

**MATH 269X - MODEL THEORY FOR ABSTRACT  
ELEMENTARY CLASSES, SPRING 2018  
LECTURE NOTES**

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1. NOTATION

We use the letter  $\tau$  for a vocabulary,  $K$  for a class of structures. For  $M$  a  $\tau$ -structure, we write  $|M|$  for its universe and  $\|M\|$  for the cardinality of its universe. We often abuse notation and write for example  $a \in M$  instead of  $a \in |M|$ . We write  $M \subseteq N$  for  $M$  is a substructure of  $N$ .

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For  $I, A$  sets, we let  ${}^I A$  be the set of functions from  $I$  to  $A$  (we think of them as  $I$ -indexed sequences of elements of  $A$ ). We write  $\bar{a}$  for a sequence of elements. We write  ${}^{<\infty} A$  for  $\bigcup_{\kappa} {}^\kappa A$ , where  $\kappa$  ranges over all ordinals. For  $\bar{a} \in {}^I A$  and  $I_0 \subseteq I$ , we write  $\bar{a} \upharpoonright I_0$  for the restriction of  $\bar{a}$  to  $I_0$ ,  $\ell(\bar{a}) = I$  (usually used when  $I$  is an ordinal),  $\text{dom}(\bar{a}) = I$ , and  $\text{ran}(\bar{a})$  be the range of  $\bar{a}$ : the set of elements in the sequence.

For  $\lambda$  a cardinal, we write  $[A]^\lambda$  for the subsets of  $A$  of cardinality  $\lambda$ . Similarly,  $[A]^{<\lambda}$  denotes the subsets of  $A$  of cardinality less than  $\lambda$ .

## 2. UNIVERSAL CLASSES

We start by studying a simple model-theoretic framework. It was first studied by Tarski under the assumption that the vocabulary is finite [Tar54].

**Definition 2.1** (Tarski). A *universal class* is a class  $K$  of structures in a fixed vocabulary  $\tau = \tau(K)$  that is fixed under isomorphisms, substructures, and unions of chains (according to the substructure relation).

**Example 2.2.** The class of all fields, of all locally finite groups, of all vector spaces over  $\mathbb{Q}$  are universal classes. The class of all algebraically closed fields is not ( $\mathbb{Q}$  is a subfield of  $\mathbb{C}$  which is not algebraically closed).

In the definition, we could have required closure under directed unions instead of just unions of chains. However it turns out that this follows. This is due to Iwamura [Iwa44]:

**Exercise 2.3.** Let  $K$  be a universal class. Let  $\langle M_i : i \in I \rangle$  be a directed (according to substructure) system in  $K$ . Then  $\bigcup_{i \in I} M_i \in K$ .

The following is an important basic result about universal classes. We will see it generalizes (in some sense) to AECs.

**Definition 2.4.** Call a  $\tau$ -structure  $M$  *finitely-generated* if there exists a *finite* subset  $A \subseteq |M|$  such that  $M$  is the closure of  $A$  under its functions.

**Theorem 2.5.** Let  $K$  be a universal class in a vocabulary  $\tau$  and let  $M$  be a  $\tau$ -structure. The following are equivalent:

- (1)  $M \in K$ .
- (2)  $M_0 \in K$  for all finitely-generated substructures  $M_0$  of  $M$ .

*Proof.* If  $M \in K$ , then by closure under substructure any substructure of it is in  $K$  as well. Conversely, if all finitely-generated substructures of  $M$ , then they form a directed system in  $K$  whose union is  $M$ , hence by Exercise 2.3 we have  $M \in K$ .  $\square$

There is a correspondence between universal classes and classes axiomatized by universal sentences in infinitary logics. When the vocabulary is finitary (and relational), this was observed by Tarski [Tar54] (in this case universal classes correspond to classes of models of a universal first-order theory). Tarski's proof generalizes.

**Definition 2.6.** We call an  $\mathbb{L}_{\infty, \omega}$ -sentence *universal* if it is of the form  $\forall x_0 \dots \forall x_{n-1} \psi(\bar{x})$ , where  $\psi$  is quantifier-free.

**Theorem 2.7** (Tarski's presentation theorem). Let  $K$  be a class of structures in some vocabulary  $\tau$ . The following are equivalent:

- (1) There is a set  $\Gamma$  of quantifier-free (first-order) types such that  $K$  is the class of all  $\tau$ -structures omitting  $\Gamma$ .
- (2)  $K$  is the class of models of a universal  $\mathbb{L}_{\infty, \omega}$  theory.
- (3)  $K$  is a universal class.

*Proof.*

- (1) implies (2): Assume that  $K$  is the class of  $\tau$ -structures omitting  $\Gamma$ . For each  $p(\bar{x}) \in \Gamma$ , let  $\phi_p$  be the sentence  $\forall \bar{x} \bigvee_{\psi \in p} \neg \psi(\bar{x})$ . Let  $T := \{\phi_p \mid p \in \Gamma\}$ . It is easy to check that  $K$  is the class of models of  $T$ .
- (2) implies (3): This is straightforward to check.
- (3) implies (1): Let  $K_0$  be the class of  $\tau$ -structures that are finitely generated and are *not* in  $K$ . For each  $M_0 \in K_0$ , let  $p_{M_0}(\bar{x})$  be a type coding it. That is, for any  $N$ , if  $N \models p[\bar{a}]$ , then  $N$  is generated by  $\bar{a}$  and  $N \cong M_0$ . Let  $\Gamma := \{p_{M_0} \mid M_0 \in K_0\}$ . We claim that  $K$  is the set of  $\tau$ -structures omitting  $\Gamma$ . To see this, first notice that any member of  $K$  omits  $\Gamma$  by closure under substructure. Conversely, if  $M$  omits  $\Gamma$ , then any finitely-generated substructure of  $M$  omits  $\Gamma$ , hence is in  $K$ . By Theorem 2.5,  $M \in K$ .

□

**Remark 2.8.** The proof of Tarski's presentation theorem shows that any universal class  $K$  is axiomatized by a universal  $\mathbb{L}_{(2^{|\tau(K)| + \aleph_0})^+, \omega}$  theory.

The following concept was somewhat implicit in Definition 2.4:

**Definition 2.9.** Let  $K$  be a universal class. For  $M \in K$  and  $A \subseteq |M|$ , let  $\text{cl}^M(A)$  be the closure of  $A$  under the functions of  $M$ . Equivalently,  $\text{cl}^M(A)$  is the intersection of all  $M_0 \subseteq M$  which contain  $A$ . Note that  $\text{cl}^M(A)$  is a substructure of  $M$ , hence is itself in  $K$ .

**2.1. Tameness in universal classes.** It is natural to ask how much of the compactness theorem is lost in the setup of universal classes. We have seen that locally finite groups are universal classes, so clearly we cannot expect the compactness theorem to hold in full generality. However, consider the following interesting consequence of compactness:

**Exercise 2.10.** Let  $T$  be a first-order theory. Let  $\mathfrak{C}$  be a monster model for  $T$  (i.e. it is  $\lambda$ -saturated, where  $\lambda$  is much bigger than any of the other objects appearing in the statement). Let  $\alpha$  be an ordinal and let  $\bar{a}, \bar{b} \in {}^\alpha \mathfrak{C}$ . The following are equivalent:

- (1)  $\mathfrak{C} \models \phi[\bar{a}] \leftrightarrow \phi[\bar{b}]$  for all first-order formulas  $\phi$ .
- (2) There exists an automorphism of  $\mathfrak{C}$  taking  $\bar{a}$  to  $\bar{b}$ .

In other words, syntactic first-order types contain the same information as “semantic” types (defined in terms of orbit of a monster model). Is there a version of such a statement for universal classes? Note that universal classes may fail the amalgamation property (e.g. locally finite groups do [Neu60]), so it may not be possible

to build a monster model in this case. Further, first-order types are not the right notion here, since they are not necessarily preserved by substructure. Quantifier-free types should be used and we then have the following result, due to Will Boney, which appears in [Vas17c, 3.7].

**Theorem 2.11** (Boney). Let  $K$  be a universal class. Let  $M_1, M_2 \in K$  and let  $\bar{a}_\ell \in {}^\alpha M_\ell$ ,  $\ell = 1, 2$ . The following are equivalent:

- (1) For any quantifier-free formula  $\phi$ ,  $M_1 \models \phi[\bar{a}_1]$  if and only if  $M_2 \models \phi[\bar{a}_2]$ .
- (2) There exists  $f : \text{cl}^{M_1}(\bar{a}_1) \cong \text{cl}^{M_2}(\bar{a}_2)$  such that  $f(\bar{a}_1) = \bar{a}_2$ .

*Proof.* (2) implies (1) is obvious: quantifier-free formulas are preserved by taking substructures and isomorphisms. We show (1) implies (2). For each  $I \subseteq \alpha$  and  $\ell = 1, 2$ , write  $M_\ell^I := \text{cl}^{M_\ell}(\bar{a}_\ell \restriction I)$ . We will build by induction on  $|I|$  maps  $f_I : M_1^I \cong M_2^I$  such that  $f_I(\bar{a}_1 \restriction I) = \bar{a}_2 \restriction I$ . This will clearly be enough: take  $I = \alpha$ .

This is possible: for  $I$  finite,  $M_\ell^I$  is coded by its quantifier-free type, hence such a map exists by equality of the quantifier-free types of  $\bar{a}_1 \restriction I$  and  $\bar{a}_2 \restriction I$ . Now if  $|I|$  is infinite, observe that for  $I_0 \subseteq J_0 \subseteq I$  with  $|I_0| + |J_0| < |I|$ ,  $f_{I_0} \subseteq f_{J_0}$ . This is because we know that  $f_{J_0}(\bar{a}_1 \restriction I_0) = \bar{a}_2 \restriction I_0 = f_{I_0}(\bar{a}_1 \restriction I_0)$  and for any  $b \in M_1^{I_0}$ ,  $b = \sigma(\bar{a}_1 \restriction I_0)$ , for  $\sigma$  a term (*this is the key feature of universal classes used in the proof*). Thus  $f_{J_0}(b) = \sigma(f_{J_0}(\bar{a}_1 \restriction I_0)) = \sigma(f_{I_0}(\bar{a}_1 \restriction I_0)) = f_{I_0}(\sigma(\bar{a}_1 \restriction I_0)) = f_{I_0}(b)$ . Therefore  $f_I := \bigcup_{I_0 \subseteq I, |I_0| < |I|} f_{I_0}$  is a directed union of a system of isomorphisms, and therefore an isomorphism itself. By definition, it must take  $\bar{a}_1 \restriction I$  to  $\bar{a}_2 \restriction I$ , as desired.  $\square$

**Remark 2.12.** We could have added a parameter set  $A$  contained in both  $M_1$  and  $M_2$ , but this is not needed: one can take  $\bar{a}_1$  and  $\bar{a}_2$  to include an enumeration of it.

We will later see that this result says in technical terms, than “universal classes are fully ( $< \aleph_0$ )-tame and short over the empty set”. A little less formally, orbital types in universal classes are determined by their finite restrictions.

### 3. ABSTRACT ELEMENTARY CLASSES AND THE PRESENTATION THEOREM

Not all elementary classes are universal (algebraically closed fields are one example). Thus the framework of universal classes is limited. Shelah introduced in the late 70s AECs as a semantic framework encompassing in particular classes of models of  $\mathbb{L}_{\infty, \omega}(Q)$  (the paper that introduced them was [She87a], but Shelah lectured on them many years before 1987). We will first give the definition of an abstract class (due to Grossberg).

**Definition 3.1.** An *abstract class* is a pair  $\mathbf{K} = (K, \leq_{\mathbf{K}})$ , where  $K$  is a class of structures in a fixed vocabulary  $\tau = \tau(\mathbf{K})$  and  $\leq_{\mathbf{K}}$  is a partial order,  $M \leq_{\mathbf{K}} N$  implies  $M \subseteq N$ , and both  $K$  and  $\leq_{\mathbf{K}}$  respect isomorphisms. Any abstract class admits a notion of **K-embedding**: these are functions  $f : M \rightarrow N$  such that  $f : M \cong f[M]$  and  $f[M] \leq_{\mathbf{K}} N$ . We sometimes think of  $\mathbf{K}$  as the category whose objects are elements in  $K$  and whose morphisms are **K-embeddings**.

We often do not distinguish between  $K$  and  $\mathbf{K}$ . For  $\lambda$  a cardinal, we will write  $\mathbf{K}_\lambda$  for the restriction of  $\mathbf{K}$  to models of cardinality  $\lambda$ . Similarly define  $\mathbf{K}_{\geq \lambda}$  or more generally  $\mathbf{K}_S$ , where  $S$  is a class of cardinals. We will also use the following notation:

**Notation 3.2.** For  $\mathbf{K}$  an abstract class and  $N \in \mathbf{K}$ , write  $\mathcal{P}_{\mathbf{K}}(N)$  for the set of  $M \in \mathbf{K}$  with  $M \leq_{\mathbf{K}} N$ . Similarly define  $\mathcal{P}_{\mathbf{K}_\lambda}(N)$ ,  $\mathcal{P}_{\mathbf{K}_{<\lambda}}(N)$ , etc.

For an abstract class  $\mathbf{K}$ , we denote by  $\mathbb{I}(\mathbf{K})$  the number of models in  $\mathbf{K}$  up to isomorphism (i.e. the cardinality of  $\mathbf{K}/\cong$ ). We write  $\mathbb{I}(\mathbf{K}, \lambda)$  instead of  $\mathbb{I}(\mathbf{K}_\lambda)$ . When  $\mathbb{I}(\mathbf{K}) = 1$ , we say that  $\mathbf{K}$  is *categorical*. We say that  $\mathbf{K}$  is *categorical in  $\lambda$*  if  $\mathbf{K}_\lambda$  is categorical, i.e.  $\mathbb{I}(\mathbf{K}, \lambda) = 1$ .

We say that  $\mathbf{K}$  has *amalgamation* if for any  $M_0 \leq_{\mathbf{K}} M_\ell$ ,  $\ell = 1, 2$  there is  $M_3 \in \mathbf{K}$  and  $\mathbf{K}$ -embeddings  $f_\ell : M_\ell \rightarrow M_3$ ,  $\ell = 1, 2$ .  $\mathbf{K}$  has *joint embedding* if any two models can be  $\mathbf{K}$ -embedded in a common model.  $\mathbf{K}$  has *no maximal models* if for any  $M \in \mathbf{K}$  there exists  $N \in \mathbf{K}$  with  $M \leq_{\mathbf{K}} N$  and  $M \neq N$  (we write  $M <_{\mathbf{K}} N$ ). Localized concepts such as *amalgamation in  $\lambda$*  mean that  $\mathbf{K}_\lambda$  has amalgamation.

**Definition 3.3** (Shelah). An *abstract elementary class (AEC)* is an abstract class  $\mathbf{K}$  in a finitary vocabulary satisfying:

- (1) Coherence: if  $M_0, M_1, M_2 \in \mathbf{K}$ ,  $M_0 \subseteq M_1 \leq_{\mathbf{K}} M_2$  and  $M_0 \leq_{\mathbf{K}} M_2$ , then  $M_0 \leq_{\mathbf{K}} M_1$ .
- (2) Tarski-Vaught axioms: if  $\delta$  is a limit ordinal,  $\langle M_i : i < \delta \rangle$  is a  $\leq_{\mathbf{K}}$ -increasing chain and  $M := \bigcup_{i < \delta} M_i$ , then:
  - (a)  $M \in \mathbf{K}$ .
  - (b)  $M_j \leq_{\mathbf{K}} M$  for all  $j < \delta$ .
  - (c) Smoothness: if  $N \in \mathbf{K}$  is such that  $M_i \leq_{\mathbf{K}} N$  for all  $i < \delta$ , then  $M \leq_{\mathbf{K}} N$ .
- (3) Löwenheim-Skolem-Tarski (LST) axiom: there exists a cardinal  $\lambda \geq |\tau(\mathbf{K})| + \aleph_0$  such that for any  $N \in \mathbf{K}$  and any  $A \subseteq |N|$ , there exists  $M \in \mathcal{P}_{\mathbf{K}_{\lambda+|A|}}(N)$  such that  $A \subseteq |M|$  and  $M \leq_{\mathbf{K}} N$ . We write  $\text{LS}(\mathbf{K})$  for the least such  $\lambda$ .

Similarly to Exercise 2.3, the following holds:

**Exercise 3.4.** Let  $\mathbf{K}$  be an AEC. Then the Tarski-Vaught axioms holds for directed systems. That is, let  $\langle M_i : i \in I \rangle$  be a  $\leq_{\mathbf{K}}$ -directed system. Let  $M := \bigcup_{i \in I} M_i$ . Then:

- (1)  $M \in \mathbf{K}$ .
- (2)  $M_i \leq_{\mathbf{K}} M$  for all  $i \in I$ .
- (3) Smoothness: if  $N \in \mathbf{K}$  is such that  $M_i \leq_{\mathbf{K}} N$  for all  $i \in I$ , then  $M \leq_{\mathbf{K}} N$ .

**Example 3.5.**

- (1)  $\mathbf{K} = (\text{Mod}(T), \preceq)$ , where  $T$  is any first-order theory, is an AEC with  $\text{LS}(\mathbf{K}) = |\tau(T)| + \aleph_0$ .
- (2)  $\mathbf{K} = (K, \subseteq)$ , where  $K$  is a universal class, is an AEC with  $\text{LS}(\mathbf{K}) = |\tau(K)| + \aleph_0$ . We may abuse notation and call also such a  $\mathbf{K}$  a universal class (or even a universal AEC).
- (3)  $\mathbf{K} = (\text{Mod}(\psi), \preceq_\Phi)$ , where  $\psi \in \mathbb{L}_{\infty, \omega}$  and  $\Phi$  is a fragment containing  $\psi$ , is an AEC with  $\text{LS}(\mathbf{K}) \leq |\Phi| + |\tau(\psi)| + \aleph_0$ .

- (4) For a fixed infinite cardinal  $\lambda$ , the class of well-orderings of type at most  $\lambda^+$  ordered by being an initial segment is an AEC  $\mathbf{K}$  with  $\text{LS}(\mathbf{K}) = \lambda$ .
- (5) The class of well-orderings ordered by being an initial segment is not an AEC (it fails the Löwenheim-Skolem-Tarski axiom).
- (6) The class of well-orderings ordered by being a subordering is not an AEC (it fails to be closed under chains).
- (7) See more examples in [BV17, §3].

How are AECs related to universal classes? The following result of Shelah says that any AEC is the reduct of a universal class [She09a, 1.9(1)] (the presentation we give combines [Vas17c, §2] and [LRVc, 6.4]):

**Theorem 3.6** (Shelah’s presentation theorem). Let  $\mathbf{K}$  be an AEC with vocabulary  $\tau = \tau(\mathbf{K})$ . Then there exists a universal class  $\mathbf{K}^+$  in an expansion  $\tau^+$  of  $\tau$  with  $|\tau^+| = \text{LS}(\mathbf{K})$  and such that the reduct map is a faithful functor from  $\mathbf{K}^+$  into  $\mathbf{K}$  which is surjective on objects. In other words:

- (1) For any  $M \in \mathbf{K}$ , there exists  $M^+ \in \mathbf{K}^+$  such that  $M^+ \upharpoonright \tau = M$ .
- (2) For any  $M^+ \subseteq N^+$  both in  $\mathbf{K}^+$ , letting  $M := M^+ \upharpoonright \tau$ ,  $N := N^+ \upharpoonright \tau$ , we have that  $M, N \in \mathbf{K}$  and  $M \leq_{\mathbf{K}} N$ .

**Corollary 3.7.** For any AEC  $\mathbf{K}$ , there exists a universal  $\mathbb{L}_{(2^{\text{LS}(\mathbf{K})})^+, \omega}$ -sentence  $\psi$  in an expansion of  $\tau(\mathbf{K})$  such that the models in  $\mathbf{K}$  are exactly the  $\tau(\mathbf{K})$ -reducts of models of  $\psi$ .

*Proof.* By Theorem 3.6, Tarski’s presentation Theorem 2.7, and Remark 2.8.  $\square$

To prove Theorem 3.6, the following notion will be useful [Vas17c, 2.9]:

**Definition 3.8.** Let  $\mathbf{K}$  be an abstract class and let  $N \in \mathbf{K}$ . We say  $\mathcal{F}$  is a *set of Skolem functions for  $N$*  if:

- (1)  $\mathcal{F}$  is a non-empty set, and each element  $f$  of  $\mathcal{F}$  is a function from  $N^n$  to  $N$ , for some  $n < \omega$ .
- (2) For all  $A \subseteq |N|$ ,  $M := \mathcal{F}[A] := \bigcup \{f[A] \mid f \in \mathcal{F}\}$  is such that  $M \leq_{\mathbf{K}} N$  and contains  $A$ .

**Remark 3.9.** Let  $\mathbf{K}$  be an AEC, let  $N \in \mathbf{K}$ ,  $\mathcal{F}$  be a set of Skolem functions for  $N$ , and  $A \subseteq |N|$ . Then (by the smoothness axiom) the closure of  $A$  under the functions in  $\mathcal{F}$  is also a  $\mathbf{K}$ -substructure of  $N$  containing  $A$ .

**Lemma 3.10.** Let  $\mathbf{K}$  be an AEC. For any  $N \in \mathbf{K}$ , there exists a set  $\mathcal{F}$  of Skolem functions for  $N$  with  $|\mathcal{F}| = \text{LS}(\mathbf{K})$ .

*Proof.* We build  $\langle N_s \mid s \in [N]^{<\aleph_0} \rangle$  such that for each  $s, t \in [N]^{<\aleph_0}$ :

- (1)  $N_s \in \mathcal{P}_{\mathbf{K} \leq \text{LS}(\mathbf{K})}(N)$ .
- (2)  $s \subseteq |N_s|$ .
- (3)  $s \subseteq t$  implies  $N_s \leq_{\mathbf{K}} N_t$ .

This is possible by inductive applications of the LST and coherence axioms. This is enough: for each  $s \in [N]^{<\aleph_0}$ , let  $\{a_i^s : i < \text{LS}(\mathbf{K})\}$  be an enumeration (possibly with repetitions) of  $N_s$ . Now for each  $n < \omega$ , each  $i < \text{LS}(\mathbf{K})$ , and each  $\bar{a} \in {}^n N$ ,

we let  $f_i^n(\bar{a})$  be  $a_i^{\text{ran}(\bar{a})}$ . Let  $\mathcal{F} := \{f_i^n : i < \text{LS}(\mathbf{K}), n < \omega\}$ . This is as desired: let  $A \subseteq |N|$  and let  $M := \mathcal{F}[A]$ . Then it is easy to check that  $M = \bigcup_{s \in [A]^{<\aleph_0}} N_s$ . Note that  $\langle N_s : s \in [A]^{<\aleph_0} \rangle$  is a directed system and since  $N_s \leq_{\mathbf{K}} N$  for all  $s$ , it follows from the smoothness axiom that  $M \leq_{\mathbf{K}} N$ .  $\square$

*Proof of Theorem 3.6.* Let  $\tau^+$  consist of  $\tau \cup \{f_i^n : i < \text{LS}(\mathbf{K}), n < \omega\}$ , where  $f_i^n$  is a new function symbol of arity  $n$ . Let  $K^+$  be class of  $\tau^+$ -structures  $M^+$  such that  $M_0^+ \upharpoonright \tau \leq_{\mathbf{K}} M^+ \upharpoonright \tau$  for any  $M_0^+ \subseteq M^+$ . Let  $\mathbf{K}^+ := (K^+, \subseteq)$ . It is easy to check that  $\mathbf{K}^+$  is a universal class and by definition, (2) is satisfied. To see (1), let  $M \in \mathbf{K}$ . By Lemma 3.10,  $M$  has a set of Skolem functions  $\mathcal{F}$ . Expand  $M$  to  $M^+ := (M, g)_{g \in \mathcal{F}}$ . Then by definition of Skolem functions,  $M^+ \in \mathbf{K}^+$ .  $\square$

#### 4. ABSTRACT ELEMENTARY CLASSES WITH INTERSECTIONS

The following generalizes Definition 2.9:

**Definition 4.1.** For  $\mathbf{K}$  an AEC,  $N \in \mathbf{K}$  and  $A \subseteq |N|$ , let  $\text{cl}^N(A) := \bigcap \{M \in \mathbf{K} \mid M \leq_{\mathbf{K}} N, A \subseteq |M|\}$ . We see it as a  $\tau(\mathbf{K})$ -substructure of  $N$ .

**Exercise 4.2.** Let  $\mathbf{K}$  be an AEC,  $M \leq_{\mathbf{K}} N$  be in  $\mathbf{K}$ , and  $A, B \subseteq |N|$ .

- (1) Invariance: If  $f : N \cong N'$ , then  $f[\text{cl}^N(A)] = \text{cl}^{N'}(f[A])$ .
- (2) Monotonicity 1:  $A \subseteq \text{cl}^N(A)$ .
- (3) Monotonicity 2:  $A \subseteq B$  implies  $\text{cl}^N(A) \subseteq \text{cl}^N(B)$ .
- (4) Monotonicity 3: If  $A \subseteq |M|$ , then  $\text{cl}^N(A) \subseteq \text{cl}^M(A)$ .
- (5) Idempotence:  $\text{cl}^N(M) = M$  and  $\text{cl}^N(\text{cl}^N(A)) = \text{cl}^N(A)$ .

The notion of having (or admitting) intersections is introduced for AECs in [BS08, 1.2] and further studied in [Vas17c, §2].

**Definition 4.3.** Let  $\mathbf{K}$  be an abstract class,  $N \in \mathbf{K}$ , and  $A \subseteq |N|$ .

- (1) We say that  $N$  has intersections over  $A$  if  $\text{cl}^N(A) \leq_{\mathbf{K}} N$ .
- (2) We say that  $N$  has intersections if it has intersections over all  $A \subseteq |N|$ .
- (3) We say that  $\mathbf{K}$  has intersections if all  $N \in \mathbf{K}$  have intersections.

**Remark 4.4.** Formally,  $\text{cl}^N(A)$  also depends on  $\mathbf{K}$  but usually  $\mathbf{K}$  is clear from context. We may write  $\text{cl}_{\mathbf{K}}^N(A)$  to make  $\mathbf{K}$  explicit.

**Exercise 4.5.** Let  $\mathbf{K}$  be an AEC and let  $N \in \mathbf{K}$ . The following are equivalent:

- (1)  $N$  has intersections.
- (2) For any non-empty  $S \subseteq \mathcal{P}_{\mathbf{K}}(N)$ ,  $\bigcap S \leq_{\mathbf{K}} N$ .

**Example 4.6.** Any universal class has intersections. Algebraically closed fields also have intersections. See more examples in [Vas17c, 2.6]. On the other hand, the class of dense linear orderings without endpoints (ordered by suborder) does not have intersections. Indeed, working in side  $(\mathbb{Q}, <)$ , for each  $n \in [1, \omega)$ ,  $(\frac{-1}{n}, \frac{1}{n})_{\mathbb{Q}}$  is a dense linear ordering without endpoints, but the intersections is  $\{0\}$  which has an endpoint. Now apply Exercise 4.5.

**Definition 4.7.** Let  $\mathbf{K}$  be an AEC. Let  $M \in \mathbf{K}$  and let  $A \subseteq |M|$  be a set.  $M$  is *minimal over  $A$*  if whenever  $M \leq_{\mathbf{K}} N$  and  $M' \leq_{\mathbf{K}} N$  contains  $A$ , then  $M' = M$ .  $M$  is *minimal over  $A$  in  $N$*  if  $M \leq_{\mathbf{K}} N$  and this holds whenever  $N' \leq_{\mathbf{K}} N$ .

The following characterization of having intersections is [Vas17c, 2.11]:

**Theorem 4.8.** Let  $\mathbf{K}$  be an AEC and let  $N \in \mathbf{K}$ . The following are equivalent:

- (1)  $N$  admits intersections.
- (2) There is an operator  $\text{cl} := \text{cl}^N : \mathcal{P}(|N|) \rightarrow \mathcal{P}(|N|)$  such that for all  $A, B \subseteq |N|$  and all  $M \leq_{\mathbf{K}} N$ :
  - (a)  $\text{cl}(A) \leq_{\mathbf{K}} N$ .
  - (b)  $A \subseteq \text{cl}(A)$ .
  - (c)  $A \subseteq B$  implies  $\text{cl}(A) \subseteq \text{cl}(B)$ .
  - (d)  $\text{cl}(M) = M$ .
- (3) For each  $A \subseteq |N|$ , there is a unique minimal model over  $A$  in  $N$ .
- (4) There is a set  $\mathcal{F}$  of Skolem functions for  $N$  such that:
  - (a)  $|\mathcal{F}| \leq \text{LS}(\mathbf{K})$ .
  - (b) For all  $M \leq_{\mathbf{K}} N$ , we have  $\mathcal{F}[M] = M$ .

Moreover the operator  $\text{cl}^N : \mathcal{P}(|N|) \rightarrow \mathcal{P}(|N|)$  with the properties in (2) is unique and if it exists then it has the following characterizations:

- $\text{cl}^N(A) = \bigcap \{M \leq_{\mathbf{K}} N \mid A \subseteq |M|\}$ .
- $\text{cl}^N(A) = \mathcal{F}[A]$ , for any set of Skolem functions  $\mathcal{F}$  for  $N$  such that  $\mathcal{F}[M] = M$  for all  $M \leq_{\mathbf{K}} N$ .
- $\text{cl}^N(A)$  is the unique minimal model over  $A$  in  $N$ .

*Proof.*

- (1) implies (2): Let  $\text{cl}^N(A) := \bigcap \{M \leq_{\mathbf{K}} N \mid A \subseteq |M|\}$ . Even without hypotheses on  $N$ , (2b), (2c), and (2d) are satisfied. Since  $N$  admits intersections, (2a) is also satisfied.
- (2) implies (3): Let  $A \subseteq |N|$ . Let  $\text{cl}$  be as given by (2). Let  $M := \text{cl}(A)$ . By (2a),  $M \leq_{\mathbf{K}} N$ . By (2b),  $A \subseteq |M|$ . Moreover if  $M' \leq_{\mathbf{K}} N$  contains  $A$ , then by (2c),  $|M| \subseteq |\text{cl}(M')|$  but by (2d),  $\text{cl}(M') = M'$ . Thus by coherence and (2a)  $M \leq_{\mathbf{K}} M'$ . This shows both that  $M$  is minimal over  $A$  and that it is unique.
- (3) implies (4): We slightly change the proof of Lemma 3.10 as follows: in the construction of the  $N_s$ 's, let  $N_s$  be the unique minimal model over  $s$  in  $N$ . Now let  $\mathcal{F}$  be as obtained by the rest of the construction there. Let  $A \subseteq |N|$ . We claim that  $\mathcal{F}[A]$  is minimal over  $A$  in  $N$ . This shows in particular that  $\mathcal{F}$  is as required.

Let  $M := \mathcal{F}[A]$ . Since  $\mathcal{F}$  is a set of Skolem functions,  $M \leq_{\mathbf{K}} N$  and  $M$  contains  $A$ . Moreover,  $M = \bigcup_{s \in [A]^{<\aleph_0}} N_s$ . Now if  $M' \leq_{\mathbf{K}} N$  contains  $A$ , then for all  $s \in [A]^{<\aleph_0}$ ,  $s \in [M']^{<\aleph_0}$ , so as  $N_s$  is minimal over  $s$  in  $N$ ,  $N_s \leq_{\mathbf{K}} M'$ . It follows that  $M \leq_{\mathbf{K}} M'$ , so  $M = M'$ .

- (4) implies (1): Let  $\mathcal{F}$  be as given by (4). Let  $A \subseteq |N|$ . Let  $M := \mathcal{F}[A]$ . By definition of Skolem functions,  $M$  contains  $A$  and  $M \leq_{\mathbf{K}} N$ . We claim



that  $M = \bigcap \{M' \leq_{\mathbf{K}} N \mid A \subseteq |M'|\}$ . Indeed, if  $M' \leq_{\mathbf{K}} N$  contains  $A$ , then by the hypothesis on  $\mathcal{F}$ ,  $M = \mathcal{F}[A] \subseteq \mathcal{F}[M'] = M'$ .

The moreover part follows from the arguments above.  $\square$

**Exercise 4.9** ([Vasa, 3.6]). Let  $\mathbf{K}$  be an AEC. Show that if  $N$  has intersections for all  $N \in \mathbf{K}_{\leq \text{LS}(\mathbf{K})}$ , then  $\mathbf{K}$  has intersections.

We obtain the following properties of the closure operator, which complement Exercise 4.2.

**Theorem 4.10.** Let  $\mathbf{K}$  be an AEC with intersections, let  $M \leq_{\mathbf{K}} N$  and let  $A \subseteq |M|$ .

- (1) Monotonicity 3:  $\text{cl}^M(A) = \text{cl}^N(A)$ .
- (2) (Finite character) For any  $b \in \text{cl}^N(A)$ , there exists a finite  $A_0 \subseteq A$  such that  $b \in \text{cl}^N(A_0)$ .

*Proof.* Finite character follows from the characterization of  $\text{cl}^N$  in terms of Skolem functions (Theorem 4.8). For monotonicity 3, let  $M_0 := \text{cl}^N(A)$ . We have  $M_0 \leq_{\mathbf{K}} N$  since  $N$  admits intersections over  $A$ . Since  $M \leq_{\mathbf{K}} N$  contains  $A$ , we must have  $|M_0| \subseteq |M|$ . By coherence,  $M_0 \leq_{\mathbf{K}} M$ . Now  $M_0$  is the unique minimal model over  $A$  in  $N$ , so it must be minimal in  $M$  as well, and hence  $M_0 = \text{cl}^M(A)$ .  $\square$

**Remark 4.11.** There is a generalization of Tarski's presentation Theorem 2.7 to AECs with intersections [BV].

## 5. $\mu$ -AECs AND ACCESSIBLE CATEGORIES

The following naturally generalizes the definition of an AEC to classes that are only closed under sufficiently directed unions:

**Definition 5.1** ([BGL<sup>+</sup>16, 2.2]). Let  $\mu$  be a regular cardinal. A  $\mu$ -abstract elementary class (or  $\mu$ -AEC for short) is an abstract class  $\mathbf{K}$  (where we allow here the vocabulary to be  $(< \mu)$ -ary) satisfying:

- (1) Coherence: if  $M_0, M_1, M_2 \in \mathbf{K}$ ,  $M_0 \subseteq M_1 \leq_{\mathbf{K}} M_2$  and  $M_0 \leq_{\mathbf{K}} M_2$ , then  $M_0 \leq_{\mathbf{K}} M_1$ .
- (2) Tarski-Vaught axioms: if  $\langle M_i : i \in I \rangle$  is a  $\mu$ -directed system (where  $I$  is  $\mu$ -directed if every subset of  $I$  of size strictly less than  $\mu$  has a least upper bound) and  $M := \bigcup_{i \in I} M_i$ , then:
  - (a)  $M \in \mathbf{K}$ .
  - (b)  $M_i \leq_{\mathbf{K}} M$  for all  $i \in I$ .
  - (c) Smoothness: if  $N \in \mathbf{K}$  is such that  $M_i \leq_{\mathbf{K}} N$  for all  $i \in I$ , then  $M \leq_{\mathbf{K}} N$ .
- (3) Löwenheim-Skolem-Tarski (LST) axiom: there exists a cardinal  $\lambda \geq |\tau(\mathbf{K})| + \mu$  such that  $\lambda = \lambda^{< \mu}$  and for any  $N \in \mathbf{K}$  and any  $A \subseteq |N|$ , there exists  $M \in \mathcal{P}_{\mathbf{K}_{\lambda+|A|}^{< \mu}}(N)$  such that  $A \subseteq |M|$ . We write  $\text{LS}(\mathbf{K})$  for the least such  $\lambda$ .

**Remark 5.2.** Technically,  $\text{LS}(\mathbf{K})$  depends on  $\mu$ , but this should not cause any problems, so we remove this from the notation.

Note that, in contrast to Exercise 2.3, asking only that the class be closed under *chains* of cofinality at least  $\mu$  is a significantly weaker condition:

**Exercise 5.3** ([AR94, 1.c.(2)]). For  $n < \omega$ , let  $P_n$  be the ordinal  $\omega_n + 1$ , ordered as usual. Let  $Q := \prod_{1 \leq n < \omega} P_n$  and let  $P$  be the subposet of  $Q$  consisting of those sequences  $(x_n)_{n < \omega}$  with only finitely many  $n < \omega$  so that  $x_n = \omega_n$ .

- (1) Check that  $Q$  is a complete lattice.
- (2) Check that  $P$  is closed (in  $Q$ ) under joins of chains of uncountable cofinality.
- (3) Check that  $P$  is not closed under joins of  $\aleph_1$ -directed sets. *Hint*: Consider  $\prod_{1 \leq n < \omega} \omega_n$ .

The coherence axiom also has the following stronger form:

**Exercise 5.4.** Show that the coherence axiom is equivalent to the following statement: for  $M_0, M_1, M_2 \in \mathbf{K}$  with  $|M_0| \subseteq |M_1| \subseteq |M_2|$ , if  $M_0 \leq_{\mathbf{K}} M_2$  and  $M_1 \leq_{\mathbf{K}} M_2$ , then  $M_0 \leq_{\mathbf{K}} M_1$ .

**Example 5.5.**

- (1) AECs are exactly the  $\aleph_0$ -AECs.
- (2) The class of well-orderings ordered by being a suborder is an  $\aleph_1$ -AEC.
- (3) The class of well-founded models of ZFC, ordered by elementary substructure, is an  $\aleph_1$ -AEC.
- (4) The class of well-orderings ordered by being an initial segment is *not* a  $\mu$ -AEC for any  $\mu$  (the LST axiom fails).
- (5) The class of all Banach spaces (ordered by being a closed subspace) is an  $\aleph_1$ -AEC.
- (6) The class of all  $\mu$ -complete Boolean algebras (ordered by being a subalgebra) is a  $\mu$ -AEC. However the class of all complete Boolean algebras is not.
- (7) The class of models of any  $\mathbb{L}_{\infty, \mu}$  sentence can be made into a  $\mu$ -AEC by ordering it with elementarity according to a fragment.
- (8) See more examples in [BGL<sup>+</sup>16, §2].

Accessible categories were introduced by Lair [Lai81] (he called them “catégorie modelable”). The standard textbooks on them are [MP89, AR94] (see also the following basic references on category theory [AHS04, Lan98]). One can see them as axiomatizing the category-theoretic essence of classes of models of  $\mathbb{L}_{\infty, \infty}$  sentences:

**Definition 5.6.** Let  $\mathcal{K}$  be a category and let  $\lambda$  be a regular cardinal.

- (1) An object  $M$  is  $\lambda$ -*presentable* if its hom-functor  $\mathcal{K}(M, -) : \mathcal{K} \rightarrow \mathbf{Set}$  preserves  $\lambda$ -directed colimits. Put another way,  $M$  is  $\lambda$ -presentable if for any morphism  $f : M \rightarrow N$  with  $N$  a  $\lambda$ -directed colimit  $\langle \phi_\alpha : N_\alpha \rightarrow N \rangle$  with diagram maps  $\phi_{\alpha\beta} : N_\alpha \rightarrow N_\beta$ ,  $f$  factors essentially uniquely through one of the  $N_\alpha$ . That is,  $f = \phi_\alpha f_\alpha$  for some  $f_\alpha : M \rightarrow N_\alpha$ , and if  $f = \phi_\beta f_\beta$  as well, there is  $\gamma > \alpha, \beta$  such that  $\phi_{\gamma\alpha} f_\alpha = \phi_{\gamma\beta} f_\beta$ .
- (2)  $\mathcal{K}$  is  $\lambda$ -*accessible* if it has  $\lambda$ -directed colimits and  $\mathcal{K}$  contains a set  $S$  of  $\lambda$ -presentable objects such that every object of  $\mathcal{K}$  is isomorphic to a  $\lambda$ -directed colimit of objects in  $S$ .
- (3)  $\mathcal{K}$  is *accessible* if it is  $\lambda'$ -accessible for some regular cardinal  $\lambda'$ .

Intuitively, an accessible category is a category with all sufficiently directed colimits and such that every object can be written as a highly directed colimit of “small” objects. Here “small” is interpreted in terms of *presentability*, a notion of size that makes sense in any (possibly non-concrete) category. In the category of sets, of course, a set is  $\lambda$ -presentable if and only if its cardinality is less than  $\lambda$ ; in an AEC  $\mathbf{K}$ , the same is true for all regular  $\lambda > \text{LS}(\mathbf{K})$ . More generally:

**Exercise 5.7.** Let  $\mathbf{K}$  be a  $\mu$ -AEC, let  $\lambda = \lambda^{<\mu} \geq \text{LS}(\mathbf{K})$ , and let  $M \in \mathbf{K}$ . Show that  $M$  is  $\lambda^+$ -presentable if and only if  $\|M\| \leq \lambda$ .

When  $\lambda < \lambda^{<\mu}$ , presentability still gives a natural notion of size in several categories. For example, in Banach spaces it corresponds to the *density character* [LR17, 3.1].

From Exercise 5.7, it is easy to see the following:

**Exercise 5.8.** Prove that if  $\mathbf{K}$  is a  $\mu$ -AEC, then it is an  $\text{LS}(\mathbf{K})^+$ -accessible category.

There are examples of accessible categories that are *not* (equivalent to)  $\mu$ -AECs. The simplest one is the category of sets (where the morphisms are functions). The problem is that the morphisms need not be monomorphisms. If we assume that all morphisms are mono, then we will see (Theorem 5.21) that we do in some sense have a  $\mu$ -AEC. Before proving this, we take a second look at presentability. First, we prove the following generalization of the fact that a small union of small sets is not too big:

**Lemma 5.9.** Let  $\mathcal{K}$  be a  $\lambda$ -accessible category. Then any  $\lambda$ -directed colimit of at most  $\theta$ -many  $\lambda$ -presentable objects is  $(\theta + \lambda)^+$ -presentable.

*Proof.* Let  $M$  be a  $\lambda$ -directed colimit  $\langle \phi_i : M_i \rightarrow M, i \in I \rangle$ , where  $|I| \leq \theta$  and each  $M_i$  is  $\lambda$ -presentable. Let  $\mu := (\theta + \lambda)^+$ . Let  $f : M \rightarrow N$  be a morphism, with  $N$  a  $\mu$ -directed colimit of objects  $\langle N_j : j \in J \rangle$ . Let  $f_i := f \phi_i$ . By  $\lambda$ -presentability of  $M_i$ ,  $f_i$  factors (essentially uniquely) through some  $N_{j_i}$ ,  $j_i \in J$ . Now there are at most  $\theta$ -many  $j_i$ 's, so since  $J$  is  $\mu$ -directed, there is  $j \in J$  with  $j_i \leq j$  for all  $i \in I$ . It follows that  $f$  must factor through  $N_j$ , showing that  $M$  is  $\mu$ -presentable.  $\square$

Recall that a *retract* is a map  $f : M \rightarrow N$  such that there is  $g : N \rightarrow M$  so that  $fg$  is the identity on  $N$ . We also say that  $N$  is a *retract* of  $M$ . In the category of sets, retracts are exactly the surjections. The following is easy to check:

**Exercise 5.10.** Prove that if  $f_1 : M \rightarrow N_1$  and  $f_2 : M \rightarrow N_2$  are retracts, as witnessed by  $g_1$  and  $g_2$ , and  $g_1 f_1 = g_2 f_2$ , then  $N_1$  and  $N_2$  are isomorphic. Conclude that there is only a set (up to isomorphism) of retracts of any given object  $M$ .

The following follows from the definition of  $\lambda$ -presentability and playing with morphisms:

**Exercise 5.11.** Let  $\mathcal{K}$  be a  $\lambda$ -accessible category and let  $S$  be a set of  $\lambda$ -presentable objects such that any object in  $\mathcal{K}$  is a  $\lambda$ -directed colimit of members of  $S$ . Prove that any  $\lambda$ -presentable object is a retract of a member of  $S$ . Thus  $\mathcal{K}$  has only a set (up to isomorphism) of  $\lambda$ -presentable objects. Conversely, show that a retract of a  $\mu$ -presentable object is  $\mu$ -presentable, for any regular  $\mu \geq \lambda$ .

Toward understanding presentability further, we prove a technical lemma saying when an object resolves into a sufficiently directed colimit. We will use the following definitions:

**Definition 5.12.** For  $\mu$  a cardinal,  $\mu^*$  is  $\mu^+$  if  $\mu$  is successor, and  $\mu$  if  $\mu$  is limit.

**Definition 5.13.** For  $\kappa, \mu$  infinite cardinals, we say that  $\mu$  is  $\kappa$ -closed if  $\theta^{<\kappa} < \mu$  for all  $\theta < \mu$ .

**Definition 5.14.** For  $\lambda$  an uncountable cardinal, we call an object  $M$  in a category  $\mathcal{K}$   $(< \lambda)$ -presentable if it is  $\lambda_0$ -presentable for some regular  $\lambda_0 < \lambda$ .

The following is given by the proof of [MP89, 2.3.10]. It is stated as [LRVb, 3.8].

**Lemma 5.15.** Let  $\kappa < \mu \leq \lambda$  be cardinals with  $\kappa$  and  $\mu$  regular and  $\text{cf}(\lambda) \geq \mu$ . Let  $\mathcal{K}$  be a category with  $\kappa$ -directed colimits. If  $M \in \mathcal{K}$  is a  $\kappa$ -directed colimit of  $(< \lambda)$ -presentable objects and  $\mu$  is  $\kappa$ -closed, then  $M$  is a  $\mu$ -directed colimit of  $(< \lambda + \mu^*)$ -presentable objects.

*Proof sketch.* Suppose that  $M$  is a  $\kappa$ -directed colimit of the  $(< \lambda)$ -presentable objects  $\langle M_i : i \in I \rangle$ . Since  $\mu$  is  $\kappa$ -closed, any subset of  $I$  of cardinality strictly less than  $\mu$  is contained inside a  $\kappa$ -directed subset of  $I$  of cardinality strictly less than  $\mu$ . Thus the set  $\mathbb{P}$  of all  $\kappa$ -directed subsets of  $I$  of cardinality strictly less than  $\mu$  is  $\mu$ -directed. For  $s \in \mathbb{P}$ , let  $M_s$  be the colimit of the  $M_i$ 's with  $i \in s$ . Now the induced system  $\langle M_s : s \in \mathbb{P} \rangle$  has  $M$  as its colimit and:

- (1)  $\mu$ -directed, since  $\mathbb{P}$  is  $\mu$ -directed.
- (2) Made of  $(< \lambda + \mu^*)$ -presentable objects.

□

We deduce several interesting results:

**Theorem 5.16.** Let  $\mathcal{K}$  be a  $\lambda$ -accessible category. If  $\mu > \lambda$  is a  $\lambda$ -closed regular cardinal, then  $\mathcal{K}$  is  $\mu$ -accessible.

*Proof.* Directly from Lemma 5.15. □

**Remark 5.17.** We cannot in general remove the assumption that  $\mu$  is  $\lambda$ -closed from Theorem 5.16 (see [AR94, 2.11]). In fact, for  $\mu > 2^{<\lambda}$  regular, the statements “ $\mu$  is  $\lambda$ -closed” and “every  $\lambda$ -accessible category is  $\mu$ -accessible” are equivalent (see [LR17, 4.11] or [LRVb, 2.6]).

**Theorem 5.18.** Let  $\mathcal{K}$  be an accessible category. Then:

- (1) Any object of  $\mathcal{K}$  is  $\lambda$ -presentable, for some  $\lambda$ .
- (2) For any regular cardinal  $\lambda$ , there is only a set (up to isomorphism) of  $\lambda$ -presentable objects.

*Proof.* Let  $\mu$  be such that  $\mathcal{K}$  is  $\mu$ -accessible. Let  $S$  be a set of  $\mu$ -presentable objects so that any object is isomorphic to a  $\mu$ -directed colimit of members of  $S$ . It follows from Lemma 5.9 that any object must be  $\lambda$ -presentable, for some  $\lambda$ . This proves the first item. For the second, Exercise 5.11 shows that there is only a set of  $\mu$ -presentable objects. By Theorem 5.16,  $\mathcal{K}$  is moreover  $\lambda$ -accessible for arbitrarily large  $\lambda$ , so the result follows. □

As mentioned before, in the category of sets, an object is  $\lambda$ -presentable if and only if its cardinality is strictly less than  $\lambda$ . Thus the least cardinal  $\lambda$  such that an object is  $\lambda$ -presentable (we call this the *presentability rank*) is always a successor. The following question of Beke and Rosický [BR12] remains open:

**Question 5.19.** For a fixed accessible category, is every *high-enough* presentability rank a successor?

We can give the following approximation [LRVb, 3.11]:

**Theorem 5.20.** Let  $\mathcal{K}$  be a  $\lambda$ -accessible category. If  $\mu > \lambda$  is weakly inaccessible and  $\lambda$ -closed, then any  $\mu$ -presentable object is  $(< \mu)$ -presentable.

*Proof.* Let  $M$  be  $\mu$ -presentable. By definition,  $M$  can be resolved into a  $\lambda$ -directed colimit of  $\lambda$ -presentable objects, hence of  $(< \mu)$ -presentables. By Lemma 5.15,  $M$  can be resolved into a  $\mu$ -directed colimit of  $(< \mu)$ -presentable objects. By  $\mu$ -presentability of  $M$ , this means that  $M$  is a retract of a  $(< \mu)$ -presentable object, hence is itself  $(< \mu)$ -presentable, as desired.  $\square$

Note that assuming the singular cardinal hypothesis, every weakly inaccessible above  $2^{<\lambda}$  is  $\lambda$ -closed. Since Solovay showed that the singular cardinal hypothesis holds above certain large cardinals (see [Sol74] or [Jec03, 20.8]) it follows that Question 5.19 has a positive answer assuming a large cardinal axiom (a proper class of strongly compact cardinals).

**5.1. From accessible category to  $\mu$ -AEC.** We now aim to show<sup>1</sup>:

**Theorem 5.21** ([BGL<sup>+</sup>16, 4.5]). For any  $\mu$ -accessible category  $\mathcal{K}$  whose morphisms are monomorphisms,  $\mathcal{K}$  is equivalent to a  $\mu$ -AEC.

Recall that two categories  $\mathcal{K}_1$  and  $\mathcal{K}_2$  are *equivalent* if there is a functor  $F : \mathcal{K}_1 \rightarrow \mathcal{K}_2$  which is:

- (1) Full: its restriction to sets of the form  $\text{Hom}(M, N)$  is onto  $\text{Hom}(FM, FN)$ .
- (2) Faithful: its restriction to sets of the form  $\text{Hom}(M, N)$  is injective.
- (3) Essentially surjective: any object  $N$  in  $\mathcal{K}_2$  is isomorphic to  $FM$  for some object  $M$  in  $\mathcal{K}_1$ .

This is weaker than an isomorphism of category, but preserves all reasonable category-theoretic notions. Intuitively, we allow isomorphic objects inside the category to be identified. One example to keep in mind is that the category of a single object with only the identity morphism is equivalent (but not isomorphic) to the category of all singleton sets.

The proof of Theorem 5.21 proceeds in two steps. The first shows that  $\mathcal{K}$  is equivalent to a certain accessible category of structures. The second shows that this category must actually be a  $\mu$ -AEC. Let us implement the first step. For  $\tau$  a vocabulary, we denote by  $\text{Emb}(\tau)$  the category whose objects are  $\tau$ -structures and whose morphisms are injective homomorphisms.

---

<sup>1</sup>It was known since Rosický's thesis [Ros83, Ros81] that accessible categories are classes of models of certain  $\mathbb{L}_{\infty, \infty}$  sentence, but seeing them as  $\mu$ -AEC is more direct.

**Lemma 5.22** ([BGL<sup>+</sup>16, 4.8]). Let  $\mathcal{K}$  be a  $\lambda$ -accessible category whose morphisms are monomorphism. Then there is a (finitary) vocabulary  $\tau$  and a functor  $E : \mathcal{K} \rightarrow \text{Emb}(\tau)$  which is full and faithful and preserves  $\lambda$ -directed colimits.

*Proof.* Let  $\mathcal{K}_0$  be a small full subcategory of  $\mathcal{K}$  containing (up to isomorphism) all the  $\lambda$ -presentable objects. For each  $M \in \mathcal{K}_0$ , let  $S_M$  be a unary relation symbol and for each morphism  $f$  in  $\mathcal{K}_0$ , let  $\underline{f}$  be a binary function symbol. The vocabulary  $\tau$  will consist of all such  $S_M$  and  $\underline{f}$ . Now map each  $M \in \mathcal{K}$  to the following  $\tau$ -structure  $EM$ :

- (1) Its universe are the morphisms  $g : M_0 \rightarrow M$ , where  $M_0 \in \mathcal{K}_0$ .
- (2) For each  $M_0 \in \mathcal{K}_0$ ,  $S_{M_0}^{EM}$  is the set of morphisms  $g : M_0 \rightarrow M$ .
- (3) For each morphism  $f : M_0 \rightarrow M_1$  of  $\mathcal{K}_0$ , and each  $g : M_1 \rightarrow M$ ,  $\underline{f}^{EM}(g) = gf$ . When  $g \notin S_{M_1}^{EM}$ , just let  $\underline{f}^{EM}(g) = g$ .

Map each morphism  $f : M \rightarrow N$  to the function  $\bar{f} : EM \rightarrow EN$  given by  $\bar{f}(g) = fg$ . That  $E$  is full and faithful and preserves  $\lambda$ -directed colimits is a long but crucial exercise in diagram chasing (closely related to the Yoneda lemma). For example, to see that  $E$  is full, assume first that  $M \in \mathcal{K}_0$ . Then  $\text{id}_M$  is a morphism in  $\mathcal{K}_0$  so given  $g : EM \rightarrow EN$ , we can let  $f := g(\text{id}_M)$  and it turns out that  $E(f) = g$ . When  $M$  is not  $\lambda$ -presentable, resolve it into a  $\lambda$ -directed colimit of  $\lambda$ -presentable objects.  $\square$

The second step shows that any coherent abstract class which looks like an accessible category is in fact a  $\mu$ -AEC. First, it is not too hard to show (using resolutions into directed systems again) that only a weak version of the LST axiom suffices:

**Exercise 5.23.** Let  $\mathbf{K}$  be an abstract class satisfying all the axioms of a  $\mu$ -AEC except possibly the LST axiom. Let  $\theta \geq \mu + |\tau(\mathbf{K})|$  be such that:

- (1)  $\theta$  is  $\mu$ -closed.
- (2)  $\text{cf}(\theta) \geq \mu$ .
- (3) For any  $M \in \mathbf{K}$  and any  $A \subseteq |M|$  with  $|A| < \theta$ , there exists  $M_0 \in \mathcal{P}_{\mathbf{K}_{<\theta}}(M)$  with  $A \subseteq |M_0|$ .

Then  $\mathbf{K}$  is a  $\mu$ -AEC with  $\text{LS}(\mathbf{K}) \leq \theta$ .

**Lemma 5.24.** Let  $\mathbf{K}$  be an abstract class satisfying the coherence axiom and let  $\mu$  be a regular cardinal. Assume that  $\mathbf{K}$  is  $\mu$ -accessible and further the  $\mu$ -directed colimits are concrete (given by unions, i.e. they are the same as in  $\text{Emb}(\tau(\mathbf{K}))$ ). Let  $C$  be the set of cardinals  $\lambda$  such that for any  $M \in \mathbf{K}$ ,  $\|M\| < \lambda$  if and only if  $M$  is  $(< \lambda)$ -presentable. Then  $C$  is closed unbounded. In particular,  $\mathbf{K}$  is a  $\mu$ -AEC.

*Proof.*  $C$  is clearly closed. Now given any cardinal  $\lambda$ , there is (up to isomorphism) only a set of  $\lambda^+$ -presentable objects (Theorem 5.18) and only a set of objects of cardinality  $\lambda$ . Thus there is a cardinal  $\lambda'$  such that any  $\lambda^+$ -presentable object has cardinality strictly less than  $\lambda'$  and any object of cardinality at most  $\lambda$  is  $(< \lambda')$ -presentable. Thus given any cardinal  $\lambda_0$ , we can build an increasing sequence  $\langle \lambda_i : i < \omega \rangle$  such that for any  $i < \omega$ , any  $\lambda_i^+$ -presentable object has cardinality strictly less than  $\lambda_{i+1}$  and any object of cardinality  $\lambda_i$  is  $(< \lambda_{i+1})$ -presentable. Now by construction  $\sup_{i < \omega} \lambda_i$  is in  $C$ . Thus  $C$  is unbounded.

To see the “in particular” part, we have to prove the LST axiom. Pick  $\theta \in C$  a limit cardinal such that  $\theta$  is  $\mu$ -closed and  $\text{cf}(\theta) \geq \mu + |\tau(\mathbf{K})|$ . Now let  $M \in \mathbf{K}$  and let  $A \subseteq |M|$  with  $|A| < \theta$  be given. Let  $\theta_0 := ((|A| + \aleph_0)^{<\mu})^+$ . Note that  $\theta_0$  is  $\mu$ -closed so by Theorem 5.16,  $\mathbf{K}$  is  $\theta_0$ -accessible. Thus  $M$  is a  $\theta_0$ -directed colimit of  $\theta_0$ -presentable objects  $\langle M_i : i \in I \rangle$ . Since  $\theta_0$ -directed colimits are concrete, this implies that  $A$  is contained inside some  $M_i$ . Now by definition of  $C$ ,  $M_i$  has cardinality strictly less than  $\theta$ . This shows that the hypotheses of Exercise 5.23 are satisfied.  $\square$

*Proof of Theorem 5.21.* Let  $\mathcal{K}$  be a  $\mu$ -accessible category whose morphisms are monomorphisms. By Lemma 5.22, there is a vocabulary  $\tau$  such that  $\mathcal{K}$  is equivalent to a full subcategory of  $\text{Emb}(\tau)$  which is closed under  $\mu$ -directed colimits inside  $\text{Emb}(\tau)$ . Equivalently, it is closed under  $\mu$ -directed unions. Closing such a category under isomorphism, we obtain an abstract class  $\mathbf{K}$  (the ordering is just substructure) which satisfies the hypotheses of Lemma 5.24, hence is a  $\mu$ -AEC.  $\square$

## 6. $\mu$ -AECs AND INFINITARY LOGICS

Makkai and Paré [MP89, 3.2.3, 3.3.5, 4.3.2] have shown (refining an argument of Rosický) that any  $\lambda$ -accessible category is equivalent to a category of models of an  $\mathbb{L}_{\infty, \lambda}$ -sentence (the morphisms are homomorphisms). In this section, we prove results around that neighborhood for  $\mu$ -AECs.

We first review the following semantic characterization of elementary equivalence.

**Definition 6.1.** Let  $M$  and  $N$  be  $\tau$ -structures. We call  $f$  a *partial isomorphism* from  $M$  to  $N$  if:

- (1)  $f$  is a function from a subset of  $|M|$  to a subset of  $|N|$ .
- (2) For any enumeration  $\bar{a}$  of the domain of  $f$  and any first-order quantifier-free formula  $\phi$ ,  $M \models \phi[\bar{a}]$  if and only if  $N \models \phi[\bar{a}]$ .

**Definition 6.2.** Let  $M$  and  $N$  be  $\tau$ -structures and let  $\theta$  be an infinite cardinal. A  $\theta$ -*forth system* from  $M$  to  $N$  is a set  $\mathcal{F}$  such that:

- (1)  $\mathcal{F} \neq \emptyset$ .
- (2) Any member  $f$  of  $\mathcal{F}$  is a partial isomorphism from  $M$  to  $N$ .
- (3) For any  $f \in \mathcal{F}$ ,  $|\text{dom}(f)| < \theta$ .
- (4) For any  $f \in \mathcal{F}$  and any  $A \subseteq \text{dom}(f)$ ,  $f \upharpoonright A \in \mathcal{F}$ .
- (5) For any  $f \in \mathcal{F}$  and any  $A \subseteq |M|$  with  $|A| < \lambda$ , there exists  $g \in \mathcal{F}$  with  $f \subseteq g$  and  $A \subseteq \text{dom}(g)$ .

We say that  $\mathcal{F}$  is a  $\theta$ -*back and forth system* from  $M$  to  $N$  if it is a  $\theta$ -forth system and  $\{f^{-1} \mid f \in \mathcal{F}\}$  is  $\theta$ -forth system from  $N$  to  $M$ .

We write  $M \equiv_{\infty, \theta}^* N$  if there is a  $\theta$ -back and forth system from  $M$  to  $N$ .

The following result is due to Karp for  $\mathbb{L}_{\infty, \omega}$ , see [Kar65]. A good basic reference on such theorems (and on  $\mathbb{L}_{\infty, \infty}$  in general) is [Dic75].

**Theorem 6.3.** Let  $M$  and  $N$  be  $\tau$ -structures and let  $\theta$  be an infinite cardinal. The following are equivalent:

- (1)  $M \equiv_{\infty, \theta} N$ .
- (2)  $M \equiv_{\infty, \theta}^* N$ .

*Proof.*

- (1) implies (2): Let  $\mathcal{F}$  be the set of partial functions  $f$  from  $|M|$  to  $|N|$  whose domain has cardinality strictly less than  $\theta$ , and such that for any enumeration  $\bar{a}$  of their domain and any  $\mathbb{L}_{\infty, \theta}$ -formula  $\phi$ ,  $M \models \phi[\bar{a}]$  if and only if  $N \models \phi[f(\bar{a})]$ . We claim that  $\mathcal{F}$  is as desired. By symmetry, it suffices to show it is a  $\theta$ -forth system. Since  $M \equiv_{\infty, \theta} N$ , the empty map is in  $\mathcal{F}$ , hence  $\mathcal{F}$  is not empty. Clearly, any member of  $\mathcal{F}$  is a partial isomorphism from  $M$  to  $N$  whose domain has cardinality strictly less than  $\theta$ . If  $f \in \mathcal{F}$  and  $A \subseteq \text{dom}(f)$ , then by definition  $f \upharpoonright A \in \mathcal{F}$ . Now let  $f \in \mathcal{F}$  and let  $A \subseteq |M|$ . Let  $\bar{a}$  be an enumeration of  $A$  and let  $\bar{a}_0$  be an enumeration of  $\text{dom}(f)$ . For a cardinal  $\mu$ , let  $p_\mu$  be the class of formulas  $\psi(\bar{x}, \bar{y}) \in \mathbb{L}_{\mu, \theta}$  such that  $M \models \psi[\bar{a}, \bar{a}_0]$ . We have that  $M \models \exists \bar{x} \bigwedge_{\psi \in p_\mu} \psi[\bar{x}, \bar{a}_0]$ . Thus  $N \models \exists \bar{x} \bigwedge_{\psi \in p_\mu} \psi[\bar{x}, f(\bar{a}_0)]$ . Let  $\bar{b}^\mu$  be a witness. Now  $N$  is a set, so there must exist a proper class  $C$  of cardinals such that  $\mu, \mu' \in C$  implies  $\bar{b} := \bar{b}^\mu = \bar{b}^{\mu'}$ . Let  $g$  send  $\bar{a}$  to  $\bar{b}$ . It is easy to check that this works.
- (2) implies (1): We show that for any  $\mathbb{L}_{\infty, \theta}$ -formula  $\phi(\bar{x})$ , any  $\bar{a} \in {}^{<\theta}M$ , and any  $f \in \mathcal{F}$  whose domain contains  $\bar{a}$ ,  $M \models \phi[\bar{a}]$  if and only if  $N \models \phi[f(\bar{a})]$ . We proceed by induction on  $\phi$ . When  $\phi$  is atomic, this is because  $f$  is a partial isomorphism. When  $\phi$  is a conjunction or negation, this is similarly easy. Assume that  $\phi = \exists \bar{y} \psi(\bar{y}, \bar{x})$ . We show that  $M \models \phi[\bar{a}]$  implies  $N \models \phi[f(\bar{a})]$ , and the converse follows from the symmetric definition of a back and forth system. So let  $\bar{b} \in {}^{<\theta}M$  be such that  $M \models \psi[\bar{b}, \bar{a}]$ . Let  $g \in \mathcal{F}$  extend  $f$  such that the domain of  $g$  contains  $\bar{b}$ . By the induction hypothesis,  $N \models \psi[g(\bar{b}), g(\bar{a})]$ . Thus  $N \models \phi[g(\bar{a})]$ . Since  $g(\bar{a}) = f(\bar{a})$ , we are done.

□

The proof can be refined to yield:

**Exercise 6.4.** Show that if  $\theta$  is regular one can replace (1) by “ $M \equiv_{\lambda, \theta} N$ ”, where  $\lambda := ((2 + \|M\| + \|N\|)^{<\theta})^+$ .

**Exercise 6.5** (Scott). Let  $\theta$  be regular and let  $M$  be a  $\tau$ -structure. Let  $\lambda := ((2 + \|M\|)^{<\theta})^+$ . Show that there exists an  $\mathbb{L}_{\lambda, \theta}$ -sentence  $\phi$  such that for any  $\tau$ -structure  $N$ ,  $N \models \phi$  implies  $M \equiv_{\infty, \theta} N$ .

The following consequence is interesting:

**Corollary 6.6.** Let  $\theta$  be an infinite cardinal of cofinality  $\aleph_0$  and let  $M$  and  $N$  be  $\tau$ -structures of cardinality  $\theta$ . If  $M \equiv_{\infty, \theta} N$ , then  $M \cong N$ .

*Proof.* By Theorem 6.3,  $M \equiv_{\infty, \theta}^* N$ . Let  $\mathcal{F}$  witness it. Write  $|M| = \bigcup_{n < \omega} A_n$ ,  $|N| = \bigcup_{n < \omega} B_n$  with  $|A_n| + |B_n| < \theta$ . This is possible by the cofinality assumption. Finally, build an increasing chain  $\langle f_n : n < \omega \rangle$  of elements of  $\mathcal{F}$  such that  $A_n \subseteq \text{dom}(f_{n+1})$  and  $B_n \subseteq \text{ran}(f_{n+1})$  for all  $n < \omega$ . This is possible since  $\mathcal{F}$  is a  $\theta$ -back and forth system.

□



We can also deduce that AECs are closed under infinitary elementary equivalence. This was observed independently by Kueker [Kue08] and Shelah [She09a, IV.1.11]. First, we prove a lemma:

**Lemma 6.7.** Let  $\mathbf{K}$  be an AEC and let  $M$  be a  $\tau$ -structure. If  $D$  is a set such that:

- (1) For all  $M_0 \in D$ ,  $M_0 \in \mathbf{K}_{\leq \text{LS}(\mathbf{K})}$  and  $M_0 \subseteq M$ .
- (2) For all  $M_0 \in D$  and all  $A \in [M]^{\leq \text{LS}(\mathbf{K})}$ , there is  $M_1 \in D$  such that  $M_0 \leq_{\mathbf{K}} M_1$  and  $A \subseteq |M_1|$ .

Then  $M \in \mathbf{K}$  and  $M_0 \leq_{\mathbf{K}} M$  for all  $M_0 \in D$ .

*Proof.* First we show:

Claim: If  $M_0$  and  $M_1$  are in  $D$ , there exists  $M_2 \in \mathbf{K}$  such that  $M_0 \leq_{\mathbf{K}} M_2$  and  $M_1 \leq_{\mathbf{K}} M_2$ .

Proof of Claim: For  $\ell = 0, 1$ , we build  $\langle M_\ell^i : i < \omega \rangle \leq_{\mathbf{K}}$ -increasing in  $D$  such that  $M_\ell^0 = M_\ell$  and  $|M_{1-\ell}^i| \subseteq |M_\ell^{i+1}|$  for all  $i < \omega$ . This is possible by the assumptions on  $D$ . Now let  $M_2 := \bigcup_{i < \omega} M_0^i = \bigcup_{i < \omega} M_1^i$ .  $\dagger_{\text{Claim}}$

Now we build  $\langle M_s : s \in [M]^{\leq \text{LS}(\mathbf{K})} \rangle$  a sequence of models in  $D$  such that  $s \subseteq t$  implies  $|M_s| \subseteq |M_t|$  and  $s \subseteq |M_s|$  for all  $s, t \in [M]^{\leq \text{LS}(\mathbf{K})}$ . This is possible by the assumptions on  $D$ . Now let  $s, t \in [M]^{\leq \text{LS}(\mathbf{K})}$  be such that  $s \subseteq t$ . Then  $|M_s| \subseteq |M_t|$  and by the claim, there is  $M' \in \mathbf{K}$  such that  $M_s \leq_{\mathbf{K}} M'$  and  $M_t \leq_{\mathbf{K}} M'$ . By coherence, this implies that  $M_s \leq_{\mathbf{K}} M_t$ . Thus  $\langle M_s : s \in [M]^{\leq \text{LS}(\mathbf{K})} \rangle$  is a directed system in  $\mathbf{K}$  whose union is  $M$ , so  $M \in \mathbf{K}$  and it follows from the proof that  $M_0 \leq_{\mathbf{K}} M$  for all  $M_0 \in D$ .  $\square$

**Theorem 6.8.** Let  $\mathbf{K}$  be an AEC and let  $M \in \mathbf{K}$ . Let  $N$  be a  $\tau(\mathbf{K})$ -structure. If  $M \equiv_{\infty, \text{LS}(\mathbf{K})^+} N$ , then  $N \in \mathbf{K}$ .

*Proof.* By Theorem 6.3, there is an  $\text{LS}(\mathbf{K})^+$ -back and forth system  $\mathcal{F}$  from  $M$  to  $N$ . Let

$$D := \{f[M_0] \mid f \in \mathcal{F}, M_0 \in \mathcal{P}_{\mathbf{K}_{\leq \text{LS}(\mathbf{K})}}(M)\}$$

It suffices to observe that  $D$  satisfies the hypotheses of Lemma 6.7 (where  $M$  there is  $N$  here). Indeed, by closure of  $\mathbf{K}$  under isomorphisms, any member of  $D$  is a member of  $\mathbf{K}_{\leq \text{LS}(\mathbf{K})}$ . Moreover if  $f[M_0] \in D$  and  $A \in [N]^{\leq (\text{LS}(\mathbf{K}))}$ , we can use the axioms of back and forth to extend  $f$  to  $g$  whose range contains  $A$ , and moreover  $M_1 := \text{dom}(g) \leq_{\mathbf{K}} M$ . By coherence,  $M_0 \leq_{\mathbf{K}} M_1$ . By closure of  $\leq_{\mathbf{K}}$  under isomorphisms,  $f[M_0] \leq_{\mathbf{K}} g[M_1]$ , and by definition  $g[M_1] \in D$ .  $\square$

**Question 6.9.** Does Theorem 6.8 generalize to  $\mu$ -AECs?

To better understand the relationship between infinitary logics and  $\mu$ -AECs, the following concept is useful. The idea is to expand the  $\mu$ -AECs with predicate that “do not add any information” in the sense that the expansion is already uniquely determined by the structure. The definition appears in [Vas16, 3.1].

**Definition 6.10.** Let  $\mathbf{K}$  be an abstract class. A *functorial expansion* of  $\mathbf{K}$  is an abstract class  $\mathbf{K}^+$  in a vocabulary  $\tau(\mathbf{K}^+)$  expanding  $\tau(\mathbf{K})$  such that the reduct map is an isomorphism of category from  $\mathbf{K}^+$  onto  $\mathbf{K}$ . That is:

- (1) If  $M^+ \leq_{\mathbf{K}^+} N^+$ , then  $M^+ \upharpoonright \tau(\mathbf{K}) \leq_{\mathbf{K}} N^+ \upharpoonright \tau(\mathbf{K})$ .
- (2) If  $M \in \mathbf{K}$ , there is a unique expansion  $M^+ \in \mathbf{K}^+$  such that  $M^+ \upharpoonright \tau(\mathbf{K}) = M$ .
- (3) If  $f : M \rightarrow N$  is a  $\mathbf{K}$ -embedding then the induced map  $f^+ : M^+ \rightarrow N^+$  also is.

We call a functorial expansion  $(< \mu)$ -ary if its vocabulary is  $(< \mu)$ -ary.

**Remark 6.11.** If  $\mathbf{K}^+$  is a functorial expansion of  $\mathbf{K}$ , then  $M^+ \leq_{\mathbf{K}^+} N^+$  holds if and only if  $M^+ \upharpoonright \tau(\mathbf{K}) \leq_{\mathbf{K}} N^+ \upharpoonright \tau(\mathbf{K})$ . Thus a functorial expansion is entirely determined by its class of models.

**Remark 6.12.** If  $\mathbf{K}^+$  is a  $(< \mu)$ -ary functorial expansion of a  $\mu$ -AEC  $\mathbf{K}$ , then  $\mathbf{K}^+$  is a  $\mu$ -AEC with  $\text{LS}(\mathbf{K}^+) = \text{LS}(\mathbf{K})$ .

**Example 6.13.**

- (1)  $\mathbf{K}$  is a functorial expansion of  $\mathbf{K}$ .
- (2) If  $\mathbf{K}$  is an elementary class (ordered with elementary substructure), we can add a relation symbol for each first-order formula and obtain a functorial expansion, called the *Morleyization* of  $\mathbf{K}$ .
- (3) The expansion given by Shelah's presentation Theorem 3.6 is not functorial (unless the starting class is a universal class itself). This is because the reduct functor is not necessarily full.

Another example of a functorial expansion, to be defined later, is the *orbital (or Galois) Morleyization*, which consists in adding a relation symbol for each orbital type. In this section, the following functorial expansion will play an important role:

**Definition 6.14.** Let  $\mathbf{K}$  be a  $\mu$ -AEC. The *substructure functorial expansion* of  $\mathbf{K}$  is the abstract class  $\mathbf{K}^+$  defined as follows:

- (1)  $\tau(\mathbf{K}^+) = \tau(\mathbf{K}) \cup \{P\}$ , where  $P$  is an  $\text{LS}(\mathbf{K})$ -ary predicate.
- (2)  $M^+ \in \mathbf{K}^+$  if and only if  $M^+ \upharpoonright \tau(\mathbf{K}) \in \mathbf{K}$  and for any  $\bar{a} \in {}^{\text{LS}(\mathbf{K})}M^+$ ,  $P^{M^+}(\bar{a})$  holds if and only if  $\text{ran}(\bar{a}) \leq_{\mathbf{K}} M^+ \upharpoonright \tau(\mathbf{K})$ , where we see  $\text{ran}(\bar{a})$  as a  $\tau(\mathbf{K})$ -structure.
- (3) For  $M^+, N^+ \in \mathbf{K}^+$ ,  $M^+ \leq_{\mathbf{K}^+} N^+$  if and only if  $M^+ \upharpoonright \tau(\mathbf{K}) \leq_{\mathbf{K}} N^+ \upharpoonright \tau(\mathbf{K})$ .

**Exercise 6.15.** Check that the substructure functorial expansion is indeed a functorial expansion.

The substructure functorial expansion has a number of nice properties.

**Definition 6.16.** We call an abstract class  $\mathbf{K}$  *model-complete* if for  $M, N \in \mathbf{K}$ ,  $M \leq_{\mathbf{K}} N$  if and only if  $M \subseteq N$ .

Note that a model complete abstract class does *not* have to be closed under substructure (the class of algebraically closed fields is one example).

The following criteria to prove model-completeness is a directed system argument:

**Exercise 6.17.** Let  $\mathbf{K}$  be a  $\mu$ -AEC and let  $M, N \in \mathbf{K}$ . Suppose that  $M \subseteq N$ . The following are equivalent:

- (1)  $M \leq_{\mathbf{K}} N$ .
- (2) For any  $M_0 \in \mathcal{P}_{\mathbf{K}_{\leq \text{LS}(\mathbf{K})}}(M)$ ,  $M_0 \leq_{\mathbf{K}} N$ .

The substructure functorial expansion is model-complete:

**Theorem 6.18.** Let  $\mathbf{K}$  be a  $\mu$ -AEC. Then the substructure functorial expansion of  $\mathbf{K}$  is model-complete.

*Proof.* Let  $\mathbf{K}^+$  be the substructure functorial expansion of  $\mathbf{K}$ . For  $M \in \mathbf{K}$ , write  $M^+$  for the expansion of  $M$  to  $\mathbf{K}^+$ . Let  $M, N \in \mathbf{K}$  and assume that  $M^+ \subseteq N^+$ . We have to see that  $M \leq_{\mathbf{K}} N$ . For this, we use the equivalent condition of Exercise 6.17. Let  $M_0 \in \mathcal{P}_{\mathbf{K}_{\leq \text{LS}(\mathbf{K})}}(M)$ . We have to see that  $M_0 \leq_{\mathbf{K}} N$ . Let  $\bar{a}$  be an enumeration of  $M_0$ . We have that  $M^+ \models P[\bar{a}]$  (where  $P$  is the additional predicate in  $\tau(\mathbf{K})^+$ ), so  $N^+ \models P[\bar{a}]$  (as  $M^+$  is a substructure of  $N^+$ ). This means that  $M_0 \leq_{\mathbf{K}} N$ , as desired.  $\square$

The substructure functorial expansion of a  $\mu$ -AEC can be axiomatized (a variation of this is due to Baldwin and Boney [BB17]). Since the ordering is trivial by the previous result, this gives that any  $\mu$ -AEC is isomorphic (as a category) to the category of models of an  $\mathbb{L}_{\infty, \infty}$  sentence, where the morphisms are injective homomorphisms.

**Theorem 6.19.** Let  $\mathbf{K}$  be a  $\mu$ -AEC and let  $\mathbf{K}^+$  be its substructure functorial expansion. There is an  $\mathbb{L}_{(2^{\text{LS}(\mathbf{K})})^+, \text{LS}(\mathbf{K})^+}$  sentence  $\phi$  such that  $\mathbf{K}^+$  is the class of models of  $\phi$ .

*Proof.* First note that for each  $M_0 \in \mathbf{K}_{\leq \text{LS}(\mathbf{K})}$ , there is a sentence  $\psi_{M_0}(\bar{x})$  of  $\mathbb{L}_{\text{LS}(\mathbf{K})^+, \text{LS}(\mathbf{K})^+}$  coding its isomorphism type, i.e. whenever  $M \models \phi[\bar{a}]$ , then  $\bar{a}$  is an enumeration of an isomorphic copy of  $M_0$ . Similarly, whenever  $M_0, M_1$  are in  $\mathbf{K}_{\leq \text{LS}(\mathbf{K})}$  with  $M_0 \leq_{\mathbf{K}} M_1$ , there is  $\psi_{M_0, M_1}(\bar{x}, \bar{y})$  that codes that  $(\bar{x}, \bar{y})$  is isomorphic to  $(M_0, M_1)$  (so in particular  $\bar{x} \leq_{\mathbf{K}} \bar{y}$ ). Let  $S$  be a complete set of members of  $\mathbf{K}_{\leq \text{LS}(\mathbf{K})}$  (i.e. any other model is isomorphic to it) and let  $T$  be a complete set of pairs  $(M_0, M_1)$ , with each in  $\mathbf{K}_{\leq \text{LS}(\mathbf{K})}$ , such that  $M_0 \leq_{\mathbf{K}} M_1$ . Now define the following:

$$\begin{aligned} \phi_1 &= \forall \bar{x} \exists \bar{y} \left( \left( \bigvee_{M_0 \in S} \psi_{M_0}(\bar{y}) \right) \wedge \bar{x} \subseteq \bar{y} \wedge P(\bar{y}) \right) \\ \phi_2 &= \forall \bar{x} \forall \bar{y} \left( (\bar{x} \subseteq \bar{y} \wedge P(\bar{x}) \wedge P(\bar{y})) \rightarrow \bigvee_{(M_0, M_1) \in T} \psi_{M_0, M_1}(\bar{x}, \bar{y}) \right) \\ \phi &= \phi_1 \wedge \phi_2 \end{aligned}$$

Where  $\bar{x} \subseteq \bar{y}$  abbreviates the obvious formula. This works. First, any  $M^+ \in \mathbf{K}^+$  satisfies  $\phi_1$  by the LST axiom and satisfies  $\phi_2$  by the coherence axiom. Conversely, if  $M \models \phi$ , then we can build a  $\mu$ -directed system  $\langle M_s : s \in [M]^{<\mu} \rangle$  in  $\mathbf{K}$  such that

$s \subseteq |M_s|$  and  $M_s \in \mathbf{K}_{\leq \text{LS}(\mathbf{K})}$  for all  $s \in [M]^{<\mu}$ . We then get that  $\bigcup_{s \in [M]^{<\mu}} M_s = M \in \mathbf{K}$  by closure under  $\mu$ -directed systems. A similar directed system argument shows that  $M^{M^+}(\bar{a})$  holds if and only if  $\text{ran}(\bar{a}) \leq_{\mathbf{K}} M$ , so  $M^+ \in \mathbf{K}^+$ .  $\square$

The following shows that elementary equivalence is preserved when passing to functorial expansions of AECs. This is because back and forth systems are preserved:

**Lemma 6.20.** Let  $\mathbf{K}$  be an AEC. let  $\mathbf{K}^+$  be a  $(< \text{LS}(\mathbf{K})^+)$ -ary functorial expansion of  $\mathbf{K}$ . Let  $M, N \in \mathbf{K}$  and let  $M^+, N^+$  be their respective expansions to  $\mathbf{K}^+$ . If  $\mathcal{F}$  is an  $\text{LS}(\mathbf{K})^+$ -back and forth system from  $M$  to  $N$ , then it is an  $\text{LS}(\mathbf{K})^+$ -back and forth system from  $M^+$  to  $N^+$ .

*Proof.* For any  $M_0 \in \mathbf{K}$ , write  $M_0^+$  for its expansion to  $\mathbf{K}^+$ . Let  $f \in \mathcal{F}$ . Using the axioms of a back and forth system and the LST axiom, one can pick  $g \in \mathcal{F}$  such that  $f \subseteq g$  and  $M_0 := \text{dom}(g) \leq_{\mathbf{K}} M$ . Let  $N_0 := g[M_0]$ . Since  $M_0 \cong N_0$ ,  $N_0 \in \mathbf{K}$ . Moreover by the proof of Theorem 6.8,  $N_0 \leq_{\mathbf{K}} N$ . Now by definition of a functorial expansion, we must have  $M_0^+ \leq_{\mathbf{K}^+} M^+$  and  $N_0^+ \leq_{\mathbf{K}^+} N^+$  and moreover  $g$  is a  $\mathbf{K}^+$ -isomorphism. It follows that  $f$  is itself a partial isomorphism from  $M^+$  to  $N^+$ . Since  $f$  was arbitrary, this shows that  $\mathcal{F}$  is indeed a back and forth system from  $M^+$  to  $N^+$ .  $\square$

As a consequence, we deduce a relationship between the ordering of the class and infinitary elementary equivalence:

**Theorem 6.21.** Let  $\mathbf{K}$  be an AEC. Let  $M \in \mathbf{K}$ . If  $M \preceq_{\mathbb{L}_{\infty, \text{LS}(\mathbf{K})^+}} N$ , then  $M \leq_{\mathbf{K}} N$ .

*Proof.* By Theorem 6.8,  $N \in \mathbf{K}$ . We use Exercise 6.17. Let  $M_0 \in \mathcal{P}_{\mathbf{K}_{\leq \text{LS}(\mathbf{K})}}(M)$ . Let  $\bar{a}$  be an enumeration of  $M_0$ . We have that  $(M, \bar{a}) \equiv_{\infty, \text{LS}(\mathbf{K})^+} (N, \bar{a})$ . By Theorem 6.3, there is an  $\text{LS}(\mathbf{K})^+$ -back and forth system  $\mathcal{F}$  from  $(M, \bar{a})$  to  $(N, \bar{a})$ . By Lemma 6.20 it is also a back and forth system from  $M^+$  to  $N^+$ , and hence it is easy to check from  $(M^+, \bar{a})$  to  $(N^+, \bar{a})$ , where  $M^+$  and  $N^+$  denote the expansions of  $M$  and  $N$  in the substructure functorial expansion. By Theorem 6.3 again, this implies that  $P^{M^+}(\bar{a})$  holds if and only if  $P^{N^+}(\bar{a})$  holds. Since  $M_0 \leq_{\mathbf{K}} M$ , we have that  $P^{M^+}(\bar{a})$ , so  $P^{N^+}(\bar{a})$ , so  $M_0 \leq_{\mathbf{K}} N$ , as desired.  $\square$

There are converses to Theorem 6.21 when  $M$  and  $N$  are sufficiently saturated. For example, in a first-order theory  $T$ , if  $M$  and  $N$  are saturated of cardinality  $\lambda$  and  $M \preceq N$ , then  $M \preceq_{\mathbb{L}_{\infty, \lambda}} N$  (exercise). The following beautiful argument of Shelah uses Fodor's lemma to provide some kind of analog even when there is no obvious notion of saturated (see [BGL<sup>+</sup>16, 6.8] for a generalization to certain  $\mu$ -AECs).

**Theorem 6.22** (Shelah, [She09a, IV.1.12(1)]). Let  $\mathbf{K}$  be an AEC, let  $\theta$  be regular and let  $\lambda = \lambda^{<\theta} \geq \text{LS}(\mathbf{K})$ . Assume that  $\mathbf{K}$  is categorical in  $\lambda$  and let  $M, N \in \mathbf{K}_{\geq \lambda}$ . If  $M \leq_{\mathbf{K}} N$ , then  $M \preceq_{\mathbb{L}_{\infty, \theta}} N$ .

*Proof.* A directed systems argument (exercise) establishes that it suffices to prove it when  $M, N \in \mathbf{K}_{\lambda}$ . We now prove by induction on  $\phi(\bar{x}) \in \mathbb{L}_{\infty, \theta}$  that for any  $M, N \in \mathbf{K}_{\lambda}$  with  $M \leq_{\mathbf{K}} N$  and any  $\bar{a} \in {}^{<\theta}M$ ,  $M \models \phi[\bar{a}]$  if and only if  $N \models \phi[\bar{a}]$ . This is easy when  $\psi$  is atomic (since  $\leq_{\mathbf{K}}$  extends substructure) and when  $\phi$  is

a conjunction or a negation. We prove what happens when  $\phi = \exists \bar{y} \psi(\bar{x}, \bar{y})$ . If  $M \models \phi[\bar{a}]$ , then  $N \models \phi[\bar{a}]$  as well. Now suppose that  $N \models \phi[\bar{a}]$ . We build an increasing continuous chain  $\langle M_i : i < \lambda^+ \rangle$  and  $\langle f_i : i < \lambda^+ \rangle$  such that for all  $i < \lambda^+$ :

- (1)  $M_i \in \mathbf{K}_\lambda$ .
- (2)  $f_i : N \cong M_{i+1}$  is such that  $f_i[M] = M_i$ .

This is possible by categoricity in  $\lambda$  and some renaming. Now let  $\bar{a}_i := f_i(\bar{a})$ . Note that since  $\bar{a} \in {}^{<\theta}M$ , we have that  $\bar{a}_i \in {}^{<\theta}M_i$ . Let  $S := \{i < \lambda^+ \mid \text{cf}(i) \geq \theta\}$ . This is a stationary set, and for each  $i \in S$ , there exists  $j_i < i$  such that  $\bar{a}_i \in {}^{<\theta}M_{j_i}$ . Thus the map  $i \mapsto j_i$  is regressive so by Fodor's lemma there exists  $S_0 \subseteq S$  stationary and  $j < \lambda^+$  such that for any  $i \in S_0$ ,  $j_i = j$ . Since  $\lambda = \lambda^{<\theta}$  and  $|S_0| = \lambda^+$ , there exists  $\bar{a}' \in {}^{<\theta}M_j$  and  $S_1 \subseteq S_0$  of cardinality  $\lambda^+$  and such that  $i \in S_1$  implies  $\bar{a}_i = \bar{a}'$ . Let  $i \in S_1$ . Since  $N \models \phi[\bar{a}]$ , we have (applying  $f_i$ ) that  $M_{i+1} \models \phi[\bar{a}']$ . Thus there exists  $\bar{b} \in {}^{<\theta}M_{i+1}$  such that  $M_{i+1} \models \psi[\bar{b}, \bar{a}']$ . Pick  $i' \in S_1$  such that  $i + 1 < i'$ . By the induction hypothesis,  $M_{i'} \models \psi[\bar{b}, \bar{a}']$ . Applying  $f_{i'}^{-1}$  to this statement (and the definition of  $S_1$ ),  $M \models \psi[f_{i'}^{-1}(\bar{b}), \bar{a}]$ , hence  $M \models \phi[\bar{a}]$ , as desired.  $\square$

## 7. ORBITAL TYPES

In any abstract class, one can define a semantic notion of type (loosely, this is the finest possible notion of types that preserves  $\mathbf{K}$ -embeddings). They were introduced by Shelah [She87b]. The name “Galois type” is used a lot in the literature, but we prefer Shelah's terminology of “orbital type” for reasons that will soon become apparent.

**Definition 7.1.** Let  $\mathbf{K}$  be an abstract class. We define an equivalence relation  $\equiv (= \equiv^{\mathbf{K}})$  on pairs  $(\bar{a}, M)$ , where  $M \in \mathbf{K}$  and  $\bar{a} \in {}^{<\infty}M$  as follows:  $\equiv$  is the intersection of all equivalence relations  $E$  on such pairs satisfying:

If  $f : M \rightarrow N$  is a  $\mathbf{K}$ -embedding, then  $(\bar{a}, M)E(f(\bar{a}), N)$ .

For  $N_1, N_2 \in \mathbf{K}$ ,  $A \subseteq N_1 \cap N_2$ , and  $\bar{b}_\ell \in {}^{<\infty}N_\ell$ , we write  $(\bar{a}_1, N_1) \equiv_A (\bar{a}_2, N_2)$  if for some (equivalently, any) enumeration  $\bar{a}$  of  $A$ ,  $(\bar{a}_1 \bar{a}, N_1) \equiv (\bar{a}_2 \bar{a}, N_2)$ . For  $N \in \mathbf{K}$ ,  $\bar{b} \in {}^{<\infty}N$  and  $A \subseteq |N|$ , we let  $\mathbf{tp}(\bar{b}/A; N)$  denote the  $\equiv_A$ -equivalence class of  $(\bar{b}, N)$ . When  $\mathbf{K}$  is not clear from context, we may write  $\mathbf{tp}_{\mathbf{K}}(\bar{b}/A; N)$ .

A more explicit definition is:

**Exercise 7.2.** Let  $\mathbf{K}$  be an abstract class. Show that  $\equiv^{\mathbf{K}}$  is the transitive closure of the relation  $E_{\text{at}}$  defined by  $(\bar{b}_1, N_1)E_{\text{at}}(\bar{b}_2, N_2)$  if and only if there exists  $N \in \mathbf{K}$ ,  $f_\ell : N_\ell \rightarrow N$  such that  $f_1(\bar{b}_1) = f_2(\bar{b}_2)$ .

From this and a diagram chase, we obtain an easier definition for abstract classes with amalgamation:

**Exercise 7.3.** Let  $\mathbf{K}$  be an abstract class with amalgamation. Show that  $\equiv^{\mathbf{K}} = E_{\text{at}}$ , where  $E_{\text{at}}$  is defined in the previous exercise. Deduce that  $\mathbf{tp}(\bar{b}_1/A; N_1) = \mathbf{tp}(\bar{b}_2/A; N_2)$  if and only if there exists  $N \in \mathbf{K}$  and  $f_\ell : N_\ell \xrightarrow{A} N$  such that  $f_1(\bar{b}_1) = f_2(\bar{b}_2)$ .

One can also prove an easier characterization in AECs with intersections:

**Exercise 7.4.** Let  $\mathbf{K}$  be an AEC with intersections. Show that  $\mathbf{tp}(\bar{b}_1/A; N_1) = \mathbf{tp}(\bar{b}_2/A; N_2)$  if and only if there exists  $f : \text{cl}^{N_1}(A\bar{b}_1) \cong_A \text{cl}^{N_2}(A\bar{b}_2)$  such that  $f(\bar{b}_1) = \bar{b}_2$ . *Hint: first show that  $(\bar{b}_1, N_1)E_{at}(\bar{b}_2, N_2)$  implies there is  $f : \text{cl}^{N_1}(\bar{b}_1) \cong \text{cl}^{N_2}(\bar{b}_2)$  sending  $\bar{b}_1$  to  $\bar{b}_2$ , then use Exercise 7.2.*

**Example 7.5.**

- (1) Let  $\mathbf{K}$  be an elementary class (ordered by elementary substructure). Then orbital types coincide with the usual syntactic types. More precisely, if  $N_1, N_2 \in \mathbf{K}$ ,  $A \subseteq N_1 \cap N_2$ ,  $\bar{b}_\ell \in {}^{<\infty}N_\ell$ , the following are equivalent:
  - (a)  $\mathbf{tp}(\bar{b}_1/A; N_1) = \mathbf{tp}(\bar{b}_2/A; N_2)$ .
  - (b) For any  $\mathbb{L}_{\omega, \omega}$  formula  $\phi$ ,  $N_1 \models \phi[\bar{b}_1]$  if and only if  $N_2 \models \phi[\bar{b}_2]$ .
 This follows from Exercise 2.10. In particular, orbital types are exactly orbits of the monster model under the action of its automorphism group. We will soon generalize this last fact to any AEC with amalgamation.
- (2) Let  $\mathbf{K}$  be a universal class. By Exercise 7.4 and Theorem 2.11, orbital types are exactly the same as the quantifier-free types.

It will be convenient to have some notation to talk about orbital types.

**Definition 7.6.** Let  $\mathbf{K}$  be an abstract class.

- (1) Let  $N \in \mathbf{K}$ ,  $A \subseteq |N|$ , and  $\alpha$  be an ordinal. Define:

$$\mathbf{S}^\alpha(A; N) := \{\mathbf{tp}(\bar{b}/A; N) \mid \bar{b} \in {}^\alpha |N|\}$$

- (2) For  $M \in \mathbf{K}$  and  $\alpha$  an ordinal, let:

$$\mathbf{S}^\alpha(M) := \{p \mid \exists N \in \mathbf{K} : M \leq_{\mathbf{K}} N \text{ and } p \in \mathbf{S}^\alpha(M; N)\}$$

- (3) For  $\alpha$  an ordinal, let:

$$\mathbf{S}^\alpha(\emptyset) := \bigcup_{N \in \mathbf{K}} \mathbf{S}^\alpha(\emptyset; N)$$

When  $\alpha = 1$ , we omit it. Similarly define  $\mathbf{S}^{<\alpha}$ , where  $\alpha$  is allowed to be  $\infty$ . When  $\mathbf{K}$  is not clear from context, we may write  $\mathbf{S}_{\mathbf{K}}^\alpha$ , etc.

**Remark 7.7.** When  $\alpha$  is an ordinal,  $\mathbf{S}^\alpha(M)$  and  $\mathbf{S}^\alpha(\emptyset)$  could a priori be proper classes. However in reasonable cases (e.g. when  $\mathbf{K}$  is a  $\mu$ -AEC) they are sets. For example when  $\mathbf{K}$  is a  $\mu$ -AEC, an upper bound for  $|\mathbf{S}^\alpha(M)|$  is  $2^{(\|M\| + \alpha + \text{LS}(\mathbf{K}))^{<\mu}}$ .

**Definition 7.8.** Let  $\mathbf{K}$  be an abstract class and let  $p$  be an orbital type.

- (1) Let  $\ell(p)$  and  $\text{dom}(p)$  be the unique  $\alpha$  and  $A$  such that there exists  $N \in \mathbf{K}$  so that  $p \in \mathbf{S}^\alpha(A; N)$ .
- (2) We say that  $p$  is *realized in  $N$  (by  $\bar{b}$ )* if  $p = \mathbf{tp}(\bar{b}/\text{dom}(p); N)$ . Similarly define type omission.
- (3) For  $A \subseteq \text{dom}(p)$ , we let  $p \upharpoonright A$  be  $\mathbf{tp}(\bar{b}/A; N)$  for some (any)  $\bar{b}$  and  $N$  such that  $p$  is realized by  $\bar{b}$  in  $N$ .
- (4) We say that an orbital type  $q$  is an *extension* of  $p$  if  $\text{dom}(p) \subseteq \text{dom}(q)$  and  $q \upharpoonright \text{dom}(p) = p$ .

- (5) If  $p = \text{tp}(\bar{b}/M; N)$ ,  $M \leq_{\mathbf{K}} N$ , and  $f : M \cong M'$ , we let  $f(p)$  be  $\text{tp}(g(\bar{b})/M'; N')$  for some (any) extension  $g : N \cong N'$  of  $f$ .

**7.1. Model-homogeneous and universal models.** Even without a notion of type, one can make the following definitions:

**Definition 7.9.** Let  $\mathbf{K}$  be an abstract class, let  $M \in \mathbf{K}$ , and let  $\lambda$  be an infinite cardinal.

- (1)  $M$  is  $\lambda$ -universal if any  $N \in \mathbf{K}_{<\lambda}$   $\mathbf{K}$ -embeds into  $M$ . When  $\lambda = \|M\|^+$ , we omit it.
- (2)  $M$  is  $\lambda$ -model-homogeneous if for any  $M_0 \leq_{\mathbf{K}} N_0$  both in  $\mathbf{K}_{<\lambda}$ , if  $M_0 \leq_{\mathbf{K}} M$  then there exists  $f : N_0 \xrightarrow{M_0} N$ . When  $\lambda = \|M\|$ , we omit it.

Let us note for later use that there is a weaker definition of being model-homogeneous which suffices:

**Exercise 7.10.** Let  $\mathbf{K}$  be an AEC with amalgamation. Let  $M \in \mathbf{K}$  and let  $\lambda > \text{LS}(\mathbf{K})$ . The following are equivalent:

- (1)  $M$  is  $\lambda$ -model-homogeneous.
- (2) For any  $M_0 \in \mathcal{P}_{\mathbf{K}_{<\lambda}}(M)$  and any  $N_0 \in \mathbf{K}_{\|M_0\|+\text{LS}(\mathbf{K})}$  with  $M_0 \leq_{\mathbf{K}} N_0$ , there exists  $f : N_0 \xrightarrow{M_0} N$ .
- (3) For any  $M_0 \in \mathcal{P}_{\mathbf{K}_{<\lambda}}(M)$  and any  $N_0 \in \mathbf{K}_{\leq\lambda}$  with  $M_0 \leq_{\mathbf{K}} N_0$ , there exists  $f : N_0 \xrightarrow{M_0} N$ .

In an AEC with amalgamation and joint embedding, it is reasonably easy to create such models via a general exhaustion argument:

**Exercise 7.11.** Let  $\mathbf{K}$  be an AEC with amalgamation and let  $M \in \mathbf{K}$ . Let  $\lambda > \text{LS}(\mathbf{K})$ .

- (1) For any  $\theta \geq \|M\| + 2$  with  $\theta = \theta^{<\lambda}$ , there exists a  $\lambda$ -model-homogeneous  $N \in \mathbf{K}$  with  $M \leq_{\mathbf{K}} N$ .
- (2) If  $\mathbf{K}$  has joint embedding, any  $\lambda$ -model-homogeneous model is  $\lambda^+$ -universal.

Moreover, the model-homogeneous universal model is unique (in a fixed cardinality) if it exists:

**Exercise 7.12.** Let  $\mathbf{K}$  be an AEC with amalgamation. Let  $M, N \in \mathbf{K}$  be model-homogeneous of the same cardinality  $\lambda > \text{LS}(\mathbf{K})$ . Let  $M_0 \in \mathcal{P}_{\mathbf{K}_{<\lambda}}(M)$  and let  $f : M_0 \rightarrow N$ . Then there exists an isomorphism  $g : M \cong N$  extending  $f$ .

Let us call a *monster model* in an AEC  $\mathbf{K}$  a proper class-sized  $\tau(\mathbf{K})$ -structure  $\mathfrak{C}$  such that there exists  $\langle \mathfrak{C}_i : i \in \text{OR} \rangle$  increasing in  $\mathbf{K}$  with  $\mathfrak{C}_i$   $(|i| + \text{LS}(\mathbf{K})^+)$ -model-homogeneous and  $(|i| + \text{LS}(\mathbf{K})^+)$ -universal for all  $i \in \text{OR}$ . Note that if it exists,  $\mathfrak{C}$  must be unique up to isomorphism. We abuse notation and think of  $\mathfrak{C}$  as a member of  $\mathbf{K}$ .

**Exercise 7.13.** Let  $\mathbf{K}$  be an AEC. Then  $\mathbf{K}$  has a monster model if and only if  $\mathbf{K}$  has amalgamation, joint embedding, and arbitrarily large models.

Orbital types are actually orbits (under the action of an automorphism group) when their equality is computed inside a model-homogeneous model (in particular in the monster model).

**Exercise 7.14.** Let  $\mathbf{K}$  be an AEC with amalgamation. Let  $M \in \mathbf{K}$  be model-homogeneous and let  $\bar{b}_1, \bar{b}_2 \in {}^\alpha M$  with  $\alpha < \|M\|$ . Then  $\mathbf{tp}(\bar{b}_1/\emptyset; M) = \mathbf{tp}(\bar{b}_2/\emptyset; M)$  if and only if there is an automorphism of  $M$  sending  $\bar{b}_1$  to  $\bar{b}_2$ .

**7.2. Model-homogeneous is equivalent to saturated.** Using orbital types, one can define a notion related to being model-homogeneous:

**Definition 7.15.** Let  $\mathbf{K}$  be an AEC with amalgamation, let  $M \in \mathbf{K}$  and let  $\lambda > \text{LS}(\mathbf{K})$ . We say that  $M$  is  $\lambda$ -saturated if for any  $M_0 \in \mathcal{P}_{\mathbf{K}_{<\lambda}}(M)$ , any  $p \in \mathbf{S}(M_0)$  is realized inside  $M$ .

**Exercise 7.16.** Show that in an AEC with amalgamation, any  $\lambda$ -model-homogeneous model is  $\lambda$ -saturated.

We will prove the following converse, due to Shelah [She09a, II.1.14] (originally proven in [She87b]). This provides some justification for using orbital types, as it tells us that model-homogeneous models can be built “element by element”.

**Theorem 7.17.** Let  $\mathbf{K}$  be an AEC with amalgamation. Let  $\lambda > \text{LS}(\mathbf{K})$  and let  $M \in \mathbf{K}$ . If  $M$  is  $\lambda$ -saturated, then  $M$  is  $\lambda$ -model-homogeneous.

*Proof.* By Exercise 7.10, it suffices to show that for all  $M_0 \in \mathcal{P}_{\mathbf{K}_{<\lambda}}(M)$  and all  $N \in \mathbf{K}_{\|M_0\|+\text{LS}(\mathbf{K})}$  with  $M_0 \leq_{\mathbf{K}} N$ , there is  $f : N \xrightarrow{M_0} M$ . Let  $\mu := \|N\| + \text{LS}(\mathbf{K})$  and let  $\langle a_i : i < \mu \rangle$  be an enumeration of  $|N|$  (possibly with repetitions). We build  $\langle N_i^0 : i \leq \mu \rangle$ ,  $\langle N_i^1 : i \leq \mu \rangle$  increasing continuous in  $\mathbf{K}_{\leq \mu}$  and  $\langle f_i : i \leq \mu \rangle$  increasing continuous such that for all  $i < \mu$ :

- (1)  $f_i : N_i^0 \rightarrow M$ .
- (2)  $N_0^0 = M_0$ ,  $N_0^1 = N$ ,  $f_0 = \text{id}_{M_0}$ .
- (3)  $N_i^0 \leq_{\mathbf{K}} N_i^1$ .
- (4)  $a_i \in N_{i+1}^0$ .

This is enough: By (4), we have that  $|N| \subseteq |N_\mu^0|$ . Since  $N \leq_{\mathbf{K}} N_\mu^1$  and  $N_\mu^0 \leq_{\mathbf{K}} N_\mu^1$ , coherence implies that  $N \leq_{\mathbf{K}} N_\mu^0$ . Let  $f := f_\mu \upharpoonright N$ . Then  $f$  is the desired  $\mathbf{K}$ -embedding of  $N$  inside  $M$  fixing  $M_0$ .

This is possible: The base case has already been specified and at limits we take unions. Suppose now that  $i = j + 1$  and stage  $j$  has been implemented. Since  $N \leq_{\mathbf{K}} N_j^1$ ,  $a_i \in N_j^1$ . Let  $q_i := \mathbf{tp}(a_i/N_j^0; N_j^1)$ . Let  $M_j := f_j[N_j^0]$  and let  $g : N_j^1 \cong M_j'$  be an extension of  $f_j$ . Let  $p_i := \mathbf{tp}(g(a_i)/M_j; M_j')$  (so  $p_i = f(q_i)$ , see Definition 7.8). By assumption,  $p_i$  is realized in  $M$ , say by  $b_i$ . Thus there exists  $M_j'' \in \mathbf{K}_{\leq \mu}$  with  $M_j' \leq_{\mathbf{K}} M_j''$  and  $h : M \xrightarrow{M_j} M_j''$  such that  $h(b_i) = g(a_i)$ . Let  $g' : N_i^1 \cong M_j''$  be an extension of  $g$ . Let  $M_i \in \mathcal{P}_{\mathbf{K}_{\leq \mu}}(M)$  be such that  $M_j \leq_{\mathbf{K}} M_i$  and  $b_i \in M_i$ . Let  $N_i^0 := (g')^{-1}h[M_i]$ . Let  $f_i := h^{-1}g' \upharpoonright N_i^0$ .

□

**Remark 7.18.** It suffices to assume that  $\mathbf{K}_{<(\lambda+\|M\|)}$  has amalgamation.



We deduce a more local technical lemma which will have several other interesting consequences:

**Definition 7.19.** For  $\mathbf{K}$  an abstract class and  $M, N \in \mathbf{K}$ , we say that  $N$  is *universal over  $M$*  if  $M \leq_{\mathbf{K}} N$  and whenever  $M' \in \mathbf{K}$  is such that  $M \leq_{\mathbf{K}} M'$  and  $\|M'\| = \|M\|$ , there is  $f : M' \xrightarrow[M]{} N$ .

**Lemma 7.20** (The universal extension construction lemma). Let  $\mathbf{K}$  be an abstract class satisfying all the axioms of AECs except perhaps the LST axiom. Assume that  $\mathbf{K}$  has amalgamation. Let  $\lambda$  be a cardinal and let  $\langle M_i : i \leq \lambda \rangle$  be an increasing continuous chain in  $\mathbf{K}$  such that  $\lambda = \|M_0\|$ . If for any  $i < \lambda$ , any  $p \in \mathbf{S}(M_i)$  is realized in  $M_{i+1}$ , then  $M_\lambda$  is universal over  $M_0$ .

*Proof.* Exactly as in the proof of Theorem 7.17. We require that  $f_i : N_i^0 \cong M_i$ .  $\square$

**Definition 7.21.** Let  $\mathbf{K}$  be an AEC and let  $\lambda \geq \text{LS}(\mathbf{K})$ . We say that  $\mathbf{K}$  is *stable in  $\lambda$*  if  $|\mathbf{S}(M)| \leq \lambda$  for all  $M \in \mathbf{K}_\lambda$ .

**Corollary 7.22.** Let  $\mathbf{K}$  be an AEC and let  $\lambda \geq \text{LS}(\mathbf{K})$ . Assume that  $\mathbf{K}_\lambda$  has amalgamation and  $\mathbf{K}$  is stable in  $\lambda$ . For any  $M \in \mathbf{K}_\lambda$ , there exists  $N \in \mathbf{K}_\lambda$  such that  $N$  is universal over  $M$ .

*Proof.* Build an increasing continuous chain  $\langle M_i : i \leq \lambda \rangle$  in  $\mathbf{K}_\lambda$  such that  $M_0 = M$  and  $M_{i+1}$  realizes all types over  $M_i$ . This is possible by stability in  $\lambda$ . This is enough: by Lemma 7.20,  $M_\lambda$  is universal over  $M_0$ .  $\square$

## 8. TAMENESS

Tameness is a locality property for orbital types first isolated by Grossberg and VanDieren [GV06]. Type-shortness is a generalization introduced by Will Boney [Bon14]. We only give two variations here.

**Definition 8.1.** Let  $\mathbf{K}$  be an abstract class and let  $\kappa$  be an infinite cardinal.

- (1)  $\mathbf{K}$  is  *$(< \kappa)$ -tame* if for any two distinct orbital types  $p, q \in \mathbf{S}(M)$  there exists  $A \in [M]^{<\kappa}$  such that  $p \upharpoonright A \neq q \upharpoonright A$ .
- (2)  $\mathbf{K}$  is  *$(< \kappa)$ -short* if for any two  $M_1, M_2 \in \mathbf{K}$ ,  $\bar{b}_\ell \in {}^\alpha M_\ell$ , if  $\text{tp}(\bar{b}_1/\emptyset; M_1) \neq \text{tp}(\bar{b}_2/\emptyset; M_2)$ , then there exists  $I \subseteq \alpha$  with  $|I| < \kappa$  such that  $\text{tp}(\bar{b}_1 \upharpoonright I/\emptyset; M_1) \neq \text{tp}(\bar{b}_2 \upharpoonright I/\emptyset; M_2)$ .

When we omit the  $\kappa$ , we meant “for some  $\kappa$ ”.

The difference between tameness and shortness is the length of the types involved and their domains (tameness is for types of length one over models). In the literature,  $(< \kappa)$ -short is called “fully  $(< \kappa)$ -tame and type-short over  $\emptyset$ ”. The following is not difficult to show:

**Exercise 8.2.** If  $\mathbf{K}$  is  $(< \kappa)$ -short, then  $\mathbf{K}$  is  $(< \kappa)$ -tame.

We have seen (Exercise 2.10 and Theorem 2.11) that elementary and universal classes are both  $(< \aleph_0)$ -short. The following is a trivial non-example:

**Example 8.3.** Let  $\mathbf{K} = (K, \leq_{\mathbf{K}})$  be defined by  $K := \{M \mid M \cong (\mathbb{Q}, <)\}$  and  $M \leq_{\mathbf{K}} N$  if and only if  $M, N \in K$  and  $M = N$ . Then  $\mathbf{K}$  is *not*  $(< \aleph_0)$ -short, since  $\mathbf{tp}(1/(0, 1); \mathbb{Q}) \neq \mathbf{tp}(2/(0, 1); \mathbb{Q})$  (there is no automorphism of  $\mathbb{Q}$  sending 1 to 2 fixing  $(0, 1)$ ) but all the finite restrictions of these types are equal.

There are various less trivial examples of non-tameness [BS08, BK09]. The following is due to Will Boney [Bon14]:

**Theorem 8.4.** Let  $\mathbf{K}$  be an AEC and let  $\kappa > \text{LS}(\mathbf{K})$  be a strongly compact cardinal. Then  $\mathbf{K}$  is  $(< \kappa)$ -short.

Boney’s proof uses closure of AECs under sufficiently-complete ultraproducts (which follows from the presentation theorem and the fact that reducts commute with ultraproducts). Later Lieberman and Rosický [LR16, 5.2] found a different proof using an older category-theoretic result of Makkai and Paré. We present here yet another proof (unpublished) which uses compactness of  $\mathbb{L}_{\kappa, \kappa}$  directly. The proof actually generalizes to  $\mu$ -AECs, just like the ones mentioned earlier (see also [BGL<sup>+</sup>16, §5]).

*Proof of Theorem 8.4.* We more generally prove the statement for any  $\mu$ -AEC. We will assume for notational simplicity that  $\mathbf{K}$  has amalgamation (more precisely that  $\equiv^{\mathbf{K}} = E_{\text{at}}$ ) but if  $\mathbf{K}$  does not have amalgamation a similar proof (with more coding) also gives the result. Since shortness is invariant under taking functorial expansions, we may assume without loss of generality (Theorems 6.18 and 6.19) that  $\mathbf{K}$  is axiomatized by an  $\mathbb{L}_{\kappa, \kappa}$ -sentence  $\phi$  and  $\mathbf{K}$  is model-complete. Let  $\tau := \tau(\mathbf{K})$ .

Let  $M_1, M_2 \in \mathbf{K}$ . Let  $\bar{b}_\ell \in {}^\alpha M_\ell$ . Suppose that  $(\bar{b}_1 \upharpoonright I, M_1) \equiv (\bar{b}_2 \upharpoonright I, M_2)$  for all  $I \in [\alpha]^{<\kappa}$ . Without loss of generality,  $M_1 \cap M_2 = \emptyset$ . Let  $\tau_\ell$  be  $\tau$  expanded with new constants symbols  $\langle c_a : a \in M_\ell \rangle$ . Let  $M_\ell^+$  be the expansions of  $M_\ell$  to  $\tau_\ell$ . Let  $T_\ell$  be the  $\mathbb{L}_{\kappa, \kappa}$ -quantifier-free diagram of  $M_\ell^+$ . Let  $B_\ell$  be the range of  $\bar{b}_\ell$  and let  $f$  be a map sending  $\bar{b}_1$  to  $\bar{b}_2$ . Let  $T$  be the  $\mathbb{L}_{\kappa, \kappa}$ -theory  $\{\phi\} \cup T_1 \cup T_2 \cup \{c_b = c_{f(b)} \mid b \in B_1\}$ . It suffices to prove that  $T$  is consistent. By the compactness theorem for  $\mathbb{L}_{\kappa, \kappa}$ , it suffices to prove that  $T$  is  $(< \kappa)$ -consistent. This is given by the assumption that  $\mathbf{tp}(\bar{a}_1 \upharpoonright I; M_1) = \mathbf{tp}(\bar{a}_2 \upharpoonright I; M_2)$  for any  $I \in [\alpha]^{<\kappa}$ : any  $M$  witnessing this will (in a suitable expansion) model  $T$ .  $\square$

Recently, Boney and Unger [BU17] (building on earlier work of Shelah [She]) found an example of an AEC  $\mathbf{K}$  which is tame if and only if there is an (almost) strongly compact above  $\text{LS}(\mathbf{K})$ . Thus the statement “every AEC is tame” is a large cardinal axiom.

The following characterization of shortness in terms of functorial expansion appears in [Vas16]. We first expand  $\mathbf{K}$  with a symbol for each orbital type:

**Definition 8.5.** Let  $\mathbf{K}$  be an abstract class. The  $(< \kappa)$ -orbital Morleyization of  $\mathbf{K}$  is given by adding an  $\ell(p)$ -ary relation symbol  $R_p$  for each  $p \in \mathbf{S}^{<\kappa}(\emptyset)$  and expanding each  $M \in \mathbf{K}$  to  $M^+$  with  $R_p^{M^+}(\bar{b})$  holding if and only if  $\mathbf{tp}(\bar{b}/\emptyset; M) = p$ .

**Exercise 8.6.** Prove that the  $(< \kappa)$ -orbital Morleyization of  $\mathbf{K}$  is a functorial expansion.

**Exercise 8.7.** Let  $\mathbf{K}$  be an abstract class. The following are equivalent:

- (1)  $\mathbf{K}$  is  $(< \kappa)$ -short.

- (2) The map sending each  $p = \mathbf{tp}(\bar{b}/\emptyset; M) \in \mathbf{S}^{<\infty}(\emptyset)$  to the quantifier-free type of  $\bar{b}$  inside  $M^+$  is an injection, where  $M^+$  is the expansion of  $M$  in the  $(<\kappa)$ -orbital Morleyization.

## 9. AMALGAMATION FROM DIAMOND

The following result is due to Shelah [She87a].

**Theorem 9.1.** Let  $\mathbf{K}$  be an AEC and let  $\lambda \geq \text{LS}(\mathbf{K})$ . Assume  $2^\lambda < 2^{\lambda^+}$ . If  $\mathbf{K}$  is categorical in  $\lambda$  and  $\mathbb{I}(\mathbf{K}, \lambda^+) < 2^{\lambda^+}$ , then  $\mathbf{K}_\lambda$  has amalgamation.

Here,  $\mathbb{I}(\mathbf{K}, \lambda^+)$  denotes the number of models of cardinality  $\lambda^+$  up to isomorphism. The hypothesis that  $2^\lambda < 2^{\lambda^+}$  is in general needed: there is an example with  $\lambda = \aleph_0$  where Martin's axiom plus  $\aleph_1 < 2^{\aleph_0}$  implies that the example is categorical in both  $\aleph_0$  and  $\aleph_1$  yet fails amalgamation [She09a, §1.6]. We will prove Theorem 9.1 using a stronger hypothesis than  $2^\lambda < 2^{\lambda^+}$  known as the diamond principle:

**Definition 9.2.** For an uncountable regular cardinal  $\lambda$ ,  $\diamond_\lambda$  is the statement that there exists a sequence  $\langle A_i : i < \lambda \rangle$  such that  $A_i \subseteq i$  and for any  $X \subseteq \lambda$ , the set  $\{i < \lambda \mid X \cap i = A_i\}$  is stationary.

If  $V = L$ ,  $\diamond_\lambda$  holds for any uncountable regular  $\lambda$  (this is due to Jensen, who also introduced  $\diamond$ ; see [Kun80, VI.5.2]). On the other hand,  $\diamond_\lambda$  implies that  $2^{<\lambda} = \lambda$  (since any bounded subset of  $\lambda$  must be equal to some  $A_i$ ). Thus  $\diamond_\lambda$  is independent of ZFC (at least when  $\lambda$  is a successor cardinal). We will use the following form of  $\diamond$ :

**Exercise 9.3.** Let  $\lambda$  be an uncountable regular cardinal. Then  $\diamond_\lambda$  is equivalent to:

There are  $\{\eta_\alpha, \nu_\alpha : \alpha \rightarrow 2 \mid \alpha < \lambda\}, \{g_\alpha : \alpha \rightarrow \alpha \mid \alpha < \lambda\}$  such that for all  $\eta, \nu : \lambda \rightarrow 2, g : \lambda \rightarrow \lambda$ ,  $\{\alpha < \lambda \mid \eta_\alpha = \eta \upharpoonright \alpha, \nu_\alpha = \nu \upharpoonright \alpha, g_\alpha = g \upharpoonright \alpha\}$  is stationary.

Before proving Theorem 9.1, we need one more fact:

**Exercise 9.4.** Let  $\mathbf{K}$  be an AEC and let  $\lambda > \text{LS}(\mathbf{K})$  be a regular cardinal. Let  $M, N \in \mathbf{K}_\lambda$  and let  $f : M \cong N$ . Let  $\langle M_i : i \leq \lambda \rangle, \langle N_i : i \leq \lambda \rangle$  be increasing continuous resolutions of  $M, N$  respectively (in particular,  $M_\lambda = M, N_\lambda = N$ ,  $\|M_i\| + \|N_i\| < \lambda$  for all  $i < \lambda$ ). Then the set of ordinals  $\alpha < \lambda$  such that  $f \upharpoonright M_\alpha : M_\alpha \cong N_\alpha$  is a club.

*Proof of Theorem 9.1 assuming  $\diamond_{\lambda^+}$ .* Suppose that  $\mathbf{K}_\lambda$  fails to have amalgamation. Using categoricity in  $\lambda$ , it is easy to see that  $\mathbf{K}_\lambda$  has no maximal models. Fix  $\langle \eta_\alpha, \nu_\alpha, g_\alpha : \alpha < \lambda^+ \rangle$  as given by Exercise 9.3 (where  $\lambda$  there stands for  $\lambda^+$  here)

Build a strictly increasing continuous tree  $\{M_\eta \mid \eta \in {}^{<\lambda^+}2\}$  such that:

- (1)  $|M_\eta| \subseteq \lambda^+$  for all  $\eta \in {}^{<\lambda^+}2$
- (2) If  $|M_{\eta_\delta}| = \delta$ ,  $\eta_\delta \neq \nu_\delta$ , and  $g_\delta : M_{\eta_\delta} \cong M_{\nu_\delta}$  is an isomorphism, then it cannot be extended to an embedding of  $M_{\eta_\delta \smallfrown i}$  into  $M_\nu$  for all  $\nu \supseteq \nu_\delta \smallfrown j$ ,  $\nu \in {}^{<\lambda^+}2$ , for all  $i, j \in 2$ .

This is enough: We show that  $M_\eta \not\cong M_\nu$  for  $\eta \neq \nu \in {}^{\lambda^+}2$ . Suppose for a contradiction that  $f : M_\eta \cong M_\nu$  is an isomorphism. Note that  $\{\alpha < \lambda^+ \mid |M_\alpha| = \alpha\}$  is club, and so is  $\{\alpha < \lambda^+ \mid f \restriction M_{\eta \restriction \alpha} : M_{\eta \restriction \alpha} \cong M_{\nu \restriction \alpha}\}$ . Thus using diamond, there is a stationary set of  $\delta < \lambda^+$  such that  $\eta \restriction \delta \neq \nu \restriction \delta$ ,  $\eta_\delta = \eta \restriction \delta$ ,  $\nu_\delta = \nu \restriction \delta$ ,  $g_\delta = f \restriction \delta$ ,  $\delta = |M_{\eta_\delta}| = |M_{\nu_\delta}|$ , and  $g_\delta : M_{\eta_\delta} \cong M_{\nu_\delta}$ . But  $f$  extends  $g_\delta$  and restricts to an embedding of  $M_{\eta \restriction \eta(\delta)}$  into  $M_{\nu \restriction \gamma}$ , for some  $\lambda^+ > \gamma > \delta$  sufficiently large. This contradicts the first property of the construction.

This is possible: Take any  $M_{< \gamma} \in \mathbf{K}$  with  $|M_{< \gamma}| = \lambda$  for the base case, and take unions at limits. Now if one wants to define  $M_{\eta \restriction l}$  for  $\eta \in {}^{\delta}2$  (assuming by induction that  $M_\nu$  for all  $\nu \in {}^{\leq \delta}2$  have been defined) take any two strict extensions, unless  $|M_\eta| = \delta$ ,  $\eta_\delta \neq \nu_\delta$ ,  $g_\delta : M_{\eta_\delta} \cong M_{\nu_\delta}$  is an isomorphism, and either  $\eta = \eta_\delta$ , or  $\eta = \nu_\delta$ . We show what to do when  $\eta = \eta_\delta$ . The other case is symmetric.

By failure of amalgamation and categoricity, we know that there exists  $M^1, M^2$  extensions of  $M_{\eta_\delta}, M_{\nu_\delta}$  respectively such that there is *no*  $N \in \mathbf{K}$  and  $f_\ell : M^\ell \rightarrow N$  commuting with  $g_\delta$ . Now let  $M_{\eta_\delta \restriction l}, M_{\nu_\delta \restriction l}$  be two copies of  $M^1, M^2$  respectively.  $\square$

## 10. EXISTENCE FROM SUCCESSIVE CATEGORICITY

The goal of this section is to prove:

**Theorem 10.1** (Shelah, [She87a]). Let  $\psi \in \mathbb{L}_{\omega_1, \omega}$ . If  $\psi$  is categorical in  $\aleph_0$  and  $\aleph_1$ , then  $\psi$  has a model of cardinality  $\aleph_2$ .

This result has a long history, recalled e.g. at the beginning of [She09a, Chapter I]. There is also an exposition by Makowsky [BFB85, Chapter XX].

The proof and some more work also answers negatively Baldwin's question: is there an  $\mathbb{L}(Q)$ -formula with exactly one uncountable model?

We will more generally prove the result for any  $\text{PC}_{\aleph_0}$ -AEC:

**Definition 10.2.** An AEC  $\mathbf{K}$  is  $\text{PC}_{\aleph_0}$  if  $\text{LS}(\mathbf{K}) = \aleph_0$  and there is a countable expansion  $\tau^+$  of  $\tau(\mathbf{K})$  and a universal  $\mathbb{L}_{\omega_1, \omega}(\tau^+)$ -sentence  $\phi$  such that  $\mathbf{K} = \text{Mod}(\phi) \restriction \tau(\mathbf{K})$  and if  $N^+ \models \phi$  and  $M^+ \subseteq N^+$ , then  $M^+ \restriction \tau(\mathbf{K}) \leq_{\mathbf{K}} N^+ \restriction \tau(\mathbf{K})$ .

By Skolemizing, we get that for any  $\mathbb{L}_{\omega_1, \omega}$  formula  $\psi$  and any countable fragment  $\Phi$  containing  $\psi$ , the AEC  $(\text{Mod}(\psi), \preceq_\Phi)$  is  $\text{PC}_{\aleph_0}$ . This is the main example to keep in mind but there are others. In fact [BL16, 3.3]:

**Fact 10.3.** An AEC  $\mathbf{K}$  with  $\text{LS}(\mathbf{K}) = \aleph_0$  is  $\text{PC}_{\aleph_0}$  if and only if  $\{(M, N) \mid M, N \in \mathbf{K}_{\leq \aleph_0}, M \leq_{\mathbf{K}} N\}$  is analytic (when seen as a set of reals).

Our goal is:

**Theorem 10.4.** Let  $\mathbf{K}$  be a  $\text{PC}_{\aleph_0}$  AEC. If  $\mathbf{K}$  is categorical in  $\aleph_0$  and  $\aleph_1$ , then  $\mathbf{K}$  has a model of cardinality  $\aleph_2$ .

Note that it is open whether for an arbitrary AEC  $\mathbf{K}$  and a  $\lambda \geq \text{LS}(\mathbf{K})$ , categoricity in  $\lambda$  and  $\lambda^+$  implies existence in  $\lambda^{++}$ . The best approximation is due to Shelah [She01] who proved assuming some set-theoretic hypothesis that categoricity in *three* (not two) successive cardinals implies existence in the next.

To prove Theorem 10.4, we need some sufficient conditions for existence of models. The following is basic:

**Exercise 10.5.** Let  $\mathbf{K}$  be an AEC and let  $\lambda \geq \text{LS}(\mathbf{K})$ . Assume that  $\mathbf{K}$  is categorical in  $\lambda$ . The following are equivalent:

- (1)  $\mathbf{K}_{\lambda^+} \neq \emptyset$ .
- (2) There exists  $M, N \in \mathbf{K}_\lambda$  such that  $M <_{\mathbf{K}} N$ .
- (3)  $\mathbf{K}_\lambda$  has no maximal models.

We will apply Exercise 10.5 with  $\lambda = \aleph_1$ . What we need now is a criteria in  $\aleph_0$  to ensure that there exists a pair  $(M, N)$  in  $\aleph_1$  with  $M <_{\mathbf{K}} N$ . The following does the trick:

**Definition 10.6.** Let  $\mathbf{K}$  be an abstract class.

- (1) We call an orbital type  $p$  of length one *algebraic* if whenever  $p = \text{tp}(a/M; N)$ ,  $a \in |M|$ . We write  $\mathbf{S}^{\text{na}}(M)$  for the class of non-algebraic types over  $M$ , and  $\mathbf{K}^{3,\text{na}}$  for the class of non-algebraic triples: triples  $(a, M, N)$  such that  $M \leq_{\mathbf{K}} N$  and  $a \in |N| \setminus |M|$ .
- (2) We say that a triple  $(a, M, N) \in \mathbf{K}^{3,\text{na}}$  has the *extension property* if for any  $M' \in \mathbf{K}$  with  $M \leq_{\mathbf{K}} M'$ , there exists  $q \in \mathbf{S}(M')$  such that  $q \upharpoonright M = \text{tp}(a/M; N)$ .
- (3) For  $(a, M, N), (b, M', N') \in \mathbf{K}^{3,\text{na}}$ , write  $(a, M, N) <_{\mathbf{K}} (b, M', N')$  if  $a = b$ ,  $N \leq_{\mathbf{K}} N'$ , and  $M <_{\mathbf{K}} M'$ .
- (4) We say that  $(a, M, N) \in \mathbf{K}^{3,\text{na}}$  has the *weak extension property* if it is *not*  $<_{\mathbf{K}}$ -maximal in  $\mathbf{K}^{3,\text{na}}$ .
- (5) We say that  $\mathbf{K}$  has the *[weak] extension property* if every triple in  $\mathbf{K}^{3,\text{na}}$  has the [weak] extension property.

We write  $\mathbf{K}_\lambda^{3,\text{na}}$  for  $\mathbf{K}_\lambda^{3,\text{na}}$ .

The weak extension property is all we will use. However we mentioned the extension property as well, since it seems more natural. We have the following relationship between the two:

**Exercise 10.7.** Let  $\mathbf{K}$  be an abstract class with no maximal models. Show that the extension property implies the weak extension property.

**Lemma 10.8.** Let  $\mathbf{K}$  be an AEC and let  $\lambda \geq \text{LS}(\mathbf{K})$ . If  $\mathbf{K}_\lambda^{3,\text{na}} \neq \emptyset$  and  $\mathbf{K}_\lambda$  has the weak extension property, then there exists  $M, N \in \mathbf{K}_{\lambda^+}$  such that  $M <_{\mathbf{K}} N$ .

*Proof.* Build a  $<_{\mathbf{K}}$ -increasing chain  $\langle (a, M_i, N_i) : i < \lambda^+ \rangle$  in  $\mathbf{K}_\lambda^{3,\text{na}}$ . Take any member of  $\mathbf{K}_\lambda^{3,\text{na}}$  for the base case. At successors, use the weak extension property and at limits take unions. In the end, let  $M := \bigcup_{i < \lambda^+} M_i$ ,  $N := \bigcup_{i < \lambda^+} N_i$ . Since the chain was strictly increasing, both  $M$  and  $N$  are in  $\mathbf{K}_{\lambda^+}$ . By definition of  $\mathbf{K}_\lambda^{3,\text{na}}$ ,  $a \in N_i \setminus M_i$  for any  $i < \lambda^+$ . Thus  $a \in N \setminus M$ , so  $M <_{\mathbf{K}} N$ , as desired.  $\square$

A partial converse to Lemma 10.8 holds: if  $\mathbf{K}$  is categorical in  $\lambda$  and has no maximal models in  $\lambda^+$ , then  $\mathbf{K}_\lambda$  has the weak extension property [She09b, VI.1.9]. In particular, if  $\mathbf{K}$  is categorical in  $\lambda$  and  $\lambda^+$ , then  $\mathbf{K}_{\lambda^{++}} \neq \emptyset$  if and only if  $\mathbf{K}_\lambda$  has the weak extension property.

We will be done once we have proven the following:

**Theorem 10.9.** Let  $\mathbf{K}$  be a  $\text{PC}_{\aleph_0}$  AEC. Assume that  $\mathbf{K}$  is categorical in  $\aleph_0$  and  $\mathbf{K}_{\aleph_0}$  has no maximal models. If  $\mathbf{K}_{\aleph_0}$  does *not* have the weak extension property, then  $\mathbb{I}(\mathbf{K}, \aleph_1) = 2^{\aleph_1}$ .

*Proof of Theorem 10.4.* By Exercise 10.5 (with  $\lambda$  there standing for  $\aleph_0$  here),  $\mathbf{K}_{\aleph_0}$  has no maximal models. By Theorem 10.9,  $\mathbf{K}_{\aleph_0}$  must have the weak extension property. By Lemma 10.8, there exists  $M, N \in \mathbf{K}_{\aleph_1}$  such that  $M <_{\mathbf{K}} N$ . By Exercise 10.5 (with  $\lambda$  there standing for  $\aleph_1$  here), this implies that  $\mathbf{K}_{\aleph_2} \neq \emptyset$ .  $\square$

There are two keys to Theorem 10.9. The first (and the only place where we use that  $\mathbf{K}$  is  $\text{PC}_{\aleph_0}$ ) is a consequence of the undefinability of well-orderings in such classes:

**Lemma 10.10** (The magic lemma). Let  $\mathbf{K}$  be a  $\text{PC}_{\aleph_0}$  AEC. If  $\mathbf{K}_{\aleph_1} \neq \emptyset$ , then there exists a *strictly decreasing continuous chain*  $\langle M_i : i \leq \omega \rangle$  in  $\mathbf{K}_{\aleph_0}$ . That is,  $M_j <_{\mathbf{K}} M_i$  for all  $i < j \leq \omega$  and  $M_\omega = \bigcap_{i < \omega} M_i$ .

The second key to Theorem 10.9 is Solovay's splitting theorem [Jec03, 8.10].

**Fact 10.11.** Let  $\lambda$  be a regular uncountable cardinal and let  $S$  be a stationary subset of  $\lambda$ . Then there exists  $\lambda$ -many pairwise disjoint stationary subsets of  $S$ .

**Exercise 10.12.** Show that for any regular cardinal  $\lambda$  and any stationary set  $S$  of  $\lambda$ , there exists  $\langle S_i : i < 2^\lambda \rangle$  stationary subsets of  $S$  such that  $i \neq j$  implies that  $S_i \Delta S_j$  is stationary. *Hint: let  $\langle T_j : j < \lambda \rangle$  be pairwise disjoint stationary subsets of  $\lambda$  and for a non-empty  $A \subseteq \lambda$ , consider  $S_A := \bigcup_{j \in A} T_j$ .*

Assuming the magic lemma for now, let us prove the theorem.

*Proof of Theorem 10.9.* For each stationary subset  $S$  of  $\aleph_1$ , we build an increasing continuous chain  $\langle M_i^S : i \leq \aleph_1 \rangle$  and  $\langle a_i^S : i \in S \rangle$  such that for all  $i < \aleph_1$ :

- (1)  $M_i^S \in \mathbf{K}_{\aleph_0}$ .
- (2) If  $i \in S$ , then  $(a_i^S, M_i^S, M_{i+1}^S) \in \mathbf{K}_{\aleph_0}^{3, \text{na}}$  and is  $<_{\mathbf{K}}$ -maximal.
- (3) If  $i \notin S$ , then there exists a strictly decreasing continuous chain  $\langle N_j : j \leq \omega \rangle$  in  $\mathbf{K}_{\aleph_0}$  such that  $M_i^S = N_\omega$  and  $M_{i+1}^S = N_0$ .

This is possible: At limits, take unions. For  $i = 0$ , take any  $M_0^S \in \mathbf{K}_{\aleph_0}$ . Assume now that  $\langle M_j^S : j \leq i \rangle$  and  $\langle a_j^S : j < i \rangle$  have been defined. We show how to define  $M_{i+1}^S$  and  $a_i^S$ . There are two cases.

- If  $i \in S$ , fix  $(a, M, N) \in \mathbf{K}_{\aleph_0}^{3, \text{na}}$  which is  $<_{\mathbf{K}}$ -maximal (exists by assumption). Let  $f : M \cong M_i^S$ , and extend it to  $g : N \cong M_{i+1}^S$ . Let  $a_i^S := f(a)$ .
- If  $i \notin S$ , then by Lemma 10.10 and categoricity, we can get a strictly decreasing continuous chain  $\langle N_j : j \leq \omega \rangle$  in  $\mathbf{K}_{\aleph_0}$  such that  $N_\omega = M_i^S$ . Let  $M_{i+1}^S := N_0$ .

This is enough: We claim that if  $S$  and  $T$  are stationary subsets of  $\aleph_1$  such that  $S \Delta T$  is stationary, then  $M_{\aleph_1}^S \not\cong M_{\aleph_1}^T$ . This suffices by Exercise 10.12. So let  $S$  and  $T$  be such that  $S \Delta T$  is stationary, and suppose for a contradiction that

$f : M_{\aleph_1}^S \cong M_{\aleph_1}^T$ . By Exercise 9.4, there exists a club  $C$  such that  $i \in C$  implies that  $f_i := f \upharpoonright M_i^S$  is an isomorphism from  $M_i^S$  onto  $M_i^T$ . Suppose without loss of generality that  $S \setminus T$  is stationary, and let  $i \in (S \setminus T) \cap C$ .

Since  $i \in S$ , we have that  $(a_i^S, M_i^S, M_{i+1}^S)$  is in  $\mathbf{K}_{\aleph_0}^{3, \text{na}}$  and  $<_{\mathbf{K}}$ -maximal. Let  $i' < \aleph_1$  be such that  $f[M_{i+1}^S] \leq_{\mathbf{K}} M_{i'}^T$  and  $i' > i + 1$ . Since  $i \notin T$ , one can fix a strictly decreasing continuous chain  $\langle N_j : j \leq \omega \rangle$  such that  $M_i^T = N_\omega$ ,  $M_{i+1}^T = N_1$ , and  $M_{i'}^T = N_0$ . Since  $f(a_i^S) \in M_{i'}^T$  and  $f(a_i^S) \notin M_i^T = \bigcap_{j < \omega} N_j$ , there must exist a least  $\ell < \omega$  such that  $f(a_i^S) \in N_\ell \setminus N_{\ell+1}$ . This shows that  $(f(a_i^S), M_i^T, f[M_{i+1}^S]) <_{\mathbf{K}} (f(a_i^S), N_{\ell+1}, M_{i'}^T)$ , so  $(f(a_i^S), M_i^T, f[M_{i+1}^S])$  is not  $<_{\mathbf{K}}$ -maximal, so  $(a_i^S, M_i^S, M_{i+1}^S)$  is not  $<_{\mathbf{K}}$ -maximal, a contradiction.  $\square$

It remains to prove the magic lemma. For this we will use undefinability of uncountable well-orderings in  $\mathbb{L}_{\omega_1, \omega}$ , due to Lopez-Escobar and (independently) Morley. To state it in a general setup, we introduce some notation.

**Definition 10.13.** For  $\lambda$  an infinite cardinal and  $\mu \geq 1$ , let  $\delta(\lambda, \mu)$  be the least ordinal  $\delta$  such that whenever  $T \subseteq \mathbb{L}_{\lambda^+, \omega}$  is such that:

- (1)  $|T| \leq \mu$  and  $|\tau(T)| \leq \lambda$ .
- (2)  $\tau(T)$  contains a predicate  $P$  and a binary relation  $<$  such that in any model  $M$  of  $T$ ,  $(P^M, <^M \upharpoonright P^M)$  is a linear order.
- (3) For every  $\alpha < \delta$ , there exists  $M \models T$  with  $\alpha$  an initial segment of  $(P^M, <^M \upharpoonright P^M)$ .

Then there exists  $M \models T$  where  $(P^M, <^M \upharpoonright P^M)$  is ill-founded.

We let  $\delta(\lambda) := \delta(\lambda, 1)$ .

The proofs of the following facts can all be found in [She90, §VIII.5]:

**Fact 10.14.**

- (1)  $\lambda^+ \leq \delta(\lambda, \mu) \leq (2^\lambda)^+$  and  $\delta(\lambda) < (2^\lambda)^+$ .
- (2)  $\delta(\lambda) = \lambda^+$  whenever  $\lambda$  is a strong limit of cofinality  $\aleph_0$ . In particular,  $\delta(\aleph_0) = \omega_1$ .
- (3)  $\delta(\lambda) > \lambda^+$  whenever  $\text{cf}(\lambda) > \aleph_0$

Note that Theorem 10.4 generalizes to any AEC  $\mathbf{K}$  with  $\text{LS}(\mathbf{K}) = \lambda$  which is the reduct of a universal  $\mathbb{L}_{\lambda^+, \omega}$ -sentence with  $\delta(\lambda) = \lambda^+$ .

*Proof of Lemma 10.10.* Let  $\langle N_i : i < \aleph_1 \rangle$  be strictly increasing continuous in  $\mathbf{K}_{\aleph_0}$ . This exists: take a resolution of the model in  $\mathbf{K}_{\aleph_1}$ . Now let  $\chi$  be a “big” cardinal, and let  $\tau^+$  be a countable expansion of the vocabulary of set theory (it should have “enough” symbols, in a sense that the proof will tell us). Let  $\mathfrak{B}$  be a  $\tau^+$ -expansion of  $(V_\chi, \in)$  ( $V_\chi$  is the  $\chi$ th level of the cumulative hierarchy). In particular, we require that there is a binary function symbol  $f \in \tau^+$  where  $f^\mathfrak{B} : \omega_1 \rightarrow \mathbf{K}$ ,  $f(i) := N_i$ . We also ask that  $\mathfrak{B}$  contains the definition of the AEC  $\mathbf{K}$ , that the natural numbers of  $\mathfrak{B}$  are well-founded, and that  $\tau^+$  also contains bijections from the natural number to every  $N_i$ . Code all this data into an  $\mathbb{L}_{\omega_1, \omega}(\tau^+)$ -sentence  $\phi$ . Now since  $\delta(\aleph_0) = \omega_1$ , there exists  $\mathfrak{B}^*$  an *ill-founded* model of set theory such that  $\mathfrak{B}^* \equiv \mathfrak{B}$ , and  $\mathfrak{B}^* \models \phi$ . Let  $\langle \alpha_n : n < \omega \rangle$  be a strictly decreasing sequence of “countable ordinals” in  $\mathfrak{B}^*$ .

Let  $I := \{\alpha \in \mathfrak{B}^* \mid \forall n < \omega : \alpha < \alpha_n\}$ . This is a linear order, hence a directed system. Let  $M_n := f^{\mathfrak{B}^*}(\alpha_n)$ . Then since enough of the definition of  $\mathbf{K}$  is reflected,  $M_n \in \mathbf{K}$  and  $M_{n+1} <_{\mathbf{K}} M_n$  for  $n < \omega$ . For  $\alpha \in I$ , let  $M_\alpha := f(\alpha)$ . We then also have that  $M_\alpha \in \mathbf{K}$  and  $M_\alpha \leq_{\mathbf{K}} M_n$  for all  $n < \omega$ . Let  $M := \bigcup_{\alpha \in I} M_\alpha$ . Clearly,  $M \leq_{\mathbf{K}} M_n$  for all  $n < \omega$ . Now if  $a \in \bigcap_{n < \omega} M_n$ , then  $\mathfrak{B}^*$  thinks there is a least ordinal  $\beta$  such that  $a \in f(\beta)$ . Now  $\beta < \alpha_n$  for all  $n < \omega$ , so  $\beta \in I$ , hence  $a \in M_\beta$ , so  $a \in M$ . This means that  $M = \bigcap_{n < \omega} M_n$ , as desired.  $\square$

## 11. EHRENFUCHT-MOSTOWSKI MODELS AND STABILITY

Everywhere in this section,  $\mathbf{K}$  is an AEC.

We go very fast on the material related to EM models, since there is already a lot of details about them in [Bal09], and Will Boney covered them extensively in the past semester. Most of the results from this section can be found in [Vas17b], see there for attribution.

We will not use EM models too much in this course. The goal of this section is to use them to prove that (in an AEC with amalgamation) categoricity implies stability and failure of the order property. In this section, we will use EM models to prove a local superstability condition. Once we have proven these properties, we will start out by assuming them and develop a local classification theory.

We will use the notation from [She09a, Chapter IV]:

**Definition 11.1.** [She09a, Definition IV.0.8] Let  $\mathbf{K}$  be an AEC. For  $\mu \geq \text{LS}(\mathbf{K})$ , let  $\Upsilon_\mu[\mathbf{K}]$  be the set of  $\Phi$  proper for linear orders (that is,  $\Phi$  is a set  $\{p_n : n < \omega\}$ , where  $p_n$  is an  $n$ -variable quantifier-free type in a fixed vocabulary  $\tau(\Phi)$  and the types in  $\Phi$  can be used to generate a  $\tau(\Phi)$ -structure  $\text{EM}(I, \Phi)$  for each linear order  $I$ ; that is,  $\text{EM}(I, \Phi)$  is the closure under the functions of  $\tau(\Phi)$  of the universe of  $I$  and for any  $i_0 < \dots < i_{n-1}$  in  $I$ ,  $i_0 \dots i_{n-1}$  realizes  $p_n$ ) with:

- (1)  $|\tau(\Phi)| \leq \mu$ .
- (2) If  $I$  is a linear order of cardinality  $\lambda$ ,  $\text{EM}_{\tau(\mathbf{K})}(I, \Phi) \in \mathbf{K}_{\lambda + |\tau(\Phi)| + \text{LS}(\mathbf{K})}$ , where  $\tau(\mathbf{K})$  is the vocabulary of  $\mathbf{K}$  and  $\text{EM}_{\tau(\mathbf{K})}(I, \Phi)$  denotes the reduct of  $\text{EM}(I, \Phi)$  to  $\tau(\mathbf{K})$ . Here we are implicitly also assuming that  $\tau(\mathbf{K}) \subseteq \tau(\Phi)$ .
- (3) For  $I \subseteq J$  linear orders,  $\text{EM}_{\tau(\mathbf{K})}(I, \Phi) \leq_{\mathbf{K}} \text{EM}_{\tau(\mathbf{K})}(J, \Phi)$ .

We call  $\Phi$  as above an *EM blueprint*.

While the definitions are somewhat technical, it turns out the only thing that really matters is that the map  $I \mapsto \text{EM}_\tau(I, \Phi)$  is a faithful functor from the category of linear orders into  $\mathbf{K}$ .

The following follows from Shelah's presentation theorem (Theorem 3.6) and Morley's omitting type theorem. We will use it without explicit mention. See for example [Bal09, Appendix A]

**Notation 11.2.**  $h(\mu) := \beth_{(2^\mu)^+}$ .

**Fact 11.3.** Let  $\mathbf{K}$  be an AEC. The following are equivalent:

- (1)  $\mathbf{K}$  has arbitrarily large models.
- (2)  $\mathbf{K}_{\geq \mu} \neq \emptyset$  for all  $\mu < h(\text{LS}(\mathbf{K}))$ .



- (3) For any  $\mu \geq \text{LS}(\mathbf{K})$ ,  $\Upsilon_\mu[\mathbf{K}] \neq \emptyset$ .
- (4) For some  $\mu \geq \text{LS}(\mathbf{K})$ ,  $\Upsilon_\mu[\mathbf{K}] \neq \emptyset$ .

We have the following important property of types computed inside EM models:

**Lemma 11.4.** Let  $\Phi$  be an EM blueprint. Let  $M_\ell := \text{EM}_\tau(I_\ell, \Phi)$ . Let  $\bar{a}_\ell \in {}^{<\infty}I_\ell$ . If  $\bar{a}_1$  and  $\bar{a}_2$  are isomorphic as linear orders, then for any sequence of  $\tau(\Phi)$ -term  $\bar{\rho}$ ,  $(\bar{\rho}(\bar{a}_1), M_1) \equiv^{\mathbf{K}} (\bar{\rho}(\bar{a}_2), M_2)$ .

*Proof.* We have that  $\text{EM}(\bar{a}_1; \Phi)$  and  $\text{EM}(\bar{a}_2; \Phi)$  are isomorphic, hence  $\text{EM}_\tau(\bar{a}_1; \Phi)$  and  $\text{EM}_\tau(\bar{a}_2; \Phi)$  also are.  $\square$

The following concept is key:

**Definition 11.5.** Let  $\Phi$  be an EM blueprint. Let  $I, J$  be linear orders, let  $\delta$  be a limit ordinal and let  $\langle \bar{a}_j : j \in J \rangle$  be a sequence. We say that  $\langle \bar{a}_j : j \in J \rangle$  is  $(\Phi, I)$ -strictly indiscernible if:

- (1)  $J$  is infinite.
- (2) For some  $\alpha$ , for all  $j \in J$ ,  $\bar{a}_j \in {}^\alpha \text{EM}_\tau(I, \Phi)$ .
- (3) There exists a sequence  $\langle \bar{a}'_j : j \in J \rangle$  and a sequence of terms  $\bar{\rho}$  such that  $\bar{a}_j = \bar{\rho}(\bar{a}'_j)$  for all  $j \in J$  and  $\langle \bar{a}'_j : j \in J \rangle$  is quantifier-free indiscernible in the vocabulary of linear orders inside  $I$ .

We call  $\langle \bar{a}_j : j \in J \rangle$   $(\Phi, I)$ -strictly indiscernible over  $A$  if  $\langle \bar{a}_j \bar{a} : j \in J \rangle$  is  $(\Phi, I)$ -strictly indiscernible for some (any) enumeration  $\bar{a}$  of  $A$ .

Using compactness for linear orders, one can prove that strictly indiscernible sequences can be extended:

**Exercise 11.6.** If  $\langle \bar{a}_j : j \in J \rangle$  is a  $(\Phi, I)$ -strictly indiscernible, then for any  $J' \supseteq J$ , there exists  $I' \supseteq I$  and  $\langle \bar{a}_j : j \in J' \rangle$  which is  $(\Phi, I')$ -strictly indiscernible.

We prove that inside EM models generated by well-ordered sets, one can extract strict indiscernibles. This appears as [She99, Claim 4.15]:

**Theorem 11.7** (Strict indiscernible extraction). Let  $\mathbf{K}$  be an AEC with arbitrarily large models and let  $\text{LS}(\mathbf{K}) < \theta \leq \lambda$  be cardinals with  $\theta$  regular. Let  $\kappa < \theta$  be a (possibly finite) cardinal. Let  $\Phi \in \Upsilon_{\text{LS}(\mathbf{K})}[\mathbf{K}]$  be an EM blueprint for  $\mathbf{K}$ .

Let  $N := \text{EM}_{\tau(\mathbf{K})}(\lambda, \Phi)$ . Let  $M \in \mathbf{K}_{\leq \text{LS}(\mathbf{K})}$  be such that  $M \leq_{\mathbf{K}} N$ . Let  $\langle \bar{a}_i : i < \theta \rangle$  be a sequence of distinct elements such that for all  $i < \theta$ ,  $\bar{a}_i \in {}^\kappa N$ .

If  $\theta_0^\kappa < \theta$  for all  $\theta_0 < \theta$ , then there exists  $w \subseteq \theta$  with  $|w| = \theta$  such that  $\langle \bar{a}_i : i \in w \rangle$  is  $(\Phi, \lambda)$ -strictly indiscernible over  $M$ .

*Proof.* First we claim that one can assume without loss of generality that  $\kappa < \text{LS}(\mathbf{K})$ . Assume that the statement of the lemma has been proven for that case. If  $\kappa > \text{LS}(\mathbf{K})$  one can replace  $\mathbf{K}$  with  $\mathbf{K}_{\geq \kappa}$  (and increase  $M$ ) so assume that  $\kappa \leq \text{LS}(\mathbf{K})$ . Now if  $\kappa = \text{LS}(\mathbf{K})$ , then  $2^{\text{LS}(\mathbf{K})} = \kappa^\kappa < \theta$  so we can replace  $\mathbf{K}$  by  $\mathbf{K}_{\geq \text{LS}(\mathbf{K})^+}$  and work there. Thus assume without loss of generality that  $\kappa < \text{LS}(\mathbf{K})$ .

Pick  $u \subseteq \lambda$  such that  $|u| = \theta$ ,  $M \leq_{\mathbf{K}} N_0 := \text{EM}_{\tau(\mathbf{K})}(u, \Phi)$ , and  $\bar{a}_i \in {}^\kappa N_0$  for all  $i < \theta$ . Increasing  $M$  if necessary, we can assume without loss of generality that  $M = \text{EM}_{\tau(\mathbf{K})}(u', \Phi)$  for some  $u' \subseteq u$  with  $|u'| = \text{LS}(\mathbf{K})$ .

For each  $i < \theta$ , we can also pick  $u_i \subseteq u$  with  $|u_i| < \kappa^+ + \aleph_0$  such that  $\bar{a}_i \in {}^\kappa |\text{EM}_{\tau(\mathbf{K})}(u_i, \Phi)|$ . Without loss of generality  $u = u' \cup \bigcup_{i < \theta} u_i$ . By the pigeonhole principle, we can without loss of generality fix an ordinal  $\alpha < \kappa^+ + \aleph_0$  such that  $\text{otp}(u_i) = \alpha$  for all  $i < \theta$ . List  $u_i$  in increasing order as  $\bar{u}_i := \langle u_{i,j} : j < \alpha \rangle$ . By pruning further (using that  $\text{LS}(\mathbf{K})^\kappa < \theta$ ), we can assume without loss of generality that for each  $i, i' < \theta$  and  $j < \alpha$ , the  $u'$ -cut of  $u_{i,j}$  and  $u_{i',j}$  are the same (i.e. for any  $\gamma \in u'$ ,  $\gamma < u_{i,j}$  if and only if  $\gamma < u_{i',j}$ ).

Pruning again with the  $\Delta$ -system lemma, we can assume without loss of generality that  $\langle u_i : i < \theta \rangle$  forms a  $\Delta$ -system (see Definition II.1.4 and Theorem II.1.6 in [Kun80]; at that point we are using that  $\theta_0^\kappa < \theta$  for all  $\theta_0 < \theta$ ). In fact, the proof of the  $\Delta$ -system lemma shows that we can make sure that in the end  $\langle \bar{u}_i : i < \theta \rangle$  is indiscernible over  $u'$  in the vocabulary of linear orders.

Now list  $\bar{a}_i$  as  $\langle a_{i,j} : j < \kappa \rangle$ . Fix  $i < \theta$ . Since  $\bar{a}_i \in {}^\kappa |\text{EM}_{\tau(\mathbf{K})}(u_i)|$ , for each  $j < \kappa$  there exists a  $\tau(\Phi)$ -term  $\rho_{i,j}$  of arity  $n := n_{i,j}$  and  $j_0^{i,j} < \dots < j_{n-1}^{i,j} < \alpha$  such that  $a_{i,j} = \rho_{i,j} \left( u_{i,j_0^{i,j}} \dots u_{i,j_{n-1}^{i,j}} \right)$ . By the pigeonhole principle applied to the map  $i \mapsto \langle (\rho_{i,j}, n_{i,j}, j_0^{i,j}, \dots, j_{n-1}^{i,j}) : j < \kappa \rangle$  (using that  $\text{LS}(\mathbf{K})^\kappa < \theta$ ), we can assume without loss of generality that these depend only on  $j$ , i.e.  $\rho_{i,j} = \rho_j$ ,  $n_{i,j} = n_j$ , and  $j_\ell^{i,j} = j_\ell^j$ .

Let  $\bar{u}'$  be an enumeration of  $u'$ , and let  $\bar{a}'_i := \bar{u}_i \bar{u}'$ . Then  $\langle \bar{a}'_i : i < \theta \rangle$  witnesses the strict indiscernibility of  $\langle \bar{a}_i : i < \theta \rangle$ .  $\square$

We deduce stability inside an EM model generated by a well-ordering:

**Corollary 11.8.** Let  $\Phi$  be an EM blueprint. Let  $\lambda > \text{LS}(\mathbf{K})$ ,  $\kappa > 0$  be a cardinal, and let  $M := \text{EM}_\tau(\lambda, \Phi)$ . For any  $A \subseteq |M|$ ,  $|\mathbf{S}^\kappa(A; M)| \leq (|A| + \text{LS}(\mathbf{K}))^\kappa$ .

*Proof.* Let  $\theta := ((|A| + \text{LS}(\mathbf{K}))^\kappa)^+$ . Let  $\langle \bar{a}_i : i < \theta \rangle$  be an arbitrary sequence of elements of  ${}^\kappa M$ . By Theorem 11.7 and Lemma 11.4, there exists in particular  $i < j < \theta$  such that  $\mathbf{tp}(\bar{a}_i/A; M) = \mathbf{tp}(\bar{a}_j/A; M)$ . This implies that  $|\mathbf{S}^\kappa(A; M)| < \theta$ , as desired.  $\square$

**Corollary 11.9.** Let  $\mathbf{K}$  be an AEC with arbitrarily large models. Let  $\lambda > \text{LS}(\mathbf{K})$ . If  $\mathbf{K}_{<\lambda}$  has amalgamation and no maximal models and  $\mathbf{K}$  is categorical in  $\lambda$ , then  $\mathbf{K}$  is stable in every  $\mu \in [\text{LS}(\mathbf{K}), \lambda)$ .

*Proof.* Let  $\Phi \in \Upsilon_{\text{LS}(\mathbf{K})}[\mathbf{K}]$  be an EM blueprint for  $\mathbf{K}$ . Let  $\mu \in [\text{LS}(\mathbf{K}), \lambda)$ . Let  $M_0 \in \mathbf{K}_\mu$ . Let  $\langle p_i : i < \mu^+ \rangle$  be types over  $M_0$ . By amalgamation, we can make sure they are all realized inside a fixed  $M_1 \in \mathbf{K}_{\leq \mu^+}$  with  $M_0 \leq_{\mathbf{K}} M_1$ . Since  $\mathbf{K}$  has no maximal models, there exists  $M_2 \in \mathbf{K}_\lambda$  with  $M_1 \leq_{\mathbf{K}} M_2$ . By categoricity,  $M_2 \cong \text{EM}_\tau(\lambda, \Phi)$ . Applying Corollary 11.8 and taking an isomorphic image, we get that  $|\mathbf{S}(M_0; M_2)| \leq \mu$ , hence that there must exist  $i < j$  so that  $p_i = p_j$ . This proves that  $|\mathbf{S}(M_0)| \leq \mu$ .  $\square$

Another interesting application concerns the order property:

**Definition 11.10** (Order property, Definition 4.3 in [She99]).

- (1) Let  $N \in \mathbf{K}$  and let  $I$  be a linear order. We say that  $N$  has the  $(\kappa, \mu)$ -order property of type (or of length)  $I$  if there exists a sequence  $\langle \bar{a}_i : i \in I \rangle$  and a set  $A$  such that  $\bar{a}_i \in {}^\kappa N$  for every  $i \in I$ ,  $A \subseteq |N|$ ,  $|A| \leq \mu$ , and for every  $i_0 < i_1, j_0 < j_1$  in  $I$ ,  $\text{tp}(\bar{a}_{i_0} \bar{a}_{i_1} / A; N) \neq \text{tp}(\bar{a}_{j_1} \bar{a}_{j_0} / A; N)$ .
- (2) We say that  $\mathbf{K}$  has the  $(\kappa, \mu)$ -order property of type  $I$  if some  $N \in \mathbf{K}$  has it.
- (3) We say that  $\mathbf{K}$  has the  $(\kappa, \mu)$ -order property if it has the  $(\kappa, \mu)$ -order property of type  $I$  for every linear order  $I$ .
- (4) When  $\mu = 0$ , we omit it and talk of the  $\kappa$ -order property.

**Remark 11.11.** For  $T$  a first-order theory and  $\mathbf{K}$  its corresponding AEC of models, the following are equivalent:

- (1)  $T$  is unstable.
- (2)  $\mathbf{K}$  has the  $(\kappa, 0)$ -order property, for some  $\kappa < \aleph_0$ .
- (3)  $\mathbf{K}$  has the  $(\kappa, \mu)$ -order property, for some cardinals  $\kappa$  and  $\mu$ .

The following can be obtained by building an order property indexed by a linear order  $I$  with a dense subset  $I_0$  smaller than  $I$  (for example, given  $\lambda$ , let  $\mu$  be least such that  $\lambda = 2^{<\mu} < 2^\mu$  and let  $I_0 := {}^{<\mu}2$ ,  $I := {}^\mu 2$ , both ordered by the lexicographical ordering):

**Exercise 11.12.** [BGKV16, Fact 5.13] If  $\mathbf{K}$  has the  $(\kappa, \mu)$ -order property, then there exists  $M \in \mathbf{K}$  and  $A \subseteq |M|$  such that  $|A| = \mu$  but  $|\mathbf{S}^\kappa(A; M)| > \mu$ .

An easy consequence of Theorem 11.7 is that if a long-enough order property holds, then we can assume that the sequence witnessing it is strictly indiscernible, and hence extend it:

**Corollary 11.13.** Let  $\mathbf{K}$  be an AEC with arbitrarily large models and let  $\text{LS}(\mathbf{K}) < \lambda$ . Let  $\kappa < \lambda$  be a (possibly finite) cardinal. Let  $\Phi \in \Upsilon_{\text{LS}(\mathbf{K})}[\mathbf{K}]$  be an EM blueprint for  $\mathbf{K}$ .

Let  $N := \text{EM}_{\tau(\mathbf{K})}(\lambda, \Phi)$ . If  $N$  has the  $(\kappa, \text{LS}(\mathbf{K}))$ -order property of length  $(\text{LS}(\mathbf{K}))^\kappa$  and  $\text{LS}(\mathbf{K})^\kappa < \lambda$ , then  $\mathbf{K}$  has the  $(\kappa, \text{LS}(\mathbf{K}))$ -order property (of any length).

*Proof.* Set  $\theta := (\text{LS}(\mathbf{K}))^\kappa$ . Fix  $\langle \bar{a}_i : i < \theta \rangle$  and  $A$  witnessing that  $N$  has the  $(\kappa, \text{LS}(\mathbf{K}))$ -order property of length  $\theta$ . Using the Löwenheim-Skolem-Tarski axiom, pick  $M \in \mathbf{K}_{\text{LS}(\mathbf{K})}$  such that  $A \subseteq |M|$  and  $M \leq_{\mathbf{K}} N$ . By Theorem 11.7, there exists  $w \subseteq \theta$  such that  $|w| = \theta$  and  $\langle \bar{a}_i : i \in w \rangle$  is  $(\Phi, \lambda)$ -strictly indiscernible over  $M$ . Now check that any extension of that sequence (Exercise 11.6) witnesses the order property.  $\square$

**Remark 11.14.** By appending an enumeration of the base set to each element of the sequence, we get that the  $(\kappa, \mu)$ -order property implies the  $(\kappa + \mu)$ -order property. However Corollary 11.13 applies more easily to the  $(\kappa, \mu)$ -order property: think for example of the case  $\kappa < \aleph_0$ , when we always have that  $\text{LS}(\mathbf{K})^\kappa = \text{LS}(\mathbf{K}) < \lambda$ .

**Corollary 11.15.** Let  $\mathbf{K}$  be an AEC with arbitrarily large models. Let  $\lambda > \text{LS}(\mathbf{K})$  and assume that  $\mathbf{K}_{<\lambda}$  has amalgamation, no maximal models, and  $\mathbf{K}$  is categorical

in  $\lambda$ . Let  $\mu \in [\text{LS}(\mathbf{K}), \lambda)$  and let  $\kappa > 0$  be a cardinal. If  $\mu^\kappa < \lambda$ , then  $\mathbf{K}$  does *not* have the  $(\kappa, \mu)$ -order property of length  $(\mu^\kappa)^+$ .

*Proof.* Let  $\Phi \in \Upsilon_{\text{LS}(\mathbf{K})}[\mathbf{K}]$  be an EM blueprint. Replacing  $\mu$  by  $\mu^\kappa$  if necessary, we can assume without loss of generality that  $\mu = \mu^\kappa$ . Replacing  $\mathbf{K}$  by  $\mathbf{K}_{\geq \mu}$  if necessary, we can also assume without loss of generality that  $\mu = \text{LS}(\mathbf{K})$ . Suppose that  $\mathbf{K}$  has the  $(\kappa, \mu)$ -order property of length  $\mu^+$ . Then there exists  $M \in \mathbf{K}_\lambda$  which has the  $(\kappa, \mu)$ -order property of length  $\mu^+$ . By categoricity, we can assume without loss of generality that  $M = \text{EM}_\tau(\lambda, \Phi)$ . By Theorem 11.13,  $\mathbf{K}$  has the  $(\kappa, \mu)$ -order property. By Exercise 11.12,  $\mathbf{K}$  must be unstable in  $\mu$ , contradicting Corollary 11.9.  $\square$

We summarize the results of this section in one corollary:

**Corollary 11.16.** Let  $\mathbf{K}$  be an AEC with arbitrarily large models. Let  $\lambda > \text{LS}(\mathbf{K})$ . Assume that  $\mathbf{K}_{< \lambda}$  has amalgamation and no maximal models, and  $\mathbf{K}$  is categorical in  $\lambda$ . Let  $\mu \in [\text{LS}(\mathbf{K}), \lambda)$ . Then:

- (1)  $\mathbf{K}$  is stable in  $\mu$ .
- (2)  $\mathbf{K}$  does *not* have the  $(2, \mu)$ -order property of length  $\mu^+$ .

## 12. SUPERSTABILITY FROM CATEGORICITY

Everywhere in this section,  $\mathbf{K}$  is still an AEC. Most of the material here is derived from [BGVV17]. We want to start studying independence notions. The following definition is a starting point:

**Definition 12.1** ([She99, 3.2]).  $p \in \mathbf{S}(N)$   $\lambda$ -splits over  $M$  if there exists  $N_\ell \in \mathcal{P}_{\mathbf{K}_\lambda}(N)$  such that  $M \leq_{\mathbf{K}} N_\ell \leq_{\mathbf{K}} N$ ,  $\ell = 1, 2$  and  $f : N_1 \cong_M N_2$  such that  $f(p \upharpoonright N_1) \neq p \upharpoonright N_2$ . When  $\lambda = \|N\| = \|M\|$ , we may omit it.

The following are basic properties of splitting:

**Exercise 12.2.** Let  $\mathbf{K}$  be an AEC. Let  $\lambda \geq \text{LS}(\mathbf{K})$ . Assume that  $\mathbf{K}_\lambda$  has amalgamation. Let  $M \leq_{\mathbf{K}} M' \leq_{\mathbf{K}} N$  all be in  $\mathbf{K}_\lambda$ .

- (1) Invariance: if  $p \in \mathbf{S}(N)$  does not split over  $M$  and  $f : N \cong N'$ , then  $f(p)$  does not split over  $f[M]$ .
- (2) Monotonicity: if  $p \in \mathbf{S}(N)$  does not split over  $M$ , then  $p \upharpoonright M'$  does not split over  $M$  and  $p$  does not split over  $M'$ .

We are interested in studying the following local character properties that splitting may have:

**Definition 12.3.** Let  $\mathbf{K}$  be an AEC,  $\lambda \geq \text{LS}(\mathbf{K})$ .

- (1) For a limit ordinal  $\alpha < \lambda^+$ , splitting has *weak universal local character at  $\alpha$  in  $\mathbf{K}_\lambda$*  if for any increasing continuous sequence  $\langle M_i \in \mathbf{K}_\lambda \mid i \leq \alpha \rangle$  and any type  $p \in \mathbf{S}(M_\alpha)$ , if  $M_{i+1}$  is universal over  $M_i$  for each  $i < \alpha$ , then there is some  $i_0 < \alpha$  such that  $p \upharpoonright M_{i_0+1}$  does not split over  $M_{i_0}$ .

- (2) For a limit ordinal  $\alpha < \lambda^+$ , splitting has *strong universal local character at  $\alpha$  in  $\mathbf{K}_\lambda$*  if for any increasing continuous sequence  $\langle M_i \in \mathbf{K}_\lambda \mid i \leq \alpha \rangle$  and any type  $p \in \mathbf{S}(M_\alpha)$ , if  $M_{i+1}$  is universal over  $M_i$  for each  $i < \alpha$ , then there is some  $i_0 < \alpha$  such that  $p$  does not split over  $M_{i_0}$ .

Weak universal local character at some  $\alpha$  can be obtained by building a tree of types. The argument was presented by Will Boney last semester. It is due to Shelah [She99, I.3.3].

**Fact 12.4.** Let  $\mathbf{K}$  be an AEC. Let  $\lambda \geq \text{LS}(\mathbf{K})$ . Assume that  $\mathbf{K}_\lambda$  has amalgamation. If  $\mathbf{K}$  is stable in  $\lambda$ , then in  $\mathbf{K}_\lambda$  splitting has weak universal local character at every cardinal  $\sigma < \lambda^+$  such that  $2^\sigma > \lambda$ .

At this point, the reader may forget the definition of splitting. All that we will use are the properties just listed.

We want to derive the strong version of universal local character. In fact, we give a name to setups where it holds.

**Definition 12.5.** We call an AEC  $\mathbf{K}$   $\lambda$ -*superstable* if:

- (1)  $\lambda \geq \text{LS}(\mathbf{K})$ .
- (2)  $\mathbf{K}_\lambda$  is not empty, has amalgamation, joint embedding, and no maximal models.
- (3)  $\mathbf{K}$  is stable in  $\lambda$ .
- (4) In  $\mathbf{K}_\lambda$ , splitting has strong universal local character at every limit ordinal  $\alpha < \lambda^+$ .

We will show the following result, essentially due to Shelah and Villaveçes [SV99] (with some small gaps fixed in [BGVV17]):

**Theorem 12.6.** Let  $\mathbf{K}$  be an AEC. Let  $\lambda \geq \text{LS}(\mathbf{K})$ . If:

- (1)  $\mathbf{K}_\lambda$  has amalgamation, joint embedding, and no maximal models.
- (2)  $\mathbf{K}$  is stable in  $\lambda$ .
- (3) There is an EM blueprint  $\Phi \in \Upsilon_{\leq \lambda}[\mathbf{K}]$  such that any  $\text{EM}_\tau(\lambda^+, \Phi)$  is universal in  $\mathbf{K}_{\lambda^+}$ .

Then  $\mathbf{K}$  is  $\lambda$ -superstable.

**Remark 12.7.** In fact, stability in  $\lambda$  follows from the other hypotheses by Corollary 11.8.

**Corollary 12.8** (Superstability from categoricity). Let  $\mathbf{K}$  be an AEC with arbitrarily large models. Let  $\lambda > \text{LS}(\mathbf{K})$ . Assume that  $\mathbf{K}_{<\lambda}$  has amalgamation and no maximal models. If  $\mathbf{K}$  is categorical in  $\lambda$ , then  $\mathbf{K}$  is  $\mu$ -superstable for any  $\mu \in [\text{LS}(\mathbf{K}), \lambda)$ .

*Proof.* Let  $\mu \in [\text{LS}(\mathbf{K}), \lambda)$ . By Corollary 11.9,  $\mathbf{K}$  is stable in  $\mu$ . By categoricity in  $\lambda$ , it is easy to see that  $\mathbf{K}_\mu$  is not empty and has joint embedding. To apply Theorem 12.6, it suffices to find an EM blueprint  $\Phi$  such that  $\text{EM}_\tau(\mu^+, \Phi)$  is universal in  $\mathbf{K}_{\mu^+}$ . First fix an EM blueprint  $\Psi \in \Upsilon_{\text{LS}(\mathbf{K})}[\mathbf{K}]$ . Note that any  $M \in \mathbf{K}_{\mu^+}$  embeds into  $\text{EM}_\tau(\lambda, \Psi)$  by categoricity. Thus it embeds into some  $\text{EM}_\tau(I, \Psi)$ , with  $I \subseteq \lambda$

$|I| \leq \mu^+$ . Thus it embeds into  $\text{EM}_\tau(\alpha, \Psi)$ , where  $\alpha := \text{otp}(I) < \mu^{++}$ . Now note that  ${}^{<\omega}\mu^+$  (ordered by the lexicographical ordering) contains a copy of  $\alpha$  for each  $\alpha < \mu^{++}$ , see [Bal09, Claim 15.5]. Thus  $\text{EM}_\tau({}^{<\omega}\mu^+, \Psi)$  is universal in  $\mathbf{K}_{\mu^+}$ . We can now change the blueprint  $\Psi$  to a new blueprint  $\Phi$  (in expanded vocabulary) such that  $|\tau(\Phi)| \leq \mu$  and  $\text{EM}_\tau(I, \Phi) = \text{EM}_\tau({}^{<\omega}I, \Psi)$  for any linear order  $I$ . In particular,  $\text{EM}_\tau(\mu^+, \Phi)$  is universal in  $\mathbf{K}_{\mu^+}$ , as desired.  $\square$

**12.1. The proof of Theorem 12.6.** For the rest of this section, we assume:

**Hypothesis 12.9.**

- (1)  $\mathbf{K}$  is an AEC.
- (2)  $\lambda \geq \text{LS}(\mathbf{K})$ .
- (3)  $\mathbf{K}_\lambda$  is not empty, has joint embedding, amalgamation, and no maximal models.
- (4)  $\mathbf{K}$  is stable in  $\lambda$ .

Recall that this implies (Corollary 7.22) that we can construct universal extensions in  $\mathbf{K}_\lambda$ . The following definition is crucial and will be explored much more later:

**Definition 12.10.** Let  $\delta < \lambda^+$  be a limit ordinal. We call  $M$   $(\lambda, \delta)$ -*limit* over  $M_0$  if there exists an increasing continuous chain  $\langle N_i : i \leq \delta \rangle$  in  $\mathbf{K}_\lambda$  such that  $N_0 = M_0$ ,  $N_\delta = N$ , and  $N_{i+1}$  is universal over  $N_i$  for all  $i < \delta$ . We say that  $M$  is  $(\lambda, \delta)$ -*limit* if it is  $(\lambda, \delta)$ -limit over some  $M_0$ . When we just say “limit” we mean “ $(\lambda, \delta)$ -limit” for some  $\lambda$  and  $\delta$  (but  $\lambda$  is fixed in this section).

An easy back and forth argument gives:

**Exercise 12.11.** Let  $M_\ell$  be  $(\lambda, \delta_\ell)$ -limit,  $\ell = 1, 2$ . Assume that  $\text{cf}(\delta_1) = \text{cf}(\delta_2)$ . Then:

- (1)  $M_1 \cong M_2$ .
- (2) If there is  $M_0$  such that  $M_\ell$  is  $(\lambda, \delta_\ell)$ -limit over  $M_0$  for  $\ell = 1, 2$ , then  $M_1 \cong_{M_0} M_2$ .

We start by stating more local character properties that splitting may have:

**Definition 12.12.** Let  $\alpha < \lambda^+$  be a limit ordinal.

- (1) Splitting has *universal continuity at  $\alpha$*  if for any increasing continuous sequence  $\langle M_i \in \mathbf{K}_\lambda \mid i \leq \alpha \rangle$  and any type  $p \in \mathbf{S}(M_\alpha)$ , if for each  $i < \alpha$   $M_{i+1}$  is universal over  $M_i$  and  $p \restriction M_i$  does not split over  $M_0$ , then  $p$  does not split over  $M_0$ .
- (2) For  $\delta < \lambda^+$  a limit, splitting has *no  $\delta$ -limit alternations at  $\alpha$*  if for any increasing continuous sequence  $\langle M_i \in \mathbf{K}_\lambda \mid i \leq \alpha \rangle$  with  $M_{i+1}$   $(\lambda, \delta)$ -limit over  $M_i$  for all  $i < \alpha$  and any type  $p \in \mathbf{S}(M_\alpha)$ , there exists  $i < \alpha$  such that the following fails:  $p \restriction M_{2i+1}$  splits over  $M_{2i}$  and  $p \restriction M_{2i+2}$  does not split over  $M_{2i+1}$ . If this fails, we say that splitting has  $\delta$ -limit alternations at  $\alpha$ .

We start by showing that having nice EM models implies that splitting has these two properties. We will then show completely locally that universal continuity, no alternations, and weak universal local character imply strong universal local character.

**Lemma 12.13.** Assume  $\mathbf{K}$  has an EM blueprint  $\Phi$  with  $|\tau(\Phi)| \leq \lambda$  such that every  $M \in \mathbf{K}_{[\lambda, \lambda^+]}$  embeds inside  $\text{EM}_\tau(\lambda^+, \Phi)$ . Let  $\alpha < \lambda^+$  be a regular cardinal. Then:

- (1) Splitting has universal continuity at  $\alpha$ .
- (2) If in addition  $\alpha < \lambda$ , then for any limit  $\gamma < \mu^+$ , splitting has no  $\gamma$ -limit alternations at  $\alpha$ .

*Proof.* Let  $\langle M_i \mid i \leq \alpha \rangle$  and  $p$  be as in the definition of universal continuity or  $\gamma$ -limit alternations. For simplicity, let us assume that in the case of universal continuity  $M_0$  is  $(\lambda, \gamma)$ -limit and that  $M_{i+1}$  is  $(\lambda, \gamma)$ -limit over  $M_i$  for all  $i < \alpha$ . The general case can be obtained from this after some renaming.

Let  $S_\alpha^{\lambda^+} := \{\delta < \lambda^+ \mid \text{cf}(\delta) = \alpha\}$ . We say that  $\bar{C} = \langle C_\delta \mid \delta \in S_\alpha^{\lambda^+} \rangle$  is an  $S_\alpha^{\lambda^+}$ -club sequence if each  $C_\delta \subseteq \delta$  is club. Clearly, club sequences exist: just take  $C_\delta := \delta$  (this will be enough for proving universal continuity). Shelah [She94] proves the existence of club-guessing club sequences in ZFC under various hypotheses (the specific result that we use will be stated later, see Fact 12.14). We will describe a construction of a sequence of models  $\bar{N}(\bar{C})$  based on a club sequence and then plug in the necessary club sequence in each case.

Given an  $S_\alpha^{\lambda^+}$ -club sequence  $\bar{C}$ , enumerate  $C_\delta \cup \{\delta\}$  in increasing order as  $\langle \beta_{\delta, j} \mid j \leq \alpha \rangle$ .

**Claim:** Let  $\gamma < \lambda^+$  be a limit ordinal. We can build increasing, continuous  $\bar{N}(\bar{C}) = \langle N_i \in \mathbf{K}_\lambda \mid i < \lambda^+ \rangle$  such that for all  $i < \lambda^+$ :

- (1)  $N_{i+1}$  is  $(\lambda, \gamma)$ -limit over  $N_i$ ;
- (2) when  $i \in S_\alpha^{\lambda^+}$ , there is  $g_i : M_\alpha \cong N_i$  such that  $g_i[M_j] = N_{\beta_{i, j}}$  for all  $j \leq \alpha$ ; and:
- (3) when  $i \in S_\alpha^{\lambda^+}$ , there is  $a_i \in N_{i+1}$  that realizes  $g_i(p)$ .

**Proof of Claim:** Build the increasing continuous chain of models as follows: start with any  $N_0 \in \mathbf{K}_\lambda$ . Given an  $N_i$ , build  $N_{i+1}$  to be  $(\lambda, \gamma)$ -limit over it. At limits, take unions. At limits  $i$  of cofinality  $\alpha$ , use the uniqueness of  $(\lambda, \gamma)$ -limits models to find the desired isomorphisms: the weak version gives  $M_0 \cong M_{\beta_{i, 0}}$ , and the strong (over the base) version allows this isomorphism to be extended to get an isomorphism  $g_i$  between  $\langle M_j \mid j \leq \alpha \rangle$  and  $\langle N_{\beta_{i, j}} \mid j \leq \alpha \rangle$  as described. Since  $N_{i+1}$  is universal over  $N_i$ , we there is some  $a_i \in N_{i+1}$  that realizes  $g_i(p)$ .  $\upharpoonright_{\text{Claim}}$

By assumption, we may assume that  $N := \bigcup_{i < \lambda^+} N_i \leq_{\mathbf{K}} \text{EM}_\tau(\lambda^+, \Phi)$ . Thus, we can write  $a_i = \rho_i(\gamma_1^i, \dots, \gamma_{n(i)}^i)$  with:

$$\gamma_1^i < \dots < \gamma_{m(i)}^i < i \leq \gamma_{m(i)+1}^i < \dots < \gamma_{n(i)}^i < \lambda^+$$

Now we begin to prove each part of the lemma. In each, we will find  $i_1 < i_2 \in S_\alpha^{\lambda^+}$  such that  $\text{tp}(a_{i_1}/N_{i_1}; N)$  and  $\text{tp}(a_{i_2}/N_{i_1}; N)$  are both the same (because of the EM structure) and different (because they exhibit different splitting behavior), which is our contradiction.

- (1) Assume that  $p \upharpoonright M_j$  does not split over  $M_0$ , for all  $j < \alpha$ .

Let  $\bar{C}$  be an  $S_\alpha^{\lambda^+}$ -club sequence, and set  $\langle N_i \in \mathbf{K}_\lambda \mid i < \lambda^+ \rangle = \bar{N}(\bar{C})$  as in the Claim (the value of  $\gamma$  doesn't matter here, e.g. take  $\gamma := \omega$ ). By Fodor's Lemma, there is a stationary subset  $S^* \subseteq S_\alpha^{\lambda^+}$ , a term  $\rho_*$ ,  $m_*, n_* < \omega$  and ordinals  $\gamma_0^*, \dots, \gamma_{n_*}^*, \beta_{*,0}$  such that:

For every  $i \in S^*$ , we have  $\rho_i = \rho_*$ ;  $n(i) = n_*$ ;  $m(i) = m_*$ ;  $\gamma_j^i = \gamma_j^*$  for  $j \leq m_*$ ; and  $\beta_{i,0} = \beta_{*,0}$ .

Set  $E := \{\delta < \lambda^+ \mid \delta \text{ is limit and } N_\delta \leq_{\mathbf{K}} \text{EM}_\tau(\delta, \Phi)\}$ . This is a club. Let  $i_1 < i_2$  both be in  $S^* \cap E$ . Then we have:

$$\begin{aligned} \mathbf{tp}(a_{i_1}/N_{i_1}) &= \mathbf{tp}(\rho_*(\gamma_1^*, \dots, \gamma_{m_*}^*, \gamma_{m_*+1}^{i_1}, \dots, \gamma_{n_*}^{i_1})/N \cap \text{EM}_\tau(i_1, \Phi)) \\ &= \mathbf{tp}(\rho_*(\gamma_1^*, \dots, \gamma_{m_*}^*, \gamma_{m_*+1}^{i_2}, \dots, \gamma_{n_*}^{i_2})/N \cap \text{EM}_\tau(i_1, \Phi)) \\ &= \mathbf{tp}(a_{i_2}/N_{i_1}) \end{aligned}$$

where all the types are computed inside  $N$ . This is because the only differences between  $a_{i_1}$  and  $a_{i_2}$  lie entirely above  $i_1$ .

We have that  $g_{i_1} : (N_{i_1}, N_{\beta_{*,0}}) \cong (M_\alpha, M_0)$  and that  $p$  splits over  $M_0$ . Thus,  $\mathbf{tp}(a_{i_1}/N_{i_1}) = g_{i_1}(p)$  splits over  $N_{\beta_{*,0}}$ . On the other hand,  $C_{i_2}$  is cofinal in  $i_2$ , so there is  $j < \alpha$  such that  $\beta_{i_2,j} > i_1$  and, thus,  $N_{i_1} \leq_{\mathbf{K}} N_{\beta_{i_2,j}}$ . Again,  $g_{i_2} : (N_{\beta_{i_2,j}}, N_{\beta_{*,0}}) \cong (M_j, M_0)$  and  $p \upharpoonright M_j$  does not split over  $M_0$  by assumption. Thus,  $\mathbf{tp}(a_{i_2}/N_{\beta_{i_2,j}}) = g_{i_2}(p \upharpoonright M_j)$  does not split over  $N_{\beta_{*,0}}$ . By monotonicity,  $\mathbf{tp}(a_{i_2}/N_{i_1})$  does not split over  $N_{\beta_{*,0}}$ . Thus,  $\mathbf{tp}(a_{i_1}/N_{i_1}) \neq \mathbf{tp}(a_{i_2}/N_{i_1})$ , a contradiction.

- (2) Let  $\chi$  be a big-enough cardinal and create an increasing, continuous elementary chain of models of set theory  $\langle \mathfrak{B}_i \mid i < \lambda^+ \rangle$  such that for all  $i < \lambda^+$ :

- (a)  $\mathfrak{B}_i \prec (H(\chi), \in)$ ;
- (b)  $\|\mathfrak{B}_i\| = \lambda$ ;
- (c)  $\mathfrak{B}_0$  contains, as elements<sup>2</sup>,  $\Phi$ ,  $\text{EM}(\lambda^+, \Phi)$ ,  $\langle g_i \mid i < \lambda^+ \rangle$ ,  $\lambda^+$ ,  $\langle N_i \mid i < \lambda^+ \rangle$ ,  $S_\alpha^{\lambda^+}$ ,  $\langle a_i \mid i \in S_\alpha^{\lambda^+} \rangle$ , and each  $f \in \tau(\Phi)$ ; and
- (d)  $\mathfrak{B}_i \cap \lambda^+$  is an ordinal.

We will use the following fact which was originally proven in [She94, III.2] (or see [AM10, Theorem 2.17] for a short proof).

**Fact 12.14.** Let  $\lambda$  be a cardinal such that  $\text{cf}(\lambda) \geq \theta^{++}$  for some regular  $\theta$  and let  $S \subseteq S_\theta^\lambda$  be stationary. Then there is a  $S$ -club sequence  $\langle C_\delta \mid \delta \in S \rangle$  such that, if  $E \subseteq \lambda$  is club, then there are stationarily many  $\delta \in S$  such that  $C_\delta \subseteq E$ .

We have that  $\alpha < \lambda$ , so we can apply Fact 12.14 with  $\lambda, \theta, S$  there standing for  $\lambda^+, \alpha, S_\alpha^{\lambda^+}$  here. Let  $\bar{C}$  be the  $S_\alpha^{\lambda^+}$ -club sequence that the fact gives. Let  $\langle N_i \in \mathbf{K}_\lambda \mid i < \lambda^+ \rangle = \bar{N}(\bar{C})$  be as in the Claim. Note that  $E := \{i < \lambda^+ \mid \mathfrak{B}_i \cap \lambda^+ = i\}$  is a club. By the conclusion of Fact 12.14, there is some  $i_2 \in S_\alpha^{\lambda^+}$  such that  $C_{i_2} \subseteq E$ . We have  $a_{i_2} = \rho_{i_2}(\gamma_1^{i_2}, \dots, \gamma_{n(i_2)}^{i_2})$ , with:

<sup>2</sup>When we say that  $\mathfrak{B}_0$  contains a sequence as an element, we mean that it contains the function that maps an index to its sequence element.



$$\gamma_1^{i_2} < \dots < \gamma_{m(i_2)}^{i_2} < i_2 \leq \gamma_{m(i_2)+1}^{i_2} < \dots < \gamma_{n(i_2)}^{i_2}$$

Since the  $\beta_{i_2,j}$ 's enumerate a cofinal sequence in  $i_2$ , we can find  $j < \alpha$  such that  $\gamma_{m(i_2)}^{i_2} < \beta_{i_2,2j+1} < i_2$ . Recall that we have  $p \upharpoonright M_{2j+2}$  does not split over  $M_{2j+1}$  by assumption. Then  $(H(\chi), \in)$  satisfies the following formulas with parameters exactly the objects listed in item (2c) above and ordinals below  $\beta_{i_2,2j+2}$ :

$$\begin{aligned} \exists x, y_{m(i_2)+1}, \dots, y_{n(i_2)} \cdot (& \text{"} x \in S_\alpha^{\lambda^+} \text{"} \\ & \wedge \text{"} x > \beta_{i_2,2j+1} \text{"} \wedge \text{"} y_k \in (x, \lambda^+) \text{" are increasing ordinals"} \\ & \wedge \text{"} a_x = \rho_{i_2}(\gamma_1^{i_2}, \dots, \gamma_{m(i_2)}^{i_2}, y_{m(i_2)+1}, \dots, y_{n(i_2)}) \text{"} \\ & \wedge \text{"} N_x \subseteq \text{EM}(x, \Phi) \text{"}) \end{aligned}$$

This is witnessed by  $x = i_2$  and  $y_k = \gamma_k^{i_2}$ . By elementarity,  $\mathfrak{B}_{\beta_{i_2,2j+2}}$  satisfies this formula as it contains all the parameters. Let  $i_1 \in (\beta_{i_2,2j+1}, \lambda^+) \cap \mathfrak{B}_{\beta_{i_2,2j+2}} = (\beta_{i_2,2j+1}, \beta_{i_2,2j+2})^3$  witness this, along with  $\gamma'_{m(i_2)+1} < \dots < \gamma'_{n(i_2)} < \lambda^+$ . Then we have:

$$a_{i_1} = \rho_{i_2}(\gamma_1^{i_2}, \dots, \gamma_{m(i_2)}^{i_2}, \gamma'_{m(i_2)+1}, \dots, \gamma'_{n(i_2)})$$

with  $\beta_{i_2,2j+1} < \gamma_{m(i_2)+1}$ . We want to compare  $\mathbf{tp}(a_{i_2}/N_{i_1})$  and  $\mathbf{tp}(a_{i_1}/N_{i_1})$ .

- From the elementarity, we get that  $N_{i_1} \subseteq \text{EM}_\tau(i_1, \Phi)$ . We also know that  $i_1 < \beta_{i_2,2j+2} < \gamma_{m(i_2)+1}^{i_2}, \gamma'_{m(i_2)+1}$ . Thus, as before, the types are equal.
- We know that  $p \upharpoonright M_{2j+2}$  does not split over  $M_{2j+1}$ . Thus,  $\mathbf{tp}(a_{i_2}/N_{\beta_{i_2,2j+2}})$  does not split over  $N_{\beta_{i_2,2j+1}}$ . Since we have  $N_{\beta_{i_2,2j+1}} \leq_{\mathbf{K}} N_{i_1} \leq_{\mathbf{K}} N_{\beta_{i_2,2j+2}}$ , this gives  $\mathbf{tp}(a_{i_2}/N_{i_1})$  does not split over  $N_{\beta_{i_2,2j+1}}$ .
- We have  $\beta_{i_2,2j+1} < i_1$ , so there is some  $k < \alpha$  such that  $\beta_{i_2,2j+1} < \beta_{i_1,k} < i'$ . By assumption,  $p$  splits over  $M_k$ . Thus  $g_{i_1}(p)$  splits over  $N_{\beta_{i_1,k}}$ . Therefore  $\mathbf{tp}(a_{i_1}/N_{i_1})$  splits over  $N_{\beta_{i_2,2j+1}} \leq_{\mathbf{K}} N_{\beta_{i_1,k}}$ .

As before, these three statements contradict each other.

□

In Lemma 12.13, we are missing  $\gamma$ -limit alternations at  $\lambda$ . This is fixed by:

**Lemma 12.15.** For any limit  $\delta < \lambda^+$ , splitting has no  $\delta$ -limit alternations at  $\lambda$ .

*Proof.* By Fact 12.4, splitting has weak universal local character at  $\lambda$ . Fix  $\langle M_i : i \leq \lambda \rangle$ ,  $\delta, p$  as in the definition of having no  $\delta$ -limit alternations and apply weak universal local character to the chain  $\langle M_{2i} : i \leq \lambda \rangle$ . □

We now outline how we are going to prove strong universal local character. We already have weak universal local character, continuity, and no alternations. Three important basic results are

<sup>3</sup>The equality here is the key use of club guessing.

- It suffices to prove strong universal local character at regular cardinals (Lemma 12.16.(1)).
- Continuity together with weak local character imply strong local character at *regular length* (Lemma 12.16.(2)); and
- It does not matter whether in the definition of weak and strong universal local character we require “ $M_{i+1}$  limit over  $M_i$ ” or “ $M_{i+1}$  universal over  $M_i$ ,” and the length of the limit models does not matter (Lemma 12.16.(3)).

The second of these is proven by contradiction, and the first and third are straightforward.

Assume for a moment we have strong universal local character at some limit length  $\gamma$ . Let us try to prove weak universal local character at (say)  $\omega$  (then we can use the second basic result to get the strong version, assuming continuity). By the third basic result, we can assume we are given an increasing continuous sequence  $\langle M_n : n \leq \omega \rangle$  with  $M_{n+1}$   $(\lambda, \gamma)$ -limit over  $M_n$  for all  $n < \omega$  and  $p \in \mathbf{S}(M_\omega)$ . By the strong universal local character assumption we know that  $p \restriction M_{n+1}$  does not split over some intermediate model between  $M_n$  and  $M_{n+1}$ , so if we assume that  $p \restriction M_{n+1}$  splits over  $M_n$  for all  $n < \omega$ , we will end up getting alternations. This is the essence of Lemma 12.16.(6).

Thus to prove strong universal local character at *all* cardinals, it is enough to obtain it at *some* cardinal. Fortunately, we already know weak universal local character holds at some  $\sigma$  (Fact 12.4). If  $\sigma$  is regular we are done by the second basic result, but unfortunately  $\sigma$  could be singular (if  $\lambda$  is singular and  $2^{<\lambda} = \lambda$ , e.g.  $\lambda = \beth_\omega$ ). In this case Lemma 12.16.(5) (using Lemma 12.16.(4) as an auxiliary claim) shows that failure of strong universal local character at  $\sigma$  implies alternations, even when  $\sigma$  is singular.

**Lemma 12.16.** Let  $\alpha < \mu^+$  be a regular cardinal,  $\sigma < \mu^+$  be a (not necessarily regular) cardinal, and  $\delta < \mu^+$  be a limit ordinal.

- (1) If splitting has strong universal character at all regular cardinals below  $\lambda^+$ , then splitting has strong universal character at all limits below  $\lambda^+$ .
- (2) If splitting has universal continuity at  $\alpha$  and weak universal local character at  $\alpha$ , then splitting has strong universal local character at  $\alpha$ .
- (3) We obtain an equivalent definition of weak [strong] universal local character at  $\sigma$ , if we ask in addition that “ $M_{i+1}$  is  $(\mu, \delta)$ -limit over  $M_i$ ” for all  $i < \sigma$ .
- (4) Assume that splitting has weak universal local character at  $\sigma$ . Let  $\langle M_i : i \leq \sigma \rangle$  be increasing continuous in  $\mathbf{K}_\lambda$  with  $M_{i+1}$  universal over  $M_i$  for all  $i < \sigma$ . For any  $p \in \mathbf{S}(M_\sigma)$  there exists a *successor*  $i < \sigma$  such that  $p \restriction M_{i+1}$  does not split over  $M_i$ .
- (5) If splitting has universal continuity at  $\sigma$ , weak universal local character at  $\sigma$ , and no  $\delta$ -limit alternations at  $\omega$ , then splitting has strong universal local character at  $\sigma$ .
- (6) Assume that splitting has strong universal local character at  $\sigma$ . If splitting does *not* have weak universal local character at  $\alpha$ , then splitting has  $\sigma$ -limit alternations at  $\alpha$ .

*Proof.*

- (1) Straightforward by cofinality consideration and the monotonicity of splitting.
- (2) Suppose that  $\langle M_i : i \leq \alpha \rangle$ ,  $p$  is a counterexample.

**Claim:** For each  $i < \alpha$ , there exists  $j_i \in (i, \alpha)$  such that  $p \restriction M_{j_i}$  splits over  $M_i$ .

**Proof of Claim:** If  $i < \alpha$  is such that for all  $j \in (i, \alpha)$ ,  $p \restriction M_j$  does not split over  $M_i$ , then applying universal continuity at  $\alpha$  on the chain  $\langle M_k : k \in [i, \alpha] \rangle$  we would get that  $p$  does not split over  $M_i$ , contradicting the choice of  $\langle M_i : i \leq \alpha \rangle$ ,  $p$ .  $\uparrow$ Claim

Now define inductively for  $i \leq \alpha$ ,  $k_0 := 0$ ,  $k_{i+1} := j_{k_i}$ , and when  $i$  is limit  $k_i := \sup_{j < i} k_j$ . Note that  $\langle k_i : i \leq \alpha \rangle$  is strictly increasing continuous and  $i < \alpha$  implies  $k_i < \alpha$  (this uses regularity of  $\alpha$ ; when  $\alpha$  is singular, see (5)).

Apply weak universal local character to the chain  $\langle M_{k_i} : i \leq \alpha \rangle$  and the type  $p$ . We get that there exists  $i < \alpha$  such that  $p \restriction M_{k_{i+1}}$  does not split over  $M_{k_i}$ . This is a contradiction since  $k_{i+1} = j_{k_i}$  and we chose  $j_{k_i}$  so that  $p \restriction M_{j_{k_i}}$  splits over  $M_{k_i}$ .

- (3) We prove the result for weak universal local character, and the proof for the strong version is similar. Fix  $\langle M_i^0 : i \leq \sigma \rangle$ ,  $p$  witnessing failure of weak universal local character at  $\sigma$ . We build a witness of failure  $\langle M_i : i \leq \sigma \rangle$ ,  $p$  such that  $M_\sigma = M_\sigma^0$ , and  $M_{i+1}$  is  $(\lambda, \delta)$ -limit over  $M_i$  for each  $i < \alpha$ . Using existence of universal extensions, we can extend each  $M_i^0$  to  $M_i^*$  that is  $(\lambda, \delta)$ -limit over  $M_i^0$ . Since  $M_{i+1}^0$  is universal over  $M_i^0$ , we can find  $f_i : M_{i+1}^* \xrightarrow{M_i^0} M_{i+1}^0$ . Now set  $M_i^1 := M_i^0$  for  $i \leq \sigma$  limit or 0 and  $M_{i+1}^1 := f_i[M_{i+1}^*]$ . This is an increasing continuous chain with  $M_{i+1}^1$   $(\lambda, \delta)$ -limit over  $M_i^1$ . Let  $M_i := M_{2i}^1$ .

This works: if there was an  $i < \sigma$  such that  $p \restriction M_{i+1}$  does not split over  $M_i$ , this would mean that  $p \restriction M_{2i+2}^1$  does not split over  $M_{2i}^1$ , but since  $M_{2i}^1 \leq_{\mathbf{K}} M_{2i+1}^0 \leq_{\mathbf{K}} M_{2i+2}^0 \leq_{\mathbf{K}} M_{2i+2}^1$ , we have by monotonicity that  $p \restriction M_{2i+2}^0$  does not split over  $M_{2i+1}^0$ , a contradiction.

- (4) Apply weak universal local character to the chain  $\langle M_{2i} : i < \sigma \rangle$  to get  $j < \sigma$  such that  $p \restriction M_{2j+2}$  does not split over  $M_{2j}$ . By monotonicity, this implies that  $p \restriction M_{2j+2}$  does not split over  $M_{2j+1}$ . Let  $i := 2j + 1$ .
- (5) Suppose not, and let  $\langle M_i : i \leq \sigma \rangle$ ,  $p$  be a counterexample. By (3), without loss of generality  $M_{i+1}$  is  $(\lambda, \delta)$ -limit over  $M_i$  for all  $i < \delta$ . As in the proof of (2), for each  $\beta < \sigma$ , there exists  $j_i \in [i, \beta)$  such that  $p \restriction M_{j_i}$  splits over  $M_i$ . On the other hand, applying (4) to the chain  $\langle M_j : j \in [j_i, \beta] \rangle$ , for each  $i < \beta$ , there exists a successor ordinal  $k_i \geq j_i$  such that  $p \restriction M_{k_{i+1}}$  does not split over  $M_{k_i}$ . Define by induction on  $n \leq \omega$ ,  $m_0 := 0$ ,  $m_{2n+1} := k_{m_{2n}}$ ,  $m_{2n+2} := k_{m_{2n}} + 1$ , and  $m_\omega := \sup_{n < \omega} m_n$ . By construction, the sequence  $\langle M_{m_n} : n \leq \omega \rangle$  witnesses that splitting has  $\delta$ -limit alternations at  $\omega$  (we use that  $k_i$  is a successor to see that each model is  $(\lambda, \gamma)$ -limit over the previous ones), a contradiction.
- (6) Let  $\gamma := \sigma \cdot \sigma$ . By (3), there exists  $\langle M_i : i \leq \alpha \rangle$ ,  $p$  witnessing failure of weak universal local character at  $\alpha$  such that for all  $i < \alpha$ ,  $M_{i+1}$  is  $(\lambda, \gamma)$ -limit

over  $M_i$ . Let  $\langle M_{i,j} : j \leq \gamma \rangle$  witness that  $M_{i+1}$  is  $(\lambda, \gamma)$ -limit over  $M_i$  (i.e. it is increasing continuous with  $M_{i,j+1}$  universal over  $M_{i,j}$  for all  $j < \gamma$ ,  $M_{i,0} = M_i$ , and  $M_{i,\delta} = M_{i+1}$ ). By strong universal local character at  $\sigma$ , for all  $i < \alpha$ , there exists  $j_i < \gamma$  such that  $p \upharpoonright M_{i+1}$  does not fork over  $M_{i,j_i}$ . By replacing  $j_i$  by  $j_i + \sigma$  if necessary we can assume without loss of generality that  $\text{cf}(j_i) = \text{cf}(\sigma)$ .

Observe also that for any  $i < \alpha$ ,  $p \upharpoonright M_{i+1,j_i}$  splits over  $M_i$  (using monotonicity and the assumption that  $p \upharpoonright M_{i+1}$  splits over  $M_i$ ). Therefore  $\langle M_0, M_{1,j_1}, M_2, M_{3,j_3}, \dots \rangle$ ,  $p$  witness that splitting has  $\sigma$ -limit alternations at  $\alpha$ .

□

*Proof of Theorem 12.6.* By Fact 12.4, splitting has weak universal local character at some  $\sigma < \lambda^+$ . By lemma 12.13 and 12.15, splitting also has continuity and no  $\delta$ -limit alternations for every limit  $\alpha, \delta < \lambda^+$ .

By Lemma 12.16(5), splitting has strong universal character at  $\sigma$ . By Lemma 12.16(6), splitting has weak universal character at every regular cardinal  $\alpha < \lambda^+$ . By Lemma 12.16(2), splitting has strong universal character at every regular cardinal  $\alpha < \lambda^+$ . This suffices by Lemma 12.16(1). □

### 13. SUPERSTABILITY AND UNIQUENESS

The material of this section is mostly from [Vas17a].

Recall the definition of  $\lambda$ -superstability (Definition 12.5). We want to study the following independence notion:

**Definition 13.1.** Let  $\mathbf{K}$  be a  $\lambda$ -superstable AEC. For  $M \leq_{\mathbf{K}} N$  both in  $\mathbf{K}_\lambda$ ,  $p \in \mathbf{S}(N)$ , we say that  $p$  *does not  $\lambda$ -fork over*  $(M_0, M)$  if  $M_0 \in \mathbf{K}_\lambda$  is such that  $M$  is universal over  $M_0$  and  $p$  does not  $\lambda$ -split over  $M_0$ . We say that  $p$  *does not  $\lambda$ -fork over*  $M$  if there exists  $M_0$  such that  $p$  does not  $\lambda$ -fork over  $(M_0, M)$ . When  $\lambda$  is clear from context, we omit it.

It turns out nonforking is much better behaved than nonsplitting. In fact in all known cases it defines a canonical superstable-like independence notion. We will see that in the elementary case it coincides with first-order nonforking, at least over limit models.

The following will prove very useful and will be used without comments:

**Exercise 13.2.** Let  $\mathbf{K}$  be a  $\lambda$ -superstable AEC and let  $M_0 \leq_{\mathbf{K}} M_1 \leq_{\mathbf{K}} M_2$  all be in  $\mathbf{K}_\lambda$ . If  $M_1$  is universal over  $M_0$ , then  $M_2$  is universal over  $M_0$ . If  $M_2$  is universal over  $M_1$ , then  $M_2$  is universal over  $M_0$ .

The following are basic properties of nonforking:

**Lemma 13.3.** Let  $\mathbf{K}$  be a  $\lambda$ -superstable AEC.

- (1) (Invariance) If  $p \in \mathbf{S}(N)$  does not fork over  $(M_0, M)$  and  $f : N \cong N'$ , then  $f(p)$  does not fork over  $(f[M_0], f[M])$ .
- (2) (Monotonicity) If  $p \in \mathbf{S}(N)$  does not fork over  $(M_0, M)$ , then:

- (a) Whenever  $M_0 \leq_{\mathbf{K}} M' \leq_{\mathbf{K}} N$  is such that  $M'$  is universal over  $M_0$ , then  $p$  does not fork over  $(M_0, M')$ .
- (b) Whenever  $M \leq_{\mathbf{K}} N' \leq_{\mathbf{K}} N$ , then  $p \upharpoonright N'$  does not fork over  $(M_0, M)$ .
- (c) Whenever  $M_0 \leq_{\mathbf{K}} M'_0 \leq_{\mathbf{K}} M$  is such that  $M$  is universal over  $M'_0$ , then  $p$  does not fork over  $(M'_0, M)$ .
- (3) (Universal local character) If  $\delta < \lambda^+$  is limit,  $\langle M_i : i \leq \delta \rangle$  are increasing continuous in  $\mathbf{K}_\lambda$  with  $M_{i+1}$  universal over  $M_i$  for all  $i < \delta$ , then for any  $p \in \mathbf{S}(M_\delta)$ , there exists  $i < \delta$  such that  $p$  does not fork over  $(M_i, M_{i+1})$ .
- (4) (Weak uniqueness) If  $p, q \in \mathbf{S}(N)$  both do not fork over  $(M_0, M)$  and  $p \upharpoonright M = q \upharpoonright M$ , then  $p = q$ .
- (5) (Weak extension) If  $p \in \mathbf{S}(N)$  does not fork over  $(M_0, M)$  and  $N' \geq_{\mathbf{K}} N$  is in  $\mathbf{K}_\lambda$ , then there exists  $q \in \mathbf{S}(N')$  that extends  $p$  and does not fork over  $(M_0, M)$ .

*Proof.* The first three properties are immediate from the definition. For weak uniqueness, we know that both  $p$  and  $q$  do not split over  $M_0$ . By universality of  $M$  over  $M_0$ , fix  $f : N \xrightarrow{M_0} M$ . Let  $N_1 := N$ ,  $N_2 := f[N]$ . Using the definition of nonsplitting, we must have that  $f(p) = p \upharpoonright N_2$  and  $f(q) = q \upharpoonright N_2$ . Since  $N_2 \leq_{\mathbf{K}} M$  and  $p \upharpoonright M = q \upharpoonright M$ , we have that  $p \upharpoonright N_2 = q \upharpoonright N_2$ . Thus  $f(p) = f(q)$ , and so  $p = q$ .

For weak extension, we first prove it when  $M$  is  $(\lambda, \omega)$ -limit over  $M_0$ . In this case we can fix  $\langle M_i : i \leq \omega \rangle$  increasing continuous such that  $M_{i+1}$  is universal over  $M_i$  and  $M_\omega = M$ . Fix  $f : N' \xrightarrow{M_1} N$  and let  $q := f^{-1}(p \upharpoonright f[N'])$ . By invariance,  $q$  does not split over  $M_0$ , hence does not fork over  $(M_0, M)$ . It remains to see that  $q \upharpoonright N = p$ . Indeed by monotonicity,  $q \upharpoonright N$  does not fork over  $(M_0, M)$ , hence over  $(M_0, M_1)$ . Also,  $p$  does not fork over  $(M_0, M_1)$ , and since  $f$  fixes  $M_1$ ,  $q \upharpoonright M_1 = p \upharpoonright M_1$ . By weak uniqueness,  $q \upharpoonright N = p$ .

Now if  $M$  is not  $(\lambda, \omega)$ -limit over  $M_0$ , use universality to find  $M' \in \mathbf{K}_\lambda$ ,  $M_0 \leq_{\mathbf{K}} M' \leq_{\mathbf{K}} M$  such that  $M'$  is  $(\lambda, \omega)$ -limit over  $M_0$ . By monotonicity,  $p$  does not fork over  $(M_0, M')$ . By the previous case, there is an extension  $q \in \mathbf{S}(N')$  of  $p$  that does not fork over  $(M_0, M')$ , hence by monotonicity over  $(M_0, M)$ .  $\square$

In superstable AECs, increasing sequences of types have upper bounds. This is an improvement on:

**Exercise 13.4** ([Bal09, 11.1]). Let  $\mathbf{K}$  be an AEC and let  $\lambda \geq \text{LS}(\mathbf{K})$ . Let  $\langle M_i : i \leq \omega \rangle$  be an increasing continuous chain in  $\mathbf{K}_\lambda$ . Let  $\langle p_i : i < \omega \rangle$  be an increasing chain of types with  $p_i \in \mathbf{S}(M_i)$  for all  $i < \omega$ . If  $\mathbf{K}_\lambda$  has amalgamation, then there exists  $p_\omega \in \mathbf{S}(M_\omega)$  such that  $p_\omega$  extends  $p_i$  for all  $i < \omega$ . *Hint: draw a picture of what the definition of orbital type gives, and take a direct limit.*

**Lemma 13.5.** Assume that  $\mathbf{K}$  is  $\lambda$ -superstable. Let  $\delta < \lambda^+$  be a limit ordinal and let  $\langle M_i : i \leq \delta \rangle$  be increasing continuous in  $\mathbf{K}_\lambda$  with  $M_{i+1}$  universal over  $M_i$  for all  $i < \delta$ . Suppose we are given an increasing chain of types  $\langle p_i : i < \delta \rangle$  such that  $p_i \in \mathbf{S}(M_i)$  for all  $i < \delta$ . Then there exists a unique  $p_\delta \in \mathbf{S}(M_\delta)$  such that  $p_\delta \upharpoonright M_i = p_i$  for all  $i < \delta$ .

*Proof.* Without loss of generality,  $\delta$  is regular. To see uniqueness, suppose that  $p, q \in \mathbf{S}(M_\delta)$  both extend all the  $p_i$ 's. By universal local character and monotonicity, there exists  $i < \delta$  such that both  $p$  and  $q$  do not fork over  $(M_i, M_{i+1})$ . Since  $p \upharpoonright M_{i+1} = q \upharpoonright M_{i+1}$ , we must have  $p = q$ . Let us now prove existence. If  $\delta = \omega$ , the conclusion is given by Exercise 13.4, so assume that  $\delta > \omega$ . Using universal local character, for each limit  $i < \delta$  there exists  $j_i < i$  such that  $p_i$  does not fork over  $M_{j_i}$ . By Fodor's lemma, there exists a stationary  $S \subseteq \delta$  and a  $j < \delta$  such that  $p_i$  does not fork over  $M_j$  for all  $i \in S$ . Since  $S$  is unbounded and the  $p_i$ 's are increasing,  $p_i$  does not fork over  $M_j$  for all  $i \in [j, \delta)$ . Let  $q \in \mathbf{S}(M_\delta)$  be an extension of  $p_{j+1}$  that does not fork over  $(M_j, M_{j+1})$ . By weak uniqueness,  $q \upharpoonright M_i = p_i$  for all  $i \in [j+1, \delta)$ , as desired.  $\square$

The following property will be crucial in many places. It will be improved later.

**Definition 13.6.** We say that two orbital types  $p \in \mathbf{S}(M)$  and  $q \in \mathbf{S}(N)$  are *conjugate over  $A$*  if  $A \subseteq M \cap N$  and there exists  $f : M \cong_A N$  such that  $f(p) = q$ .

**Lemma 13.7** (The conjugation property). Let  $\mathbf{K}$  be a  $\lambda$ -superstable AEC. Let  $\delta < \lambda^+$  be a limit ordinal. Let  $M_0 \leq_{\mathbf{K}} M \leq_{\mathbf{K}} N$  be such that  $M$  is  $(\lambda, \delta)$ -limit over  $M_0$  and  $N$  is  $(\lambda, \delta)$ -limit over  $M$ . If  $p \in \mathbf{S}(N)$  does not fork over  $(M_0, M)$ , then  $p$  and  $p \upharpoonright M$  are conjugates over  $M_0$ .

*Proof.* Since  $M$  is  $(\lambda, \delta)$ -limit over  $M_0$ , there exists  $M_1 \leq_{\mathbf{K}} M$  which is universal over  $M_0$  and such that  $M$  is  $(\lambda, \delta)$ -limit over  $M_1$ . It is easy to check that  $N$  is also  $(\lambda, \delta)$ -limit over  $M_1$ . By uniqueness of limit models of the same length, there exists  $f : N \cong_{M_1} M$ . Now  $f(p) \in \mathbf{S}(M)$  and  $f(p)$  does not fork over  $(M_0, f[M])$ , hence (by monotonicity) over  $(M_0, M_1)$ . We also have that  $p \upharpoonright M$  does not fork over  $(M_0, M_1)$  and  $f(p) \upharpoonright M_1 = p \upharpoonright M_1$ , since  $f$  fixes  $M_1$ . Thus  $p \upharpoonright M = f(p)$ , as desired.  $\square$

We aim to get rid of the witness  $M_0$  in the definition of nonforking, at least when working over limit models. For this, we want to prove the following result (due to the author):

**Theorem 13.8** (The uniqueness theorem). Let  $\mathbf{K}$  be a  $\lambda$ -superstable AEC. Let  $M \leq_{\mathbf{K}} N$  both be limit models in  $\mathbf{K}_\lambda$ . If  $p, q \in \mathbf{S}(N)$  do not fork over  $M$  and  $p \upharpoonright M = q \upharpoonright M$ , then  $p = q$ .

To see the difficulty, expand the definition: if  $p$  does not fork over  $M$ , there is  $M_p$  such that  $p$  does not fork over  $(M_p, M)$ . Similarly, there is  $M_q$  such that  $q$  does not fork over  $(M_q, M)$ . If we knew that  $M_p = M_q$ , or at least that one was contained in the other, then we would be able to use weak uniqueness and be done. However there is no reason this should be the case (think of the case of the AEC of all sets,  $M$  is some infinite sets, and  $M = M_p \cup M_q$ , where  $M_p$  and  $M_q$  are disjoint). Thus we will prove the uniqueness theorem by contradiction: a type that does not satisfy the conclusion of the uniqueness theorem will be called *bad*, and the theory of bad types will turn out to be so nice that there cannot be any bad type.

Throughout the rest of this section, we assume:

**Hypothesis 13.9.**  $\mathbf{K}$  is a  $\lambda$ -superstable AEC.

For technical reason, we close the concept of “failing the conclusion of the uniqueness theorem” under nonforking extensions. This leads to the following technical concept:

**Definition 13.10.** Let  $M \in \mathbf{K}_\lambda$  be limit. We define by induction on  $n < \omega$  what it means for a type  $p \in \mathbf{S}(M)$  to be  $n$ -bad:

- (1)  $p$  is 0-bad if there exists a limit model  $N \in \mathbf{K}_\lambda$  with  $M \leq_{\mathbf{K}} N$  and  $q_1, q_2 \in \mathbf{S}(N)$  such that:
  - (a) Both  $q_1$  and  $q_2$  extend  $p$ .
  - (b)  $q_1 \neq q_2$ .
  - (c) Both  $q_1$  and  $q_2$  do not fork over  $M$ .
- (2) For  $n < \omega$ ,  $p$  is  $(n+1)$ -bad if there exists a limit model  $M_0 \in \mathbf{K}_\lambda$  with  $M_0 \leq_{\mathbf{K}} M$  such that  $p \upharpoonright M_0$  is  $n$ -bad and  $p$  does not fork over  $M_0$ .
- (3)  $p$  is bad if  $p$  is  $n$ -bad for some  $n < \omega$ .

The following is an easy consequence of the definition (in fact the definition is tailored exactly to make this work):

**Remark 13.11.** Let  $M \leq_{\mathbf{K}} N$  both be limit in  $\mathbf{K}_\lambda$ . If  $p \in \mathbf{S}(N)$  does not fork over  $M$  and  $p \upharpoonright M$  is bad, then  $p$  is bad.

We now proceed to develop some the theory of bad types. In the end, we will conclude that this contradicts stability in  $\lambda$ , hence there cannot be any bad types. The next two lemmas are crucial: bad types are closed under unions of universal chains, and any bad type has two distinct bad extensions.

**Lemma 13.12.** Assume that  $\mathbf{K}$  is  $\lambda$ -superstable. Let  $\delta < \lambda^+$  be a limit ordinal. Let  $\langle M_i : i \leq \delta \rangle$  be an increasing continuous chain of limit models in  $\mathbf{K}_\lambda$  with  $M_{i+1}$  limit over  $M_i$  for all  $i < \delta$ . Let  $\langle p_i : i \leq \delta \rangle$  be an increasing chain of types, with  $p_i \in \mathbf{S}(M_i)$  for all  $i < \delta$ . If  $p_i$  is bad for all  $i < \delta$ , then  $p_\delta$  is bad.

*Proof.* By universal local character, there exists  $i < \delta$  such that  $p_\delta$  does not fork over  $M_i$ . By assumption,  $p \upharpoonright M_i$  is bad, so by Remark 13.11  $p_\delta$  is also bad, as desired.  $\square$

**Lemma 13.13.** Let  $M \in \mathbf{K}_\lambda$  be a limit model. If  $p \in \mathbf{S}(M)$  is bad, then there exists a limit model  $N$  in  $\mathbf{K}_\lambda$  with  $M \leq_{\mathbf{K}} N$  and  $q_1, q_2 \in \mathbf{S}(N)$  such that:

- (1) Both  $q_1$  and  $q_2$  extend  $p$ .
- (2)  $q_1 \neq q_2$ .
- (3) Both  $q_1$  and  $q_2$  are bad.

*Proof.* By definition,  $p$  is  $n$ -bad for some  $n < \omega$ . We proceed by induction on  $n$ .

- If  $n = 0$ , this is the definition of being 0-bad (note that  $q_1$  and  $q_2$  from Definition 13.10 are bad because they are nonforking extensions of the bad type  $p$ , see Remark 13.11)
- If  $n = m + 1$ , let  $M_0 \in \mathbf{K}_\lambda$  be a limit model such that  $M_0 \leq_{\mathbf{K}} M$ ,  $p$  does not fork over  $M_0$ , and  $p \upharpoonright M_0$  is  $m$ -bad. Pick  $M'_0$  such that  $p$  does not fork over  $(M'_0, M_0)$ . Let  $M'_1$  be  $(\mu, \omega)$ -limit over  $M'_0$  with  $M'_1 \leq_{\mathbf{K}} M_0$ . By monotonicity,  $p$  does not fork over  $(M'_0, M'_1)$ . Let  $M^*$  be  $(\lambda, \omega)$ -limit over

$M$  (hence over  $M'_1$ ). Let  $q \in \mathbf{S}(M^*)$  be an extension of  $p$  that does not fork over  $(M'_0, M)$ , hence over  $(M'_0, M'_1)$ . By Lemma 13.7,  $q$  and  $p \upharpoonright M'_1$  are conjugate over  $M'_0$ . Now by the induction hypothesis, there exists a limit model  $N^*$  extending  $M_0$  and two distinct bad extensions of  $p \upharpoonright M_0$  to  $N^*$ . These are also extensions of  $p \upharpoonright M'_1$ , so the result follows from the fact that  $q$  and  $p \upharpoonright M'_1$  are conjugate over  $M'_0$ .

□

The following nominally stronger version of Lemma 13.13 (where  $N$  is fixed first) is the one that we will use to show that there are no bad types:

**Lemma 13.14.** Let  $M$  be a limit model in  $\mathbf{K}_\lambda$  and let  $N$  be limit over  $M$ . If  $p \in \mathbf{S}(M)$  is bad, then there exists  $q_1, q_2 \in \mathbf{S}(N)$  such that:

- (1) Both  $q_1$  and  $q_2$  extend  $p$ .
- (2)  $q_1 \neq q_2$ .
- (3) Both  $q_1$  and  $q_2$  are bad.

*Proof.* By Lemma 13.13, there exists  $N' \in \mathbf{K}_\lambda$  limit with  $M \leq_{\mathbf{K}} N'$  and  $q'_1, q'_2 \in \mathbf{S}(N')$  distinct bad extensions of  $p$ . Use universality of  $N$  to pick  $f : N' \xrightarrow{M} N$ . For  $\ell = 1, 2$ , let  $q''_\ell := f(q'_\ell)$ . Clearly,  $q''_1, q''_2$  are still distinct bad extensions of  $p$ . Now for  $\ell = 1, 2$ , let  $q_\ell \in \mathbf{S}(N)$  be an extension of  $q''_\ell$  that does not fork over  $f[N']$  (use universal local character and extension). Then  $q_1$  and  $q_2$  are as desired (they are bad because they are nonforking extensions of the bad types  $q''_1, q''_2$ , see Remark 13.11). □

**Lemma 13.15.** If  $\mathbf{K}$  is  $\lambda$ -superstable, then there are no bad types.

*Proof.* Suppose for a contradiction that there is a limit model  $M$  in  $\mathbf{K}_\lambda$  and a bad type  $p \in \mathbf{S}(M)$ . Fix an increasing continuous chain  $\langle M_i : i \leq \lambda \rangle$  with  $M_0 = M$  and  $M_{i+1}$  limit over  $M_i$  for all  $i < \lambda$ . We build a tree of types  $\langle p_\eta : \eta \in {}^{<\lambda}2 \rangle$  satisfying:

- (1)  $p_{<} = p$ .
- (2) For all  $\eta \in {}^{\leq \lambda}2$ ,  $p_\eta \in \mathbf{S}(M_{\ell(\eta)})$ .
- (3) For all  $\nu \leq \eta \in {}^{\leq \lambda}2$ ,  $p_\eta$  is an extension of  $p_\nu$ .
- (4) For all  $\eta \in {}^{\leq \lambda}2$ ,  $p_\eta$  is bad.
- (5) For all  $\eta \in {}^{<\lambda}2$ ,  $p_{\eta \smallfrown 0} \neq p_{\eta \smallfrown 1}$ .

This is enough: The requirements give that for all  $\eta, \nu \in {}^\lambda 2$ ,  $\eta \neq \nu$  implies  $p_\eta \neq p_\nu$ . Therefore  $|\mathbf{S}(M_\lambda)| = 2^\lambda > \lambda$ , contradicting stability in  $\lambda$ .

This is possible: We proceed by induction on  $\ell(\eta)$ . The base case has already been specified. At limits, we use Lemma 13.5 and Lemma 13.12. At successors, we use Lemma 13.14. □

*Proof of Theorem 13.8.* Otherwise, this would mean that  $p \upharpoonright M$  is 0-bad, contradicting Lemma 13.15. □

We now restate the properties of nonforking over limit models, without the witness.



**Corollary 13.16.** Let  $\mathbf{K}$  be a  $\lambda$ -superstable AEC. We have the following properties of nonforking:

- (1) Invariance: if  $M \leq_{\mathbf{K}} N$  are both limit models in  $\mathbf{K}_{\lambda}$  and  $p \in \mathbf{S}(N)$  does not fork over  $M$ , then if  $f : N \cong N'$ ,  $f(p)$  does not fork over  $f[M]$ .
- (2) Monotonicity: if  $M \leq_{\mathbf{K}} M' \leq_{\mathbf{K}} N' \leq_{\mathbf{K}} N$  are all limit models in  $\mathbf{K}_{\lambda}$ ,  $p \in \mathbf{S}(N)$  does not fork over  $M$ , then  $p \upharpoonright N'$  does not fork over  $M'$ .
- (3) Universal local character: if  $\delta < \lambda^+$ ,  $\langle M_i : i \leq \delta \rangle$  is an increasing continuous chain of limit models in  $\mathbf{K}_{\lambda}$ , with  $M_{i+1}$  universal over  $M_i$  for all  $i < \delta$ , then for any  $p \in \mathbf{S}(M_{\delta})$ , there exists  $i < \delta$  such that  $p$  does not fork over  $M_i$ .
- (4) Uniqueness: if  $M \leq_{\mathbf{K}} N$  are both limit models in  $\mathbf{K}_{\lambda}$  and  $p, q \in \mathbf{S}(N)$  do not fork over  $M$ , then  $p \upharpoonright M = q \upharpoonright M$  implies  $p = q$ .
- (5) Existence: Let  $M \leq_{\mathbf{K}} N$  both be limit models in  $\mathbf{K}_{\lambda}$ . If  $p \in \mathbf{S}(M)$ , then there exists  $q \in \mathbf{S}(N)$  such that  $q$  does not fork over  $M$  and  $q$  extends  $p$ .

*Proof.* By Lemma 13.3 and Theorem 13.8. □

#### 14. CANONICITY OF FORKING

In this section, we show that the exact definition of forking we take is immaterial: at least over limit models, any two definitions satisfying some basic properties must be the same. In particular, nonforking in the sense defined here must in a superstable elementary class coincide (over limit models) with the usual nonforking.

**Theorem 14.1** (The canonicity theorem). Let  $\mathbf{K}$  be a  $\lambda$ -superstable AEC. Assume we have a relation “ $p$  is free over  $M$ ” for a type  $p \in \mathbf{S}(N)$  and  $M, N \in \mathbf{K}_{\lambda}$  *limit models*. If this relation satisfies invariance, monotonicity, universal local character, uniqueness, and extension (in the sense of Corollary 13.16), then  $p$  is free over  $M$  if and only if  $p$  does not fork over  $M$  (in the sense of Definition 13.1).

*Proof.* We first show that if  $M \leq_{\mathbf{K}} N$  are both limit models in  $\mathbf{K}_{\lambda}$  and  $p$  is free over  $M$ , then  $p$  does not split over  $M$ . Indeed, let  $N_1, N_2 \in \mathbf{K}_{\lambda}$  with  $M \leq_{\mathbf{K}} N_1, N_2 \leq_{\mathbf{K}} N$  and let  $f : N_1 \cong_M N_2$ . Let  $N'$  be limit over  $N$ , and extend  $f$  to an automorphism  $g$  of  $N'$ . We have that  $p$  is free over  $M$ , hence so is  $g(p)$ . We know that  $g(p) \upharpoonright M = p \upharpoonright M$  since  $f$  fixes  $M$ . Thus  $g(p) \upharpoonright N = p \upharpoonright N$  by uniqueness. In particular,  $f(p \upharpoonright N_1) = p \upharpoonright N_2$ .

It follows that if  $M \leq_{\mathbf{K}} N$  are both limit models in  $\mathbf{K}_{\lambda}$ , then whenever  $p \in \mathbf{S}(N)$  is free over  $M$ , we have that  $p$  does not fork over  $M$ . Indeed, we can write  $M = \bigcup_{i < \delta} M_i$ , with  $\langle M_i : i < \delta \rangle$  an increasing continuous chain of limit models in  $\mathbf{K}_{\lambda}$ ,  $M_{i+1}$  universal over  $M_i$  for all  $i < \delta$ . By universal local character, there exists  $i < \delta$  such that  $p \upharpoonright M$  is free over  $M_i$ . By uniqueness and extension, we must also have that  $p$  is free over  $M_i$ . Thus  $p$  does not split over  $M_i$ , hence  $p$  does not fork over  $M$ .

Assume now that  $p \in \mathbf{S}(N)$  does not fork over  $M$  (where as before  $M \leq_{\mathbf{K}} N$  and  $M, N \in \mathbf{K}_{\lambda}$  are limit models). Note that  $p \upharpoonright M$  is free over  $M$  (by universal local character and monotonicity). Pick  $q \in \mathbf{S}(N)$  such that  $q$  extends  $p \upharpoonright M$  and  $q$  is free over  $M$ . By what has been said before,  $q$  does not fork over  $M$ . By uniqueness of nonforking  $p = q$ . Thus  $p$  is free over  $M$ , as desired. □

**Remark 14.2.** The proof of Lemma 13.3 shows that extension (over limit models) follows abstractly from the other properties. Thus it is not necessary to assume that freeness satisfies extension in Theorem 14.1.

## 15. SUPERSTABILITY AND SYMMETRY

We start by proving some auxiliary properties of forking in a superstable AEC:

**Theorem 15.1.** Let  $\mathbf{K}$  be a  $\lambda$ -superstable AEC. Then nonforking satisfies:

- (1) Transitivity: Let  $M_0 \leq_{\mathbf{K}} M_1 \leq_{\mathbf{K}} M_2$  all be limit models in  $\mathbf{K}_\lambda$ . Let  $p \in \mathbf{S}(M_2)$  and assume that  $p$  does not fork over  $M_1$  and  $p \upharpoonright M_1$  does not fork over  $M_0$ . Then  $p$  does not fork over  $M_0$ .
- (2) Continuity: Let  $\delta < \lambda^+$  be a limit ordinal and let  $\langle M_i : i \leq \delta \rangle$  be an increasing continuous chain of limit models with  $M_{i+1}$  universal over  $M_i$ . Let  $\langle p_i : i < \delta \rangle$  be given such that for all  $i < \delta$ ,  $p_i \in \mathbf{S}(M_i)$ , and  $p_i$  is a nonforking extension of  $p_0$ . Then there exists a unique  $p_\delta \in \mathbf{S}(M_\delta)$  such that  $p_\delta$  does not fork over  $M_0$ . In particular,  $p_\delta$  extends each  $p_i$ .

*Proof.*

- (1) Let  $q \in \mathbf{S}(M_2)$  be a nonforking extension of  $p \upharpoonright M_0$ . By monotonicity,  $q \upharpoonright M_1$  does not fork over  $M_0$ , hence by uniqueness  $p \upharpoonright M_1 = q \upharpoonright M_1$ . By monotonicity again,  $q$  does not fork over  $M_1$ . By uniqueness again,  $q = p$ , so  $p$  does not fork over  $M_0$ .
- (2) Let  $p_\delta \in \mathbf{S}(M_\delta)$  be the nonforking extension of  $p_0$ . By uniqueness,  $p_\delta$  also extends each  $p_i$ .

□

The following lemma will also be useful:

**Lemma 15.2.** Let  $\mathbf{K}$  be a  $\lambda$ -superstable AEC. Let  $M \leq_{\mathbf{K}} N \leq_{\mathbf{K}} N'$  all be limits in  $\mathbf{K}_\lambda$ . Let  $a \in N'$  and let  $\bar{b}_1, \bar{b}_2 \in {}^\lambda N$ . If  $\mathbf{tp}(a/N; N')$  does not fork over  $M$  and  $\mathbf{tp}(\bar{b}_1/M; N) = \mathbf{tp}(\bar{b}_2/M; N)$ , then  $\mathbf{tp}(a\bar{b}_1/M; N') = \mathbf{tp}(a\bar{b}_2/M; N')$ .

*Proof.* Extending  $N$  if necessary (using extension and transitivity), we can assume without loss of generality that  $N$  is limit over  $M$ , and in fact limit over an extension of  $M$  containing  $\bar{b}_1\bar{b}_2$ . We can then find  $f \in \text{Aut}_M(N)$  such that  $f(\bar{b}_1) = \bar{b}_2$ . Let  $p := \mathbf{tp}(a/N; N')$ . Since  $p$  does not fork over  $M$ , it does not split over  $M$ , hence  $f(p) = p$ . Assume without loss of generality that  $N'$  is limit over  $N$  and  $f$  extends to an automorphism  $g$  of  $N'$ , we then have that  $\mathbf{tp}(a/N; N') = \mathbf{tp}(g(a)/N; N')$ , hence there exists an automorphism  $h$  of  $N'$  such that  $h(g(a)) = a$  and  $h$  fixes  $N$ . Now if  $h' := hg$ , we have that  $h'(\bar{b}_1) = \bar{b}_2$  and  $h'(a) = a$ . Thus  $\mathbf{tp}(a\bar{b}_1/M; N') = \mathbf{tp}(h'(a)h'(\bar{b}_1)/M; N') = \mathbf{tp}(a\bar{b}_2/M; N')$ , as desired. □

We aim to investigate the following property:

**Definition 15.3.** Let  $\mathbf{K}$  be a  $\lambda$ -superstable AEC. We say that  $\mathbf{K}$  has  $\lambda$ -symmetry if for any two limit model  $M \leq_{\mathbf{K}} N$  in  $\mathbf{K}_\lambda$  and any  $a, b \in N$ , the following are equivalent:

- (1) There exists  $M_b \leq_{\mathbf{K}} N_b$  in  $\mathbf{K}_\lambda$  both limits such that  $N \leq_{\mathbf{K}} N_b$ ,  $M \leq_{\mathbf{K}} M_b$ ,  $b \in M_b$ , and  $\text{tp}(a/M_b; N_b)$  does not fork over  $M$ .
- (2) There exists  $M_a \leq_{\mathbf{K}} N_a$  in  $\mathbf{K}_\lambda$  both limits such that  $N \leq_{\mathbf{K}} N_a$ ,  $M \leq_{\mathbf{K}} M_a$ ,  $a \in M_a$ , and  $\text{tp}(b/M_a; N_a)$  does not fork over  $M$ .

The first-order version of this symmetry property would say that  $\text{tp}(a/Mb)$  does not fork over  $M$  if and only if  $\text{tp}(b/Ma)$  does not fork over  $M$  (Shelah established that this holds in any stable theory). Playing with monotonicity a little bit, one sees that Definition 15.3 says essentially this. We will see that symmetry is useful in order to understand limit models better. The following is open:

**Question 15.4.** Does  $\lambda$ -symmetry follow from  $\lambda$ -superstability?

We will establish that it follows from categoricity by showing that failure of symmetry implies an instance of the order property [VV17, 5.7]:

**Theorem 15.5.** Let  $\mathbf{K}$  be a  $\lambda$ -superstable AEC which does *not* have  $\lambda$ -symmetry. Then  $\mathbf{K}$  has the  $(2, \lambda)$ -order property of length  $\lambda^+$ .

*Proof.* To simplify the notation, let us work inside a saturated model  $\mathfrak{C}$  of cardinality  $\lambda^+$ . We assume that all the models we work with are inside  $\mathfrak{C}$  and all the types are also computed inside  $\mathfrak{C}$ . We will establish that  $\mathfrak{C}$  has the  $(2, \lambda)$ -order property of length  $\lambda^+$ . Fix  $M \in \mathbf{K}_\lambda$  limit, and  $a, b \in \mathfrak{C}$  witnessing failure of symmetry. Say there exists  $M_b \in \mathbf{K}_\lambda$  limit such that  $M \leq_{\mathbf{K}} M_b$ ,  $b \in M_b$ , and  $\text{tp}(a/M_b)$  does not fork over  $M$ , but there is *no*  $M_a \in \mathbf{K}_\lambda$  limit such that  $M \leq_{\mathbf{K}} M_a$ ,  $a \in M_a$ , and  $\text{tp}(b/M_a)$  does not fork over  $M$ .

We build increasing continuous  $\langle N_\alpha : \alpha < \lambda^+ \rangle$  and  $\langle a_\alpha, b_\alpha, N'_\alpha : \alpha < \lambda^+ \rangle$  by induction so that for all  $i < \lambda^+$ :

- (1)  $N_i, N_i \in \mathbf{K}_\lambda$  are limits.
- (2)  $N_0$  is limit over  $M_b$  and  $a \in N_0$ .
- (3)  $\text{tp}(a_i/M_b) = \text{tp}(a/M_b)$  and  $a_i \in N_{i+1}$ .
- (4)  $\text{tp}(b_i/M) = \text{tp}(b/M)$  and  $b_i \in N'_i$ .
- (5)  $N'_i$  is limit over  $N_i$  and  $N_{i+1}$  is limit over  $N'_i$ .
- (6)  $\text{tp}(a_i/N'_i)$  and  $\text{tp}(b_i/N_i)$  both do not fork over  $M$ .

This is possible. Let  $N_0$  be any model in  $\mathbf{K}_\lambda$  containing  $a$  that is limit over  $M_b$ . At  $i$  limits, let  $N_i := \bigcup_{j < i} N_j$ . Now assume inductively that  $N_j$  has been defined for  $j \leq i$ , and  $a_j, b_j, N_j$  have been defined for  $j < i$ . By extension, find  $q \in \mathbf{S}(N_i)$  that does not fork over  $M$  and extends  $\text{tp}(b/M)$ . Let  $b_i$  realize  $q$  and pick  $N'_i$  limit over  $N_i$  containing  $b_i$ . Now by extension again, find  $q' \in \mathbf{S}(N'_i)$  that does not fork over  $M$  and extends  $\text{tp}(a/M)$ . Let  $a_i$  realize  $q'$  and pick  $N_{i+1}$  limit over  $N'_i$  containing  $a_i$ .

This is enough. We show that for  $i, j < \lambda^+$ :

- (1)  $\text{tp}(ab_j/M) \neq \text{tp}(ab/M)$
- (2)  $\text{tp}(a_ib/M) = \text{tp}(ab/M)$
- (3) If  $i < j$ ,  $\text{tp}(a_ib_j/M) = \text{tp}(ab_j/M)$ .
- (4) If  $i \geq j$ ,  $\text{tp}(a_ib_j/M) = \text{tp}(a_ib/M)$ .

For (1), observe that  $a \in N_0 \subseteq N_j$  and  $\mathbf{tp}(b_j/N_j)$  does not fork over  $M$ . Therefore by monotonicity  $N_j$  witnesses that there exists  $M_a \in \mathbf{K}_\lambda$  limit containing  $a$  so that  $\mathbf{tp}(b_j/M_a)$  explicitly does not fork over  $M$ . By failure of symmetry and invariance, we must have that  $\mathbf{tp}(ab_j/M) \neq \mathbf{tp}(ab/M)$ .

For (2), use the assumption that  $b \in M_b$  together with clause (3) of the construction.

For (3), suppose  $i < j$ . We know that  $\mathbf{tp}(b_j/N_j)$  does not fork over  $M$ . Since  $i < j$ ,  $a, a_i \in N_j$ , and  $\mathbf{tp}(a/M) = \mathbf{tp}(a_i/M)$ , we must have by Lemma 15.2 that  $\mathbf{tp}(a_ib_j/M) = \mathbf{tp}(ab_j/M)$ .

For (4), suppose  $i \geq j$ . We know that  $\mathbf{tp}(a_i/N'_i)$  does not fork over  $M$ . Since  $i \geq j$ ,  $b, b_j \in N'_i$ , and  $\mathbf{tp}(b/M) = \mathbf{tp}(b_j/M)$ , we must have by Lemma 15.2 again that  $\mathbf{tp}(a_ib_j/M) = \mathbf{tp}(a_ib/M)$ .

Now by (1) and (3),  $\mathbf{tp}(a_ib_j/M) \neq \mathbf{tp}(ab/M)$  when  $i < j$ . Similarly, by (2) and (4),  $\mathbf{tp}(a_ib_j/M) = \mathbf{tp}(ab/M)$  when  $i \geq j$ . Letting  $\bar{c}_i := a_ib_i$  for  $i < \lambda^+$ , this tells us that the sequence  $\langle \bar{c}_i : i < \lambda^+ \rangle$  witnesses the  $(2, \lambda)$ -order property of length  $\lambda^+$ .  $\square$

**Remark 15.6.** The converse of Theorem 15.5 is also true, see [LRVa, 9.7].

**Corollary 15.7** ([Vas17b, 4.8]). Let  $\mathbf{K}$  be an AEC with arbitrarily large models and let  $\mu > \text{LS}(\mathbf{K})$ . If  $\mathbf{K}_{<\mu}$  has amalgamation and no maximal models and  $\mathbf{K}$  is categorical in  $\mu$ , then for every  $\lambda \in [\text{LS}(\mathbf{K}), \mu)$ ,  $\mathbf{K}$  is  $\lambda$ -superstable and has  $\lambda$ -symmetry.

*Proof.* Fix  $\lambda \in [\text{LS}(\mathbf{K}), \mu)$ . By Theorem 12.8,  $\mathbf{K}$  is  $\lambda$ -superstable. By Corollary 11.16,  $\mathbf{K}$  does not have the  $(2, \lambda)$ -order property of length  $\lambda^+$ . By Theorem 15.5,  $\mathbf{K}$  has  $\lambda$ -symmetry.  $\square$

Limit models are, in a sense, a local analog of saturated models. Indeed, at least when  $\lambda$  is a regular cardinal, a  $(\lambda, \lambda)$ -limit model will be (say in an AEC with amalgamation and  $\lambda > \text{LS}(\mathbf{K})$ ) saturated. It is natural to ask whether a  $(\lambda, \delta)$ -limit model is also saturated, for  $\delta < \lambda$ . That is, are we building the same saturated models if we only take a chain of universal extension of length  $\delta$  instead of going all the way to  $\lambda$ ? More precisely:

**Definition 15.8.** Let  $\mathbf{K}$  be a  $\lambda$ -superstable AEC. We say that  $\mathbf{K}$  has *uniqueness of limit models in  $\lambda$*  if whenever  $M_0, M_1, M_2 \in \mathbf{K}_\lambda$  are such that  $M_1$  is limit over  $M_0$  and  $M_2$  is limit over  $M_0$ , then  $M_1 \cong_{M_0} M_2$ .

The interesting part of the definition is when  $M_1$  is  $(\lambda, \delta_1)$ -limit over  $M_0$ ,  $M_2$  is  $(\lambda, \delta_2)$ -limit over  $M_0$  and  $\text{cf}(\delta_1) \neq \text{cf}(\delta_2)$ . In this case it is no longer clear how to carry out a back and forth argument to build the isomorphism. We will prove the following result, essentially due to VanDieren [Van16a] and Shelah [She09a, II.4.8] in slightly different contexts:

**Theorem 15.9.** Let  $\mathbf{K}$  be a  $\lambda$ -superstable AEC. If  $\mathbf{K}$  has  $\lambda$ -symmetry, then  $\mathbf{K}$  has uniqueness of limit models in  $\lambda$ .

It is again not known whether the assumption of  $\lambda$ -symmetry is necessary. It is known that in a first-order stable theory, uniqueness of limit models holds if and only if the theory is superstable [GVV16, 6.1].

Before proving Theorem 15.9, we give the following consequence, which connects uniqueness of limit models to chains of saturated models. It was first proven by VanDieren [Van16b] but the proof we give is from [Vasb, 3.3].

**Theorem 15.10.** Let  $\mathbf{K}$  be a  $\lambda$ -superstable AEC. If  $\mathbf{K}$  is  $\lambda^+$ -superstable and has uniqueness of limit models in  $\lambda^+$ , then the union of any increasing chain of  $\lambda^+$ -saturated models is  $\lambda^+$ -saturated.

*Proof.* We first prove:

Claim: If  $M \leq_{\mathbf{K}} N$  are both in  $\mathbf{K}_{\lambda^+}$  and saturated and  $M^0 \in \mathcal{P}_{\mathbf{K}_{\lambda}}(M)$ ,  $N^0 \in \mathcal{P}_{\mathbf{K}_{\lambda}}(N)$  are both limits such that  $M^0 \leq_{\mathbf{K}} N^0$  and  $p \in \mathbf{S}(N^0)$  does not fork over  $M^0$ , then  $p$  is realized inside  $M$ . That is, there exists  $a \in M$  such that  $p = \mathbf{tp}(a/N^0; N)$ .

Proof of Claim: Suppose not. We build  $\langle M_i : i \leq \omega \rangle$ ,  $\langle N_i : i \leq \omega \rangle$  increasing continuous such that for all  $i < \omega$ :

- (1)  $M_0 = M^0$ ,  $N_0 = N$ .
- (2)  $M_i \in \mathbf{K}_{\lambda}$ ,  $M_i$  is limit,  $N_i \in \mathbf{K}_{\lambda^+}$ ,  $N_i$  is saturated,  $M_i \leq_{\mathbf{K}} N_i$ .
- (3)  $N_{i+1}$  is universal over  $N_i$ .
- (4) The nonforking extension  $p$  to  $M_{i+1}$  is *not* realized in  $N_i$ .

(In what follows, we write  $p_{M'}$  for the nonforking extension of  $p$  to some fixed  $M' \geq_{\mathbf{K}} M$ .)

This is enough:  $N_{\omega}$  is  $(\lambda^+, \omega)$ -limit, hence by uniqueness of limit models in  $\lambda^+$ , it is also  $(\lambda^+, \lambda^+)$ -limit, hence saturated. Thus, there must exist  $a \in M_{\omega}$  such that  $p_{M_{\omega}} = \mathbf{tp}(a/M_{\omega}; N_{\omega})$ . But then there exists  $i < \omega$  such that  $a \in N_i$  so  $p_{M_{\omega}}$ , hence  $p_{M_{i+1}}$ , is realized in  $N_i$ , a contradiction.

This is possible: the base case is already specified. Then let  $M_1 := N^0$  and pick  $N_1 \in \mathbf{K}_{\lambda^+}$  saturated such that  $N_1$  is universal over  $N$ . Now given  $\langle M_j, N_j : j \leq i+1 \rangle$ , note that by saturation  $N_i$  and  $N_{i+1}$  are isomorphic over  $M_i$ . Pick  $f : N_i \cong_{M_i} N_{i+1}$ . Let  $g : N_{i+1} \cong N'_{i+2}$  be an extension of  $f$  (so in particular  $N_{i+1} \leq_{\mathbf{K}} N'_{i+2}$ ). Since  $p_{M_{i+1}}$  is not realized inside  $N_i$ ,  $g(p_{M_{i+1}})$  is not realized inside  $N_{i+1}$ , and note that  $g(p_{M_{i+1}})$  is the nonforking extension of  $p_{M_i}$ , hence of  $p$ , by uniqueness and the fact that  $g$  fixes  $M_i$ . Pick  $M_{i+2} \in \mathcal{P}_{\mathbf{K}_{\lambda}}(N'_{i+2})$  limit such that  $g[M_{i+1}] \leq_{\mathbf{K}} M_{i+2}$  and  $M_{i+1} \leq_{\mathbf{K}} M_{i+2}$ . Finally, pick some  $N_{i+2} \in \mathbf{K}_{\lambda^+}$  saturated such that  $N_{i+2}$  is universal over  $N'_{i+2}$ .  $\uparrow_{\text{Claim}}$

Now let  $\langle M_i : i < \delta \rangle$  be an increasing chain of  $\lambda^+$ -saturated models. Let  $M_{\delta} := \bigcup_{i < \delta} M_i$ . We can assume without loss of generality that  $\delta = \text{cf}(\delta)$ . If  $\delta \geq \lambda^+$ , then an easy cofinality argument gives the result, so assume that  $\delta < \lambda$ . It is enough to see that every submodel of  $M_{\delta}$  of cardinality  $\lambda^+$  is saturated, so assume without loss of generality again that  $M_i \in \mathbf{K}_{\lambda^+}$  (so in particular is saturated) for all  $i < \delta$ . Now let  $M \in \mathcal{P}_{\mathbf{K}_{\lambda}}(M_{\delta})$  and let  $p \in \mathbf{S}(M)$ . Find  $\langle M_i^0 : i < \delta \rangle$  be an increasing chain of limit models in  $\mathbf{K}_{\lambda}$  such that for all  $i < \delta$ ,  $M_{i+1}^0$  is universal over  $M_i^0$ ,  $M_i^0 \leq_{\mathbf{K}} M_i$ , and  $M_i^0$  contains  $M \cap M_i$ . Let  $M_{\delta}^0 := \bigcup_{i < \delta} M_i^0$ . Then by coherence  $M \leq_{\mathbf{K}} M_{\delta}^0$ . Let  $q \in \mathbf{S}(M_{\delta}^0)$  be any extension of  $p$ . By universal local character, there is  $i < \delta$  such that  $q$  does not fork over  $M_i^0$ . By the claim,  $q$  is realized inside  $M_i$ , hence inside  $M_{\delta}$ , hence  $p$  is also realized in  $M_{\delta}$ , as desired.  $\square$

Modulo Theorem 15.9, we deduce the following structural properties of categorical AECs with amalgamation [Vas17b, 5.7]:

**Corollary 15.11.** Let  $\mathbf{K}$  be an AEC with arbitrarily large models and let  $\mu > \text{LS}(\mathbf{K})$ . Assume that  $\mathbf{K}_{<\mu}$  has amalgamation and no maximal models. If  $\mathbf{K}$  is categorical in  $\mu$ , then:

- (1) For any  $\lambda \in (\text{LS}(\mathbf{K}), \mu]$ , the union of any increasing chain of  $\lambda$ -saturated models is  $\lambda$ -saturated. Moreover, there is a saturated model of cardinality  $\lambda$ .
- (2) The model of cardinality  $\mu$  is saturated.

*Proof.* The second part obviously follows from the first (taking  $\lambda = \mu$ ). Now let  $\lambda \in [\text{LS}(\mathbf{K}), \mu)$ . By Corollary 15.7,  $\mathbf{K}$  is  $\lambda$ -superstable and has  $\lambda$ -symmetry. By Theorem 15.9, this implies that  $\mathbf{K}$  has uniqueness of limit models in  $\lambda$ . Thus if  $\lambda > \text{LS}(\mathbf{K})$  and  $\lambda$  is a successor, Theorem 15.10 implies that unions of chains of  $\lambda$ -saturated models are saturated. If  $\lambda$  is limit, then the same result holds, as a model is  $\lambda$ -saturated if and only if it is  $\lambda_0^+$ -saturated for all  $\lambda_0 < \lambda$ . We also have that any limit model is saturated, hence in this case there is a saturated model in  $\lambda$ .

Next, we show that the union of an increasing chain of  $\mu$ -saturated models is  $\mu$ -saturated. If  $\mu$  is limit, this again follows from what was previously established, so assume that  $\mu$  is a successor. In this case, it is straightforward to establish that there is a saturated model of cardinality  $\mu$ : simply use stability below  $\mu$  to build  $\langle M_i : i < \mu \rangle$  increasing continuous in  $\mathbf{K}_{<\mu}$  with  $M_{i+1}$  universal over  $M_i$ , and observe that  $\bigcup_{i < \mu} M_i$  is saturated by regularity of  $\mu$  and has cardinality  $\mu$ . Once we know that the model of cardinality  $\mu$  is saturated, it follows that any model of larger cardinality is  $\mu$ -saturated as well, and hence in particular the union of any chain of  $\mu$ -saturated models is  $\mu$ -saturated.

It remains to establish that the model of cardinality  $\mu$  is saturated when  $\mu$  is limit. But this is straightforward given what we have established already: simply build  $\langle M_i : i < \mu \rangle$  increasing continuous in  $\mathbf{K}_{<\mu}$  such that each  $M_i$  is saturated. Then the union will be  $\mu_0$ -saturated for each  $\mu_0 < \mu$ , hence (since  $\mu$  is limit) saturated.  $\square$

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