CATEGORICITY AND MULTIDIMENSIONAL DIAGRAMS

SAHARON SHELAH AND SEBASTIEN VASEY

Dedicated to the memory of Oren Kolman.

Abstract. We study multidimensional diagrams in independent amalgamation in the framework of abstract elementary classes (AECs). We use them to prove the eventual categoricity conjecture for AECs, assuming a large cardinal axiom. More precisely, we show assuming the existence of a proper class of strongly compact cardinals that an AEC which has a single model of *some* high-enough cardinality will have a single model in *any* high-enough cardinal. Assuming a weak version of the generalized continuum hypothesis, we also establish the eventual categoricity conjecture for AECs with amalgamation.

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1. Introduction

1.1. **General background and motivation.** One of the most important result of modern model theory is Morley's categoricity theorem [Mor65]: a countable first-order theory categorical in *some* uncountable cardinal is categorical in *all* uncountable cardinals. Here, we call a theory (or more generally a class of structures) *categorical* in a cardinal λ if it has (up to isomorphism) exactly one model of cardinality λ. Morley's theorem generalizes the fact that algebraically closed fields of characteristic zero are categorical in all uncountable cardinals.

The machinery developed in the proof of Morley's theorem has proven to be important, both for pure model theory and for applications to other fields of mathematics (on the former, see e.g. the first author's book [She90]; on the latter, see e.g. [Bou99]). Specifically, the proof of Morley's theorem shows that there is a notion of independence generalizing algebraic independence (as well as linear independence) for the models of the theory in question. This was later precisely generalized and studied by the first author, who called this notion *forking*. Forking has now become a central concept of model theory.

It is natural to look for extensions of Morley's theorem, the real goal being to look for generalizations of *forking* to other setups. The first author has established [She74] that for *any* (not necessarily countable) first-order theory T, if T is categorical in *some* $\mu > |T|$, then T is categorical in *all* $\mu' > |T|$. One possible next step would be to prove versions of Morley's theorem for infinitary logics, like $\mathbb{L}_{\omega_1,\omega}$ (countably-many conjunctions are allowed within a single formula). Indeed, already in the late seventies the first-author conjectured (see [She83a, Conjecture 2]):

Conjecture 1.1 (Categoricity conjecture for $\mathbb{L}_{\omega_1,\omega}$, strong version)**.** Let ψ be an $\mathbb{L}_{\omega_1,\omega}$-sentence. If ψ is categorical in *some* $\mu \geq \beth_{\omega_1}$, then ψ is categorical in *all* $\mu' \geq \beth_{\omega_1}$.

Forty years down the road, and despite a lot of partial approximations (see the history given in the introduction of [Vas17d]), Conjecture 1.1 is still open. The main difficulty is that the compactness theorem fails, hence one has to work much more locally. The interest is that one can derive weak version of compactness from the categoricity assumption itself, often combined with large cardinals or combinatorial set-theoretic assumptions. Thus any proof is likely to exhibit a rich interplay between set theory and model theory.

The threshold $\beth_{\omega_1}$ appears in Conjecture 1.1 because it is provably best possible (for lower thresholds, there is a standard counterexample due to Morley, see [AV, 4.1]). Nevertheless, the spirit of the conjecture is captured by the following eventual version:

Conjecture 1.2 (Categoricity conjecture for $\mathbb{L}_{\omega_1,\omega}$, eventual version)**.** There is a threshold cardinal θ such that if ψ is an $\mathbb{L}_{\omega_1,\omega}$-sentence categorical in *some* $\mu \geq \theta$, then ψ is categorical in *all* $\mu' \geq \theta$.

1.2. **Eventual categoricity from large cardinals.** Conjecture 1.2 is also still open. In the present paper, we prove it assuming that a large cardinal axiom holds. More precisely and generally:

Corollary 14.7. Let κ be a strongly compact cardinal and let ψ be an $\mathbb{L}_{\kappa,\omega}$-sentence. If ψ is categorical in *some* $\mu \geq \beth_{(2^\kappa)^+}$, then ψ is categorical in *all* $\mu' \geq \beth_{(2^\kappa)^+}$.

This solves Question 6.14(1) in the first author's list of open problems [She00] and improves on a result of Makkai and the first author [MS90] who proved the statement of Corollary 14.7 under the additional assumption that the initial categoricity cardinal μ is a *successor* cardinal. We see this as quite a strong assumption, since it allows one to work directly with models of a single cardinality μ_0 (where $\mu = \mu_0^+$). The gap between [MS90] and Corollary 14.7 is further reflected by the several additional levels of sophistication that we use to prove the latter. For example, we work in the framework of abstract elementary classes (AECs). This was introduced by the first author [She87] as a general semantic framework encompassing in particular classes axiomatized by "reasonable" logics. The following version of Conjecture 1.2 has been stated there [She09a, N.4.2]:

Conjecture 1.3 (Eventual categoricity conjecture for AECs)**.** An AEC categorical in *some* high-enough cardinal is categorical in *all* high-enough cardinals.

In this paper, we more generally prove Conjecture 1.3 assuming a large cardinal axiom (a proper class of strongly compact cardinals). Again, Conjecture 1.3 was known in the special case where the starting categoricity cardinal is a successor (a result of Boney [Bon14]). The second author had also proven it for AECs closed under intersections [Vas17d, 1.7].

1.3. **Multidimensional diagrams.** The proof of Corollary 14.7 only tangentially uses large cardinals. We have already mentioned that one key step in the proof is the move to AECs: this is done because of generality, but also because the framework has a lot of closure, allowing us for example to take subclasses of saturated models and still remain inside an AEC. The other key step is the use of multidimensional diagrams in independent amalgamation. Essentially, the idea is to generalize [She83a, She83b] (which discussed $\mathbb{L}_{\omega_1,\omega}$ only) to AECs. This was started in the first author's two volume book [She09a, She09b], where the central concept of a *good λ-frame* was introduced. Roughly, an AEC $\mathfrak{K}$ has a good λ-frame if it has several basic structural properties in λ (amalgamation, no maximal models, and stability), and also has a superstable-like notion of forking defined for models of cardinality λ. The idea here is that everything is assumed only for models of cardinality λ. The program, roughly, is to develop the theory of good frames so that one can start with a good λ-frame, extend it to a good λ^+-frame, and keep going until the structure of the whole class is understood.

In order to carry out this program in practice, one has to prove that the good λ-frame has more structural properties. Specifically, one first proves (under some conditions) that a notion of *nonforking amalgamation* can be defined (initially, the definition of a good frame only requires a notion of nonforking for types of single elements, but here we get one for types of models). Thus one has at least a *two-dimensional* notion of nonforking (it applies to squares of models). From this, one seeks to define a three-dimensional notion (for cubes of models), and then inductively an n-dimensional notion. It turns out that once one has a good n-dimensional notion for all $n < \omega$ (still for models of cardinality λ), then one can lift this to the whole class. In [She83a, She83b], an $\mathbb{L}_{\omega_1,\omega}$ sentence with all the good n-dimensional properties was called *excellent*, so we naturally call an AEC with those properties an *excellent* AEC. It turns out that this is truly a generalization of [She83a, She83b], and in fact another consequence of the methods of this paper is Theorem 14.12, which generalizes [She83a, She83b] to any $\mathrm{PC}_{\aleph_0}$-AEC (and hence in particular to $\mathbb{L}_{\omega_1,\omega}(Q)$, see [She09a, II.3.5]; it had been a longstanding open question whether such a generalization was possible).

The use of multidimensional diagrams has had several more applications. For example, the *main gap theorem* of the first author [She85] is a theorem about countable first-order theory but was proven using multidimensional diagrams. Thus we believe that the systematic study of multidimensional diagrams undertaken in this paper is really its key contribution (in fact, a main gap theorem for certain AECs now also seems within reach, although we do not explore this here). Let us note that multidimensional diagrams for AECs were already studied in [She09a, III.12], but we do not rely on this here. See later for how the present paper relates to this reference.

1.4. **Eventual categoricity in AECs with amalgamation.** We mention another application of multidimensional diagram, which is really why this paper was started. In [She09a, Theorem IV.7.12], it was asserted that the eventual categoricity conjecture holds for AECs with amalgamation, assuming a weak version of the generalized continuum hypothesis (GCH). However, the proof relied on a claim of [She09a, III.12] that was not proven there, but promised to appear in [She]. See also [Vas17b, Section 11] for an exposition of the proof, modulo the claim.

We prove this missing claim in the present paper (Corollary 14.4). As a consequence, we obtain the consistency of the eventual categoricity conjecture in AECs with amalgamation:

Corollary 14.10. Assume $2^\theta < 2^{\theta^+}$ for all cardinals θ. Let $\mathfrak{K}$ be an AEC with amalgamation. If $\mathfrak{K}$ is categorical in *some* $\mu \geq \beth_{\left(2^{\aleph_{\mathrm{LS}(\mathfrak{K})^+}}\right)^+}$, then $\mathfrak{K}$ is categorical in *all* $\mu' \geq \beth_{\left(2^{\aleph_{\mathrm{LS}(\mathfrak{K})^+}}\right)^+}$.

Note that a similar-looking result starting with categoricity in a *successor* cardinal was established (in ZFC) in [She99]. There however, only a downward transfer was obtained (i.e. nothing was said about categoricity *above* the starting categoricity cardinal μ). Moreover the threshold cardinal was bigger. After the present paper was written, the second author [Vas] established that the threshold cardinal in Corollary 14.10 can be lowered further[1], all the way to $\beth_{\left(2^{\mathrm{LS}(\mathfrak{K})}\right)^+}$.

Note also that is is known (see for example [MS90, 1.13] or [Bon14, 7.3]) that AECs categorical above a strongly compact cardinal have (eventual) amalgamation. Thus Corollary 14.10 can be seen as a generalization of Corollary 14.7. However, the two results are formally incompatible since there are no cardinal arithmetic assumptions in the hypotheses of Corollary 14.7. By a result of Kolman and the first author [SK96], some amount of amalgamation already follows from categoricity above a *measurable* cardinal. In fact, this amount of amalgamation is enough to carry out the proof of Corollary 14.10, see Theorem 14.9. Thus the present paper also establishes that Conjecture 1.3 is consistent with a proper class of measurable cardinals.[2]

1.5. **Other approaches to eventual categoricity.** The eventual categoricity conjecture is known to hold under several additional assumptions (in addition to those already mentioned). Grossberg and VanDieren [GV06c, GV06a] established an *upward* categoricity transfer from categoricity in a successor in AECs (with amalgamation) that they called *tame* (we use the term "local" here, preferred by the first author). Roughly, orbital types in such AECs are determined by their restrictions to small sets. This is a very desirable property which is known to follow from large cardinals [Bon14].

Recently, the second author [Vas17d, Vas17e] established the eventual categoricity conjecture for *universal classes*: classes of models axiomatized by a universal $\mathbb{L}_{\infty,\omega}$ sentence (or alternatively, classes of models closed under isomorphism, substructure, and unions of chains). The proof is in ZFC and the threshold is low (not a large cardinal). Further, the starting cardinal does not need to be a successor. To prove this result, the second author established eventual categoricity more generally in AECs that have amalgamation are local, and in addition have *primes* over models of the form Ma [Vas18].

Now, it is folklore that any excellent AEC (for the purpose of this discussion, an AEC satisfying strong multidimensional amalgamation properties) should have amalgamation, be local, and have primes. Thus the main contribution of the present paper is to establish excellence for the setups mentioned above (large cardinals or amalgamation with weak GCH). Then one can essentially just quote the categoricity transfer of the second author.

Note however that even with additional locality assumptions, the known categoricity transfers for AECs are proven using local approaches (mainly good frames or

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[1] The reasons for the appearance of the strange cardinal $\beth_{\left(2^{\aleph_{\mathrm{LS}(\mathfrak{K})^+}}\right)^+}$ are perhaps best explained by the proof itself. The short version is that $\aleph_{\mathrm{LS}(\mathfrak{K})^+}$ is the minimal limit cardinal with cofinality greater than $\mathrm{LS}(\mathfrak{K})$.

[2] In fact, the machinery of this paper was designed to use a measurable rather than a strongly compact in Corollary 14.7. However, some technical points remain on how exactly to do this, so for simplicity we focus on the strongly compact case and leave the measurable case to future works.

a close variant[3]), and transferring the local structure upward using the locality assumptions. The present paper takes the local approach further by using it to *prove* locality conditions, through excellence. Of course, in the present paper we pay a price: cardinal arithmetic or large cardinal assumptions. It remains open whether paying such a price is necessary, or whether excellence (and therefore eventual categoricity) can be proven in ZFC.

1.6. **How to prove excellence.** While the present paper is admittedly complex, relying on several technical frameworks to bootstrap itself, we can readily describe the main idea of the proof of excellence. A starting point is a result of the first author [She09a, I.3.8]: assuming $2^\lambda < 2^{\lambda^+}$ (the weak diamond, see [DS78]), an AEC categorical in λ and λ^+ must have amalgamation in λ. This can be rephrased as follows: the 1-dimensional amalgamation property in λ holds provided that a strong version of the 0-dimensional amalgamation property holds in λ^+ (we think of the 0-dimensional amalgamation property as joint embedding).

A key insight (a variant of which already appears in [She09a, III.12.30]) is that this argument can be repeated by looking at *diagrams* (or *systems*) of models rather than single models (we do not quite get an AEC, but enough of the properties hold). We obtain, roughly, that $(n+1)$-dimensional amalgamation in λ follows from a strong version of n-dimensional amalgamation in λ^+. Since we are talking about *nonforking* amalgamation, we call these properties "uniqueness" properties rather than "amalgamation" properties, but the idea remains. Parametrizing (and forgetting about the "strong" part), we have that (λ^+, n)-uniqueness implies $(\lambda, n+1)$-uniqueness.

Now, it was already known how to obtain $(\lambda, 1)$, and even $(\lambda, 2)$-uniqueness. In fact, it is known (under reasonable assumptions) how to obtain a λ such that $(\lambda^{+n}, 2)$-uniqueness holds for all $n < \omega$ (such a setup is called an *ω-successful frame*). Assuming $2^{\lambda^{+m}} < 2^{\lambda^{+(m+1)}}$ for all $m < \omega$, we obtain from the earlier discussion used with each λ^{+m} that $(\lambda^{+m}, 3)$-uniqueness holds for all $m < \omega$. Continuing like this, we get (λ^{+m}, n)-uniqueness for all $n, m < \omega$.

The implementation of this sketch is complicated by the fact that there are other properties to consider, including extension properties. Further, we have to ensure a *strong* uniqueness property (akin to categoricity) in the successor cardinal, so the induction is much more complicated.

Forgetting this for the moment, let us explain how, assuming large cardinals, one can remove the cardinal arithmetic assumption. For this we assume more generally that we have an increasing sequence $\langle \lambda_m : n < \omega \rangle$ of cardinals such that for all $m < \omega$, $2^{<\lambda_{m+1}} = 2^{\lambda_m} < \lambda_{m+1}$ (in other words, λ_{m+1} is least such that $2^{\lambda_m} < 2^{\lambda_{m+1}}$). Let $\lambda_\omega := \sup_{m<\omega} \lambda_m$. Assume that $(\lambda, 2)$-uniqueness holds for all $\lambda \in [\lambda_0, \lambda_\omega)$. In the previous setup, we had that $\lambda_{m+1} = \lambda_m^+$. Now, as in [SV99, 1.2.4], we can generalize the first author's aforementioned result on getting amalgamation from two successive categoricity to: whenever $\lambda < \lambda'$ and $2^\lambda = 2^{<\lambda'} < 2^{\lambda'}$ and categoricity in λ' implies there exists an amalgamation base in $[\lambda, \lambda')$. Essentially, in some $\mu \in [\lambda, \lambda')$, we will have amalgamation. The n-dimensional version of this

[3]See [Vas17b] on how the Grossberg-VanDieren result can be proven using good frames.

will essentially be that if (λ', n)-uniqueness holds, then $(\mu, n+1)$-uniqueness holds for *some* $\mu \in [\lambda, \lambda')$.

So far, we have not used large cardinals. However, we use them to "fill in the gaps" and make sure that we have amalgamation in every cardinals below μ. So let $\mu_0 < \mu$. Fix a triple (M_0, M_1, M_2) of size μ_0 to amalgamate. Since we have large cardinals, we can take ultraproducts by a sufficiently complete ultrafilters to obtain an extension (M_0', M_1', M_2') of this triple that has size μ. Now using amalgamation in μ we can amalgamate this new triple, and hence get an amalgam of the original triple as well. The suitable n-dimensional version of this argument can be carried out, and we similarly obtain (λ, n)-uniqueness for all $\lambda \in [\lambda_0, \lambda_\omega)$ and $n < \omega$.

1.7. **Structure of the paper.** We assume that the reader has a solid knowledge of AECs, including [Bal09] (and preferably at least parts of the first three chapters of [She09a]) but also the more recent literature. We have tried to repeat the relevant background but still, to keep the paper to a manageable size we had to heavily quote and use existing arguments.

Schematically, for each n section n discusses a certain framework, call it framework n. Framework n is more powerful than framework $n-1$. Section n discusses how to get into framework n, starting with some instances of framework $n-1$. Then enough properties of framework n are also proven so that one can develop the properties of framework $n+1$ in the next section. We have tried to minimize the cross-section dependencies: the reader can often forget about previous framework and focus on the current one. At the end of the paper, an index will also help the reader getting back to previous definitions.

Specifically, the paper is organized as follows. After some notational preliminaries, AECs and more generally abstract classes are discussed (Section 3). Then a special framework that will be used to deal with large cardinals, compact AECs, is introduced (Section 4). We do some combinatorics, in particular the weak diamond argument alluded to earlier, in Section 5. The next framework, good frames, is discussed in Section 6. We then move on to two-dimensional independence notions (Section 7), and multidimensional independence notions (Section 8). Some properties of multidimensional independence notions are then proven that are purely combinatorial, in the sense of only requiring finitely many steps (no arguments involving unions of chains and resolutions), see Sections 9, 10. One then moves on to looking at multidimensional independence notions with also requirements on chains of systems (Section 11). After a section on primes (Section 12), we discuss excellent AECs (Section 13) and conclude with the main theorems of this paper (Section 14).

Each section begins with a short overview of its organization and main results.

2. Preliminaries and semilattices

Our notation is standard, and essentially follows [She09a] and [SV]. For a structure M, we write $|M|$ for its universe and $\|M\|$ for its cardinality. We imitate this convention for ordered classes (see Definition 3.1 and Notation 3.2): $\mathfrak{K}$ will denote a pair $(K, \leq_{\mathfrak{K}})$, where K is a class of structures and $\leq_{\mathfrak{K}}$ is a partial order on K. We will sometimes write $|\mathfrak{K}|$ for K, but may identify $\mathfrak{K}$ and K when there is no danger of confusion.

We write $[A]^\mu$ (respectively $[A]^{<\mu}$) for the set of all subsets of A of cardinality μ (respectively strictly less than μ).

Recall that a (meet) *semilattice* is a partial order $(I, \leq)$ so where every two elements have a meet (i.e. a greatest lower bound). We will often identify I with $(I, \leq)$. We write $s \wedge t$ for the meet of two elements $s, t \in I$. In this paper, I will almost always be finite, and in this case we write $\bot$ for $\wedge I$ (when I is not empty), the least element of I. We will also use interval notation: for $u \leq v$ both in I, we write $[u, v]$ for the partial order with universe $\{w \in I \mid u \leq w \leq v\}$, similarly for (u, v), (u, ∞), and other interval notations.

When I and J are semilattices, we write $I \subseteq J$ to mean that I is a *subsemilattice* of J. In particular, the meet operation in I and J coincides. Notice that any initial segment of a semilattice is a subsemilattice.

Given I and J partial orders, we write $I \times J$ for their product ordered lexicographically: $(i_1, j_1) \leq (i_2, j_2)$ if and only if $i_1 \leq i_2$ and $j_1 \leq j_2$. When I and J are semilattices, $I \times J$ is a semilattice: the meet is given by the meet of each coordinate.

For u a set, we write $\mathcal{P}(u)$ for the semilattice $(\mathcal{P}(u), \subseteq)$, where the meet operation is given by $s \wedge t = s \cap t$ and $\bot = \emptyset$. We write $\mathcal{P}^-(u)$ for the semilattice on $\mathcal{P}(u) \backslash \{u\}$ induced by $\mathcal{P}(u)$. Very often, we will have $u = n = \{0, 1, \ldots, n-1\}$. More generally, the semilattices we will work with will usually be finite initial segment of $\mathcal{P}(\omega)$.

Note that for any set u, $\mathcal{P}(u) \cong \mathcal{P}(|u|)$, and $\mathcal{P}^-(u) \cong \mathcal{P}^-(|u|)$. Moreover, $\mathcal{P}(n) \times \mathcal{P}(m) \cong \mathcal{P}(n+m)$ for $n, m < \omega$.

3. Abstract classes and skeletons

The definition of an abstract class gives the minimal properties we would like any class of structures discussed in this paper to satisfy. We also discuss the notion of a *skeleton* of an abstract class: a subclass that captures many of the essential properties of the original class but may be more manageable.

We study special kind of abstract classes: *fragmented AECs* are the main new concept: they are AECs, except that the chain axiom is weakened so that chains that jump cardinalities may not be closed under unions. Class of saturated models in a superstable first-order theory are an example, and we will use them extensively to deal with such setups.

We finish this section with a review of some standard definitions and results on AECs.

Let us start with the definition of an abstract class (due to Rami Grossberg).

Definition 3.1. An *abstract class* is a pair $\mathfrak{K} = (K, \leq_{\mathfrak{K}})$ such that K is a class of structures in a fixed (here always finitary) vocabulary $\tau = \tau(\mathfrak{K})$, $\leq_{\mathfrak{K}}$ is a partial order on K extending τ-substructure, and $\mathfrak{K}$ is closed under isomorphisms.

We often do not distinguish between $\mathfrak{K}$ and its underlying class, writing for example "$M \in \mathfrak{K}$". When we need to be precise, we will use the following notation:

Notation 3.2. Let $\mathfrak{K} = (K, \leq_{\mathfrak{K}})$ be an abstract class.

(1) We write $|\mathfrak{K}|$ for the underlying class K of $\mathfrak{K}$.
(2) Let $K_0 \subseteq K$ be a class of structures which is closed under isomorphisms. We let $\mathfrak{K} \upharpoonright K_0 := (K_0, \leq_{\mathfrak{K}}\upharpoonright K_0)$, where $\leq_{\mathfrak{K}}\upharpoonright K_0$ denotes the restriction of $\leq_{\mathfrak{K}}$ to K_0.

We will also use the following standard notation:

Notation 3.3. Let $\mathfrak{K}$ be an abstract class and let Θ be a class of cardinals. We write $\mathfrak{K}_\Theta$ for the abstract class $\mathfrak{K} \upharpoonright \{M \in \mathfrak{K} \mid \|M\| \in \Theta\}$. That is, it is the abstract class whose underlying class is the class of models of $\mathfrak{K}$ of size in Θ, ordered by the restriction of the ordering of $\mathfrak{K}$. We write $\mathfrak{K}_\lambda$ instead of $\mathfrak{K}_{\{\lambda\}}$, $\mathfrak{K}_{\leq\lambda}$ instead of $\mathfrak{K}_{[0,\lambda]}$, etc.

We define amalgamation, orbital types, stability, etc. for abstract classes as in the preliminaries of [Vas16c] but we mostly use the notation and terminology of [She09a]. In particular we write $\mathbf{tp}_{\mathfrak{K}}(\bar{b}/A; N)$ for the orbital type of $\bar{b}$ over A in N as computed inside $\mathfrak{K}$. Usually, $\mathfrak{K}$ will be clear from context, so we will omit it.

The following locality notion (also called tameness) will play an important role. It is implicit already in [She99], but is it considered as a property of AECs per say by Grossberg and VanDieren in [GV06b].

Definition 3.4. Let $\mathfrak{K}$ be an abstract class and let κ be an infinite cardinal.

We say that $\mathfrak{K}$ is $(< \kappa)$-*local* if for any $M \in \mathfrak{K}$ and any $p, q \in \mathscr{S}(M)$, if $p \upharpoonright A = q \upharpoonright A$ for any $A \in [M]^{<\kappa}$, then $p = q$. We say that $\mathfrak{K}$ is κ-*local* if it is $(< \kappa^+)$-local.

We will also use the following standard notions of saturation:

Definition 3.5. Let $\mathfrak{K}$ be an abstract class and let $M \leq_{\mathfrak{K}} N$ both be in $\mathfrak{K}$.

(1) We say that N is *universal over* M if whenever N_0 is such that $M \leq_{\mathfrak{K}} N_0$ and $\|N_0\| = \|M\|$, we have that there exists $f : N_0 \xrightarrow[M]{} N$.
(2) We say that N is (λ, δ)-*brimmed over* M if there exists an increasing continuous chain $\langle M_i : i \leq \delta \rangle$ in $\mathfrak{K}_\lambda$ such that $M_0 = M$, $M_\delta = M$, and M_{i+1} is universal over M_i for all $i < \delta$.
(3) We say that N is *brimmed over* M if it is (λ, δ)-brimmed over M for some λ and δ.
(4) We say that N is *brimmed over* A (for $A \subseteq |N|$ a set) if there exists $M' \in \mathfrak{K}$ such that $M' \leq_{\mathfrak{K}} N$, $A \subseteq |M'|$, and N is brimmed over M'.
(5) We say that N is *brimmed* if it is brimmed over some M'.
(6) We say that M is λ-*model-homogeneous* if whenever $M_0 \leq_{\mathfrak{K}} N_0$ are both in $\mathfrak{K}$ with $M_0 \leq_{\mathfrak{K}} M$ and $\|N_0\| < \lambda$, there exists $f : N_0 \xrightarrow[M_0]{} M$. We say that M is *model-homogeneous* if it is $\|M\|$-model-homogeneous.
(7) We say that M is λ-*saturated* if for any $M_0 \leq_{\mathfrak{K}} M$ with $M_0 \in \mathfrak{K}_{<\lambda}$, any $p \in \mathscr{S}(M_0)$ is realized inside M. We say that M is *saturated* if it is $\|M\|$-saturated.

We give the definition of a few more useful general concepts:

Definition 3.6. The *domain* $\text{dom}(\mathfrak{K})$ of an abstract class $\mathfrak{K}$ is the class of cardinals λ such that $\mathfrak{K}_\lambda \neq \emptyset$.

Definition 3.7. A *sub-abstract class* of an abstract class $\mathfrak{K} = (K, \leq_{\mathfrak{K}})$ is an abstract class $\mathfrak{K}^* = (K^*, \leq_{\mathfrak{K}^*})$ such that $K^* \subseteq K$ and for $M, N \in K^*$, $M \leq_{\mathfrak{K}^*} N$ implies $M \leq_{\mathfrak{K}} N$.

The definition of a skeleton is due to the second author [Vas16a, 5.3]. We have further weakened the requirement on chains appearing there.

Definition 3.8. A *skeleton* of an abstract class $\mathfrak{K}$ is a sub-abstract class $\mathfrak{K}^*$ of $\mathfrak{K}$ such that:

(1) For any $M \in \mathfrak{K}$, there exists $N \in \mathfrak{K}^*$ with $M \leq_{\mathfrak{K}} N$.
(2) For any $M, N \in \mathfrak{K}^*$ with $M \leq_{\mathfrak{K}} N$, there exists $N' \in \mathfrak{K}^*$ such that $M \leq_{\mathfrak{K}^*} N'$ and $N \leq_{\mathfrak{K}^*} N'$.

Examples of skeletons will be given later (see Fact 3.24). A simple example the reader can keep in mind is the class of $\aleph_0$-saturated models of a first-order theory, ordered either by elementary substructure or by being "universal over".

The following apparent strengthening of the definition of a skeleton will be useful.

Lemma 3.9. Let $\mathfrak{K}^*$ be a skeleton of $\mathfrak{K}$. Let $M \in \mathfrak{K}$, $n < \omega$, and let $\langle M_i : i < n \rangle$ be a (not necessarily increasing) sequence of elements of $\mathfrak{K}^*$ such that $M_i \leq_{\mathfrak{K}} M$ for all $i < n$. Then there exists $N \in \mathfrak{K}^*$ such that $M \leq_{\mathfrak{K}} N$ and $M_i \leq_{\mathfrak{K}^*} N$ for all $i < n$.

Proof. We work by induction on n. If $n = 0$, use (1) in the definition of a skeleton to find $N \in \mathfrak{K}^*$ with $M \leq_{\mathfrak{K}} N$. Assume now that $n = m + 1$. By the induction hypothesis, let $N_0 \in \mathfrak{K}^*$ be such that $M \leq_{\mathfrak{K}} N_0$ and $M_i \leq_{\mathfrak{K}^*} N_0$ for all $i < m$. Now apply (2) in the definition of a skeleton, with M, N there standing for M_m, N_0 here. We get $N \in \mathfrak{K}^*$ such that $N_m \leq_{\mathfrak{K}^*} N$ and $N_0 \leq_{\mathfrak{K}^*} N$. By transitivity of $\leq_{\mathfrak{K}^*}$, we have that $M_i \leq_{\mathfrak{K}^*} N$ for all $i < m$, so N is as desired. $\square$

We now give the definition of several properties that an abstract class may have. In particular, it could be a fragmented AEC, or even just an AEC.

Definition 3.10. Let $\mathfrak{K}$ be an abstract class.

(1) Let $\text{LS}(\mathfrak{K})$ be the least cardinal $\lambda \geq |\tau(\mathfrak{K})| + \aleph_0$ such that for any $M \in \mathfrak{K}$ and any $A \subseteq |M|$, there is $M_0 \in \mathfrak{K}$ with $M_0 \leq_{\mathfrak{K}} M$, $A \subseteq |M_0|$, and $\|M_0\| \leq |A| + \lambda$. When such a λ does not exist, we write $\text{LS}(\mathfrak{K}) = \infty$.
(2) Let $[\lambda_1, \lambda_2)$ be an interval of cardinals and let δ be a limit ordinal. We say that $\mathfrak{K}$ is $([\lambda_1, \lambda_2), \delta)$-*continuous* if whenever $\langle M_i : i < \delta \rangle$ is an increasing chain in $\mathfrak{K}_{[\lambda_1, \lambda_2)}$, then $M_\delta := \bigcup_{i<\delta} M_i$ is such that $M_\delta \in \mathfrak{K}$ and $M_0 \leq_{\mathfrak{K}} M_\delta$. We say that $\mathfrak{K}$ is $[\lambda_1, \lambda_2)$-*continuous* if it is $([\lambda_1, \lambda_2), \delta)$-continuous for all limit ordinals δ. We say that $\mathfrak{K}$ is *continuous* if it is $\text{dom}(\mathfrak{K})$-continuous.
(3) We say that $\mathfrak{K}$ is *coherent* if whenever $M_0, M_1, M_2 \in \mathfrak{K}$, if $|M_0| \subseteq |M_1| \subseteq |M_2|$, $M_0 \leq_{\mathfrak{K}} M_2$, and $M_1 \leq_{\mathfrak{K}} M_2$, then $M_0 \leq_{\mathfrak{K}} M_1$.
(4) $\mathfrak{K}$ is a *fragmented abstract elementary class (AEC)* if it satisfies:
    (a) (Coherence) $\mathfrak{K}$ is coherent.

(b) (Löwenheim-Skolem-Tarski axiom) $\mathrm{LS}(\mathfrak{K}) < \infty$.
(c) (Restricted chain axioms) Let $\langle M_i : i < \delta \rangle$ be an increasing chain in $\mathfrak{K}$. Let $M_\delta := \bigcup_{i<\delta} M_i$.
  (i) *If* $\|M_i\| = \|M_\delta\|$ for some $i < \delta$, then:
    (A) $M_\delta \in \mathfrak{K}$.
    (B) $M_0 \leq_{\mathfrak{K}} M_\delta$.
  (ii) (Smoothness) If $N \in \mathfrak{K}$ is such that $M_i \leq_{\mathfrak{K}} N$ for all $i < \delta$, then $M_\delta \in \mathfrak{K}$ and $M_\delta \leq_{\mathfrak{K}} N$.

(5) $\mathfrak{K}$ is an *abstract elementary class (AEC)* if it is a continuous fragmented AEC.
(6) Define also the notion of a *[very] weak [fragmented] AEC*, where "weak" means that the smoothness axiom may not hold and "very weak" means that the coherence axiom may also not hold.

The following are examples of fragmented AECs. The generalization of the first one to AECs will play an important role in the present paper. The second will not be studied here, but we give it as an additional motivation.

Example 3.11. Let T be a superstable first-order theory and let $\mathfrak{K}$ be its class of saturated models, ordered by elementary substructure. Then $\mathfrak{K}$ is a fragmented AEC with $\mathrm{LS}(\mathfrak{K}) \leq 2^{|T|}$.

Example 3.12. Let ϕ be a complete $\mathbb{L}_{\omega_1,\omega}(Q)$-sentence ($Q$ is the quantifier "there exists uncountably many") and let Φ be a countable fragment of $\mathbb{L}_{\omega_1,\omega}(Q)$ containing ϕ. Assume that for every $\psi(\bar{x}) \in \Phi$, there exists a relation $R_\psi(\bar{x})$ such that $\psi \models \forall \bar{x} \psi(\bar{x}) \leftrightarrow R_\psi(\bar{x})$. Let K be the class of models M of $\mathbb{L}_{\omega,\omega}$ consequences of ϕ which also model ψ if M uncountable. Order it by $M \leq_{\mathfrak{K}} N$ if and only if $M \preceq N$ and for any $\psi(x, \bar{y}) \in \Phi$, if $M \models R_{\neg Qx\psi(x,\bar{y})}[\bar{a}]$, then $\psi(M, \bar{a}) = \psi(N, \bar{a})$. Then $\mathfrak{K}$ is a fragmented AEC with $\mathrm{LS}(\mathfrak{K}) = \aleph_0$.

In any AEC, models of big cardinality can be resolved as an increasing union of smaller models. We make this into a definition:

Definition 3.13. Let $\mathfrak{K}$ be an abstract class and let $M_0, M \in \mathfrak{K}$ with $M_0 <_{\mathfrak{K}} M$. We say that M is *δ-resolvable over M_0* if there exists a strictly increasing continuous chain $\langle N_i : i \leq \delta \rangle$ such that:

(1) $N_0 = M_0$, $N_\delta = M$.
(2) $\|M_i\| = \|M_0\| + |i|$ for all $i \leq \delta$.

We say that M is *resolvable over M_0* if it is δ-resolvable over M_0 for some limit ordinal δ. We say that M is *resolvable* if it is resolvable over M_0 for every $M_0 \in \mathfrak{K}$ with $M_0 \leq_{\mathfrak{K}} M$ and $\|M_0\| < \|M\|$. We say that $\mathfrak{K}$ is *resolvable* if any $M \in \mathfrak{K}$ is resolvable.

Remark 3.14. For any fragmented AEC $\mathfrak{K}$, $\mathfrak{K}_{\geq \mathrm{LS}(\mathfrak{K})}$ is resolvable (this uses that we do not require $\|M_i\| = \|M_\delta\|$ in the smoothness axiom).

It is known that any class of structures is contained in a smallest AEC. Indeed, by [She09a, I.2.19]:

Fact 3.15. Let $\{\mathfrak{K}_i : i \in I\}$ be a non-empty collection of AECs, all in the same vocabulary τ. Then $\bigcap_{i \in I} \mathfrak{K}_i$ (defined as $(\bigcap_{i\in I} |\mathfrak{K}_i|, \bigcap_{i \in I} \leq_{\mathfrak{K}_i})$) is an AEC.

So:

Fact 3.16. For any abstract class $\mathfrak{K}$, there exists a unique smallest AEC $\mathfrak{K}^*$ such that $\mathfrak{K}$ is a sub-abstract class of $\mathfrak{K}^*$.

Proof. Take the intersection of the collection of all AECs that are sub-abstract classes of $\mathfrak{K}$. Note that this collection is non-empty: it contains the AEC of all $\tau(\mathfrak{K})$-structures, ordered with substructure. $\square$

Definition 3.17. Let $\mathfrak{K}$ be an abstract class. We call the AEC $\mathfrak{K}^*$ given by Fact 3.16, the AEC *generated by* $\mathfrak{K}$.

AECs are uniquely determined by their restrictions of size λ:

Fact 3.18 ([She09a, II.1.23])**.** Let $\mathfrak{K}^1$ and $\mathfrak{K}^2$ be AECs with $\lambda := \text{LS}(\mathfrak{K}^1) = \text{LS}(\mathfrak{K}^2)$. If $\mathfrak{K}^1_{\le\lambda} = \mathfrak{K}^2_{\le\lambda}$, then $\mathfrak{K}^1 = \mathfrak{K}^2$.

We now look into how much of this uniqueness is carried over in fragmented AECs. We will specifically look at totally categorical fragmented AECs (this is not such a strong assumption, since classes of saturated models are examples).

First, only part of the fragmented AEC suffices in order to specify the AEC it generates.

Lemma 3.19. Let $\mathfrak{K}$ be a fragmented AEC with $\mathfrak{K}_{<\text{LS}(\mathfrak{K})} = \emptyset$. Then the AEC generated by $\mathfrak{K}$ is the same as the AEC generated by $\mathfrak{K}_{\text{LS}(\mathfrak{K})}$.

Proof. Let $\lambda := \text{LS}(\mathfrak{K})$. Let $\mathfrak{K}^1$ be the AEC generated by $\mathfrak{K}$ and let $\mathfrak{K}^2$ be the AEC generated by $\mathfrak{K}_{\le\lambda}$. Since $\text{LS}(\mathfrak{K}) = \lambda$, it is easy to check that $\text{LS}(\mathfrak{K}^1) = \lambda$. Of course, we also have that $\text{LS}(\mathfrak{K}^2) = \lambda$. Moreover, $\mathfrak{K}^1_\lambda = \mathfrak{K}^2_\lambda$. By Fact 3.18 and Lemma 3.19, $\mathfrak{K}^1 = \mathfrak{K}^2$. $\square$

Toward studying categoricity in fragmented AECs, we look at homogeneous models. As in [She09a, I.2.5], we have:

Fact 3.20. Let $\mathfrak{K}$ be a fragmented AEC and let $M_0 \le_{\mathfrak{K}} M_\ell$, $\ell = 1, 2$. If $\|M_1\| = \|M_2\| > \text{LS}(\mathfrak{K})$ and both M_1 and M_2 are model-homogeneous, then $M_1 \cong_{M_0} M_2$.

We obtain:

Theorem 3.21. Let $\mathfrak{K}$ be a fragmented AEC with amalgamation and $\mathfrak{K}_{<\text{LS}(\mathfrak{K})} = \emptyset$. Assume that for every $\lambda \in \text{dom}(\mathfrak{K})$, $\mathfrak{K}$ is categorical in λ and stable in λ. Then any $M \in \mathfrak{K}_{>\text{LS}(\mathfrak{K})}$ is model-homogeneous.

Proof. Let $\lambda \in \text{dom}(\mathfrak{K}) \cap (\text{LS}(\mathfrak{K}), \infty)$ and let $M \in \mathfrak{K}_\lambda$. Pick $M_0 \le_{\mathfrak{K}} N_0$ in $\mathfrak{K}_{<\lambda}$ with $M_0 \le_{\mathfrak{K}} M$. By categoricity and stability in λ, M is (λ, λ)-brimmed. If λ is regular, this means that we can find $M^0 \in \mathfrak{K}_\lambda$ such that $M_0 \le_{\mathfrak{K}} M^0$ and M is universal over M^0. Now by amalgamation, there exists $N^0 \in \mathfrak{K}_\lambda$ and $f : N_0 \underset{M_0}{\longrightarrow} N^0$ such that $M^0 \le_{\mathfrak{K}} N^0$. Now by universality let $g : N^0 \underset{M^0}{\longrightarrow} M$. Then gf embeds N_0 into M over M_0.

If λ is singular, let $\mu := \|N_0\|$. Pick $M^0 \le_{\mathfrak{K}} M$ with $M^0 \in \mathfrak{K}_{\mu^+}$ and M^0 containing M_0. By the previous case, N_0 embeds into M^0 (and hence into M) over M_0, as desired. $\square$

Theorem 3.22 (Canonicity of categorical fragmented AECs)**.** Let $\mathfrak{K}^1$ and $\mathfrak{K}^2$ be fragmented AECs with $\Theta := \mathrm{dom}(\mathfrak{K}^1) = \mathrm{dom}(\mathfrak{K}^2)$ and $\mathrm{LS}(\mathfrak{K}^1) = \mathrm{LS}(\mathfrak{K}^2)$. Suppose that for $\ell = 1, 2$, $\mathfrak{K}^\ell$ has amalgamation and is stable and categorical in every $\lambda \in \Theta$. If $\mathfrak{K}^1_{\leq \mathrm{LS}(\mathfrak{K}^1)} = \mathfrak{K}^2_{\leq \mathrm{LS}(\mathfrak{K}^2)}$, then $\mathfrak{K}^1 = \mathfrak{K}^2$.

Proof. Without loss of generality, $\mathfrak{K}^\ell_{<\mathrm{LS}(\mathfrak{K}^\ell)} = \emptyset$ for $\ell = 1, 2$. Let $\lambda > \mathrm{LS}(\mathfrak{K})$ be such that $\lambda \in \Theta$ and assume inductively that we know $\mathfrak{K}^1_{<\lambda} = \mathfrak{K}^2_{<\lambda}$. We have that the model M^ℓ of cardinality λ in $\mathfrak{K}^\ell$ is model-homogeneous (Theorem 3.21) for $\ell = 1, 2$. By categoricity and the inductive assumption, there exists $M_\ell \in \mathfrak{K}^\ell$ of cardinality strictly less than λ with $M_\ell \leq_{\mathfrak{K}} M^\ell$ and $f : M_1 \cong M_2$. Now use the proof of Fact 3.20 (the usual back and forth argument) to conclude that $M^1 \cong M^2$. This implies that $|\mathfrak{K}^1_\lambda| = |\mathfrak{K}^2_\lambda|$. To see that $\mathfrak{K}^1_\lambda = \mathfrak{K}^2_\lambda$ (that is, the orderings also coincide), observe that the AECs generated by $\mathfrak{K}^1$ and $\mathfrak{K}^2$ must be the same (Fact 3.18 and Lemma 3.19), hence the orderings must coincide on all models there. $\square$

Definition 3.23. Let $\mathfrak{K}$ be an abstract class.

(1) We write $\mathfrak{K}^{\lambda\text{-sat}}$ for the abstract class $\mathfrak{K} \upharpoonright \{M \in \mathfrak{K} \mid M \text{ is } \lambda\text{-saturated}\}$ of λ-saturated models in $\mathfrak{K}$ ordered by the restriction of the ordering of $\mathfrak{K}$.
(2) We write $\mathfrak{K}^{\mathrm{sat}}$ for the abstract class $\mathfrak{K} \upharpoonright \{M \in \mathfrak{K} \mid M \text{ is saturated}\}$.
(3) We write $\mathfrak{K}^{\mathrm{brim}}$ for the abstract class with $|\mathfrak{K}^{\mathrm{brim}}| = \{M \in \mathfrak{K} \mid M \text{ is brimmed}\}$ ordered by $M \leq_{\mathfrak{K}^{\mathrm{brim}}} N$ if and only if $M \leq_{\mathfrak{K}} N$ and one of the following conditions holds:
  (a) $M = N$.
  (b) $\|M\| < \|N\|$.
  (c) N is brimmed over M (so $\|M\| = \|N\|$).

It is known [Vas17c, 5.7] that an AEC with amalgamation categorical in a high-enough cardinal will satisfy a weak version of locality, and unions of chains of λ-saturated models will be λ-saturated there. This gives several examples of fragmented AECs and skeletons. Note that these results have a long history (detailed at the reference), with contributions of, among others, both the first and second author.

Fact 3.24 (Structure of categorical AECs with amalgamation)**.** Let $\mathfrak{K}$ be an AEC with arbitrarily large models. Let $\mu > \mathrm{LS}(\mathfrak{K})$. Assume that $\mathfrak{K}$ is categorical in μ and $\mathfrak{K}_{<\mu}$ has amalgamation and no maximal models.

(1) If $\mu \geq h(\mathrm{LS}(\mathfrak{K}))$, then there exists $\chi \in (\mathrm{LS}(\mathfrak{K}), \mu)$ such that $\mathfrak{K}^{\mathrm{sat}}_{(\chi,\mu)}$ is χ-local.
(2) Let $\lambda \leq \mu$. If $\mathrm{cf}(\lambda) \geq \mathrm{LS}(\mathfrak{K})^+$, then there exists $\chi \in (\mathrm{LS}(\mathfrak{K}), \lambda)$ such that $\mathfrak{K}^{\mathrm{sat}}_{(\chi,\lambda)}$ is χ-local.
(3) For any $\lambda \in (\mathrm{LS}(\mathfrak{K}), \mu]$, $\mathfrak{K}^{\lambda\text{-sat}}$ is an AEC with Löwenheim-Skolem-Tarski number λ. In particular, $\mathfrak{K}^{\mathrm{sat}}_{(\mathrm{LS}(\mathfrak{K}),\mu]}$ is a fragmented AEC with Löwenheim-Skolem-Tarski number $\mathrm{LS}(\mathfrak{K})^+$. Moreover, $\mathfrak{K}^{\mathrm{sat}}_{(\mathrm{LS}(\mathfrak{K}),\mu]}$ is categorical in every $\lambda \in (\mathrm{LS}(\mathfrak{K}), \mu)$ and it is a skeleton of $\mathfrak{K}_{(\mathrm{LS}(\mathfrak{K}),\mu]}$.
(4) Let $\Theta := [\mathrm{LS}(\mathfrak{K}), \mu)$. Then:
  - $\mathfrak{K}^{\mathrm{brim}}_\Theta$ is a fragmented very weak AEC which is $[\mathrm{LS}(\mathfrak{K}), \lambda)$-continuous and categorical in λ for every $\lambda \in \Theta$.
  - $\mathfrak{K}^{\mathrm{brim}}_\Theta$ is a skeleton of $\mathfrak{K}_\Theta$.

- $|\mathfrak{K}^{\text{brim}}_{(\text{LS}(\mathfrak{K}),\mu)}| = |\mathfrak{K}^{\text{sat}}_{(\text{LS}(\mathfrak{K}),\mu)}|$ and $\mathfrak{K}^{\text{brim}}_{(\text{LS}(\mathfrak{K}),\mu)}$ is a sub-abstract class of $\mathfrak{K}^{\text{sat}}_{(\text{LS}(\mathfrak{K}),\mu)}$.

Regarding categoricity transfers we will use the following two known results. The first can be traced back to the presentation theorem of the first author. See [Vas17b, 9.2] for a full proof.

Fact 3.25 (Morley's omitting type theorem for AECs)**.** Let $\mathfrak{K}$ be an AEC with amalgamation and let $\mu > \text{LS}(\mathfrak{K})$. If every model in $\mathfrak{K}_\mu$ is $\text{LS}(\mathfrak{K})^+$-saturated, then there exists $\chi < h(\text{LS}(\mathfrak{K}))$ such that every model in $\mathfrak{K}_{\geq \chi}$ is $\text{LS}(\mathfrak{K})^+$-saturated.

The second is due to the second author. It is a consequence of an omitting type theorem of the first author (see the reference below for a history).

Fact 3.26 ([Vas17b, 9.8])**.** Let $\mathfrak{K}$ be an $\text{LS}(\mathfrak{K})$-local AEC with amalgamation, and arbitrarily large models. If $\mathfrak{K}$ is categorical in *some* $\mu > \text{LS}(\mathfrak{K})$, then $\mathfrak{K}$ is categorical in a proper class of cardinals.

4. Compact abstract elementary classes

Compact AECs are an axiomatization of the AECs "essentially below κ" introduced by Boney [Bon14, 2.10]. They encompass classes of models of an $\mathbb{L}_{\kappa,\omega}$ sentence, κ a strongly compact, and more generally AECs closed under κ-complete ultraproducts, κ a strongly compact cardinal below (yes, below!) the Löwenheim-Skolem-Tarski number. The reader who does not want to process yet another abstract definition can with little loss think of $\mathfrak{K}_{\geq \kappa}$, where $\mathfrak{K}$ is an AEC and $\kappa > \text{LS}(\mathfrak{K})$ is a strongly compact cardinal[4].

Definition 4.1. Let $\mathfrak{K}$ be an AEC and let κ be a strongly compact cardinal (we allow $\kappa = \aleph_0$). We call $\mathfrak{K}$ *κ-compact* if:

(1) $\kappa \leq \text{LS}(\mathfrak{K})$ and $\mathfrak{K}_{<\text{LS}(\mathfrak{K})} = \emptyset$.
(2) $\mathfrak{K}$ is closed under κ-complete ultraproducts. More precisely, if U is a κ-complete ultrafilter on some index set I and $\langle M_i : i \in I \rangle$ is a sequence in $\mathfrak{K}$, then $\prod_{i \in I} M_i / U$ is in $\mathfrak{K}$. Moreover if $\langle N_i : i \in I \rangle$ is another sequence in $\mathfrak{K}$ and $M_i \leq_{\mathfrak{K}} N_i$ for all $i \in I$, then $\prod_{i \in I} M_i / U \leq_{\mathfrak{K}} \prod_{i \in I} N_i / U$.

We call $\mathfrak{K}$ *compact* if it is κ-compact for some strongly compact cardinal κ.

Immediately from the definition, we have:

Remark 4.2. Let $\mathfrak{K}$ be a κ-compact AEC.

(1) $\mathfrak{K}$ has no maximal models (just take an ultrapower).
(2) For any cardinal λ, $\mathfrak{K}_{\geq \lambda}$ is a κ-compact AEC.
(3) The conclusion of [Bon14, 4.3] holds.

The two main examples are:

Fact 4.3.

[4] With a little bit of care, one could adapt the definitions to work also with *almost* strongly compact cardinals, see [BTR17]. Since there are no obvious benefits, we have not bothered.

(1) Let κ be a strongly compact cardinal (we allow $\kappa = \aleph_0$) and let T be a theory in $\mathbb{L}_{\kappa,\omega}$. Let Φ be a fragment containing T. Let $\mathfrak{K}$ be the class of models of T, ordered by $M \leq_{\mathfrak{K}} N$ if and only if $M \preceq_\Phi N$. Then $\mathfrak{K}_{\geq\kappa}$ is a κ-compact AEC with $\text{LS}(\mathfrak{K}_{\geq\kappa}) \leq |\Phi| + |\tau(\Phi)| + \kappa$.
(2) Let $\mathfrak{K}$ be an AEC and let $\kappa > \text{LS}(\mathfrak{K})$ be a strongly compact cardinal. Then $\mathfrak{K}_{\geq\kappa}$ is κ-compact.

The following is an important property of compact AECs. Using the terminology of [Bon14, 3.3], it should be called "fully $(<\kappa)$-tame and short over the empty set". We prefer the shorter "$(<\kappa)$-short" here.

Definition 4.4. We say that an abstract class $\mathfrak{K}$ is $(<\kappa)$-*short* if for any $M_1, M_2 \in \mathfrak{K}$, any ordinal α, and any $\bar{a}_\ell \in {}^\alpha M_\ell$, $\ell = 1, 2$, if $\mathbf{tp}(\bar{a}_1 \restriction I/\emptyset; M_1) = \mathbf{tp}(\bar{a}_2 \restriction I/\emptyset; M_2)$ for any $I \in [\alpha]^{<\kappa}$, then $\mathbf{tp}(\bar{a}_1/\emptyset; M_1) = \mathbf{tp}(\bar{a}_2/\emptyset; M_2)$. We say that $\mathfrak{K}$ is κ-*short* if it is $(<\kappa^+)$-short. We say that $\mathfrak{K}$ is *short* if it is $(<\kappa)$-short for some κ.

Remark 4.5 ([Bon14, 3.4])**.** If an abstract class $\mathfrak{K}$ is $(<\kappa)$-short, then it is $(<\kappa)$-local (put an enumeration of M as part of both $\bar{b}_1$ and $\bar{b}_2$).

Fact 4.6 ([Bon14, 4.5])**.** If $\mathfrak{K}$ is a κ-compact AEC, then $\mathfrak{K}$ is $(<\kappa)$-short.

Finally, we note that categorical compact AECs have amalgamation:

Fact 4.7. Let $\mathfrak{K}$ be a compact AEC. If $\mathfrak{K}$ is categorical in some $\mu > \text{LS}(\mathfrak{K})$, then $\mathfrak{K}$ has amalgamation.

Proof. As in [SK96], $\mathfrak{K}_{<\mu}$ has amalgamation. Now as in the proof of [BB17, 2.3], we can use sufficiently complete fine ultrafilters to push amalgamation up and get that $\mathfrak{K}$ has amalgamation. □

Thus we obtain that categorical compact AECs already have quite a lot of structure:

Corollary 4.8. Let $\mathfrak{K}$ be a compact AEC. If $\mathfrak{K}$ is categorical in *some* $\mu > \text{LS}(\mathfrak{K})$, then $\mathfrak{K}$ has amalgamation, no maximal models, is $(<\text{LS}(\mathfrak{K}))$-short, and is categorical in a proper class of cardinals.

Proof. Say $\mathfrak{K}$ is κ-compact. By Fact 4.7, $\mathfrak{K}$ has amalgamation. By Remark 4.2, $\mathfrak{K}$ has no maximal models. By Fact 4.6, $\mathfrak{K}$ is $(<\kappa)$-short, hence by Remark 4.5 it is $\text{LS}(\mathfrak{K})$-local. Now apply Fact 3.26. □

5. Combinatorics of abstract classes

In addition to a few technical lemmas, the main result of this section is a generalization of the "amalgamation from categoricity and weak diamond" theorem of the first author [She09a, I.3.8]. The proof is essentially the same, but the axioms of AECs that were not used are dropped (as in the study of nice categories in [GS83, §4]), and an additional parametrization in the spirit of classification theory over a predicate [PS85, She86] is introduced: we consider the ϕ-amalgamation property to mean essentially that we can amalgamate while fixing the set defined by the formula ϕ.

First, we recall a generally useful folklore fact about increasing continuous chains reflecting on a club:

Lemma 5.1. Let λ be a regular uncountable cardinal. Let $\langle A_i : i \le \lambda \rangle$, $\langle B_i : i \le \lambda \rangle$ be increasing continuous sequences of sets such that:

(1) $A_\lambda = B_\lambda$.
(2) For all $i < \lambda$, $|A_i| + |B_i| < \lambda$.

Then $C := \{i < \lambda \mid A_i = B_i\}$ is club.

Proof. C is clearly closed by continuity of the chains. To see unboundedness, let $i < \lambda$. We build $\langle j_k : k < \omega \rangle$ an increasing sequence of ordinals below λ such that $j_0 = i$, and for all $k < \omega$, $B_{j_k} \subseteq A_{j_{k+1}}$ and $A_{j_k} \subseteq B_{j_{k+1}}$. This is straightforward using regularity of λ, $A_\lambda = B_\lambda$, $|A_j| + |B_j| < \lambda$ for all j, and the fact that the chains are increasing. Now by continuity of the chains, $j := \sup_{k<\omega} j_k$ is in C, as desired. $\square$

We use this to prove a result about chain of structures that are not even necessarily ordered by substructure:

Definition 5.2. For M and N τ-structures, $M \subseteq^* N$ (M is a *weak substructure of* N) if $|M| \subseteq |N|$ and for all relations symbols R in τ, $R^M \subseteq R^N$.

Lemma 5.3. Let λ be a regular uncountable cardinal. Let τ be a vocabulary and let $\langle M_i : i \le \lambda \rangle$, $\langle N_i : i \le \lambda \rangle$ be $\subseteq^*$-increasing continuous sequences of τ-structures. Assume:

(1) $M_\lambda = N_\lambda$.
(2) $|\tau| < \lambda$.
(3) $\|M_i\| + \|N_i\| < \lambda$ for all $i < \lambda$.

Then the set $C := \{i < \lambda \mid M_i = N_i\}$ is club.

Proof. Expanding M_i and N_i if necessary, we may assume without loss of generality that there is a unary predicate P in τ such that $P^{M_i} = M_i$ and $P^{N_i} = N_i$ for all $i < \lambda$. For each $R \in \tau$, the set $C_R := \{i < \lambda \mid R^{M_i} = R^{N_i}\}$ is club by Lemma 5.1. Now observe that $C = \bigcap_{R \in \tau} C_R$. $\square$

The following technical criteria for being resolvable (Definition 3.13) is used in the proof of Lemma 11.9.

Lemma 5.4. Let $\mathfrak{K}$ be an abstract class. Assume:

(1) $\text{LS}(\mathfrak{K}) < \infty$, $\mathfrak{K}_{<\text{LS}(\mathfrak{K})} = \emptyset$.
(2) For any $\lambda \in \text{dom}(\mathfrak{K})$, $\mathfrak{K}$ is $[\text{LS}(\mathfrak{K}), \lambda)$-continuous.
(3) (Coherence for models of different sizes) For $M_0, M_1, M_2 \in \mathfrak{K}$ with $M_0 \le_{\mathfrak{K}} M_2$, $M_1 \le_{\mathfrak{K}} M_2$, $|M_0| \subseteq |M_1|$, *and* $\|M_0\| < \|M_1\| < \|M_2\|$, we have that $M_0 \le_{\mathfrak{K}} M_1$.
(4) (Smoothness for big extensions) If $\langle M_i : i \le \delta \rangle$ is increasing and $N \in \mathfrak{K}$ is such that $M_i \le_{\mathfrak{K}} N$ for all $i < \delta$ and $\|M_\delta\| < \|N\|$, then $M_\delta \le_{\mathfrak{K}} N$.
(5) (Resolvability for successors) If $\lambda \ge \text{LS}(\mathfrak{K})$, $M \le_{\mathfrak{K}} N$ are such that $M \in \mathfrak{K}_\lambda$ and $N \in \mathfrak{K}_{\lambda^+}$, then there exists $\langle M_i : i < \lambda^+ \rangle$ increasing continuous in $\mathfrak{K}_\lambda$ such that $M_0 = M$ and $\bigcup_{i<\lambda^+} M_i = N$.

Then $\mathfrak{K}$ is resolvable.

Proof. Let $M_0 \leq_{\mathfrak{K}} M$ be given with $\|M_0\| < \|M\|$. Write $\lambda := \|M_0\|$, $\mu := \|M\|$. We show by induction on μ that M is resolvable over M. If $\mu = \lambda^+$, this is the assumption. Assume now that $\lambda^+ < \mu$. If μ is a successor, say μ_0^+, use that $\mathrm{LS}(\mathfrak{K}) \leq \lambda$ to pick M_1 such that $M_1 \leq_{\mathfrak{K}} M$, $|M_0| \subseteq |M_1|$, and $\lambda < \|M_1\| = \mu_0$. By coherence for models of different sizes, $M_0 \leq_{\mathfrak{K}} M_1$. By the induction hypothesis, there is a resolution $\langle M_{0,i} : i < \delta_0 \rangle$ of M_1 over M_0. By assumption, there is also a resolution $\langle M_{1,i} : i < \delta_1 \rangle$ of M over M_1. Now concatenate these two resolutions.

Thus we can assume that μ is limit. Let $\delta := \mathrm{cf}(\mu)$ and let $\langle A_i : i < \delta \rangle$ be an increasing sequence of sets such that $|A_i| < \mu$ for all $i < \delta$ and $\bigcup_{i<\mu} A_i = M$. We build $\langle M_i : i < \delta \rangle$ increasing continuous such that for all $i < \delta$:

(1) $M_0 = M$.
(2) $A_i \subseteq |M_{i+1}|$ for all $i < \delta$.
(3) $\|M_i\| < \|M_{i+1}\| < \|M\|$.

This is possible using the coherence and smoothness assumptions. This is enough. At the end, $\bigcup_{i<\delta} M_i = M$. We can then take a resolution of M_{i+1} over M_i for each $i < \delta$ (using the induction hypothesis), and concatenate all these resolutions to obtain the final desired resolution. $\square$

We will use the following form of the weak diamond principle (the proof is the same as in [DS78, 6.1]; or see the appendix of [She98]).

Fact 5.5. Let λ be an infinite cardinal and suppose there is an infinite cardinal $\lambda_0 < \lambda$ such that $2^{\lambda_0} = 2^{<\lambda} < 2^\lambda$. Then ($\lambda$ is regular uncountable and) for every sequence $\langle f_\eta \in {}^\lambda\lambda : \eta \in {}^\lambda 2 \rangle$, there exists $\eta \in {}^\lambda 2$ such that the set

$$S_\eta := \{\delta < \lambda \mid \exists \nu \in {}^\lambda 2 : f_\eta \restriction \delta = f_\nu \restriction \delta, \eta \restriction \delta = \nu \restriction \delta, \eta \restriction (\delta + 1) \neq \nu \restriction (\delta + 1)\}$$

is stationary.

The following notions relativize the usual amalgamation, categoricity, and universality properties to a formula ϕ:

Definition 5.6. Let $\mathfrak{K}$ be an abstract class in a vocabulary $\tau = \tau(\mathfrak{K})$ and let $\phi(x)$ be a quantifier-free $\mathbb{L}_{\omega,\omega}(\tau)$-formula.

(1) For $M, N \in \mathfrak{K}$, we say M and N are *ϕ-equal* if $\phi(M) = \phi(N)$ and $R^M \restriction \phi(M) = R^N \restriction \phi(N)$ for every relation and function symbols R of τ. In other words, the (partial) τ-structures induced by ϕ on M and N are equal.
(2) A *ϕ-span* is a triple (M_0, M_1, M_2) such that $M_0 \leq_{\mathfrak{K}} M_\ell$, $\ell = 1, 2$, and M_1 and M_2 are ϕ-equal.
(3) A *ϕ-amalgam* of a ϕ-span (M_0, M_1, M_2) is a triple (N, f_1, f_2) such that $N \in \mathfrak{K}$ and $f_\ell : M_\ell \xrightarrow[M_0]{} N$ are such that $f_1 \restriction \phi(M_1) = f_2 \restriction \phi(M_2)$.
(4) We say that M is a *ϕ-amalgamation base (in $\mathfrak{K}$)* if $M \in \mathfrak{K}$ and every ϕ-span (M, M_1, M_2) has a ϕ-amalgam.
(5) We say that $\mathfrak{K}$ has the *ϕ-amalgamation property* if every $M \in \mathfrak{K}$ is a ϕ-amalgamation base.
(6) We say that $N \in \mathfrak{K}$ is *ϕ-universal* if whenever $M \in \mathfrak{K}$ and $f : \phi[M] \cong \phi[N]$, there exists $g : M \to N$ which extends f.

(7) We say that $\mathfrak{K}$ has *ϕ-uniqueness* if whenever $M_0, M_1, M_2 \in \mathfrak{K}$ are such that $M_0 <_{\mathfrak{K}} M_\ell$, $\ell = 1, 2$, there is $f : \phi(M_1) \cong_{\phi(M_0)} \phi(M_2)$.
(8) We say that $\mathfrak{K}$ is *ϕ-categorical* if whenever $M, N \in \mathfrak{K}$, we have that $\phi(M) \cong \phi(N)$.

Remark 5.7.

(1) Setting $\phi(x)$ to be $x \neq x$, we get back the usual definitions of the amalgamation property and of a universal model.
(2) Setting $\phi(x)$ to be $x = x$, we get back the usual definition of categoricity.

Theorem 5.8. Let $\mathfrak{K}$ and $\mathfrak{K}^*$ be abstract classes such that $|\mathfrak{K}| = |\mathfrak{K}^*|$ and $M \leq_{\mathfrak{K}} N$ implies $M \leq_{\mathfrak{K}^*} N$. Let ϕ be a quantifier-free $\mathbb{L}_{\omega,\omega}(\tau(\mathfrak{K}))$-formula, and let $\Theta = [\lambda_1, \lambda_2]$ be an interval of cardinals. Assume $2^{\lambda_1} = 2^{<\lambda_2} < 2^{\lambda_2}$. If:

(1) $\mathfrak{K}$ is $[\lambda_1, \lambda_2)$-continuous.
(2) There is a ϕ-universal model in $\mathfrak{K}^*_{\lambda_2}$.
(3) $\mathfrak{K}_{\lambda_2}$ is ϕ-categorical.
(4) For any $\lambda \in [\lambda_1, \lambda_2)$:
    (a) $\mathfrak{K}_\lambda$ has ϕ-uniqueness.
    (b) For any ϕ-span $\bar{M} = (M_0, M_1, M_2)$ in $\mathfrak{K}_\lambda$, if $\bar{M}$ has a ϕ-amalgam in $\mathfrak{K}^*$, then $\bar{M}$ has a ϕ-amalgam in $\mathfrak{K}_\lambda$.

Then for any $M \in \mathfrak{K}_{[\lambda_1, \lambda_2)}$, there exists $N \in \mathfrak{K}_{[\lambda_1, \lambda_2)}$ such that $M \leq_{\mathfrak{K}} N$ and N is a ϕ-amalgamation base in $\mathfrak{K}_{\|N\|}$.

Proof. Suppose not. We build an increasing continuous tree $\langle M_\eta : \eta \in {}^{\leq \lambda_2}2 \rangle$ such that for all $\eta \in {}^{<\lambda_2}2$:

(1) $M_{\langle\rangle} = M$.
(2) $M_\eta \in \mathfrak{K}_{\lambda_1 + \ell(\eta)}$.
(3) $|M_\eta| \subseteq \lambda_2$.
(4) $\phi(M_\eta) = \phi(M_\nu)$ for all $\nu \in {}^{\ell(\eta)}2$.
(5) In $\mathfrak{K}_{\lambda_1 + \ell(\eta)}$, $(M_\eta, M_{\eta^\frown 0}, M_{\eta^\frown 1})$ is a ϕ-span that has no ϕ-amalgam.

This is possible: use the assumption that the conclusion fails to take care of requirement (5), and use ϕ-uniqueness and some renaming to take care of requirements (3) and (4). This is enough: it is easy to check that we must have that $M_\eta <_{\mathfrak{K}} M_{\eta^\frown \ell}$ for all $\ell \in 2$ and $\eta \in {}^{<\lambda_2}2$. Thus $M_\eta \in \mathfrak{K}_{\lambda_2}$ for each $\eta \in {}^{\lambda_2}2$.

Let A be the $\tau(\mathfrak{K})$-structure induced by $\phi(M_\eta)$ for some (or any, see requirement (4)) $\eta \in {}^{\lambda_2}2$. Let $\mathfrak{C} \in \mathfrak{K}^*_{\lambda_2}$ be ϕ-universal in $\mathfrak{K}^*_{\lambda_2}$. By ϕ-categoricity, pick $h : A \cong \phi(N)$. For each $\eta \in {}^{\lambda_2}2$, pick $g_\eta : M_\eta \to \mathfrak{C}$ a $\mathfrak{K}^*$-embedding extending h. We have that $|M_\eta| \subseteq \lambda_2$, so extend each g_η arbitrarily to $f_\eta \in {}^{\lambda_2}\lambda_2$.

By Fact 5.5, there exists $\eta \in {}^{\lambda_2}2$ such that the set S_η described there is stationary. Let $C_\eta := \{\delta < \lambda_2 \mid |M_{\eta \upharpoonright \delta}| \subseteq \delta\}$. This is a club (by Lemma 5.1 applied to $\langle M_{\eta \upharpoonright \delta} : \delta < \lambda_2 \rangle$, $\langle M_{\eta \upharpoonright \delta} \cap \delta : \delta < \lambda_2 \rangle$), so let $\delta \in S_\eta \cap C_\eta$. We then must have that there is $\nu \in {}^{\lambda_2}2$ such that $\eta_0 := \eta \upharpoonright \delta = \nu \upharpoonright \delta$, $\eta \upharpoonright (\delta + 1) \neq \nu \upharpoonright (\delta + 1)$, and (in particular), g_η and g_ν agree on M_{η_0}. Moreover, g_η and g_ν also extend h, so must in particular agree on $\phi(M_{\eta \upharpoonright (\delta+1)})$. This means that $(N, g_\eta \upharpoonright M_{\eta \upharpoonright (\delta+1)}, g_\nu \upharpoonright M_{\nu \upharpoonright (\delta+1)})$ is a ϕ-amalgam of $(M_\eta, M_{\eta \upharpoonright (\delta+1)}, M_{\nu \upharpoonright (\delta+1)})$ in $\mathfrak{K}^*$. By assumption (4b), this means there is such a ϕ-amalgam in $\mathfrak{K}_{\lambda_1 + \delta}$, a contradiction to requirement (5). $\square$

Remark 5.9. Setting $\mathfrak{K} = \mathfrak{K}^*$, $\lambda_1 = \mathrm{LS}(\mathfrak{K}) = \lambda$, $\lambda_2 = \mathrm{LS}(\mathfrak{K})^+$, $\phi(x) := x \neq x$, we get back the following result of the first author [She09a, I.3.8]: if an AEC $\mathfrak{K}$ with $\lambda := \mathrm{LS}(\mathfrak{K})$ is categorical in λ, has a universal model in λ^+, and $2^\lambda < 2^{\lambda^+}$, then $\mathfrak{K}$ has amalgamation in λ.

6. Good frames

Good frames are a local notion of "bare bone" superstability, introduced by the first author [She09a, Chapter II]. Essentially, an AEC $\mathfrak{K}$ has a *good λ-frame* if it looks superstable in λ in the sense that $\mathfrak{K}_\lambda$ has some reasonable structural properties (like amalgamation), and there is a forking-like notion for types of singletons over models.

In this section, we give the definition we will use and state some fundamental results (most of them known or folklore) about good frames.

The definition we give here allows good frames in several cardinals (as in [Vas16b, 2.21]) but using fragmented AECs (so also allowing "shrinking frames", [Vas17b, Appendix A]). The frames in this paper will always be type-full (i.e. all nonalgebraic types will be basic), so we will drop the adjective and ignore basic types in the definition.

Definition 6.1. A *(type-full) good frame* is a pair $\mathfrak{s} = (\mathfrak{K}, F)$, where:

(1) $\mathfrak{K}$ is a fragmented AEC such that:
   (a) $\mathfrak{K} \neq \emptyset$.
   (b) $\mathfrak{K}_{<\mathrm{LS}(\mathfrak{K})} = \emptyset$.
   (c) $\mathfrak{K}$ has amalgamation, joint embedding, and no maximal models.
   (d) For every $M \in \mathfrak{K}$, there is $N \in \mathfrak{K}$ such that N is universal over M and $\|N\| = \|M\|$.
(2) F is a binary relation taking as input pairs (p, M), where p is an orbital type and M is a model in $\mathfrak{K}$. We write *p does not fork over M* (or *p does not $\mathfrak{s}$-fork over M*) instead of $F(p, M)$ and require that F satisfies the following:
   (a) If p does not fork over M, then $p \in \mathscr{S}(N)$ for some $N \geq_{\mathfrak{K}} M$.
   (b) Invariance: if $f : N \cong N'$ and $p \in \mathscr{S}(N)$ does not fork over M, then $f(p)$ does not fork over $f[M]$.
   (c) Monotonicity: if $p \in \mathscr{S}(N)$ does not fork over M and $M \leq_{\mathfrak{K}} M' \leq_{\mathfrak{K}} N$, then $p \upharpoonright M'$ does not fork over M and p does not fork over M'.
   (d) Disjointness: if $p \in \mathscr{S}(N)$ does not fork over M, then $p \upharpoonright M$ is algebraic if and only if p is algebraic.
   (e) Extension: for any $M \leq_{\mathfrak{K}} N$ and any $p \in \mathscr{S}(M)$ there exists $q \in \mathscr{S}(N)$ such that q extends p and q does not fork over M.
   (f) Uniqueness: for any $M \leq_{\mathfrak{K}} N$ and any $p, q \in \mathscr{S}(N)$, if both p and q do not fork over M and $p \upharpoonright M = q \upharpoonright M$, then $p = q$.
   (g) Local character: If $\langle M_i : i \leq \delta \rangle$ is increasing continuous in $\mathfrak{K}$ and $p \in \mathscr{S}(M_\delta)$, then there exists $i < \delta$ such that p does not fork over M_i.
   (h) Symmetry: If $p = \mathbf{tp}(a/N; N')$ does not fork over M and $b \in |N|$, then there exists $M', N'' \in \mathfrak{K}$ such that $N' \leq_{\mathfrak{K}} N''$, $M \leq_{\mathfrak{K}} M'$, $a \in M'$, and $\mathbf{tp}(b/M'; N'')$ does not fork over M.

We write $\mathbf{K}_{\mathfrak{s}}$ for the class of the frame $\mathfrak{s}$. We say that $\mathfrak{s}$ is *on* $\mathbf{K}^*$ if $\mathbf{K}_{\mathfrak{s}} = \mathbf{K}^*$. The *domain* of a good frame $\mathfrak{s}$ is the domain of $\mathbf{K}_{\mathfrak{s}}$ (see Definition 3.6). For $\lambda \in \text{dom}(\mathfrak{s})$, we say that $\mathfrak{s}$ is *categorical in* λ if $\mathbf{K}_{\mathfrak{s}}$ is categorical in λ. We say that $\mathfrak{s}$ is *categorical* if it is categorical in *all* $\lambda \in \text{dom}(\mathfrak{s})$. We also define restrictions to smaller classes of models such as $\mathfrak{s}_\lambda$ in the natural way. We say that $\mathfrak{s}$ is a *good λ-frame* if $\mathfrak{s} = \mathfrak{s}_\lambda$.

We will use the following construction of a good frame:

Fact 6.2. Let $\mathbf{K}$ be an AEC with arbitrarily large models. Let $\mu > \text{LS}(\mathbf{K})$ and let $\theta \geq \mu$. Assume that $\mathbf{K}_{<\theta}$ has amalgamation and no maximal models. Assume further that $\mathbf{K}$ is categorical in μ and $\mathbf{K}^{\text{sat}}_{(\text{LS}(\mathbf{K}),\theta)}$ is $\text{LS}(\mathbf{K})$-local. Then there is a (categorical) good frame on $\mathbf{K}^{\text{sat}}_{(\text{LS}(\mathbf{K}),\theta)}$.

Proof. By [Vas17c, 5.7(1)], $\mathbf{K}$ is $\text{LS}(\mathbf{K})$-superstable and has $\text{LS}(\mathbf{K})$-symmetry. The result now follows from the proof of [Vas17b, A.3]. $\square$

Remark 6.3. Combining local character, transitivity, and uniqueness, we have that whenever $\mathfrak{s}$ is a good frame, then $\mathbf{K}_{\mathfrak{s}}$ is $\min(\text{dom}(\mathfrak{s}))$-local.

Regarding disjointness, it follows from the other properties if the frame is categorical:

Lemma 6.4. If $\mathfrak{s}$ satisfies all the properties of good frames, except perhaps disjointness, and $\mathfrak{s}$ is categorical, then $\mathfrak{s}$ satisfies disjointness as well.

Proof. By the conjugation property [She09a, III.1.21] (whose proof never uses disjointness). $\square$

We now state and prove canonicity of the framework. First, frames with the same restriction in their low cardinals are the same:

Lemma 6.5. Let $\mathfrak{s}$ and $\mathfrak{t}$ be good frames with $\mathbf{K}_{\mathfrak{s}} = \mathbf{K}_{\mathfrak{t}}$. Let $\lambda := \min(\text{dom}(\mathfrak{s}))$. If $\mathfrak{s}_\lambda = \mathfrak{t}_\lambda$, then $\mathfrak{s} = \mathfrak{t}$.

Proof. Let $\mathbf{K} := \mathbf{K}_{\mathfrak{s}} = \mathbf{K}_{\mathfrak{t}}$. Let $M \leq_{\mathbf{K}} N$ be in $\mathbf{K}$ and let $p \in \mathscr{S}(N)$. Assume that p does not $\mathfrak{s}$-fork over M. We show that p does not $\mathfrak{t}$-fork over M, and the converse is symmetric. First, by local character and transitivity there exists $M_0 \in \mathbf{K}_\lambda$ such that p does not $\mathfrak{s}$-fork over M_0. In particular (by monotonicity), for every $N_0 \in \mathbf{K}_\lambda$ with $M_0 \leq_{\mathbf{K}} N_0 \leq_{\mathbf{K}} N$, $p \upharpoonright N_0$ does not $\mathfrak{s}$-fork over M_0. Since $\mathfrak{s}_\lambda = \mathfrak{t}_\lambda$, $p \upharpoonright N_0$ does not $\mathfrak{t}$-fork over M_0. Now pick $N_0' \in \mathbf{K}_\lambda$ such that p does not $\mathfrak{t}$-fork over N_0' and (by monotonicity), enlarge it so that $M_0 \leq_{\mathbf{K}} N_0'$. Then by transitivity, p does not $\mathfrak{t}$-fork over M_0, hence (by monotonicity) over M, as desired. $\square$

Remark 6.6. Disjointness and symmetry are not used in the proof. Regarding local character, we only use that any type does not fork over a model of size λ.

Fact 6.7 (Canonicity of categorical good frames)**.** Let $\mathfrak{s}$ and $\mathfrak{t}$ be categorical good frames with $\text{dom}(\mathfrak{s}) = \text{dom}(\mathfrak{t})$. Let $\lambda := \min(\text{dom}(\mathfrak{s}))$. If $(\mathbf{K}_{\mathfrak{s}})_\lambda = (\mathbf{K}_{\mathfrak{t}})_\lambda$, then $\mathfrak{s} = \mathfrak{t}$.

Proof. By [Vas16a, 9.7], $\mathfrak{s}_\lambda = \mathfrak{t}_\lambda$. By canonicity of categorical fragmented AECs (Theorem 3.22), $\mathbf{K}_{\mathfrak{s}} = \mathbf{K}_{\mathfrak{t}}$. By Lemma 6.5, $\mathfrak{s} = \mathfrak{t}$. $\square$

Categoricity of a good frame may seem a strong assumption. However, we have:

Fact 6.8. Let $\mathfrak{s}$ be a good frame and let $\lambda := \min(\text{dom}(\mathfrak{s}))$. If $\mathfrak{s}$ is categorical in λ, then there is a categorical good frame $\mathfrak{t}$ with $(\mathfrak{K}_{\mathfrak{t}})_\lambda = (\mathfrak{K}_{\mathfrak{s}})_\lambda$.

Proof. By the proof of [Vas17b, A.2]. □

We recall the following result about categoricity transfers in good frames:

Fact 6.9. Let $\mathfrak{s}$ be a good frame with fragmented AEC $\mathfrak{K}$. Let $\Theta := \text{dom}(\mathfrak{K})$, let $\lambda := \min(\Theta)$, and let $\mu \in \Theta$. Assume that $\mathfrak{K}$ is categorical in λ.

Let $\mathfrak{K}^*$ be the AEC generated by $\mathfrak{K}_\mu$. If $\mathfrak{K}^*$ is categorical in μ^+, then:

(1) $\mathfrak{K}$ is θ-continuous for every $\theta \in \Theta$.
(2) $\mathfrak{K}$ is categorical in every $\mu' \in \Theta$.

Proof. By Fact 6.8, we might as well assume that $\mathfrak{s}$ is categorical. We are then in the setup of [Vas17b, Theorem A.9], which gives the result we want. □

We will also use the following upward frame transfer:

Definition 6.10 ([She09a, II.2.4,II.2.5])**.** Let $\mathfrak{s}$ be a good λ-frame and let $\mathfrak{K}$ be the AEC generated by $\mathfrak{K}_{\mathfrak{s}}$. We let $\mathfrak{s}^{up} := (\mathfrak{K}, F)$, where F is the following binary relation on pairs (p, M), with p is an orbital type and M is a model in $\mathfrak{K}$: $F(p, M)$ if and only if $p \in \mathscr{S}(N)$ for some $M \leq_{\mathfrak{K}} N$, and there exists $M_0 \in \mathfrak{K}_{\mathfrak{s}}$ such that $M_0 \leq_{\mathfrak{K}} M$ and for all $N_0 \in \mathfrak{K}_{\mathfrak{s}}$ with $M_0 \leq_{\mathfrak{K}} N_0 \leq_{\mathfrak{K}} N$, $p \upharpoonright N_0$ does not $\mathfrak{s}$-fork over M_0.

Fact 6.11. Let $\mathfrak{s}$ be a good λ-frame.

(1) [She09a, II.2.11] $\mathfrak{s}^{up}$ satisfies all the axioms from the definition of a type-full good frame, except for (1c),(1d), (2e), (2f), and (2h) in Definition 6.1.
(2) [BV17, 6.9] If $\mathfrak{K}_{\mathfrak{s}^{up}}$ is λ-local and has amalgamation, then $\mathfrak{s}^{up}$ is a good frame.

7. Two-dimensional independence notions

While good frames describe a forking-like relation for types of *singletons* over models, two-dimensional independence notions describe a forking-like relations for types of models over models. At that point, it seems more convenient to think of such a relation as a 4-ary relation on squares of models. Squares in nonforking amalgamation form the simplest nontrivial example of an independent system (a concept defined in the next section).

Here, we state the important properties of two-dimensional independence relations. We look also at their canonicity (Theorem 7.14). Two key questions are when a two-dimensional independence relation can be built from a good frame, and when a two-dimensional independence relation for models of size λ implies the existence of a good λ^+-frame (these themes are present already in [She09a, II, III]). Frames that are well-behaved in these respects are called *extendible* (Definition 7.15). A frame that can be extended ω-many steps is called $(< \omega)$-*extendible.* These correspond (but are slightly more convenient to work with than) the ω-successful good frames in [She09a, III]. We give conditions under which such frames exist, both in the compact (Fact 7.21) and non-compact (Fact 7.16) cases. For the latter result, the weak diamond (a weakening of the generalized continuum hypothesis) is assumed.

Definition 7.1. Let $\mathfrak{K}$ be an abstract class. A *two-dimensional independence relation (or notion) on* $\mathfrak{K}$ is a 4-ary relation $\downarrow$ on $\mathfrak{K}$ satisfying:

(1) $\downarrow(M_0, M_1, M_2, M_3)$ implies $M_0 \leq_{\mathfrak{K}} M_\ell \leq_{\mathfrak{K}} M_3$ for $\ell = 1, 2$. We write $M_1 \overset{M_3}{\underset{M_0}{\downarrow}} M_2$ instead of $\downarrow(M_0, M_1, M_2, M_3)$.
(2) If $M_0 \leq_{\mathfrak{K}} M_\ell \leq_{\mathfrak{K}} M_3$, $\ell = 1, 2$ and $f : M_3 \to M_3'$ is a $\mathfrak{K}$-embedding, then $M_1 \overset{M_3}{\underset{M_0}{\downarrow}} M_2$ if and only if $f[M_1] \overset{M_3'}{\underset{f[M_0]}{\downarrow}} f[M_2]$.
(3) Monotonicity: if $M_1 \overset{M_3}{\underset{M_0}{\downarrow}} M_2$ and $M_0 \leq_{\mathfrak{K}} M_1' \leq_{\mathfrak{K}} M_1$, then $M_1' \overset{M_3}{\underset{M_0}{\downarrow}} M_2$.
(4) Disjointness: if $M_1 \overset{M_3}{\underset{M_0}{\downarrow}} M_2$, then $M_1 \cap M_2 = M_0$.
(5) Symmetry: if $M_1 \overset{M_3}{\underset{M_0}{\downarrow}} M_2$, then $M_2 \overset{M_3}{\underset{M_0}{\downarrow}} M_1$.
(6) Transitivity: if $M_1 \overset{M_3}{\underset{M_0}{\downarrow}} M_2$ and $M_3 \overset{M_5}{\underset{M_2}{\downarrow}} M_4$, then $M_1 \overset{M_5}{\underset{M_0}{\downarrow}} M_4$.
(7) Extension: whenever $M_0 \leq_{\mathfrak{K}} M_\ell$, $\ell = 1, 2$, there exists $M_3 \in \mathfrak{K}$ and $f_\ell : M_\ell \xrightarrow[M_0]{} M_3$ such that $f_1[M_1] \overset{M_3}{\underset{M_0}{\downarrow}} f_2[M_2]$.
(8) Uniqueness: whenever $M_1^\ell \overset{M_3^\ell}{\underset{M_0^\ell}{\downarrow}} M_2^\ell$ for $\ell = 0, 1$ and $f_k : M_k^0 \cong M_k^1$, $k < 3$ are such that $f_0 \subseteq f_1$, $f_0 \subseteq f_2$, then there exists $M_3^2 \in \mathfrak{K}$ with $M_3^1 \leq_{\mathfrak{K}} M_3^2$ and and $f_3 : M_3^0 \to M_3^2$ such that $f_k \subseteq f_3$ for all $k < 3$.

Definition 7.2. Let $\downarrow$ be a two-dimensional independence notion on $\mathfrak{K}$. If $\mathfrak{s}$ is a good frame on $\mathfrak{K}$, we say that $\downarrow$ *respects* $\mathfrak{s}$ if whenever $M_1 \overset{M_3}{\underset{M_0}{\downarrow}} M_2$ and $a \in |M_1|$, we have that $\mathbf{tp}(a/M_2; M_3)$ does not $\mathfrak{s}$-fork over M_0.

As in Lemma 6.4, note that disjointness is not really needed in some cases:

Lemma 7.3. Assume that $\downarrow$ satisfies all the properties of a two-dimensional independence notion on $\mathfrak{K}$, except perhaps for disjointness. Assume that $\downarrow$ respects a good frame $\mathfrak{s}$ on $\mathfrak{K}$. Then $\downarrow$ satisfies disjointness.

Proof. Because $\mathfrak{s}$ satisfies disjointness. □

It is sometimes useful to extend $\downarrow$ to take sets on the left and right hand sides. This is the content of the next definition, versions of which were already considered by both the first author [She09a, II.6.35] and the second author (in joint work with Boney, Grossberg, and Kolesnikov) [BGKV16]. At this level of generality, this is studied in recent work of Lieberman, Rosický, and the second author [LRV, 8.2].

Definition 7.4. Let $\downarrow$ be a two-dimensional independence relation on $\mathfrak{K}$.

(1) Define a 4-ary relation $\overline{\perp}$ as follows: $A \overset{N}{\underset{M}{\overline{\perp}}} B$ holds if and only if $M \leq_{\mathfrak{K}} N$, $A, B \subseteq |N|$, and there exists $M_1, M_2, M_3 \in \mathfrak{K}$ such that $N \leq_{\mathfrak{K}} M_3$, $A \subseteq |M_1|$, $B \subseteq |M_2|$, and $M_1 \overset{M_3}{\underset{M_0}{\perp}} M_2$.

(2) For p an orbital type, we say that p *does not fork over* M if $M \in \mathfrak{K}$, and there exists b, A, M_3 such that $p = \mathbf{tp}(b/A; M_3)$ and $b \overset{M_3}{\underset{M}{\overline{\perp}}} A$.

(3) Define a binary relation $F = F(\perp)$ taking as input pairs (p, M), where p is an orbital type and M is a model in $\mathfrak{K}$ as follows: pFM if $p \in \mathscr{S}(N)$ for some $N \geq_{\mathfrak{K}} M$ and p does not fork over M. We let $\mathfrak{s}(\perp) := (\mathfrak{K}, F(\perp))$.

Properties of $\perp$ generalize to $\overline{\perp}$ as follows:

Fact 7.5 ([LRV, 8.4,8.5])**.** Let $\perp$ be a two-dimensional independence relation on $\mathfrak{K}$.

(1) Let $M_0 \leq_{\mathfrak{K}} M_\ell \leq_{\mathfrak{K}} M_3$ for $\ell = 1, 2$. Then $M_1 \overset{M_3}{\underset{M_0}{\perp}} M_2$ if and only if $M_1 \overset{M_3}{\underset{M_0}{\overline{\perp}}} M_2$.

(2) (Preservation under $\mathfrak{K}$-embeddings) Given $M_0 \leq_{\mathfrak{K}} M_3$, $A, B \subseteq UM_3$, and $f : M_3 \to N_3$, we have that $A \overset{M_3}{\underset{M_0}{\overline{\perp}}} B$ if and only if $f[A] \overset{N_3}{\underset{f[M_0]}{\overline{\perp}}} f[B]$.

(3) (Monotonicity) If $A \overset{M_3}{\underset{M_0}{\overline{\perp}}} B$ and $A_0 \subseteq A$, $B_0 \subseteq B$, then $A_0 \overset{M_3}{\underset{M_0}{\overline{\perp}}} B_0$.

(4) (Normality) $A \overset{M_3}{\underset{M_0}{\overline{\perp}}} B$ if and only if $AM_0 \overset{M_3}{\underset{M_0}{\overline{\perp}}} BM_0$.

(5) (Base monotonicity) If $A \overset{M_3}{\underset{M_0}{\overline{\perp}}} B$, $M_0 \leq_{\mathfrak{K}} M_2 \leq_{\mathfrak{K}} M_3$, and $UM_2 \subseteq B$, then $A \overset{M_3}{\underset{M_2}{\overline{\perp}}} B$.

(6) (Extension) Whenever $M \leq_{\mathfrak{K}} N$ and $p \in \mathscr{S}^{<\infty}(M)$, there exists $q \in \mathscr{S}^{<\infty}(N)$ extending p such that q does not fork over M.

(7) (Symmetry) $A \overset{N}{\underset{M}{\overline{\perp}}} B$ holds if and only if $B \overset{N}{\underset{M}{\overline{\perp}}} A$ holds.

(8) (Uniqueness) Given $p, q \in \mathscr{S}^{<\infty}(B; N)$ with $M \leq_{\mathfrak{K}} N$ and $UM \subseteq B \subseteq UN$, if $p \restriction M = q \restriction M$ and p, q do not fork over M, then $p = q$.

(9) (Transitivity) If $M_0 \leq_{\mathfrak{K}} M_2 \leq_{\mathfrak{K}} M_3$, $A \overset{M_3}{\underset{M_0}{\overline{\perp}}} M_2$ and $A \overset{M_3}{\underset{M_2}{\overline{\perp}}} B$, then $A \overset{M_3}{\underset{M_0}{\overline{\perp}}} B$.

In a general two-dimensional independence relation $\perp$, $\mathfrak{s}(\perp)$ may not induce a good frame (because e.g. such a relation also exists in strictly stable first-order theories). We call the ones that do (and satisfy a few more convenient properties) *good*:

Definition 7.6. A two-dimensional independence notion $\perp$ on $\mathfrak{K}$ is *good* if it satisfies the following properties:

(1) $\mathfrak{s}(\perp)$ is a good frame on $\mathfrak{K}$ (in particular, $\mathfrak{K}$ is a fragmented AEC).

(2) Long transitivity: if δ is a limit ordinal, $\langle M_i : i \le \delta \rangle$, $\langle N_i : i \le \delta \rangle$ are increasing continuous in $\mathfrak{K}$ and $N_i \overset{N_{i+1}}{\underset{M_i}{\mathop{\smile\!\!\!\!\!\!|}}} M_{i+1}$ for all $i < \delta$, then $N_0 \overset{N_\delta}{\underset{M_0}{\mathop{\smile\!\!\!\!\!\!|}}} M_\delta$.
(3) Local character: if $M \le_{\mathfrak{K}} N$ and $A \subseteq |N|$, there exists $M_0 \le_{\mathfrak{K}} N_0$ such that $N_0 \overset{N}{\underset{M_0}{\mathop{\smile\!\!\!\!\!\!|}}} M$, $A \subseteq |N_0|$, and $\|N_0\| \le |A| + \mathrm{LS}(\mathfrak{K})$.

Note that the local character property of two-dimensional independence notions is vacuous in case the relation is on $\mathfrak{K}_\lambda$. In this case, the following replacement is useful (this is related to the definition of successful good$^+$ in [She09a, III], see [BV, 2.14]; the "reflects down" terminology appears in [Vas17b, 3.7(2)]):

Definition 7.7. A good two-dimensional independence notion $\mathop{\smile\!\!\!\!\!\!|}$ on $\mathfrak{K}$ *reflects down* if for any $\lambda \in \mathrm{dom}(\mathfrak{K})$ and any two increasing continuous chains $\langle M_i : i < \lambda^+ \rangle$, $\langle N_i : i < \lambda^+ \rangle$ in $\mathfrak{K}_\lambda$, there is a club $C \subseteq \lambda^+$ such that for any $i < j$ in C, $N_i \overset{N_j}{\underset{M_i}{\mathop{\smile\!\!\!\!\!\!|}}} M_j$.

Remark 7.8. If $\mathop{\smile\!\!\!\!\!\!|}$ is a good two-dimensional independence notion on $\mathfrak{K}$ and $\lambda, \lambda^+ \in \mathrm{dom}(\mathfrak{K})$, then by local character $\mathop{\smile\!\!\!\!\!\!|} \restriction \mathfrak{K}_\lambda$ reflects down.

We will use the following very useful fact about good two-dimensional independence notions: a union of independent brimmed squares is brimmed, in the following sense:

Fact 7.9 ([She09a, II.6.29])**.** Let $\mathop{\smile\!\!\!\!\!\!|}$ be a good two-dimensional independence notion on $\mathfrak{K}$. Let $\langle M_i : i \le \delta \rangle$, $\langle N_i : i \le \delta \rangle$ be increasing continuous in $\mathfrak{K}$ such that $N_i \overset{N_{i+1}}{\underset{M_i}{\mathop{\smile\!\!\!\!\!\!|}}} M_{i+1}$ for all $i < \delta$. If N_{i+1} is brimmed over $M_i \cup N_i$ (see Definition 3.5(4)) for all $i < \delta$, then N_δ is brimmed over $M_\delta \cup N_0$.

Regarding brimmed models, we also have that one can resolve them in a nice way:

Fact 7.10 ([She09a, III.1.17])**.** Let $\mathop{\smile\!\!\!\!\!\!|}$ be a good two-dimensional independence notion on $\mathfrak{K}$. Let $\lambda \in \mathrm{dom}(\mathfrak{K})$ be such that $\lambda^+ \in \mathrm{dom}(\mathfrak{K})$. Let $M, N \in \mathfrak{K}_{\lambda^+}$ be brimmed such that N is brimmed over M. Then there exists $\langle M_i : i < \lambda^+ \rangle, \langle N_i : i < \lambda^+ \rangle$ increasing continuous resolutions of M and N in $\mathfrak{K}_\lambda$ such that M_i is brimmed and N_i is brimmed over M_i for all $i < \lambda^+$.

Being good and reflecting down is useful, but one sometimes want more than the long transitivity property. This is the content of the next definition:

Definition 7.11. A two-dimensional independence notion $\mathop{\smile\!\!\!\!\!\!|}$ on $\mathfrak{K}$ is *very good* if it is good, reflects down, and satisfies in addition *strong continuity*: whenever $\langle M_\ell^i : i \le \delta \rangle$ are increasing continuous in $\mathfrak{K}$, $\ell < 4$, and $M_1^i \overset{M_3^i}{\underset{M_0^i}{\mathop{\smile\!\!\!\!\!\!|}}} M_2^i$ for all $i < \delta$, then $M_1^\delta \overset{M_3^\delta}{\underset{M_0^\delta}{\mathop{\smile\!\!\!\!\!\!|}}} M_2^\delta$.

We now proceed to show that very good categorical two-dimensional independence notions are canonical. This is essentially [BGKV16], but since the setup here is not as global as there, we use a slightly different road.

Lemma 7.12. If $\overset{1}{\mathop{\smile}\limits^{|}}$ and $\overset{2}{\mathop{\smile}\limits^{|}}$ are very good two-dimensional independence notions on $\mathfrak{K}$ and $\overset{1}{\mathop{\smile}\limits^{|}} \upharpoonright \mathfrak{K}_\lambda = \overset{2}{\mathop{\smile}\limits^{|}} \upharpoonright \mathfrak{K}_\lambda$ for all $\lambda \in \operatorname{dom}(\mathfrak{K})$, then $\overset{1}{\mathop{\smile}\limits^{|}} = \overset{2}{\mathop{\smile}\limits^{|}}$.

Proof. Assume that $\overset{1}{\mathop{\smile}\limits^{|}}(M_0, M_1, M_2, M_3)$. We show that $\overset{2}{\mathop{\smile}\limits^{|}}(M_0, M_1, M_2, M_3)$, and the converse will be symmetric. We proceed by induction on $\|M_3\|$. If $\|M_0\| = \|M_3\|$, then by assumption $\overset{2}{\mathop{\smile}\limits^{|}}(M_0, M_1, M_2, M_3)$. Assume now that $\|M_0\| < \|M_3\|$. Let $\delta := \|M_3\|$. For $\ell = 1, 2, 3$, build $\langle M_\ell^i : i \leq \delta \rangle$ increasing continuous such that for all $i \leq \delta$:

(1) $\|M_\ell^i\| = \|M_0\| + |i|$ for $\ell = 1, 2, 3$.
(2) $nf^1(M_0, M_1^i, M_2^i, M_3^i)$.
(3) $M_\ell^\delta = M_\ell$ for $\ell = 1, 2, 3$.

This is possible using monotonicity and the fact that $\mathfrak{K}$ is a fragmented AEC. This is enough: by the induction hypothesis, $\overset{1}{\mathop{\smile}\limits^{|}}(M_0, M_1^i, M_2^i, M_3^i)$ for all $i < \delta$. By strong continuity, $\overset{2}{\mathop{\smile}\limits^{|}}(M_0, M_1, M_2, M_3)$, as desired. $\square$

Fact 7.13. Let $\mathfrak{K}$ be fragmented AEC and let $\lambda \geq \operatorname{LS}(\mathfrak{K})$. Let $\overset{1}{\mathop{\smile}\limits^{|}}$ and $\overset{2}{\mathop{\smile}\limits^{|}}$ be two good two-dimensional independence notions on $\mathfrak{K}_\lambda$ which reflect down. If $\mathfrak{K}$ is categorical in λ, then $\overset{1}{\mathop{\smile}\limits^{|}} = \overset{2}{\mathop{\smile}\limits^{|}}$.

Proof. Let $\mathfrak{s}^\ell := \mathfrak{s}(\overset{\ell}{\mathop{\smile}\limits^{|}})$. By canonicity of categorical good frames (Fact 6.7), $\mathfrak{s}^1 = \mathfrak{s}^2$, so write $\mathfrak{s} := \mathfrak{s}^1$. By [Vas17b, 3.11], $\mathfrak{s}$ has the existence property for uniqueness triples (see [She09a, II.5.3(3)]). By [She09a, II.6.3(3)], $\overset{1}{\mathop{\smile}\limits^{|}} = \overset{2}{\mathop{\smile}\limits^{|}}$. $\square$

Theorem 7.14 (Canonicity)**.** If $\overset{1}{\mathop{\smile}\limits^{|}}$ and $\overset{2}{\mathop{\smile}\limits^{|}}$ are two very good two-dimensional independence notions on $\mathfrak{K}$ and $\mathfrak{K}$ is categorical in every $\lambda \in \operatorname{dom}(\mathfrak{K})$, then $\overset{1}{\mathop{\smile}\limits^{|}} = \overset{2}{\mathop{\smile}\limits^{|}}$.

Proof. By Fact 7.13, $\overset{1}{\mathop{\smile}\limits^{|}} \upharpoonright \mathfrak{K}_\lambda = \overset{2}{\mathop{\smile}\limits^{|}} \upharpoonright \mathfrak{K}_\lambda$ for every $\lambda \in \operatorname{dom}(\mathfrak{K})$. Now apply Lemma 7.12. $\square$

The next topic is when (very) good two-dimensional independence notions exist. It is natural to construct them from good frames, since we already have sufficient conditions for the existence of good frames (Fact 6.2). A good frame that can be extended to a good two-dimensional independence relation which reflects down will be called *extendible* (this is essentially equivalent to "successful good$^+$" in [She09a, III.1], but has a shorter definition):

Definition 7.15. We say a good frame $\mathfrak{s}$ is *extendible* if there is a good two-dimensional independence notion $\mathop{\smile}\limits^{|}$ on $\mathfrak{K}_{\mathfrak{s}}$ which reflects down. We say that $\mathfrak{s}$ is *very good* if in addition $\mathop{\smile}\limits^{|}$ is very good.

We will use the following sufficient condition for a good frame to be extendible:

Fact 7.16. Let $\mathfrak{s}$ be a good frame and let $\lambda \in \text{dom}(\mathfrak{s})$. If $2^\lambda < 2^{\lambda^+}$, $\mathfrak{s}$ is categorical in λ, and $\lambda^+ \in \text{dom}(\mathfrak{s})$, then $\mathfrak{s}_\lambda$ is extendible.

Proof. By [Vas17b, E.8] (the main idea of the argument is due to the first author, see [She09a, p. 798]), $\mathfrak{s}_\lambda$ has what is called the existence property for uniqueness triples. By [She09a, II.6.34], there is a two-dimensional independence relation $\perp$ on $\mathfrak{K}_\lambda$ which satisfies long transitivity and respects $\mathfrak{s}_\lambda$. Now, any time a type does not fork $\mathfrak{s}(\perp)$-fork, this implies that it does not $\mathfrak{s}_\lambda$-fork as $\perp$ respects $\mathfrak{s}_\lambda$. But we know that $\mathfrak{s}$ satisfies uniqueness and $\mathfrak{s}(\perp)$ satisfies extension, see Fact 7.5, so by the proof of [BGKV16, 4.1], $\mathfrak{s} = \mathfrak{s}(\perp)$. We have shown that $\perp$ is good (since we only consider it as a relation on $\mathfrak{K}_\lambda$, local character is not relevant).

It remains to see that $\perp$ reflects down. Since $\mathfrak{s}$ is a good frame and $\lambda^+ \in \text{dom}(\mathfrak{s})$, we have that $(\mathfrak{K}_\mathfrak{s})_{[\lambda, \lambda^+]}$ is λ-local (Remark 6.3). Thus we can apply [Jar16, 7.15] to obtain that $\perp$ reflects down, as desired. □

Note also that being extendible implies that the frame itself can be extended (this is the main idea of the end of Chapter II in [She09a]):

Fact 7.17. If $\mathfrak{s}$ is an extendible categorical good λ-frame, then there exists a (unique) categorical good frame $\mathfrak{t}$ such that $\text{dom}(\mathfrak{t}) = \{\lambda, \lambda^+\}$ and $\mathfrak{t}_\lambda = \mathfrak{s}_\lambda$.

Proof. Let $\mathfrak{K}$ be the AEC generated by $\mathfrak{K}_\mathfrak{s}$. By [She09a, III.1.6(2)] (or see [JS13, 10.1.9]), there is a good λ^+-frame $\mathfrak{s}^+$ on $\mathfrak{K}^{\text{sat}}_{\lambda^+}$. Since by [She09a, III.1.10], $\mathfrak{K}^{\text{sat}}_{\lambda^+}$ is λ-local, nonforking in $\mathfrak{s}^+$ must be generated by nonforking in $\mathfrak{s}$ (i.e. $\mathfrak{s}^+ = \mathfrak{s}^{up} \upharpoonright \mathfrak{K}^{\text{sat}}_{\lambda^+}$). Thus letting $\mathfrak{t} := \mathfrak{s}^{up} \upharpoonright (\mathfrak{K}^{\text{sat}}_{\lambda^+} \cup \mathfrak{K}_\lambda)$, we get the result. □

Definition 7.18. For $\mathfrak{s}$ an extendible categorical good λ-frame, we write $\mathfrak{s}^+$ for $\mathfrak{t}_{\lambda^+}$, where $\mathfrak{t}$ is as given by Fact 7.17.

Since we now know how to extend a frame to the next cardinal, it makes sense to define when one can do this successively:

Definition 7.19 ([She09a, III.1.12])**.** Let $\mathfrak{s}$ be a categorical good λ-frame. We define by induction on $n < \omega$ what it means for $\mathfrak{s}$ to be n-extendible as well as a good λ^{+n}-frame $\mathfrak{s}^{+n}$ as follows:

(1) $\mathfrak{s}$ is always 0-extendible and $\mathfrak{s}^{+0} = \mathfrak{s}$.
(2) $\mathfrak{s}$ is $(n+1)$-extendible if it is n-extendible and $\mathfrak{s}^{+n}$ is extendible.
(3) If $\mathfrak{s}$ is $(n+1)$-extendible, let $\mathfrak{s}^{+(n+1)} := (\mathfrak{s}^{+n})^+$.

We say that $\mathfrak{s}$ is $(<\omega)$-*extendible* if $\mathfrak{s}$ is n-extendible for all $n < \omega$.

When a good λ-frame is $(<\omega)$-extendible, then after taking its successor a few times, it becomes very good. Moreover after this is done one can "connect" forking between the cardinals, getting a good $[\lambda, \lambda^{<\omega})$-frame:

Fact 7.20. Let $\mathfrak{s}$ be a categorical good λ-frame.

(1) If $\mathfrak{s}$ is ω-successful and good$^+$ (in the sense of [She09a, III.1]), then $\mathfrak{s}$ is $(<\omega)$-extendible.
(2) If $\mathfrak{s}$ is 4-extendible, then $\mathfrak{s}^{+3}$ is very good.

(3) If $\mathfrak{s}$ is very good and 2-extendible, then $\mathfrak{s}^+$ is very good.
(4) If $\mathfrak{s}$ is very good and $(< \omega)$-extendible, then there exists a (unique) very good categorical frame $\mathfrak{t}$ such that $\text{dom}(\mathfrak{t}) = [\lambda, \lambda^{+\omega})$ and $\mathfrak{t}_{\lambda^{+n}} = \mathfrak{s}^{+n}$ for all $n < \omega$.

Proof.

(1) Essentially follows from [BV, 2.14].
(2) By [She09a, III.8.19] (see the proof of [Vas16a, 12.14]).
(3) Also by [She09a, III.8.19].
(4) Let $\mathfrak{t} := \mathfrak{s}^{up} \upharpoonright (\mathfrak{K}_{\mathfrak{s}} \cup \mathfrak{K}^{\text{sat}}_{[\lambda^+, \lambda^{+\omega})})$. As in the proof of Fact 7.17, we have enough locality for nonforking to connect, so this works (see also [Vas17b, Appendix A]).

$\square$

While in general, it is not known how to build extendible good frames without using the weak diamond (Fact 7.16), in the compact case it is possible:

Fact 7.21. Let $\mathfrak{K}$ be a compact AEC. If $\mathfrak{K}$ is categorical in some $\mu > \text{LS}(\mathfrak{K})$, then there exists a (unique) categorical very good frame on $\mathfrak{K}^{\text{sat}}_{\geq \text{LS}(\mathfrak{K})^{+6}}$.

Proof. Let κ be such that $\mathfrak{K}$ is κ-compact. By Corollary 4.8, $\mathfrak{K}$ has amalgamation, no maximal models, is $(< \kappa)$-short, and is categorical in a proper class of cardinals. By essentially the main result of [Vas16a], and more precisely by [Vas17d, A.16], $\mathfrak{K}^{\lambda\text{-sat}}$ is what is called there fully good. Here, we have set $\lambda := (\text{LS}(\mathfrak{K})^{<\kappa})^{+5}$. Since κ is strongly compact, we have by a result of Solovay (see the proof of [Jec03, 20.8]) that $\text{LS}(\mathfrak{K})^{<\kappa} \leq \text{LS}(\mathfrak{K})^+$ (of course this also holds if $\kappa = \aleph_0$). Thus $\lambda \leq \text{LS}(\mathfrak{K})^{+6}$. Checking the definition of fully good in [Vas16a, 8.4], we see that this implies that $\mathfrak{K}^{\text{sat}}_{\geq \text{LS}(\mathfrak{K})^{+6}}$ carries a very good two-dimensional independence relation, as desired. $\square$

8. Multidimensional independence

We define here the main notion of this paper: multidimensional independence relations. This section contains mostly definitions and easy lemmas. A result of importance is how to build multidimensional independence relations from the two-dimensional relations (Theorem 8.20).

8.1. **Systems.** We start by defining what is meant by a system of models.

Definition 8.1. For $\mathfrak{K}$ an abstract class and $I = (I, \leq)$ a partial order, an $(I, \mathfrak{K})$-*system* is a sequence $\mathbf{m} = \langle M_u : u \in I \rangle$ such that $u \leq v$ implies $M_u \leq_{\mathfrak{K}} M_v$. A $\mathfrak{K}$-*system* is an $(I, \mathfrak{K})$-system for some I. For $\mathbf{m}$ a $\mathfrak{K}$-system, we write $I(\mathbf{m})$ for the unique I such that $\mathbf{m}$ is an $(I, \mathfrak{K})$-system. When $\mathfrak{K}$ is clear from context, we omit it from all the above definitions.

We will often be interested in systems where all extensions are strict. In fact, a strengthening of this, being proper, is more useful (one can see it as an abstract version of Definition 3.5(4), when we take A to consist of the union of several models):

Definition 8.2. Let $\mathbf{m} = \langle M_u : u \in I \rangle$ be an I-system. Let $\mathfrak{K}^*$ be a skeleton of $\mathfrak{K}$ (the reader can think of $\mathfrak{K}^* = \mathfrak{K}$ at first reading).

(1) Let $u \in I$. We say that u is *$\mathfrak{K}^*$-proper in* $\mathbf{m}$ if $M_u \in \mathfrak{K}^*$ and there exists $N \in \mathfrak{K}^*$ such that:
   (a) For all $v \in I$, $v < u$ implies $M_v \leq_{\mathfrak{K}} N <_{\mathfrak{K}^*} M_u$.
   (b) For all $v \in I$, if $v < u$ and $M_v \in \mathfrak{K}^*$, then $M_v \leq_{\mathfrak{K}^*} N$.
(2) For I_0 a sub-order of I, we say that $\mathbf{m}$ is *$(I_0, \mathfrak{K}^*)$-proper* if every $u \in I_0$ is $\mathfrak{K}^*$-proper in $\mathbf{m}$. When $\mathfrak{K}^* = \mathfrak{K}$ and $I_0 = I$, we omit them.

Note that a system where all extensions are strict may not be proper:

Example 8.3. Let $\mathfrak{K}$ be the AEC of all infinite sets, ordered by subset. Let $A \subseteq B$ be two countably infinite sets with $B \backslash A$ infinite. Partition $B \backslash A$ into two infinite sets C_1 and C_2 and let $A_\ell := A \cup C_\ell$. Then the $\mathcal{P}(2)$-system (A, A_1, A_2, B) is not proper but all the extensions are strict.

The next two lemmas are easy properties of proper systems: being proper does not depend on the exact indexing set, and proper systems can be built.

Lemma 8.4. Let $I_0 \subseteq I_1 \subseteq I$ all be partial orders and let $\mathbf{m}$ be an I-system. Let $\mathfrak{K}^*$ be a skeleton of $\mathfrak{K}$. If $\mathbf{m}$ is $(I_0, \mathfrak{K}^*)$-proper, then $\mathbf{m} \restriction I_1$ is $(I_0, \mathfrak{K}^*)$-proper.

Proof. Straightforward. $\square$

Lemma 8.5. Let $\mathfrak{K}^*$ be a skeleton of $\mathfrak{K}$. Let $\mathbf{m} = \langle M_u : u \in I \rangle$ be an I-system with I a finite partial order and let v be maximal in I. If $\mathfrak{K}$ has no maximal models, then there exists $N \in \mathfrak{K}$ such that $M_v \leq_{\mathfrak{K}} N$ and v is $\mathfrak{K}^*$-proper in $\mathbf{m} \restriction (I \backslash \{v\}) \frown \langle N \rangle$.

Proof. Using that $\mathfrak{K}$ has no maximal models, let $M \in \mathfrak{K}$ be such that $M_v <_{\mathfrak{K}} M$. Now apply Lemma 3.9 with M, $\langle M_i : i < n \rangle$ there standing for M, $\{M_u \in \mathfrak{K}^* \mid u < v\}$ here. $\square$

It is now time to define how systems can relate to each other, in particular how they can be isomorphic, extensions, etc.

Definition 8.6.

(1) Let $I \subseteq J$ be partial orders and let $\mathbf{m} = \langle M_u : u \in J \rangle$ be a J-system. Then $\mathbf{m} \restriction I := \langle M_u : u \in I \rangle$.
(2) For $\mathbf{m}_\ell = \langle M_u^\ell : u \in I \rangle$, $\ell = 1, 2$ both I-systems, say $\mathbf{m}_1 \leq_{\mathfrak{K}} \mathbf{m}_2$ if $M_u^1 \leq_{\mathfrak{K}} M_u^2$ for all $u \in I$. We say that $\mathbf{m}_2$ is a *disjoint* extension of $\mathbf{m}_1$ (written $\mathbf{m}_1 \leq_{\mathfrak{K}}^d \mathbf{m}_2$) if $\mathbf{m}_1 \leq_{\mathfrak{K}} \mathbf{m}_2$ and $M_u^2 \cap M_v^1 \subseteq M_u^1$ for all $u, v \in I$.
(3) [She09a, III.12.14(4)] For $k < \omega$, $\mathbf{m}_\ell = \langle M_u^\ell : u \in I \rangle$, $\ell < k$ all I-systems with $\mathbf{m}_0 \leq_{\mathfrak{K}} \mathbf{m}_1 \leq_{\mathfrak{K}} \ldots \leq_{\mathfrak{K}} \mathbf{m}_{k-1}$, let $\mathbf{m} := \mathbf{m}_0 * \mathbf{m}_1 * \ldots * \mathbf{m}_{k-1}$ be the system defined as follows:
   (a) It is indexed by the partial order $J := I \times k$ ordered lexicographically, so $\mathbf{m} = \langle M_u : u \in J \rangle$.
   (b) For all $u \in I$ and $i < k$, $M_{(u,i)} = M_u^i$.
(4) For $\mathbf{m}_\ell = \langle M_u^\ell : u \in I \rangle$, $\ell = 1, 2$ both I-systems, we say f is a *system embedding* from $\mathbf{m}_1$ to $\mathbf{m}_2$ and write $f : \mathbf{m}_1 \to \mathbf{m}_2$ if $f = \langle f_u : u \in I \rangle$, for

all $u \in I$, f_u is a $\mathfrak{K}$-embedding from M_u^1 to M_u^2, and $u \leq v$ implies $f_u \subseteq f_v$.

(5) For $\mathbf{m}_\ell = \langle M_u^\ell : u \in I \rangle$, $\ell = 1, 2$ both I-systems, we say f is an *isomorphism* from $\mathbf{m}_1$ to $\mathbf{m}_2$ and write $f : \mathbf{m}_1 \cong \mathbf{m}_2$ if $f = \langle f_u : u \in I \rangle$ is a system embedding from $\mathbf{m}_1$ to $\mathbf{m}_2$ and for all $u \in I$, f_u is an isomorphism from M_u^1 onto M_u^2.
(6) For $\pi : I_1 \cong I_2$ an isomorphism of partial orders, $\mathbf{m} = \langle M_u : u \in I_1 \rangle$ an I_1-system, let $\pi(\mathbf{m})$ denote the I_2-system $\langle M_{\pi^{-1}(v)} : v \in I_2 \rangle$.
(7) A system $\mathbf{m} = \langle M_u : u \in I \rangle$ is called *disjoint* if whenever $u, v, w, u^* \in I$ are such that $u \leq v \leq u^*$ and $u \leq w \leq u^*$, then $M_u \subseteq M_v \cap M_w$.
(8) A system $\mathbf{m} = \langle M_u : u \in I \rangle$ is called *fully disjoint* if whenever $u, v, w \in I$ are such that $u \leq v$ and $u \leq w$, then $M_u \subseteq M_v \cap M_w$.

Remark 8.7. Every disjoint system is isomorphic to a fully disjoint system.

It is often useful to code systems as an abstract class:

Definition 8.8. Let $\mathfrak{K}$ be an abstract class and let I be a partial order. The *vocabulary of* $(\mathfrak{K}, I)$*-systems* is $\tau^I := \tau(\mathfrak{K}) \cup \{P_i : i \in I\}$, where each P_i is a new unary predicate. The *abstract class of* $(\mathfrak{K}, I)$*-system* is the abstract class $\mathfrak{K}^I = (K^I, \leq_{\mathfrak{K}^I})$ defined as follows:

(1) K^I consists of all the τ^I-structures M such that $\langle M^{P_i} : i \in I \rangle$ forms a fully disjoint $(\mathfrak{K}, I)$-system (we see M^{P_i} as a $\tau(\mathfrak{K})$-structure). We identify the elements of K^I with the corresponding systems.
(2) $\mathbf{m}_1 \leq_{\mathfrak{K}^I} \mathbf{m}_2$ if and only if $\mathbf{m}_1 \leq_{\mathfrak{K}}^d \mathbf{m}_2$ (see Definition 8.6(2)).

8.2. **Multidimensional independence relations.** Multidimensional independence relations consist of systems indexed by a certain class of semilattices. For reasons that will become apparent, we require that this class has a certain amount of closure:

Definition 8.9. A class $\mathcal{I}$ of semilattices is called *closed* if:

(1) $\mathcal{I}$ is closed under isomorphisms.
(2) $\mathcal{I}$ is closed under taking initial segments.
(3) For any $I \in \mathcal{I}$ and any $u \in I \backslash \{\bot\}$, $[u, \infty)_I \times \{0, 1\} \in \mathcal{I}$.

The main examples of a closed class of semilattice are:

Lemma 8.10. Let $n < \omega$ and let $\mathcal{I}$ be the class of all initial semilattices isomorphic to an initial segments of $\mathcal{P}(n)$. Then $\mathcal{I}$ is closed.

Proof. The third condition of Definition 8.9 is the only non-trivial one. Let $I \in \mathcal{I}$, and identify it with an initial segment of $\mathcal{P}(n)$. Let $u \in I \backslash \{\bot\}$ (so $u \neq \emptyset$). Then $[u, \infty)_I$ is the set of all subsets of $\mathcal{P}(n)$ extending u. As a semilattice, it is isomorphic to $\mathcal{P}(n \backslash u) \cong \mathcal{P}(n - |u|)$. Thus $[u, \infty)_I \times \{0, 1\} \cong \mathcal{P}(n - |u|) \times \mathcal{P}(1) \cong \mathcal{P}(n - |u| + 1)$, which since $|u| \geq 1$ is an initial segment of $\mathcal{P}(n)$. $\square$

A multidimensional independence relation consists of an abstract class $\mathfrak{K}$, a closed class of finite semilattices, and a class of systems indexed by these semilattices. Several properties are required, akin to the requirements in the definition of a two-dimensional independence relation. A complication arises in the monotonicity properties: there are several and may not always seem natural. Nevertheless they will hold for the relation we construct and are all used in the next section. Note that we do not know whether a multidimensional independence relations are canonical (i.e. whether under reasonable conditions there can be at most one satisfying this list of axiom).

Definition 8.11. $\mathrm{NF}_{\mathfrak{i}}$—see multidimensional independence relation$\mathcal{I}_{\mathfrak{i}}$—see multidimensional independence relation A *multidimensional independence relation (or notion)* is a triple $\mathfrak{i} = (\mathfrak{K}, \mathcal{I}, \mathrm{NF})$, where $\mathfrak{K} = \mathfrak{K}_{\mathfrak{i}}$ is an abstract class, $\mathcal{I} = \mathcal{I}_{\mathfrak{i}}$ is a class of semilattices, and $\mathrm{NF} = \mathrm{NF}_{\mathfrak{i}}$ is a class of $(\mathfrak{K}, I)$-systems with $I \in \mathcal{I}$. We call the members of NF *independent systems*. We require the following properties:

(1) $\mathcal{I}$ is a *closed* class of *finite* semilattices.
(2) Invariance: If $\mathbf{m}$ and $\mathbf{m}'$ are systems and $f : \mathbf{m} \cong \mathbf{m}'$, then $\mathrm{NF}(\mathbf{m})$ if and only if $\mathrm{NF}(\mathbf{m}')$.
(3) Nontriviality: $\mathrm{NF} \neq \emptyset$ and for every $I \in \mathcal{I}$ and every $M \in \mathfrak{K}$, there exists an independent I-system containing M.
(4) Disjointness: Every independent system is disjoint (see Definition 8.6(7)).
(5) Symmetry: If $I_1, I_2 \in \mathcal{I}$, $\pi : I_1 \cong I_2$ is an isomorphism, and $\mathbf{m}$ is an I_1-system, then $\mathrm{NF}(\mathbf{m})$ if and only if $\mathrm{NF}(\pi(\mathbf{m}))$, where $\pi(\mathbf{m})$ is the I_2-system naturally induced by π (see Definition 8.6(6)).
(6) Transitivity: If $\mathbf{m}_0 \leq_{\mathfrak{K}} \mathbf{m}_1 \leq_{\mathfrak{K}} \mathbf{m}_2$, $\mathbf{m}_0 * \mathbf{m}_1$ is independent, and $\mathbf{m}_1 * \mathbf{m}_2$ is independent, then $\mathbf{m}_0 * \mathbf{m}_2$ is independent.
(7) Monotonicity 1: If $\mathbf{m}$ is an I-system, $\mathrm{NF}(\mathbf{m})$, and $J \subseteq I$ is such that $J \in \mathcal{I}$, then $\mathrm{NF}(\mathbf{m} \upharpoonright J)$.
(8) Monotonicity 2: If $\mathbf{m} = \langle M_u : u \in I \rangle$ is an I-system, v is a maximal element in I, and $M_v \leq_{\mathfrak{K}} N$, then $\mathbf{m}$ is independent if and only if $\langle M_u : u \in I \backslash \{v\} \rangle \frown \langle N \rangle$ is independent.
(9) Monotonicity 3: If $\mathbf{m}$ is an I-system, then $\mathbf{m}$ is independent if and only if for any maximal $v \in I$, $\mathbf{m} \upharpoonright \{u \in I : u \leq v\}$ is independent.
(10) Monotonicity 4: Let $I_1, I_2 \subseteq I$ all be in $\mathcal{I}$. Let $\mathbf{m}$ be an I-system. If:
    (a) I_1 is an initial segment of I.
    (b) $I_1 \cap I_2$ is cofinal in I_1.
    (c) $I_1 \cup I_2 = I$.
    (d) $\mathbf{m} \upharpoonright I_1$ and $\mathbf{m} \upharpoonright I_2$ are independent.

    Then $\mathbf{m}$ is independent.
(11) Monotonicity 5: Let $I \in \mathcal{I}$, let $u \in I \backslash \{\bot\}$, and let $I_0 := [u, \infty)_I$. Let $\mathbf{m}$, $\mathbf{m}'$ be I-systems, $\mathbf{m}_0 := \mathbf{m} \upharpoonright I_0$ be independent, and $\mathbf{m}'_0 := \mathbf{m}' \upharpoonright I_0$ be such that $\mathbf{m}_0 \leq_{\mathfrak{K}} \mathbf{m}'_0$ and $\mathbf{m}_0 * \mathbf{m}'_0$ is independent. Assume that $\mathbf{m} \upharpoonright (I \backslash I_0) = \mathbf{m}' \upharpoonright (I \backslash I_0)$. Then $\mathbf{m}$ is independent if and only if $\mathbf{m}'$ is independent.

The natural restrictions can be defined on a multidimensional independence relation:

Definition 8.12. Let $\mathfrak{i} = (\mathfrak{K}, \mathcal{I}, \mathrm{NF})$ be a multidimensional independence relation.

(1) For $\mathcal{J} \subseteq \mathcal{I}$ a class of partial orders, we let $\mathfrak{i} \upharpoonright \mathcal{J}$ denote the multidimensional independence relation $\mathfrak{j} = (\mathfrak{K}, \mathcal{J}, \mathrm{NF} \upharpoonright \mathcal{J})$, where $\mathrm{NF} \upharpoonright \mathcal{J}$ denotes the class of systems in NF that are J-systems for $J \in \mathcal{J}$.

(2) For $\mathfrak{K}^*$ a sub-abstract class of $\mathfrak{K}$, let $\mathfrak{i} \upharpoonright \mathfrak{K}^*$ denote the multidimensional independence relation $\mathfrak{j} = (\mathfrak{K}^*, \mathcal{I}, \mathrm{NF} \upharpoonright \mathfrak{K}^*)$, where $\mathrm{NF} \upharpoonright \mathfrak{K}^*$ denotes the class of systems in NF that are also $\mathfrak{K}^*$-systems. We may write $\mathfrak{i}_\lambda$ instead of $\mathfrak{i} \upharpoonright \mathfrak{K}_\lambda$.

The next lemma shows that, under some closure conditions on $\mathcal{I}$, transitivity follows from the other axioms:

Lemma 8.13. Let $\mathfrak{i}$ be a multidimensional independence relation. Let $\mathbf{m}_0, \mathbf{m}_1, \mathbf{m}_2$ be I-systems with $I \in \mathcal{I}_\mathfrak{i}$. If $\mathbf{m}_0 \leq_{\mathfrak{K}_\mathfrak{i}} \mathbf{m}_1 \leq_{\mathfrak{K}_\mathfrak{i}} \mathbf{m}_2$, $\mathbf{m}_0 * \mathbf{m}_1$ is independent, $\mathbf{m}_1 * \mathbf{m}_2$ is independent, and $I \times 3 \in \mathcal{I}_\mathfrak{i}$, then $\mathbf{m}_0 * \mathbf{m}_1 * \mathbf{m}_2$ is independent. In particular, if $\mathcal{I}_\mathfrak{i}$ is closed under products the transitivity axiom follows from the others.

Proof. Say the $\mathbf{m}_\ell$'s are I-systems. Let $J := I \times 3$. Let $\mathbf{m} := (\mathbf{m}_0 * \mathbf{m}_1 * \mathbf{m}_2$. We check that $\mathbf{m}$ is independent by using monotonicity 4: let $I_1 := I \times \{0, 1\}$ and let $I_2 := I \times \{1, 2\}$. We have that $\mathbf{m} \upharpoonright I_1 = \mathbf{m}_1 * \mathbf{m}_2$ and $\mathbf{m} \upharpoonright I_2 = \mathbf{m}_2 * \mathbf{m}_3$. Both are independent by assumption. Moreover, $I_1 \cup I_2 = J$, I_1 is an initial segment of J, and $I_1 \cap I_2 = I \times \{1\}$ is cofinal in I_1. Thus monotonicity 4 implies that $\mathbf{m}$ is independent. The "in particular" part follows from monotonicity 1. $\square$

One wants to order systems so that if $\mathbf{m}_2$ extends $\mathbf{m}_1$, then the entire diagram that this forms is independent. This is formalized by looking at $\mathbf{m}_1 * \mathbf{m}_2$:

Definition 8.14. Let $\mathfrak{i}$ be a multidimensional independence relation. We define a binary relation $\leq_\mathfrak{i}$ on $\mathrm{NF}_\mathfrak{i}$ as follows: $\mathbf{m}_1 \leq_\mathfrak{i} \mathbf{m}_2$ if and only if $\mathbf{m}_1 \leq_\mathfrak{K} \mathbf{m}_2$ (so in particular $I(\mathbf{m}_1) = I(\mathbf{m}_2)$) and $\mathrm{NF}_\mathfrak{i}(\mathbf{m}_1 * \mathbf{m}_2)$. Let $\mathfrak{K}_{\mathfrak{i},I}$ be the sub-abstract class of $\mathfrak{K}^I$ (Definition 8.8) consisting of all independent I-systems and ordered by $\leq_\mathfrak{i}$.

One may want in addition to require that the system $\mathbf{m}_1 * \mathbf{m}_2$ is *proper*:

Definition 8.15. Let $\mathfrak{i}$ be a multidimensional independence relation. Let $I \in \mathcal{I}_\mathfrak{i}$. We let $\mathfrak{K}_{\mathfrak{i},I}^{\mathrm{proper}}$ be the class of *proper* independent I-systems ordered by $\mathbf{m}_1 \leq_{\mathfrak{K}_{\mathfrak{i},I}^{\mathrm{proper}}} \mathbf{m}_2$ if and only if either $\mathbf{m}_1 = \mathbf{m}_2$ or $\mathbf{m}_1 * \mathbf{m}_2$ is a *proper* independent system.

Absent from the properties in Definition 8.11 were any kind of extension or uniqueness. We define these now. Note that existence says there *is* a system, extension that an *existing* system can be extended. Strong uniqueness will hold only in classes of brimmed models, where also the extensions are brimmed: it says in particular that there is only one proper system indexed by a given lattice. Uniqueness is the property that will hold in the original class: it says that each independent system has at most one completion, in the sense that any two completions amalgamate.

Definition 8.16. Let $\mathfrak{i} = (\mathfrak{K}, \mathcal{I}, \mathrm{NF})$ be a multidimensional independence relation. The following are additional properties that $\mathfrak{i}$ may have:

(1) Existence: For any $I \in \mathcal{I}$ there exists a proper independent I-system.
(2) Extension: For any $I, J \in \mathcal{I}$ with I an initial segment of J, if $\mathbf{m}$ is an independent I-system, then there exists a $(J \backslash I)$-proper independent J-system $\mathbf{m}'$ such that $\mathbf{m} \cong \mathbf{m}' \upharpoonright I$.
(3) Strong uniqueness: Let $\mathbf{m}_1$, $\mathbf{m}_2$ be independent J-systems and let I be an initial segment of J with $I \in \mathcal{I}$. Let $f : \mathbf{m}_1 \upharpoonright I \cong \mathbf{m}_2 \upharpoonright I$. If $\mathbf{m}_1$ and $\mathbf{m}_2$

are $(J \backslash I)$-proper (Definition 8.2), then there exists $g : \mathbf{m}_1 \cong \mathbf{m}_2$ extending f.
(4) Uniqueness: Let $\mathbf{m}_1$, $\mathbf{m}_2$ be independent J-systems and let I be an initial segment of J with $J = I \cup \{v\}$, $I < v$. Let $f : \mathbf{m}_1 \restriction I \cong \mathbf{m}_2 \restriction I$. Then there exists an independent J-system $\mathbf{m}$ and systems embeddings (see Definition 8.6(4) $g_\ell : \mathbf{m}_\ell \to \mathbf{m}$, $\ell = 1, 2$ such that g_ℓ extends f for $\ell = 1, 2$.

Any reasonable multidimensional independence relation induces a two-dimensional independence relation:

Definition 8.17. Let $\mathfrak{i}$ be a multidimensional independence relation with $\mathcal{P}(2) \in \mathcal{I}_{\mathfrak{i}}$. Define a 4-ary relation $\mathop{\smile}\limits^{\mid} = \mathop{\smile}\limits^{\mid}(\mathfrak{i})$ by $M_1 \overset{M_3}{\underset{M_0}{\mathop{\smile}\limits^{\mid}}} M_2$ if and only if (M_0, M_1, M_2, M_3) (seen as a system indexed by $\mathcal{P}(2)$) is an independent system.

Lemma 8.18. If $\mathfrak{i}$ is a multidimensional independence relation with existence, extension, and uniqueness such that $\mathcal{P}(2) \in \mathcal{I}_{\mathfrak{i}}$, then $\mathop{\smile}\limits^{\mid}(\mathfrak{i})$ (from Definition 8.17) is a two-dimensional independence notion on $\mathfrak{K}_{\mathfrak{i}}$.

Proof. The least trivial axioms to prove are monotonicity and symmetry. Symmetry can be obtained by using the symmetry property of multidimensional independence relations with the automorphism of $\mathcal{P}(2)$ swapping $\{0\}$ and $\{1\}$. Monotonicity actually follows from the other axioms, see [LRV, 3.23]. $\square$

8.3. **Building a multidimensional independence relation.** Starting from a two-dimensional independence relation, one can also build a multidimensional independence relation in the natural way:

Definition 8.19. Let $\mathop{\smile}\limits^{\mid}$ be a two-dimensional independence notion on an abstract class $\mathfrak{K}$. Let $\mathfrak{i} = \mathfrak{i}(\mathop{\smile}\limits^{\mid})$ be the following multidimensional independence relation:

(1) $\mathfrak{K}_{\mathfrak{i}} = \mathfrak{K}$.
(2) $\mathcal{I}_{\mathfrak{i}}$ is the class of all finite semilattices.
(3) For $I \in \mathcal{I}_{\mathfrak{i}}$ and $\mathbf{m} = \langle M_u : u \in I \rangle$ an I-system, $\mathbf{m} \in \mathrm{NF}_{\mathfrak{i}}$ if and only if whenever $u_1, u_2, v \in I$ and $u_1 \leq v$, $u_2 \leq v$, then $M_{u_1} \overset{M_v}{\underset{M_{u_1 \wedge u_2}}{\mathop{\smile}\limits^{\mid}}} M_{u_2}$.

Theorem 8.20. If $\mathop{\smile}\limits^{\mid}$ is a two-dimensional independence notion on an abstract class $\mathfrak{K}$ with no maximal models, then $\mathfrak{i} = \mathfrak{i}(\mathop{\smile}\limits^{\mid})$ is a multidimensional independence notion such that $\mathop{\smile}\limits^{\mid}(\mathfrak{i}) = \mathop{\smile}\limits^{\mid}$. Moreover, $\mathfrak{i}$ restricted to the class of initial segments of $\mathcal{P}(2)$ has existence, extension, and uniqueness.

Proof. Most of the axioms follow directly from the definitions. We only prove:

- Monotonicity 4: Let I_1, I_2, $\mathbf{m} = \langle M_u : u \in I \rangle$ be as in the statement of monotonicity 4. Let $u_1, u_2, v \in I$ with $u_1 \leq v$, $u_2 \leq v$. We have to see that $M_{u_1} \overset{M_v}{\underset{M_{u_1 \wedge u_2}}{\mathop{\smile}\limits^{\mid}}} M_{u_2}$. If both u_1 and u_2 are in I_1, or both are in I_2, then this is immediate from the assumptions that $\mathbf{m} \restriction I_1$ and $\mathbf{m} \restriction I_2$ are both independent (and preservation of $\mathop{\smile}\limits^{\mid}$ under $\mathfrak{K}$-embeddings). Assume now that one is in $I_1 \backslash I_2$ and the other in $I_2 \backslash I_1$. By symmetry, without loss of generality $u_1 \in I_1 \backslash I_2$, $u_2 \in I_2 \backslash I_1$.

By assumption, $I_1 \cap I_2$ is maximal in I and $I = I_1 \cup I_2$, so there exists $u' \in I_2$ such that $u_1 \leq u'$. Let $u_1' := \wedge\{u' \in I_2 \cap I_1 \mid u_1 \leq u'\}$. In other words, u_1' is the minimal element of $I_1 \cap I_2$ such that $u_1 \leq u_1'$. By minimality, we must have that $u_1' \leq v$ (otherwise, consider $u_1'' := u_1' \wedge v$; since I_1 is an initial segment, this is in $I_1 \wedge I_2$ and is still above u_1). By the previous paragraph, we have that $M_{u_1'} \overset{M_v}{\underset{M_{u_1' \wedge u_2}}{\perp}} M_{u_2}$. Also by the previous paragraph (recalling that I_1 is an initial segment, so $u_1' \wedge u_2 \in I_1 \cap I_2$), $M_{u_1} \overset{M_{u_1'}}{\underset{M_{u_1 \wedge u_1' \wedge u_2}}{\perp}} M_{u_1' \wedge u_2}$. By transitivity for $\perp$, $M_{u_1} \overset{M_v}{\underset{M_{u_1 \wedge u_1' \wedge u_2}}{\perp}} M_{u_2}$. Now note that $u_1 \leq u_1'$ by assumption, hence $u_1 \wedge u_1' \wedge u_2 = u_1 \wedge u_2$, as desired.

- Monotonicity 5: Fix the data given by the statement of monotonicity 5. Write $\mathbf{m} = \langle M_u : u \in I \rangle$ and $\mathbf{m}' = \langle M_u' : u \in I \rangle$.
  - We first show that $\mathbf{m}'$ independent implies that $\mathbf{m}$ is independent. Let $u_1, u_2, u_3 \in I$ be such that $u_1, u_2 \leq u_3$. We want to see that $M_{u_1} \overset{M_{u_3}}{\underset{M_{u_1 \wedge u_2}}{\perp}} M_{u_2}$. There are several cases:
    * If $u_1, u_2 \notin I_0$, then $u_1 \wedge u_2 \notin I_0$ (as $I \backslash I_0 \subseteq I$), hence the result follows directly from the assumption that $M_{u_1}' \overset{M_{u_3}'}{\underset{M_{u_1 \wedge u_2}'}{\perp}} M_{u_2}'$ and ambient monotonicity.
    * If $u_1, u_2 \in I_0$, then apply transitivity.
    * If $u_1 \in I_0$, $u_2 \notin I_0$, and $u_1 \wedge u_2 \in I_0$, apply transitivity again.
    * If $u_1 \in I_0$, $u_2 \notin I_0$, and $u_1 \wedge u_2 \notin I_0$, use monotonicity.
  - Assume now that $\mathbf{m}$ is independent. Let $u_1, u_2, u_3 \in I$ be such that $u_1, u_2 \leq u_3$. We want to see that $M_{u_1}' \overset{M_{u_3}'}{\underset{M_{u_1 \wedge u_2}'}{\perp}} M_{u_2}'$ and we again split into cases. The cases $u_1, u_2 \notin I_0$ and $u_1, u_2 \in I_0$ are similar to before. If $u_1 \in I_0$, $u_2 \notin I_0$, then $u_3 \in I_0$ since I_0 is and end segment. Thus we can apply transitivity and we are done.
- Transitivity: follows from Lemma 8.13.

That $\mathfrak{i}$ restricted to the class of initial segments of $\mathcal{P}(2)$ has existence, extension, and uniqueness follows from the corresponding properties of $\perp$: we only have to note that a $\mathcal{P}(2)$-system $\langle M_u : u \in \mathcal{P}(2) \rangle$ is independent if and only if $M_{\{0\}} \overset{M_2}{\underset{M_\emptyset}{\perp}} M_{\{1\}}$, by the symmetry axiom of $\perp$. □

9. Finite combinatorics of multidimensional independence

In this section, we assume:

Hypothesis 9.1. $\mathfrak{i} = (\mathfrak{K}, \mathcal{I}, \mathrm{NF})$ is a multidimensional independence relation.

We consider properties of $\mathfrak{i}$ that are finitary in nature. That is, they do not depend on any kind of closure under unions of chains. This allows us to work at a high level of generality (for example, $\mathfrak{K}$ is just assumed to be an abstract class). A crucial question is whether properties (such as extension or uniqueness) that hold for brimmed systems (that is, for systems consisting of brimmed models, ordered by being "brimmed over") will also hold for the not necessarily brimmed other systems. We prove general positive results in this direction here (Theorems 9.12 and 9.15). The statements use the notion of a skeleton. This is also a place where we use that $\mathcal{I}$ is closed (Definition 8.9), as well as the monotonicity axioms in Definition 8.11.

We start by proving a few easy consequences of the definition of a multidimensional independence relation. First, $\mathcal{P}(1)$-systems are trivial:

Lemma 9.2. $M \leq_{\mathfrak{K}} N$ both be in $\mathfrak{K}$. If $\mathcal{P}(1) \in \mathcal{I}$, then $\langle M, N \rangle$ is an independent I-system.

Proof. By nontriviality, there is an independent $\mathcal{P}(1)$-system $\mathbf{m}$ containing M. Write $\mathbf{m} = \langle M, N' \rangle$. By monotonicity 2, $\langle M, M \rangle$ is independent. By monotonicity 2 again, $\langle M, N \rangle$ is independent. $\square$

It is also easy to see that existence is weaker than extension:

Lemma 9.3. If $\mathfrak{i}$ has extension, then $\mathfrak{i}$ has existence.

Proof. Let $I \in \mathcal{I}$. Start with the empty system and extend it to a proper independent I-system. $\square$

Uniqueness is also weaker than strong uniqueness:

Lemma 9.4. If $\mathfrak{i}$ has strong uniqueness and $\mathfrak{K}$ has no maximal models, then $\mathfrak{i}$ has uniqueness.

Proof. Given two models M_1, M_2 to amalgamate over an independent system $\mathbf{m}$, take a strict extension M_ℓ' of M_ℓ and use strong uniqueness to see that M_1' is isomorphic to M_2' over $\mathbf{m}$, hence M_1 and M_2 amalgamate over $\mathbf{m}$. $\square$

9.1. **Skeletons of independence relations.** It is natural to consider what happens to $\mathfrak{i}$ when restricting it to a skeleton of $\mathfrak{K}$. We overload terminology and also call such restrictions *skeletons* of $\mathfrak{i}$:

Definition 9.5. Let $\mathfrak{i}$ and $\mathfrak{i}^*$ be independence relations. We say that $\mathfrak{i}^*$ is a *skeleton* of $\mathfrak{i}$ if:

(1) $\mathfrak{K}_{\mathfrak{i}^*}$ is a skeleton of $\mathfrak{K}_{\mathfrak{i}}$.
(2) $\mathfrak{i}^* = \mathfrak{i} \upharpoonright \mathfrak{K}_{\mathfrak{i}^*}$.

Remark 9.6. If $\mathfrak{K}^*$ is a skeleton of $\mathfrak{K}$, then it is straightforward to check that $\mathfrak{i} \upharpoonright \mathfrak{K}^*$ will be an independence relation.

We will consider the following localized versions of extension and uniqueness:

Definition 9.7. Let $\mathbf{m}$ be an independent I-system and let $\mathfrak{K}^*$ be a skeleton of $\mathfrak{K}$.

(1) We call $\mathbf{m}$ a $\mathfrak{K}^*$-*extension base in* $\mathfrak{i}$ if for any $J \in \mathcal{I}$ such that $J = I \cup \{v\}$ and $I < v$, there exists an independent J-system $\mathbf{m}'$ such that v is $\mathfrak{K}^*$-proper in $\mathbf{m}'$ and $\mathbf{m} \cong \mathbf{m}' \upharpoonright I$. When $\mathfrak{K}^* = \mathfrak{K}$ and $\mathfrak{i}$ is clear from context, we omit them and call $\mathbf{m}$ an *extension base*.
(2) We call $\mathbf{m}$ a $\mathfrak{K}^*$-*strong uniqueness base in* $\mathfrak{i}$ if for any $J \in \mathcal{I}$ such that $J = I \cup \{v\}$ and $I < v$ and any two independent J-systems $\mathbf{m}_1, \mathbf{m}_2$ such that $\mathbf{m}_1 \upharpoonright I = \mathbf{m}_2 \upharpoonright I = \mathbf{m}$ and v is $\mathfrak{K}^*$-proper in both $\mathbf{m}_1$ and $\mathbf{m}_2$, we have that $\mathbf{m}_1 \cong_{\mathbf{m}} \mathbf{m}_2$. As before, when $\mathfrak{K}^* = \mathfrak{K}$ and $\mathfrak{i}$ is clear from context, we omit them.
(3) We call $\mathbf{m}$ a *uniqueness base in* $\mathfrak{i}$ if for any $J \in \mathcal{I}$ such that $J = I \cup \{v\}$ and $I < v$ and any two independent J-systems $\mathbf{m}_1, \mathbf{m}_2$ such that $\mathbf{m}_1 \upharpoonright I = \mathbf{m}_2 \upharpoonright I = \mathbf{m}$, we have that there is an $\mathfrak{i}$-independent J-system $\mathbf{m}^*$ and $f_\ell : \mathbf{m}_\ell \underset{\mathbf{m}}{\longrightarrow} \mathbf{m}^*$ systems embeddings for $\ell = 1, 2$. When $\mathfrak{i}$ is clear from context, we omit it.

It turns out that the parameter $\mathfrak{K}^*$ in the definition of a $\mathfrak{K}^*$-extension base can safely be omitted. This will be used without comments:

Lemma 9.8. Let $\mathbf{m}$ be an $\mathfrak{i}$-independent system and let $\mathfrak{K}^*$ be a skeleton of $\mathfrak{K}$. Let $\mathfrak{i}^* := \mathfrak{i} \upharpoonright \mathfrak{K}^*$.

(1) $\mathbf{m}$ is a $\mathfrak{K}$-extension base in $\mathfrak{i}$ if and only if $\mathbf{m}$ is a $\mathfrak{K}^*$-extension base in $\mathfrak{i}$.
(2) If $\mathbf{m}$ is $\mathfrak{i}^*$-independent, then $\mathbf{m}$ is a $\mathfrak{K}^*$-extension base in $\mathfrak{i}^*$ if and only if $\mathbf{m}$ is a $\mathfrak{K}^*$-extension base in $\mathfrak{i}$.

Proof. Use monotonicity 2 and the proof of Lemma 8.5. □

Of course, having extension is the same as all independent systems being extension bases, and similarly for the other properties:

Lemma 9.9. $\mathfrak{i}$ has extension [uniqueness] [strong uniqueness] if and only if every independent system is an extension [uniqueness] [strong uniqueness] base.

Proof. Straightforward. □

Note also that the uniqueness properties transfers down from $\mathfrak{i}$ to $\mathfrak{i}^*$:

Lemma 9.10. Let $\mathfrak{i}^*$ be a skeleton of $\mathfrak{i}$. If $\mathfrak{i}$ has [strong] uniqueness, then $\mathfrak{i}^*$ has [strong] uniqueness.

Proof. Straightforward using the basic properties of a skeleton. □

The extension property also transfers. In fact, more can be said:

Lemma 9.11. Let $\mathfrak{i}^*$ be a skeleton of $\mathfrak{i}$. Write $\mathfrak{K} := \mathfrak{K}_{\mathfrak{i}}$, $\mathfrak{K}^* := \mathfrak{K}_{\mathfrak{i}^*}$. Assume that $\mathfrak{i}$ has extension. Let $I, J \in \mathcal{I}$ with I an initial segment of J. Let $\mathbf{m}$ be an $\mathfrak{i}$-independent I-system. There exists an $\mathfrak{i}$-independent J-system $\mathbf{m}'$ such that:

(1) $\mathbf{m} \cong \mathbf{m}' \upharpoonright J$.
(2) $\mathbf{m}'$ is $(J \backslash I)$-$\mathfrak{K}^*$-proper (see Definition 8.2).

In particular, $\mathfrak{i}^*$ has extension.

Proof. As in the proof of Lemma 8.5. $\square$

We now turn to the harder problem of getting the properties in $\mathfrak{i}$ if we know them in the skeleton. For extension, this can be done provided that the underlying class is the same (the ordering may be different). The example to keep in mind is $\mathfrak{K}$ being the class of saturated models of cardinality λ in a given AEC, and $\mathfrak{K}^*$ consisting of the same class, but ordered by "being brimmed over". Note however that the theorem below also has content when $\mathfrak{i} = \mathfrak{i}^*$: it says that it is enough to prove that *proper* systems are extension bases.

Theorem 9.12. Let $\mathfrak{i}^*$ be a skeleton of $\mathfrak{i}$ such that $|\mathfrak{K}_{\mathfrak{i}^*}| = |\mathfrak{K}_{\mathfrak{i}}|$. If every proper $\mathfrak{i}^*$-independent system is an extension base, then $\mathfrak{i}$ has extension.

Proof. Write $\mathfrak{K} := \mathfrak{K}_{\mathfrak{i}}$, $\mathfrak{K}^* := \mathfrak{K}_{\mathfrak{i}^*}$. For $\mathbf{m}$ an $\mathfrak{i}$-independent I-system, let $k(\mathbf{m})$ be the number of $u \in I$ which are *not* $\mathfrak{K}^*$-proper in $\mathbf{m}$. We prove that every $\mathfrak{i}$-independent system $\mathbf{m}$ is an extension base by induction on $k(\mathbf{m})$.

Let $\mathbf{m}$ be an independent I-system and let $J = I \cup \{v\} \in \mathcal{I}$, with $I < v$. If $k(\mathbf{m}) = 0$, then $\mathbf{m}$ is an extension base by assumption. Assume now that $k(\mathbf{m}) > 0$. Let $u \in I$ be such that u is *not* $\mathfrak{K}^*$-proper in $\mathbf{m}$. Note that $u \neq \bot$. Let $I_0 := [u, \infty)_I$ and let $\mathbf{m}_0 := \mathbf{m} \restriction I_0$. Note that:

(1) $I_0 \subseteq I$ and $I_0 \times \{0, 1\} \in \mathcal{I}$ (by (1) in the definition of a multidimensional independence relation), hence also $I_0 \in \mathcal{I}$ (it is isomorphic to an initial segment of $I_0 \times \{0, 1\}$).
(2) $I \backslash I_0$ is an initial segment of I, hence is in $\mathcal{I}$.
(3) $k(\mathbf{m}_0) < k(\mathbf{m})$, since u is minimal in I_0, hence, since $|\mathfrak{K}^*| = |\mathfrak{K}|$, $\mathfrak{K}^*$-proper in $\mathbf{m}_0$ (we are also using that properness is preserved when passing to subsystems, i.e. Lemma 8.4).

Let $\mathbf{m}_0'$ be an $\mathfrak{i}^*$-independent proper I_0-system such that $\mathbf{m}_0 \leq_{\mathfrak{i}} \mathbf{m}_0'$. To see that this exists, we apply (the proof of) Lemma 9.11, where $\mathfrak{i}, \mathfrak{i}^*, I, J$ there stand for $\mathfrak{i} \restriction \mathrm{NF}^*, \mathfrak{i}^* \restriction \mathrm{NF}^*, I_0 \times \{0\}, I_0 \times \{0, 1\}$ here, and we have set NF^* to be the class of $\mathfrak{i}$-independent systems $\mathbf{m}^*$ with $k(\mathbf{m}^*) < k(\mathbf{m})$. We obtain an $(I_0 \times \{1\}, \mathfrak{K}^*)$-proper independent system $\mathbf{m}'$ with $\mathbf{m}' \restriction I_0 \times \{0\} = \mathbf{m}_2$ (we identify I_0 with $I_0 \times \{0\}$). Now let $\mathbf{m}_0' := \mathbf{m} \restriction I_2 \times \{1\}$.

Write $\mathbf{m}_0' = \langle N_u : u \in I_2 \rangle$. For $u \in I$, let $M_u' := M_u$ if $u \notin I_2$ and $M_u' = N_u$ otherwise. Let $\mathbf{m}' := \langle M_u' : u \in I \rangle$. By monotonicity 5, $\mathbf{m}'$ is $\mathfrak{i}$-independent. Moreover, $k(\mathbf{m}') < k(\mathbf{m})$ because u is proper in $\mathbf{m}'$. By the induction hypothesis, $\mathbf{m}'$ must be an extension base. Let $\mathbf{m}''$ be an independent J-system such that v is proper in $\mathbf{m}''$ and there is $f : \mathbf{m}' \cong \mathbf{m}'' \restriction I$. Let $\mathbf{m}'''$ be the unique J-system so that $\mathbf{m}''' \restriction \{v\} = \mathbf{m}'' \restriction \{v\}$ and $f \restriction \mathbf{m} \cong \mathbf{m}''' \restriction I$. By monotonicity 5, $\mathbf{m}'''$ is independent, and hence witnesses that $\mathbf{m}$ is an extension base. $\square$

This implies a way to obtain extension from existence, provided that the skeleton satisfies strong uniqueness (in case $\mathfrak{K}_{\mathfrak{i}^*}$ consists of brimmed models, ordered by being brimmed over, it turns out that in reasonable cases strong uniqueness will hold in $\mathfrak{i}^*$, though there is no hope of it holding in $\mathfrak{i}$).

Corollary 9.13. Let $\mathfrak{i}^*$ be a skeleton of $\mathfrak{i}$ with $|\mathfrak{K}_{\mathfrak{i}^*}| = |\mathfrak{K}_{\mathfrak{i}}|$. If $\mathfrak{i}^*$ has strong uniqueness and existence, then $\mathfrak{i}$ has extension.

Proof. By Theorem 9.12, it suffices to show that every proper $\mathfrak{i}^*$-independent I-system $\mathbf{m}$ is an extension base. So let $J = I \cup \{v\} \in \mathcal{I}$ with $I < v$. By existence, there is a proper $\mathfrak{i}^*$-independent J-system $\mathbf{m}^*$. By strong uniqueness, $m^* \upharpoonright I$ is isomorphic to $\mathbf{m}$, and clearly $\mathbf{m}^* \upharpoonright I$ is an extension base, so $\mathbf{m}$ must be one as well. $\square$

We now turn to studying under what conditions uniqueness in a skeleton implies uniqueness in the original independence relation. We are unable to prove the full analog of Theorem 9.12, so we make the additional assumption that $\mathcal{I}$ is closed under certain products. The following is the key lemma:

Lemma 9.14. Let $\mathfrak{i}^*$ be a skeleton of $\mathfrak{i}$. Assume that $\mathfrak{i}$ has extension. Let $J \in \mathcal{I}_{\mathfrak{i}}$ be such that $J = I \cup \{v\}$, for $I < v$. If $J \times \{0,1\} \in \mathcal{I}_{\mathfrak{i}}$ and any proper $\mathfrak{i}^*$-independent I-system is a uniqueness base, then any $\mathfrak{i}$-independent I-system is a uniqueness base.

Proof. Let $\mathbf{m} = \langle M_u : u \in I \rangle$ be an $\mathfrak{i}$-independent I-system. Let $\mathbf{m}_1, \mathbf{m}_2$ both be $\mathfrak{i}$-independent J-systems, $\mathbf{m}_\ell = \langle M_u^\ell : u \in J \rangle$, $\ell = 1, 2$ and without loss of generality $\mathbf{m}_1 \upharpoonright I = \mathbf{m}_2 \upharpoonright I = \mathbf{m}$. We want to find an amalgam of M_v^1 and M_v^2 fixing $\mathbf{m}$.

Let $J' := J \times \{0,1\}$, $I' := I \times \{0,1\}$. Identify J with $J \times \{0\}$. By Lemma 9.11, we can find an I'-independent system $\mathbf{m}'$ such that $\mathbf{m}' \upharpoonright I = \mathbf{m}$ which is $(I' \backslash I)$-$\mathfrak{K}^*$-proper. For $\ell = 1, 2$, let $\mathbf{m}'_\ell$ be the $I' \cup \{(v, 0)\}$-system obtained by adding M_v^ℓ to $\mathbf{m}'$. By monotonicity 2 and 3, $\mathbf{m}'_\ell$ is still independent. By extension and some renaming, one can find a J'-independent system $\mathbf{m}''_\ell$ such that $\mathbf{m}''_\ell \upharpoonright (I' \cup \{(v, 0)\}) = \mathbf{m}'_\ell$. Moreover, we can arrange that $\mathbf{m}''_\ell$ is $(J' \backslash J)$-$\mathfrak{K}^*$-proper. Now let $J^* := J \times \{1\}$, $I^* := I \times \{1\}$. We have that $\mathbf{m}^* := \mathbf{m}''_1 \upharpoonright I^* = \mathbf{m}''_2 \upharpoonright I^*$. Moreover for $\ell = 1, 2$, $\mathbf{m}^*_\ell := \mathbf{m}''_\ell \upharpoonright J^*$ is a proper $\mathfrak{i}^*$-independent system by construction. Thus it is a uniqueness base, so $\mathbf{m}^*_1$ and $\mathbf{m}^*_2$ amalgamate over $\mathbf{m}^*$. This implies that M_v^1 and M_v^2 amalgamate over $\mathbf{m}$, as desired. $\square$

Theorem 9.15. Assume that $\mathcal{I}_{\mathfrak{i}}$ is closed under products. Let $\mathfrak{i}^*$ be a skeleton of $\mathfrak{i}$ with $|\mathfrak{K}_{\mathfrak{i}}| = |\mathfrak{K}_{\mathfrak{i}^*}|$. If $\mathfrak{i}^*$ has strong uniqueness and existence, then $\mathfrak{i}$ has uniqueness and extension.

Proof. By Corollary 9.13, $\mathfrak{i}$ has extension. We have to see that for every $I \in \mathcal{I}$, every $\mathfrak{i}$-independent I-system is a uniqueness base but this is immediate from Lemma 9.14. $\square$

The following more local result will be used in the proof of Theorem 10.16. It says roughly that if a uniqueness base is changed by making one of the models on the boundary smaller, then it is still a uniqueness base. A version of it appears in [She09a, III.12.26].

Lemma 9.16. Let $\mathbf{m} = \langle M_u : u \in I \rangle$ be an independent I-system. Let $u^* \in I$ be maximal. Let $N_{u^*} \in \mathfrak{K}$ be such that $M_{u^*} \leq_{\mathfrak{K}} N_{u^*}$. Let $\mathbf{m}^* := \mathbf{m} \upharpoonright (I \backslash \{u^*\}) \frown N_{u^*}$. If $\mathop{\smile\!\!\!\!|}(\mathfrak{i})$ has extension and $\mathbf{m}^*$ is a uniqueness base, then $\mathbf{m}$ is a uniqueness base.

Proof. Let $J = I \cup \{v\} \in \mathcal{I}$, $I < v$. Let $\mathbf{m}_\ell = \langle M_u^\ell : u \in J \rangle$ for $\ell = 1, 2$ be independent systems with $\mathbf{m}_\ell \upharpoonright I = \mathbf{m}$ for $\ell = 1, 2$. We want to amalgamate M_v^1 and M_v^2 over $\mathbf{m}$. Let $\mathop{\smile\!\!\!\!|} := \mathop{\smile\!\!\!\!|}(\mathfrak{i})$.

Using the extension property for $\mathop{\smile}\limits^{|}$, find f_ℓ, N_v^ℓ, $\ell = 1, 2$ such that $f_\ell : N_{u^*} \xrightarrow[M_{u^*}]{} N_v^\ell$ and $M_v^\ell \overset{N_v^\ell}{\underset{M_{u^*}}{\mathop{\smile}\limits^{|}}} f_\ell[N_{u^*}]$. Let $\mathbf{m}_\ell' := \langle (M_u^\ell)' : u \in J\rangle$ be defined by $(M_u^\ell)' := M_u$ if $u \in I \backslash \{u^*\}$, $(M_{u^*}^\ell)' := f_\ell[N_{u^*}]$, and $(M_v^\ell)' := N_v^\ell$. By monotonicity 4, $\mathbf{m}_\ell'$ is an independent system.

Let $f := f_2 f_1^{-1} \upharpoonright (M_{u^*}^1)'$. Then $f : \mathbf{m}_1' \upharpoonright I \cong \mathbf{m}_2' I$ and since $\mathbf{m}_1' \cong \mathbf{m}^*$ and $\mathbf{m}^*$ is a uniqueness base, we know that there exists g extending f amalgamating N_v^1 and N_v^2. Such an f also amalgamates M_v^1 and M_v^2 over $\mathbf{m}$, as desired. $\square$

10. The multidimensional amalgamation properties

Throughout this section, we continue to assume:

Hypothesis 10.1. $\mathfrak{i} = (\mathfrak{K}, \mathcal{I}, \mathop{\smile}\limits^{|})$ is a multidimensional independence relation.

We study the existence and uniqueness properties restricted to systems indexed by $\mathcal{P}(n)$, for some $n < \omega$. We show (Lemma 10.9, Theorem 10.10) that once we have those, we can (under reasonable conditions), obtain the properties for systems indexed by other sets. We also give a relationship between extension and existence using strong uniqueness, akin to Corollary 9.13 but stronger: we only use strong uniqueness for smaller systems. This is Theorem 10.11. Finally, we give sufficient conditions for the n-dimensional properties in terms of amalgamation properties inside classes of n-dimensional systems. The most important of these result is Theorem 10.16.

It will be convenient to restrict oneself to systems indexed by finite initial segments of $\mathcal{P}(\omega)$. Independence relations considering only such systems will be called *okay*. We then define the n-dimensional properties by saying that they must hold for systems indexed by initial segments of $\mathcal{P}(n)$.

Definition 10.2.

(1) For $n < \omega$, let $\mathcal{I}_n$ be the class of all partial orders isomorphic to an initial segment of $\mathcal{P}(n)$. Let $\mathcal{I}_{<\omega} := \bigcup_{n<\omega} \mathcal{I}_n$.
(2) We say that $\mathfrak{i}$ is *okay* if $\mathcal{I} \subseteq \mathcal{I}_{<\omega}$. We say that $\mathfrak{i}$ is *n-okay* if it is okay and $\mathcal{P}(n) \in \mathcal{I}_{\mathfrak{i}}$.
(3) For P a property from Definition 8.16 and $n < \omega$, we say that $\mathfrak{i}$ has n-P if $\mathfrak{i}$ is n-okay and $\mathfrak{i} \upharpoonright \mathcal{I}_n$ has P. We say that $\mathfrak{i}$ has $(< n)$-P if it has m-P for all $m < n$.
(4) For P a property, λ an infinite cardinal and $n < \omega$, we say that $\mathfrak{i}$ has (λ, n)-P if $\mathfrak{i} \upharpoonright \mathfrak{K}_\lambda$ has n-P. We say that $\mathfrak{i}$ has $(\lambda, < n)$-P if it has (λ, m)-P for all $m < n$. Similarly define variations like the (Θ, n)-P, for Θ an interval of cardinals.

Remark 10.3. Since $\mathfrak{i}$ is closed under initial segments, if $\mathfrak{i}$ is n-okay then $\mathcal{I}_n \subseteq \mathcal{I}$, so $\mathfrak{i}$ is m-okay for any $m < n$.

Note that $\mathfrak{i} \upharpoonright \mathcal{I}_{<\omega}$ is always an okay independence relation, so replacing $\mathfrak{i}$ by $\mathfrak{i} \upharpoonright \mathcal{I}_{<\omega}$, we assume for the rest of this section that $\mathfrak{i}$ is okay:

Hypothesis 10.4. $\mathfrak{i}$ is *okay*.

It is easy to characterize the low-dimensional amalgamation properties in terms of familiar properties:

Lemma 10.5.

(1) $\mathfrak{i}$ has 0-existence if and only if $\mathfrak{i}$ has 0-extension if and only if $\mathfrak{K} \neq \emptyset$.
(2) $\mathfrak{i}$ has strong 0-uniqueness if and only if $\mathfrak{K}$ has at most one model up to isomorphism.
(3) $\mathfrak{i}$ has 0-uniqueness if and only if $\mathfrak{K}$ has joint embedding.
(4) $\mathfrak{i}$ has 1-extension if and only if $\mathfrak{i}$ is 1-okay, $\mathfrak{K}$ is not empty, and $\mathfrak{K}$ has no maximal models.
(5) $\mathfrak{i}$ has 1-uniqueness if and only if $\mathfrak{i}$ is 1-okay and $\mathfrak{K}$ has amalgamation and joint embedding.
(6) If $\mathfrak{i}$ has 2-extension, then $\mathfrak{K}$ has disjoint amalgamation.

Proof. Immediate. $\square$

Interestingly, strong 1-uniqueness together with uniqueness already implies strong uniqueness. In case we are considering classes of brimmed models ordered by being brimmed over, strong 1-uniqueness is exactly the uniqueness of brimmed models, a key property studied in many papers [SV99, Van06, Van13, GVV16, Van16].

Lemma 10.6. If $\mathfrak{i}$ has strong 1-uniqueness and uniqueness, then $\mathfrak{i}$ has strong uniqueness.

Proof. We use Lemma 9.9. Let $\mathbf{m}_\ell = \langle M_u^\ell : u \in J \rangle$, $\ell = 1, 2$ be independent J-systems. Let v be a top element of J and let $I := J \backslash \{v\}$. Assume that $\mathbf{m} := \mathbf{m}_1 \upharpoonright I = \mathbf{m}_2 \upharpoonright I$ and v is proper in both $\mathbf{m}_1$ and $\mathbf{m}_2$. We show that M_v^1 and M_v^2 are isomorphic over $\mathbf{m}$. By definition of being proper, we can pick $N_v^\ell <_{\mathfrak{K}} M_v^\ell$ so that $M_u^\ell \leq_{\mathfrak{K}} N_v^\ell$, $\ell = 1, 2$. By uniqueness, there exists $M \in \mathfrak{K}$ and $f_\ell : N_v^\ell \to M$ with f_ℓ fixing $\mathbf{m}$, $\ell = 1, 2$. If $|J| \leq 2$, there is nothing to prove so assume that $|J| > 2$. Then M_v^1 must be a proper extension of M_u^1 for each $u < v$. By strong 0-uniqueness, $\mathfrak{K}$ must have no maximal models. Thus we can pick $N \in \mathfrak{K}$ with $M <_{\mathfrak{K}} N$. By strong 1-uniqueness and some renaming there exists $g_\ell : M_v^\ell \cong N$, with g_ℓ extending f_ℓ, $\ell = 1, 2$. In the end, $g := g_2^{-1} g_1$ is the desired isomorphism. $\square$

We now want to show that it is enough to prove that $\mathcal{P}^-(n)$-systems are extension bases to get n-extension. To this end, the following invariance of a semilattice will be useful:

Definition 10.7. For $I \in \mathcal{I}_{<\omega}$ and $u \in I$, let $n(I, u)$ be the cardinality of $f(u)$, where $f : I \to \mathcal{P}(\omega)$ is any semilattice embedding such that $f[I]$ is an initial segment of $\mathcal{P}(\omega)$. Let $n(I)$ be the maximum of $n(I, u)$ for all $u \in I$. When I is empty, we specify $n(\emptyset) = -1$.

Lemma 10.8. Let $n < \omega$ and $I, J \in \mathcal{I}_{<\omega}$ be non-empty.

(1) If $I \subseteq J$, then $n(I) \leq n(J)$.
(2) $n(\mathcal{P}(n)) = n$.
(3) $n(\mathcal{P}^-(n)) = n - 1$.
(4) $n(I \times \{0, 1\}) = n(I) + 1$.
(5) If $I \in \mathcal{I}_n$, then $n(I) \leq n$.

Proof. Straightforward. □

Lemma 10.9. Let $I, J \in \mathcal{I}_{<\omega}$ and let I be an initial segment of J. Let $n := n(J)$.

(1) If for every $m \leq n$, every proper $\mathcal{P}^-(m)$-system is an extension base, then every proper independent I-system can be extended to a proper independent J-system.
(2) If for every $m \leq n$, every proper $\mathcal{P}^-(m)$-system is a strong uniqueness base, then for any proper independent J-systems $\mathbf{m}_1, \mathbf{m}_2$ and any isomorphism $f : \mathbf{m}_1 \restriction I \cong \mathbf{m}_2 \restriction I$, there is an isomorphism $g : \mathbf{m}_1 \cong \mathbf{m}_2$ extending f.

Proof. Identify J with an initial segment of $\mathcal{P}(\omega)$.

(1) Let $\mathbf{m}$ be a proper independent I-system. Let $u \in J \backslash I$ be such that $J_0 := I \cup \{u\}$ is an initial segment of J. It is enough to show that $\mathbf{m}$ can be extended to a proper independent J_0-system. Let $m := |u|$. Note that $m \leq n$. Let $\pi : u \to m$ be a bijection. Since J_0 is an initial segment of $\mathcal{P}(\omega)$, We have that π induces an isomorphism from $(-\infty, u] = \mathcal{P}(u)$ onto $\mathcal{P}(m)$. In particular, $\mathbf{m} \restriction \mathcal{P}^-(u)$ is an extension base. Now extend it to $\mathcal{P}(u)$ and use monotonicity to argue that this induces an extension of $\mathbf{m}$ to J_0.
(2) Similar.

□

Theorem 10.10. Let $n < \omega$ and assume that $\mathfrak{i}$ is n-okay. If for every $m \leq n$, every proper $\mathcal{P}^-(m)$-system is an extension base, then $\mathfrak{i}$ has n-extension.

Proof. Directly from Theorem 9.12 and Lemma 10.9. □

The following result will be useful to obtain extension from existence and strong uniqueness for lower-dimensional systems:

Theorem 10.11. Let $n < \omega$ and assume that $\mathfrak{i}$ is n-okay. If $\mathfrak{i}$ has strong $(< n)$-uniqueness, then $\mathfrak{i}$ has n-existence if and only if $\mathfrak{i}$ has n-extension.

Proof. If $\mathfrak{i}$ has n-extension, then by Lemma 9.3 it always has n-existence (this does not use strong uniqueness). Assume now that $\mathfrak{i}$ has n-existence. By Theorem 10.10, it suffices to show that for every $m \leq n$, every proper $\mathcal{P}^-(m)$-system is an extension base. Let $m \leq n$ and let $\mathbf{m}$ be a proper $\mathcal{P}^-(m)$-system. By n-existence, there is a proper $\mathcal{P}(m)$-system $\mathbf{m}'$. By strong $(< n)$-uniqueness and Lemma 10.9, $\mathbf{m}' \restriction \mathcal{P}^-(m) \cong \mathbf{m}$. Since $\mathbf{m}' \restriction \mathcal{P}^-(m)$ is an extension base, so is $\mathbf{m}$. □

We would like to establish a criteria to obtain uniqueness from a kind of amalgamation in the class $\mathfrak{K}^{\text{proper}}_{\mathfrak{i}, \mathcal{P}(n)}$ (Definition 8.15). It turns out that a slightly finer version of $\mathfrak{K}^{\text{proper}}_{\mathfrak{i}, \mathcal{P}(n)}$ is useful for this purpose: the idea is that we require the interior of the system to be brimmed but allow the boundary to not extend the rest in brimmed way (of course this is formulated abstractly using skeletons).

Definition 10.12. Let $n < \omega$ and assume that $\mathfrak{i}$ is n-okay. Let $\mathfrak{i}^*$ be a skeleton of $\mathfrak{i}$. Let K^* be the class of proper $\mathfrak{i}$-independent $\mathcal{P}(n)$-systems $\mathbf{m}$ such that $\mathbf{m} \restriction \mathcal{P}^-(n)$ is $\mathfrak{i}^*$-independent. Let $\mathfrak{K}^{\text{proper}}_{\mathfrak{i}, \mathfrak{i}^*, \mathcal{P}(n)}$ be the abstract class whose underlying class is

K^* and whose ordering is $\mathbf{m}_1 \leq_{\mathfrak{K}^{\text{proper}}_{\mathfrak{i},\mathfrak{i}^*,\mathcal{P}(n)}} \mathbf{m}_2$ if and only if $\mathbf{m}_1 = \mathbf{m}_2$ or whenever $\mathbf{m}_\ell = \mathbf{m}_\ell^- \frown M_\ell$ with $\mathbf{m}_\ell^- := \mathbf{m}_\ell \upharpoonright \mathcal{P}^-(n)$, $\ell = 1, 2$, we have that $\mathbf{m}_1 \leq_{\mathfrak{i}} \mathbf{m}_2$, M_1 and M_2 are in $\mathfrak{K}_{\mathfrak{i}^*}$, $M_1 <_{\mathfrak{K}_{\mathfrak{i}^*}} M_2$, and $\mathbf{m}_1 \upharpoonright \mathcal{P}^-(n) <_{\mathfrak{K}^{\text{proper}}_{\mathfrak{i}^*,\mathcal{P}^-(n)}} \mathbf{m}_2 \upharpoonright \mathcal{P}^-(n)$.

Remark 10.13. We have that $\mathfrak{K}^{\text{proper}}_{\mathfrak{i}^*,\mathcal{P}(n)} \subseteq \mathfrak{K}^{\text{proper}}_{\mathfrak{i},\mathfrak{i}^*,\mathcal{P}(n)} \subseteq \mathfrak{K}^{\text{proper}}_{\mathfrak{i},\mathcal{P}(n)}$. In particular, $\mathfrak{K}^{\text{proper}}_{\mathfrak{i},\mathfrak{i},\mathcal{P}(n)} = \mathfrak{K}^{\text{proper}}_{\mathfrak{i},\mathcal{P}(n)}$.

Definition 10.14. For $n < \omega$, we let $\phi_n(x)$ be the formula in the vocabulary of systems (Definition 8.8) which holds inside a $\mathcal{P}(n)$-system $\langle M_u : u \in \mathcal{P}(n) \rangle$ if and only if $a \in \bigcup_{u \subsetneq n} M_u$.

The following easy relationships hold between the n-dimensional properties and classes of proper n-dimensional systems. See Definition 5.6 for the definitions of ϕ-categoricity, ϕ-amalgamation, ϕ-uniqueness, etc.

Lemma 10.15. Let $n < \omega$. Assume that $\mathfrak{i}$ is $(n+1)$-okay and let $\mathfrak{i}^*$ be a skeleton of $\mathfrak{i}$.

(1) For any $\mathbf{m}_1 \in \mathfrak{K}^{\text{proper}}_{\mathfrak{i},\mathfrak{i}^*,\mathcal{P}(n)}$, there exists $\mathbf{m}_2 \in \mathfrak{K}^{\text{proper}}_{\mathfrak{i}^*,\mathcal{P}(n)}$ such that $\mathbf{m}_1 \leq_{\mathfrak{K}_{\mathfrak{i}}} \mathbf{m}_2$ and $\mathbf{m}_1 \upharpoonright \mathcal{P}^-(n) = \mathbf{m}_2 \upharpoonright \mathcal{P}^-(n)$.
(2) If $\mathfrak{i}^*$ has strong $(< n)$-uniqueness, then $\mathfrak{K}^{\text{proper}}_{\mathfrak{i},\mathfrak{i}^*,\mathcal{P}(n)}$ is ϕ_n-categorical.
(3) If $\mathfrak{i}^*$ has strong n-uniqueness, then any $\mathbf{m} \in \mathfrak{K}^{\text{proper}}_{\mathfrak{i}^*,\mathcal{P}(n)}$ is ϕ_n-universal in $(|\mathfrak{K}^{\text{proper}}_{\mathfrak{i},\mathfrak{i}^*,\mathcal{P}(n)}|, \leq^d_{\mathfrak{K}_{\mathfrak{i}}})$.
(4) If $\mathfrak{i}^*$ has strong n-uniqueness, then $\mathfrak{K}^{\text{proper}}_{\mathfrak{i},\mathfrak{i}^*,\mathcal{P}(n)}$ has ϕ_n-uniqueness.
(5) Let $\bar{\mathbf{m}} = (\mathbf{m}_0, \mathbf{m}_1, \mathbf{m}_2)$ be a ϕ-span in $\mathfrak{K}^{\text{proper}}_{\mathfrak{i},\mathfrak{i}^*,\mathcal{P}(n)}$, and write $\mathbf{m}_\ell = \langle M_u^\ell : u \in \mathcal{P}(n) \rangle$, $\ell = 0, 1, 2$. Then $\bar{\mathbf{m}}$ has a ϕ-amalgam in $\mathfrak{K}^{\text{proper}}_{\mathfrak{i},\mathfrak{i}^*,\mathcal{P}(n)}$ if and only if there exists $N \in \mathfrak{K}_{\mathfrak{i}}$ and $f_\ell : M^\ell_{\mathcal{P}(n)} \xrightarrow[\mathbf{m}_0]{} N$, $\ell = 1, 2$, such that $f_1 \upharpoonright M_u^1 = f_2 \upharpoonright M_u^2$ for all $u \in \mathcal{P}^-(n)$.
(6) If $\mathbf{m}$ is a ϕ_n-amalgamation base in $\mathfrak{K}^{\text{proper}}_{\mathfrak{i},\mathfrak{i}^*,\mathcal{P}(n)}$, then for any $\mathbf{m}' \in \mathfrak{K}^{\text{proper}}_{\mathfrak{i},\mathfrak{i}^*,\mathcal{P}(n)}$ with $\mathbf{m} \leq_{\mathfrak{K}^{\text{proper}}_{\mathfrak{i},\mathfrak{i}^*,\mathcal{P}(n)}} \mathbf{m}'$, we have that $(\mathbf{m} * \mathbf{m}') \upharpoonright \mathcal{P}^-(n+1)$ is a uniqueness base in $\mathfrak{i}$.

Proof.

(1) By the basic properties of a skeleton.
(2) Since $\mathfrak{i}^*$ has strong $(< n)$-uniqueness, any two $\mathfrak{i}^*$-independent $\mathcal{P}^-(n)$-systems are isomorphic.
(3) By the first part and strong n-uniqueness.
(4) Expanding the definitions, it suffices to show that if $\mathbf{m}_1$, $\mathbf{m}_2$ are two $\mathcal{P}^-(n+1)$-systems such that $\mathbf{m}_1 \upharpoonright \mathcal{P}^-(n) = \mathbf{m}_2 \upharpoonright \mathcal{P}^-(n)$, then there is an isomorphism $f : \mathbf{m}_1 \cong \mathbf{m}_2$ fixing $\mathbf{m}_1 \upharpoonright \mathcal{P}^-(n)$. This is implied by strong n-uniqueness, because $n(\mathcal{P}^-(n+1)) = n$.
(5) By monotonicity.
(6) Expand the definitions.

$\square$

The following result (used in the proof of Lemma 11.16) will be crucial: under the assumption of density of certain amalgamation bases in the class of $(n+1)$-dimensional systems, it implies that the *whole* independence relation has $(n+1)$-uniqueness. More precisely, we also assume that $n \geq 2$, work inside a skeleton, and assume that the skeleton has strong n-uniqueness and $(n+1)$-extension (in the case of interest, these will hold).

Theorem 10.16. Let $n \in [2, \omega)$. Let $\mathfrak{i}$ be a multidimensional independence notion and let $\mathfrak{i}^*$ be a skeleton of $\mathfrak{i}$. If:

(1) $\mathfrak{i}^*$ has $(n+1)$-extension and strong n-uniqueness.
(2) For any $\mathbf{m}_0 \in \mathfrak{K}^{\text{proper}}_{\mathfrak{i}^*, \mathcal{P}(n)}$, there exists a ϕ_n-amalgamation base $\mathbf{m}_1 \in \mathfrak{K}^{\text{proper}}_{\mathfrak{i}, \mathfrak{i}^*, \mathcal{P}(n)}$ such that $\mathbf{m}_0 \leq_{\mathfrak{K}^{\text{proper}}_{\mathfrak{i}, \mathfrak{i}^*, \mathcal{P}(n)}} \mathbf{m}_1$.

Then $\mathfrak{i}^*$ has $(n+1)$-uniqueness.

Proof. By Theorem 9.12, $\mathfrak{i}$ has $(n+1)$-extension. By Lemma 9.14, the proof of Lemma 10.6, and Lemma 10.9, it suffices to show that every proper $\mathfrak{i}^*$-independent $\mathcal{P}^-(n+1)$-system is a uniqueness base. Note that $n(\mathcal{P}^-(n+1)) = n$, so by Lemma 10.9 again, any two proper $\mathfrak{i}^*$-independent $\mathcal{P}^-(n+1)$-systems are isomorphic. Thus it suffices to show that *some* proper $\mathfrak{i}^*$-independent $\mathcal{P}^-(n+1)$-system is a uniqueness base.

Pick any $\mathbf{m}_0 \in \mathfrak{K}^{\text{proper}}_{\mathfrak{i}^*, \mathcal{P}(n)}$ (exists by extension in $\mathfrak{i}^*$). By assumption, there is a ϕ_n-amalgamation base $\mathbf{m}_1$ in $\mathfrak{K}^{\text{proper}}_{\mathfrak{i}, \mathfrak{i}^*, \mathcal{P}(n)}$ which extends $\mathbf{m}_0$. By $(n+1)$-extension, there exists $\mathbf{m}_2 \in \mathfrak{K}^{\text{proper}}_{\mathfrak{i}, \mathfrak{i}^*, \mathcal{P}(n)}$ such that $\mathbf{m}_1 <_{\mathfrak{K}^{\text{proper}}_{\mathfrak{i}, \mathfrak{i}^*, \mathcal{P}(n)}} \mathbf{m}_2$. By Lemma 10.15(6), $\mathbf{m} := (\mathbf{m}_1 * \mathbf{m}_2) \upharpoonright \mathcal{P}^-(n+1)$ is a uniqueness base. Write $\mathbf{m} = \langle M_u : u \in \mathcal{P}^-(n+1) \rangle$ and let $\mathbf{m}' = \langle M'_u : u \in \mathcal{P}^-(n+1) \rangle$ be defined as follows: let $I_0 := \mathcal{P}^-(n+1) \backslash \{n\}$, $\mathbf{m}' \upharpoonright I_0 := (\mathbf{m}_0 * \mathbf{m}_2) \upharpoonright I_0$, $M'_n = M_n$. It is easy to check that $\mathbf{m}'$ is an independent system and a uniqueness base as well: anytime a model N extends all the elements of $\mathbf{m}'$, it extends all the elements of $\mathbf{m}$.

Now let $\mathbf{m}^*$ be a $\mathcal{P}^-(n+1)$-system defined as follows: $\mathbf{m}^* \upharpoonright I_0 = \mathbf{m}' \upharpoonright I_0$ and $\mathbf{m}^* \upharpoonright \{n\} = \mathbf{m}_0 \upharpoonright \{n\}$. Note that then $\mathbf{m}^* \upharpoonright \mathcal{P}(n) = \mathbf{m}_0$, and it is easy to check that $\mathbf{m}^*$ is a proper $\mathfrak{i}^*$-independent system. Moreover, $\mathbf{m}^* \upharpoonright \{n\} \leq_{\mathfrak{K}} \mathbf{m}' \upharpoonright \{n\}$ by definition of $\mathbf{m}'$, so by Lemma 9.16, $\mathbf{m}^*$ is a uniqueness base, as desired. $\square$

11. Continuous independence relations

This section is central. We define here the conditions a multidimensional independence relation should satisfy to be well-behaved with respect to increasing chains. In particular, it should be closed under union of those. We call such multidimensional independence relations *very good*. They are the ones we really want to study. We prove the *construction theorem* (Theorem 11.3), which says that from a very good two-dimensional independence relation, we can build a very good multidimensional independence relation.

We study an important subclass of independent systems: the *brimmed* ones (Definition 11.7). We show that they are quite well-behaved under unions and resolvability (Lemma 11.9). There are some technical issues here, since we do not know that the class of n-dimensional brimmed systems is closed under unions until we have proven $(n+1)$-dimensional uniqueness, but we want to use the former in the proof of the latter. We use the class $\mathfrak{K}^{\text{proper}}_{\mathfrak{i},\mathfrak{i}^*,\mathcal{P}(n)}$ (Definition 10.12) to get around this.

We prove the n-dimensional properties by studying the class of brimmed systems and using the weak diamond (or a strongly compact cardinal). Crucial is the *stepping up lemma* (Lemma 11.16) which shows how to go from the n-dimensional properties to the $(n+1)$-dimensional properties (for brimmed systems). It leads directly to the *brimmed excellence theorem* (Theorem 11.17) giving sufficient conditions for the n-dimensional properties. Of course, once we have the properties for brimmed systems we will be able to get them for regular systems using the results of Section 9.

The first definition is the multidimensional analog of Definition 7.11.

Definition 11.1. We say that a multidimensional independence relation $\mathfrak{i}$ has *strong continuity* if whenever $\langle \mathbf{m}_i : i \leq \delta \rangle$ is a $\leq_{\mathfrak{K}}$-increasing continuous chain of $(I, \mathfrak{K}_{\mathfrak{i}})$-systems and $\mathbf{m}_i$ is independent for all $i < \delta$, then $\mathbf{m}_\delta$ is independent.

Definition 11.2. Let $\mathfrak{i}$ be a multidimensional independence relation. We say that $\mathfrak{i}$ is *very good* if:

(1) $\mathfrak{i}$ is n-okay for all $n < \omega$ (so $\mathcal{I}_{\mathfrak{i}} = \mathcal{I}_{<\omega}$), see Definition 10.2.
(2) $\perp(\mathfrak{i})$ is very good.
(3) $\mathfrak{i}$ has strong continuity.
(4) For any $I \in \mathcal{I}$, both $\mathfrak{K}_{\mathfrak{i},I}$ and the class $\mathfrak{K}^*$ of independent I-system ordered by $\leq^d_{\mathfrak{K}}$ (see Definition 8.6(2)) are fragmented AECs. Moreover the AECs they generate have the same models[5].

Very good multidimensional independence relations are obtained by taking a very good two-dimensional independence notion and applying the construction described in Definition 8.19 (and proven to have some good properties in Theorem 8.20).

Theorem 11.3 (The construction theorem)**.** If $\perp$ is a very good two-dimensional independence notion on $\mathfrak{K}$, then $\mathfrak{i}(\perp) \upharpoonright \mathcal{I}_{<\omega}$ (see Definition 10.2) is a very good multidimensional independence notion.

Proof. Let $\mathfrak{i} := \mathfrak{i}(\perp) \upharpoonright \mathcal{I}_{<\omega}$. By Theorem 8.20, $\mathfrak{i}$ is a multidimensional independence notion. By definition, $\mathfrak{i}$ is n-okay for all $n < \omega$ and $\perp(\mathfrak{i}) = \perp$ (see the statement of Theorem 8.20), which is very good by assumption. It also follows from the definition of $\mathfrak{i}$ that it has strong continuity, since $\perp$ has it. It remains to see that for any $I \in \mathcal{I}$, both $\mathfrak{K}_{\mathfrak{i},I}$ and $\mathfrak{K}^*$ (the class of independent I-systems ordered by $\leq^d_{\mathfrak{K}}$) are fragmented AECs that generate the same models. Now $\mathfrak{K}$ is a fragmented AEC because $\perp$ is very good. Thus by strong continuity it follows immediately that $\mathfrak{K}^*$ is a fragmented AEC, and that $\mathfrak{K}_{\mathfrak{i},I}$ is a fragmented AEC similarly follows from the definition of $\perp$. It remains to check that those two classes generate the same class of models. Let $\lambda := \min(\Theta)$, let $\mathfrak{K}^1$ be the AEC generated by $\mathfrak{K}_{\mathfrak{i},I}$, and let $\mathfrak{K}^2$ be the AEC generated by $\mathfrak{K}^*$. Note that $|\mathfrak{K}^1_\lambda| = |\mathfrak{K}^2_\lambda|$, and $\leq_{\mathfrak{K}^1_\lambda} = \leq_{(\mathfrak{K}_{\mathfrak{i},I})_\lambda}$, $\leq_{\mathfrak{K}^2} = \leq^d_{\mathfrak{K}}$. Since $\leq_{\mathfrak{K}^*}$ extends $\leq_{\mathfrak{K}_{\mathfrak{i},I}}$, we have that $|\mathfrak{K}^1| \subseteq |\mathfrak{K}^2|$. To see the other direction, let $\mathbf{m} \in \mathfrak{K}^2$. We claim that there exists $\mathbf{m}_0$ in $\mathfrak{K}^1_\lambda$ such that for any subset A of $\mathbf{m}$ of size λ there is $\mathbf{m}_1 \in \mathfrak{K}^1_\lambda$ with $\mathbf{m}_0 \leq_{\mathfrak{K}^1} \mathbf{m}_1$, $\mathbf{m}_1 \leq_{\mathfrak{K}^2} \mathbf{m}$, and $\mathbf{m}_1$ containing A. This

[5]This is a multidimensional analog of reflecting down (Definition 7.7).

will suffice by the usual directed system argument. Suppose that the claim fails. We build $\langle \mathbf{m}_i : i < \lambda^+ \rangle$ $\leq_{\mathfrak{K}^2}$-increasing continuous in $\mathfrak{K}^1_\lambda$ such that for all $i < \lambda$, $\mathbf{m}_i \not\leq_{\mathfrak{K}^1} \mathbf{m}_{i+1}$. This is possible by failure of the claim. This is enough: we know that $\perp$ is very good, hence reflects down. From the definition of $\mathfrak{i}$, we get that there must be a club of $i < \lambda^+$ such that $\mathbf{m}_i \leq_{\mathfrak{i},I} \mathbf{m}_{i+1}$, a contradiction. $\square$

Note that a very good multidimensional independence relation has underlying class a fragmented AEC, which may possibly "shrink" (e.g. it could be a class of saturated models). The next lemma shows that the AEC generated by this fragmented AEC also carries a multidimensional independence relation (which is quite nice, but may fail to be very good because of the requirement that $\perp(\mathfrak{i})$ be very good, hence has extension and uniqueness):

Definition 11.4. We call a multidimensional independence relation $\mathfrak{i}$ *almost very good* if:

(1) $\mathfrak{i}_{\min(\text{dom}(\mathfrak{i}))}$ is very good.
(2) $\mathfrak{i}$ satisfies all the properties in the definition of very good, except that $\perp(\mathfrak{i})$ may not be very good.

Lemma 11.5. Let $\mathfrak{i}$ be a very good multidimensional independence notion. Let $\lambda := \min(\text{dom}(\mathfrak{K}_\mathfrak{i}))$. Then there exists a (unique) multidimensional independence relation $\mathfrak{i}'$ such that:

(1) $\lambda = \min(\text{dom}(\mathfrak{K}_{\mathfrak{i}'}))$ and $\mathfrak{i}_\lambda = \mathfrak{i}'_\lambda$.
(2) $\mathfrak{i}'$ is almost very good.

Proof. Let $\mathfrak{K}$ be the AEC generated by $(\mathfrak{K}_\mathfrak{i})_\lambda$. More generally, for each $I \in \mathcal{I}$, let $\mathfrak{K}_I$ be the AEC generated by $(\mathfrak{K}_{\mathfrak{i},I})_\lambda$. We let $\mathfrak{i}'$ be defined as follows: $\mathfrak{K}_{\mathfrak{i}'} = \mathfrak{K}$, $\mathcal{I}_{\mathfrak{i}'} = \mathcal{I}_\mathfrak{i}$, and for any $(\mathfrak{K}, I)$-system $\mathbf{m}$, $\mathbf{m}$ is $\mathfrak{i}'$-independent if and only if $\mathbf{m} \in \mathfrak{K}_I$. It is a straightforward (but long) to check that $\mathfrak{i}'$ is indeed the desired multidimensional independence relation. $\square$

Note that if we have all the n-dimensional properties at a fixed cardinal λ, we can transfer them up. More precisely:

Lemma 11.6. Let $\mathfrak{i}$ be an almost very good multidimensional independence relation with domain $[\lambda_1, \lambda_2]$ (for $\lambda_1 < \lambda_2$) and let $n < \omega$.

(1) If $\mathfrak{i}$ has $([\lambda_1, \lambda_2), n+1)$-extension, then $\mathfrak{i}$ has $([\lambda_1, \lambda_2], n)$-extension.
(2) If $\mathfrak{i}$ has $([\lambda_1, \lambda_2), n+1)$-uniqueness and $([\lambda_1, \lambda_2), n+1)$-extension, then $\mathfrak{i}$ has $([\lambda_1, \lambda_2], n)$-uniqueness.

In particular, if $\mathfrak{i}$ has $(\lambda_1, < \omega)$-extension and $(\lambda_1, < \omega)$-uniqueness, then $\mathfrak{i}$ has extension and uniqueness.

Proof. This is quite standard, and the definitions are tailored to make this work, so we do not replay the arguments here. See [She83b, §5]. $\square$

Brimmed systems are candidates for strong uniqueness, so are an important object of study:

Definition 11.7. Let $\mathfrak{i}$ be a very good multidimensional independence relation. We define a multidimensional independence relation $\mathfrak{i}^{\text{brim}}$ as follows:

(1) $\mathfrak{K}_{\mathfrak{i}^{\text{brim}}}$ consist of $\mathfrak{K}_{\mathfrak{i}}^{\text{brim}}$, the class of brimmed models in $\mathfrak{K}_{\mathfrak{i}}$, ordered by being brimmed over (see Definition 3.23).
(2) $\mathcal{I}_{\mathfrak{i}^{\text{brim}}} = \mathcal{I}_{\mathfrak{i}}$.
(3) $\text{NF}_{\mathfrak{i}^{\text{brim}}} = \text{NF}_{\mathfrak{i}} \upharpoonright \mathfrak{K}_{\mathfrak{i}^{\text{brim}}}$.

We call a proper $\mathfrak{i}^{\text{brim}}$-independent system a *brimmed* $\mathfrak{i}$-independent system.

Definition 11.8. Let $\mathfrak{i}$ be a very good multidimensional independence relation and let $\mathfrak{i}^*$ be a skeleton of $\mathfrak{i}$. For $I \in \mathcal{I}_{\mathfrak{i}}$, we let $\mathfrak{K}_{\mathfrak{i},\mathfrak{i}^*,I}^{\text{proper},*}$ be the sub-abstract class of $\mathfrak{K}_{\mathfrak{i},\mathfrak{i}^*,I}^{\text{proper}}$ (Definition 10.12) with the same ordering but consisting only of the systems where all models have the same cardinality. Similarly define $\mathfrak{K}_{\mathfrak{i}^*,I}^{\text{proper},*}$ to be the sub-abstract class of $\mathfrak{K}_{\mathfrak{i}^*,I}^{\text{proper}}$ (Definition 8.15) consisting of systems with all models of the same cardinality.

The class of brimmed systems satisfies the following properties:

Lemma 11.9. Let $\mathfrak{i}$ be a very good multidimensional independence relation with domain Θ.

(1) $\mathfrak{i}^{\text{brim}}$ is a skeleton of $\mathfrak{i}$.
(2) For any $\lambda \in \Theta$, $\mathfrak{i}^{\text{brim}}$ has strong $(\lambda, 2)$-uniqueness and $(\lambda, 2)$-existence.
(3) For any subinterval Θ_0 of Θ, $(\mathfrak{i}_{\Theta_0})^{\text{brim}} = \left(\mathfrak{i}^{\text{brim}}\right)_{\Theta_0}$.
(4) For any $I \in \mathcal{I}_{\mathfrak{i}}$, $\mathfrak{K}_{\mathfrak{i}^{\text{brim}},I}^{\text{proper},*}$ is a resolvable abstract class with LST number $\min(\Theta)$.
(5) Let $\Theta^- := \{\lambda \in \Theta \mid \lambda^+ \in \Theta\}$.
    (a) Let $I \in \mathcal{I}_{\mathfrak{i}}$ and let $n := n(I)$. If for any $\lambda \in \Theta^-$, we have that $\mathfrak{i}^{\text{brim}}$ has strong $(\lambda, n+1)$-uniqueness and $(\lambda, n+1)$-extension, then $\mathfrak{K}_{\mathfrak{i}^*,I}^{\text{proper},*}$ is Θ^--continuous.
    (b) Let $n < \omega$. If for any $\lambda \in \Theta^-$, we have that $\mathfrak{i}^{\text{brim}}$ has strong (λ, n)-uniqueness and (λ, n)-extension, then $\mathfrak{K}_{\mathfrak{i},\mathfrak{i}^*,\mathcal{P}(n)}^{\text{proper},*}$ is Θ^--continuous.

Proof.

(1) Directly from the definitions and the (easy to check) fact that $\mathfrak{K}_{\mathfrak{i}}^{\text{brim}}$ is a skeleton of $\mathfrak{K}_{\mathfrak{i}}$.
(2) Because $\mathfrak{i}$ has it.
(3) Immediate from the definitions.
(4) We leave it to the reader to verify that the LST number is $\min(\Theta)$. We prove that $\mathfrak{K}_{\mathfrak{i}^{\text{brim}},I}^{\text{proper},*}$ is resolvable. We check that the conditions of Lemma 5.4 are satisfied. All the conditions are easy, except for resolvability for successors. So let $\mathbf{m} \in \mathfrak{K}_{\mathfrak{i}^{\text{brim}},I}^{\text{proper},*}$ have cardinality λ^+ and let $\mathbf{m}_0 \leq_{\mathfrak{K}_{\mathfrak{i}^{\text{brim}},I}^{\text{proper},*}} \mathbf{m}_1$ have cardinality λ, with $\lambda, \lambda^+ \in \Theta$. Since $\mathfrak{K}_{\mathfrak{i},I}$ is a fragmented AEC, it is resolvable, hence there exists $\langle \mathbf{m}_i : i < \lambda^+ \rangle$ a resolution of $\mathbf{m}$ over $\mathbf{m}_0$ in $\mathfrak{K}_{\mathfrak{i},I}$. Write $\mathbf{m}_i = \langle M_u^i : u \in I \rangle$. What are we missing to make this a resolution in $\mathfrak{K}_{\mathfrak{i}^{\text{brim}},I}^{\text{proper},*}$? We want the $\mathbf{m}_i$'s to be brimmed, and in fact we also want $\mathbf{m}_i * \mathbf{m}_{i+1}$ to be brimmed. From the properties of being brimmed, it is in fact enough to show that there is a club C of λ^+ such that

$\langle \mathbf{m}_i : i \in C \rangle$ is an increasing chain in $\mathfrak{K}^{\text{proper},*}_{\mathfrak{i}^{\text{brim}},I}$. To get this club, first get a club C_1 (using Lemma 5.3) such that for $i < j$ both in C_1, M_u^i is brimmed and M_u^j is brimmed over M_u^i (this is possible since $M_u^{\lambda^+}$ is brimmed, hence saturated). Next, use Fact 7.10 on each pair of models in $\mathbf{m}$ and intersect clubs to get that on a club C, for all $i \in C$, M_v^i is brimmed over M_u^i whenever $u < v$. This is not enough to make $\mathbf{m}_i$ brimmed, as we need for each $u \in I$ an intermediate M such that M_u^i is brimmed over M and M is brimmed over each M_v^i, $v < u$. Now by assumption $\mathbf{m}_{\lambda^+}$ is brimmed, so for each $u \in I$ there exists M_u^* with M_u^* brimmed over $M_v^{\lambda^+}$ and $M_u^{\lambda^+}$ brimmed over M_u^*. Resolve $\mathbf{m}_{\lambda^+} \frown M_u^*$ as before to obtain a club C_u and a resolution $\langle M_u^{*,i} : i \in C_u \rangle$ of M_u^* such that $M_u^{*,i}$ is brimmed over M_v^i for each $v < u$ and M_u^i is brimmed over $M_u^{*,i}$. Note that we can further require that for $i < j$ both in C_u, $M_u^{*,j}$ is brimmed over $M_u^{*,i}$, hence the corresponding part of the system $\mathbf{m}_i * m_j$ is also brimmed. Now intersect all the C_u's and intersect further with C_1 to get the desired club.

(5) (a) Let $\langle \mathbf{m}_i : i < \delta \rangle$ be an increasing chain in $\left(\mathfrak{K}^{\text{proper},*}_{\mathfrak{i}^*,I}\right)_{\Theta^-}$. Let $\mathbf{m}_\delta := \bigcup_{i<\delta} \mathbf{m}_i$. We have that $\mathbf{m}_\delta$ is an independent system, the only issue is to prove that it is brimmed (that is, that the extensions between the models are brimmed). Once this is done, it will immediately follow that $\mathbf{m}_\delta$ extends $\mathbf{m}_0$. Write $\mathbf{m}_i = \langle M_u^i : u \in I \rangle$. By Fact 7.9, we directly have that M_v^δ is brimmed over M_u^δ for each $u < v$. However, we need more: for a fixed $u \in I$, we want M_u^δ to be brimmed over $\bigcup_{v<u} M_v^\delta$. Recall (Definition 3.5(4)) that this means that there is M such that M is brimmed over each M_v^δ, $v < u$, and M_u^δ is brimmed over M. In fact, it suffices to find M^* such that M extends each M_v^δ, $v < u$ and M_u^δ is brimmed over M^* (then pick M brimmed over M^* such that M_u^δ is brimmed over M).

We proceed by induction on δ. So we can assume that $\delta = \text{cf}(\delta)$ and the chain $\langle \mathbf{m}_i : i < \delta \rangle$ is continuous. Fix $u \in I$ and write $A_i := \bigcup_{v<u} M_v^i$. We build $\langle M_i : i \leq \delta \rangle$ an increasing continuous chain of brimmed models such that for all $i < \delta$:

(i) M_{i+1} is brimmed over M_i.
(ii) $A_i \subseteq |M_i|$.
(iii) M_u^i is brimmed over M_i.
(iv) $M_u^i \underset{M_i}{\overset{M_u^{i+1}}{\mathop{\smile}\limits^{\vert}}} M_{i+1}$ (where $\mathop{\smile}\limits^{\vert} = \mathop{\smile}\limits^{\vert}(\mathfrak{i})$).

This is enough: by Fact 7.9, M_δ is as desired. This is possible: the base case uses that $\mathbf{m}_0$ is itself a brimmed system. At limits, take unions (and use Fact 7.9). For the successor step, assume we are given $i < \delta$. Let $I_0 := \{v \in I \mid v \leq u\}$, and let $J := I_0 \cup \{v^*\}$, where v^* is a new element such that $v^* < u$ and $v < v^*$ for all $v \in I$ with $v < u$.

Let $\mathbf{m}_i^* = \langle M_v^* : v \in J \rangle$ be the J-system such that $\mathbf{m}_i^* \upharpoonright I_0 = \mathbf{m}_i \upharpoonright I_0$ and $M_{v^*}^* = M_i$. It is easy to check that this is a brimmed independent system. We now use $(n+1)$-extension to build a brimmed independent J-system $\mathbf{m}^{**}$ with $\mathbf{m}^* \leq_{\mathfrak{K}} \mathfrak{K}^{\text{proper},*}_{\mathfrak{i}^{\text{brim}},J} \mathbf{m}^*$. We do this in two steps: first we build an I-system extending $\mathbf{m}^* \upharpoonright I$ (using extension on $I \times \{0,1\}$, note that $n(I \times \{0,1\}) = n(I)+1 = n+1$), then using two-dimensional extension to complete this to an extension $\mathbf{m}^{**}$. Now by strong $(n+1)$-uniqueness (again applied in two steps), we must have that there is $f : \mathbf{m}^{**} \cong_{\mathbf{m}_i} \mathbf{m}_{i+1}$. Let $M_{i+1} := f[M^{**}_{v^*}]$, where $M^{**}_{v^*}$ is the v^*-indexed element of $\mathbf{m}^{**}$.

(b) Immediate from the previous part applied to $I = \mathcal{P}^-(n)$.

□

As hinted at before, brimmed models are well-behaved with respect to the n-dimensional properties: strong uniqueness is equivalent to uniqueness, and existence is equivalent to extension (assuming lower-dimensional uniqueness).

Lemma 11.10. Let $\mathfrak{i}$ be a very good independence relation. Let $\lambda \in \text{dom}(\mathfrak{i})$ and let $n < \omega$. Then:

(1) $\mathfrak{i}^{\text{brim}}$ has strong (λ, n)-uniqueness if and only if $\mathfrak{i}^{\text{brim}}$ has (λ, n)-uniqueness.
(2) If $\mathfrak{i}^{\text{brim}}$ has $(\lambda, < n)$-uniqueness, then $\mathfrak{i}^{\text{brim}}$ has (λ, n)-existence if and only if $\mathfrak{i}^{\text{brim}}$ has (λ, n)-extension.

Proof.

(1) By Lemma 10.6 and Lemma 11.9(2).
(2) By Theorem 10.11 and the previous part.

□

In compact classes, one can take an ultraproduct of brimmed systems and it will still be brimmed. In fact, we will see that the existence of such a uniform extension is very powerful, so we call independence relations with this property *nice*:

Definition 11.11. Let $\mathfrak{i}$ be a very good multidimensional independence relation. For $\mathbf{m}_1, \mathbf{m}_2 \in \mathfrak{K}_{\mathfrak{i},I}$, say with $\mathbf{m}_\ell = \langle M^\ell_u : u \in I\rangle$ we say that $\mathbf{m}_2$ is *brimmed over* $\mathbf{m}_1$ if $\mathbf{m}_1 \leq_{\mathfrak{i}} \mathbf{m}_2$ and M^2_u is brimmed over M^1_u for all $u \in I$.

Definition 11.12. A very good multidimensional independence relation $\mathfrak{i}$ is *nice* if:

(1) $\mathfrak{K}_{\mathfrak{i}}$ is categorical in every $\lambda \in \text{dom}(\mathfrak{i})$.
(2) For any $n < \omega$ and any two $\mathbf{m}_1$, $\mathbf{m}_2 \in \mathfrak{K}_{\mathfrak{i},\mathcal{P}(n)}$ with $\mathbf{m}_1 \restriction \mathcal{P}^-(n) = \mathbf{m}_2 \restriction \mathcal{P}^-(n)$, there exists independent $\mathbf{m}'_1, \mathbf{m}'_2$ such that:
   (a) $\mathbf{m}'_\ell$ is brimmed over $\mathbf{m}_\ell$.
   (b) $\mathbf{m}'_1 \restriction \mathcal{P}^-(n) = \mathbf{m}'_2 \restriction \mathcal{P}^-(n)$.
   (c) If $\mathbf{m}_\ell = \langle M^\ell_u : u \in \mathcal{P}(n)\rangle$, $\mathbf{m}'_\ell = \langle N^\ell_u : u \in \mathcal{P}(n)\rangle$, then $\|M^\ell_u\| = \|N^\ell_u\|$ for all $u \in \mathcal{P}(n)$ and $\ell = 1, 2$.

Compact AECs have a nice independence notion:

Theorem 11.13. Let $\mathfrak{K}$ be a compact AEC. If $\mathfrak{K}$ is categorical in some $\mu > \text{LS}(\mathfrak{K})$, then there exists a *nice* very good multidimensional independence notion $\mathfrak{i}$ with $\mathfrak{K}_{\mathfrak{i}} = \mathfrak{K}^{\text{sat}}_{\geq \text{LS}(\mathfrak{K})^{+6}}$.

Proof. By Fact 7.21, there is a categorical very good frame $\mathfrak{s}$ on $\mathfrak{K}^{\text{sat}}_{\geq \text{LS}(\mathfrak{K})^{+6}}$. Let $\mathop{\smile}\limits^{\vert}$ be the very good two-dimensional independence notion associated with $\mathfrak{s}$. Let

$\mathfrak{i} := \mathfrak{i}(\perp)$. By Theorem 11.3, $\mathfrak{i}$ is very good, we only have to check that it is nice. From the definition, $\mathfrak{K}_{\mathfrak{i}}$ is indeed categorical in all cardinals, so we have to check the second clause in the definition of being nice. Let $\mathbf{m}_1, \mathbf{m}_2$ be as there. Write $\mathbf{m}_\ell = \langle M_u^\ell : u \in \mathcal{P}(n) \rangle$. Using [Bon14, 4.11], we can build a κ-complete ultrafilter U such that for each $u \in \mathcal{P}(n)$ and $\ell = 1, 2$, the U-ultrapower N_u^ℓ of M_u^ℓ (seen as an extension of M_u^ℓ) is brimmed over M_u^ℓ. Let $\mathbf{m}_\ell' := \langle N_u^\ell : u \in \mathcal{P}(n) \rangle$. We show that $\mathbf{m}_\ell \leq_{\mathfrak{i}} \mathbf{m}_\ell'$. It is then not difficult to show, using a downward Löwenheim-Skolem-like argument, that one can take appropriate submodels of the N_u^ℓ's to be of the same size as the M_u^ℓ's and satisfy the requirements.

Essentially by [BG17, BGKV16] (and more precisely by the proof of [BGKV16, 6.8]), the two-dimensional independence relation on $\mathfrak{i}$ is given by κ-coheir (see [BG17, 3.2]). Thus it follows from the claim before the proof of [BG17, 8.2] and some forking calculus that $\mathbf{m}_\ell \leq_{\mathfrak{i}} \mathbf{m}_\ell'$. □

To find a $(\lambda, n+1)$-system, one can simply resolve a (λ^+, n)-system. This is formalized by the following lemma:

Lemma 11.14. Let $\mathfrak{i}$ be a very good multidimensional independence notion with domain $\{\lambda, \lambda^+\}$. If $\mathfrak{i}^{\mathrm{brim}}$ has (λ^+, n)-existence, then $\mathfrak{i}^{\mathrm{brim}}$ has $(\lambda, n+1)$-existence.

Proof. By (λ^+, n)-existence, pick $\mathbf{m}$ a brimmed independent $(\lambda^+, \mathcal{P}(n))$-system. By Lemma 11.9(4), $\mathfrak{K}^{\mathrm{proper},*}_{\mathfrak{i}^{\mathrm{brim}},\mathcal{P}(n)}$ is resolvable, so there exists a strictly increasing continuous chain $\langle \mathbf{m}_i : i < \lambda^+ \rangle$ in $\mathfrak{K}^{\mathrm{proper}}_{\mathfrak{i}^{\mathrm{brim}},\mathcal{P}(n)}$ where for each $i < \lambda^+$, $\mathbf{m}_i$ is a brimmed independent $(\lambda, \mathcal{P}(n))$-system, and $\mathbf{m} = \bigcup_{i<\lambda^+} \mathbf{m}_i$. Now by definition of $\leq_{\mathfrak{K}^{\mathrm{proper}}_{\mathfrak{i}^{\mathrm{brim}},\mathcal{P}(n)}}$, we must have that $\mathbf{m}_0 * \mathbf{m}_1$ is a brimmed independent $(\lambda, \mathcal{P}(n+1))$-system, as desired. □

Thus we obtain the existence properties for free if the interval of cardinals we are working in is closed under the successor operation.

Lemma 11.15. Let $\mathfrak{i}$ be a very good independence notion with domain $\Theta := [\lambda_1, \lambda_2)$. If λ_2 is a limit cardinal, then for any $\lambda \in \Theta$, $\mathfrak{i}^{\mathrm{brim}}$ has $(\lambda, < \omega)$-existence.

Proof. We prove by induction on $n < \omega$ that for any $\lambda \in \Theta$, $\mathfrak{i}^{\mathrm{brim}}$ has (λ, n)-existence. When $n = 0$, this is part of Lemma 11.9. Assume now inductively that $\mathfrak{i}^{\mathrm{brim}}$ has (λ, n)-existence for all $\lambda \in \Theta$. Let $\lambda \in \Theta$. Since λ_2 is limit, $\lambda^+ \in \Theta$. By the induction hypothesis, $\mathfrak{i}^{\mathrm{brim}}$ has (λ^+, n)-existence. By Lemma 11.14 (applied to $\mathfrak{i} \upharpoonright \{\lambda, \lambda^+\}$), $\mathfrak{i}^{\mathrm{brim}}$ has $(\lambda, n+1)$-existence, as desired. □

We now attack the harder question of obtain $(n+1)$-dimensional uniqueness from the n-dimensional properties. This is done using the weak diamond:

Lemma 11.16 (The stepping up lemma)**.** Let λ_1 and λ_2 be infinite cardinals such that $2^{\lambda_1} = 2^{<\lambda_2} < 2^{\lambda_2}$. Let $\mathfrak{i}$ be a very good independence relation with domain $\Theta := [\lambda_1, \lambda_2]$. Assume that for all $\lambda \in \Theta$, $\mathfrak{i}^{\mathrm{brim}}$ has (λ, n)-existence and (λ, n)-uniqueness. Then:

(1) For all $\lambda \in [\lambda_1, \lambda_2)$, $\mathfrak{i}^{\mathrm{brim}}$ has $(\lambda, n+1)$-existence.
(2) There exists $\lambda \in [\lambda_1, \lambda_2)$ such that $\mathfrak{i}^{\mathrm{brim}}$ has $(\lambda, n+1)$-uniqueness.

(3) If $\mathfrak{i}$ is nice and $\mathfrak{i}^{\mathrm{brim}}$ has $(\lambda_2, n+1)$-existence, then $\mathfrak{i}^{\mathrm{brim}}$ has $(\lambda, n+1)$-uniqueness for *all* $\lambda \in [\lambda_1, \lambda_2)$.

Proof.

(1) By Lemma 11.14 applied to $\mathfrak{i} \upharpoonright \{\lambda, \lambda^+\}$.
(2) By Lemma 11.10, for every $\lambda \in [\lambda_1, \lambda_2]$, $\mathfrak{i}^{\mathrm{brim}}$ has strong (λ, n)-uniqueness and (λ, n)-extension, and for every $\lambda \in [\lambda_1, \lambda_2)$, $\mathfrak{i}^{\mathrm{brim}}$ also has $(\lambda, n+1)$-extension. Write $\mathfrak{K}^* := \mathfrak{K}^{\mathrm{proper},*}_{\mathfrak{i},\mathfrak{i}^{\mathrm{brim}},\mathcal{P}(n)}$ (Definition 11.8).

Claim: There is $N \in \mathfrak{K}^*_{[\lambda_1,\lambda_2)}$ which is a ϕ_n-amalgamation base in $\mathfrak{K}^*_{\|N\|}$.

Proof of Claim: Let $\mathfrak{K}^{**}$ be $|\mathfrak{K}^*|$ ordered by $\mathbf{m} \le_{\mathfrak{K}^{**}} \mathbf{m}'$ if and only if $\mathbf{m} \le^d_{\mathfrak{K}} \mathbf{m}'$ (Definition 8.6(2)). We apply Theorem 5.8, where $\mathfrak{K}$, $\mathfrak{K}^*, \phi$ there stand for $\mathfrak{K}^*$, $\mathfrak{K}^{**}, \phi_n$ here. Note first that $\mathfrak{K}^*_{[\lambda_1,\lambda_2)}$ is not empty since $\mathfrak{i}^{\mathrm{brim}}$ has $(\lambda, n+1)$-existence for each $\lambda \in [\lambda_1, \lambda_2)$. Thus if we can check that the hypotheses of Theorem 5.8, we will obtain the conclusion of the claim. First observe that $\mathfrak{K}^*$ is $[\lambda_1, \lambda_2)$-continuous by Lemma 11.9(5). Next, there is a ϕ_n-universal model in $\mathfrak{K}^{**}_{\lambda_2}$ by Lemma 10.15(3). Further, $\mathfrak{K}^{**}_{\lambda_2}$ is ϕ_n-categorical by Lemma 10.15(2). Moreover, for any $\lambda \in [\lambda_1, \lambda_2)$, $\mathfrak{K}^*_\lambda$ has ϕ-uniqueness (Lemma 10.15(4)). Finally, for any $\lambda \in [\lambda_1, \lambda_2)$, if we can ϕ_n-amalgamate in $\mathfrak{K}^{**}$, then we can ϕ_n-amalgamate in $\mathfrak{K}^*$ by Lemma 10.15(5). $\dagger_{\mathrm{Claim}}$

Now let $\lambda := \|N\|$, and apply Theorem 10.16, where $\mathfrak{i}, \mathfrak{i}^*$ there stand for $\mathfrak{i}_\lambda, (\mathfrak{i}^{\mathrm{brim}})_\lambda$ here. We get that $\mathfrak{i}^{\mathrm{brim}}$ has $(\lambda, n+1)$-uniqueness, as desired.
(3) Assume that $\mathfrak{i}$ is nice and $\mathfrak{i}^{\mathrm{brim}}$ has $(\lambda_2, n+1)$-existence. It is enough to show that $\mathfrak{i}^{\mathrm{brim}}$ has $(\lambda_1, n+1)$-uniqueness. Then given an arbitrary $\lambda \in [\lambda_1, \lambda_2)$, we can run the same argument with $\Theta = [\lambda, \lambda_2]$. By the previous part, there is $\lambda \in [\lambda_1, \lambda_2)$ such that $\mathfrak{i}^{\mathrm{brim}}$ has $(\lambda, n+1)$-uniqueness. Let $\mathbf{m}_0$ be a brimmed independent $\mathcal{P}^-(n+1)$ system in λ_1 and let $\mathbf{m}_1, \mathbf{m}_2$ be brimmed independent $\mathcal{P}(n+1)$-systems in λ_1 such that $\mathbf{m}_1 \upharpoonright \mathcal{P}^-(n+1) = \mathbf{m}_2 \upharpoonright \mathcal{P}^-(n+1) = \mathbf{m}_0$. We build $\langle \mathbf{m}^i_\ell : i \le \lambda \rangle$ strictly increasing continuous in $\mathfrak{K}_{\mathfrak{i},\mathcal{P}(n+1)}$, such that for all $i < \lambda$:
  (a) $\mathbf{m}^0_\ell = \mathbf{m}_\ell$, $\ell = 1, 2$.
  (b) All the models in $\mathbf{m}^i_\ell$ have cardinality $\lambda_1 + |i|$.
  (c) $\mathbf{m}^{i+1}_\ell$ is brimmed over $\mathbf{m}^i_\ell$, for $\ell = 1, 2$.
  (d) $\mathbf{m}^i_1 \upharpoonright \mathcal{P}^-(n+1) = \mathbf{m}^i_2 \upharpoonright \mathcal{P}^-(n+1)$.

This is possible by definition of niceness (using that the extensions are brimmed, we can take unions at limits). Now let $\mathbf{m}^* := \mathbf{m}^\lambda_1 \upharpoonright \mathcal{P}^-(n+1) = \mathbf{m}^\lambda_2 \upharpoonright \mathcal{P}^-(n+1)$. We claim that $\mathbf{m}^*$ is a uniqueness base. From that it will directly follow that $\mathbf{m}_0$ is also a uniqueness base, as desired. We would like to use that $\mathfrak{i}^{\mathrm{brim}}$ has $(\lambda, n+1)$-uniqueness, but we can't do so directly since $\mathbf{m}^*$ may not be brimmed (all the models in the system are brimmed, but the extensions between the models may not be). We want to apply Lemma 9.14, where $\mathfrak{i}, \mathfrak{i}^*, J$ there stand for $\mathfrak{i}_\lambda \upharpoonright \mathcal{I}_{n+2}$, $(\mathfrak{i}^{\mathrm{brim}})_\lambda \upharpoonright \mathcal{I}_{n+2}, \mathcal{P}(n+1)$ here. To apply it, we need to check that $\mathfrak{i}$ has $(\lambda, n+2)$-extension.

By Lemma 9.12 and categoricity, it is enough to check that $\mathfrak{i}^{\mathrm{brim}}$ has $(\lambda, n+2)$-extension. We know already that $\mathfrak{i}^{\mathrm{brim}}$ has $(\lambda', n+1)$-existence for all $\lambda' \in [\lambda_1, \lambda_2]$ (when $\lambda' = \lambda_2$, this is a hypothesis, and when $\lambda' \in [\lambda_1, \lambda_2)$, this was derived in the first part of this lemma). In particular, $\mathfrak{i}^{\mathrm{brim}}$

has $(\lambda^+, n+1)$-existence. By Lemma 11.14, $\mathfrak{i}^{\text{brim}}$ has $(\lambda, n+2)$-existence. Since $\mathfrak{i}^{\text{brim}}$ has $(\lambda, n+1)$-uniqueness, we get by Lemma 11.10 that $\mathfrak{i}^{\text{brim}}$ has $(\lambda, n+2)$-extension, as desired.

□

We obtain two sufficient conditions to have all the n-dimensional properties: weak GCH or niceness.

Theorem 11.17 (The brimmed excellence theorem)**.** Let $\langle \lambda_n : n < \omega \rangle$ be an increasing sequence of infinite cardinals such that $2^{\lambda_n} = 2^{<\lambda_{n+1}} < 2^{\lambda_{n+1}}$ for all $n < \omega$. Let $\lambda_\omega := \sup_{n<\omega} \lambda_n$ and let $\Theta := [\lambda_0, \lambda_\omega)$. Let $\mathfrak{i}$ be a very good independence notion with domain Θ. Assume that at least one of the following conditions holds:

(1) $\mathfrak{i}$ is nice.
(2) $\lambda_{n+1} = \lambda_n^+$ for all $n < \omega$.

Then for all $\lambda \in \Theta$, $\mathfrak{i}^{\text{brim}}$ has $(\lambda, <\omega)$-existence and $(\lambda, <\omega)$-uniqueness.

Proof. By Lemma 11.15 (where λ_1, λ_2 there stand for $\lambda_0, \lambda_\omega$ here), for any $\lambda \in \Theta$, $\mathfrak{i}^{\text{brim}}$ has $(\lambda, <\omega)$-existence. Next, we prove by induction on $n < \omega$ that for all $\lambda \in \Theta$, $\mathfrak{i}^{\text{brim}}$ has (λ, n)-uniqueness. When $n = 0$, this is part of Lemma 11.9. Assume now that for all $\lambda \in \Theta$, $\mathfrak{i}$ has (λ, n)-uniqueness. Fix $k < \omega$. By Lemma 11.16 (where λ_1, λ_2 there stand for λ_k, λ_{k+1} here), $\mathfrak{i}^{\text{brim}}$ has $(\lambda, n+1)$-uniqueness for all $\lambda \in [\lambda_k, \lambda_{k+1})$. Since k was arbitrary, this shows that $\mathfrak{i}^{\text{brim}}$ has $(\lambda, n+1)$-uniqueness for all $\lambda \in \Theta$. □

12. Building primes

In this section, we show that prime models over independent systems can be built, provided that we have the n-dimensional extension and uniqueness properties. This is the *prime extension theorem*, Theorem 12.5.

To build primes, we need the following technical concept, which says that a system can in a sense only be extended in an independent way. This is similar to the definition of a *reduced tower* [SV99, 3.1.11]. The differences are that we work with systems and are slightly more local (we work inside a given system).

Definition 12.1. Let $\mathfrak{i} = (\mathfrak{K}, \mathcal{I}, \text{NF})$ be a multidimensional independence relation and let $\mathbf{m}$ be an independent I-system, with $I = J \cup \{v\}$, $J < v$. Let $\mathbf{m}_*$ be an independent J-system. We say that $\mathbf{m}$ is *reduced inside* $\mathbf{m}_*$ if:

(1) $\mathbf{m} \upharpoonright J \leq_{\mathfrak{i}} \mathbf{m}_*$.
(2) Whenever $\mathbf{m}'$ is an independent I-system and $f : \mathbf{m} \to \mathbf{m}'$ is a systems embedding, if:
    (a) f is the identity on J.
    (b) $\mathbf{m} \upharpoonright J \leq_{\mathfrak{i}} \mathbf{m}' \upharpoonright J$.
    (c) $\mathbf{m}' \upharpoonright J \leq_{\mathfrak{i}} \mathbf{m}_*$.
    Then $f(\mathbf{m}) \leq_{\mathfrak{i}} \mathbf{m}'$.

Reduced systems exist:

Lemma 12.2 (Existence of reduced systems)**.** Let $\mathfrak{i}$ be a very good multidimensional independence notion. Let $\Theta := \text{dom}(\mathfrak{K}_{\mathfrak{i}})$. Let $I = J \cup \{v\}$ be in $\mathcal{I}_{\mathfrak{i}}$, $J < v$. Assume that I is an initial segment of $\mathcal{P}(n)$. Let $\lambda \in \Theta$ be such that $\lambda^+ \in \Theta$ and $\mathfrak{i}$ has $(\lambda, n+1)$-extension.

For any independent J-system $\mathbf{m}$ in $\mathfrak{i}_{\lambda^+}$, there exists a $\leq_{\mathfrak{i}}$-increasing chain of I-systems $\langle \mathbf{m}_i : i < \lambda^+ \rangle$ in $\mathfrak{i}_\lambda$ such that each $\mathbf{m}_i$ is reduced in $\mathbf{m}$ and $\bigcup_{i<\lambda^+} \mathbf{m}_i \upharpoonright J = \mathbf{m}$.

Proof. Without loss of generality, $\lambda = \min(\Theta)$. Let $\mathfrak{i}'$ be as given by Lemma 11.5. Let $\langle a_i : i < \lambda^+ \rangle$ be an enumeration of $\mathbf{m}$. We build $\langle \mathbf{m}_i^0 : i < \lambda^+ \rangle$ a $\leq_{\mathfrak{i}}$-increasing continuous chain of independent J-systems in $\mathfrak{i}_\lambda$, $\langle N_i : i < \lambda^+ \rangle$ in $\mathfrak{K}_{\mathfrak{i}_\lambda}$, and $\langle f_{i,j} : i \leq j < \lambda^+ \rangle$ such that for all $i < \lambda^+$:

(1) $f_{i,j} : N_i \to N_j$ form a continuous directed system.
(2) $\mathbf{m}_i^0 \frown N_i$ is an independent I-system.
(3) $f_{i,j}$ fixes $\mathbf{m}_i^0$.
(4) $\mathbf{m}_{i+1}^0$ contains a_i.
(5) If $\mathbf{m}_i^0 \frown N_i$ is *not* reduced in $\mathbf{m}$, then $\mathbf{m}_{i+1}^0$, N_{i+1}, and $f_{i,i+1}$ witness it. That is, $\mathbf{m}_i^0 \frown f_{i,i+1}[N_i] \not\leq_{\mathfrak{i}} \mathbf{m}_{i+1}^0 \frown N_{i+1}$.

This is possible using the extension property and taking direct limits at limits (we use strong continuity to see that the direct limit is still an independent system). This is enough: let $\mathbf{m}_{\lambda^+}^0$, f_{i,λ^+}, N_{λ^+} be the direct limit of the system. Note that $\mathbf{m}_{\lambda^+}^0 \frown N_{\lambda^+}$ is an independent I-system in $\mathfrak{i}'$, and since $\mathfrak{K}_{\mathfrak{i}',I}$ is an AEC, this system can be $\leq_{\mathfrak{i}}$-resolved into independent systems of cardinality λ. By Lemma 5.3, this implies that there is a club $C \subseteq \lambda^+$ such that for any $i \in C$, $\mathbf{m}_i^0 \frown f_{i,\lambda^+}[N_i] \leq_{\mathfrak{i}} \mathbf{m}_{\lambda^+}^0 \frown N_{\lambda^+}$. Now if $i \in C$ and $\mathbf{m}_i^0 \frown N_i$ is *not* reduced in $\mathbf{m}$, then by property (5), $\mathbf{m}_i^0 \frown f_{i,i+1}[N_i] \not\leq_{\mathfrak{i}} \mathbf{m}_{i+1}^0 \frown N_{i+1}$. Taking the image of this statement under f_{i+1,λ^+}, we get that $\mathbf{m}_i^0 \frown f_{i,\lambda^+}[N_i] \not\leq_{\mathfrak{i}} \mathbf{m}_{i+1}^0 \frown f_{i+1,\lambda^+}[N_{i+1}]$. By monotonicity 2, this means that $\mathbf{m}_i^0 \frown f_{i,\lambda^+}[N_i] \not\leq_{\mathfrak{i}} \mathbf{m}_{i+1}^0 \frown N_{\lambda^+}$, a contradiction. Thus for any $i \in C$, $\mathbf{m}_i^0 \frown N_i$ *is* reduced in $\mathbf{m}$. Let $\langle \alpha_i : i < \lambda^+ \rangle$ be a normal enumeration of C and let $\mathbf{m}_i := \mathbf{m}_{\alpha_i}^0 \frown N_{\alpha_i}$. $\square$

We use reduced systems to build prime ones:

Definition 12.3. Let $\mathfrak{i}$ be a multidimensional independence relation.

(1) Let $I = J \cup \{v\}$ be in $\mathcal{I}_{\mathfrak{i}}$ with $J < v$, and let $\mathbf{m}$ be an independent I-system. We say that $\mathbf{m}$ is *prime* if whenever $\mathbf{m}'$ is an independent J-system with $\mathbf{m} \upharpoonright J = \mathbf{m}' \upharpoonright J$, there exists $f : \mathbf{m} \to \mathbf{m}'$ a systems embedding fixing $\mathbf{m} \upharpoonright J$.
(2) We say that $\mathfrak{i}$ has *prime extension* if for any $I = J \cup \{v\} \in \mathcal{I}_{\mathfrak{i}}$ with $J < v$ and any independent J-system $\mathbf{m}$, there is a *prime* I-system $\mathbf{m}'$ such that $\mathbf{m} \cong \mathbf{m}' \upharpoonright J$. We define variations such as (λ, n)-prime extension as in Definition 10.2.

Lemma 12.4. Let $\mathfrak{i}$ be a very good multidimensional independence notion, let $\Theta := \text{dom}(\mathfrak{K}_{\mathfrak{i}})$. Let $I = J \cup \{v\}$ be in $\mathcal{I}_{\mathfrak{i}}$, $J < v$. Assume that I is an initial segment of $\mathcal{P}(n)$. Let $\lambda \in \Theta$ be such that $\lambda^+ \in \Theta$, $\mathfrak{i}$ has $(\lambda, n+1)$-extension and

$(\lambda, n+1)$-uniqueness. If $\mathfrak{K}_{\mathfrak{i}}$ is categorical in λ^+, then any independent J-system $\mathbf{m}$ in $\mathfrak{i}_{\lambda^+}$ can be extended to a *prime* independent I-system in $\mathfrak{i}_{\lambda^+}$.

Proof. Similar to [She09a, III.4.9] (or [Vas17a, 3.6]), using Lemma 12.2 to get the resolution into reduced triples. □

Theorem 12.5 (The prime extension theorem)**.** Let $\mathfrak{i}$ be a very good multidimensional independence notion, let $\Theta := \text{dom}(\mathfrak{K}_{\mathfrak{i}})$. Let $\lambda \in \Theta$ be such that $\lambda^+ \in \Theta$ and $\mathfrak{K}_{\mathfrak{i}}$ is categorical in λ^+. If $\mathfrak{i}$ has (λ, n)-extension and (λ, n)-uniqueness, then $\mathfrak{i}$ has $(\lambda^+, < n)$-prime extension.

Proof. Immediate from Lemma 12.4. □

13. Excellent classes

In this section, we introduce the definition of an *excellent* AEC (this generalizes the definition in [She83a, She83b]). We show how to obtain excellent AECs from the setups considered earlier (Lemma 13.3) and explain why excellent classes are local and have primes (Theorem 13.9), hence (as we will see in the next section) admit categoricity transfers. We also show how to get excellence in compact AECs (Theorem 13.7) and in $(< \omega)$-extendible good frames, using the weak diamond (Theorem 13.6).

Definition 13.1. Let $\mathfrak{i}$ be a multidimensional independence relation. We call $\mathfrak{i}$ *excellent* if:

(1) $\mathfrak{K}_{\mathfrak{i}}$ is an AEC.
(2) $\mathfrak{i}$ is very good.
(3) $\mathfrak{i}$ has extension and uniqueness.

We say that an AEC $\mathfrak{K}$ is *excellent* if there exists an excellent multidimensional independence relation $\mathfrak{i}$ such that $\mathfrak{K} = \mathfrak{K}_{\mathfrak{i}}$.

Reasonable multidimensional independence relations will be local:

Lemma 13.2. Let $\mathfrak{i}$ be a multidimensional independence relation. If $\mathfrak{K}_{\mathfrak{i}}$ is an AEC, $\mathfrak{i}$ has extension and uniqueness, and $\mathfrak{i}$ satisfies the definition of being very good, except possibly for $\mathfrak{s}(\mathop{\smile}\limits^{}(\mathfrak{i}))$ being a good frame, then $\mathfrak{K}_{\mathfrak{i}}$ is $\text{LS}(\mathfrak{K}_{\mathfrak{i}})$-local.

Proof. As in [She09a, III.1.10] (see also [GK]). □

The next two lemmas tell us how to derive excellence from the conclusion of the brimmed excellence theorem (Theorem 11.17).

Lemma 13.3. Let $\mathfrak{i}$ be a very good multidimensional independence relation. Let $\lambda := \min(\text{dom}(\mathfrak{i}))$. If $\mathfrak{i}$ has $(\lambda, < \omega)$-extension and $(\lambda, < \omega)$-uniqueness, then there exists a unique excellent multidimensional independence relation $\mathfrak{i}'$ such that $(\mathfrak{i}')_\lambda = \mathfrak{i}_\lambda$. In particular, the AEC generated by $(\mathfrak{K}_{\mathfrak{i}})_\lambda$ is excellent.

Proof. Let $\mathfrak{K}$ be the AEC generated by $(\mathfrak{K}_{\mathfrak{i}})_\lambda$. Let $\mathfrak{i}'$ be as described by Lemma 11.5. It satisfies all the conditions in the definition of being very good, except perhaps for existence, uniqueness, and $\mathfrak{s}(\mathop{\smile}\limits^{}(\mathfrak{i}'))$ being a good frame. By Lemma

11.6, $\mathfrak{i}'$ has extension and uniqueness. By Lemma 13.2, $\mathfrak{K}$ is $\mathrm{LS}(\mathfrak{K})$-local. Since $\mathfrak{i}'$ has extension, $\mathfrak{K}$ has in particular amalgamation. By Fact 6.11, there is a good frame $\mathfrak{s}$ with underlying class $\mathfrak{K}$. Let $\mathfrak{t} := \mathfrak{s}(\perp(\mathfrak{i}'))$. It is clear from the definition that $\mathfrak{s}_\lambda = \mathfrak{t}_\lambda$. Moreover, $\mathfrak{t}$ satisfies most of the axioms of a good frame (Fact 7.5). The only axiom that can fail is local character. However since $\perp(\mathfrak{i}')$ is very good, it has local character, so every type does not $\mathfrak{t}$-fork over a model of size λ. By Lemma 6.5 and Remark 6.6, this implies that $\mathfrak{s} = \mathfrak{t}$, so $\mathfrak{t}$ is a good frame. Thus $\mathfrak{i}'$ is very good, and hence excellent. □

Lemma 13.4. Let $\mathfrak{i}$ be a very good multidimensional independence notion and let $\Theta := \mathrm{dom}(\mathfrak{i})$. Let $\lambda := \min(\Theta)$ and let $\mathfrak{K}$ be the AEC generated by $(\mathfrak{K}_{\mathfrak{i}})_\lambda$. If:

(1) $\mathfrak{K}$ is categorical in λ.
(2) $\mathfrak{i}^{\mathrm{brim}}$ has $(\lambda, < \omega)$-existence and $(\lambda, < \omega)$-uniqueness.

Then $\mathfrak{K}$ is excellent.

Proof. By Lemma 11.10, $\mathfrak{i}^*$ also has strong $(\lambda, < \omega)$-uniqueness. By Theorems 9.12 and 9.15, we obtain that $\mathfrak{i}$ also has $(\lambda, < \omega)$-extension and $(\lambda, < \omega)$-uniqueness. Now apply Lemma 13.3. □

Assuming a weak version of the generalized continuum hypothesis, we obtain excellence in $(< \omega)$-extendible very good frames. We will use the following notation:

Definition 13.5. For S a class of cardinals, we write $\mathrm{WGCH}(S)$ for the statement "$2^\lambda < 2^{\lambda^+}$ for all $\lambda \in S$". We write WGCH instead of WGCH(Card), where Card is the class of all cardinals.

Theorem 13.6. Let $\mathfrak{s}$ be a $(< \omega)$-extendible categorical very good λ-frame. Let $\mathfrak{K}$ be the AEC generated by $\mathfrak{K}_{\mathfrak{s}}$. If $\mathrm{WGCH}([\lambda, \lambda^{+\omega}))$ holds, then $\mathfrak{K}$ is excellent.

Proof. Let $\Theta := [\lambda, \lambda^{+\omega})$. By Fact 7.20(4), there is a very good categorical frame $\mathfrak{t}$ such that $\mathrm{dom}(\mathfrak{t}) = \Theta$ and $\mathfrak{t}_{\lambda^{+n}} = \mathfrak{s}^{+n}$ for all $n < \omega$. By definition of being very good, there is a very good two-dimensional independence notion $\perp$ on $\mathfrak{K}_{\mathfrak{t}}$.

Fix $\mathfrak{i}$ and $\mathfrak{i}^*$ satisfying the conclusion of Theorem 11.3. By Lemma 11.17, we have that $\mathfrak{i}^*$ has $(\lambda, < \omega)$-existence and $(\lambda, < \omega)$-uniqueness. Now apply Lemma 13.4. □

In a compact AEC, excellence follows from categoricity (without any cardinal arithmetic hypothesis)

Theorem 13.7. Let $\mathfrak{K}$ be a compact AEC. If $\mathfrak{K}$ is categorical in some $\mu > \mathrm{LS}(\mathfrak{K})$, then $\mathfrak{K}^{\mathrm{LS}(\mathfrak{K})^{+6}\text{-sat}}$ is an excellent AEC.

Proof. By Corollary 4.8, $\mathfrak{K}$ is categorical in a proper class of cardinals, so we might as well assume that $\mu \geq \beth_\omega(\mathrm{LS}(\mathfrak{K}))$. Fix $\mathfrak{i}$ satisfying the conclusion of Theorem 11.13. Let $\Theta := [\mathrm{LS}(\mathfrak{K})^{+6}, \beth_\omega(\mathrm{LS}(\mathfrak{K})))$. By Lemma 11.17, we have that $\mathfrak{i}^{\mathrm{brim}}$ has $(\mathrm{LS}(\mathfrak{K})^{+6}, < \omega)$-existence and $(\mathrm{LS}(\mathfrak{K})^{+6}, < \omega)$-uniqueness. The result now follows from Lemma 13.4. □

An appropriate subclass of saturated models of an excellent class will always have primes in the following sense:

Definition 13.8 ([She09a, III.3.2]). Let $\mathfrak{K}$ be an abstract class.

(1) A *prime triple* is a triple (a, M, N) such that $M \leq_{\mathfrak{K}} N$, $a \in |N|$, and whenever $\mathbf{tp}(b/M; N') = \mathbf{tp}(a/M; N)$, there exists a $\mathfrak{K}$-embedding $f : N \underset{M}{\longrightarrow} N'$ such that $f(b) = a$.
(2) $\mathfrak{K}$ *has primes* if for any $p \in \mathscr{S}(M)$ there exists a prime triple (a, M, N) such that $p = \mathbf{tp}(a/M; N)$.

Theorem 13.9 (Structure of excellent AECs). If $\mathfrak{K}$ is an excellent AEC, then $\mathfrak{K}$ is not empty, has amalgamation, joint embedding, no maximal models and is $\text{LS}(\mathfrak{K})$-local. Moreover, $\mathfrak{K}^{\text{LS}(\mathfrak{K})^+\text{-sat}}$ has primes.

Proof. Let $\lambda := \text{LS}(\mathfrak{K})$. Let $\mathfrak{i}$ be an excellent multidimensional independence relation with $\mathfrak{K}_{\mathfrak{i}} = \mathfrak{K}$. $\mathfrak{K}$ is not empty and has no maximal models because $\mathfrak{i}$ has 1-extension, and $\mathfrak{K}$ has amalgamation and joint embedding because $\mathfrak{i}$ has 1-uniqueness (see Lemma 10.5). By Lemma 13.2, $\mathfrak{K}$ is $\text{LS}(\mathfrak{K})$-local. Let $\mathfrak{i}^*$ be the restriction of $\mathfrak{i}$ to the fragmented AEC $\mathfrak{K}^*$ such that $\mathfrak{K}^*_{\leq \lambda} = \mathfrak{K}_\lambda$ and $\mathfrak{K}^*_{>\lambda} = \mathfrak{K}^{\lambda^+\text{-sat}}$. It is easy to check that $\mathfrak{i}^*$ is still a very good multidimensional independence relation with extension and uniqueness. By Theorem 12.5, $\mathfrak{i}^*$ has $(\lambda^+, < \omega)$-prime extension. By standard arguments similar to those in [She83b, §5], one gets that $\mathfrak{i}^*_{\geq \lambda^+}$ has prime extension.

Now by [She09a, III.4.9] (see also [Vas17a, 3.6]), $\mathfrak{K}^*_{\lambda^+}$ has primes. It remains to check that $\mathfrak{K}^*_{>\lambda^+}$ has primes. Let $M \in \mathfrak{K}^*_{>\lambda^+}$ and let $p \in \mathscr{S}(M)$. Let $N \in \mathfrak{K}^*$ be such that $M \leq_{\mathfrak{K}} N$ and N realizes p, say with a. By local character for $\mathop{\smile}\limits^{\textstyle\mid} := \mathop{\smile}\limits^{\textstyle\mid}(\mathfrak{i}^*)$, there exists $M_0, N_0 \in \mathfrak{K}^*_\lambda$ such that $a \in |N_0|$ and $M \overset{N}{\underset{M_0}{\mathop{\smile}\limits^{\textstyle\mid}}} N_0$. Since $\mathfrak{K}^*_{\lambda^+}$ has primes, we may pick $N_0^* \leq_{\mathfrak{K}} N_0$ such that (a, M_0, N_0) is a prime triple in $\mathfrak{K}^*$. By monotonicity, we still have $M \overset{N}{\underset{M_0}{\mathop{\smile}\limits^{\textstyle\mid}}} N_0^*$, so by prime extension (applied to the independent system $\mathbf{m}$ consisting of M_0, M, and N_0^*), there exists a prime model N^* over $\mathbf{m}$, which can be taken to be contained inside N. Now check that (a, M, N^*) is a prime triple. $\square$

14. Excellence and categoricity: the main theorems

In this section, we combine the results derived so far about excellence with known categoricity transfers to obtain the main theorems of this paper. In addition to Facts 3.25, 3.26, the facts we will use about categoricity are:

Fact 14.1. Let $\mathfrak{K}$ be an $\text{LS}(\mathfrak{K})$-local AEC with amalgamation, and arbitrarily large models. Assume that $\mathfrak{K}$ is categorical in *some* $\mu > \text{LS}(\mathfrak{K})$. Then:

(1) [Vas17b, 10.9][6] If $\mathfrak{K}$ has primes then $\mathfrak{K}$ is categorical in all $\mu' \geq \mu$.
(2) [Vas17b, 10.3,10.4][7] If $\mathfrak{K}$ is categorical in $\text{LS}(\mathfrak{K})$ and μ is a successor, then $\mathfrak{K}$ is categorical in *all* $\mu' \geq \text{LS}(\mathfrak{K})$.

We obtain the following categoricity transfer for excellent classes:

[6]The main ideas of the proof appear in [Vas17d, Vas18].

[7]The upward part of the transfer (i.e. getting categoricity in $\mu' \geq \mu$) is due to Grossberg and VanDieren [GV06c, GV06a].

Theorem 14.2. Let $\mathfrak{K}$ be an excellent AEC. If $\mathfrak{K}$ is categorical in *some* $\mu > \mathrm{LS}(\mathfrak{K})$, then there exists $\chi < h(\mathrm{LS}(\mathfrak{K}))$ such that $\mathfrak{K}$ is categorical in *all* $\mu' \geq \min(\mu, \chi)$. If in addition $\mathfrak{K}$ is categorical in $\mathrm{LS}(\mathfrak{K})$, then $\mathfrak{K}$ is categorical in all $\mu' > \mathrm{LS}(\mathfrak{K})$.

Proof. By Fact 3.26, we may assume that $\mu > \mathrm{LS}(\mathfrak{K})^+$. By Fact 14.1 and Theorem 13.9, $\mathfrak{K}^{\mathrm{LS}(\mathfrak{K})^+\text{-sat}}$ is categorical on a tail of cardinals. By Fact 3.25, $\mathfrak{K}$ is categorical in all $\mu' \geq \min(\mu, \chi)$. If $\mathfrak{K}$ is categorical in $\mathrm{LS}(\mathfrak{K})$, then it will be categorical in a successor, hence by Fact 14.1 again, $\mathfrak{K}$ will be categorical in all cardinals above $\mathrm{LS}(\mathfrak{K})$. $\square$

When working inside a $(<\omega)$-extendible categorical good λ-frame, it is known how to transfer categoricity across finite successors of λ, so we can be more precise:

Lemma 14.3. Let $\mathfrak{s}$ be a $(<\omega)$-extendible categorical good λ-frame. Let $\mathfrak{K}$ be the AEC generated by $\mathfrak{K}_{\mathfrak{s}}$. Let $n < \omega$ and let $\mathfrak{K}^*$ be the AEC generated by $\mathfrak{K}_{\mathfrak{s}^{+n}}$. If $\mathfrak{K}^*$ is excellent, then:

(1) $\mathfrak{K}$ has arbitrarily large models.
(2) If in addition $\mathfrak{K}$ is categorical in *some* $\mu > \lambda$, then $\mathfrak{K}$ is categorical in *all* $\mu' > \lambda$, $\mathfrak{K}_{\geq \mu}$ has amalgamation, no maximal models, and is μ-local.

Proof. If $\mu \leq \lambda^{+n}$, then μ is a successor cardinal, so by Fact 6.9 we get that $\mathfrak{K}$ is also categorical in $\lambda^{+(n+1)}$. Thus we can assume that $\mu > \lambda^{+n}$. Note that $\mathfrak{K}^*$ is categorical in $\lambda^{+n} = \mathrm{LS}(\mathfrak{K}^*)$. By Theorem 14.2, $\mathfrak{K}^*$ is categorical in all $\mu' \geq \lambda^{+n}$. By Fact 6.9, this implies that $\mathfrak{K}$ is categorical in all $\mu' > \lambda$. That $\mathfrak{K}_{\geq \mu}$ has amalgamation, no maximal models, and is μ-local is by Theorem 13.9. $\square$

Thus assuming some instances of the weak diamond, we obtain the following eventual categoricity transfer for $(<\omega)$-extendible frames:

Corollary 14.4. Let $\mathfrak{s}$ be a $(<\omega)$-extendible categorical good λ-frame. Let $\mathfrak{K}$ be the AEC generated by $\mathfrak{K}_{\mathfrak{s}}$. Assume $\mathrm{WGCH}([\lambda^{+n}, \lambda^{+\omega}))$ holds for some $n < \omega$. Then $\mathfrak{K}$ has arbitrarily large models and if $\mathfrak{K}$ is categorical in *some* $\mu > \lambda$, then $\mathfrak{K}$ is categorical in *all* $\mu' > \lambda$ and moreover $\mathfrak{K}_{\geq \mu}$ has amalgamation, no maximal models, and is μ-local.

Proof. Let $\mathfrak{t} := \mathfrak{s}^{+(n+3)}$. By Fact 7.20(2), $\mathfrak{t}$ is very good. By Theorem 13.6, the AEC $\mathfrak{K}^*$ generated by $\mathfrak{K}_{\mathfrak{t}}$ is excellent. Now apply Lemma 14.3. $\square$

Specializing to compact AECs, we get:

Theorem 14.5. Let $\mathfrak{K}$ be a compact AEC. Let $\mu > \mathrm{LS}(\mathfrak{K})$. If $\mathfrak{K}$ is categorical in μ, then there exists $\chi < h(\mathrm{LS}(\mathfrak{K}))$ such that $\mathfrak{K}$ is categorical in all $\mu' \geq \min(\mu, \chi)$.

Proof. By Corollary 4.8, $\mathfrak{K}$ has amalgamation, no maximal models, and is categorical in a proper class of cardinals. We will also use without comments Fact 3.24, which says in particular that $\mathfrak{K}^{\lambda\text{-sat}}$ is an AEC for any $\lambda > \mathrm{LS}(\mathfrak{K})$. By Theorem 13.7, $\mathfrak{K}^{\mathrm{LS}(\mathfrak{K})^{+6}\text{-sat}}$ is excellent. By Theorem 14.2, $\mathfrak{K}^{\mathrm{LS}(\mathfrak{K})^{+6}\text{-sat}}$ is categorical in all high-enough cardinals. Since $\mathfrak{K}$ itself is categorical in a proper class of cardinals, this implies that $\mathfrak{K}^{\mathrm{LS}(\mathfrak{K})^+\text{-sat}}$ is also categorical in all high-enough cardinals, and so in particular in a high-enough successor cardinal. Now $\mathfrak{K}^{\mathrm{LS}(\mathfrak{K})^+\text{-sat}}$ is categorical in $\mathrm{LS}(\mathfrak{K})^+ = \mathrm{LS}(\mathfrak{K}^{\mathrm{LS}(\mathfrak{K})^+\text{-sat}})$, and hence by Fact 14.1 it is categorical in *all* $\mu' \geq \mathrm{LS}(\mathfrak{K})^+$. Now apply Morley's omitting type theorem for AECs (Fact 3.25) to get the desired conclusion about categoricity in $\mathfrak{K}$. $\square$

Corollary 14.6. Let $\mathfrak{K}$ be a compact AEC. If $\mathfrak{K}$ is categorical in *some* $\mu \geq h(\mathrm{LS}(\mathfrak{K}))$, then $\mathfrak{K}$ is categorical in *all* $\mu' \geq h(\mathrm{LS}(\mathfrak{K}))$.

Proof. This is a special case of Theorem 14.5. $\square$

Corollary 14.7.

(1) Let $\mathfrak{K}$ be an AEC and let $\kappa > \mathrm{LS}(\mathfrak{K})$ be a strongly compact cardinal. If $\mathfrak{K}$ is categorical in *some* $\mu \geq h(\kappa)$, then $\mathfrak{K}$ is categorical in *all* $\mu' \geq h(\kappa)$.
(2) Let T be a theory in $\mathbb{L}_{\kappa,\omega}$, κ a strongly compact cardinal. If T is categorical in *some* $\mu \geq h(|T| + |\tau(T)| + \kappa)$, then T is categorical in *all* $\mu' \geq h(|T| + |\tau(T)| + \kappa)$.
(3) Let ψ be an $\mathbb{L}_{\kappa,\omega}$ sentence. If ψ is categorical in *some* $\mu \geq h(\kappa)$, then ψ is categorical in *all* $\lambda' \geq h(\kappa)$.

Proof. The first two parts are by Fact 4.3 and Corollary 14.6. The third part follows from the second because an $\mathbb{L}_{\kappa,\omega}$ sentence has a vocabulary of size strictly less than κ. $\square$

Regarding AECs with amalgamation, the following lemma says that (assuming WGCH) with a little bit of locality, we can transfer categoricity. Note that the proof only uses amalgamation below the categoricity cardinal, but transfers it above.

Lemma 14.8. Let $\mathfrak{K}$ be an AEC with arbitrarily large models. Assume $\mathrm{WGCH}([\mathrm{LS}(\mathfrak{K}), \mathrm{LS}(\mathfrak{K})^{+\omega}))$. Let $\mu > \mathrm{LS}(\mathfrak{K})^+$ and assume that $\mathfrak{K}_{<\max(\mu, \mathrm{LS}(\mathfrak{K})^{+\omega})}$ has amalgamation and no maximal models. If $\mathfrak{K}^{\mathrm{sat}}_{(\mathrm{LS}(\mathfrak{K}), \mathrm{LS}(\mathfrak{K})^{+\omega})}$ is $\mathrm{LS}(\mathfrak{K})$-local and $\mathfrak{K}$ is categorical in μ, then there is $\chi < h(\mathrm{LS}(\mathfrak{K}))$ such that $\mathfrak{K}$ is categorical in all $\mu' \geq \min(\mu, \chi)$.

Proof. We use without further comments (Fact 3.24) that for any $\lambda \in (\mathrm{LS}(\mathfrak{K}), \mu]$, $\mathfrak{K}^{\lambda\text{-sat}}$ is an AEC with Löwenheim-Skolem-Tarski number λ. In particular, the model of cardinality μ is saturated. By Fact 6.2, there is a (categorical) good frame $\mathfrak{s}$ on $\mathfrak{K}^{\mathrm{sat}}_{[\mathrm{LS}(\mathfrak{K})^+, \mathrm{LS}(\mathfrak{K})^{+\omega})}$. By Fact 7.16, $\mathfrak{s}$ is $(< \omega)$-extendible. By Corollary 14.4, $\mathfrak{K}^{\mathrm{LS}(\mathfrak{K})^+\text{-sat}}$ is categorical in all $\mu' \geq \mathrm{LS}(\mathfrak{K})^+$ and has amalgamation above μ. In particular, $\mathfrak{K}$ is categorical in all $\mu' \geq \mu$ and $\mathfrak{K}_{\geq \mu}$ has amalgamation. Since we knew that $\mathfrak{K}_{<\mu}$ had amalgamation, we have that $\mathfrak{K}$ has amalgamation. By Morley's omitting type theorem for AECs (Fact 3.25), there is $\chi < h(\mathrm{LS}(\mathfrak{K}))$ such that $\mathfrak{K}$ is categorical in all $\mu' \geq \min(\mu, \chi)$. $\square$

Since locality can be derived from high-enough categoricity, we obtain:

Theorem 14.9. Let $\mathfrak{K}$ be an AEC with arbitrarily large models. Let $\mu \geq \aleph_{\mathrm{LS}(\mathfrak{K})^+}$. Assume there exists unboundedly-many $\lambda < \aleph_{\mathrm{LS}(\mathfrak{K})^+}$ such that $\mathrm{WGCH}([\lambda, \lambda^{+\omega}))$ holds. If $\mathfrak{K}$ is categorical in μ and $\mathfrak{K}_{<\mu}$ has amalgamation and no maximal models, then there exists $\chi < \aleph_{\mathrm{LS}(\mathfrak{K})^+}$ such that $\mathfrak{K}$ is categorical in all $\mu' \geq \min(h(\chi), \mu)$.

Proof. By Fact 3.24(2) (where λ there stands for $\aleph_{\text{LS}(\mathfrak{K})^+}$ here), there is $\chi_0 < \aleph_{\text{LS}(\mathfrak{K})^+}$ such that $\mathfrak{K}^{\text{sat}}_{(\chi_0, \aleph_{\text{LS}(\mathfrak{K})^+})}$ is χ_0-local. By Lemma 14.8 applied to $\mathfrak{K}_{\geq \chi_0}$, we get the result. $\square$

Corollary 14.10. Assume WGCH (see Definition 13.5). Let $\mathfrak{K}$ be an AEC with amalgamation. If $\mathfrak{K}$ is categorical in *some* $\mu \geq h(\aleph_{\text{LS}(\mathfrak{K})^+})$, then $\mathfrak{K}$ is categorical in *all* $\mu' \geq h(\aleph_{\text{LS}(\mathfrak{K})^+})$.

Proof. This is a special case of Theorem 14.9. It is well-known that AECs with a model in $h(\text{LS}(\mathfrak{K}))$ have arbitrarily large models. Moreover, by considering the equivalence relation "M and N embed into a common model", we can partition the AEC into disjoint classes, each of which have joint embedding. One can then work inside the unique class with arbitrarily large models, which will have joint embedding and no maximal models. It is then easy to see that the original class must also be categorical in all high-enough cardinals. See for example [Vas16a, 10.13] for the details of this argument. $\square$

We can also obtain results from completely local hypotheses about the number of models in the λ^{+n}'s. This was already stated in [She09a, III.12.43]. Below, $\mathbb{I}(\mathfrak{K}, \lambda)$ denotes the number of models (up to isomorphism) in $\mathfrak{K}_\lambda$. On μ_{unif}, see [She09b, VII.0.4] for the definition and [She09b, VII.9.4] on what is known (morally, $\mu_{\text{unif}}(\lambda^+, 2^\lambda) = 2^{\lambda^+}$ when $2^\lambda < 2^{\lambda^+}$).

Theorem 14.11. Let $\mathfrak{K}$ be an AEC and let $\lambda \geq \text{LS}(\mathfrak{K})$ be such that $\text{WGCH}([\lambda, \lambda^{+\omega}))$ holds. Assume that $\mathfrak{K}$ is categorical in λ, λ^+, $\mathfrak{K}_{\lambda^{++}} \neq \emptyset$, and $\mathbb{I}(\mathfrak{K}, \lambda^{+n}) < \mu_{\text{unif}}(\lambda^{+n}, 2^{\lambda^{+(n-1)}})$ for all $n \in [2, \omega)$. Then $\mathfrak{K}$ is categorical in all $\mu > \lambda$.

Proof. As in the proof of [She09a, II.9.2], we get that there is an ω-successful good$^+$ λ-frame $\mathfrak{s}$ on $\mathfrak{K}_\lambda$. This implies (Fact 7.20(1)) that $\mathfrak{s}$ is $(< \omega)$-extendible. Now apply Lemma 14.3. $\square$

When $\lambda = \aleph_0$, the hypotheses can be weakened and we obtain the following generalization of the main result of [She83a, She83b]:

Theorem 14.12. Let $\mathfrak{K}$ be a $\text{PC}_{\aleph_0}$ AEC (so $\text{LS}(\mathfrak{K}) = \aleph_0$). Assume $\text{WGCH}([\aleph_0, \aleph_\omega))$. Assume that $\mathfrak{K}$ is categorical in $\aleph_0$, $1 \leq \mathbb{I}(\mathfrak{K}, \aleph_1) < 2^{\aleph_1}$, and $\mathbb{I}(\mathfrak{K}, \aleph_n) < \mu_{\text{unif}}(\aleph_n, 2^{\aleph_{n-1}})$ for all $n \in [2, \omega)$. Then $\mathfrak{K}$ has arbitrarily large models. Moreover, if $\mathfrak{K}$ is categorical in *some* uncountable cardinal, then $\mathfrak{K}$ is categorical in *all* uncountable cardinals.

Proof. As in the proof of Theorem 14.11, using [She09a, II.9.3] instead of [She09a, II.9.2]. $\square$

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E-mail address: shelah@math.huji.ac.il

URL: http://shelah.logic.at

Einstein Institute of Mathematics, Edmond J. Safra Campus, Givat Ram, The Hebrew University of Jerusalem, Jerusalem, 91904, Israel, and Department of Mathematics, Hill Center - Busch Campus, Rutgers, The State University of New Jersey, 110 Frelinghuysen Road, Piscataway, NJ 08854-8019, USA

E-mail address: sebv@math.harvard.edu

URL: http://math.harvard.edu/~sebv/

Department of Mathematics, Harvard University, Cambridge, Massachusetts, USA