is a natural notion of stability in this context. This paper's definition follows [Vasc, Definition 2.20].

Definition 2.7 (Stability)**.** Let α be a cardinal, μ be a cardinal. A model $N \in K$ is $(< \alpha)$-*stable*[5] *in* μ if for all $A \subseteq |N|$ of size $\leq \mu$, $|\mathrm{gS}^{<\alpha}(A; N)| \leq \mu$. Here and below, α-stable means $(< (\alpha^+))$-stable. We say "stable" instead of "1-stable".

K is $(< \alpha)$-*stable in* μ if every $N \in K$ is $(< \alpha)$-stable in μ. K is $(< \alpha)$-*stable* if it is $(< \alpha)$-stable in unboundedly[6] many cardinals.

A corresponding definition of the order property in AECs appears in [She99, Definition 4.3]. For simplicity, we have removed one parameter from the definition.

Definition 2.8. Let α and μ be cardinals and let K be an abstract class. A model $M \in K$ has the *α-order property of length μ* if there exists $\langle \bar{a}_i : i < \mu \rangle$ inside M with $\ell(\bar{a}_i) = \alpha$ for all $i < \mu$, such that for any $i_0 < j_0 < \mu$ and $i_1 < j_1 < \mu$, $\mathrm{gtp}(\bar{a}_{i_0}\bar{a}_{j_0}/\emptyset; N) \neq \mathrm{gtp}(\bar{a}_{j_1}\bar{a}_{i_1}/\emptyset; N)$.

M has the *$(< \alpha)$-order property of length μ* if it has the β-order property of length μ for some $\beta < \alpha$. M has the *order property of length μ* if it has the α-order property of length μ for some α.

K has the α-order of length μ if some $M \in K$ has it. *K has the order property* if it has the order property for every length.

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[4] But this need not hold if K is just a coherent AC.

[5] We could have called this *Galois* stability to distinguish it from *syntactic* stability defined at the beginning of Section 5. However it will always be clear from context which notion we refer to.

[6] Note ([GV06, Corollary 6.4]) that in a $\mathrm{LS}(K)$-tame AEC with amalgamation, this is equivalent to stability in *some* cardinal.

The following sums up all the results we will use about stability and the order property:

Fact 2.9. Let K be an AEC.

(1) [Vasc, Lemma 4.8] Let $\kappa = \beth_\kappa > \mathrm{LS}(K)$. The following are equivalent:
  (a) K has the $(< \kappa)$-order property of length κ.
  (b) K has the $(< \kappa)$-order property.
(2) [Vasc, Theorem 4.13, Fact 4.5] Assume K is $(< \kappa)$-tame and has amalgamation. The following are equivalent:
  (a) K is stable in some $\lambda \geq \kappa + \mathrm{LS}(K)$.
  (b) There exists $\mu \leq \lambda_0 < h(\kappa + \mathrm{LS}(K))$ such that K is stable in any $\lambda \geq \lambda_0$ with $\lambda = \lambda^{<\mu}$.
  (c) K does not have the order property.
  (d) There exists $\chi < h(\kappa + \mathrm{LS}(K))$ such that K does not have the $(< \kappa)$-order property of length χ.

2.3. **Independence relations.** We repeat the definition of an independence relation from [Vasb, Definition 3.6], omitting some parameters that we do not need. See the aforementioned paper for motivation and background.

Definition 2.10 (Independence relation)**.** Let α be a cardinal, λ be an infinite cardinals. A $(< \alpha, \geq \lambda)$-*independence relation* is a pair $\mathfrak{i} = (K, \mathop{\smile\!\!\!\!\!\!\,|}\,)$, where:

(1) $K = K_{\geq \lambda}$ is a coherent abstract class with amalgamation.
(2) $\mathop{\smile\!\!\!\!\!\!\,|}\,$ is a relation on quadruples of the form (M, A, B, N), where $M \leq N$ are all in K, $A \subseteq |N|$ is such that $|A \backslash |M|| < \alpha$. We write $\mathop{\smile\!\!\!\!\!\!\,|}\,(M, A, B, N)$ or $A \overset{N}{\underset{M}{\mathop{\smile\!\!\!\!\!\!\,|}\,}} B$ instead of $(M, A, B, N) \in \mathop{\smile\!\!\!\!\!\!\,|}\,$.
(3) The following properties hold:
  (a) <u>Invariance</u>: If $f : N \cong N'$ and $A \overset{N}{\underset{M}{\mathop{\smile\!\!\!\!\!\!\,|}\,}} B$, then $f[A] \overset{N'}{\underset{f[M]}{\mathop{\smile\!\!\!\!\!\!\,|}\,}} f[B]$.
  (b) <u>Monotonicity</u>: Assume $A \overset{N}{\underset{M}{\mathop{\smile\!\!\!\!\!\!\,|}\,}} B$. Then:
    (i) Ambient monotonicity: If $N' \geq N$, then $A \overset{N'}{\underset{M}{\mathop{\smile\!\!\!\!\!\!\,|}\,}} B$. If $M \leq N_0 \leq N$ and $A \cup B \subseteq |N_0|$, then $A \overset{N_0}{\underset{M}{\mathop{\smile\!\!\!\!\!\!\,|}\,}} B$.
    (ii) Left and right monotonicity: If $A_0 \subseteq A$, $B_0 \subseteq B$, then $A_0 \overset{N}{\underset{M}{\mathop{\smile\!\!\!\!\!\!\,|}\,}} B_0$.

(c) Left and right normality: If $A \overset{N}{\underset{M}{\mathop{\smile}\limits^{\vert}}} B$, then $AM \overset{N}{\underset{M}{\mathop{\smile}\limits^{\vert}}} BM$.

When α or λ are clear from context or irrelevant, we omit them and just say that $\mathfrak{i}$ is an independence relation. A $(\leq \alpha, \geq \lambda)$-independence relation is just a $(< (\alpha + 1))$-independence relation.

Notation 2.11. For $\mathfrak{i} = (K, \mathop{\smile}\limits^{\vert})$ an independence relation, we write $\mathop{\smile}\limits^{\vert}(M_0, \bar{a}, M, N)$ or $\bar{a} \overset{N}{\underset{M_0}{\mathop{\smile}\limits^{\vert}}} M$ for $\operatorname{ran}(\bar{a}) \overset{N}{\underset{M_0}{\mathop{\smile}\limits^{\vert}}} M$ (similarly when other parameters are sequences). When $p \in \operatorname{gS}^{<\infty}(M)$, we say p *does not* $\mathfrak{i}$*-fork over* M_0 (or just *does not fork over* M_0 if $\mathfrak{i}$ is clear from context) if $\bar{a} \overset{N}{\underset{M_0}{\mathop{\smile}\limits^{\vert}}} M$ whenever $p = \operatorname{gtp}(\bar{a}/M; N)$ (it is easy to check that this does not depend on the choice of representatives).

Notation 2.12. Let $\mathfrak{i} = (K, \mathop{\smile}\limits^{\vert})$ be a $(< \alpha, \geq \lambda)$-independence relation.

(1) For $\alpha_0 \leq \alpha$, $\mathfrak{i}^{<\alpha_0}$ denotes the $(< \alpha_0, \geq \lambda)$ independence relation obtained by restricting the types to have length less than α_0.
(2) For $K' \subseteq K$ a coherent AC (with the ordering inherited from K), let $\mathfrak{i} \upharpoonright K^1$ be the $(< \alpha, \geq \lambda)$-independence relation obtained by restricting the underlying class to K'. We also use the notation $\mathfrak{i}_\lambda := \mathfrak{i} \upharpoonright K_\lambda$.

Next, we recall the names of some properties that an independence relation may or may not have. This is a shortened version of [Vasb, Definition 3.10].

Definition 2.13 (Properties of independence relations)**.** Let $\mathfrak{i} := (K, \mathop{\smile}\limits^{\vert})$ be a $(< \alpha, \geq \lambda)$-independence relation.

- $\mathfrak{i}$ has *right base monotonicity* if $A \overset{N}{\underset{M}{\mathop{\smile}\limits^{\vert}}} B$ and $M \leq M' \leq N$, $|M'| \subseteq B \cup |M|$ implies $A \overset{N}{\underset{M'}{\mathop{\smile}\limits^{\vert}}} B$.
- $\mathfrak{i}$ has *right transitivity* if whenever $M_0 \leq M_1 \leq N$, $A \overset{N}{\underset{M_0}{\mathop{\smile}\limits^{\vert}}} M_1$ and $A \overset{N}{\underset{M_1}{\mathop{\smile}\limits^{\vert}}} B$ implies $A \overset{N}{\underset{M_0}{\mathop{\smile}\limits^{\vert}}} B$. *Strong right transitivity* is the same property when we do not require $M_0 \leq M_1$.
- $\mathfrak{i}$ has *right uniqueness* if whenever $M_0 \leq M \leq N_\ell$, $\ell = 1, 2$, $|M_0| \subseteq B \subseteq |M|$, $q_\ell \in \operatorname{gS}^{<\alpha}(B; N_\ell)$, $q_1 \upharpoonright M_0 = q_2 \upharpoonright M_0$, and q_ℓ does not fork over M_0, then $q_1 = q_2$. $\mathfrak{i}$ has *right uniqueness*

for unions of models if this is true when $B = \bigcup_{i<\delta} |M_i'|$, with $\langle M_i' : i < \delta \rangle$ an increasing chain in K.
- $\mathfrak{i}$ has *symmetry* if $A \overset{N}{\underset{M}{\perp}} B$ implies $B_0 \overset{N}{\underset{M}{\perp}} A$ for all $B_0 \subseteq B$ of size less than α.
- $\mathfrak{i}$ has the *right μ-model-witness property* if whenever $M \leq M' \leq N$, $A \subseteq |N|$, and $A \overset{N}{\underset{M}{\perp}} B_0$ for all $B_0 \subseteq |M'|$ of size $\leq \mu$, then $A \overset{N}{\underset{M}{\perp}} M'$.

We naturally define the "left" version of the properties, see [Vasb, Definition 3.13] for a rigorous way to do it using the dual independence relation. We will often omit "right" from the name of a property.

Definition 2.14 (Locality cardinals, Definition 3.14 in [Vasb])**.** Let $\mathfrak{i} = (K, \perp)$ be a $(< \alpha, \geq \lambda)$-independence relation. Let $\alpha_0 < \alpha$.

(1) Let $\bar{\kappa}_{\alpha_0}(\mathfrak{i})$ be the minimal cardinal $\mu \geq |\alpha_0|^+ + \lambda^+$ such that for any $M \leq N$ in K, any $A \subseteq |N|$ with $|A| \leq \alpha_0$, there exists $M_0 \leq M$ in $K_{<\mu}$ with $A \overset{N}{\underset{M_0}{\perp}} M$. When μ does not exist, we set $\bar{\kappa}_{\alpha_0}(\mathfrak{i}) = \infty$.
(2) For R a binary relation on K, Let $\kappa_{\alpha_0}(\mathfrak{i}, R)$ be the minimal cardinal $\mu \geq |\alpha_0|^+ + \aleph_0$ such that for any regular $\delta \geq \mu$, any R-increasing (that is, $i < j < \delta$ implies $M_i R M_j$) $\langle M_i : i < \delta \rangle$ in K, any N extending all the M_i's, and any $A \subseteq |N|$ of size $\leq \alpha_0$, there exists $i < \delta$ such that $A \overset{N}{\underset{M_i}{\perp}} M_\delta$. Here, we have set[7] $M_\delta := \bigcup_{i<\delta} M_\delta$. When $R = \leq$, we omit it. When μ does not exist, we set $\kappa_{\alpha_0}(\mathfrak{i}) = \infty$.

When K is clear from context, we may write $\bar{\kappa}_{\alpha_0}(\perp)$. For $\alpha_0 \leq \alpha$, we also let $\bar{\kappa}_{<\alpha_0}(\mathfrak{i}) := \sup_{\alpha_0' < \alpha_0} \bar{\kappa}_{\alpha_0'}(\mathfrak{i})$. Similarly define $\kappa_{<\alpha_0}$.

The next proposition gives the relationship between the chain (κ_α) and set ($\bar{\kappa}_\alpha$) locality cardinals. The second part (not used in the rest of this paper) is new and imitates an argument in [Bal88, Proposition 4.18]. This adds to the discussion on p. 19 of [Dru13], showing that in AECs, $\kappa_\alpha(\mathfrak{i}) < \infty$ if and only if $\bar{\kappa}_\alpha(\mathfrak{i}) < \infty$.

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[7]Recall that K is only a coherent abstract class, so may not be closed under unions of chains of length δ.

Proposition 2.15. Let $\mathfrak{i} = (K, \mathop{\copy\noforkbox}\limits)$ be a $(< \alpha, \geq \lambda)$-independence relation.

(1) If K is a μ-AEC, then $\kappa_{<\alpha}(\mathfrak{i}) \leq \bar{\kappa}_{<\alpha}(\mathfrak{i}) + \mu$.
(2) Conversely, if K is λ-closed, $\mathfrak{i}$ has base monotonicity, transitivity, and the λ-model-witness property, then $\bar{\kappa}_{<\alpha}(\mathfrak{i}) \leq (\kappa_{<\alpha}(\mathfrak{i}) + \lambda^+)^+$.

Proof.

(1) Straightforward (see e.g. the proof of [Vasb, Proposition 4.3.5]).
(2) We show that for any regular $\mu \geq \lambda^+$, if $\mu < \bar{\kappa}_{<\alpha}(\mathfrak{i})$ then $\mu < \kappa_{<\alpha}(\mathfrak{i})$. This is enough: if $\bar{\kappa}_{<\alpha}(\mathfrak{i})$ is a limit cardinal, we have shown that $\bar{\kappa}_{<\alpha}(\mathfrak{i}) \leq \kappa_{<\alpha}(\mathfrak{i}) + \lambda^+$. If $\bar{\kappa}_{<\alpha}(\mathfrak{i}) = \theta^+$, then if θ is regular, we get the result and if it is not, then it is limit and we can apply the previous sentence.

Since $\mu < \bar{\kappa}_{<\alpha}(\mathfrak{i})$, there exists $M \in K$, $p \in \text{gS}^{<\alpha}(M)$ such that for all $M_0 \leq M$ of size less than μ, p forks over M_0. Build $\langle M_i : i < \mu \rangle$ increasing such that for all $i < \mu$:
(a) $p \upharpoonright M_{i+1}$ forks over M_i.
(b) $M_i \in K_{<\mu}$.
(c) $M_i \leq M$.

<u>This is possible</u>. Assume inductively that $\langle M_j : j < i \rangle$ have been defined. If i is limit or zero, Pick any $M_i \leq M$ in $K_{<\mu}$ containing each M_j, $j < i$. If $i = j + 1$ is successor, by assumption p forks over M_j. By the witness property, we can pick $M_i \leq M$ containing M_j such that $M_i \in K_{<\mu}$ and $p \upharpoonright M_i$ forks over M_j.

<u>This is enough</u>. Let $M_\mu := \bigcup_{i<\mu} M_i$. Assume for a contradiction $\mu \geq \kappa_{\alpha_0}(\mathfrak{i})$. Then there exists $i < \mu$ such that $p \upharpoonright M_\mu$ does not fork over M_i. In particular, $p \upharpoonright M_{i+1}$ does not fork over M_i, a contradiction.

$\square$

The two main examples of independence relations we will consider are coheir and splitting. Coheir for AECs was introduced in [BG], and splitting comes from [She99]. We use the definitions from [Vasb, Section 3].

Definition 2.16.

(1) ($(< \kappa)$-coheir) Let K be an AEC with amalgamation and let $\kappa > \text{LS}(K)$. Define $\mathfrak{i}_{\kappa\text{-ch}}(K) := (K^{\kappa\text{-sat}}, \mathop{\copy\noforkbox}\limits)$ by $\mathop{\copy\noforkbox}\limits(M, A, B, N)$ if and only if $M \leq N$ are in K, $A \cup B \subseteq |N|$, and for any $\bar{a} \in {}^{<\kappa} A$

and $B_0 \subseteq |M| \cup B$ of size less than κ, there exists $\bar{a}' \in {}^{<\kappa}|M|$ such that $\operatorname{gtp}(\bar{a}/B_0; N) = \operatorname{gtp}(\bar{a}'/B_0; M)$.

(2) (λ-nonsplitting) Let K be a coherent AC with amalgamation. For λ an infinite cardinal, define $\mathfrak{i}_{\lambda\text{-ns}}(K) := (K_{\geq \lambda}, \mathop{\smile\!\!\!\!\!|}\,)$ by $\bar{a} \overset{N}{\underset{M_0}{\mathop{\smile\!\!\!\!\!|}}} B$ if and only if $\bar{a} \in {}^{<\infty}|N|$, $B \subseteq |N|$, $M \leq N$ are in $K_{\geq \lambda}$, and there exists $M \geq M_0$ and $N' \geq N$ such that $M \leq N'$, $B \subseteq |M|$, and whenever $M_0 \leq N_\ell \leq M$, $N_\ell \in K_\lambda$, $\ell = 1, 2$, and $f : N_1 \cong_{M_0} N_2$, then $f(\operatorname{gtp}(\bar{a}/N_1; N')) = \operatorname{gtp}(\bar{a}/N_2; N')$.

Fact 2.17 (Theorem 5.13 in [Vasc])**.** Let K be an AEC with amalgamation and let $\kappa > \operatorname{LS}(K)$. Assume K is $(< \kappa)$-tame and does not have the $(< \kappa)$-order property of length κ. Let $\mathfrak{i} := \mathfrak{i}_{\kappa\text{-ch}}(K)$. Then:

(1) $\mathfrak{i}$ is a $(< \infty, \geq \kappa)$-independence relation with base monotonicity, symmetry, and strong transitivity.
(2) $\mathfrak{i}^{\leq 1}$ has uniqueness for unions of models.
(3) For all α, $\bar{\kappa}_\alpha(\mathfrak{i}) \leq \left((\alpha + 2)^{<\kappa_r}\right)^+$.

2.4. **Superstability.** We recall the definition of superstability from [Vasb] using local character of nonsplitting. As there, we also recall the definition of *strong superstability* that uses local character of coheir rather than splitting. The definitions below appear in [Vasb, Definitions 10.1, 10.4]. Note that they coincide with the first-order definition (see [Vasb, Remark 10.9]). Moreover the definition of superstability is equivalent to the definition implicit in [GVV, Vasa] and explicit in [Gro02, Definition 7.12]. Strong superstability appears for the first time in [Vasb] but is equivalent to superstability if one does not care about the exact parameters, see Fact 2.21.

Definition 2.18 (Superstability)**.** Let K be an AEC.

(1) For $M, N \in K$, say $M <_{\text{univ}} N$ (N is *universal over* M) if and only if $M < N$ and whenever we have $M' \geq M$ such that $\|M'\| \leq \|N\|$, then there exists $f : M' \underset{M}{\longrightarrow} N$. Say $M \leq_{\text{univ}} N$ if and only if $M = N$ or $M <_{\text{univ}} N$.
(2) K is μ-superstable if:
    (a) $\operatorname{LS}(K) \leq \mu$.
    (b) There exists $M \in K_\mu$ such that for any $M' \in K_\mu$ there is $f : M' \to M$ with $f[M'] <_{\text{univ}} M$.
    (c) $\kappa_1(\mathfrak{i}_{\mu\text{-ns}}(K_\mu), \leq_{\text{univ}}) = \aleph_0$.
(3) K is κ-strongly μ-superstable if:
    (a) $\operatorname{LS}(K) < \kappa \leq \mu$.

(b) (2b) above holds.
(c) K does not have the $(< \kappa)$-order property of length κ.
(d) $K_\mu^{\kappa\text{-sat}}$ is dense in K_μ.
(e) $\kappa_1(\mathfrak{i}_{\kappa\text{-ch}}(K)_\mu, \leq_{\text{univ}}) = \aleph_0$.

When we omit a parameter, we mean some value of the parameter exists so that the class satisfies the parametrized definition, e.g. K is superstable if it is μ-superstable for some μ. We say K is *μ-superstable$^+$* if $K_{\geq \mu}$ is μ-superstable, μ-tame, and has amalgamation. We say K is *κ-strongly μ-superstable$^+$* if there exists $\kappa_0 < \kappa$ such that $K_{\geq \kappa_0}$ is κ-strongly μ-superstable, $(< \kappa)$-tame, and has amalgamation.

Remark 2.19. It is easy to check (see [Vasb, Fact 2.24]) that Condition 2b is equivalent to "K_μ is nonempty, has amalgamation, joint embedding, no maximal models, and is stable in μ".

Remark 2.20. While Definition 2.18 makes sense in any AEC, here we focus on tame AECs with amalgamation, and will not study what happens to Definition 2.18 without these assumptions (although this can be done, see [GVV]).

For the convenience of the reader, we recall of few facts about superstability. They all appear in [Vasb, Section 10].

Fact 2.21. Let K be an AEC.

(1) If K is [κ-strongly] μ-superstable$^+$ and $\mu' \geq \mu$, then K is [κ-strongly] μ'-superstable$^+$.
(2) If K is strongly μ-superstable$^+$, then K is μ-superstable$^+$.
(3) If K is μ-superstable$^+$ and $\kappa = \beth_\kappa > \mu$, then K is κ-strongly $(2^{<\kappa_r})^+$-superstable$^+$.
(4) If K has amalgamation, is $(< \kappa)$-tame with $\kappa = \beth_\kappa > \text{LS}(K)$, and is categorical in a $\lambda > \kappa$, then K is κ-strongly κ-superstable$^+$.
(5) If K is μ-superstable$^+$, then $K_{\geq \mu}$ has joint embedding, no maximal models, and is stable in all cardinals.
(6) If K is κ-strongly μ-superstable$^+$, then $\kappa_1(\mathfrak{i}_{\kappa\text{-ch}}(K) \upharpoonright K^{\mu^+\text{-sat}}) = \aleph_0$.

Another key consequence of superstability is the existence of a well-behaved independence relation:

Fact 2.22 (Proposition 7.4.2, Theorem 7.5, Theorem 10.8.2a in [Vasb])**.** Let K be an AEC with amalgamation and let $\lambda \geq \text{LS}(K)$. If K is λ-tame and stable in λ, then there exists a $(\leq 1, \geq \lambda^+)$-independence relation $\mathfrak{i} = (K^{\lambda^+\text{-sat}}, \perp)$ such that:

(1) $\mathfrak{i}$ has base monotonicity and transitivity.
(2) $\mathfrak{i}_{\lambda^+}$ has uniqueness.
(3) $\bar{\kappa}_1(\mathfrak{i}) = \lambda^{++}$.

If in addition K is λ-superstable, one can take $\mathfrak{i}$ such that $\kappa_1(\mathfrak{i}) = \aleph_0$.

3. Using independence calculus: the stable case

Hypothesis 3.1.

(1) $\mathfrak{i} = (K, \mathop{\smile\!\!\!\!\!\!|}\,)$ is a $(<\infty, \geq \lambda_0)$-independence relation.
(2) κ is an infinite cardinal.
(3) $\mathfrak{i}$ has base monotonicity, symmetry, and strong transitivity.
(4) $\mathfrak{i}^{\leq 1}$ has uniqueness for unions of models.
(5) For all α, $\bar{\kappa}_\alpha(\mathfrak{i}) \leq \left((\alpha + \lambda_0)^{<\kappa_r}\right)^+$

Remark 3.2. Often, we will assume that K is a κ-AEC, but when considering the superstable case K might actually be a θ-AEC for $\theta > \kappa$.

Remark 3.3. Fact 2.17 gives reasonable conditions under which Hypothesis 3.1 holds for $K = (K^0)^{\kappa\text{-sat}}$, K^0 an AEC.

Remark 3.4. Strong transitivity is used to make the relation $\overset{*}{\mathop{\smile\!\!\!\!\!\!|}}$ (Definition 3.8) well-behaved. We do not know if strong transitivity can be replaced with just transitivity.

Hypothesis 3.5. We work inside a fixed model $\mathcal{N} \in K$. Thus we assume all models and sets we consider are contained in $\mathcal{N}$ and all types we consider are realized in $\mathcal{N}$. We do *not* assume that $\mathcal{N}$ has any homogeneity. We write $A \mathop{\smile\!\!\!\!\!\!|}_M N$ for $A \overset{\mathcal{N}}{\mathop{\smile\!\!\!\!\!\!|}}_M N$. Note that it is enough to assume the hypotheses above hold relativized to $\mathcal{N}$.

Remark 3.6. As part of the definition of an independence relation, we assumed that K had amalgamation. Since we work inside a fixed model, this was not necessary.

Remark 3.7. Assume $\langle M_i : i < \delta \rangle$ is increasing. Then $\bigcup_{i<\delta} M_i$ need not be a member of K (but we will sometimes still use M_δ to denote it).

For what comes next, it will be convenient if we could say that $A \mathop{\smile\!\!\!\!\!\!|}_M B$ and $M \leq N$ implies $A \mathop{\smile\!\!\!\!\!\!|}_N B$. By base monotonicity, this holds if $|N| \subseteq B$ but in general this is not part of our assumptions (and in practice this need not hold). Thus we will close $\perp$ under this property. This is where we depart from [MS90]; there the authors used that the singular cardinal hypothesis holds above a strongly compact to prove the result corresponding to our Lemma 4.2. Here we need to be more clever.

Definition 3.8. $A \overset{*}{\underset{C}{\perp}} B$ means that there exists $M_0 \leq \mathcal{N}$, $|M_0| \subseteq C$ such that $A \underset{M_0}{\perp} B$.

Remark 3.9. $(K, \overset{*}{\perp})$ is not an independence relation (because $\overset{*}{\perp}$ is also defined when the base is a set and also because it may not satisfy the normality axioms). Nevertheless we will apply some of the notation defined in the preliminaries to it.

Definition 3.10. We write $A \overset{*}{\underset{C}{\perp}} [B]^1$ to mean that $A \overset{*}{\underset{C}{\perp}} b$ for all $b \in B$. Similarly define $[A]^1 \overset{*}{\underset{C}{\perp}} B$. Let $(\overset{*}{\perp})^1$ denote the relation defined by $A (\overset{*}{\underset{C}{\perp}})^1 B$ if and only if $A \overset{*}{\underset{C}{\perp}} [B]^1$. For α a cardinal, let $\bar{\kappa}_\alpha^1 = \bar{\kappa}_\alpha^1(\overset{*}{\perp}) := \bar{\kappa}_\alpha((\overset{*}{\perp})^1)$.

Note that $A \overset{*}{\underset{C}{\perp}} N$ implies $[A]^1 \overset{*}{\underset{C}{\perp}} N$ by monotonicity.

Proposition 3.11.

(1) $\overset{*}{\perp}$ (or pedantically the pair $(K, \overset{*}{\perp})$) satisfies invariance, monotonicity, symmetry, strong right transitivity.
(2) For all α, $\bar{\kappa}_\alpha(\overset{*}{\perp}) = \bar{\kappa}_\alpha(\perp)$, $\kappa_\alpha(\overset{*}{\perp}) = \kappa_\alpha(\perp)$.
(3) $\overset{*}{\perp}$ has *strong base monotonicity*: If $A \overset{*}{\underset{C}{\perp}} B$ and $C \subseteq C'$, then $A \overset{*}{\underset{C'}{\perp}} B$.
(4) If $A \underset{M}{\perp} B$, then $A \overset{*}{\underset{M}{\perp}} B$.
(5) If $A \overset{*}{\underset{M}{\perp}} B$ for $M \subseteq |B|$, then $A \underset{M}{\perp} B$.
(6) For all α, $\bar{\kappa}_\alpha^1(\overset{*}{\perp}) \leq \bar{\kappa}_\alpha(\perp)$.

Proof. All quickly follow from the definition. As an example, we prove that $\overset{*}{\perp}$ has strong right transitivity. Assume $A \overset{*}{\underset{M_0}{\perp}} M_1$ and $A \overset{*}{\underset{M_1}{\perp}} B$.

Then there exists $M_0' \leq M_0$ and $M_1' \leq M_1$ such that $A \underset{M_0'}{\mathop{\perp}} M_1$ and $A \underset{M_1'}{\mathop{\perp}} B$. By monotonicity for $\mathop{\perp}$, $A \underset{M_0'}{\mathop{\perp}} M_1'$. By strong right transitivity for $\mathop{\perp}$, $A \underset{M_0'}{\mathop{\perp}} B$. Thus M_0' witnesses $A \underset{M_0}{\overset{*}{\mathop{\perp}}} B$. $\square$

Proposition 3.12. Assume $\langle M_i : i < \delta \rangle$, $\langle N_i : i < \delta \rangle$ are increasing chains of models, A is a set. If $A \underset{M_i}{\overset{*}{\mathop{\perp}}} N_i$ for all $i < \delta$ and $\kappa_{|A|}(\mathop{\perp}) \leq \mathrm{cf}(\delta)$, then $A \underset{M_\delta}{\overset{*}{\mathop{\perp}}} N_\delta$, where[8] $M_\delta := \bigcup_{i<\delta} M_i$ and $N_\delta := \bigcup_{i<\delta} N_i$.

Proof. Without loss of generality, $\delta = \mathrm{cf}(\delta)$. By definition of $\kappa_{|A|}(\mathop{\perp})$, there exists $i < \delta$ such that $A \underset{N_i}{\mathop{\perp}} N_\delta$, so $A \underset{N_i}{\overset{*}{\mathop{\perp}}} N_\delta$. By strong right transitivity for $\overset{*}{\mathop{\perp}}$, $A \underset{M_i}{\overset{*}{\mathop{\perp}}} N_\delta$. By strong base monotonicity, $A \underset{M_\delta}{\overset{*}{\mathop{\perp}}} N_\delta$. $\square$

As already discussed, the reason we use $\overset{*}{\mathop{\perp}}$ is that we want to generalize [MS90, Lemma 4.17] to our context. In their proof, Makkai and Shelah use that cardinal arithmetic behaves nicely above a strongly compact, and we cannot make use of this fact here. Thus we are only able to prove this lemma for $\overset{*}{\mathop{\perp}}$ instead of $\mathop{\perp}$ (see Lemma 4.2). Fortunately, this turns out to be enough. The reader can also think of $\overset{*}{\mathop{\perp}}$ as a trick to absorb some quantifiers.

The next lemma imitates [MS90, Proposition 4.18].

Lemma 3.13. Let $\lambda > \lambda_0$ be regular and let $\langle M_i : i < \delta \rangle$ be an increasing chain with M_i λ-saturated for types of length one for all $i < \delta$. Assume that $\kappa_1(\mathop{\perp}) \leq \mathrm{cf}(\delta)$.

Assume that K is $(< \lambda)$-closed (Recall Definition 2.3) and there exists a regular cardinal $\mu < \lambda$ such that K is a μ-AEC.

If $\bar{\kappa}^1_{<\lambda}(\overset{*}{\mathop{\perp}}) \leq \lambda$, then[9] $M_\delta := \bigcup_{i<\delta} M_i$ is λ-saturated for types of length one.

Proof. Without loss of generality, $\delta = \mathrm{cf}(\delta)$. Let $A \subseteq |M_\delta|$ have size less than λ. If $\lambda \leq \delta$, then $A \subseteq |M_i|$ for some $i < \delta$ and so any type

[8]Note that M_δ and N_δ need not be members of K.

[9]Note that M_δ need not be a member of K, but our definition of saturated (Definition 2.5) still applies.

over A is realized in $M_i \subseteq |M_\delta|$. Now assume without loss of generality that $\lambda > \delta$. Replacing μ by $\mu + \delta$, we can also assume $\mu \geq \delta$. We need to show every Galois type over A is realized in M_δ. First, we build an array of models $\langle N_i^\alpha \in K_{<\lambda} : i < \delta, \alpha < \mu \rangle$ such that:

(1) For all $i < \delta$, $\langle N_i^\alpha : \alpha < \mu \rangle$ is increasing.
(2) For all $\alpha < \mu$, $\langle N_i^\alpha : i < \delta \rangle$ is increasing.
(3) For all $i < \delta$ and all $\alpha < \mu$, $N_i^\alpha \leq M_i$.
(4) $A \subseteq \bigcup_{i<\delta} |N_i^0|$.
(5) For all $\alpha < \mu$ and all $i < \delta$, $\bigcup_{i<\delta} N_i^\alpha \underset{N_i^{\alpha+1}}{\overset{*}{\perp}} [M_i]^1$.

For $\alpha < \mu$, write $N_\delta^\alpha := \bigcup_{i<\delta} N_i^\alpha$ and for $i \leq \delta$, write $N_i^\mu := \bigcup_{\alpha<\mu} N_i^\alpha$. The following is a picture of the array constructed; note that arrows denote inclusions and are *not* necessarily K-embeddings.

$$\begin{tikzcd}
M_i \arrow[r] & M_{i+1} \arrow[r] & M_\delta \\
N_i^\mu \arrow[u] \arrow[r] & N_{i+1}^\mu \arrow[u] \arrow[r] & N_\delta^\mu \arrow[u] \\
N_i^{\alpha+1} \arrow[u] \arrow[r] & N_{i+1}^{\alpha+1} \arrow[u] \arrow[r] & N_\delta^{\alpha+1} \arrow[u] \\
N_i^\alpha \arrow[u] \arrow[r] & N_{i+1}^\alpha \arrow[u] \arrow[r] & N_\delta^\alpha \arrow[u] \\
 & & A \arrow[u]
\end{tikzcd}$$

<u>This is enough</u>: Note that for $i < \delta$, N_i^μ is in K (since K is a μ-AEC and μ is regular) and has size less than λ (since $\lambda > \mu$ and is regular). Note also that since $\delta \leq \mu < \lambda$, N_δ^μ has size less than λ (but we do not claim it is in K).

Claim: For all $i < \delta$, $N_\delta^\mu \underset{N_i^\mu}{\overset{*}{\perp}} [M_i]^1$.

Proof of claim: Fix $i < \delta$ and let $a \in M_i$. Fix $j < \delta$. By (5), monotonicity, and symmetry, $a \underset{N_i^{\alpha+1}}{\overset{*}{\perp}} N_j^\alpha$ for all $\alpha < \mu$. By Proposition

3.12 applied to $\langle N_i^{\alpha+1} : \alpha < \mu\rangle$ and $\langle N_j^\alpha : \alpha < \mu\rangle$, $a \overset{*}{\underset{N_i^\mu}{\perp}} N_j^\mu$ (note that $\mu = \text{cf}(\mu) \geq \delta \geq \kappa_1(\perp)$). Since j was arbitrary, we can apply Proposition 3.12 again with the constantly N_i^μ sequence and $\langle N_j^\mu : j < \delta\rangle$ (note that $\delta = \text{cf}(\delta) \geq \kappa_1(\perp)$) to get that $a \overset{*}{\underset{N_i^\mu}{\perp}} N_\delta^\mu$. By symmetry, $N_i^\mu \overset{*}{\underset{N_i^\mu}{\perp}} a$, as desired. $\dagger_{Claim}$

Now let $p \in \text{gS}(A)$. By (4), $A \subseteq N_\delta^\mu$ so we can extend p to some $q \in \text{gS}(N_\delta^\mu)$. Since $\delta \geq \kappa_1(\perp)$, we can find $i < \delta$ such that q does not fork over N_i^μ. Since $N_i^\mu \leq M_i$, M_i is λ-saturated for types of length one, and $\|N_i^\mu\| < \lambda$, we can find $a \in M_i$ realizing $q \upharpoonright N_i^\mu$. Since by the claim $N_\delta^\mu \overset{*}{\underset{N_i^\mu}{\perp}} [M_i]^1$, we can use symmetry to conclude $a \overset{*}{\underset{N_i^\mu}{\perp}} N_\delta^\mu$, and hence (Proposition 3.11.(5)) $a \underset{N_i^\mu}{\perp} N_\delta^\mu$. By uniqueness for unions of models, a must realize q, so in particular a realizes p. This concludes the proof that M_δ is λ-saturated for types of length one.

<u>This is possible</u>: We define $\langle N_i^\alpha : i < \delta\rangle$ by induction on α. For a fixed $i < \delta$, choose any $N_i^0 \leq M_i$ in $K_{<\lambda}$ that contains $A \cap |M_i|$ (this is possible since K is $(< \lambda)$-closed). For $\alpha < \mu$ limit and $i < \delta$, pick any $N_i^\alpha \leq M_i$ containing $\bigcup_{\beta<\alpha} N_i^\beta$ which is in $K_{<\lambda}$ (again, this is possible by closure). Now assume $\alpha = \beta + 1 < \mu$, and N_i^β has been defined for $i < \delta$. Define N_i^α by induction on i. Assume N_j^α has been defined for all $j < i$. Pick N_i^α containing $\bigcup_{j<i} N_j^\alpha$ which is in $K_{<\lambda}$, satisfies $N_i^\alpha \leq M_i$ and (5). This is possible by strong base monotonicity and definition of $\bar{\kappa}^1_{<\lambda}$. □

Below, we give a more natural formulation of the hypotheses.

Theorem 3.14. Assume K is a κ-AEC. Let $\lambda > \text{LS}(K)$. Let $\langle M_i : i < \delta\rangle$ be an increasing chain with M_i λ-saturated for types of length one for all $i < \delta$. If

(1) $\text{cf}(\delta) \geq \kappa_1(\perp)$; and
(2) $\chi^{<\kappa_r} < \lambda$ for all $\chi < \lambda$,

then $\bigcup_{i<\delta} M_i$ is λ-saturated for types of length one.

Proof. Let $M_\delta := \bigcup_{i<\delta} M_i$. Note that $\lambda > \kappa_r$: since $\lambda > \text{LS}(K)$, $\lambda \geq \kappa^+$ and if $\lambda = \kappa^+$ then $\kappa^{<\kappa} < \lambda$ so $\kappa = \kappa^{<\kappa}$ hence κ is regular:

$\kappa_r = \kappa$. Let $\mu := \kappa_r$. This is a regular cardinal and we have just argued $\mu < \lambda$. Since K is a κ-AEC, K is also a μ-AEC.

Let $\chi < \lambda$ be such that $\chi^+ > \mu + \lambda_0$. We show that M_δ is χ^+-saturated. By hypothesis, $\chi^{<\kappa_r} < \lambda$, so replacing χ by $\chi^{<\kappa_r}$ if necessary, we might as well assume that $\chi = \chi^{<\kappa_r}$. We check that χ^+ satisfies the conditions of Lemma 3.13 as λ there. By assumption, $\text{cf}(\chi^+) = \chi^+ > \mu$. Also, K is χ-closed, as $\chi = \chi^{<\kappa} \geq \text{LS}(K)$.

Now by Proposition 3.11.(6), $\bar{\kappa}^1_\chi(\overset{*}{\perp}) \leq \bar{\kappa}_\chi(\perp)$. By Hypothesis 3.1, $\bar{\kappa}_\chi(\perp) \leq ((\chi + \lambda_0)^{<\kappa_r})^+ \leq (\chi^{<\kappa_r})^+ = \chi^+$. Thus $\bar{\kappa}^1_\chi(\overset{*}{\perp}) \leq \chi^+$, as needed.

Thus Lemma 3.13 applies and so M_δ is χ^+-saturated for types of length one. Since $\chi < \lambda$ was arbitrary, M_δ is λ-saturated for types of length one. $\square$

For the next corollaries to AECs, we drop our hypotheses.

Corollary 3.15. Let K be a $(< \kappa)$-tame AEC with amalgamation, $\kappa > \text{LS}(K)$. Assume K does not have the $(< \kappa)$-order property of length κ. Assume $\langle M_i : i < \delta \rangle$ is an increasing chain of λ-saturated models. If:

(1) $\text{cf}(\delta) > \text{LS}(K)^{<\kappa_r}$.
(2) $\chi^{<\kappa_r} < \lambda$ for all $\chi < \lambda$.

Then $\bigcup_{i<\delta} M_i$ is λ-saturated.

Proof. Without loss of generality, $\delta = \text{cf}(\delta) < \lambda$. By Fact 2.17, $K' := K^{\kappa\text{-sat}}$ and $\mathfrak{i} := \mathfrak{i}_{\kappa\text{-ch}}(K'_{\geq \text{LS}(K')})$ satisfy Hypothesis 3.1 with $\lambda_0 = \text{LS}(K')$. Note that $\kappa_1(\mathfrak{i}) \leq \bar{\kappa}_1(\mathfrak{i}) \leq (\text{LS}(K)^{<\kappa_r})^+$. Now use Theorem 3.14. $\square$

Remark 3.16. Compared with Fact 1.1, an extra condition is (2). Note however that if T is a first-order stable theory, then (ignoring the condition $\kappa > \text{LS}(K)$ which is not relevant when K is an elementary class) Corollary 3.15 holds with $(K, \leq) := (\text{Mod}(T), \preceq)$, $\kappa := \aleph_0$, and so we recover Harnik's result.

Corollary 3.17. Let K be a stable $(< \kappa)$-tame AEC with amalgamation, $\kappa = \beth_\kappa > \text{LS}(K)$. Let $\lambda > \kappa$ and assume $\langle M_i : i < \delta \rangle$ is an increasing chain of λ-saturated models. If:

(1) $\text{cf}(\delta) > \text{LS}(K)^{<\kappa_r}$.
(2) $\chi^{<\kappa_r} < \lambda$ for all $\chi < \lambda$.

Then $\bigcup_{i<\delta} M_i$ is λ-saturated.

Proof. By Fact 2.9, K does not have the $(<\kappa)$-order property of length κ. Now apply Corollary 3.15. $\square$

4. Using independence calculus: the superstable case

Next we show that in the *superstable* case we can remove the cardinal arithmetic condition (2) in Corollary 3.15.

Hypothesis 4.1. Same as in the previous section: Hypotheses 3.1 and 3.5.

In Theorem 3.14, we estimated $\bar{\kappa}^1_\alpha(\mathop{\smile\!\!\!\!\!\!|}\;)$ using $\bar{\kappa}_\alpha(\mathop{\smile\!\!\!\!\!\!|}\;)$. Using superstability, we can prove a better bound. This is adapted from [MS90, Lemma 4.17].

Lemma 4.2. Assume that $\kappa_1(\mathop{\smile\!\!\!\!\!\!|}\;) = \aleph_0$ and K is λ-closed for all $\lambda \geq \lambda_0$. Then for any cardinal α, $\bar{\kappa}^1_\alpha(\overset{*}{\mathop{\smile\!\!\!\!\!\!|}}\;) \leq \bar{\kappa}_{\lambda_0}(\mathop{\smile\!\!\!\!\!\!|}\;) + \alpha^+$.

Proof. Let A have size α and N be a model. We show by induction on α that there exists a $M \leq N$ with $\|M\| < \mu := \bar{\kappa}_{\lambda_0}(\mathop{\smile\!\!\!\!\!\!|}\;) + \alpha^+$ and $A \underset{M}{\overset{*}{\mathop{\smile\!\!\!\!\!\!|}}} [N]^1$. Note that $\mu > \lambda_0$.

If $\alpha \leq \lambda_0$, then apply the definition of $\bar{\kappa}_{\lambda_0}(\mathop{\smile\!\!\!\!\!\!|}\;)$ to get a $M \leq N$ with $\|M\| < \bar{\kappa}_{\lambda_0}(\mathop{\smile\!\!\!\!\!\!|}\;)$, $A \underset{M}{\overset{*}{\mathop{\smile\!\!\!\!\!\!|}}} N$, which is more than what we need.

Now, assume $\alpha > \lambda_0$, and the result has been proven for all $\alpha_0 < \alpha$. By the closure assumption, we can assume without loss of generality that A is a model. Let $\langle A_i : i < \alpha \rangle$ be an increasing resolution of A such that $A_i \in K_{<\alpha}$ for all $i < \alpha$. Now define an increasing chain $\langle M_i : i < \alpha \rangle$ such that for all $i < \alpha$:

(1) $M_i \in K_{<\mu}$.
(2) $M_i \leq N$.
(3) $A_i \underset{M_i}{\overset{*}{\mathop{\smile\!\!\!\!\!\!|}}} [N]^1$.

This is possible: For $i < \alpha$, use the induction hypothesis to find $M_i \leq N$ such that $A_i \underset{M_i}{\overset{*}{\mathop{\smile\!\!\!\!\!\!|}}} [N]^1$ and $\|M_i\| < \mu$. By strong base monotonicity of $\overset{*}{\mathop{\smile\!\!\!\!\!\!|}}$ and the closure assumption, we can assume that M_i contains $\bigcup_{j<i} M_j$.

<u>This is enough</u>: Let $M \in K$ contain $\bigcup_{i<\alpha} M_i$ and have size less than μ. We claim that $A \overset{*}{\underset{M}{\perp}} [N]^1$. Let $a \in N$. By symmetry, it is enough to see $a \overset{*}{\underset{M}{\perp}} A$. This follows from strong base monotonicity and Proposition 3.12 applied to $\langle M_i : i < \alpha \rangle$ and $\langle A_i : i < \alpha \rangle$ since $\kappa_1(\overset{*}{\perp}) = \aleph_0 \le \mathrm{cf}(\alpha)$ by Proposition 3.11.(2) and the hypothesis. $\square$

Remark 4.3. The heavy use of strong base monotonicity in the above proof was the reason for introducing $\overset{*}{\perp}$.

Theorem 4.4. Assume that $\kappa_1(\perp) = \aleph_0$ and K is λ-closed for all $\lambda \ge \lambda_0$. Let $\lambda \ge \bar{\kappa}_{\lambda_0}(\perp)$. Assume there exists a regular cardinal $\mu < \lambda$ such that K is a μ-AEC.

Let $\langle M_i : i < \delta \rangle$ be an increasing chain with M_i λ-saturated for types of length one for all $i < \delta$. Then $M_\delta := \bigcup_{i<\delta} M_i$ is λ-saturated for types of length one.

Proof. Let $\chi < \lambda$ be such that $\chi^+ \ge \mu^+ + \bar{\kappa}_{\lambda_0}(\perp)$. We claim that χ^+ satisfies the hypotheses of Lemma 3.13 (as λ there). Indeed by Lemma 4.2, $\bar{\kappa}^1_\chi(\overset{*}{\perp}) \le \bar{\kappa}_{\lambda_0} + \chi^+ = \chi^+$.

Thus Lemma 3.13 applies: M_δ is χ^+-saturated for types of length one. Since $\chi < \lambda$ was arbitrary, M_δ is λ-saturated for types of length one. $\square$

For the next corollary to AECs, we drop our hypotheses.

Corollary 4.5. Assume K is a κ-strongly μ-superstable$^+$ AEC. Let $\lambda > (\mu^{<\kappa_r})^+$ and assume $\langle M_i : i < \delta \rangle$ is an increasing chain of λ-saturated models. Then $\bigcup_{i<\delta} M_i$ is λ-saturated.

Proof. By Fact 2.21, K is stable in all $\lambda \ge \mu$. By definition of strong superstability and Fact 2.17, letting $K' := K^{\mu^+\text{-sat}}$ and $\mathfrak{i} := \mathfrak{i}_{\kappa\text{-ch}}(K) \upharpoonright K'$, Hypothesis 3.1 is satisfied with K, λ_0 there standing for K', μ^+ here. Moreover (by Fact 2.21) $\kappa_1(\mathfrak{i}) = \aleph_0$. Now it is easy to check using stability that K' is λ'-closed for all $\lambda' \ge \lambda_0 := \mu^+$. Finally, by Hypothesis 3.1 $\bar{\kappa}_{\lambda_0}(\mathfrak{i}) = (\lambda_0^{<\kappa_r})^+ = (\mu^{<\kappa_r})^{++}$. Now apply Theorem 4.4 (note that K' is a μ^+-AEC and $\mu^+ < \lambda$). $\square$

5. Averages

Hypothesis 5.1.

(1) κ is an infinite cardinal.
(2) L is a $(< \kappa)$-ary language.
(3) $\mathcal{N}$ is a fixed L-structure.
(4) We work inside $\mathcal{N}$.
(5) Hypotheses 5.2 and 5.4, see the discussion below.

Midway through, we will also assume Hypothesis 5.19.

We use the same notation and convention as [Vasc, Section 2]: We always work with *quantifier-free* $L_{\kappa,\kappa}$ formulas and types. Since we work inside $\mathcal{N}$, everything is defined relative to $\mathcal{N}$. For example $\text{tp}(\bar{c}/A)$ means $\text{tp}(\bar{c}/A; \mathcal{N})$, the *quantifier-free* $L_{\kappa,\kappa}$-type of $\bar{c}$ over A, and saturated means saturated in $\mathcal{N}$. Similarly, we write $\models \phi[\bar{b}]$ instead of $\mathcal{N} \models \phi[\bar{b}]$. By "type", we mean a member of $S^{<\infty}(A)$ for some set A. A set of formula $p(\bar{x})$ is *complete* over A if for any formula $\phi(\bar{x})$ over A, at least one of ϕ, $\neg\phi$ is in p.

Unless otherwise noted, the letters $\bar{a}$, $\bar{b}$, $\bar{c}$ denote tuples of elements *of length less than* κ. The letters A, B, C, will denote subsets of $\mathcal{N}$. We say $\langle A_i : i < \delta \rangle$ is *increasing* if $A_i \subseteq A_j$ for all $i < j < \delta$.

We say $\mathcal{N}$ is *α-stable in* λ if $|S^\alpha(A)| \leq \lambda$ for all A with $|A| \leq \lambda$. We say $\mathcal{N}$ has the *order property of length* χ if there exists a (quantifier-free) formula $\phi(\bar{x}, \bar{y})$ and elements $\langle \bar{a}_i : i < \chi \rangle$ of the same arity such that for $i, j < \chi$, $\models \phi[\bar{a}_i, \bar{a}_j]$ if and only if $i < j$.

Boldface letters like $\mathbf{I}$, $\mathbf{J}$ will always denote *sequences of tuples of the same arity* (less than κ). We assume all our sequences are indexed by *ordinals*. We sometimes treat such a sequence as a set of tuples, writing things like $\bar{a} \in \mathbf{I}$, but then we are really looking at the range of the sequence. To avoid potential mistakes, we do not necessarily assume that the elements are all distinct although it should always hold in cases of interest. We write $|\mathbf{I}|$ for the cardinality of the range, i.e. the number of distinct elements in $\mathbf{I}$.

As the reader will see, this section builds on earlier work of Shelah from [She09b, Chapter V.A]. Note that Shelah works in an arbitrary logic. We work only with quantifier-free $L_{\kappa,\kappa}$-formulas in order to be concrete and because this is the case we are interested in to translate the syntactic results to AECs.

The reader may wonder what the right notion of submodel is in this context. We could simply say that it is "subset" but this does not quite work when translating to AECs. Thus we fix a set of subsets of $\mathcal{N}$ that by definition will be the substructures of $\mathcal{N}$. We require that this set satisfies some axioms akin to those of AECs. This can be taken to be the full powerset if one is not interested in doing an AEC translation.

Hypothesis 5.2. $\mathcal{S} \subseteq \mathcal{P}(|\mathcal{N}|)$ is a fixed set of subsets of $\mathcal{N}$ satisfying:

(1) Closure under chains: If $\langle A_i : i < \delta \rangle$ is an increasing sequence of members of $\mathcal{S}$, then $\bigcup_{i<\delta} A_i$ is in $\mathcal{S}$.
(2) Löwenheim-Skolem axiom: If A is a set, there exists $A' \in \mathcal{S}$ such that $A \subseteq A'$ and $|A'| \leq (|L| + 2)^{<\kappa} + |A|$.

We exclusively use the letters M and N to denote elements of $\mathcal{S}$ and call such elements *models*. We pretend they are L-structures and write $|M|$ and $|N|$ for their universe and $\|M\|$ and $\|N\|$ for their cardinalities.

Remark 5.3. An element M of $\mathcal{S}$ is *not* required to be an L-structure. Note however that if it is κ-saturated for types of length less than κ (see below), then it will be one.

We also need to discuss the definition of saturated: define M to be λ-*saturated for types of length* α if for any $A \subseteq |M|$ of size less than λ, any $p \in S^\alpha(A)$ is realized in M. Similarly Define λ-saturated for types of length less than α. Now in the framework we are working in, μ-saturated for types of length less than κ seems to be the right notion, so we say that M *is* μ-*saturated* if it is μ-saturated for types of length κ. Unfortunately it is not clear that it is equivalent to μ-saturated for types of length one (or length less than ω), even when $\mu > \kappa$. However Remark 2.6 tells us that in case $\mathcal{N}$ comes from an AEC, then this is the case. Thus we will make the following additional assumption. Note that it is possible to work without it, but then everywhere below "stability" must be replaced by "$(< \kappa)$-stability".

Hypothesis 5.4. If $\mu > (|L| + 2)^{<\kappa}$, then whenever M is μ-saturated for types of length one, it is μ-saturated (for types of length less than κ).

Our goal in this section is to use Shelah's notion of average in this framework to prove a result about chains of saturated models. Recall:

Definition 5.5 (Definition V.A.2.6 in [She09b])**.** For $\mathbf{I}$ a sequence, χ an infinite cardinal such that $|\mathbf{I}| \geq \chi$, and A a set, define $\text{Av}_\chi(\mathbf{I}/A)$ to be the set of formulas $\phi(\bar{x})$ over A so that the set $\{\bar{b} \in \mathbf{I} \mid \models \neg\phi[\bar{b}]\}$ has size less than χ.

Note that if $|\mathbf{I}| \geq \chi$ and ϕ is a formula, then *at most one* of ϕ, $\neg\phi$ is in $\text{Av}(\mathbf{I}/A)$. Thus the average is not obviously contradictory, but we do *not* claim that there is an element in $\mathcal{N}$ realizing it. The next lemma is a simple counting argument allowing us to find such an element:

Lemma 5.6. Let $\mathbf{I}$ be a sequence with $|\mathbf{I}| \geq \chi$ and let A be a set. Let $p := \mathrm{Av}_\chi(\mathbf{I}/A)$. Assume that

$$|\mathbf{I}| > \chi + \min((|A| + |L| + 2)^{<\kappa}, |S^{\ell(p)}(A)|)$$

Then there exists $\bar{b} \in \mathbf{I}$ realizing p.

Proof. Assume first that the minimum is realized by $(|A| + |L| + 2)^{<\kappa}$. By definition of the average, for every every formula $\phi(\bar{x}) \in p$, $\mathbf{J}_\phi := \{\bar{b} \in \mathbf{I} \mid\models \neg\phi[\bar{b}]\}$ has size less than χ. Let $\mathbf{J} := \bigcup_{\phi \in p} \mathbf{J}_\phi$. Note that $|\mathbf{J}| \leq \chi + (|A| + |L| + 2)^{<\kappa}$ and by definition any $\bar{b} \in \mathbf{I} \backslash \mathbf{J}$ realizes p.

Now assume that the minimum is realized by $|S^{\ell(p)}(A)|$. Let $\mu := \chi + |S^{\ell(p)}(A)|$. By the pigeonhole principle, there exists $\mathbf{I}_0 \subseteq \mathbf{I}$ of size μ^+ such that $\bar{c}, \bar{c}' \in \mathbf{I}_0$ implies $q := \mathrm{tp}(\bar{c}/A) = \mathrm{tp}(\bar{c}'/A)$. We claim that $p \subseteq q$, which is enough: any $\bar{b} \in \mathbf{I}_0$ realizes p. If not, there exists $\phi(\bar{x}) \in p$ such that $\neg\phi(\bar{x}) \in q$. By definition of the average, fewer than χ-many elements of $\mathbf{I}$ satisfy $\neg\phi(\bar{x})$. However, $\neg\phi(\bar{x})$ is in q which means that it is realized by all the elements of $\mathbf{I}_0$ and $|\mathbf{I}_0| = \mu^+ > \chi$, a contradiction. $\square$

We know that *at most* one of ϕ, $\neg\phi$ is in the average. It is very desirable to have that *exactly* one is in, i.e. that the average is a *complete* type. This is the purpose of the next definition. Note that Shelah uses the word "convergent" instead of "averageable". We changed it because "convergent" sounds too close to "coherent", defined later, and "averageable" seems to convey the meaning better.

Definition 5.7 (Definition V.A.2.1 in [She09b])**.** A sequence $\mathbf{I}$ is said to be χ-*averageable* if $|\mathbf{I}| \geq \chi$ and for any set A, $\mathrm{Av}_\chi(\mathbf{I}/A)$ is a complete type over A.

Remark 5.8 (Monotonicity)**.** If $\mathbf{I}$ is χ-averageable, $\mathbf{J} \subseteq \mathbf{I}$, and $|\mathbf{J}| \geq \chi' \geq \chi$, then for any set A, $\mathrm{Av}_\chi(\mathbf{I}/A) = \mathrm{Av}_{\chi'}(\mathbf{J}/A)$. In particular, $\mathbf{J}$ is χ'-averageable.

Recall [She90, Lemma III.1.7.1] that if T is a first-order stable theory and $\mathbf{I}$ is an infinite sequence of indiscernibles (in its monster model), then $\mathbf{I}$ is $\aleph_0$-averageable. The proof relies heavily on the compactness theorem. We would like a replacement of the form "if $\mathcal{N}$ has some stability and $\mathbf{I}$ is nice, then it is averageable". The next result is key. It plays the same role as the ability to extract indiscernible subsequences in first-order stable theories.

Fact 5.9 (The averageable set existence theorem: V.A.2.8 in [She09b])**.** Let $\chi_0 \geq (|L| + 2)^{<\kappa}$ be such that $\mathcal{N}$ does not have the order property of length χ_0^+. Let μ be an infinite cardinal such that $\mu = \mu^{\chi_0} + 2^{2^{\chi_0}}$.

Let $\mathbf{I}$ be a sequence with $|\mathbf{I}| = \mu^+$. Then there is $\mathbf{J} \subseteq \mathbf{I}$ of size μ^+ which is χ_0-averageable.

However having to extract a subsequence every time is too much for us. One issue is with the cardinal arithmetic condition on μ: what if we have a sequence of length μ^+ when μ is a singular cardinal of low cofinality? We work toward proving a more constructive result: *coherent* sequences (defined below) are always averageable.

Definition 5.10. We say $\langle \bar{a}_i : i < \delta \rangle \frown \langle N_i : i < \delta \rangle$ is a χ-*coherent sequence for* (p, A) if:

(1) χ is an infinite cardinal, δ is a limit ordinal, A is a set, $p(\bar{x})$ is a set of formulas with $\ell(\bar{x}) < \kappa$, and there is $\alpha < \kappa$ such that for all $i < \delta$, $\bar{a}_i \in {}^{\alpha}\mathcal{N}$.
(2) $A \subseteq |N_0|$ and $|A| \leq \chi$.
(3) $\langle N_i : i < \delta \rangle$ is increasing, and each N_i is χ^+-saturated.
(4) For all $i < \delta$, $\bar{a}_i \in {}^{\alpha}N_{i+1}$ and realizes $p \upharpoonright N_i$.[10]
(5) $i < j < \delta$ implies $\bar{a}_i \neq \bar{a}_j$.
(6) $\delta \geq \chi$.
(7) For all $i < j < \delta$, $\text{tp}(\bar{a}_i / N_i) = \text{tp}(\bar{a}_j / N_i)$.

When A is empty, we omit it. When p is empty, we also omit it. We call $\langle N_i : i < \delta \rangle$ the *witnesses* to the coherence of $\mathbf{I} := \langle \bar{a}_i : i < \delta \rangle$, and when we omit them we simply mean that $\mathbf{I} \frown \langle N_i : i < \delta \rangle$ is coherent for some witnesses $\langle N_i : i < \delta \rangle$.

Remark 5.11 (Monotonicity)**.** Let $\langle \bar{a}_i : i < \delta \rangle \frown \langle N_i : i < \delta \rangle$ be χ-coherent for (p, A). If $\chi_0 \leq \chi$ is infinite, $A_0 \subseteq A$ has size $\leq \chi_0$, $p_0 \subseteq p$, and $S \subseteq \delta$ has no maximum, then $\langle \bar{a}_i : i \in S \rangle \frown \langle N_i : i \in S \rangle$ is χ_0-coherent for (p_0, A_0).

The next result tells us how to build coherent sequences inside a given model:

Lemma 5.12. Let $A \subseteq |M|$ and let $\chi \geq (|L| + 2)^{<\kappa}$ be such that $|A| \leq \chi$. Let $p \in S^{\alpha}(M)$ be nonalgebraic, and let $\mu > \chi$. If:

[10] Note that $\text{dom}(p)$ might be smaller than N_i.

(1) M is μ^+-saturated.
(2) $\mathcal{N}$ is stable in μ.

Then there exists $\langle \bar{a}_i : i < \mu^+ \rangle \frown \langle N_i : i < \mu^+ \rangle$ inside M which is χ-coherent for (p, A).

Proof. We build $\langle \bar{a}_i : i < \mu^+ \rangle$ and $\langle N_i : i < \mu^+ \rangle$ increasing such that for all $i < \mu^+$:

(1) $A \subseteq |N_0|$.
(2) $|N_i| \subseteq |M|$.
(3) $\|N_i\| \leq \mu$.
(4) N_i is χ^+-saturated.
(5) $\bar{a}_i \in {}^\alpha N_{i+1}$.
(6) $\bar{a}_i$ realizes $p \upharpoonright N_i$.

This is enough by definition of a coherent sequence (note that for all $i < \mu^+$, $\bar{a}_i \notin {}^\alpha N_i$ by nonalgebraicity of p, so $\bar{a}_i \neq \bar{a}_j$ for all $j < i$).

This is possible: assume inductively that $\langle \bar{a}_j : j < i \rangle \frown \langle N_j : j < i \rangle$ has been defined. Pick $N_i \subseteq M$ which is χ^+-saturated, has size $\leq \mu$, and contains $A \cup \bigcup_{j<i} N_j$. Such an N_i exists: simply build an increasing chain $\langle M_k : k < \chi^+ \rangle$ with $M_0 := A \cup \bigcup_{j<i} N_j$, $\|M_k\| \leq \mu$, and M_k realizing all elements of $S(\bigcup_{k'<k} M_{k'})$ (this is where we use stability in μ). Then $N_i := \bigcup_{k<\chi^+} M_k$ is as desired (we are using Hypothesis 5.4 to deduce that it is χ^+-saturated for types of length less than κ). Now pick $\bar{a}_i \in {}^\alpha M$ realizing $p \upharpoonright N_i$ (exists by saturation of M). $\square$

Before proving that coherent sequences are averageable (Theorem 5.18), we recall the definition of splitting and study how it interacts with averages.

Definition 5.13. A set of formulas p *splits over* A if there exists $\phi(\bar{x}, \bar{b}) \in p$ and $\bar{b}'$ with $\text{tp}(\bar{b}'/A) = \text{tp}(\bar{b}/A)$ and $\neg\phi(\bar{x}, \bar{b}') \in p$.

The following result is classical:

Lemma 5.14 (Uniqueness for nonsplitting)**.** Let $A \subseteq |M| \subseteq B$. Assume p, q are complete sets of formulas over B that do not split over A and M is $|A|^+$-saturated. If $p \upharpoonright M = q \upharpoonright M$, then $p = q$.

Proof. Let $\phi(\bar{x}, \bar{b}) \in p$ with $\bar{b} \in B$. We show $\phi(\bar{x}, \bar{b}) \in q$ and the converse is symmetric. By saturation[11], find $\bar{b}' \in M$ such that $\text{tp}(\bar{b}'/A) =$

[11]Note that we are really using saturation for types *of length less than* κ here.

$\mathrm{tp}(\bar{b}/A)$. Since p does not split over A, $\phi(\bar{x}, \bar{b}') \in p$. Since $p \restriction M = q \restriction M$, $\phi(\bar{x}, \bar{b}') \in q$. Since again q does not split, $\phi(\bar{x}, \bar{b}) \in q$. $\square$

We would like to study when the average is a nonsplitting extension. This is the purpose of the next definition.

Definition 5.15. $\mathbf{I}$ is *χ-based on A* if for any B, $\mathrm{Av}_\chi(\mathbf{I}/B)$ does not split over A.

The next lemma tells us *averageable* sequences are based on a set of small size.

Lemma 5.16 (Claim IV.1.23.2 in [She09a])**.** If $\mathbf{I}$ is χ-averageable and $\mathbf{J} \subseteq \mathbf{I}$ has size $\geq \chi$, then $\mathbf{I}$ is χ-based on $\mathbf{J}$.

Proof. By Remark 5.8, $\mathbf{J}$ is averageable and has the same average as $\mathbf{I}$. Let $B \supseteq \mathbf{J}$ and let $p := \mathrm{Av}_\chi(\mathbf{I}/B)$. Let $\bar{b}, \bar{b}' \in B$ be such that $\mathrm{tp}(\bar{b}/\mathbf{J}) = \mathrm{tp}(\bar{b}'/\mathbf{J})$. Assume $\phi(\bar{x}, \bar{b}) \in p$. Then since $p = \mathrm{Av}_\chi(\mathbf{J}/B)$, let $\bar{a} \in \mathbf{J}$ be such that $\models \phi[\bar{a}, \bar{b}]$. Since $\bar{a} \in \mathbf{J}$, $\models \phi[\bar{a}, \bar{b}']$. Since there are at least χ-many such $\bar{a}$'s and $\mathbf{J}$ is averageable, $\phi(\bar{x}, \bar{b}') \in p$. $\square$

Before proving the relationship between coherent and averageable, we a crucial easy lemma that demonstrates the importance of the last condition in the definition of coherence:

Lemma 5.17. Let $\langle \bar{a}_i : i < \delta \rangle \frown \langle N_i : i < \delta \rangle$ be χ-coherent *and* χ-averageable. For any $i \leq j < \delta$, $\mathrm{tp}(\bar{a}_j/N_i) = \mathrm{Av}(\mathbf{I}/N_i)$.

Proof. By (6) in Definition 5.10, it is enough to show $\mathrm{tp}(\bar{a}_i/N_i) = \mathrm{Av}(\mathbf{I}/N_i)$. Let $\phi(\bar{x})$ be a formula over N_i with $\ell(\bar{x}) = \ell(\bar{a}_i)$. Assume $\phi(\bar{x}) \in \mathrm{tp}(\bar{a}_i/N_i)$, i.e. $\models \phi[\bar{a}_i]$. Then by (6) in the definition of coherence, $\models \phi[\bar{a}_j]$ for all $j \geq i$. Thus at least χ-many elements of $\mathbf{I}$ model ϕ, so by averageability $\phi(\bar{x}) \in \mathrm{Av}(\mathbf{I}/N_i)$. Conversely, if $\phi(\bar{x}) \in \mathrm{Av}(\mathbf{I}/N_i)$, averageability implies there exists $j \in [i, \delta]$ such that $\models \phi[\bar{a}_j]$. By (6) again, $\models \phi[\bar{a}_i]$ so $\phi(\bar{x}) \in \mathrm{tp}(\bar{a}_i/N_i)$. $\square$

We are now ready to prove the relationship between coherent and averageable:

Theorem 5.18. Let $\chi_0 \geq (|L| + 2)^{<\kappa}$ be such that $\mathcal{N}$ does not have the order property of length χ_0^+. Let $\chi := \left(2^{2^{\chi_0}}\right)^+$.

If $\mathbf{I}$ is a χ_0-coherent sequence with $|\mathbf{I}| \geq \chi$, then $\mathbf{I}$ is χ-averageable.

Proof. Write $\mathbf{I} = \langle \bar{a}_i : i < \delta \rangle$ and let $\langle N_i : i < \delta \rangle$ witness the coherence. It is enough to show that any subset of $\mathbf{I}$ of size χ is χ-averageable, so without loss of generality (using Remark 5.11) $\delta = \chi$. Assume for a contradiction that $\mathbf{I}$ is *not* χ-averageable. Then there exists a formula $\phi(\bar{x})$ (over $\mathcal{N}$) and sets $S_\ell \subseteq \delta$, $\ell = 0, 1$ such that $|S_\ell| \geq \chi$ and $i \in S_\ell$ implies[12] $\models \phi^\ell[\bar{a}_i]$. By Fact 5.9, we can assume without loss of generality that $\mathbf{I}_\ell := \langle \bar{a}_i : i \in S_\ell \rangle$ is χ-averageable. Pick any $\mathbf{J}_\ell \subseteq \mathbf{I}_\ell$ of size χ_0. Let $\mathbf{J} := \mathbf{J}_0 \cup \mathbf{J}_1$, and let $i < \chi$ be such that $\mathbf{J} \subseteq |N_i|$. By Lemma 5.16, $\mathbf{I}_\ell$ is based on $\mathbf{J}$ for $\ell = 0, 1$. Let $p_\ell := \mathrm{Av}(\mathbf{I}_\ell / \mathcal{N})$. Since $\mathbf{I}_\ell$ is based on $\mathbf{J}$, p_ℓ does not split over $\mathbf{J}$. By Lemma 5.17, $p_0 \upharpoonright N_i = p_1 \upharpoonright N_i$. By assumption, N_i is χ_0^+-saturated. By uniqueness for nonsplitting (Lemma 5.14), $p_0 = p_1$. However $\phi(\bar{x}) \in p_0$ while $\neg\phi(\bar{x}) \in p_1$, contradiction. □

From now on we assume:

Hypothesis 5.19.

(1) $\chi_0 \geq (|L| + 2)^{<\kappa}$ is an infinite cardinal.
(2) $\mathcal{N}$ does not have the order property of length χ_0^+.
(3) $\chi := \left(2^{2^{\chi_0}}\right)^+$.
(4) The default parameter for averages, averageability, and coherence becomes χ. That is, coherent means χ-coherent, averageable means χ-averageable, $\mathrm{Av}(\mathbf{I}/A)$ means $\mathrm{Av}_\chi(\mathbf{I}/A)$, and based means χ-based.

Note that Theorem 5.18 and Hypothesis 5.19 imply that any coherent sequence is averageable. We will use this freely.

Before studying chains of saturated models, we need one more easy lemma telling us that the average of a sequence coherent for a nonsplitting type is a nonsplitting extension. For technical reasons having to do with the translation to AECs (Section 6), we will state it more generally for independence notions that are very close to splitting:

Definition 5.20. A *splitting-like notion* is a binary relation $R(p, A)$, where $p \in S^{<\infty}(B)$ for some set B and $A \subseteq B$, satisfying the following properties:

(1) Monotonicity: If $A \subseteq A' \subseteq B_0 \subseteq B$, $p \in S^{<\infty}(B)$, and $R(p, A)$, then $R(p \upharpoonright B_0, A')$.

[12]Where ϕ^0 stands for ϕ, ϕ^1 for $\neg\phi$.

(2) Weak uniqueness: If $A \subseteq |M| \subseteq B$, M is $(|A| + (|L| + 2)^{<\kappa})^+$-saturated, and for $\ell = 1, 2$, $q_\ell \in S^{<\infty}(B)$, $R(q_\ell, A)$, and $q_1 \upharpoonright M = q_2 \upharpoonright M$, then $q_1 = q_2$.
(3) R extends nonsplitting: If $p \in S^{<\infty}(B)$ does not split over $A \subseteq B$, then $R(p, A)$.

We also say "p does not R-split over A" instead of $R(p, A)$.

Remark 5.21. If $R(p, A)$ holds if and only if p does not split over A, then R is a splitting-like notion: monotonicity is easy to check and R is nonsplitting. Weak uniqueness is Lemma 5.14.

Lemma 5.22. Let R be a splitting-like notion. Let $p \in S^{<\kappa}(B)$ be such that p does not R-split over $A \subseteq B$ with $|A| \leq \chi$.

Let $\mathbf{I} := \langle \bar{a}_i : i < \delta \rangle \frown \langle N_i : i < \delta \rangle$ be coherent for (p, A).

If $|\bigcup_{i<\delta} N_i| \subseteq B$, then $\mathrm{Av}(\mathbf{I}/B) = p$.

Proof. Since $\mathbf{I}$ is coherent, $\mathbf{I}$ is averageable. By Lemma 5.16, we can assume without loss of generality that $\mathbf{I}$ is based on A. Thus we have that $\mathrm{Av}(\mathbf{I}/B)$ does not split over A, so it does not R-split over A. By the weak uniqueness axiom of splitting-like relations (with N_0 here standing for M there), $\mathrm{Av}(\mathbf{I}/B) = p$. □

We can now get a (completely local) result on unions of saturated models.

Theorem 5.23. Assume:

(1) $\lambda > \chi^+$ is such that $\mathcal{N}$ is stable in μ for unboundedly many $\mu < \lambda$.
(2) $\langle M_i : i < \delta \rangle$ is increasing and for all $i < \delta$, M_i is λ-saturated. Write $M_\delta := \bigcup_{i<\delta} M_i$.
(3) For any $q \in S(M_\delta)$, there exists a splitting-like notion R, $i < \delta$ and $A \subseteq |M_i|$ of size $\leq \chi$ such that q does not R-split over A.

Then M_δ is λ-saturated.

Proof. By Hypothesis 5.4, it is enough to check that M_δ is λ-saturated for types of length one. Let $p \in S(B)$, $B \subseteq |M_\delta|$ have size less than λ. Let q be an extension of p to $S(M_\delta)$. If q is algebraic, then p is realized inside M_δ so assume without loss of generality that q is not algebraic. By assumption, there exists a splitting-like notion R, $i < \delta$ and $A \subseteq |M_i|$ such that p does not R-split over A and $|A| \leq \chi$. Without loss of generality, $i = 0$.

Pick $\mu < \lambda$ such that $\mu \geq \chi^+ + |B|$ and $\mathcal{N}$ is stable in μ. Such a μ exists by the hypothesis on λ. By Lemma 5.12, there exists a sequence $\mathbf{I}$ of length μ^+ which is coherent for (q, A), with the witnesses living inside M_0.

By Lemma 5.22, $\text{Av}(\mathbf{I}/M_\delta) = q$, and so in particular $\text{Av}(\mathbf{I}/B) = q \upharpoonright B = p$. By Lemma 5.6, p is realized by an element of $\mathbf{I} \subseteq |M_0| \subseteq |M_\delta|$, as needed. $\square$

The condition (3) in Theorem 5.23 is useful in case we know that the local character cardinal for *chains* κ_α is significantly lower than the local character cardinal for *sets* $\bar{\kappa}_\alpha$. This is the case when a superstability-like condition holds. If we do not care about the local character cardinal for chains, we can state a version of Theorem 5.23 without condition (3). This relies on two more results of Shelah:

Fact 5.24.

(1) If $\mu = \mu^{\chi_0} + 2^{2^{\chi_0}}$, then $\mathcal{N}$ is $(< \kappa)$-stable in μ.
(2) Let M be χ_0^+-saturated. Then for any $p \in S^{<\kappa}(M)$, there exists $A \subseteq |M|$ of size $\leq \chi_0$ such that p does not split over A.

Proof. The first result is [She09b, Theorem V.A.1.19]. The second follows from [She09b, V.A.1.12]: one only has to observe that the condition between M and $\mathcal{N}$ there holds when M is χ_0^+-saturated. $\square$

Corollary 5.25. Assume:

(1) $\lambda > \chi^+$ is such that $\mu^{\chi_0} < \lambda$ for all $\mu < \lambda$.
(2) $\langle M_i : i < \delta \rangle$ is increasing and for all $i < \delta$, M_i is λ-saturated.

If $\text{cf}(\delta) \geq \chi_0^+$, then $\bigcup_{i<\delta} M_i$ is λ-saturated.

Proof. Fix $\alpha < \kappa$. By Fact 5.24.(1), $\mathcal{N}$ is α-stable in μ for any $\mu < \lambda$ with $\mu^{\chi_0} = \mu$ and $\mu \geq \chi$. By hypothesis, there are unboundedly many such μ's.

Let $M_\delta := \bigcup_{i<\delta} M_i$. By an easy argument using the cofinality condition on δ, M_δ is χ_0^+-saturated. By Fact 5.24.(2), for any $p \in S^{<\kappa}(M_\delta)$, there exists $A \subseteq |M_\delta|$ of size $\leq \chi_0$ such that p does not split over A. By the cofinality assumption on δ, we can find $i < \delta$ such that $A \subseteq |M_i|$. Now apply Theorem 5.23 and get the result. $\square$

Remark 5.26. The proof shows that we can still replace (1) with "$\lambda > \chi^+$ is such that $\mathcal{N}$ is stable in μ for unboundedly many $\mu < \lambda$".

We end this section with the following interesting variation: the cardinal arithmetic condition on λ is improved, and we do not even need that the M_i's be λ-saturated, only that they realize enough types from the previous M_j's.

Theorem 5.27. Assume:

(1) $\lambda > \chi$ is such that $\mu^{<\kappa} < \lambda$ for all $\mu < \lambda$ (or such that $\mathcal{N}$ is stable in μ for unboundedly many $\mu < \lambda$).
(2) M is such that for any $q \in S(M)$ there exists $\langle M_i : i < \delta \rangle$ strictly increasing so that:
    (a) $\delta \geq \lambda$ is a limit ordinal.
    (b) $M = \bigcup_{i<\delta} M_i$
    (c) For all $i < \delta$, M_i is χ^+-saturated and M_{i+1} realizes $q \restriction M_i$.
    (d) There exists a splitting-like notion R, $i < \delta$ and $A \subseteq |M_i|$ of size $\leq \chi$ such that q does not R-split over A.

Then M is λ-saturated.

Proof. By Hypothesis 5.4, it is enough to check that M is λ-saturated for types of length one. Let $p \in S(B)$, $B \subseteq |M|$ have size less than λ. Let q be an extension of p to $S(M)$. If q is algebraic, then p is realized inside M, so assume q is not algebraic. Let $\langle M_i : i < \delta \rangle$ be as given by (2) for q. Let R be a splitting-like notion for which there is $i < \delta$ and $A \subseteq |M_i|$ such that q does not R-split over A and $|A| \leq \chi$. Without loss of generality, $i = 0$.

Let $\mu := (\chi + |B|)^{<\kappa}$ (or take $\mu < \lambda$ such that $\mu \geq \chi + |B|$ and $\mathcal{N}$ is stable in μ). Note that $\mu < \lambda$. For $i < \mu^+$, let $a_i \in |M_{i+1}|$ realize $q \restriction M_i$. It is easy to check that $\mathbf{I} := \langle a_i : i < \mu^+ \rangle$ is coherent for (q, A), as witnessed by $\langle M_i : i < \mu^+ \rangle$.

By Lemma 5.22, $\mathrm{Av}(\mathbf{I}/M) = q$, and so in particular $\mathrm{Av}(\mathbf{I}/B) = q \restriction B = p$. By Lemma 5.6, p is realized by an element of $\mathbf{I} \subseteq |M_0| \subseteq |M|$, as needed. □

6. Translating to AECs

To translate the result of the previous section to AECs, we will use the *Galois Morleyization* of an AEC, a tool introduced in [Vasc]: Essentially, we expand the language of the AEC with a symbol for each Galois type. With enough tameness, Galois types then become syntactic.

Definition 6.1 (Definition 3.3 in [Vasc])**.** Let K be an abstract class and let κ be an infinite cardinal. Define an expansion $\widehat{L}$ of $L(K)$ by adding a relation symbol R_p of arity $\ell(p)$ for each $p \in \mathrm{gS}^{<\kappa}(\emptyset)$. Expand each $N \in K$ to a $\widehat{L}$-structure $\widehat{N}$ by specifying that for each $\bar{a} \in \widehat{N}$, $R_p^{\widehat{N}}(\bar{a})$ holds exactly when $\mathrm{gtp}(\bar{a}/\emptyset; N) = p$. We write $\widehat{K}^{<\kappa}$ for $\widehat{K}$. We call $\widehat{K}^{<\kappa}$ the $(<\kappa)$-*Galois Morleyization* of K.

Remark 6.2. Let K be an AEC and κ be an infinite cardinal. Then $|L(\widehat{K}^{<\kappa})| \leq |\mathrm{gS}^{<\kappa}(\emptyset)| + |L| \leq 2^{<(\kappa+\mathrm{LS}(K)^+)}$.

Fact 6.3 (Theorem 3.16 in [Vasc])**.** Let K be a $(<\kappa)$-tame abstract class, and let $M \leq N_\ell$, $a_\ell \in |N_\ell|$, $\ell = 1, 2$. Then $\mathrm{gtp}(a_1/M; N_1) = \mathrm{gtp}(a_2/M; N_2)$ if and only if[13] $\mathrm{tp}_{q\widehat{L}_{\kappa,\kappa}}(a_1/M; \widehat{N_1}) = \mathrm{tp}_{q\widehat{L}_{\kappa,\kappa}}(a_2/M; \widehat{N_2})$.

Moreover the left to right direction does not need tameness: if $M \leq N_\ell$, $\bar{a}_\ell \in {}^{<\infty}|N_\ell|$, $\ell = 1, 2$, and $\mathrm{gtp}(\bar{a}_1/M; N_1) = \mathrm{gtp}(\bar{a}_2/M; N_2)$, then $\mathrm{tp}_{q\widehat{L}_{\kappa,\kappa}}(\bar{a}_1/M; \widehat{N_1}) = \mathrm{tp}_{q\widehat{L}_{\kappa,\kappa}}(\bar{a}_2/M; \widehat{N_2})$.

Note that this implies in particular that (if K is $(<\kappa)$-tame and has amalgamation) the Galois version of saturation and stability coincide with their syntactic analog in $\widehat{K}^{<\kappa}$. There is also a nice correspondence for the order property:

Fact 6.4 (Proposition 4.4 in [Vasc])**.** Let K be an AEC. Let $\widehat{K} := \widehat{K}^{<\kappa}$. If $\widehat{N} \in \widehat{K}$ has the (syntactic) order property of length χ, then N has the (Galois) $(<\kappa)$-order property of length χ. Conversely, if $\chi \geq 2^{<(\kappa+\mathrm{LS}(K)^+)}$ and N has the (Galois) $(<\kappa)$-order property of length $(2^\chi)^+$, then $\widehat{N}$ has the (syntactic) order property of length χ.

We will use these two facts freely in this section. It remains to find an independence notion to satisfy condition (3) in Theorem 5.23. The splitting-like notion R there will be given by the following:

Definition 6.5. Let K be a coherent AC and let κ be an infinite cardinal.

(1) For $p \in \mathrm{gS}^{<\infty}(B; N)$ and $A \subseteq B$, say p *κ-explicitly does not split over A* if whenever $p = \mathrm{gtp}(\bar{c}/B; N)$, for any $\bar{b}, \bar{b}' \in {}^{<\kappa}B$, if $\mathrm{gtp}(\bar{b}/A; N) = \mathrm{gtp}(\bar{b}'/A; N)$, then $\mathrm{tp}_{q\widehat{L}_{\kappa,\kappa}}(\bar{c}\bar{b}/A; N) = \mathrm{tp}_{q\widehat{L}_{\kappa,\kappa}}(\bar{c}\bar{b}'/A; N)$, where $\widehat{L} = L(\widehat{K}^{<\kappa})$.

[13]Recall that $\mathrm{tp}_{q\widehat{L}_{\kappa,\kappa}}$ stands for quantifier-free $L_{\kappa,\kappa}$-type.

(2) Define $\mathfrak{i}_{\kappa\text{-nes}}(K) := (K, \mathop{\smile\hspace{-0.9em}\perp})$, where $A \overset{N}{\underset{M}{\mathop{\smile\hspace{-0.9em}\perp}}} B$ if and only if $M \leq N$ are in K, $A, B \subseteq |N|$, and $\text{gtp}(\bar{a}/B \cup |M|; N)$ κ-explicitly does not split over M (for some/any enumeration $\bar{a}$ of A).

Remark 6.6. This is closely related to explicit nonsplitting defined in [BGKV, Definition 3.13]. The definition there is that p explicitly does not split if and only if it κ-explicitly does not split for all κ. When K is fully $(< \kappa)$-tame and short (see [Bon14b, Definition 3.3]), this is equivalent to just asking for p to κ-explicitly not split.

Remark 6.7 (Syntactic invariance)**.** Let $\widehat{K} := \widehat{K}^{<\kappa}$. Assume $\text{tp}_{q\widehat{L}_{\kappa,\kappa}}(\bar{c}/B; N) = \text{tp}_{q\widehat{L}_{\kappa,\kappa}}(\bar{c}'/B; N)$ and $\text{gtp}(\bar{c}/B; N)$ κ-explicitly does not split over $A \subseteq B$. Then $\text{gtp}(\bar{c}'/B; N)$ κ-explicitly does not split over A.

The next result imitates [BGKV, Lemma 5.6]:

Lemma 6.8. Let $\mathfrak{i} = (K, \mathop{\smile\hspace{-0.9em}\perp})$ be a $(< \alpha, \geq \lambda)$-independence relation. Assume $\mathfrak{i}$ has base monotonicity, transitivity, $\mathfrak{i}_\lambda$ has uniqueness, and $\bar{\kappa}_{<\alpha_0}(\mathfrak{i}) = \lambda^+$, $\kappa \leq \lambda^+$. Let $M \leq N$, $|M| \subseteq B \subseteq |N|$. If $p \in \text{gS}^{<\alpha}(B; N)$ does not $\mathfrak{i}$-fork over M, then p does not $\mathfrak{i}_{\kappa\text{-nes}}(K)$-fork over M.

Proof. By monotonicity, we can assume without loss of generality that $\alpha = \alpha_0$. By local character and transitivity, there exists $M_0 \leq M$ of size λ such that p does not $\mathfrak{i}$-fork over M_0. We will show that p does not $\mathfrak{i}_{\kappa\text{-nes}}(K)$-fork over M_0 which is enough by base monotonicity for $\mathfrak{i}_{\kappa\text{-nes}}(K)$. So without loss of generality, assume $M = M_0 \in K_\lambda$.

Write $p = \text{gtp}(\bar{c}/B; N)$. Let $\bar{b}, \bar{b}' \in {}^{<\kappa}B$ be such that $\text{gtp}(\bar{b}/M; N) = \text{gtp}(\bar{b}'/M; N)$. By monotonicity, $p \restriction M\bar{b}$, $p \restriction M\bar{b}'$ do not $\mathfrak{i}$-fork over M. Let $N' \geq N$ and $f : N \underset{M}{\longrightarrow} N'$ be such that $f(\bar{b}) = \bar{b}'$. By invariance, $f(p \restriction M\bar{b})$ does not $\mathfrak{i}$-fork over M. Now using uniqueness for $\mathfrak{i}_\lambda$, $f(p \restriction M\bar{b}) = p \restriction M\bar{b}'$. The result follows. $\square$

The next technical lemma captures the essence of our translation:

Lemma 6.9. Let K be a $(< \kappa)$-tame AEC with amalgamation. Let χ_0 be such that:

(1) $\chi_0 \geq 2^{<(\kappa + \text{LS}(K)^+)}$.
(2) K does not have the $(< \kappa)$-order property of length χ_0^+.

Set $\chi := \left(2^{2^{\chi_0}}\right)^+$. Let λ be such that:

(1) $\lambda > \chi^+$.
(2) K is stable in μ for unboundedly many $\mu < \lambda$.

Let $\mathfrak{i}$ be a $(\leq 1, \geq \chi_0)$-independence relation with underlying class $K^{\chi_0\text{-sat}}$ such that:

(1) $\mathfrak{i}$ has base monotonicity and transitivity.
(2) $\mathfrak{i}_{\chi_0}$ has uniqueness.
(3) $\bar{\kappa}_1(\mathfrak{i}) = \chi_0^+$.

Then:

(1) If $\langle M_i : i < \delta \rangle$ is an increasing chain of λ-saturated models and $\text{cf}(\delta) \geq \kappa_1(\mathfrak{i})$, then $\bigcup_{i<\delta} M_i$ is λ-saturated.
(2) If $M \in K$ is such that for any $q \in \text{gS}(M)$ there exists $\langle M_i : i < \delta \rangle$ strictly increasing so that:
 (a) $\delta \geq \lambda$ and $\text{cf}(\delta) \geq \kappa_1(\mathfrak{i})$.
 (b) $M = \bigcup_{i<\delta} M_i$.
 (c) For all $i < \delta$, M_i is χ^+-saturated and M_{i+1} realizes $q \restriction M_i$.
 Then M is λ-saturated.

Proof. We prove the first statement. The proof of the second is analogous but uses Theorem 5.27 instead of Theorem 5.23. Set $M_\delta := \bigcup_{i<\delta} M_i$. Let $N \geq M_\delta$ be such that N realizes all types in $\text{gS}^{<\kappa}(M_\delta)$. We check that M_δ is λ-saturated in N. Let $\widehat{K} := \widehat{K}^{<\kappa}$ be the $(<\kappa)$-Galois Morleyization of K. Let $\mathcal{N} := \widehat{N}$. By $(<\kappa)$-tameness, it is enough to show that $\widehat{M_\delta}$ is (syntactically) λ-saturated in $\mathcal{N}$. Work inside $\mathcal{N}$ in the language of $\widehat{K}$. We also let $\mathcal{S} := \{|M| \mid M \leq N\}$. Note that $\mathcal{S}$ satisfies Hypothesis 5.2.

First observe that Hypothesis 5.19 holds as (Remark 6.2) $|L(\widehat{K})| \leq 2^{<(\kappa + \text{LS}(K)^+)}$, so χ_0 has all the required properties. Also, Hypothesis 5.4 holds by Remark 2.6. By hypothesis, $\lambda > \chi^+$. We want to use Theorem 5.23, and it remains to check that (3) there holds.

For $A \subseteq B$ and $p \in S^{<\infty}(B)$, define the relation $R(p, A)$ to hold if and only if $p = \text{tp}(\bar{c}/B)$ and $\text{gtp}(\bar{c}/B; N)$ κ-explicitly does not split over A. Note that this is well-defined by Remark 6.7. We want to check that this is a splitting-like notion (Definition 5.20). By definition of κ-explicit nonsplitting, if $p \in S^{<\infty}(B)$ does not split over $A \subseteq B$, then $R(p, A)$. Also, it is easy to check that R satisfies the monotonicity axiom. It remains to check the weak uniqueness axiom. So let M be $\mu := \left(|A| + (|L(\widehat{K})| + 2)^{<\kappa}\right)^+$-saturated, $A \subseteq |M| \subseteq B$, and for $\ell = 1, 2$, $q_\ell \in S^{<\infty}(B)$, $R(q_\ell, A)$ and $q_1 \restriction M = q_2 \restriction M$. Note that M is also μ-saturated in the Galois sense (by tameness and Remark 3). Thus we can imitate the proof of Lemma 5.14, using Galois saturation instead of syntactic saturation to get $\bar{b}'$ satisfying $\mathrm{gtp}(\bar{b}'/A) = \mathrm{gtp}(\bar{b}/A)$ (instead of just $\mathrm{tp}(\bar{b}'/A) = \mathrm{tp}(\bar{b}'/A)$ as there). The definition of κ-explicit nonsplitting then makes the proof go through.

Now let $q \in \mathrm{gS}(M_\delta)$. By definition of κ_1, there exists $i < \delta$ such that q does not $\mathfrak{i}$-fork over M_i. Now by definition of $\bar{\kappa}_1$, there exists $M \leq M_i$ of size χ_0 such that $q \upharpoonright M_i$ does not $\mathfrak{i}$-fork over M. By transitivity, q does not $\mathfrak{i}$-fork over M. By Lemma 6.8, working syntactically inside $\mathcal{N}$, q does not R-split over M. Thus (3) holds. Therefore M_δ is λ-saturated, as desired. $\square$

We obtain the following result on chains of saturated models in stable AECs:

Theorem 6.10. Let K be a $(< \kappa)$-tame AEC with amalgamation, $\kappa \geq \mathrm{LS}(K)$. If K is stable, then there exists $\chi_0 \leq \lambda_0 < h(\kappa)$ satisfying the following property:

If $\lambda \geq \lambda_0$ is such that $\mu^{<\chi_0} < \lambda$ for all $\mu < \lambda$ (or just that K is stable in μ for unboundedly many $\mu < \lambda$), then whenever $\langle M_i : i < \delta \rangle$ is an increasing chain of λ-saturated models with $\mathrm{cf}(\delta) \geq \chi_0$, we have that $\bigcup_{i<\delta} M_i$ is λ-saturated.

Proof. Using Fact 2.9, pick $\chi_{00} \leq \mu_0 < h(\kappa)$ such that:

(1) $\chi_{00}^+ \geq 2^{<(\kappa+\mathrm{LS}(K)^+)} + \kappa^+$.
(2) K is stable in any $\mu \geq \mu_0$ with $\mu = \mu^{\chi_{00}}$.
(3) K does not have the $(< \kappa)$-order property of length χ_{00}^+.
(4) K is stable in χ_{00}.

Set $\chi_0 := \chi_{00}^+$, $\lambda_0 := \mu_0^+ + (2^{2^{\chi_0}})^{+3}$. Let $\mathfrak{i}$ be as given by Fact 2.22 with λ there standing for χ_{00} here. Then $\mathfrak{i}$ satisfies the conditions of Lemma 6.9, so the result follows. $\square$

The statement becomes much nicer in superstable AECs:

Theorem 6.11. Let K be a μ-superstable$^+$ AEC. Then there exists $\lambda_0 < h(\mu^+)$ such that for any $\lambda \geq \lambda_0$:

(1) $K^{\lambda\text{-sat}}$ is an AEC with $\mathrm{LS}(K^{\lambda\text{-sat}}) = \lambda$.
(2) If $M \in K^{\lambda_0\text{-sat}}$ is such that for any $q \in \mathrm{gS}(M)$ there exists $\langle M_i : i < \lambda \rangle$ a resolution of M in $K^{\lambda_0\text{-sat}}$ such that $q \upharpoonright M_i$ is realized in M_{i+1} for all $i < \lambda$, then $M \in K^{\lambda\text{-sat}}$.

Proof.

(1) We first show that any increasing union of λ-saturated models is saturated. By definition of superstability, K is μ-tame. Let $\lambda_0 < h(\mu^+)$ be as given by the proof of Theorem 6.10 (applied to $K_{\geq \mu}$). Let $\mathfrak{i}$ be as given by Fact 2.22 with μ here standing for λ there. In particular, $\kappa_1(\mathfrak{i}) = \aleph_0$. Now apply Lemma 6.9 (note that by Fact 2.21, K is stable in any $\mu' \geq \mu$). To see that $\text{LS}(K^{\lambda\text{-sat}}) = \lambda$, imitate the proof of [She90, Theorem III.3.12].
(2) Use the second conclusion of Lemma 6.9.

$\square$

Remark 6.12. A similar proof gives a more precise result if K is more than μ-tame: If K is a $(< \kappa)$-tame AEC with amalgamation, $\text{LS}(K) \leq \kappa$, then there exists $\mu_0 < h(\kappa)$ such that if K is μ-superstable with $\mu \geq \mu_0$, then the statement of Theorem 6.11 holds for $\lambda_0 := \mu^+$.

The difference with Corollary 4.5 is that we do not need *strong* superstability. Since we only know how to get strong superstability by going above a fixed point of the Beth function (Fact 2.21), the Hanf number is also improved compared to the known cases (see the remark above). We also do not know how to prove Theorem 6.11.(2) with independence calculus.

7. On superstability in AECs

In the introduction to [She09a], Shelah points out the importance of finding a definition of superstability for AECs. He also remarks (p. 19) that superstability in AECs suffers from "schizophrenia": definitions that are equivalent in the first-order case might not be equivalent in AECs. In this section, we point out that Definition 2.18 implies several other candidate definitions of superstability. Recall from Fact 2.21 that Definition 2.18 implies that the class is stable on a tail of cardinals. We will focus on five other definitions:

(1) For every high-enough λ, the union of any increasing chain of λ-saturated models is λ-saturated. This is the focus of this paper and is equivalent to first-order superstability by [AG90, Theorem 13].
(2) The existence of a saturated model of size λ for every high-enough λ. In first-order, this is an equivalent definition of superstability by the saturation spectrum theorem (Fact 1.2).
(3) The existence of a superlimit model of size λ for every high-enough λ. This is the definition of superstability listed by Shelah in [She09a, Definition N.2.4]. Recall that a model $M \in K_\lambda$

is *superlimit* if it is universal, has an isomorphic proper extension in K_λ, and whenever $\langle M_i : i < \delta \rangle$ is increasing in K_λ, $\delta < \lambda^+$, and $M_i \cong M$ for all $i < \delta$, then $\bigcup_{i<\delta} M_i \cong M$.
(4) The existence of a good λ-frame on a subclass of saturated models (e.g. for every high-enough λ). Recall that a good frame is essentially a forking-like notion for types of length one (see [She09a, Definition II.2.1] for the formal definition). Good frames are *the* central notion in [She09a] and are described by Shelah as a "bare bone" definition of superstability.
(5) The uniqueness of limit models of size λ for every high-enough λ: Recall that a model M is (λ, δ)*-limit over* M_0 if $M_0 \leq M$ are in K_λ, $\delta < \lambda^+$ is a limit ordinal and there exists $\langle M_i : i \leq \delta \rangle$ increasing continuous such that $M_\delta = M$ and $i < \delta$ implies $M_i <_{\text{univ}} M_{i+1}$ (recall Definition 2.18). We say K_λ has *uniqueness of limit models* if for any $M_0 \in K_\lambda$, any limit $\delta_1, \delta_2 < \lambda^+$, any M_ℓ which are (λ, δ_ℓ)-limit over M_0 are isomorphic over M_0. Uniqueness of limit models is central in [She99, SV99, Van06, Van13] and is further examined in [GVV] (Theorem 7 there proves that the condition is equivalent to first-order superstability). These papers all prove the uniqueness under a categoricity (or no Vaughtian pair) assumption. In [She09a, Lemma II.4.8], uniqueness of limit models is proven from a good frame (see also [Bon14a, Theorem 9.2] for a detailed writeup). This is used in [BG] to get eventual uniqueness of limit models from categoricity, but the authors have to make an extra assumption (the extension property for coheir).

Note that some easy implications between these definitions are already known (see for example [Dru13, Corollary 2.3.12]). We now show that assuming amalgamation and tameness, if K is superstable, then all five of these conditions hold. This gives an eventual version of [Dru13, Conjecture 4.2.5], which was initially claimed in [GVV]. It also shows how to build a good frame without relying on categoricity (as opposed to all previous constructions, see [She09a, Theorem II.3.7], [Vasa, Theorem 7.3], or [Vasb, Theorem 10.16]).

Theorem 7.1. If K is a μ-superstable$^+$ AEC, then there exists $\lambda_0 < h(\mu^+)$ such that for all $\lambda \geq \lambda_0$:

(1) The union of any increasing chain of λ-saturated models in K is λ-saturated.
(2) K has a saturated model of size λ.
(3) K has a superlimit model of size λ.

(4) There exists a type-full good λ-frame with underlying class $K_\lambda^{\lambda\text{-sat}}$.
(5) K_λ has uniqueness of limit models.

Proof. Note that by Fact 2.21, $K_{\geq \mu}$ has no maximal models, joint embedding, and is stable in every cardinal. Let $\lambda_0 < h(\mu^+)$ be as given by Theorem 6.11 and let $\lambda \geq \lambda_0$. Then $K^{\lambda\text{-sat}}$ is an AEC with $\text{LS}(K^{\lambda\text{-sat}}) = \lambda$. Thus (1) and (2) hold. If M is the saturated model of size λ, then it is easy to check that M is superlimit: it is universal as $K_{\geq \mu}$ has joint embedding, it has a saturated proper extension of size λ since $\text{LS}(K^{\lambda\text{-sat}}) = \lambda$, and any increasing chain of saturated models in K_λ of length less than λ^+ has a saturated union. Thus (3) hold. To see (4), use [Vasb, Theorem 10.8.2c].

We are now ready to prove (5). As observed above, a good frame implies uniqueness of limit models. Thus $K_\lambda^{\lambda\text{-sat}}$ has uniqueness of limit models. It follows that K_λ has uniqueness of limit models: Let M_ℓ be (λ, δ_ℓ)-limit over M_0, $\ell = 1, 2$. Pick $M_0' \geq M_0$ in $K_\lambda^{\lambda\text{-sat}}$. By universality, M_ℓ is also (λ, δ_ℓ)-limit over some copy of M_0', so after some renaming we can assume without loss of generality that $M_0 = M_0'$. For $\ell = 1, 2$, build $\langle M_i' : i \leq \delta_\ell \rangle$ increasing continuous such that for all $i < \delta_\ell$, $M_i' \in K_\lambda^{\lambda\text{-sat}}$ and $M_i' <_{\text{univ}} M_{i+1}'$. This is possible (see for example [Vasb, Section 2.3]), and by a back and forth argument, $M_\ell \cong_{M_0} M_{\delta_\ell}'$. By uniqueness of limit models in $K^{\lambda\text{-sat}}$, $M_{\delta_1}' \cong_{M_0} M_{\delta_2}'$. Composing the isomorphisms, we obtain that $M_1 \cong_{M_0} M_2$. $\square$

Remark 7.2. If K is $(< \kappa)$-tame and μ-superstable$^+$, we can give a better bound on λ_0 as explained in Remark 6.12. Similarly, if K is κ-strongly μ-superstable$^+$ we can take $\lambda_0 := (\mu^{<\kappa_r})^{++}$, see Corollary 4.5.

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