INDISCERNIBLE EXTRACTION AND MORLEY SEQUENCES

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Abstract. We present a new proof of the existence of Morley sequences in simple theories. We avoid using the Erdős-Rado theorem and instead use an idea of Gaifman from the sixties. The proof shows that the basic theory of forking in simple theories can be developed inside $\left\langle H((2^{2^{|T|}})^+), \in \right\rangle$, answering a question of Grossberg, Iovino and Lessmann.

1. Introduction

Shelah [She80, Lemma 9.3] has shown that, in a simple first-order theory T, Morley sequences exist for every type. The proof proceeds by first building an independent sequence of length $\beth_{\left(2^{|T|}\right)^+}$ for the given type and then using the Erdős-Rado theorem together with Morley's method to extract the desired indiscernibles.

After slightly improving on the length of the original independent sequence [GIL02, Appendix A], Grossberg, Iovino and Lessmann observed that, in contrast, most of the theory of forking in a stable first-order theory T can be carried out inside $\langle H(\chi), \in \rangle$ for $\chi := \left(2^{2^{|T|}}\right)^+$. The authors then asked whether the same could be said about simple theories, and so in particular whether there was another way to build Morley sequences there.

We answer the question in the affirmative by showing how to build a Morley sequence from any infinite independent sequence. Our construction actually works whenever forking has what we call *dual finite character*, a property that holds in simple theories and has seldom been studied before.

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2. Preliminaries

For the rest of this paper, fix a complete first-order theory T in a language $L(T)$ and work inside its monster model $\mathfrak{C}$. We write $|T|$ for $|L(T)| + \aleph_0$. We denote by $\mathrm{Fml}(L(T))$ the set of first-order formulas in the language $L(T)$. For a tuple $\bar{a}$ in $\mathfrak{C}$ and ϕ a formula, we write $\models \phi[\bar{a}]$ instead of $\mathfrak{C} \models \phi[\bar{a}]$.

When I is a linearly ordered set, $(\bar{a}_i)_{i\in I}$ are tuples, and $i \in I$, we write $\bar{a}_{<i}$ for $(\bar{a}_j)_{j<i}$. It is often assumed without comments that all the $\bar{a}_i$s have the same (finite) arity.

One could, in the spirit of reverse mathematics [Fri74], try to find the exact proof-theoretic strength of some of the facts of simple theories, like Harnik did for stable theories [Har85, Har87]. We do not attempt this here and focus on the model theory. We let $\chi := \left(2^{2^{|T|}}\right)^+$ and choose a large-enough fragment ZFC^0 of ZFC that is satisfied by $\langle H(\chi), \in \rangle$ (we assume ZFC^0 includes the axiom of choice). We do not attempt to make ZFC^0 explicit, although we suspect a suitably modified version of Woodin's ZFC^* [Woo10] would work.

For simplicity, we will actually state and prove our results in ZFC. We point out, however, that (except for Fact 4, Corollary 5, and Proposition 6) they can all be formalized in ZFC^0 if one makes some easy changes in the statements and proofs (e.g. one will need to bound the sizes of some of the sets under consideration, and work in a more local monster model).

We assume the reader is familiar with forking. As a brief reminder, forking is an independence notion, originally developed by Shelah to solve the stability spectrum problem, which has turned out to be central in classification theory. We will use the following definition:

Definition 1 (Forking)**.** Let $\bar{b}$ be a tuple, and let A be a set. A formula $\phi(\bar{x}, \bar{b})$ *divides over* A if there is a sequence $\langle \bar{b}_i \mid i < \omega \rangle$ of indiscernibles over A such that $\bar{b}_0 = \bar{b}$, and $\{\phi(\bar{x}, \bar{b}_i) \mid i < \omega\}$ is inconsistent.

A formula $\phi(\bar{x}, \bar{b})$ *forks over* A if there are formulas $\phi_0(\bar{x}, \bar{b}_0), \ldots, \phi_{m-1}(\bar{x}, \bar{b}_{m-1})$, each dividing over A, such that $\phi(\bar{x}, \bar{b}) \vdash \bigvee_{i<m} \phi_i(\bar{x}, \bar{b}_i)$.

A type p *forks over* A if $p \vdash \phi(\bar{x}, \bar{b})$ for some formula $\phi(\bar{x}, \bar{b})$ that forks over A.

The following concepts are central:

Definition 2 (Morley sequence)**.** Let I be a linearly ordered set. Let $\mathbf{I} := \langle \bar{a}_i \mid i \in I \rangle$ be a sequence of finite tuples of the same arity. Let $A \subseteq B$ be sets, and let $p \in S(B)$ be a type that does not fork over A.

$\mathbf{I}$ is said to be an *independent sequence for p over A* if:

(1) For all $i \in I$, $\bar{a}_i \models p$.
(2) For all $i \in I$, $\mathrm{tp}(\bar{a}_i / B\bar{a}_{<i})$ does not fork over A.

$\mathbf{I}$ is said to be a *Morley sequence for p over A* if:

(1) $\mathbf{I}$ is an independent sequence for p over A.
(2) $\mathbf{I}$ is indiscernible over B.

The following fact about forking (which holds in all complete first-order theories) follows from the extension property and the compactness theorem:

Fact 3 (Existence of independent sequences)**.** Let $A \subseteq B$ be sets, and let $p \in S(B)$ be a type that does not fork over A. Let I be a linearly ordered set. Then there is an independent sequence $\mathbf{I} := \langle \bar{a}_i \mid i \in I \rangle$ for p over A.

3. Indiscernible extraction

Fact 3 tells us it is easy to build independent sequences. What about Morley sequences? If a sufficiently long sequence always contains an indiscernible subsequence, the existence of Morley sequences follows from Fact 3. This is the case in stable theories, but not in general: There are counterexamples among both simple [She85, p. 209] and dependent [KS] theories. Thus a different approach is needed in the unstable case. Shelah observed [She80, Lemma 9.3] the following:

Fact 4 (The indiscernible extraction theorem)**.** Let A be a set, and let I be a linearly ordered set. Let $\gamma := \left(2^{|T|+|A|}\right)^+$, $\mu := \beth_\gamma$, and let $\langle \bar{a}_j \mid j < \mu \rangle$ be a sequence of finite tuples of the same arity. Then there exists a sequence $\mathbf{I} := \left\langle \bar{b}_i \mid i \in I \right\rangle$, indiscernible over A such that:

For any $i_0 < \ldots < i_{n-1}$ in I, there exists $j_0 < \ldots < j_{n-1} < \mu$ so that $\mathrm{tp}(\bar{b}_{i_0} \ldots \bar{b}_{i_{n-1}} / A) = \mathrm{tp}(\bar{a}_{j_0} \ldots \bar{a}_{j_{n-1}} / A)$.

Corollary 5 (Existence of Morley sequences in arbitrary theories)**.** Let $A \subseteq B$. Let $p \in S(B)$ be a type that does not fork over A. Let I be a linearly ordered set. Then there is a Morley sequence $\mathbf{I} := \left\langle \bar{b}_i \mid i \in I \right\rangle$ for p over A.

Proof sketch. Build a long-enough independent sequence for p over A using Fact 3, then use the monotonicity, finite character and invariance properties of forking to see that the extracted sequence $\mathbf{I}$ given by Fact 4 is as desired. $\square$

It is shown in [GIL02, Theorem A.2] that the length μ of the original sequence in Fact 4 can be decreased to $\beth_{\delta(|T|+|A|)}$, where $\delta(\lambda)$ is defined to be the least ordinal not definable in the logic $L_{\lambda^+,\omega}$. The proof is set-theoretic. The main idea is to use the Erdős–Rado theorem infinitely many times inside an ill-founded model of (a large fragment of) set theory.

Observe that in the indiscernible extraction theorem, the bound $\mu := \beth_{\delta(|T|+|A|)}$ is optimal:

Proposition 6. For every infinite cardinal λ, and every $\mu < \beth_{\delta(\lambda)}$, there is a theory T with $|T| = \lambda$ such that the indiscernible extraction theorem (with $A = \emptyset$) fails for sequences of length μ.

Proof. Fix λ. Pick $\mu < \beth_{\delta(\lambda)}$. By building on Morley's idea for lower bounds of Hanf numbers (see [She90], Theorem VII.5.4), we can get a complete theory T of size λ with built-in Skolem functions, and a type p such that $\mathrm{EC}(T, p)$ (the class of all models of T omitting the type p) contains a model M of size $\geq \mu$, but no model of size $\beth_{\delta(\lambda)}$. Without loss of generality, $||M|| = \mu$. Let $\{a_j \mid j < \mu\}$ enumerate M. Work in the monster model for T. By assumption, a_j does not realize p, for all $j < \mu$.

Assume for a contradiction there exists a sequence of indiscernibles $\mathbf{I} := \left\langle b_i \mid i < \beth_{\delta(\lambda)} \right\rangle$ satisfying the conclusion of the indiscernible extraction theorem. Then in particular, b_i does not realize p for any $i < \beth_{\delta(\lambda)}$. Let $N := \mathrm{EM}(\mathbf{I})$ be the Skolem hull of $\mathbf{I}$. Then $||N|| = \beth_{\delta(\lambda)}$, so by construction, $N \notin \mathrm{EC}(T, p)$. This means there is $i_0 < \ldots < i_n < \beth_{\delta(\lambda)}$ and a term τ such that $\tau(b_{i_0}, \ldots, b_{i_n}) \models p$, so $b_{i_0} \ldots b_{i_n} \models q$, where $q(\bar{x}) := p(\tau(\bar{x}))$. But then for some $j_0 < \ldots < j_n < \mu$, $a_{j_0} \ldots a_{j_n} \models q$, i.e. $\tau(a_{j_0}, \ldots, a_{j_n}) \models p$. But $\tau(a_{j_0}, \ldots, a_{j_n}) \in M \in \mathrm{EC}(T, p)$, so $\tau(a_{j_0}, \ldots, a_{j_n})$ does not realize p, a contradiction. $\square$

Proposition 6 does not rule out a smaller upper bound μ for particular classes of theories: As was hinted at earlier, if T is stable $\mu := \left(2^{|T|+|A|}\right)^+$ is enough (see [She90, Theorem I.2.8]). We do not know if there is also a smaller bound for simple theories. Restricting the initial sequence to be independent may also give additional information.

We give a weaker version of the indiscernible extraction theorem which holds for $\mu = \omega$. This is the key to obtain existence of Morley sequences. The idea comes from an alternative proof of the Ehrenfeucht-Mostowski theorem presented in [Kei70, p. 55]. Keisler describes his proof as a syntactical version of an argument of Gaifman [Gai67].

Theorem 7 (The weak indiscernible extraction theorem)**.** Let A be a set, and let I be a linearly ordered set. Let $\langle \bar{a}_j \mid j < \omega \rangle$ be a sequence of finite tuples of the same arity m. Then there exists a sequence $\mathbf{I} := \left\langle \bar{b}_i \mid i \in I \right\rangle$, indiscernible over A such that:

For any $i_0 < \ldots < i_{n-1}$ in I, for all *finite* $q \subseteq \mathrm{tp}(\bar{b}_{i_0} \ldots \bar{b}_{i_{n-1}}/A)$, there exists $j_0 < \ldots < j_{n-1} < \omega$ so that $\bar{a}_{j_0} \ldots \bar{a}_{j_{n-1}} \models q$.

Proof. By adding new constant symbols to T, we can without loss of generality assume that $A = \emptyset$.

Let U be any nonprincipal ultrafilter on ω. Given a statement P, let $(\forall^* x)P(x)$ stand for "$\{x \in \omega \mid P(x)\} \in U$".

Add new constant symbols $\{\bar{c}_i\}_{i \in I}$ to the language, each of arity m. Consider the set:

$$\begin{aligned}\Gamma := \{\phi(\bar{c}_{i_0}, \ldots, \bar{c}_{i_{n-1}}) \mid\ & i_0 < \ldots < i_{n-1} \text{ in } I,\ \phi(\bar{x}) \in \mathrm{Fml}(L(T)),\\ & (\forall^* j_0) \ldots (\forall^* j_{n-1}) \models \phi[\bar{a}_{j_0}, \ldots, \bar{a}_{j_{n-1}}]\}\end{aligned}$$

Observe that:

(1) For all $i_0 < \ldots < i_{n-1}, i'_0 < \ldots < i'_{n-1}$ in I, $\phi(c_{i_0}, \ldots, c_{i_{n-1}}) \in \Gamma$ if and only if $\phi(c_{i'_0}, \ldots, c_{i'_{n-1}}) \in \Gamma$. This is true by construction.
(2) For all $\phi(\bar{x}) \in \mathrm{Fml}(L(T))$, and any $i_0 < \ldots < i_{n-1}$ in I, exactly one of $\phi(c_{i_0}, \ldots, c_{i_{n-1}})$ or $\neg\phi(c_{i_0}, \ldots, c_{i_{n-1}})$ is in Γ. This is because U is an ultrafilter.
(3) Γ is closed under conjunctions, as U is closed under intersections.
(4) Every formula in Γ has a model, since every set in U is nonempty.

(5) Thus by (2), (3), and (4), Γ is a *complete* and *consistent* set of formulas.

Let $M \models \Gamma$, and set $\bar{b}_i := \bar{c}_i^M$. By (1) and (5), the $\bar{b}_i$s form a sequence of indiscernibles.

Now, let $i_0 < \ldots < i_{n-1}$ in I be given, and let q be a finite subset of $\mathrm{tp}(\bar{b}_{i_0} \ldots \bar{b}_{i_{n-1}}/\emptyset)$. Without loss of generality, q consists of a single formula $\phi(\bar{x})$. Since Γ is complete, $\phi(\bar{c}_{i_0}, \ldots, \bar{c}_{i_{n-1}}) \in \Gamma$, so using that U is nonprincipal and the definition of Γ, we can find $j_0 < \ldots < j_{n-1}$ so that $\models \phi[\bar{a}_{j_0}, \ldots, \bar{a}_{j_{n-1}}]$. This proves that $\mathbf{I} := \left\langle \bar{b}_i \mid i \in I \right\rangle$ is as required. $\square$

The reader should be wary of concluding the existence of Morley sequences directly from Theorem 7 and the finite character of forking. Indeed, Theorem 7 does not give us enough invariance to imitate the proof of Corollary 5. In fact, we suspect that a sequence extracted from an independent sequence using Theorem 7 need not in general be Morley.

4. Extracting Morley sequences in simple theories

To continue, we become interested in the following property of forking:

Definition 8 (Dual finite character)**.** Forking is said to have *dual finite character (DFC)* if whenever $p := \mathrm{tp}(\bar{c}/A\bar{b})$ forks over A, there exists $\bar{a} \in A$, and a formula $\phi(\bar{x}, \bar{b}, \bar{a}) \in p$ such that $\models \phi[\bar{c}, \bar{b}', \bar{a}]$ implies $\mathrm{tp}(\bar{c}/A\bar{b}')$ forks over A.

Remark 9. If $p := \mathrm{tp}(\bar{c}/A\bar{b})$ forks over A, and $\phi(\bar{x}, \bar{b}, \bar{a}) \in p$ witnesses DFC, then by invariance, whenever $\models \phi[\bar{c}', \bar{b}', \bar{a}]$ and $\mathrm{tp}(\bar{c}'/A) = \mathrm{tp}(\bar{c}/A)$, we have $\mathrm{tp}(\bar{c}'/A\bar{b}')$ forks over A.

Dual finite character says that forking is witnessed by a formula, in a way that allows us to change the *domain* of the type under consideration. This allows us to complete the proof of existence of Morley sequences:

Theorem 10. Assume forking has DFC. Let $A \subseteq B$ be sets. Let $p \in S(B)$ be a type that does not fork over A. Let I be a linearly ordered set. Then there is a Morley sequence $\mathbf{I} := \left\langle \bar{b}_i \mid i \in I \right\rangle$ for p over A.

Proof. Use Fact 3 to build an independent sequence $\mathbf{J} := \langle \bar{a}_j \mid j < \omega \rangle$ for p over A. Let $\mathbf{I} := \left\langle \bar{b}_i \mid i \in I \right\rangle$ be indiscernible over B, as described by Theorem 7. We claim $\mathbf{I}$ is as required.

It is indiscernible over B, and for every $i \in I$, every $\bar{b}_i$ realizes p: If $\bar{b}_i \not\models p$, fix a formula $\phi(\bar{x}, \bar{b}) \in p$ so that $\models \neg\phi[\bar{b}_i, \bar{b}]$. By the defining property of $\mathbf{I}$, there exists $j < \omega$ so that $\models \neg\phi[\bar{a}_j, \bar{b}]$, so $\bar{a}_j \not\models p$, a contradiction.

It remains to see that for every $i \in I$, $p_i := \mathrm{tp}(\bar{b}_i/B\bar{b}_{<i})$ does not fork over A. Assume not, and fix $i \in I$ so that p_i forks over A. Fix $\bar{b} \in B$ and $i_0 < \ldots < i_{n-1} < i$ such that $p'_i := \mathrm{tp}(\bar{b}_i/A\bar{b}_{i_0} \ldots \bar{b}_{i_{n-1}}\bar{b})$ forks over A. Fix $\bar{a} \in A$ and $\phi(\bar{x}, \bar{b}_{i_0} \ldots \bar{b}_{i_{n-1}}\bar{b}, \bar{a}) \in p'_i$ witnessing DFC.

Find $j_0 < \ldots < j_n < \omega$ such that $\models \phi[\bar{a}_{j_n}, \bar{a}_{j_0} \ldots \bar{a}_{j_{n-1}}\bar{b}, \bar{a}]$. Since it has already been observed that $\mathrm{tp}(\bar{a}_{j_n}/A) = \mathrm{tp}(\bar{b}_i/A) = p \restriction A$, Remark 9 implies that $\mathrm{tp}(\bar{a}_{j_n}/A\bar{a}_{j_0} \ldots \bar{a}_{j_{n-1}}\bar{b})$ forks over A, contradicting the independence of $\mathbf{J}$. $\square$

A variation of DFC appears as property A.7' in [Mak84], but we haven't found any other occurrence in the literature. Makkai observed that forking symmetry implies DFC, so this is how we will define simplicity:

Definition 11. A first-order theory T is *simple* if its forking has the symmetry property, i.e. whenever $\mathrm{tp}(\bar{c}/A\bar{b})$ forks over A, $\mathrm{tp}(\bar{b}/A\bar{c})$ forks over A.

This is equivalent to T not having the tree property, or to forking having local character [Kim01, Theorem 2.4]. Moreover, the methods of [Adl09] show that the equivalence can be proven in ZFC^0, without using Morley sequences.

Lemma 12. Assume T is simple. Then forking has DFC.

Proof. Assume $p := \mathrm{tp}(\bar{c}/A\bar{b})$ fork over A. By symmetry, $q := \mathrm{tp}(\bar{b}/A\bar{c})$ forks over A. Fix $\bar{a} \in A$ and $\psi(\bar{y}, \bar{x}, \bar{z})$ such that $\psi(\bar{y}, \bar{c}, \bar{a}) \in q$ witnesses forking, i.e. if $\models \psi[\bar{b}', \bar{c}, \bar{a})$ then $\mathrm{tp}(\bar{b}'/A\bar{c})$ forks over A.

Let $\phi(\bar{x}, \bar{y}, \bar{z}) := \psi(\bar{y}, \bar{x}, \bar{z})$. Then $\phi(\bar{x}, \bar{b}, \bar{a}) \in p$, and if $\models \phi[\bar{c}, \bar{b}', \bar{a}]$, then $\models \psi[\bar{b}', \bar{c}, \bar{a}]$, so $\mathrm{tp}(\bar{b}'/A\bar{c})$ forks over A, so by symmetry, $\mathrm{tp}(\bar{c}/A\bar{b}')$ forks over A. This shows $\phi(\bar{x}, \bar{b}, \bar{a})$ witnesses DFC.

$\square$

Corollary 13 (Existence of Morley sequences in simple theories)**.** Assume T is simple. Let $A \subseteq B$ be sets. Let $p \in S(B)$ be a type that does not fork over A. Let I be a linearly ordered set. Then there is a Morley sequence $\mathbf{I} := \left\langle \bar{b}_i \mid i \in I \right\rangle$ for p over A.

Proof. Combine Lemma 12 and Theorem 10. $\square$

We do not know whether forking having DFC is equivalent to simplicity, or whether forking has DFC in a larger class of theories. The following example shows DFC does not always hold:

Example 14. Let T be the first-order theory of dense linear orderings. Let A be the set of nonzero rational numbers. Let $b_0 < c < b_1$ be positive infinitesimals (i.e. greater than zero but smaller than any positive rational). Then $p := \mathrm{tp}(c/Ab_0b_1)$ divides over A, and for any $\phi(x, b_0, b_1) \in p$, one can find nonzero rationals b'_0, b'_1 such that $\models \phi[c, b'_0, b'_1]$. But then $\mathrm{tp}(c/Ab'_0b'_1) = \mathrm{tp}(c/A)$ does not fork over A. Thus forking does not have DFC in T.

We conclude by answering the question of Grossberg, Iovino and Lessmann mentioned in the introduction. Specifically, [GIL02, Question A.1] asked whether the first three sections of their paper could be formalized inside $\langle H(\chi), \in \rangle$. Corollary 13 establishes the existence of Morley sequences (Theorem 1.14 there). The only other problematic results are the characterization of forking using the D-rank (Corollary 3.21 there), and Corollary 3.20 preceding it. To formalize these in ZFC^0, one can use the (equivalent) definition of the D-rank given in [Wag00, Definition 2.3.4], together with the proof of [Wag00, Proposition 2.3.9].

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