

# WEAK FACTORIZATION SYSTEMS AND STABLE INDEPENDENCE

MICHAEL LIEBERMAN, JIŘÍ ROSICKÝ, AND SEBASTIEN VASEY

ABSTRACT. We exhibit a bridge between the theory of *weak factorization systems*, a categorical concept used in algebraic topology and homological algebra, and the model-theoretic notion of *stable independence*. Roughly speaking, we show that the cofibrantly generated weak factorization systems (those that are, in a precise sense, generated by a set) are exactly those that give rise to stable independence notions. This two way connection yields a powerful new tool to build tame and stable abstract elementary classes. In particular, we generalize a construction of Baldwin-Eklof-Trlifaj to prove that the category of flat modules with flat monomorphisms has a stable independence notion, and explain how this connects to the fact that every module has a flat cover.

## CONTENTS

1. Introduction	2
2. Preliminaries	6
3. On subcategories of accessible categories	10
4. Independence from pushouts	12
5. Weak factorization systems and injectives	22
6. Examples	28
Appendix A. Canonicity of stable independence	37
References	43

---

*Date:* April 11, 2019

AMS 2010 Subject Classification: Primary: 18C35. Secondary: 03C45, 03C48, 03C52, 03C55, 16B50, 16B60, 55U35.

*Key words and phrases.* weak factorization system; forking; stable independence; abstract elementary class; cofibrantly generated.

The second author is supported by the Grant agency of the Czech Republic under the grant 19-00902S.

## 1. INTRODUCTION

**1.1. Background.** Classical model theory is concerned with studying classes of models of a first-order<sup>1</sup> theory. Given such a theory  $T$ , it is natural to look at its category  $\mathbf{Elem}(T)$  of models, with morphisms the elementary embeddings<sup>2</sup>. The category  $\mathbf{Elem}(T)$  is, however, not very well-behaved: because all the morphisms are monomorphisms, many limits and colimits (for example pushouts) are lacking. Thus while one can axiomatize, for example, the *class* of abelian groups in first-order logic, model theory does not really allow us to directly study the *category*  $\mathbf{Ab}$  of abelian groups, where the morphisms are group homomorphisms. This latter category is in fact complete and cocomplete. The category  $\mathbf{Ab}$  also has many other nice features, well known to category theorists (for example, it satisfies the axioms of a Grothendieck abelian category). A central preoccupation of this paper involves starting from a natural and well-behaved (and therefore not very “model-theoretic”) category  $\mathcal{K}$ , such as  $\mathbf{Ab}$ , and studying how what we know about  $\mathcal{K}$  can give us useful model-theoretic information about a subcategory (such as  $\mathbf{Elem}(T)$ ) with the same objects but a restricted class of morphisms.

The basic framework for this study will be that of *accessible categories*. Recall that a category is *accessible* if it has sufficiently directed colimits and all objects can be written as a directed colimits of a fixed set of “small” subobjects (see Definition 2.1). We call a complete and cocomplete accessible category *locally presentable*. Basic references on accessible and locally presentable categories are [MP89, AR94]. They are often used in algebraic topology and homological algebra (cf. [Lur09]) but are also closely connected with model theory (not, to be sure, in the most familiar sense—i.e. the analysis of definable sets in a fixed structure—but in the sense of *abstract model theory*, which forgets the definable structure and instead studies various *categories* of models; the latter approach aims in particular to investigate classes that are not necessarily given by first-order theories). For example, an *abstract elementary class* (AEC) [She87] can be seen as a certain kind of category that satisfies the basic properties of  $\mathbf{Elem}(T)$ . In particular, it is an accessible category with all morphisms monomorphisms. This connection between abstract model theory and accessible categories is foreseen in the introduction of the book of Makkai and Paré [MP89] but was only relatively recently made precise, see for example [Lie11, BR12, BGL<sup>+</sup>16, LRV19b].

While we have just argued that category theory has something to say about model theory, we believe the reverse is also true: working with categories where all morphisms are monomorphisms (e.g. classes of models of first-order theories, or more generally AECs) allows one to use powerful set-theoretic method as well as concrete, element by element constructions that are not available in the general case. As an example of a nontrivial model-theoretic result with some category-theoretic flavor, recall that a class of models is called *categorical* in a cardinal  $\lambda$  if it has a single model of cardinality  $\lambda$  up to isomorphism (this definition can be categorified using presentability ranks). Morley’s categoricity theorem [Mor65] says a class of models axiomatized by a countable first-order theory that is categorical in *some*

<sup>1</sup>A first-order theory is a set of “finitary” formulas like  $\forall x \exists y (xy = 1 \wedge yx = 1)$ . The exact definitions are not relevant for this paper.

<sup>2</sup>That is, injective homomorphisms preserving all formulas with parameters.

uncountable cardinal is categorical in *all* uncountable cardinals. Recent generalizations have pushed variations of this result all the way to AECs. For the benefit of the reader, we cite here a version for finitely accessible categories:

**Fact 1.1.** Assuming a large cardinal axiom<sup>3</sup> (namely, that there exists a proper class of strongly compact cardinals), if a finitely accessible category is categorical in a proper class of cardinals, then it is categorical on a tail of cardinals.

*Proof.* By [BR12, 5.8(2)], this follows from the corresponding result for AECs, due to Shelah and the third author [SV] (in fact we could have started with any abstract elementary category in the sense of [BR12, 5.3]).  $\square$

Classical examples of uncountably categorical classes are the class of vector spaces over a fixed countable field, or the class of algebraically closed fields of a fixed characteristic. In both cases, there is a notion of independence (linear independence or algebraic independence) that, via a corresponding notion of basis, implies categoricity. Modern proofs of Morley’s theorem show that this is not an accident: in any setup with categoricity there is an abstract notion of independence (dubbed *forking* by Shelah [She90]) generalizing linear and algebraic independence. Forking is now a central concept of modern model theory and its study has had an impact also on other branches of mathematics (the most well known example is Hrushovski’s proof of the Mordell-Lang conjecture for function fields, see for example [Bou99]). Nevertheless, the connections between forking and category theory had not been examined until recently.

**1.2. Stable independence.** The category-theoretic essence of forking was investigated in the authors’ recent [LRV19a], where the purely category-theoretic definition of a *stable independence notion* was given (see Definition 2.10). Roughly, a stable independence notion in a given category is a class of commutative squares (called *independent*) that satisfies certain properties: an existence property (any span amalgamates to an independent square), a uniqueness property (the independent amalgam is unique, in a certain weak sense), and a transitivity property (independent squares can be “pasted” in the same way as pushout squares). These properties turn independent squares into the morphisms of an arrow category, and we require that this arrow category itself be *accessible*. It was proven in [LRV19a] that, under reasonable conditions, there can be at most one stable independence notion (see also Theorem A.6 here), and in the case of the category of models of a first-order theory a stable independence notion exists exactly when the theory is stable (independent squares then correspond to the usual nonforking amalgams). We emphasize again that stable independence is *the* key tool in the proof of Fact 1.1.

The reader can think of stable independence as a replacement for pushouts in setups where they are not available (for example when all morphisms are monomorphisms). Indeed, in accessible categories with pushouts the class of *all* commutative squares will form a stable independence notion (Example 6.2). Thus, working toward our stated goal, it is natural to consider the following problem: suppose we start with a

---

<sup>3</sup>If we restrict ourselves to the locally  $\aleph_0$ -multipresentable categories, we can remove the large cardinal assumption. See [Vas17], [LRV19a, 6.6].

category  $\mathcal{K}$  that is very nice (locally presentable, so with pushouts), and suppose we are given a subclass  $\mathcal{M}$  of monomorphisms of  $\mathcal{K}$  (these could contain all monomorphisms, or just a subclass of the “nice” ones). What “trace” of pushouts are left in the category  $\mathcal{K}_{\mathcal{M}}$  with the same objects as  $\mathcal{K}$  but with morphisms those of  $\mathcal{M}$ ? In particular, when does  $\mathcal{K}_{\mathcal{M}}$  have a stable independence notion? This question was considered already in [LRV19a, §5], where a sufficient condition was obtained: when  $\mathcal{M}$  is the set of regular monomorphisms and  $\mathcal{K}$  has effective unions, the latter being an exactness property first introduced by Barr [Bar88]. Here we study the problem in a much broader axiomatic framework and give a condition that is both necessary and sufficient for the existence of a stable independence notion in  $\mathcal{K}_{\mathcal{M}}$ .

**1.3. Model categories and weak factorization systems.** We note that the problem of investigating “nice classes of monomorphisms” springs up in different areas of mathematics. For example, in algebraic topology a *model category* [Hov99] is a category with three distinguished classes of morphisms—the fibrations, cofibrations, and weak equivalences—which represent, roughly, the nice surjections, nice monomorphisms, and a notion of weak isomorphism (in homological algebra, the notion of a *cotorsion pair* plays a somewhat similar role, see [Hov07]). Part of the axioms of a model category is that the distinguished morphisms should form a *weak factorization system*. For example, it should be possible to write any morphism as a cofibration followed by a trivial fibration (i.e. a weak equivalence that is also a fibration). This factorization is only required to be canonical in a weak sense (hence the “weak” in “weak factorization system”).

As a simple example that may help the reader, let us consider the category of sets (with all functions as morphisms). There, the pair (epi, mono) forms a factorization system (not just a weak one): any function  $A \rightarrow B$  factors as  $A \rightarrow f[A] \rightarrow B$ , a surjection followed by an injection, and this factorization is canonical. On the other hand, it is also true that any function factors as an injection followed by a surjection! This factorization is much less canonical (existence uses the axiom of choice), but is still not completely arbitrary. This translates to the fact that (mono, epi) forms a *weak* factorization system in the category of sets.

Let us return to our problem: we start with a locally presentable category  $\mathcal{K}$  and a “nice” class of monomorphisms  $\mathcal{M}$ , and we want to obtain a stable independence notions on  $\mathcal{K}_{\mathcal{M}}$ . What properties should we require of  $\mathcal{M}$ ? First,  $\mathcal{M}$  should contain the isomorphisms and be closed under (possibly transfinite) compositions. Also, since we are going to build our stable independence notions by using pushouts (in  $\mathcal{K}$ ),  $\mathcal{M}$  should be closed—in a sense we will make precise—under pushouts as well. It turns out that if  $\mathcal{M}$  forms the left part of a weak factorization system, then it has all of these properties (Fact 5.3)! Finally, we require a technical weak “two out of three” property, called coherence, which seems to hold in most examples. A natural candidate for a stable independence notion in  $\mathcal{K}_{\mathcal{M}}$ , the  $\mathcal{M}$ -effective squares, can then be devised (Definition 4.3). The  $\mathcal{M}$ -effective squares can be shown to satisfy all the properties from the definition of a stable independence notion, except perhaps accessibility (this already establishes some important properties of  $\mathcal{K}_{\mathcal{M}}$ , for example the amalgamation property).

What is needed to establish accessibility? Our main theorem, Theorem 4.9 shows that this holds *if and only if* the class  $\mathcal{M}$  is *cofibrantly generated*. The latter

condition means, in particular, that  $\mathcal{M}$  can be generated from a subset—a set, crucially, rather than a class—using only pushouts, (transfinite) compositions, and retracts. This notion comes from topology, where the cell complexes are precisely the spaces that can be generated in this way (using only finitely many steps) from the inclusions  $\partial D^n \rightarrow D^n$  of the boundary sphere of the  $n$ -dimensional unit ball.

A basic tool in the theory of weak factorization system is the small object argument (first introduced by Quillen, [Qui67], and recalled here as Fact 5.4), which shows that *any* cofibrantly generated class of morphisms forms the left part of a weak factorization system. Thus the original hypothesis that  $\mathcal{M}$  was part of a weak factorization system is unnecessary: if  $\mathcal{M}$  is closed under pushouts and transfinite compositions, then  $\mathcal{K}_{\mathcal{M}}$  will have a stable independence notion (defined from pushout squares in  $\mathcal{K}$ ) if and only if  $\mathcal{M}$  is cofibrantly generated. In this case,  $\mathcal{M}$  will form the left part of a weak factorization system.

**1.4. Some consequences of the main theorem.** We have just seen that the main theorem establishes a tight connection between cofibrantly generated weak factorization systems and stable independence. Section 6 gives many examples of this two-way interaction at work. Let us mention here that we also get from the main theorem that  $\mathcal{K}_{\mathcal{M}}$  will be an AEC. In fact, since it has a stable independence notion, it will have amalgamation, and be stable and tame in the sense of [GV06]. Quite a lot is known about such AECs (see [Vas16, Vas18]), but the present paper is the first to give a systematic way to produce examples. We also generalize the work of Baldwin-Eklof-Trlifaj [BET07] by showing that their study of Ext-orthogonality classes of modules  ${}^{\perp}N$  (see Definition 6.15) fits naturally in the framework of this paper, and furthermore that anytime this class is an AEC it has a stable independence notion (Theorem 6.20).

An interesting particular case occurs when  $\mathcal{K}$  is the category of flat  $R$ -modules (i.e. modules  $F$  such that tensoring with  $F$  preserves exact sequences), and  $\mathcal{M}$  is the class of flat monomorphisms (i.e. monomorphisms whose cokernel is flat). For a long time, it was open whether all modules admit a flat cover (see Definition 6.25). Eklof and Trlifaj proved using set-theoretic methods [ET01, Corollary 11] that it sufficed to show that any flat module could be resolved as a chain of flat submodules where the quotient at each intermediate step is small (we will say that “ $\mathcal{K}$  has refinements”). Later, Bican, El Bashir, and Enochs proved that  $\mathcal{K}$  indeed had refinements, completing the proof of the flat cover conjecture [BBE01, Proposition 2]. Still later, the second author [Ros02] noticed that this entire proof of the flat cover conjecture really consisted in establishing that the class  $\mathcal{M}$  of flat monomorphisms was the left part of a cofibrantly generated weak factorization system. In fact, having refinements is the same as being cofibrantly generated. Using our main theorem we deduce (Theorem 6.20) the “model-theoretic content” of this proof of the flat cover conjecture: the category  $\mathcal{K}_{\mathcal{M}}$  is an AEC with a stable independence notion.

**1.5. Acknowledgments.** We thank Jan Trlifaj for helpful conversations about flat-like categories of modules (Section 6).

## 2. PRELIMINARIES

We assume familiarity with basic category theory, as expositied for example in [Lan98] or [AHS04]. Knowledge of accessible categories [MP89, AR94] would be helpful but we briefly review some terminology here. We use the letters  $\mathcal{L}$  and  $\mathcal{K}$  for categories. We write 0 for the initial object and 1 for the terminal object of a category (if they exist).

**Definition 2.1.** For a regular cardinal  $\lambda$ , an object  $A$  in a category  $\mathcal{K}$  is called  $\lambda$ -*presentable* if the functor  $\text{Hom}(A, -) : \mathcal{K} \rightarrow \mathbf{Set}$  preserves  $\lambda$ -directed colimits (i.e. colimits indexed by a  $\lambda$ -directed poset, which is a poset where every subset of cardinality strictly less than  $\lambda$  has an upper bound). A category  $\mathcal{K}$  is  $\lambda$ -*accessible* if it has  $\lambda$ -directed colimits, only a set up to isomorphism of  $\lambda$ -presentable objects, and every object is a  $\lambda$ -directed colimit of  $\lambda$ -presentable objects. The category  $\mathcal{K}$  is *locally  $\lambda$ -presentable* if it is complete and cocomplete and  $\lambda$ -accessible. A category is *accessible* [*locally presentable*] if it is  $\lambda$ -accessible [*locally  $\lambda$ -presentable*] for some  $\lambda$ . When  $\lambda = \aleph_0$ , we also talk of a finitely presentable object, and of a finitely accessible [*locally finitely presentable*] category.

We will use without comment that the index of accessibility can be raised: if  $\mathcal{K}$  is  $\mu$ -accessible and  $\lambda > \mu$  is a regular cardinal such that  $\theta^{<\mu} < \lambda$  for all  $\theta < \lambda$ , then  $\mathcal{K}$  is  $\lambda$ -accessible [AR94, 2.11, 2.13(3)]. For locally presentable categories (or more generally for accessible categories with directed colimits), the cardinal arithmetic condition is not needed [BR12, 4.1].

The following basic concepts will be used very often:

**Definition 2.2.** Let  $\mathcal{K}$  be a category.

- (1) A *monomorphism* (or *mono*, for short) is a morphism  $f$  such that  $fg_1 = fg_2$  always implies  $g_1 = g_2$ . Dually, an *epimorphism* (*epi*, for short) is a morphism  $f$  such that  $g_1f = g_2f$  implies  $g_1 = g_2$  for any  $g_1$  and  $g_2$ .
- (2) Suppose we have maps  $A \xrightarrow{i} B \xrightarrow{r} A$  with  $ri = \text{id}_A$ . Then we call  $r$  a *retraction* (or *split epimorphism*) and we call  $i$  a *section* (or *split monomorphism*). The map  $r$  witnesses that  $A$  is a *retract* of  $B$ .

Closure under certain classes of limits will play an important role here, so we take a few minutes to make the terminology clear:

**Definition 2.3.** A subcategory  $\mathcal{L}$  of a category  $\mathcal{K}$  is:

- (1) *isomorphism-closed* (or *replete*) if whenever  $A$  is an object of  $\mathcal{L}$  and  $f : A \rightarrow B$  is an isomorphism, we have that  $f$  and  $B$  are both in  $\mathcal{L}$ .
- (2) *closed under limits in  $\mathcal{K}$*  if whenever  $D : I \rightarrow \mathcal{L}$  is a diagram in  $\mathcal{L}$  and  $(A \xrightarrow{f_i} D_i)_{i \in I}$  is a limit cone in  $\mathcal{K}$ , then  $(A \xrightarrow{f_i} D_i)_{i \in I}$  is a limit cone in  $\mathcal{L}$ .

Similarly define closure under any class of (co)limits, such as colimits,  $\lambda$ -directed colimits, etc.

**Remark 2.4.** Let  $\mathcal{K}$  be a category with limits. For  $\mathcal{L}$  an isomorphism-closed subcategory of  $\mathcal{K}$  such that whenever  $D : I \rightarrow \mathcal{L}$  is a diagram in  $\mathcal{L}$  and  $(A \xrightarrow{f_i} D_i)_{i \in I}$  is a limit cone in  $\mathcal{K}$ , then  $f_i$  is a morphism in  $\mathcal{L}$  for all  $i \in I$ . Then, the following are equivalent:

- (1)  $\mathcal{L}$  is closed under limits in  $\mathcal{K}$ .
- (2)  $\mathcal{L}$  has limits.

When  $\mathcal{L}$  is a full subcategory of  $\mathcal{K}$ , these conditions hold. Similarly for other classes of (co)limits.

**Remark 2.5.** When  $r : B \rightarrow A$  is a retraction and  $i : A \rightarrow B$  is a corresponding section, the map  $f = ir : B \rightarrow B$  is an idempotent:  $ff = f$ . We say an idempotent  $f : B \rightarrow B$  *splits* if there exists maps  $r : B \rightarrow A$  and  $i : A \rightarrow B$  so that  $f = ir$  and  $ri = \text{id}_A$  (that is,  $f$  splits if its idempotence is witnessed by a retraction). If a category has  $\lambda$ -directed colimits for some  $\lambda$ , then any idempotent splits [AR94, 1.2, 1.21, 2.4]. Moreover, a full isomorphism-closed subcategory  $\mathcal{L}$  of a category  $\mathcal{K}$  so that (in  $\mathcal{L}$ ) all idempotents split must be closed under retract ([AR94, 2.5(1)]).

We recall, too, the notion of a reflective subcategory:

**Definition 2.6.** We say that a full subcategory  $\mathcal{L}$  of a category  $\mathcal{K}$  is *reflective* if the inclusion functor  $\mathcal{L} \rightarrow \mathcal{K}$  has a left adjoint. Equivalently: for every  $K$  in  $\mathcal{K}$  there is a *reflection arrow*  $r_K : K \rightarrow L$ ,  $L$  in  $\mathcal{L}$ , so that any  $K \rightarrow L'$ ,  $L'$  in  $\mathcal{L}$ , factors uniquely through  $r_K$ . We say that  $\mathcal{L}$  is a *weakly reflective subcategory* if it admits a *weak reflection arrow*  $r_K$  for each  $K$  in  $\mathcal{K}$ —as above, but removing the requirement that the factorizations be unique.

We will also investigate various operations on classes of morphisms. We use the notation from [MRV14, 2.1].

**Definition 2.7.** Let  $\mathcal{K}$  be a category and let  $\mathcal{M}$  be a class of morphisms of  $\mathcal{K}$ .

- (1)  $\text{Po}(\mathcal{M})$  is the class of pushouts of morphisms in  $\mathcal{M}$ ; that is, a morphism  $g$  is in  $\text{Po}(\mathcal{M})$  if and only if it is an isomorphism or there exists  $f \in \mathcal{M}$  and a pushout square of the form:

$$\begin{array}{ccc} C & \xrightarrow{g} & D \\ \uparrow & & \uparrow \\ A & \xrightarrow{f} & B \end{array}$$

- (2)  $\text{Tc}(\mathcal{M})$  is the class of transfinite compositions of morphisms from  $\mathcal{M}$ ; that is,  $f \in \text{Tc}(\mathcal{M})$  if and only if there exists an ordinal  $\alpha$  and morphisms  $(f_{i,j})_{i \leq j \leq \alpha}$  with  $f_{i,i+1} \in \mathcal{M}$  for all  $i < \alpha$ ,  $\langle f_{i,j} : i < j \rangle$  a colimit cocone for all limit ordinals  $j \leq \alpha$ , and  $f = f_{0,\alpha}$ .
- (3)  $\text{Rt}(\mathcal{M})$  is the class of retracts of morphisms of  $\mathcal{M}$  (where retracts are taken in the arrow category  $\mathcal{K}^2$  of morphisms of  $\mathcal{K}$ ).
- (4)  $\text{cell}(\mathcal{M}) := \text{Tc}(\text{Po}(\mathcal{M}))$  is called the class of  $\mathcal{M}$ -cellular maps.
- (5)  $\text{cof}(\mathcal{M}) := \text{Rt}(\text{cell}(\mathcal{M}))$  is called the class of  $\mathcal{M}$ -cofibrations.
- (6) We say that  $\mathcal{M}$  is *cofibrantly generated* if there exists a *subset* (i.e. not a proper class)  $\mathcal{X}$  of  $\mathcal{M}$  such that  $\mathcal{M} \subseteq \text{cof}(\mathcal{X})$ . We say that  $\mathcal{M}$  is *cellularly generated* if there exists a subset  $\mathcal{X}$  of  $\mathcal{M}$  so that  $\mathcal{M} \subseteq \text{cell}(\mathcal{X})$ .
- (7) For  $\lambda$  a regular cardinal, we write  $\mathcal{K}_\lambda$  for the full subcategory of  $\mathcal{K}$  consisting of  $\lambda$ -presentable objects. We similarly denote by  $\mathcal{K}_\lambda^2$  the full subcategory

of  $\mathcal{K}^2$  consisting of morphisms with  $\lambda$ -presentable domains and codomains. We will also write, for example,  $\mathcal{M}_\lambda := \mathcal{M} \cap \mathcal{K}_\lambda^2$ .

**Remark 2.8.** Let  $\mathcal{K}$  be a category and let  $\mathcal{M}$  be a class of morphisms in  $\mathcal{K}$ . Then  $\text{Po}(\text{Po}(\mathcal{M})) = \text{Po}(\mathcal{M})$ ,  $\text{Tc}(\text{Tc}(\mathcal{M})) = \text{Tc}(\mathcal{M})$ ,  $\text{Rt}(\text{Rt}(\mathcal{M})) = \text{Rt}(\mathcal{M})$ ,  $\text{cell}(\text{cell}(\mathcal{M})) = \text{cell}(\mathcal{M})$ , and  $\text{cof}(\text{cof}(\mathcal{M})) = \text{cof}(\mathcal{M})$ . We will also use without comment that if  $\mathcal{M}$  is closed under transfinite compositions, then it is closed under arbitrary directed colimits (Iwamura’s lemma; see e.g. [AR94, 1.7]).

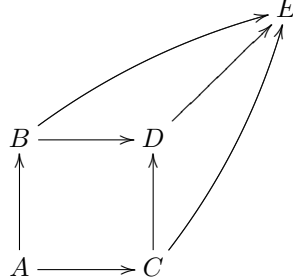
It will be very helpful later to be able to replace  $\text{cof}(\mathcal{X})$  by  $\text{cell}(\mathcal{X}^*)$ , for  $\mathcal{X}^*$  a possibly bigger set of maps. The following result is key:

**Fact 2.9** (Elimination of retracts; [MRV14, B.1]). If  $\lambda$  is a regular uncountable cardinal,  $\mathcal{K}$  is a locally  $\lambda$ -presentable category, and  $\mathcal{X}$  is a set of morphisms with  $\lambda$ -presentable domain and codomain, then  $\text{cof}(\mathcal{X}) = \text{cell}(\text{cof}(\mathcal{X}) \cap \mathcal{K}_\lambda^2)$ .

**2.1. Stable independence.** The categorical definition of a stable independence notion was introduced in [LRV19a]. We recall the main properties (and some additional ones) here.

**Definition 2.10.** Let  $\mathcal{K}$  be a category.

- (1) An *independence relation* (or *independence notion*) is a class  $\perp$  of commutative square (called  $\perp$ -*independent*, or just *independent*, squares) in  $\mathcal{K}$  such that, for any commutative diagram

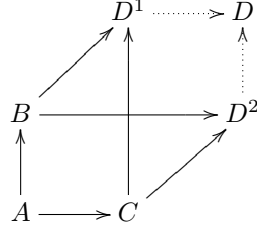


the square spanning  $A, B, C$ , and  $D$  is independent if and only if the square spanning  $A, B, C$ , and  $E$  is independent. We may label independent squares with the anchor symbol  $\perp$  (introduced by Makkai in [Mak84]).

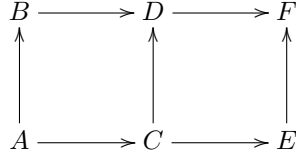
- (2) For any independence relation  $\perp$ , we can define a dual independence relation  $\overset{d}{\perp}$  which simply “swaps the ears”  $B$  and  $C$ , see [LRV19a, 3.9]. Any property of  $\perp$  has a dual, the corresponding property for  $\overset{d}{\perp}$ . Below, properties of  $\perp$  that are not self-dual will be prefixed by “right”. The corresponding left properties are then defined to be the right properties for  $\overset{d}{\perp}$ . If we drop the “left/right” prefix, this will mean both properties at once (this will usually be done in a symmetric setup).
- (3) The following are properties that an independence relation  $\perp$  might have:
  - (a) Symmetry: “swapping the ears”  $B$  and  $C$  preserves independence:
$$\perp = \overset{d}{\perp}.$$



- (b) Existence: any span can be amalgamated to an independent square.
- (c) Uniqueness: any two *independent* amalgam of the same span are equivalent: they can be amalgamated in the following way:

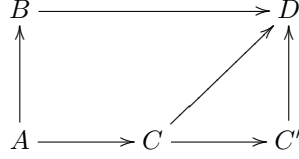


- (d) Right transitivity: Given a commutative diagram of the following form



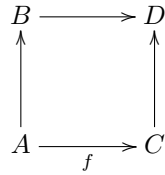
if both the left and right squares are independent, so is the outer rectangle. That is, independent squares are *closed under horizontal composition*.

- (e) Right monotonicity: consider a commutative diagram:



If the outer rectangle is independent, then so is the inner square spanning  $A, B, C$ , and  $D$ .

- (f) Right weak existence: whenever we have a commutative square



with  $f$  an isomorphism, then the square is independent.

- (4) An independence relation  $\perp$  is *weakly stable* if it is symmetric, has existence and uniqueness, and is transitive.
- (5) An independence relation  $\perp$  is *right basic* if it is right transitive and right monotonic, and has right weak existence.
- (6) Let  $\perp$  be a right transitive independence relation with right weak existence.
  - (a)  $\mathcal{K}_\perp$  is the subcategory of  $\mathcal{K}^2$  whose objects are arrows, and whose morphisms are  $\perp$ -independent squares (with composition on the right). Note that we have modified our notation from that of [LRV19a]:  $\mathcal{K}_\perp$  is precisely the category denoted by  $\mathcal{K}_{NF}$  there, e.g. in [LRV19a, 3.16].

- (b) For  $\lambda$  a regular cardinal, we say that  $\perp$  is  $\lambda$ -continuous if  $\mathcal{K}_\perp$  has  $\lambda$ -directed colimits. We say that  $\perp$  is  $\lambda$ -accessible if  $\mathcal{K}_\perp$  is  $\lambda$ -accessible. Finally, we say that  $\perp$  is accessible if it is  $\lambda$ -accessible for some  $\lambda$ .
- (7) An independence relation  $\perp$  is *stable* if it is weakly stable and accessible.

The following known implications between the properties will be used without comment.

**Fact 2.11.** Let  $\perp$  be an independence relation on a category  $\mathcal{K}$ .

- (1) [LRV19a, 3.12] If  $\perp$  has existence, then it has weak existence.
- (2) [LRV19a, 3.18] If  $\perp$  is right transitive and has right weak existence, then independence is invariant under isomorphisms of squares.
- (3) [LRV19a, 3.20] If  $\perp$  is right transitive and has existence and uniqueness, then  $\perp$  is right basic (hence, in particular, right monotonic).

**Fact 2.12.** Let  $\perp$  be a right basic independence relation on a category  $\mathcal{K}$ .

- (1) [LRV19a, 3.26] If  $\perp$  is  $\lambda$ -continuous, then  $\mathcal{K}$  has  $\lambda$ -directed colimits and the embedding of  $\mathcal{K}_\perp$  into  $\mathcal{K}^2$  preserves  $\lambda$ -directed colimits.
- (2) [LRV19a, 3.27] If  $\perp$  is  $\lambda$ -accessible, then  $\mathcal{K}$  is  $\lambda$ -accessible.

We emphasize that under mild conditions (say if the underlying category has directed colimits) stable independence is canonical: there can be at most one such relation. In fact, if there is a stable independence notion  $\perp$  then any *weakly stable* independence notion has to be  $\perp$ . A model-theoretic proof of this nontrivial fact for AECs appears in [BGKV16]. The proof was generalized in [LRV19a, 9.1] to obtain canonicity in any accessible category with directed colimits when all morphisms are monos. In the appendix to the present paper, we give a purely category-theoretic version of the proof (see Theorem A.6 there) which dispenses with the assumption that all morphisms are monos.

Finally, we note that the existence of a stable independence notion on an accessible category with all morphisms monos has strong consequences for the structure of the class. Per [BGL<sup>+</sup>16, 4.10], such categories are equivalent to  $\mu$ -abstract elementary classes (a useful generalization of AECs—which correspond to the case  $\mu = \aleph_0$ —the details of which are not essential here), existence implies the amalgamation property, and by [LRV19a, 8.16], any  $\mu$ -AEC with a stable independence relation will be stable and tame.

### 3. ON SUBCATEGORIES OF ACCESSIBLE CATEGORIES

The classes of morphisms that we will consider in the present paper will always satisfy the following property:

**Definition 3.1.** Let  $\mathcal{K}$  be a category. A class of morphisms  $\mathcal{M}$  in  $\mathcal{K}$  will be called *normal* if all isomorphisms are in  $\mathcal{M}$  and  $\mathcal{M}$  is closed under composition.

**Remark 3.2.** Note that any normal class of morphisms  $\mathcal{M}$  induces an isomorphism-closed subcategory  $\mathcal{K}_\mathcal{M}$  of  $\mathcal{K}$ , whose objects are those in  $\mathcal{K}$ , and whose morphisms are precisely those of  $\mathcal{M}$ . Conversely, any isomorphism-closed subcategory of  $\mathcal{K}$  with the same objects as  $\mathcal{K}$  induces a normal family of morphisms.

The following condition will play an important role:

**Definition 3.3.** A class  $\mathcal{M}$  of morphisms in a category  $\mathcal{K}$  is *coherent* if whenever  $f$  and  $g$  are composable morphisms,  $gf \in \mathcal{M}$  and  $g \in \mathcal{M}$ , then  $f \in \mathcal{M}$ . We say that  $\mathcal{M}$  is *left cancellable* if  $gf \in \mathcal{M}$  implies  $f \in \mathcal{M}$ .

**Remark 3.4.** If  $\mathcal{M}$  is a normal and coherent class of morphisms in a category  $\mathcal{K}$  and an object  $A$  of  $\mathcal{K}$  is  $\lambda$ -presentable in  $\mathcal{K}$ , then  $A$  is  $\lambda$ -presentable in  $\mathcal{K}_{\mathcal{M}}$ .

In this section, we investigate conditions under which  $\mathcal{K}_{\mathcal{M}}$  is accessible, or even a  $\mu$ -abstract elementary class (see [BGL<sup>+</sup>16]). The following closure properties will of course be key:

**Definition 3.5** ([AR94, 2.35]). For  $\lambda$  a regular cardinal, a full subcategory  $\mathcal{L}$  of  $\mathcal{K}$  is called  *$\lambda$ -accessibly embedded* if it is full and closed under  $\lambda$ -directed colimits in  $\mathcal{K}$ . We say that  $\mathcal{L}$  is *accessibly embedded* if it is  $\lambda$ -accessibly embedded for some regular  $\lambda$ .

**Definition 3.6.** Let  $\mathcal{K}$  be a category, let  $\mathcal{M}$  be a normal class of morphisms, and let  $\lambda$  be a regular cardinal.

- (1) We say that  $\mathcal{M}$  is  *$\lambda$ -continuous* if  $\mathcal{K}$  has  $\lambda$ -directed colimits and  $\mathcal{K}_{\mathcal{M}}$  is closed under  $\lambda$ -directed colimits in  $\mathcal{K}$  (see Definition 2.3).
- (2) We say that  $\mathcal{M}$  is  *$\lambda$ -accessible* if  $\mathcal{M}$  is  $\lambda$ -continuous and both  $\mathcal{K}$  and  $\mathcal{K}_{\mathcal{M}}$  are  $\lambda$ -accessible. We say that  $\mathcal{M}$  is *accessible* if it is  $\lambda$ -accessible for some  $\lambda$ .

**Remark 3.7.** If  $\mathcal{K}$  is a category, a right basic independence relation  $\perp$  on  $\mathcal{K}$  is  $\lambda$ -continuous (in the sense of Definition 2.10(6b)) if and only if the morphisms of  $\mathcal{K}_{\perp}$  are  $\lambda$ -continuous (in the sense of the previous definition) in the arrow category  $\mathcal{K}^2$ . This follows from Fact 2.12. Similarly,  $\perp$  is  $\lambda$ -accessible if and only if the morphisms of  $\mathcal{K}_{\perp}$  are  $\lambda$ -accessible in  $\mathcal{K}^2$ .

The notion of a  $\lambda$ -pure morphism will be useful, since—in reasonable cases—a class of morphisms containing the  $\lambda$ -pure ones for some  $\lambda$  will be accessible (and all our examples will satisfy this condition). We recall the definition and a few facts.

**Definition 3.8.** For  $\lambda$  a regular cardinal, a morphism  $A \xrightarrow{f} B$  in a category  $\mathcal{K}$  is  *$\lambda$ -pure* if whenever we are given  $A_0 \xrightarrow{f_0} B_0 \xrightarrow{h} B$  and  $A_0 \xrightarrow{g} A$  so that  $fg = hf_0$  and both  $A_0$  and  $B_0$  are  $\lambda$ -presentable, then there exists  $B_0 \xrightarrow{d} A$  so that  $df_0 = g$ . When  $\lambda = \aleph_0$ , we omit it and simply say that  $f$  is *pure*.

**Fact 3.9.**

- (1) When  $\lambda_1 \leq \lambda_2$ , any  $\lambda_2$ -pure morphism is  $\lambda_1$ -pure.
- (2)  $\lambda$ -pure morphisms form a normal and left cancellable class. In particular, any split monomorphism is  $\lambda$ -pure.
- (3) In a  $\lambda$ -accessible category, any  $\lambda$ -pure morphism is a monomorphism.
- (4) If  $\mathcal{K}$  is an accessible category with  $\lambda$ -directed colimits, then the class of  $\lambda$ -pure morphisms is accessible and  $\lambda$ -continuous.
- (5) If  $\mathcal{K}$  is an accessible category with  $\lambda$ -directed colimits and  $\mathcal{M}$  is a normal and  $\lambda$ -continuous class of morphisms that contains all  $\mu$ -pure morphisms for some  $\mu$ , then  $\mathcal{M}$  is accessible.

*Proof.* The first two parts are immediate, the third is [AR94, 2.29], and the fourth is [AR94, 2.34]. The fifth part can be derived as follows:  $\mathcal{K}_{\mathcal{M}}$  will have  $\lambda$ -directed colimits by  $\lambda$ -continuity, and given  $A$  an object of  $\mathcal{K}_{\mathcal{M}}$ , there exists a regular cardinal  $\mu$  such that  $A$  can be written as a directed colimit of  $\mu$ -presentable objects in  $\mathcal{K}$ , where the morphisms are all  $\mu$ -pure. By Remark 3.4, the objects will be  $\mu$ -presentable also in  $\mathcal{K}_{\mathcal{M}}$ .  $\square$

If  $\mathcal{M}$  is accessible, it is natural to ask when  $\mathcal{K}_{\mathcal{M}}$  will be (equivalent to) a  $\lambda$ -AEC. This is one place where coherence comes in. The following result is essentially due to Beke and the second author [BR12, 5.7]. It will mostly be used when  $\lambda = \aleph_0$  (in which case  $\mathcal{K}_{\mathcal{M}}$  will be an AEC).

**Fact 3.10.** Let  $\mathcal{K}$  be a category and let  $\mathcal{M}$  be a normal,  $\lambda$ -continuous, and accessible class of morphisms of  $\mathcal{K}$ . If:

- (1)  $\mathcal{K}$  is a  $\lambda$ -accessibly embedded subcategory of a  $\lambda$ -accessible category.
- (2) All morphisms in  $\mathcal{M}$  are monos.
- (3)  $\mathcal{M}$  is coherent.

Then  $\mathcal{K}_{\mathcal{M}}$  is equivalent to a  $\lambda$ -AEC.

*Proof.* Suppose  $\mathcal{K}$  is a  $\lambda$ -accessibly embedded subcategory of a  $\lambda$ -accessible category  $\mathcal{K}^*$ . Then the embedding of  $\mathcal{K}_{\mathcal{M}}$  into  $\mathcal{K}^*$  satisfies the conditions in [BR12, 5.7] (this is stated there for  $\lambda = \aleph_0$ , but the proof generalizes — see [LRV19a, 7.2]).  $\square$

**Remark 3.11.** The proof shows that if  $\mathcal{K}$  is a  $\lambda$ -accessible and  $\lambda$ -accessibly embedded subcategory of a  $\lambda$ -accessible category  $\mathcal{K}^*$  and  $\mathcal{K}^*$  is itself a category of structures (that is, a subcategory of  $\mathbf{Emb}(\tau)$  for some  $\lambda$ -ary vocabulary  $\tau$ , where  $\mathbf{Emb}(\tau)$  denotes the category of  $\tau$ -structures with injective homomorphisms), then  $\mathcal{K}_{\mathcal{M}}$  will be an actual  $\lambda$ -AEC (i.e. there is no need to take an equivalence). See also [LRV19a, 7.2].

#### 4. INDEPENDENCE FROM PUSHOUTS

In this section, we start with a normal class  $\mathcal{M}$  of morphisms and try to build a stable independence notion in  $\mathcal{K}_{\mathcal{M}}$ . We will assume that  $\mathcal{M}$  is coherent and closed under pushouts. A normal class of morphisms closed under pullbacks is called a clan in [Joy], so we dualize<sup>4</sup> this terminology here:

**Definition 4.1.** Let  $\mathcal{K}$  be a category.

- (1) A class  $\mathcal{M}$  of morphisms is a *coclan* if it is normal (Definition 3.1), the pushout of two morphisms with at least one in  $\mathcal{M}$  always exists, and  $\mathcal{M}$  is closed under pushouts:  $\text{Po}(\mathcal{M}) = \mathcal{M}$  (see Definition 2.7).
- (2) If  $\mathcal{M}$  is a coherent coclan that is closed under retracts (i.e.  $\text{Rt}(\mathcal{M}) = \mathcal{M}$ , see Definition 2.7), we call it a *nice class of morphisms*. We say it is *almost nice* if it is a coherent coclan, but not necessarily closed under retracts.

<sup>4</sup>In fact, this is a slight abuse of terminology: Joyal’s definition of clan assumes that all objects are fibrant with respect to the chosen class of morphisms. The dual of his notion would be, in our terminology, a *cofibrant coclan*.

**Remark 4.2.** Closure under retracts holds under very reasonable conditions: if  $\mathcal{M}$  is a normal and coherent class of morphisms containing all split monos, then a diagram chase shows that it is closed under retracts. In particular, normal left cancellable classes of morphisms, as well as normal coherent classes of morphisms containing all  $\mu$ -pure morphisms for some  $\mu$  (see Fact 3.9), are closed under retracts.

Given a family  $\mathcal{M}$ , the  $\mathcal{M}$ -effective squares, defined below, will be our candidate definition of independence: the special case where  $\mathcal{M}$  is the class of regular monomorphisms was investigated in [LRV19a, §5], but here we take a much broader view. When  $\mathcal{M}$  is almost nice, we can prove that  $\mathcal{M}$ -effective squares will be weakly stable (recall that this means all the axioms from the definition of stable independence are satisfied except perhaps accessibility, see Definition 2.10(4)).

**Definition 4.3.** Let  $\mathcal{M}$  be a class of morphisms in a category  $\mathcal{K}$ . An  $\mathcal{M}$ -effective square is a (commutative) square:

$$\begin{array}{ccc} B & \xrightarrow{h} & D \\ f \uparrow & & \uparrow k \\ A & \xrightarrow{g} & C \end{array}$$

where all the morphisms are in  $\mathcal{M}$ , the pushout  $P$  of  $f$  and  $g$  exists, and the induced morphism  $P \rightarrow D$  is in  $\mathcal{M}$ . That is, all the morphisms in the diagram below are in  $\mathcal{M}$ :

$$\begin{array}{ccccc} & & & & D \\ & & & \nearrow h & \\ B & \xrightarrow{\quad} & P & \xrightarrow{\quad} & D \\ f \uparrow & & \uparrow \lrcorner & & \uparrow k \\ A & \xrightarrow{g} & C & \searrow & \\ & & & & \end{array}$$

**Theorem 4.4.** If  $\mathcal{M}$  is an almost nice class of morphisms in a category  $\mathcal{K}$ , then  $\mathcal{M}$ -effective squares form a weakly stable independence relation in  $\mathcal{K}_{\mathcal{M}}$ .

*Proof.* We first check that  $\mathcal{M}$ -effective squares (from now on referred to only as effective squares) form an independence notion. Assume that  $(A, B, C, D)$  is a commutative square<sup>5</sup> in  $\mathcal{K}_{\mathcal{M}}$  and we are given a morphism  $D \rightarrow E$  in  $\mathcal{M}$ . If  $(A, B, C, D)$  is effective, then closure of  $\mathcal{M}$  under composition yields that  $(A, B, C, E)$  is effective. Conversely, if  $(A, B, C, E)$  is effective, then the map  $P \rightarrow E$  from the pushout is in  $\mathcal{M}$  by assumption, and also  $D \rightarrow E$  is in  $\mathcal{M}$ , so by coherence also the map  $P \rightarrow D$  is in  $\mathcal{M}$ . Thus  $(A, B, C, D)$  is effective.

<sup>5</sup>We occasionally economize by not explicitly naming the morphisms involved, when there is no danger of confusion.

This concludes the proof that effective squares form an independence notion. Of course, the relation is also symmetric. Existence follows from closure under pushouts (and the fact that the identity map is an isomorphism, hence in  $\mathcal{M}$ ). In order to prove the uniqueness property, consider effective squares  $(A, B, C, D^1)$  and  $(A, B, C, D^2)$  with the same span  $B \leftarrow A \rightarrow C$ . Form the pushout

$$\begin{array}{ccc} B & \longrightarrow & P \\ \uparrow & & \uparrow \\ A & \longrightarrow & C \end{array}$$

and take the induced morphisms  $P \rightarrow D^1$  and  $P \rightarrow D^2$ . They are in  $\mathcal{M}$  by effectiveness. Then the pushout

$$\begin{array}{ccc} D^1 & \longrightarrow & D \\ \uparrow & & \uparrow \\ P & \longrightarrow & D^2 \end{array}$$

amalgamates the starting diagram.

To prove transitivity, consider:

$$\begin{array}{ccccc} B & \longrightarrow & D & \longrightarrow & F \\ \uparrow & & \uparrow & & \uparrow \\ f \uparrow & & & & \\ A & \xrightarrow{g} & C & \xrightarrow{g'} & E \end{array}$$

where both squares are effective. We have to show that the outer rectangle is effective. Thus we have to show that the induced morphism  $p : P \rightarrow F$  from the pushout

$$\begin{array}{ccc} B & \longrightarrow & P \\ \uparrow & & \uparrow \\ f \uparrow & & \lrcorner \\ A & \xrightarrow{g'g} & E \end{array}$$

is in  $\mathcal{M}$ . This pushout is a composition of pushouts

$$\begin{array}{ccccc} B & \xrightarrow{u} & Q & \xrightarrow{u'} & P \\ \uparrow & & \uparrow & & \uparrow \\ f \uparrow & & v \uparrow & & v' \uparrow \\ A & \xrightarrow{g} & C & \xrightarrow{g'} & E \end{array}$$

Recalling the left square of the starting diagram, we have an induced morphism  $q : Q \rightarrow D$ . Consider the pushout

$$\begin{array}{ccc} D & \longrightarrow & P' \\ \uparrow q & & \uparrow \bar{q} \\ Q & \xrightarrow{u'} & P \end{array}$$

Since the left square of the starting diagram is effective,  $q$  is in  $\mathcal{M}$  and thus  $\bar{q}$  is in  $\mathcal{M}$ . Composing this pushout with the right pushout square in the diagram above it, we obtain the pushout

$$\begin{array}{ccc} D & \longrightarrow & P' \\ \uparrow & & \uparrow \\ C & \longrightarrow & E \end{array}$$

The right square in the starting diagram is effective, so the induced morphism  $p' : P' \rightarrow F$  is in  $\mathcal{M}$ . Thus  $p = p'\bar{q}$  is in  $\mathcal{M}$ . □

**Remark 4.5.** Coherence was only used in the proof that  $\mathcal{M}$ -effective squares formed an independence notion (specifically, in the proof that the top right corner can be made “smaller”). Instead of coherence, we could also have assumed the dual property, cocomherence: indeed we know in the proof that the maps  $C \rightarrow D$  and  $C \rightarrow P$  are in  $\mathcal{M}$ , so cocomherence would give us immediately that  $P \rightarrow D$  is in  $\mathcal{M}$ . Note however that if  $\mathcal{M}$  is a class of monomorphisms, cocomherence is too strong an assumption: if a section  $i : A \rightarrow B$  is in  $\mathcal{M}$ , cocomherence would imply that the corresponding retract  $r : B \rightarrow A$  is in  $\mathcal{M}$ , and so  $r$  would have to be an isomorphism.

We now investigate when effective squares will be accessible. First, some notation:

**Definition 4.6.** For  $\mathcal{K}$  a category and  $\mathcal{M}$  an almost nice class of morphisms in  $\mathcal{K}$ , we write  $\mathcal{K}_{\mathcal{M}, \downarrow}$  for  $(\mathcal{K}_{\mathcal{M}})_{\downarrow}$ , the category whose objects are the morphisms in  $\mathcal{M}$  and whose arrows are the  $\mathcal{M}$ -effective squares.

**Remark 4.7.** By Fact 2.11 and Theorem 4.4,  $\mathcal{K}_{\mathcal{M}, \downarrow}$  is isomorphism-closed in  $(\mathcal{K}_{\mathcal{M}})^2$ .

Continuity of effective squares follows immediately from continuity of the corresponding class of morphisms:

**Lemma 4.8.** If  $\mathcal{M}$  is an almost nice and  $\lambda$ -continuous class of morphisms, then the independence relation given by  $\mathcal{M}$ -effective squares is  $\lambda$ -continuous.

*Proof.* Let  $\mathcal{M}$  be an almost nice  $\lambda$ -continuous class of morphisms on a category  $\mathcal{K}$ . Let  $D : I \rightarrow \mathcal{K}_{\mathcal{M}, \downarrow}$  be a  $\lambda$ -directed diagram where  $Di$  is  $f_i : A_i \rightarrow B_i$ . Let  $g : A \rightarrow B$  be a colimit of  $D$  in  $(\mathcal{K}_{\mathcal{M}})^2$ . For each  $i \in I$ , the pushout of the colimit

coprojection  $A_i \rightarrow A$  along  $f_i$ , i.e.

$$\begin{array}{ccc} A & \xrightarrow{g} & P \\ \uparrow & & \uparrow \\ A_i & \xrightarrow{f_i} & B_i \end{array}$$

is a  $\lambda$ -directed colimit of pushouts

$$\begin{array}{ccc} A_{i'} & \xrightarrow{g_{i'}} & P_{i'} \\ \uparrow & & \uparrow \\ A_i & \xrightarrow{f_i} & B_i \end{array}$$

Thus the induced morphism  $p : P \rightarrow B$  is a  $\lambda$ -directed colimit of induced morphisms  $p_{i'} : P_{i'} \rightarrow B_{i'}$ . Since  $\mathcal{M}$  is  $\lambda$ -continuous, it follows that  $p \in \mathcal{M}$ . This shows that all the maps of the cocone  $(f_i \rightarrow g)_{i \in I}$  are independent squares. Similarly, one can check that this is a colimit cocone in  $\mathcal{K}_{\mathcal{M}, \downarrow}$ . Thus  $\mathcal{K}_{\mathcal{M}, \downarrow}$  is closed under  $\lambda$ -directed colimits in  $(\mathcal{K}_{\mathcal{M}})^2$ . By Remark 3.7 this gives the result.  $\square$

The next result, the main theorem of this paper, characterizes when effective squares form a stable independence notion in terms of cofibrant generation of the corresponding class of morphisms. We assume closure under retracts and work in a locally presentable category, but these hypotheses can be removed if we happen to know that the class of morphisms can be generated without retracts, see Remark 4.10.

To go from stable independence to cofibrant generation, we require a technical result from [LRV, §9] concerning the existence of *filtrations*. Recall that the *presentability rank* of an object  $A$  is the least regular cardinal  $\lambda$  such that  $A$  is  $\lambda$ -presentable. We say that  $A$  is *filtrable* if it can be written as the directed colimit of a chain of objects with lower presentability rank than  $A$ . We say that  $A$  is *almost filtrable* if it is a retract of such a chain. The chain is *smooth* if directed colimits are taken at every limit ordinal. By [LRV, 9.12], in any accessible category with directed colimits, there exists a regular cardinal  $\lambda$  such that any object with presentability rank at least  $\lambda$  is almost filtrable (and the chain in the filtration can be chosen to be smooth). We say that the category is *almost well  $\lambda$ -filtrable*.

**Theorem 4.9** (Main theorem). Let  $\mathcal{K}$  be a locally presentable category and let  $\mathcal{M}$  be a nice and  $\aleph_0$ -continuous class of morphisms in  $\mathcal{K}$ . The following are equivalent:

- (1)  $\mathcal{K}_{\mathcal{M}}$  has a stable independence notion.
- (2)  $\mathcal{M}$ -effective squares form a stable independence notion in  $\mathcal{K}_{\mathcal{M}}$
- (3)  $\mathcal{M}$  is accessible and cofibrantly generated in  $\mathcal{K}$ .

*Proof.*



- (1) implies (2): If  $\mathcal{K}_{\mathcal{M}}$  has a stable independence notion, then canonicity (Theorem A.6 – note that  $\mathcal{K}$  has directed colimits, since  $\mathcal{M}$  is  $\aleph_0$ -continuous) together with Theorem 4.4 ensures that it is given by  $\mathcal{M}$ -effective squares.
- (2) implies (3): Assume that  $\mathcal{K}_{\mathcal{M}}$  has a stable independence  $\perp$  given by  $\mathcal{M}$ -effective squares. Thus  $\mathcal{K}_{\mathcal{M},\downarrow}$  is accessible and has directed colimits (by Lemma 4.8). By Fact 2.12,  $\mathcal{K}_{\mathcal{M}}$  is accessible, so  $\mathcal{M}$  is accessible. Using the preceding discussion, pick a regular uncountable cardinal  $\lambda$  such both  $\mathcal{K}$  and  $\mathcal{K}_{\mathcal{M},\downarrow}$  are  $\lambda$ -accessible and almost well  $\lambda$ -filtrable. Following Definition 2.7, let  $\mathcal{M}_{\lambda}$  be the collection of morphisms in  $\mathcal{M}$  whose domains and codomains are  $\lambda$ -presentable (in  $\mathcal{K}$ ). We will show that for each infinite cardinal  $\mu$ ,  $\mathcal{M}_{\mu^+} \subseteq \text{cof}(\mathcal{M}_{\lambda})$ . We proceed by induction on  $\mu$ . When  $\mu < \lambda$ , this is trivial, so assume that  $\mu \geq \lambda$ . Note that, playing with pushouts, it is straightforward to check that the  $\mu^+$ -presentable objects in  $\mathcal{K}_{\mathcal{M},\downarrow}$  are exactly the morphisms of  $\mathcal{M}_{\mu^+}$ .

Every morphism  $h$  in  $\mathcal{M}_{\mu^+}$  must be a retract of a filtrable object in  $\mathcal{K}_{\mathcal{M},\downarrow}$ . Now, retracts in  $\mathcal{K}_{\mathcal{M},\downarrow}$  are retracts in  $\mathcal{K}^2$ , so since we are looking at  $\text{cof}(\mathcal{M}_{\lambda})$  it suffices to show that any morphism  $h$  in  $\mathcal{M}_{\mu^+}$  which *is* filtrable in  $\mathcal{K}_{\mathcal{M},\downarrow}$  is in  $\text{cof}(\mathcal{M}_{\lambda})$ . So take such a morphism. Write  $h = h_0 : K_0 \rightarrow L$ . We will show that  $h_0 \in \text{cof}(\mathcal{M}_{\lambda})$ . Express  $h_0$  as a colimit of a smooth chain of morphisms  $t_{0i} \in \text{cof}(\mathcal{M}_{\lambda})$ ,  $i < \text{cf}(\mu)$ , between  $(< \mu^+)$ -presentable objects in  $\mathcal{K}_{\mathcal{M},\downarrow}$ .

$$\begin{array}{ccc} K_0 & \xrightarrow{h_0} & L \\ k_{0i} \uparrow & & \uparrow l_{0i} \\ K_{0i} & \xrightarrow{t_{0i}} & L_{0i} \end{array}$$

Form a pushout

$$\begin{array}{ccc} K_0 & \xrightarrow{h_{01}} & K_1 \\ k_{00} \uparrow & & \uparrow \bar{k}_{00} \\ K_{00} & \xrightarrow{t_{00}} & L_{00} \end{array}$$

and take the induced morphism  $h_1 : K_1 \rightarrow L$ . Since the starting square is effective,  $h_1$  is in  $\mathcal{M}$ . Note also that  $K_1$  is  $\mu^+$ -presentable. We have a commutative square

$$\begin{array}{ccc} K_1 & \xrightarrow{h_1} & L \\ h_{01}k_{01} \uparrow & & \uparrow l_{01} \\ K_{01} & \xrightarrow{t_{01}} & L_{01} \end{array}$$

because  $h_1 h_{01} k_{01} = h_0 k_{01} = l_{01} t_{01}$ . We can express  $h_1$  as a colimit of a smooth chain of morphisms  $t_{1i} \in \text{cof}(\mathcal{M}_\lambda)$ ,  $1 \leq i < \text{cf}(\mu)$ , between  $< \mu^+$ -presentable objects in  $\mathcal{K}_{\mathcal{M}, \downarrow}$  which are above  $t_{01}$

$$\begin{array}{ccc}
 K_1 & \xrightarrow{h_1} & L \\
 \uparrow k_{1i} & & \uparrow l_{1i} \\
 K_{1i} & \xrightarrow{t_{1i}} & L_{1i} \\
 \uparrow & & \uparrow \\
 K_{01} & \xrightarrow{t_{01}} & L_{01}
 \end{array}$$

Form a pushout

$$\begin{array}{ccc}
 K_1 & \xrightarrow{h_{12}} & K_2 \\
 \uparrow k_{11} & & \uparrow \bar{k}_{11} \\
 K_{11} & \xrightarrow{t_{11}} & L_{11}
 \end{array}$$

and take the induced morphisms  $h_2 : K_2 \rightarrow L$ . Again, by effectivity,  $h_2$  is in  $\mathcal{M}$ . In

$$K_0 \xrightarrow{h_{01}} K_1 \xrightarrow{h_{12}} K_2 \xrightarrow{h_2} L$$

we put  $h_{02} = h_{12} h_{01}$  and continue transfinitely. This means that for  $i < \text{cf}(\mu)$  we express  $h_i$  as a colimit of a smooth chain of morphisms  $t_{ij} \in \text{cof}(\mathcal{M}_\lambda)$ ,  $i \leq j < \text{cf}(\mu)$ , between  $(< \mu^+)$ -presentable objects in  $\mathcal{K}_{\mathcal{M}, \downarrow}$  which are above  $t_{0i}$

$$\begin{array}{ccc}
 K_i & \xrightarrow{h_i} & L \\
 \uparrow k_{i+1,j} & & \uparrow l_{i+1,j} \\
 K_{i+1,j} & \xrightarrow{t_{i+1,j}} & L_{i+1,j} \\
 \uparrow & & \uparrow \\
 K_{0i} & \xrightarrow{t_{0i}} & L_{0i}
 \end{array}$$

Form a pushout

$$\begin{array}{ccc}
 K_i & \xrightarrow{h_{i,i+1}} & K_{i+1} \\
 \uparrow k_{i+1,i+1} & & \uparrow \bar{k}_{i+1,i+1} \\
 K_{i+1,i+1} & \xrightarrow{t_{i+1,i+1}} & L_{i,i+1}
 \end{array}$$

and take the induced morphisms  $h_{i+1} : K_{i+1} \rightarrow L$ . By effectivity,  $h_{i+1}$  is in  $\mathcal{M}$ . We put  $h_{k,i+1} = h_{i,i+1}h_{ik}$ . At limit steps we take colimits. Then by construction  $L = K_{\text{cf}(\mu)}$  and  $h_0$  is the transfinite composition of  $(h_{ij})_{i < j < \text{cf}(\mu)}$ . We have just observed that each  $h_{ij}$  is in  $\text{cof}(\mathcal{M}_\lambda)$ , so  $h_0$  also is.

- (3) implies (1): Assume that  $\mathcal{M}$  is accessible and cofibrantly generated in  $\mathcal{K}$ . Let  $\mathcal{X}$  be a subset of  $\mathcal{M}$  so that  $\mathcal{M} = \text{cof}(\mathcal{X})$ . Let  $\lambda$  be a big-enough uncountable regular cardinal such that  $\mathcal{K}$  and  $\mathcal{K}_\mathcal{M}$  are  $\lambda$ -accessible, and all the morphisms in  $\mathcal{X}$  have  $\lambda$ -presentable domain and codomain. We claim that  $\mathcal{K}_{\mathcal{M},\downarrow}$  is  $\lambda$ -accessible. First,  $\mathcal{K}_{\mathcal{M},\downarrow}$  is closed under directed colimits in  $\mathcal{K}_\mathcal{M}$  by Lemma 4.8. Now let  $\mathcal{M}_\lambda$  be the class of morphisms in  $\mathcal{M}$  with  $\lambda$ -presentable domain and codomain and let  $\mathcal{M}^*$  be the class of morphisms in  $\mathcal{M}$  that are  $\lambda$ -directed colimit (in  $\mathcal{K}_{\mathcal{M},\downarrow}$ ) of morphisms in  $\mathcal{M}_\lambda$ . It suffices to see that  $\mathcal{M}^* = \mathcal{M}$ .
  - First, any pushout of a morphism in  $\mathcal{M}_\lambda$  is in  $\mathcal{M}^*$ . Consider such a pushout

$$\begin{array}{ccc} K & \xrightarrow{h} & L \\ \uparrow k_0 & & \uparrow l_0 \\ K_0 & \xrightarrow{h_0} & L_0 \end{array}$$

where  $K_0$  and  $L_0$  are  $\lambda$ -presentable. Then  $K$  is a  $\lambda$ -directed colimits of  $\lambda$ -presentable objects  $K_i$  above  $K_0$  in  $\mathcal{K}_\mathcal{M}$ . Consider pushouts

$$\begin{array}{ccc} K & \xrightarrow{h} & L \\ \uparrow & & \uparrow \\ K_i & \xrightarrow{h_i} & L_i \\ \uparrow & & \uparrow \\ K_0 & \xrightarrow{h_0} & L_0 \end{array}$$

It is easy to check that the  $L_i$ 's are also  $\lambda$ -presentable and that  $h = \text{colim } h_i$  in  $\mathcal{K}_{\mathcal{M},\downarrow}$ . Thus  $h \in \mathcal{M}^*$ .

- Second,  $\mathcal{M}^*$  is closed under compositions of morphisms from  $\text{Po}_\lambda$  where  $\text{Po}_\lambda$  consists of pushouts of morphisms from  $\mathcal{M}_\lambda$ . Let  $f : K \rightarrow L$  and  $g : L \rightarrow M$  belong to  $\text{Po}_\lambda$ . As above,  $f$  is a  $\lambda$ -directed colimit (in  $\mathcal{K}_{\mathcal{M},\downarrow}$ ),  $(k_i, l_i) : f_i \rightarrow f$  of  $f_i \in \mathcal{M}_\lambda$ ,  $f_i : K_i \rightarrow L_i$ . Moreover,  $g$  is a pushout of  $g_0 : L_0 \rightarrow M_0$  having  $L_0$  and  $M_0$  both  $\lambda$ -presentable. Without loss of generality, we can assume that  $L_0 \rightarrow L$  factors through

the  $L_i$ . We then take pushouts as above

$$\begin{array}{ccc}
 L & \xrightarrow{g} & M \\
 \uparrow & & \uparrow \\
 L_i & \xrightarrow{g_i} & M_i \\
 \uparrow & & \uparrow \\
 L_0 & \xrightarrow{g_0} & M_0
 \end{array}$$

This shows that  $gf$  is a  $\lambda$ -directed colimit of the  $g_i f_i$ 's in  $\mathcal{K}_{\mathcal{M}, \downarrow}$ .

- Third,  $\mathcal{M}^*$  is closed under transfinite compositions of morphisms from  $\text{Po}_\lambda$ . Let  $(f_{ij})_{i,j \leq \alpha}$  be such a transfinite composition. At limit steps,  $f_{0i}$  is the following directed colimit in  $\mathcal{K}_{\mathcal{M}, \downarrow}$ :

$$\begin{array}{ccc}
 K_0 & \xrightarrow{f_{0i}} & K_i \\
 \uparrow \text{id} & & \uparrow f_{ji} \\
 K_0 & \xrightarrow{f_{0j}} & K_j
 \end{array}$$

This shows that  $f_{0,i}$  is in  $\mathcal{M}^*$  (we used that  $\downarrow$  has weak existence, see Fact 2.11).

We have shown that any transfinite composition of pushouts from  $\mathcal{M}_\lambda$  is in  $\mathcal{M}^*$ . That is,  $\text{cell}(\mathcal{M}_\lambda) = \text{Tc}(\text{Po}(\mathcal{M}_\lambda)) \subseteq \mathcal{M}^*$ . Since  $\mathcal{M}$  is closed under pushouts, retracts, and transfinite compositions,  $\text{cof}(\mathcal{X}) \cap \mathcal{K}_\lambda^2 \subseteq \mathcal{M}_\lambda$ . By Fact 2.9, it follows that  $\mathcal{M} = \text{cof}(\mathcal{X}) = \text{cell}(\mathcal{M}_\lambda)$ . We deduce that  $\mathcal{M} = \mathcal{M}^*$ , as desired.

□

**Remark 4.10.** Only the implication (3) implies (1) uses that  $\mathcal{K}$  is locally presentable (to eliminate retracts by appealing to Fact 2.9). Otherwise if we know that  $\mathcal{M}$  is cellularly generated, it is enough to assume that  $\mathcal{K}$  is accessible. Similarly, the fact that  $\mathcal{M}$  is nice rather than almost nice (i.e. closed under retracts) is only used to prove (3) implies (1).

Often, it is natural to look not at all objects, but just those objects  $A$  so that  $0 \rightarrow A$  is in  $\mathcal{M}$ :

**Definition 4.11.** Let  $\mathcal{K}$  be a category and  $\mathcal{M}$  a class of morphisms in  $\mathcal{K}$ .

- (1) If  $\mathcal{K}$  has a terminal object  $1$ , let  $\mathcal{F}(\mathcal{M})$  denote the full subcategory of  $\mathcal{K}$  determined by the objects  $B$  so that the map  $B \rightarrow 1$  is in  $\mathcal{M}$ . These objects are often called the  *$\mathcal{M}$ -fibrant objects*.
- (2) Dually, if  $\mathcal{K}$  has an initial object  $0$ , let  $\mathcal{C}(\mathcal{M})$  denote the full subcategory of  $\mathcal{K}$  determined by the objects  $A$  so that the map  $0 \rightarrow A$  is in  $\mathcal{M}$ . These objects are often called the  *$\mathcal{M}$ -cofibrant objects*.

**Notation 4.12.** For the sake of brevity, given a class of morphisms  $\mathcal{M}$ , we denote by  $\mathcal{M}_0$  the class of morphisms in  $\mathcal{M}$  whose domains and codomains are  $\mathcal{M}$ -cofibrant; that is,

$$\mathcal{M}_0 = \mathcal{M} \cap \text{Mor}(\mathcal{C}(\mathcal{M}))$$

**Remark 4.13.**

- (1) If  $\mathcal{M}$  is a coherent class of morphisms in a category  $\mathcal{K}$  with an initial object, then  $\mathcal{C}(\mathcal{M})$  is closed under “ $\mathcal{M}$ -subobjects”: if  $A \rightarrow B$  is in  $\mathcal{M}$  and  $B \in \mathcal{C}(\mathcal{M})$ , then coherence implies that  $A \in \mathcal{C}(\mathcal{M})$ . In particular:
  - (a) If  $\mathcal{M}$  is normal and accessible in  $\mathcal{K}$ , then  $\mathcal{M}_0$  is normal and accessible in  $\mathcal{C}(\mathcal{M})$ .
  - (b) If  $\mathcal{K}_{\mathcal{M}}$  has a stable independence notion, then its restriction to  $\mathcal{C}(\mathcal{M})_{\mathcal{M}_0}$  is a stable independence notion.
- (2) If  $\mathcal{M}$  is [almost] nice in  $\mathcal{K}$ , then  $\mathcal{M}_0$  is [almost] nice in  $\mathcal{C}(\mathcal{M})$ .
- (3) If  $\mathcal{K}$  is a locally presentable category,  $\mathcal{M}$  is nice,  $\aleph_0$ -continuous, and the class of cofibrant maps  $0 \rightarrow A$  in  $\mathcal{M}$  is cofibrantly generated by some subset of  $\mathcal{M}$ , then by [MRV14, 5.2],  $\mathcal{C}(\mathcal{M})_{\mathcal{M}_0}$  is accessible.

We have the following version of Theorem 4.9 for cofibrant objects:

**Theorem 4.14.** Let  $\mathcal{K}$  be a locally presentable category and let  $\mathcal{M}$  be a nice and  $\aleph_0$ -continuous class of morphisms in  $\mathcal{K}$ . The following are equivalent:

- (1)  $\mathcal{C}(\mathcal{M})_{\mathcal{M}_0}$  has a stable independence notion.
- (2)  $\mathcal{M}_0$ -effective squares form a stable independence notion in  $\mathcal{C}(\mathcal{M})_{\mathcal{M}_0}$ .
- (3)  $\mathcal{M}_0$  is cofibrantly generated in  $\mathcal{C}(\mathcal{M})$ .

*Proof.* Similar to the proof of Theorem 4.9, using Remark 4.13.  $\square$

**$\mathcal{M}$ -effective unions.** We note that in many cases, the  $\mathcal{M}$ -effective squares will be pullback squares:

**Fact 4.15** ([Rin72], [AHS04, 11.15]). Let  $\mathcal{M}$  be an almost nice class of monomorphisms. If:

- (1) A pullback of two morphisms in  $\mathcal{M}$  is again in  $\mathcal{M}$ .
- (2) Every epimorphism in  $\mathcal{M}$  is an isomorphism.

Then every  $\mathcal{M}$ -effective square is a pullback square.

Conversely, it is natural to ask whether every pullback square is  $\mathcal{M}$ -effective. When  $\mathcal{M}$  is the class of regular monomorphisms, categories with this property are said to have *effective unions*, a condition isolated by Barr [Bar88]. The connections of this special case with stable independence were investigated in [LRV19a, §5], where it was shown that having effective unions implies that effective squares form a stable independence notion. We show that the definition can be naturally parameterized by  $\mathcal{M}$  (this was done already for pure morphisms in [BR07, 2.2]), and the corresponding results generalized.

**Definition 4.16.** Let  $\mathcal{M}$  be an almost nice class of morphisms in a category  $\mathcal{K}$ . We say that  $\mathcal{K}$  has  *$\mathcal{M}$ -effective unions* if

- (1) The pullback of any two morphisms in  $\mathcal{M}$  with common codomain exists and the projections are again in  $\mathcal{M}$ .
- (2) Any pullback square with morphisms in  $\mathcal{M}$  is  $\mathcal{M}$ -effective.

**Remark 4.17.** In a  $\lambda$ -accessible category  $\mathcal{K}$ ,  $\lambda$ -directed colimits commute with existing limits. The verification is analogous to [AR94, 1.59] because the canonical functor  $E : \mathcal{K} \rightarrow \mathbf{Set}^{\mathcal{A}^{\text{op}}}$  preserves  $\lambda$ -directed colimits and existing limits; here  $\mathcal{A}$  is a representative full subcategory of  $\lambda$ -presentable objects.

**Theorem 4.18.** Assume that  $\mathcal{M}$  is an almost nice class of morphisms in an accessible category  $\mathcal{K}$ . If  $\mathcal{K}$  has  $\mathcal{M}$ -effective unions, then  $\mathcal{M}$  is accessible if and only if  $\mathcal{M}$ -effective squares form a stable independence notion in  $\mathcal{K}_{\mathcal{M}}$ .

*Proof.* If there is a stable independence notion in  $\mathcal{K}_{\mathcal{M}}$ , then by Remark 3.7,  $\mathcal{M}$  is accessible. Let us prove the converse. Pick a regular cardinal  $\lambda$  such that  $\mathcal{M}$  is  $\lambda$ -accessible. By Theorem 4.4,  $\mathcal{M}$ -effective squares form a weakly stable independence notion and by Lemma 4.8 this independence notion is  $\lambda$ -continuous. It remains to see that  $\mathcal{K}_{\mathcal{M}, \downarrow}$  is accessible. Consider an object  $C \rightarrow D$  of  $\mathcal{K}_{\mathcal{M}, \downarrow}$ . Since  $\mathcal{M}$  is  $\lambda$ -accessible,  $D$  can be written as a  $\lambda$ -directed colimit  $\langle D_i : i \in I \rangle$  of  $\lambda$ -presentable objects. Let  $C_i$  be the pullback of  $C$  and  $D_i$  over  $D$ . Then the resulting maps  $C_i \rightarrow D_i$  form a  $\lambda$ -directed system. Following 4.17, the pullback functor is accessible so must preserve arbitrarily large presentability ranks. Thus there is a bound on the presentability rank of  $C_i$  that depends only on  $\lambda$ . This shows that  $\mathcal{K}_{\mathcal{M}, \downarrow}$  is accessible.  $\square$

Note that, as opposed to Theorem 4.9, we did *not* need to assume that  $\mathcal{M}$  was  $\aleph_0$ -continuous (nor that  $\mathcal{K}$  was locally presentable, nor that  $\mathcal{M}$  was closed under retracts). However, a category may fail to have effective unions even if the effective squares form a stable independence notion (this is the case for example in locally finite graphs with regular monos, see [LRV19a, 5.7]).

As a corollary, we obtain a quick proof that having effective unions implies cofibrant generation. This had been done “by hand” before for several special classes of morphisms [Bek00, 1.12], [BR07, 2.4].

**Corollary 4.19.** If  $\mathcal{M}$  is an almost nice, accessible, and  $\aleph_0$ -continuous class of morphisms in a category with  $\mathcal{M}$ -effective unions, then  $\mathcal{M}$  is cofibrantly generated.

*Proof.* By Theorem 4.18  $\mathcal{M}$ -effective squares form a stable independence notion, so Theorem 4.9 (and Remark 4.10) implies that  $\mathcal{M}$  is cofibrantly generated.  $\square$

## 5. WEAK FACTORIZATION SYSTEMS AND INJECTIVES

In this section, we recall the definition of a weak factorization system (and related concepts), and investigate the connection to stable independence implied by Theorem 4.9.

**Definition 5.1.** Let  $\mathcal{K}$  be a category and let  $f : A \rightarrow B, g : C \rightarrow D$  be morphisms. We say that  $f$  has the *left lifting property with respect to*  $g$  (and  $g$  has the *right lifting property with respect to*  $f$ ) and write  $f \square g$  if for any commutative square as below, there exists a diagonal  $d$  making both triangles commute.

$$\begin{array}{ccc}
A & \xrightarrow{\quad} & C \\
\downarrow f & \nearrow d & \downarrow g \\
B & \xrightarrow{\quad} & D
\end{array}$$

For a set  $\mathcal{M}$  of morphisms, we write  $\mathcal{M}^\square$  for the class of all morphisms  $g$  such that  $f\square g$  for all  $f \in \mathcal{M}$ . Similarly, define  ${}^\square\mathcal{M}$ .

**Definition 5.2.** A *weak factorization system* in a category  $\mathcal{K}$  consists of a pair of classes of morphisms  $(\mathcal{M}, \mathcal{N})$  such that:

- (1) Any morphism  $h$  of  $\mathcal{K}$  can be written as  $h = gf$ , where  $f \in \mathcal{M}$  and  $g \in \mathcal{N}$ .
- (2)  $\mathcal{M} = {}^\square\mathcal{N}$  and  $\mathcal{N} = \mathcal{M}^\square$ .

We say that the weak factorization system  $(\mathcal{M}, \mathcal{N})$  is *cofibrantly generated* if  $\mathcal{M}$  is cofibrantly generated.

An *orthogonal factorization system* (often just called a *factorization system*) is defined just as a weak factorization system, except that we require the diagonal map  $d$  in Definition 5.1 to be unique. A weak factorization system  $(\mathcal{M}, \mathcal{N})$  is *functorial* if the factorization can be performed functorially: there exist functors  $M : \mathcal{K}^2 \rightarrow \mathcal{M}$ ,  $N : \mathcal{K}^2 \rightarrow \mathcal{N}$  such that for any morphism  $h$  of  $\mathcal{K}$ ,  $h = (Nh)(Mh)$ .

While the concept of a weak factorization system is relatively new (the term itself appears in, e.g. [AHRT02]), they have been studied—in other, more or less recognizable forms—for some time. It is essentially due to [Rin70], translated into contemporary terminology, that we have the following:

**Fact 5.3.** If  $(\mathcal{M}, \mathcal{N})$  is a weak factorization system in a cocomplete category  $\mathcal{K}$ , then:

- (1)  $\mathcal{M} \cap \mathcal{N}$  consists of exactly the isomorphisms.
- (2)  $\mathcal{M}$  is closed under pushouts, transfinite compositions, and retracts.

The following very useful fact tells us we can build a weak factorization system by starting with an arbitrary *set* of morphisms and closing under pushouts, transfinite compositions, and retracts:

**Fact 5.4** (The small object argument; e.g. [Bek00, 1.3]). If  $\mathcal{K}$  is a locally presentable category and  $\mathcal{X}$  is any *set* of morphisms in  $\mathcal{K}$ , then  $(\text{cof}(\mathcal{X}), \mathcal{X}^\square)$  is a (cofibrantly generated) functorial weak factorization system.

**Remark 5.5.** Any orthogonal factorization system is functorial, and a consequence of Fact 5.4 is that cofibrantly generated weak factorization systems are also functorial.

Using the small object argument, we immediately obtain the following characterization of stable independence in terms of cofibrantly generated weak factorization systems (recall from Definition 4.11 that  $\mathcal{C}(\mathcal{M})$  denotes the full subcategory of  $\mathcal{M}$ -cofibrant objects, and that  $\mathcal{M}_0$  denotes the class of  $\mathcal{M}$ -morphisms between  $\mathcal{C}(\mathcal{M})$ -objects):

**Corollary 5.6.** Let  $\mathcal{K}$  be a locally presentable category, and let  $(\mathcal{M}, \mathcal{N})$  be a weak factorization system in  $\mathcal{K}$ . If  $\mathcal{M}$  is coherent, then the following are equivalent:

- (1)  $\mathcal{M}_0$  is cofibrantly generated.
- (2)  $\mathcal{C}(\mathcal{M})_{\mathcal{M}_0}$  has a stable independence notion.

*Proof.* By Fact 5.3,  $\mathcal{M}$  is nice and  $\aleph_0$ -continuous. Now apply Theorem 4.14.  $\square$

Weak factorization systems imply that injectives are well-behaved:

**Definition 5.7.** Let  $\mathcal{M}$  be a class of morphisms in a category  $\mathcal{K}$ .

- (1) An object  $A$  is  $\mathcal{M}$ -*injective* if for any  $f : A_0 \rightarrow A$  and  $g : A_0 \rightarrow B$  with  $g \in \mathcal{M}$ , there exists  $h : B \rightarrow A$  with  $hg = f$  (the dual notion would be that of a projective object).
- (2) We say that  $\mathcal{K}$  *has enough  $\mathcal{M}$ -injectives* if for any object  $A$  of  $\mathcal{K}$ , there exists an  $\mathcal{M}$ -injective  $A'$  and a morphism  $A \rightarrow A'$  in  $\mathcal{M}$ .
- (3) We say that  $\mathcal{K}$  *has accessibly-enough  $\mathcal{M}$ -injectives* if  $\mathcal{K}$  has enough  $\mathcal{M}$ -injectives and the  $\mathcal{M}$ -injectives form an accessible and accessibly-embedded full subcategory of  $\mathcal{K}$ .

**Remark 5.8.** Let  $\mathcal{K}$  be a locally presentable category and let  $(\mathcal{M}, \mathcal{N})$  be a weak factorization system.

- (1) An  $\mathcal{M}$ -injective is exactly an object  $A$  such that the unique map  $A \rightarrow 1$  is in  $\mathcal{M}^\square$ . Recalling Definition 4.11, this means precisely that the  $\mathcal{M}$ -injectives are exactly the objects of  $\mathcal{F}(\mathcal{M}^\square) = \mathcal{F}(\mathcal{N})$ . Consequently,  $\mathcal{K}$  has enough  $\mathcal{M}$ -injectives.
- (2) If an object  $A$  is  $\mathcal{M}$ -injective, then  $A$  is  $\mathcal{H}$ -injective for any  $\mathcal{H} \subseteq \mathcal{M}$ . Moreover,  $A$  is  $\text{cof}(\mathcal{M})$ -injective (Fact 5.3 and the previous point).
- (3) If  $(\mathcal{M}, \mathcal{N})$  is a functorial [weak] factorization system, then the  $\mathcal{M}$ -injective objects  $\mathcal{F}(\mathcal{N})$  form a [weakly] reflective full subcategory of  $\mathcal{K}$ , where a [weak] reflection arrow is any arrow  $A \xrightarrow{f} B$ , with  $B \in \mathcal{F}(\mathcal{N})$  and  $f \in \mathcal{M}$  (see Definition 2.6).
- (4) If  $(\mathcal{M}, \mathcal{N})$  is cofibrantly generated, then  $\mathcal{K}$  has accessibly-enough  $\mathcal{M}$ -injectives. This follows from the two previous points and [AR94, 4.7].

We conclude that existence of a stable independence notion implies that the injectives are well-behaved:

**Corollary 5.9.** Let  $\mathcal{K}$  be a locally presentable category and let  $\mathcal{M}$  be a nice and  $\aleph_0$ -continuous class of morphisms in  $\mathcal{K}$ . If  $\mathcal{K}_{\mathcal{M}}$  has a stable independence notion, then  $(\mathcal{M}, \mathcal{M}^\square)$  is a cofibrantly generated weak factorization system and  $\mathcal{K}$  has accessibly-enough  $\mathcal{M}$ -injectives.

*Proof.* By Theorem 4.9,  $\mathcal{M}$  is cofibrantly generated, say by a set  $\mathcal{X} \subseteq \mathcal{M}$ . Thus  $\text{cof}(\mathcal{X}) = \mathcal{M}$  and  $\mathcal{X}^\square = \mathcal{M}^\square$ . Thus by Fact 5.4,  $(\mathcal{M}, \mathcal{M}^\square)$  is a cofibrantly generated weak factorization system. By Remark 5.8(4),  $\mathcal{K}$  has accessibly-enough  $\mathcal{M}$ -injectives.  $\square$



**Reflections on coherence.** Recall from Definition 4.1, that a *coclan* is a class of morphisms containing all isomorphisms and closed under composition and pushouts. A key hypothesis throughout this paper is that the coclan should be *coherent* (Definition 3.3): if  $gf \in \mathcal{M}$  and  $g \in \mathcal{M}$ , then  $f \in \mathcal{M}$ . We now investigate this property further and link it to the theory of factorization systems.

First note that any coclan can be closed to a coherent one:

**Definition 5.10.** For  $\mathcal{M}$  a class of morphisms in  $\mathcal{K}$ , the *coherent closure*  $\widehat{\mathcal{M}}$  is the set of morphisms  $f : A \rightarrow B$  so that there exists  $g : B \rightarrow C$  in  $\mathcal{M}$  with  $gf \in \mathcal{M}$ .

**Lemma 5.11.** If  $\mathcal{M}$  is a coclan, then  $\widehat{\mathcal{M}}$  is a coherent coclan.

*Proof.* It is easy to check that  $\widehat{\mathcal{M}}$  is coherent, closed under pushouts, and contains all isomorphisms. It remains to check closure under composition. Assume that  $f_1 : K_1 \rightarrow K_2$  and  $f_2 : K_2 \rightarrow K_3$  belong to  $\widehat{\mathcal{M}}$ , i.e., there are  $g_1 : K_2 \rightarrow L_1$  and  $g_2 : K_3 \rightarrow L_2$  in  $\mathcal{M}$  such that  $g_1 f_1, g_2 f_2 \in \mathcal{M}$ . Consider the pushout

$$\begin{array}{ccc} L_1 & \xrightarrow{h_1} & P_1 \\ \uparrow g_1 & & \uparrow h_2 \\ K_2 & \xrightarrow{g_2 f_2} & L_2 \end{array}$$

We have that  $h_1, h_2 \in \mathcal{M}$ , so

$$h_1 g_1 f_1 = h_2 g_2 f_2 f_1$$

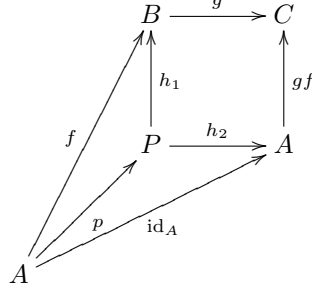
is in  $\mathcal{M}$ . Since also  $h_2 g_2 \in \mathcal{M}$ , we have  $f_2 f_1 \in \widehat{\mathcal{M}}$ . □

The following is a simple criterion for checking that a class of morphisms is coherent. Unfortunately, it relies on closure under pullbacks, which holds for the right rather than the left part of a weak factorization system.

**Theorem 5.12.** Let  $\mathcal{M}$  be a normal class of morphisms in a category  $\mathcal{K}$  with pullbacks. Assume that any pullback of two morphisms in  $\mathcal{M}$  is in  $\mathcal{M}$ . The following are equivalent:

- (1)  $\mathcal{M}$  is coherent.
- (2) Any split mono whose corresponding split epi is in  $\mathcal{M}$  must also be in  $\mathcal{M}$ .  
That is, if  $A \xrightarrow{i} B \xrightarrow{r} A$  is such that  $ir = \text{id}_A$  and  $r \in \mathcal{M}$ , then  $i \in \mathcal{M}$ .

*Proof.* (1) clearly implies (2):  $\text{id}_A \in \mathcal{M}$  because  $\mathcal{M}$  contains all isomorphisms. It remains to see that (2) implies (1). Assume (2), and let  $A \xrightarrow{f} B \xrightarrow{g} C$  be given with  $gf$  and  $g$  in  $\mathcal{M}$ . We have the following commutative diagram



where  $P$  is a pullback and  $A \xrightarrow{p} P$  the induced map. By assumption,  $h_1$  and  $h_2$  are in  $\mathcal{M}$ . Since  $h_2 p = id_A$  and  $h_2 \in \mathcal{M}$ , the second condition gives that  $p \in \mathcal{M}$ . We also have that  $f = h_1 p$ , so since  $\mathcal{M}$  is closed under composition,  $f \in \mathcal{M}$ .  $\square$

**Remark 5.13.**

- (1) A similar proof shows that if  $\mathcal{M}$  is closed under pullbacks, then  $\mathcal{M}$  is left cancellable if and only if  $\mathcal{M}$  contains all split monomorphisms.
- (2) If all morphisms of  $\mathcal{M}$  are monomorphisms, then (2) from Theorem 5.12 automatically holds: any split epi  $r$  in  $\mathcal{M}$  must be an isomorphism, hence the corresponding split mono  $i$  must be an isomorphism as well.
- (3) The right part of a weak factorization system is always closed under pullbacks (this is the dual of Fact 5.3). Thus if  $\mathcal{M}$  is the right part of a weak factorization system and  $\mathcal{M}$  consists of monomorphisms, then it is coherent. It is also true that the right part of an *orthogonal* factorization system is always coherent, and will be left cancellable when consisting only of monomorphisms [AHS04, 14.9].

Whether the class of morphisms on the *left* of a weak factorization system is coherent is tied to whether the morphisms can be generated by a reflection. Recall from Remark 5.8(3) that any [weak] factorization system has a natural [weakly] reflective subcategory (Definition 2.6) given by its fibrant objects. We will prove that  $\mathcal{M}$  is coherent if and only if  $\mathcal{M}$  is the inverse image of its coherent closure under the reflector. In fact we prove it in a more general context:

**Lemma 5.14.** Let  $\mathcal{K}$  be a category, let  $\mathcal{L}$  be a full subcategory of  $\mathcal{K}$ , and let  $\mathcal{M}$  be a class of morphisms of  $\mathcal{K}$  that is closed under composition. Let  $I : \mathcal{L} \rightarrow \mathcal{K}$  be the inclusion functor, let  $R : \mathcal{K} \rightarrow \mathcal{L}$  be a functor with  $R[\mathcal{M}] \subseteq \mathcal{M}$ , and let  $id_{\mathcal{K}} \xrightarrow{r} I \circ R$  be a natural transformation associating to each object  $A$  of  $\mathcal{K}$  an arrow  $A \xrightarrow{r_A} RA$  in  $\mathcal{M}$ .

Then  $\mathcal{M} = R^{-1}(\widehat{\mathcal{M}})$  if and only if  $\mathcal{M}$  is coherent.

*Proof.* Assume  $A \xrightarrow{m} B$  is in  $\mathcal{M}$ . Then  $RA \xrightarrow{Rm} RB$  is also in  $\mathcal{M}$ , hence in  $\widehat{\mathcal{M}}$ . Thus  $\mathcal{M} \subseteq R^{-1}(\widehat{\mathcal{M}})$  (this part does not depend on coherence of  $\mathcal{M}$ ).

Assume now that  $\mathcal{M}$  is coherent. Let  $A \xrightarrow{f} B$  be in  $R^{-1}(\widehat{\mathcal{M}})$ . This means that  $RA \xrightarrow{Rf} RB$  is in  $\widehat{\mathcal{M}}$ . Because  $\mathcal{M}$  is coherent,  $\widehat{\mathcal{M}} = \mathcal{M}$ , hence  $Rf \in \mathcal{M}$ . By naturality we have a commutative diagram

$$\begin{array}{ccc}
RA & \xrightarrow{Rf} & RB \\
\uparrow r_A & & \uparrow r_B \\
A & \xrightarrow{f} & B
\end{array}$$

By assumption,  $r_A$  and  $r_B$  are in  $\mathcal{M}$ , and we have just argued that  $Rf \in \mathcal{M}$ . By closure of  $\mathcal{M}$  under composition,  $(Rf)r_A \in \mathcal{M}$ , so by coherence also  $f \in \mathcal{M}$ .

Conversely, assume that  $\mathcal{M} = R^{-1}(\widehat{\mathcal{M}})$ . Suppose we are given  $A \xrightarrow{f} B \xrightarrow{m} C$  with  $m \in \mathcal{M}$  and  $mf \in \mathcal{M}$ . We have to see that  $f \in \mathcal{M}$ . Applying  $R$  to this situation, we obtain a diagram

$$\begin{array}{ccccc}
RA & \xrightarrow{Rf} & RB & \xrightarrow{Rm} & RC \\
\uparrow r_A & & \uparrow r_B & & \uparrow r_C \\
A & \xrightarrow{f} & B & \xrightarrow{m} & C
\end{array}$$

We have that  $Rm \in \mathcal{M}$ , and since  $mf \in \mathcal{M}$  we also have that  $(Rm)(Rf) \in \mathcal{M}$ . Thus  $Rf \in \widehat{\mathcal{M}}$ , and so  $f \in R^{-1}(\widehat{\mathcal{M}}) = \mathcal{M}$ , as desired.  $\square$

**Theorem 5.15.** Let  $(\mathcal{M}, \mathcal{N})$  be a functorial weak factorization system in a category  $\mathcal{K}$  with a terminal object. Let  $R$  be a weak reflector selecting weak reflection arrows as described by Remark 5.8(3). Then  $\mathcal{M} = R^{-1}(\widehat{\mathcal{M}})$  if and only if  $\mathcal{M}$  is coherent.

*Proof.* Immediate from Lemma 5.14 (used with  $\mathcal{L}$  standing for the full subcategory of  $\mathcal{K}$  with objects in  $\mathcal{F}(\mathcal{N})$ ).  $\square$

**Remark 5.16.** In the setup of Theorem 5.15, if  $(\mathcal{M}, \mathcal{N})$  is a factorization system (i.e. not only a weak one), then it is called *reflective* if  $\mathcal{M}$  is coherent (see [CHK85]). Theorem 5.15 gives some justification for adopting that terminology also for functorial weak factorization system with  $\mathcal{M}$  coherent (perhaps they could be called *weakly reflective*). Indeed, it is a consequence of Theorem 5.15 (already observed in [RT07, 3.2]) that a factorization system  $(\mathcal{M}, \mathcal{N})$  is reflective if and only if  $\mathcal{M} = R^{-1}(\text{Iso})$ . In fact, a map  $Rf$  is in  $\widehat{\mathcal{M}}$  if and only if it is an isomorphism. To see this, assume that we have a diagram

$$\begin{array}{ccc}
RA & \xrightarrow{Rf} & RB \\
\uparrow r_A & & \uparrow r_B \\
A & \xrightarrow{f} & B
\end{array}$$

with  $Rf \in \widehat{\mathcal{M}}$ . By definition of  $\widehat{\mathcal{M}}$ , there exist maps  $RB \xrightarrow{g} C$  in  $\mathcal{M}$  such that  $g(Rf)$  is also in  $\mathcal{M}$ . Factor the map  $C \rightarrow 1$  as  $C \xrightarrow{h} C' \rightarrow 1$ , where  $h \in \mathcal{M}$  and

$C' \rightarrow 1$  is in  $\mathcal{N}$ . Then the map  $A \xrightarrow{hg(Rf)r_A} C'$  is in  $\mathcal{M}$ , hence is a reflection of  $A$ . By uniqueness of reflections,  $RA \cong C'$ . Similarly,  $RB \cong C'$ , so  $RA \cong RB$  are both reflections of  $A$  and  $Rf$  must be an isomorphism.

## 6. EXAMPLES

**Trivial examples.** The examples below describe the minimal and maximal choices for  $\mathcal{M}$ :

**Example 6.1.** Let  $\mathcal{K}$  be any category with pushouts. Let  $\mathcal{M}$  be the class of all isomorphisms. Then  $\mathcal{M}$  is normal and coherent. It is also a coclan, and closed under retracts, so it is nice. By Theorem 4.4,  $\mathcal{M}$ -effective squares give a weakly stable independence relation on  $\mathcal{K}_{\mathcal{M}}$ : the independent squares are those consisting only of isomorphisms. Except in trivial cases,  $\mathcal{M}$  will of course not be accessible.

**Example 6.2** ([LRV19a, 3.31(8)]). Let  $\mathcal{K}$  be an accessible category with pushouts and let  $\mathcal{M}$  be the class of all morphisms. Then any commutative square is  $\mathcal{M}$ -effective, so  $\mathcal{K}_{\mathcal{M}, \downarrow} = \mathcal{K}^2$  which is also accessible. Thus the class of all commutative squares gives a stable independence relation in  $\mathcal{K}$ . More generally, if  $\mathcal{K}$  is an accessible category where any two amalgams of the same span are equivalent (in the sense given by the uniqueness axiom of stable independence, see Definition 2.10), then the class of all commutative squares gives a stable independence relation. This holds for example if  $\mathcal{K}$  has weak pushouts or if  $\mathcal{K}$  has a terminal object.

**Regular monomorphisms.** A monomorphism is *regular* if it is an equalizer of a pair of morphisms. A diagram chase shows that regular monos are normal and coherent (this can also be obtained from Theorem 5.12, since regular monos are closed under pullbacks). Moreover, every split mono is regular ([AHS04, 7.59(1)]), so, by Remark 4.2, regular monos are closed under retracts. For  $\mathcal{K}$  a category, let  $\mathcal{K}_{reg}$  denote the subcategory with the same objects as  $\mathcal{K}$ , but only regular monos as morphisms; that is,  $\mathcal{K}_{reg} = \mathcal{K}_{\mathcal{M}}$ , where  $\mathcal{M}$  is precisely the class of regular monomorphisms in  $\mathcal{K}$ . By [AR94, 1.62], regular monos are  $\lambda$ -continuous in any locally  $\lambda$ -presentable category. By [AR94, 2.31], in a locally  $\lambda$ -presentable category,  $\lambda$ -pure morphisms are regular. It follows from Fact 3.9(5) that in any locally  $\lambda$ -presentable category, regular monos are accessible and  $\lambda$ -continuous. By Fact 3.10, then,  $\mathcal{K}_{reg}$  is equivalent to a  $\lambda$ -AEC.

In order to apply the results of Section 4, we assume  $\lambda = \aleph_0$  and require that regular monomorphisms be closed under pushouts. A locally presentable category  $\mathcal{K}$  is called *coregular* if the latter condition holds (specific examples are given below). If  $\mathcal{K}$  is locally presentable and coregular, then regular monos are nice, so Theorem 4.4 gives that (regular-)effective squares form a weakly stable independence relation in  $\mathcal{K}_{reg}$  (this was observed already in [LRV19a, 5.6]). In order to get a stable independence relation, Theorem 4.9 tells us that we need regular monos to be cofibrantly generated. A sufficient condition for this is having (regular-)effective unions (Definition 4.16) and in this case we get a stable independence relation from Theorem 4.18. (Note that the terminology “effective” was developed for the regular case, [Bar88], so we omit the “(regular-)” for the remainder of this subsection.) On the other hand, there are examples without effective unions that are still cofibrantly generated. In other examples, we can use our knowledge about stable independence

to deduce that regular monos cannot be cofibrantly generated. We repeat the examples given in [LRV19a] here to see how they shed light on our results.

**Example 6.3.**

- (1) [LRV19a, 3.31(6), 4.8(2)] The category **Gra** of graphs (i.e. symmetric and reflexive binary relations) with homomorphisms is locally presentable and coregular. Regular monomorphisms correspond to full subgraph embeddings. It is well known to model theorists that the category **Gra<sub>reg</sub>** does *not* have a stable independence notion. A quick way to see this is to exhibit two different weakly stable independence notions (if there were a stable independence relation, this would contradict canonicity, Theorem A.6). One weakly stable independence notion is given by effective squares, which turns out to mean that two graphs  $B$  and  $C$  are independent over a base graph  $A$  if there are no cross edges from  $B$  to  $C$  except inside  $A$ . Another weakly stable independence relation would require that all possible cross edges from  $B - A$  to  $C - A$  are present. Using Theorem 4.9, we conclude that regular monomorphisms are not cofibrantly generated in **Gra**.
- (2) [LRV19a, 4.8(3), 4.8(4)] The categories **Grp** of groups, **Ban** of Banach spaces, and **Bool** of boolean algebras do *not* have a stable independence relation when restricted to their regular monos. This can be seen model theoretically, or via Corollary 5.9: these categories do not have accessibly enough regular regular injectives. In **Bool**, for example, the regular injectives—equivalently, the injectives, as regular monos and monos coincide—are precisely the complete Boolean algebras, which do not form an accessible category.
- (3) [LRV19a, 4.8(1)] Any Grothendieck topos (like the category of sets) or any Grothendieck abelian category (like  $R\text{-}\mathbf{Mod}$ , the category of  $R$ -modules) has effective unions of regular monos. Thus they have a stable independence relation when restricted to regular monos (which are just monos in this case).
- (4) [LRV19a, 4.8(5)] The category of Hilbert spaces has effective unions, hence also has a stable independence notion when restricted to its regular monos (which are just isometries).
- (5) [LRV19a, 5.7] For  $n$  a natural number, the category of graphs whose vertices have degree at most  $n$  does not have effective unions (for the same reason that effective squares are not accessible in **Gra**), but still has a stable independence notion.

**Monomorphisms.** A diagram chase shows that monos are normal and left cancellable (hence, in particular, closed under retracts, Remark 4.2). For  $\mathcal{K}$  a category, let  $\mathcal{K}_{mono}$  denote the subcategory with the same objects as  $\mathcal{K}$ , but with only monos as morphisms; that is,  $\mathcal{K}_{mono} = \mathcal{K}_{\mathcal{M}}$ , where  $\mathcal{M}$  is the class of monos in  $\mathcal{K}$ . By the proof of [LRV19a, 6.2], monos are  $\lambda$ -continuous in any  $\lambda$ -accessible category. Fact 3.9 implies that any  $\lambda$ -pure morphism is a mono in a  $\lambda$ -accessible category, and it follows that in any  $\lambda$ -accessible category, monos are accessible and  $\lambda$ -continuous. By Fact 3.10, then,  $\mathcal{K}_{mono}$  is equivalent to a  $\lambda$ -AEC.

In order to apply the results of Section 4, we assume  $\lambda = \aleph_0$  and need also the monomorphisms to be closed under pushouts. By Fact 5.3, this will hold if monos

form the left part of a weak factorization system. The following basic tool tells us that it is enough to check that the category has enough injectives:

**Fact 6.4** ([AHRT02, 1.6]). Let  $\mathcal{K}$  be a category with finite products. Let  $\mathcal{M}$  be a left cancellable class of morphisms (Definition 3.3). The following are equivalent:

- (1)  $(\mathcal{M}, \mathcal{M}^\square)$  is a weak factorization system.
- (2)  $\mathcal{K}$  has enough  $\mathcal{M}$ -injectives.

Having enough injectives is a well known condition which again holds for example in a Grothendieck abelian category or a Grothendieck topos (see Example 6.3(3)). The following is another interesting example, which should be contrasted with Example 6.3(1):

**Example 6.5.** The category **Gra** of graphs and graph homomorphisms is locally presentable. The monomorphisms in this case are (not necessarily full) subgraph embeddings. Injectives (with respect to monos) are the complete graphs, so **Gra** has enough injectives. Moreover monos are cofibrantly generated by  $\emptyset \rightarrow 1$  and  $1 + 1 \rightarrow 2$ , where 1 is a vertex and 2 is an edge. By Theorem 4.9, **Gra**<sub>mono</sub> has a stable independence relation.

The existence of enough injectives does *not* however guarantee that there will be a stable independence notion:

**Example 6.6.** Recall the discussion of **Bool**, the category of Boolean algebras, in Example 6.3(2). The injectives—which are precisely the same as the regular injectives, i.e. complete Boolean algebras—do not form an accessible category, so Corollary 5.9 implies that **Bool**<sub>mono</sub> does not have a stable independence relation.

Does having *accessibly*-enough injectives guarantee the existence of a stable independence notion? This is unclear for monomorphisms, but fails for split monomorphisms:

**Example 6.7** ([Ros17, 2.6]). In **Ab**, the category of abelian groups, (SplitMono, SplitEpi) is a weak factorization system which has accessibly-enough injectives (in fact any abelian group is injective with respect to split monos), and any object is cofibrant. However, this weak factorization system is not cofibrantly generated.

**Pure morphisms.** For  $\lambda$  a regular cardinal, let  $\mathcal{K}_{\lambda\text{-pure}}$  denote category with the same objects as  $\mathcal{K}$  but only  $\lambda$ -pure morphisms (see Definition 3.8). By Fact 3.9,  $\lambda$ -pure morphisms are normal, left cancellable,  $\lambda$ -continuous and accessible. As usual, Fact 3.10 implies that the category  $\mathcal{K}_{\lambda\text{-pure}}$  is equivalent to a  $\lambda$ -AEC.

In order to apply the results of Section 4, we assume again that  $\lambda = \aleph_0$  and need  $\lambda$ -pure morphisms to be closed under pushouts. This is true in any locally  $\lambda$ -presentable category [AR04, 15(i)]. Thus we always get a weakly stable independence relation in this case (Theorem 4.4). Again, Theorem 4.18 gives a sufficient condition for the existence of a stable independence relation: having effective pure unions. Categories with this property are a central focus of [BR07] where effective pure unions are shown to imply cofibrant generation of pure monomorphisms. We do not know at present whether, e.g. **Ab** satisfies this condition. Note that if  $\mathcal{K}_{\aleph_0\text{-pure}}$  has stable independence (which should be given by effective pure squares)

then  $\mathcal{K}$  has enough pure injectives (Corollary 5.9). In universal algebra, these are called *equationally compact algebras* (see [Grä08, Appendix 6]), and in module theory they are called *algebraically compact modules* (see [EM02, §V.1]).

**Model categories.** A *model category* is a complete and cocomplete category  $\mathcal{K}$  together with three classes of morphisms,  $\mathcal{C}$  (the cofibrations),  $\mathcal{F}$  (the fibrations), and  $\mathcal{W}$  (the weak equivalences) such that:

- (1)  $\mathcal{W}$  has the two-out-of-three property: given two composable morphisms  $f$  and  $g$ , if two of  $f$ ,  $g$ ,  $gf$  are in  $\mathcal{W}$ , so is the third.
- (2)  $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$  and  $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$  are weak factorization systems (the maps in  $\mathcal{F} \cap \mathcal{W}$  are called *trivial fibrations* and those in  $\mathcal{C} \cap \mathcal{W}$  are called *trivial cofibrations*).

A model category is called *cofibrantly generated* provided that both of the weak factorization systems above are cofibrantly generated.

**Example 6.8.** Given a weak factorization system  $(\mathcal{C}, \mathcal{F})$  on a complete and cocomplete category  $\mathcal{K}$ , a trivial model category on  $\mathcal{K}$  can be obtained by setting  $\mathcal{W}$  to be the class of all morphisms. For a while, it was open whether there were any examples of model categories that are *not* cofibrantly generated. The paper [AHRT02] was one of the first to come up with examples. From the result of this paper, we can find many others: for example, the trivial model category on **Gra** with cofibrations the regular monos is not cofibrantly generated (Example 6.3(1)).

**Example 6.9.** The category  $\mathcal{K}$  of simplicial sets is a locally finitely presentable category. Taking the cofibrations to be the monomorphisms, the fibrations to be the Kan fibrations, and the weak equivalences to be the weak homotopy equivalences, we obtain a cofibrantly generated model category. As before, we deduce that  $\mathcal{K}_{mono}$  is an AEC with stable independence.

**Remark 6.10.** We note that the same pattern holds whenever we have a model structure in which the cofibrations are precisely the monos and all objects are cofibrant. This covers a number of examples, including:

- (1) For a given ring  $R$ , the *stable module category* over  $R$  is obtained from  $R\text{-Mod}$  by identifying homomorphisms whose difference factors through a projective module: when  $R$  is Frobenius, such categories are examples of the desired kind. This notion of “stable” corresponds, in this case, to stability of the corresponding model structure—yet another sense of the word (see, e.g. [Hov99, §2.2]). From the above, we also have stability in the sense of this piece: the subcategory induced by the monos has a stable independence relation.
- (2) The injective model structure on a category of chain complexes over a ring  $R$ ,  $\text{Ch}(R)$ , [Hov99, 2.3.13]. Hence the subcategory of  $\text{Ch}(R)$  whose morphisms are precisely the level-by-level monos has a stable independence relation.
- (3) The Cisinski model structure on a Grothendieck topos—in particular, on the category of presheaves,  $\text{PSh}(\mathcal{C})$ , over a small category  $\mathcal{C}$  (see [Cis06, §1.2, 1.3]). Here again, there is an independence relation on the appropriate induced subcategory of  $\text{PSh}(\mathcal{C})$ .

**Flat-like categories of modules.** For a fixed (associative and unital) ring  $R$ , let  $R\text{-}\mathbf{Mod}$  denote the category of (left)  $R$ -modules with homomorphisms. It is a locally finitely presentable category. To ensure that we are able to work meaningfully with quotients while still insisting our morphisms are (good) monos, we make the following definition:

**Definition 6.11.** We say that a subcategory  $\mathcal{L}$  of  $R\text{-}\mathbf{Mod}$  is *flat-like* if:

- (1)  $\mathcal{L}$  is an isomorphism-closed full subcategory of  $R\text{-}\mathbf{Mod}$ .
- (2)  $\mathcal{L}$  contains  $R$  (seen as an  $R$ -module).
- (3) For  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  a short exact sequence with  $C \in \mathcal{L}$ , we have that  $A \in \mathcal{L}$  if and only if  $B \in \mathcal{L}$ .
- (4)  $\mathcal{L}$  is closed under directed colimits in  $R\text{-}\mathbf{Mod}$ .

For  $\mathcal{L}$  a flat-like subcategory of  $R\text{-}\mathbf{Mod}$ , let  $\mathcal{M}_{\mathcal{L}}$  denote the class of monomorphisms in  $R\text{-}\mathbf{Mod}$  whose cokernel is in  $\mathcal{L}$ . That is, a monomorphism  $A \xrightarrow{f} B$  is in  $\mathcal{M}_{\mathcal{L}}$  if and only if  $B/f[A]$  is in  $\mathcal{L}$ .

**Remark 6.12.** Categories satisfying properties similar to those in Definition 6.11 are considered already in [Ros02], but there it is *not* assumed that  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  exact with  $B, C \in \mathcal{L}$  implies  $A \in \mathcal{L}$  (this corresponds to coherence). This additional hypothesis is used to build the stable independence, see also Remark 6.22.

**Remark 6.13.** Let  $\mathcal{L}$  be a flat-like subcategory of  $R\text{-}\mathbf{Mod}$ .

- (1) The  $\mathcal{M}_{\mathcal{L}}$ -cofibrant objects (Definition 4.11) are exactly the objects of  $\mathcal{L}$ ; that is,  $\mathcal{C}(\mathcal{M}_{\mathcal{L}}) = \mathcal{L}$ .
- (2) By Remark 2.5, closure under directed colimits implies that  $\mathcal{L}$  is closed under retracts in  $R\text{-}\mathbf{Mod}$ . Concretely, this means that  $A \oplus B \in \mathcal{L}$  implies  $A \in \mathcal{L}$ . In particular, since  $0 \oplus R \in \mathcal{L}$ , the zero module is in  $\mathcal{L}$ .
- (3)  $\mathcal{L}$  is closed under sums: it contains the zero module and if  $A \in \mathcal{L}$  and  $B \in \mathcal{L}$ , then  $0 \rightarrow A \rightarrow A \oplus B \rightarrow B \rightarrow 0$  is an exact sequence, so  $A \oplus B \in \mathcal{L}$ . Infinite sums are handled using closure under directed colimits.
- (4) The previous two points show that  $\mathcal{L}$  contains all free modules and hence (since a module  $P$  is projective exactly when there is  $A$  with  $P \oplus A$  free) all projective modules. Since any flat module is a directed colimit of projective modules,  $\mathcal{L}$  also contains all flat modules.

**Example 6.14.**

- (1)  $R\text{-}\mathbf{Mod}$  itself is a flat-like subcategory of  $R\text{-}\mathbf{Mod}$  and  $\mathcal{M}_{R\text{-}\mathbf{Mod}}$  is the class of all monomorphisms.
- (2) Flat modules form the smallest flat-like subcategory of  $R\text{-}\mathbf{Mod}$  (by Remark 6.13(4), and e.g. [Rot09, 3.46, 3.48, 7.4]).

We recall the following definition (see e.g. [ET00]):

**Definition 6.15.** Given a class  $K$  of  $R$ -modules, we define its Ext-orthogonality class  ${}^{\perp}K$  as follows<sup>6</sup>:

<sup>6</sup>We could have defined  ${}^{\perp}K$  as in e.g. [ET00] by requiring only the vanishing of  $\text{Ext}^1$ . Let us write  ${}^{\perp_1}K$  for this definition. When  $R$  is a left hereditary ring, the two notions coincide. However



$${}^{\perp}K = \{A : \text{Ext}^i(A, N) = 0 \text{ for all } 1 \leq i < \omega \text{ and all } N \in K\}$$

Roughly speaking,  ${}^{\perp}K$  is the collection of  $R$ -modules that do not admit nontrivial extensions by modules in  $K$ . We write  ${}^{\perp}N$  instead of  ${}^{\perp}\{N\}$ .

**Remark 6.16.** For  $K$  a class of pure injective modules,  ${}^{\perp}K$  is a flat-like subcategory of  $R\text{-Mod}$  (as outlined in, for example, [BET07, §1]). Conversely, if  $K$  is a class of modules and  ${}^{\perp}K$  is a flat-like subcategory of  $R\text{-Mod}$ , then by Remark 6.13(4)  ${}^{\perp}K$  contains all flat modules hence all the modules in  $K$  are cotorsion<sup>7</sup>. When  $K$  is the class of *all* pure injective modules,  ${}^{\perp}K$  is exactly the class of flat modules (see [EJ00, 5.3.22, 7.1.4]).

The following is observed in more generality in [Ros02, 4.2]. We give some details for the convenience of the reader.

**Lemma 6.17.** If  $\mathcal{L}$  is a flat-like subcategory of  $R\text{-Mod}$ , then  $\mathcal{M}_{\mathcal{L}}$  is nice and  $\aleph_0$ -continuous.

*Proof.* Write  $\mathcal{M} := \mathcal{M}_{\mathcal{L}}$ .

- $\mathcal{M}$  contains all isomorphisms: the cokernel of an isomorphism is the zero module, which is in  $\mathcal{L}$  by Remark 6.13.
- $\mathcal{M}$  is closed under composition: assume we have  $A \xrightarrow{f} B \xrightarrow{g} C$  with  $f, g$  in  $\mathcal{M}$ , let's identify  $f$  and  $g$  with the corresponding inclusion morphisms. Then  $B/A$  and  $C/B$  are in  $\mathcal{L}$  and we have a short exact sequence  $0 \rightarrow B/A \rightarrow C/A \rightarrow C/B \rightarrow 0$ . It follows that  $C/A$  is in  $\mathcal{L}$ , hence that  $gf \in \mathcal{M}$ .
- $\mathcal{M}$  is coherent: suppose we have modules  $A \subseteq B \subseteq C$ , and  $C/A, C/B$  are in  $\mathcal{M}$ . Again, we have the short exact sequence  $0 \rightarrow B/A \rightarrow C/A \rightarrow C/B \rightarrow 0$ , and so it follows that  $B/A$  is in  $\mathcal{L}$ , as desired.
- $\mathcal{M}$  is closed under pushouts: the pushout  $g$  of a monomorphism  $f$  is a monomorphism, and  $\text{coker}(f)$  is isomorphic to  $\text{coker}(g)$ .
- $\mathcal{M}$  is closed under retracts: if  $f$  is a retract of  $g$ , then  $\text{coker}(f)$  must be a retract of  $\text{coker}(g)$ , so the result follows from closure of  $\mathcal{L}$  under retracts (Remark 6.13(2)).
- $\mathcal{M}$  is  $\aleph_0$ -continuous: this is proven as in [BET07, 1.5]. In details, let  $\langle A_i : i \leq \delta \rangle$  be an increasing continuous chain of modules, with  $A_j/A_i$  in  $\mathcal{L}$  for all  $i < j < \delta$  (we think of all the maps as inclusions, and assume unions have been taken at limits). First observe that for all  $i < \delta$ ,  $A_\delta/A_i$  is in  $\mathcal{L}$  as well, since it is simply the directed colimit of  $A_j/A_i$  for  $j < \delta$ . Now if  $A_\delta$  is a submodule of a module  $B$ , and  $B/A_i$  is in  $\mathcal{L}$  for all  $i < \delta$ , then  $B/A$  is the colimit of the diagram consisting of the canonical maps  $B/A_i \rightarrow B/A_j$ ,  $i < j < \delta$ .

□

---

in general, we may not have that if  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  is exact with  $B, C \in {}^{\perp}K$ , then  $A \in {}^{\perp}K$ . That is,  ${}^{\perp}K$  may not be *resolving* in the sense of [GT06, 2.2.8]. In fact by a result of Garcia Rozas,  ${}^{\perp}K$  is resolving if and only if  ${}^{\perp}K = {}^{\perp_1}K$  (see [GT06, 2.2.10]).

<sup>7</sup>A module  $M$  is *cotorsion* if  $\text{Ext}(F, M) = 0$  for all flat modules  $F$ , see [EJ00, 5.3.22].

It is natural to ask when a flat-like subcategory  $\mathcal{L}$  is an actual AEC (or, equivalently by Fact 3.10, just an accessible category) when restricted to  $\mathcal{M}_{\mathcal{L}}$ . When  $\mathcal{L}$  is of the form  ${}^{\perp}N$  for a single  $R$ -module  $N$ , this was characterized in [BET07, 1.17] using the following key notion (also called being *deconstructible* e.g. in [ŠT12, 2.3]):

**Definition 6.18** ([ET00, Definition 1]). For  $\theta$  a regular cardinal and  $\mathcal{L}$  a flat-like subcategory of  $R\text{-Mod}$  closed under directed colimits, we say that  $\mathcal{L}$  *has  $\theta$ -refinements* if any object of  $\mathcal{L}$  can be written as the union of an increasing smooth chain  $\langle A_i : i < \alpha \rangle$  of submodules, with  $A_0$  the zero module and for all  $i < \alpha$ ,  $A_{i+1}/A_i$  in  $\mathcal{L}$  and  $\theta$ -presentable (we call such a chain a  *$\theta$ -refinement* of  $A$ ). We say that  $\mathcal{L}$  *has refinements* if it has  $\theta$ -refinements for some regular cardinal  $\theta$ .

**Remark 6.19.** Let  $\mathcal{L}$  be a flat-like subcategory of  $R\text{-Mod}$ .

- (1) For  $\theta > |R| + \aleph_0$ , an object of  $R\text{-Mod}$  is  $\theta$ -presentable if and only if it has cardinality strictly less than  $\theta$ .
- (2) Given a  $\theta$ -refinement  $\langle A_i : i < \alpha \rangle$  of  $A = A_\alpha$ , the inclusion  $A_i \rightarrow A_j$  for  $i \leq j \leq \alpha$  is always in  $\mathcal{M}_{\mathcal{L}}$ . Further, each  $A_i$ ,  $i \leq \alpha$  is in  $\mathcal{L}$ . This follows by induction, using the definition of a flat-like subcategory.

We now show that any flat-like subcategory with refinements is, with the appropriate class of morphisms, an AEC with stable independence. Indeed, having refinements is just another disguise for the appropriate morphisms being cofibrantly generated. This answers [BET07, 4.1(1)], which asked what one could say about tameness and stability in the AEC  ${}^{\perp}N$  defined there. The answer is: if we know that  ${}^{\perp}N$  is an AEC, then it will *always* be stable and tame.

Below, note that only the implication (2) implies (3) is new: the equivalence between having refinements and the appropriate class being an AEC is just another version of the module-theoretic fact that Kaplansky classes closed under extensions and directed colimits are deconstructible, see [ŠT12, 2.5]. Nevertheless, our proof of (1) implies (5) is different than (for example) the one in [BET07], which uses the Hill lemma (a module-theoretic version of the fat small object argument, [MRV14]).

**Theorem 6.20.** Let  $\mathcal{L}$  be a flat-like subcategory of  $R\text{-Mod}$ , and  $\mathcal{M} = \mathcal{M}_{\mathcal{L}}$ . For the sake of brevity (Notation 4.12), we set  $\mathcal{M}_0 = \mathcal{M} \cap \text{Mor}(\mathcal{L})$ . The following are equivalent:

- (1)  $\mathcal{L}$  has refinements.
- (2)  $\mathcal{M}$  is cofibrantly generated by a subset of  $\mathcal{M}_0$ .
- (3)  $\mathcal{L}_{\mathcal{M}_0}$  has a stable independence notion.
- (4)  $\mathcal{L}_{\mathcal{M}_0}$  is accessible.
- (5)  $\mathcal{L}_{\mathcal{M}_0}$  is an AEC.

*Proof.*

- (1) implies (2): Assume that  $\mathcal{L}$  has  $\theta$ -refinements for some  $\theta$ . By the proof of [Ros02, 4.5],  $\mathcal{M}$  is cofibrantly generated by a set of maps  $f$  so that  $0 \rightarrow A \xrightarrow{f} F \rightarrow B \rightarrow 0$  is a short exact sequence,  $F$  is a free module, and  $B$  is a  $\theta$ -presentable object of  $\mathcal{L}$ . Since  $F$  is free,  $F \in \mathcal{L}$  as well, hence  $A \in \mathcal{L}$ . Thus  $f \in \mathcal{M}_0$ .
- (2) implies (3): Apply Theorem 4.14.

- (3) implies (4): By Fact 2.12.
- (4) implies (5): By Fact 3.10 and Remark 3.11. Notice that the ambient category here is  $R\text{-}\mathbf{Mod}$ , which is locally finitely presentable, hence finitely accessible.
- (5) implies (1): As in the proof of [BET07, 1.17(2)].

□

Using (4), it is easy to give more characterizations of existence of refinements. Let us say that a flat-like subcategory  $\mathcal{L}$  of  $R\text{-}\mathbf{Mod}$  is *closed under  $\lambda$ -pure quotients [subobjects]* if whenever  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  is a short exact sequence,  $B \in \mathcal{L}$ , and  $A \rightarrow B$  is  $\lambda$ -pure, then  $C \in \mathcal{L}$  [ $A \in \mathcal{L}$ ].

**Theorem 6.21.** Let  $\mathcal{L}$  be a flat-like subcategory of  $R\text{-}\mathbf{Mod}$ . The following are equivalent:

- (1)  $\mathcal{L}$  has refinements.
- (2)  $\mathcal{L}$  is accessible.
- (3)  $\mathcal{L}$  is closed under  $\lambda$ -pure quotients for some  $\lambda$ .
- (4)  $\mathcal{L}$  is closed under  $\lambda$ -pure subobjects for some  $\lambda$ .

*Proof.* Let  $\mathcal{M} := \mathcal{M}_{\mathcal{L}}$ ,  $\mathcal{M}_0 := \mathcal{M} \cap \text{Mor}(\mathcal{L})$ .

- (1) implies (2):  $\mathcal{L}$  is an  $\aleph_0$ -accessibly embedded full subcategory of  $R\text{-}\mathbf{Mod}$ , an accessible category. In particular,  $\mathcal{L}$  has directed colimits and any object of  $\mathcal{L}$  that is  $\lambda$ -presentable in  $R\text{-}\mathbf{Mod}$  is  $\lambda$ -presentable in  $\mathcal{L}$ . Now, by Theorem 6.20,  $\mathcal{L}_{\mathcal{M}_0}$  is accessible. Fix a regular cardinal  $\lambda$  such that  $\mathcal{L}_{\mathcal{M}_0}$  is  $\lambda$ -accessible. The inclusion functor  $F : \mathcal{L}_{\mathcal{M}_0} \rightarrow R\text{-}\mathbf{Mod}$  is  $\lambda$ -accessible, hence by the uniformization theorem ([AR94, 2.19]) there exists a regular  $\mu \geq \lambda$  such that  $F$  is  $\mu$ -accessible and preserves  $\mu$ -presentable objects. We then have that every object of  $\mathcal{L}$  can, in  $\mathcal{L}_{\mathcal{M}_0}$  be written as a  $\mu$ -directed colimit of  $\mu$ -presentable objects in  $\mathcal{L}_{\mathcal{M}_0}$ . By what has just been said,  $\mu$ -presentable objects in  $\mathcal{L}_{\mathcal{M}_0}$  are  $\mu$ -presentable in  $R\text{-}\mathbf{Mod}$ , and hence  $\mu$ -presentable in  $\mathcal{L}$ . This shows that  $\mathcal{L}$  is  $\mu$ -accessible.
- (2) implies (3): Since  $\mathcal{L}$  is closed under directed colimits in  $R\text{-}\mathbf{Mod}$ , it is accessibly embedded in  $R\text{-}\mathbf{Mod}$ , so apply [AR04, Proposition 13].
- (3) implies (4): Immediate from the definition of a flat-like subcategory.
- (4) implies (2): By [AR94, 2.36], any accessibly-embedded subcategory of an accessible category closed under  $\lambda$ -pure subobjects for some  $\lambda$  is accessible.
- (3) implies (1): Assume that  $\mathcal{L}$  is closed under  $\lambda$ -pure quotients. We have seen this implies closure under  $\lambda$ -pure subobjects. The proof of Fact 3.9(5) then implies that  $\mathcal{L}_{\mathcal{M}_0}$  is accessible. By Theorem 6.20,  $\mathcal{L}$  has refinements.

□

**Remark 6.22.** Compared to [Ros02, 4.5], we are using only the weaker hypothesis of closure under  $\lambda$ -pure quotients rather than  $\aleph_0$ -pure quotients. On the other hand, we are assuming that  $\mathcal{M}_{\mathcal{L}}$  is coherent (see Remark 6.12).

Under a large cardinal axiom, we can even immediately deduce that *every* flat-like category has refinements. This partially answers [BET07, 1.21(4)]: under Vopěnka's principle, the cotorsion modules  $N$  such that  ${}^\perp N$  is an AEC are exactly those such that  ${}^\perp N$  is closed under directed colimits in  $R\text{-}\mathbf{Mod}$ .

**Corollary 6.23.** Assuming Vopěnka's principle, every flat-like subcategory of  $R\text{-}\mathbf{Mod}$  has refinements.

*Proof.* By [AR94, 6.17] (using the large cardinal hypothesis), every accessibly embedded subcategory of an accessible category is accessible. Since flat-like subcategories are  $\aleph_0$ -accessibly embedded by definition, we can apply Theorem 6.21 to get the result.  $\square$

**Example 6.24.** The class of all modules, the class of flat modules, and more generally the class  ${}^\perp K$  for  $K$  a class of pure injective modules is closed under  $\aleph_0$ -pure quotient, hence has refinements by Theorem 6.21 (this was already known, see [BBE01, Proposition 2], [ET00, Theorem 8]). Other examples with refinements appear in [ET00].

For the convenience of the reader, we end by recalling why the existence of refinements implies the existence of covers. Note however that stronger results than stated here are known: per [Bas06], if  $\mathcal{L}$  is any accessible full subcategory of  $R\text{-}\mathbf{Mod}$  closed under sums and directed colimits, then every module has an  $\mathcal{L}$ -cover.

**Definition 6.25.** For  $\mathcal{L}$  a class of modules, a  $\mathcal{L}$ -precover of a module  $M$  is a map  $A \xrightarrow{f} M$  such that  $A \in \mathcal{L}$  and for any  $A' \in \mathcal{L}$ , any map  $A' \rightarrow M$  factors through  $f$ . A  $\mathcal{L}$ -cover of  $M$  is a  $\mathcal{L}$ -precover  $A \xrightarrow{f} M$  such that any endomorphism  $g : A \rightarrow A$  satisfying  $f = fg$  is an automorphism.

**Fact 6.26** ([Xu96, 2.2.12]). If  $\mathcal{L}$  is a flat-like subcategory of  $R\text{-}\mathbf{Mod}$  and a module  $M$  has an  $\mathcal{L}$ -precover, then it has an  $\mathcal{L}$ -cover.

**Fact 6.27** ([Ros02, §1]). If  $\mathcal{L}$  is a flat-like subcategory of  $R\text{-}\mathbf{Mod}$  and  $\mathcal{M}_{\mathcal{L}}$  forms the left part of a weak factorization system, then any module has an  $\mathcal{L}$ -cover.

*Proof.* By Fact 6.26, it is enough to show that every module has an  $\mathcal{L}$ -precover. Let  $M$  be a module. Factor the map  $0 \rightarrow M$  as  $0 \rightarrow A \xrightarrow{f} M$ . Then  $A \in \mathcal{L}$  and  $f$  is a  $\mathcal{L}$ -precover: for any map  $g : A' \rightarrow M$  with  $A' \in \mathcal{L}$ , the map  $0 \rightarrow A'$  is in  $\mathcal{M}_{\mathcal{L}}$ , hence we can apply the lifting property to the diagram

$$\begin{array}{ccc} A' & \xrightarrow{g} & M \\ \uparrow & & \uparrow f \\ 0 & \longrightarrow & A \end{array}$$

$\square$

**Fact 6.28** ([Ros02, 4.5]). If  $\mathcal{L}$  is a flat-like subcategory of  $R\text{-}\mathbf{Mod}$  which has refinements, then every module has an  $\mathcal{L}$ -cover.

*Proof.* Let  $\mathcal{M} := \mathcal{M}_{\mathcal{L}}$ . By Theorem 6.20,  $\mathcal{M}$  is cofibrantly generated. By Fact 5.4,  $\mathcal{M}$  forms the left part of a weak factorization system, so apply Fact 6.27.  $\square$

Using Vopěnka's principle, we get refinements, and hence covers, for free. This was known already for even weaker hypotheses than flat-like: closure under sums and directed colimits, see before 3.3 in [Bas06].

**Corollary 6.29.** Assuming Vopěnka's principle, if  $\mathcal{L}$  is a flat-like subcategory of  $R\text{-Mod}$ , then every module has an  $\mathcal{L}$ -cover.

*Proof.* By Corollary 6.23 and Fact 6.28.  $\square$

## APPENDIX A. CANONICITY OF STABLE INDEPENDENCE

We prove here canonicity of stable independence without the hypothesis, present in [LRV19a, 9.1] that all morphisms are monomorphisms. Our proof is a category-theoretic version of the argument in [BGKV16] which shows somewhat more transparently what is going on there. The key notion is that of an independent sequence:

**Definition A.1.** Let  $\mathcal{K}$  be a category and let  $\perp$  be an independence notion on  $\mathcal{K}$ . Let  $f : M_0 \rightarrow M$  be a morphism in  $\mathcal{K}$ . An  $\perp$ -independent sequence for  $f$  consists of a nonzero ordinal  $\alpha$  and morphisms  $(f_i)_{i \leq \alpha}$  and  $(g_{i,j})_{i \leq j \leq \alpha}$  such that for  $i \leq j \leq k \leq \alpha$ :

- $f = f_0$  and  $N_0 = M$ .
- $f_i : M \rightarrow N_i$  for  $0 < i$ .
- $g_{i,j} : N_i \rightarrow N_j$ .
- $g_{j,k}g_{i,j} = g_{i,k}$ ,  $g_{i,i} = \text{id}_{N_i}$ .
- When  $i < j$ , the following square commutes and, when  $j < \alpha$ , is  $\perp$ -independent:

$$\begin{array}{ccc} M & \xrightarrow{f_j} & N_j \\ \uparrow f_0 & & \uparrow g_{i,j} \\ M_0 & \xrightarrow{g_{0,i}f_0} & N_i \end{array}$$

We call  $\alpha$  the *length* of the sequence. For a regular cardinal  $\lambda$ , we say the independent sequence is  $\lambda$ -smooth if whenever  $\text{cf}(i) \geq \lambda$ ,  $N_i$  is the colimit of the system  $(g_{j,k})_{j \leq k < i}$ . We say it is smooth if it is  $\aleph_0$ -smooth.

For example, an independent sequence of length one for  $f : M_0 \rightarrow M$  consists of  $f_0 = f$ ,  $f_1 : M \rightarrow N_1$ ,  $g_{0,1} : M = N_0 \rightarrow N_1$  such that  $f_1 f_0 = g_{0,1} f_0$ . Since there are no independence requirements, it is essentially just the morphism  $f_0$  (the additional data is only relevant when  $\alpha$  is limit; we could have taken  $N_1 = N_0 = M$ ,  $f_1 = \text{id}_M$ ). More interestingly, an independent sequence of length two consists essentially (because  $N_0 = M$  and  $g_{0,0} f_0 = f_0$ ) of an independent square:

$$\begin{array}{ccc}
M & \xrightarrow{f_1} & N_1 \\
\uparrow f_0 & & \uparrow g_{0,1} \\
M_0 & \xrightarrow{f_0} & M
\end{array}$$

Thus it consists of two “independent copies” of  $M$ .

An independent sequence of length three will look like:

$$\begin{array}{ccccc}
& & N_2 & & \\
& & \swarrow g_{1,2} & & \\
& & N_1 & & \\
& \nearrow f_2 & \nearrow f_1 & \nwarrow g_{0,1} & \\
M & & M & & M \\
& \nwarrow f_0 & \nwarrow f_0 & \nearrow f_0 & \\
& & M_0 & & 
\end{array}$$

where the inner diamond  $(M_0, M, M, N_1)$  and the outer diamond  $(M_0, M, N_1, N_2)$  is independent (in fact, if  $\perp$  is monotonic, all commutative subsquares of the diagram will be independent). Essentially, the leftmost “copy” of  $M$  is independent of the two rightmost copies (in fact it is independent of  $N_1$ ).

Existence allows us to build independent sequences. Recall that a category  $\mathcal{K}$  has *chain bounds* if any chain has a compatible cocone.

**Lemma A.2.** If  $\mathcal{K}$  has  $\lambda$ -directed colimits, chain bounds, and  $\perp$  is a monotonic independence notion with existence, then for any morphism  $f : M_0 \rightarrow M$  and any ordinal  $\alpha$ , there exists a  $\lambda$ -smooth independent sequence for  $f$  of length  $\alpha$ . More generally, any independent sequence of length  $\alpha_0 < \alpha$  extends to one of length  $\alpha$  (in the natural sense).

*Proof.* By repeated use of existence. □

The following local character lemma will be handy:

**Lemma A.3.** Let  $\mathcal{K}$  be a category,  $\perp$  an independence relation such that  $\mathcal{K}_\perp$  is a  $\lambda$ -accessible category. Let  $(M_i \rightarrow N_i)_{i < \lambda^+}$  be a system of  $\lambda^+$ -presentable objects in  $\mathcal{K}^2$  with colimit  $M \rightarrow N$ . Then there exists  $i < \lambda^+$  such that the square

$$\begin{array}{ccc}
N_i & \longrightarrow & N \\
\uparrow & & \uparrow \\
M_i & \longrightarrow & M
\end{array}$$

is independent.

*Proof.* Write  $I$  for  $\lambda^+$  with the usual ordering. By taking colimits at ordinals of cofinality  $\lambda$  and adding them to the system, we can assume without loss of generality that the system is  $\lambda$ -smooth: for any  $i \in I$  of cofinality  $\lambda$ ,  $M_i$  is the colimit of  $(M_{i_0})_{i_0 < i}$ .

Let  $(M'_j \rightarrow N'_j)_{j \in J}$  be a  $\lambda^+$ -directed system of  $\lambda^+$ -presentable objects whose colimit in  $\mathcal{K}_\downarrow$  is  $M \rightarrow N$ ; we know that  $\mathcal{K}_\downarrow$  is  $\lambda^+$ -accessible. We build  $(i_\alpha, j_\alpha)_{\alpha < \lambda}$  such that for all  $\alpha < \lambda$ :

- (1)  $i_\alpha \in I, j_\alpha \in J$ .
- (2)  $i_\alpha < i_{\alpha+1}$ .
- (3) The map from  $M_{i_\alpha} \rightarrow N_{i_\alpha}$  to  $M \rightarrow N$  factors through  $M'_{j_\alpha} \rightarrow N'_{j_\alpha}$ .
- (4) The map from  $M'_{j_\alpha} \rightarrow N'_{j_\alpha}$  to  $M \rightarrow N$  factors through  $M_{i_{\alpha+1}} \rightarrow N_{i_{\alpha+1}}$ .

This is possible since  $I$  and  $J$  are  $\lambda^+$ -directed and  $M_i \rightarrow N_i, M'_j \rightarrow N'_j$  are  $\lambda^+$ -presentable. Now, let  $i := \sup_{\alpha < \lambda} i_\alpha$ . The colimit in  $\mathcal{K}^2$  of  $(M_{i_\alpha} \rightarrow N_{i_\alpha})_{\alpha < \lambda}$  and  $(M'_{j_\alpha} \rightarrow N'_{j_\alpha})_{\alpha < \lambda}$  coincide and by  $\lambda$ -smoothness must be  $M_i \rightarrow N_i$ . By assumption, for all  $\alpha < \lambda$ , the square

$$\begin{array}{ccc} N'_{j_\alpha} & \longrightarrow & N \\ \uparrow & & \uparrow \\ M'_{j_\alpha} & \longrightarrow & M \end{array}$$

is independent. Since  $\mathcal{K}_\downarrow$  has  $\lambda$ -directed colimits, this means that the square

$$\begin{array}{ccc} N_i & \longrightarrow & N \\ \uparrow & & \uparrow \\ M_i & \longrightarrow & M \end{array}$$

is also independent. □

A much simpler result than the canonicity theorem is:

**Lemma A.4.** Assume  $\mathcal{K}$  is a category,  $\overset{1}{\perp}, \overset{2}{\perp}$  are independence notions such that  $\overset{1}{\perp} \subseteq \overset{2}{\perp}$ ,  $\overset{1}{\perp}$  has existence, and  $\overset{2}{\perp}$  has uniqueness. Then  $\overset{1}{\perp} = \overset{2}{\perp}$ .

*Proof.* Given a square  $M_0, M_1, M_2, M_3$  that is  $\overset{2}{\perp}$ -independent, use existence for  $\overset{1}{\perp}$  to  $\overset{1}{\perp}$ -amalgamate the span  $M_0 \rightarrow M_1, M_0 \rightarrow M_2$ , giving maps  $M_1 \rightarrow M'_3, M_2 \rightarrow M'_3$ . Now by uniqueness for  $\overset{2}{\perp}$ , the amalgam involving  $M_3$  and the one involving  $M'_3$  must be equivalent, hence  $M_0, M_1, M_2, M_3$  is also  $\overset{1}{\perp}$ -independent. □

We can now prove the canonicity theorem. The idea is to use a generalization of the fact that, in a vector space, if  $I$  is linearly independent and  $a$  is a vector, there

exists a finite subset  $I_0 \subseteq I$  such that  $(I - I_0) \cup \{a\}$  is independent. Thus we can remove a small subset of  $I$  and get something independent.

**Lemma A.5.** Assume  $\mathcal{K}$  has chain bounds, and  $\overset{1}{\perp}, \overset{2}{\perp}$  are independence notions with existence such that:

- (1)  $\overset{1}{\perp}$  is right monotonic.
- (2)  $\overset{2}{\perp}$  is transitive, left monotonic, and right accessible.

Then any span has an amalgam that is both  $\overset{1}{\perp}$ -independent and  $\overset{2}{\perp}$ -independent. In particular, if  $\overset{1}{\perp}$  has uniqueness then  $\overset{1}{\perp} \subseteq \overset{2}{\perp}$ .

*Proof.* Consider a span  $M_0 \xrightarrow{f_0} M, M_0 \xrightarrow{f'_0} M'$ . Fix a regular cardinal  $\lambda$  such that

$\mathcal{K}_{\downarrow^2}$  (the arrow category induced by  $\overset{2}{\perp}$ ) is  $\lambda$ -accessible and  $M_0, M, M', f_0, f'_0$  are  $\lambda$ -presentable in all relevant categories.

Using Lemma A.2, build a  $(\overset{2}{\perp})^d$ -independent sequence for  $f_0, (f_i : M \rightarrow N_i)_{i \leq \lambda^+}, (g_{i,j} : N_i \rightarrow N_j)_{i \leq j \leq \lambda^+}$ , where  $N_{\lambda^+}$  is the colimit of  $(N_i)_{i < \lambda^+}$ . Observe that

$$f_{\lambda^+} f_0 = g_{0,\lambda^+} f_0.$$

Along the way, we ensure that  $N_i$  is  $\lambda^+$ -presentable for  $i < \lambda^+$ . Now  $\overset{1}{\perp}$ -amalgamate the span  $M_0 \rightarrow N_{\lambda^+}, M_0 \rightarrow M'$ , giving an  $\overset{1}{\perp}$ -independent square:

$$\begin{array}{ccc} M' & \xrightarrow{h'} & N'_{\lambda^+} \\ f'_0 \uparrow & & \uparrow h \\ M_0 & \xrightarrow{g_{0,\lambda^+} f_0} & N_{\lambda^+} \end{array}$$

with  $N'_{\lambda^+}$  a  $\lambda^{++}$ -presentable object. Reworking the proof of [Ros97, Lemma 1]—which requires directed colimits—to use the chain bounds available to us here, we can write  $N'_{\lambda^+}$  as a colimit of  $\lambda^+$ -presentables  $(g'_{i,j} : N'_i \rightarrow N'_j)_{i \leq j < \lambda^+}$ , where:

- (1) There is an arrow  $h_i : N_i \rightarrow N'_i$  for each  $i < \lambda^+$ .
- (2) The  $N'_i$  lie above  $M'$ , in the sense that  $h' : M' \rightarrow N'_{\lambda^+}$  factors as

$$M' \xrightarrow{u_i} N'_i \xrightarrow{g'_{i,\lambda^+}} N'_{\lambda^+}$$

and, moreover, that the morphisms  $h' f'_0 = h g_{0,\lambda^+} f_0 : M_0 \rightarrow N'_{\lambda^+}$  factor identically through  $g'_{i,\lambda^+}$ , i.e.

$$h_i f_i f_0 = u_i f'_0.$$

Here we use  $\lambda$ -presentability of  $M_0, M'$ , and  $\lambda^+$ -directedness of the chain.

Then  $h$  is a colimit of the  $h_i$  in  $\mathcal{K}^2$  and by Lemma A.3, there exists  $i < \lambda^+$  such that the square



$$\begin{array}{ccc}
N'_i & \xrightarrow{g'_{i,\lambda+}} & N'_{\lambda+} \\
\uparrow h_i & & \uparrow h \\
N_i & \xrightarrow{g_{i,\lambda+}} & N_{\lambda+}
\end{array}$$

is  $\overset{2}{\perp}$ -independent. By definition of an  $(\overset{2}{\perp})^d$ -independent sequence, the square

$$\begin{array}{ccc}
N_i & \xrightarrow{g_{i,\lambda+}} & N_{\lambda+} \\
\uparrow f_i f_0 & & \uparrow f_{\lambda+} \\
M_0 & \xrightarrow{f_0} & M
\end{array}$$

is  $\overset{2}{\perp}$ -independent. By left transitivity, we obtain that the following is  $\overset{2}{\perp}$ -independent.

$$\begin{array}{ccc}
N'_i & \xrightarrow{g'_{i,\lambda+}} & N'_{\lambda+} \\
\uparrow h_i f_i f_0 & & \uparrow h f_{\lambda+} \\
M_0 & \xrightarrow{f_0} & M
\end{array}$$

A chase through the diagrams above reveals that

$$g'_{i,\lambda+} h_i f_i f_0 = h' f'_0 = h g_{0,\lambda+} f_0 = h f_{\lambda+} f_0,$$

meaning that the outer square and the large upper triangle in the following diagram commute:

$$\begin{array}{ccccc}
& & g'_{i,\lambda+} & & \\
& & \longrightarrow & & \\
N'_i & & & & N'_{\lambda+} \\
& \nwarrow u_i & & \nearrow h' & \\
& & M' & & \\
& \nearrow f'_0 & & & \\
M_0 & \xrightarrow{f_0} & & & M \\
& & & & \uparrow h f_{\lambda+}
\end{array}$$

Thus the square

$$\begin{array}{ccc}
N'_i & \xrightarrow{g'_{i,\lambda+}} & N'_{\lambda+} \\
\uparrow u_i f'_0 & & \uparrow h f_{\lambda+} \\
M_0 & \xrightarrow{f_0} & M
\end{array}$$

is  $\overset{2}{\perp}$ -independent.

By left monotonicity for  $\overset{2}{\downarrow}$ , then, the following is also  $\overset{2}{\downarrow}$ -independent:

$$\begin{array}{ccc} M' & \xrightarrow{h'} & N'_{\lambda+} \\ f'_0 \uparrow & & \uparrow hf_{\lambda+} \\ M_0 & \xrightarrow{f_0} & M \end{array}$$

Note, however, that the morphism from  $M$  to  $N'_{\lambda+}$  in the diagram above is not the same as the one in the  $\overset{1}{\downarrow}$ -amalgam of  $M_0 \rightarrow N_{\lambda+}$ ,  $M_0 \rightarrow M'$ . In fact, we have a diagram of the form:

$$\begin{array}{ccccc} M' & & \xrightarrow{h'} & & N'_{\lambda+} \\ f'_0 \uparrow & & & & \uparrow h \\ M_0 & \xrightarrow{f_0} & M & \xrightarrow{g_0, \lambda^+} & N_{\lambda+} \\ & \searrow f_0 & & \nearrow f_{\lambda+} & \\ & & M & & \end{array}$$

where the upper rectangle is  $\overset{1}{\downarrow}$ -independent and the outer “square”  $(f'_0, f_0, hf_{\lambda+}, h')$  is  $\overset{2}{\downarrow}$ -independent. By right monotonicity for  $\overset{1}{\downarrow}$ , we get that  $(f'_0, f_0, hf_{\lambda+}, h')$  is also  $\overset{1}{\downarrow}$ -independent. Thus it is the desired amalgam of  $f'_0, f_0$ .  $\square$

**Theorem A.6** (The canonicity theorem). Assume  $\mathcal{K}$  has chain bounds, and  $\overset{1}{\downarrow}$ ,  $\overset{2}{\downarrow}$  are independence notions with existence and uniqueness such that:

- (1)  $\overset{1}{\downarrow}$  is right monotonic.
- (2)  $\overset{2}{\downarrow}$  is transitive and right accessible.

Then  $\overset{1}{\downarrow} = \overset{2}{\downarrow}$ . In particular,  $\mathcal{K}$  has at most one stable independence notion.

*Proof.* Combine Lemmas A.4 and A.5. Note that right monotonicity for  $\overset{2}{\downarrow}$  follows from existence, uniqueness, and transitivity (Fact 2.11).  $\square$

**Corollary A.7.** Assume  $\mathcal{K}$  has chain bounds. If  $\downarrow$  is a transitive and right accessible independence notion with existence and uniqueness, then  $\downarrow$  is a stable independence notion. In particular, it is symmetric.

*Proof.* It suffices to see that  $\downarrow = \overset{d}{\downarrow}$ . For this, apply Theorem A.6 with  $\overset{1}{\downarrow} = \downarrow$  and  $\overset{2}{\downarrow} = \overset{d}{\downarrow}$  ( $\overset{d}{\downarrow}$  is right monotonic by Fact 2.11).  $\square$

**Remark A.8.** Instead of chain bounds, it suffices to be able to build the appropriate independent sequences. See [LRV19a, 9.6].

## REFERENCES

- [AHRT02] Jiří Adámek, Horst Herrlich, Jiří Rosický, and Walter Tholen, *Weak factorization systems and topological functors*, Applied Categorical Structures **10** (2002), 237–249.
- [AHS04] Jiří Adámek, Horst Herrlich, and George E. Strecker, *Abstract and concrete categories*, online edition ed., 2004, Available from <http://katmat.math.uni-bremen.de/acc/>.
- [AR94] Jiří Adámek and Jiří Rosický, *Locally presentable and accessible categories*, London Math. Society Lecture Notes, Cambridge University Press, 1994.
- [AR04] Jiří Adámek and Jiří Rosický, *On pure quotients and pure subobjects*, Czechoslovak Mathematical Journal **54** (2004), no. 129, 623–636.
- [Bar88] Michael Barr, *On categories with effective unions*, Categorical Algebra and its Applications (F. Borceux, ed.), Lecture Notes in Mathematics, vol. 1348, Springer, 1988, pp. 19–35.
- [Bas06] Robert El Bashir, *Covers and directed colimits*, Algebras and Representation Theory **9** (2006), no. 5, 423–430.
- [BBE01] Ladislav Bican, Robert El Bashir, and Edgar E. Enochs, *All modules have flat covers*, Bulletin of the London Mathematical Society **33** (2001), 385–390.
- [Bek00] Tibor Beke, *Sheafifiable homotopy model category*, Mathematical Proceedings of the Cambridge Philosophical society **129** (2000), 447–475.
- [BET07] John T. Baldwin, Paul C. Eklof, and Jan Trlifaj,  ${}^{\perp}N$  as an abstract elementary class, Annals of Pure and Applied Logic **149** (2007), 25–39.
- [BGKV16] Will Boney, Rami Grossberg, Alexei Kolesnikov, and Sebastien Vasey, *Canonical forking in AECs*, Annals of Pure and Applied Logic **167** (2016), no. 7, 590–613.
- [BGL<sup>+</sup>16] Will Boney, Rami Grossberg, Michael J. Lieberman, Jiří Rosický, and Sebastien Vasey,  $\mu$ -Abstract elementary classes and other generalizations, The Journal of Pure and Applied Algebra **220** (2016), no. 9, 3048–3066.
- [Bou99] Elisabeth Bouscaren (ed.), *Model theory and algebraic geometry: An introduction to E. Hrushovski’s proof of the geometric Mordell-Lang conjecture*, Lecture Notes in Mathematics, Springer-Verlag, 1999.
- [BR07] Francis Borceux and Jiří Rosický, *Purity in algebra*, Algebra Universalis **56** (2007), 17–35.
- [BR12] Tibor Beke and Jiří Rosický, *Abstract elementary classes and accessible categories*, Annals of Pure and Applied Logic **163** (2012), 2008–2017.
- [CHK85] Charles Cassidy, Michel Hébert, and Gregory Maxwell Kelly, *Reflective subcategories, localizations and factorization systems*, Journal of the Australian Mathematical society (Series A) **38** (1985), 287–329.
- [Cis06] Denis-Charles Cisinski, *Les préfaisceaux comme modèles des types d’homotopie*, Astérisque **308** (2006), xxiv+392 pp.
- [EJ00] Edgar E. Enochs and Overtoun M. G. Jenda, *Relative homological algebra*, De Gruyter expositions in mathematics, vol. 30, De Gruyter, 2000.
- [EM02] Paul C. Eklof and Alan Mekler, *Almost free modules: set-theoretic methods*, revised ed., North Holland Mathematical library, vol. 46, North-Holland, 2002.
- [ET00] Paul C. Eklof and Jan Trlifaj, *Covers induced by Ext*, Journal of Algebra **231** (2000), 640–651.
- [ET01] ———, *How to make Ext vanish*, Bulletin of the London Mathematical Society **33** (2001), 41–51.
- [Grä08] George A. Grätzer, *Universal algebra*, second edition ed., Springer-Verlag, 2008.
- [GT06] Rüdiger Göbel and Jan Trlifaj, *Approximations and endomorphism algebras of modules*, Expositions in mathematics, vol. 41, De Gruyter, 2006.
- [GV06] Rami Grossberg and Monica VanDieren, *Galois-stability for tame abstract elementary classes*, Journal of Mathematical Logic **6** (2006), no. 1, 25–49.
- [Hov99] Mark Hovey, *Model categories*, Mathematical surveys and monographs, vol. 63, American Mathematical Society, 1999.

- [Hov07] ———, *Cotorsion pairs and model categories*, Interactions between homotopy theory and algebra, Contemporary Mathematics, vol. 436, American Mathematical Society, 2007, pp. 277–296.
- [Joy] André Joyal, *Notes on clans and tribes*, Preprint. URL: <https://arxiv.org/abs/1710.10238v1>.
- [Lan98] Saunders Mac Lane, *Categories for the working mathematician*, second ed., Graduate Texts in Mathematics, vol. 5, Springer, 1998.
- [Lie11] Michael J. Lieberman, *Category-theoretic aspects of abstract elementary classes*, Annals of Pure and Applied Logic **162** (2011), no. 11, 903–915.
- [LRV] Michael J. Lieberman, Jiří Rosický, and Sebastien Vasey, *Set-theoretic aspects of accessible categories*, Preprint. URL: <https://arxiv.org/abs/1902.06777v2>.
- [LRV19a] ———, *Forking independence from the categorical point of view*, Advances in Mathematics **346** (2019), 719–772.
- [LRV19b] ———, *Universal abstract elementary classes and locally multipresentable categories*, Proceedings of the American Mathematical Society **147** (2019), no. 3, 1283–1298.
- [Lur09] Jacob Lurie, *Higher topos theory*, Annals of Mathematics Studies, no. 170, Princeton University Press, 2009.
- [Mak84] Michael Makkai, *A survey of basic stability theory, with particular emphasis on orthogonality and regular types*, Israel Journal of Mathematics **49** (1984), no. 1, 181–238.
- [Mor65] Michael Morley, *Categoricity in power*, Transactions of the American Mathematical Society **114** (1965), 514–538.
- [MP89] Michael Makkai and Robert Paré, *Accessible categories: The foundations of categorical model theory*, Contemporary Mathematics, vol. 104, American Mathematical Society, 1989.
- [MRV14] Michael Makkai, Jiří Rosický, and Lukáš Vokřínek, *On a fat small object argument*, Advances in Mathematics **254** (2014), 49–68.
- [Qui67] D. G. Quillen, *Homotopical algebra*, Lecture Notes in Mathematics, no. 43, Springer-Verlag, Berlin, 1967.
- [Rin70] Claus Michael Ringel, *Diagonalisierungspaare I.*, Mathematische Zeitschrift **117** (1970), 249–266.
- [Rin72] ———, *The intersection property of amalgamations*, Journal of Pure and Applied Algebra **2** (1972), 341–342.
- [Ros97] Jiří Rosický, *Accessible categories, saturation and categoricity*, The Journal of Symbolic Logic **62** (1997), no. 3, 891–901.
- [Ros02] ———, *Flat covers and factorizations*, Journal of Algebra **253** (2002), 1–13.
- [Ros17] ———, *Accessible model categories*, Applied Categorical Structures **25** (2017), 187–196.
- [Rot09] Joseph J. Rotman, *An introduction to homological algebra*, second ed., Universitext, Springer, 2009.
- [RT07] Jiří Rosický and Walter Tholen, *Factorization, fibration and torsion*, Journal of Homotopy and Related Structures **2** (2007), no. 2, 295–314.
- [She87] Saharon Shelah, *Classification of non elementary classes II. Abstract elementary classes*, Classification Theory (Chicago, IL, 1985) (John T. Baldwin, ed.), Lecture Notes in Mathematics, vol. 1292, Springer-Verlag, 1987, pp. 419–497.
- [She90] ———, *Classification theory and the number of non-isomorphic models*, 2nd ed., Studies in logic and the foundations of mathematics, vol. 92, North-Holland, 1990.
- [ŠT12] Jan Šároch and Jan Trlifaj, *Kaplansky classes, finite character and  $\aleph_1$ -projectivity*, Forum Mathematicum **24** (2012), 1091–1109.
- [SV] Saharon Shelah and Sebastien Vasey, *Categoricity and multidimensional diagrams*, Preprint. URL: <http://arxiv.org/abs/1805.06291v2>.
- [Vas16] Sebastien Vasey, *Infinitary stability theory*, Archive for Mathematical Logic **55** (2016), 567–592.
- [Vas17] ———, *Shelah’s eventual categoricity conjecture in universal classes: part II*, Selecta Mathematica **23** (2017), no. 2, 1469–1506.
- [Vas18] ———, *Toward a stability theory of tame abstract elementary classes*, Journal of Mathematical Logic **18** (2018), no. 2, 1850009.
- [Xu96] Jinzhong Xu, *Flat covers of modules*, Lecture Notes in Mathematics, Springer, 1996.

*E-mail address:* `lieberman@math.muni.cz`

*URL:* `http://www.math.muni.cz/~lieberman/`

DEPARTMENT OF MATHEMATICS AND STATISTICS, FACULTY OF SCIENCE, MASARYK UNIVERSITY,  
BRNO, CZECH REPUBLIC

INSTITUTE OF MATHEMATICS, FACULTY OF MECHANICAL ENGINEERING, BRNO UNIVERSITY OF  
TECHNOLOGY, BRNO, CZECH REPUBLIC

*E-mail address:* `rosicky@math.muni.cz`

*URL:* `http://www.math.muni.cz/~rosicky/`

DEPARTMENT OF MATHEMATICS AND STATISTICS, FACULTY OF SCIENCE, MASARYK UNIVERSITY,  
BRNO, CZECH REPUBLIC

*E-mail address:* `sebv@math.harvard.edu`

*URL:* `http://math.harvard.edu/~sebv/`

DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY, CAMBRIDGE, MASSACHUSETTS, USA