Lattice Coding for Strongly Secure Compute-and-Forward in a Bidirectional Relay

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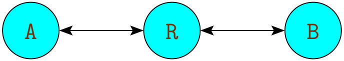
Bidirectional relay



No direct link between user nodes. All links are wireless with unit gain and AWGN.

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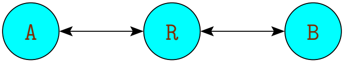

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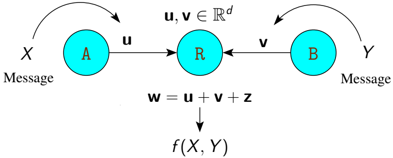
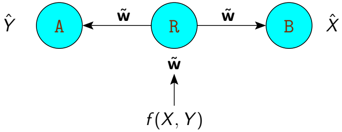


Figure: MAC phase



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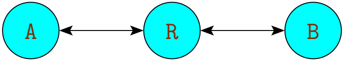

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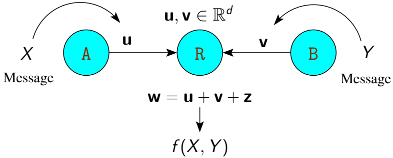

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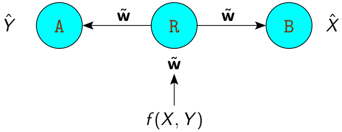

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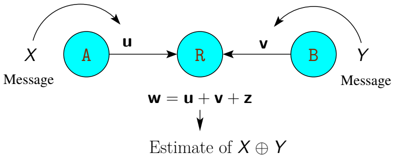
where $\mathbf{z}$ is AWGN with mean zero and variance σ^2.

- Relay acts as passive eavesdropper.
- We want strong secrecy: $\mathcal{I}(X; \mathbf{w}), \mathcal{I}(Y; \mathbf{w}) \to 0$ as $d \to \infty$.

Wilson et al. (2010), Nazer and Gastpar (2011).

Messages X and Y are mapped to elements of a suitably chosen finite Abelian group $(\mathbb{G}, \oplus)$.

Here, $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$, and $f(X, Y) = X \oplus Y$.



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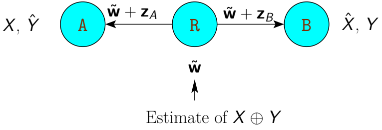

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Compute-and-forward

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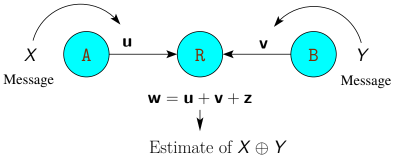

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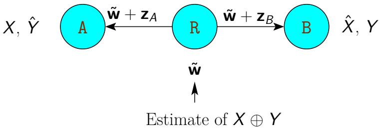

Figure: Broadcast phase

- We assume X and Y are uniformly distributed over the set of messages.
- Then, $(X \oplus Y) \perp\!\!\!\perp X$ and $(X \oplus Y) \perp\!\!\!\perp Y$.

Prior work using nested lattice coding

Secure Compute-and-forward:

- Weak secrecy using random binning: He and Yener, *"Providing secrecy using lattice codes,"* Allerton '08.
- Strong secrecy using universal hash functions: He and Yener, *"Strong secrecy and reliable byzantine detection in the presence of an untrusted relay,"* IEEE Trans. Inf. Theory '12.
- Perfect secrecy using well chosen pmfs satisfying an average power constraint: Kashyap et al., *"Secure Computation in a Bidirectional Relay,"* ISIT '12.

Gaussian wiretap channel:

- Semantic security using nested lattice codes and sampled Gaussian pmfs: Ling et al., *"Semantically Secure Lattice Codes for the Gaussian Wiretap Channel,"* arXiv:1210.6673.

Lattices

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_d$ be linearly independent vectors in $\mathbb{R}^d$. Then the set $\Lambda = \{\sum_{i=1}^{d} a_i \mathbf{v}_i : a_i \in \mathbb{Z}\}$ is called a lattice.

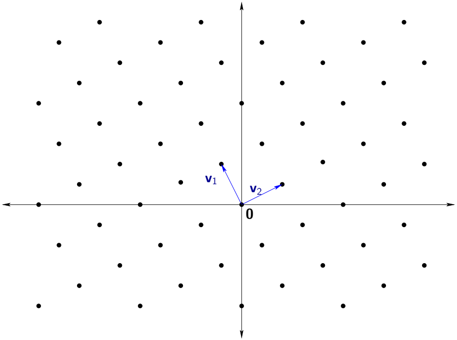

Figure: A lattice in $\mathbb{R}^2$.

Define the nearest neighbour quantizer for Λ as $Q_\Lambda(\mathbf{x}) := \arg\min_{\lambda\in\Lambda}\|\mathbf{x}-\lambda\|$. The fundamental Voronoi region of Λ is defined as $\mathcal{V}(\Lambda) := \{\mathbf{y} : Q_\Lambda(\mathbf{y}) = \mathbf{0}\}$.

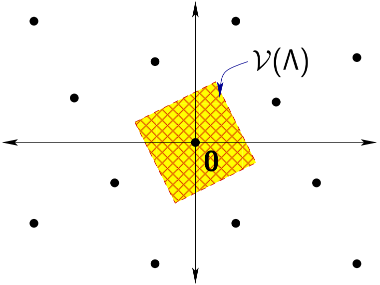

Figure: Fundamental Voronoi region of Λ, $\mathcal{V}(\Lambda)$.

$$\mathrm{vol}(\Lambda) := \mathrm{vol}\big(\mathcal{V}(\Lambda)\big).$$

Illustration: Nested lattices

If Λ and Λ_0 are lattices in $\mathbb{R}^d$ with $\Lambda_0 \subset \Lambda$, then Λ_0 is said to be nested within Λ, or Λ_0 is a sublattice of Λ.

Λ is called the fine lattice and Λ_0 is called the coarse lattice.

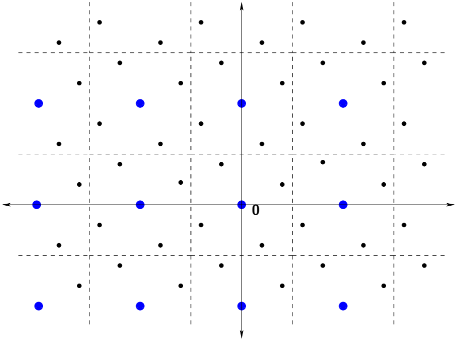

Figure: The blue dots indicate the coarse lattice Λ_0.

Cosets and coset representatives

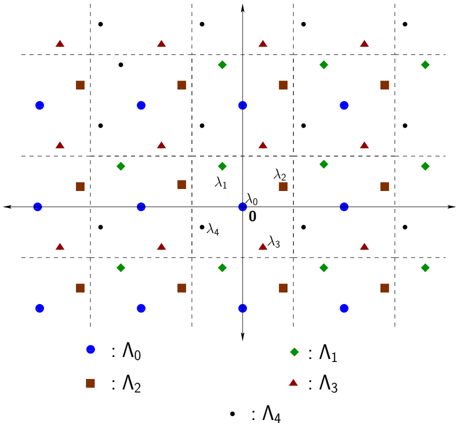

Figure: λ_i is the coset representative of Λ_i within $\mathcal{V}(\Lambda_0)$.

Basic idea to get secrecy

- Fix nested lattice pair (Λ, Λ_0) (e.g. $(\mathbb{Z}, 2\mathbb{Z})$).
- Cosets correspond to different messages. Given message Λ_i, transmit random point from Λ_i.

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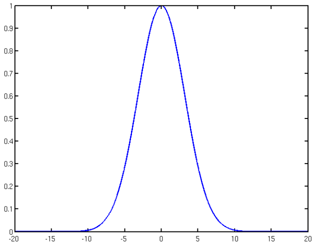

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- Cosets correspond to different messages. Given message Λ_i, transmit random point from Λ_i.
- Select a pdf $f(\cdot)$ over $\mathbb{R}^d$.
- Probability of transmitting $\mathbf{u} \in \Lambda_j$ is $f(\mathbf{u})/(\sum_{\mathbf{v}\in\Lambda_j} f(\mathbf{v}))$.

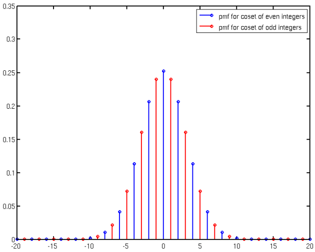

Figure: pmfs for the nested lattice pair $(\mathbb{Z}, 2\mathbb{Z})$.

- Don't use nested lattice shaping.
- Nested lattice shaping: Codewords chosen from $\Lambda \cap \mathcal{V}(\Lambda_0)$.
- Different choices of $f(\cdot)$ gives different secrecy properties![a]

[a] *"Secure compute-and-forward in a bidirectional relay,"* online: http://ece.iisc.ernet.in/~shashank/publications.html

Notation and definitions

For any $\mathbf{x}, \mathbf{z} \in \mathbb{R}^d$, and real $\kappa > 0$, we define

$$g_{\kappa,\mathbf{x}}(\mathbf{z}) := \frac{1}{\left(\sqrt{2\pi}\kappa\right)^n} e^{-\frac{\|\mathbf{z}-\mathbf{x}\|^2}{2\kappa^2}}, \tag{1}$$

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Observation: The function

$$p(\mathbf{u}) = \begin{cases} \frac{g_\kappa(\mathbf{u})}{g_\kappa(\Lambda)}, & \mathbf{u} \in \Lambda \\ 0 & \text{otherwise.} \end{cases}$$

is a probability mass function supported over Λ.

Coding scheme

Our objective

We want a randomized coding scheme $X \to \mathbf{u}$, $Y \to \mathbf{v}$ such that

(S1) $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$, and $(X, \mathbf{u}) \perp\!\!\!\perp (Y, \mathbf{v})$.

(S2) $\mathbf{u} + \mathbf{v}$ must determine $X \oplus Y$ (for a suitably defined $\oplus$).

(S3) $\mathcal{I}(X; \mathbf{u} + \mathbf{v})$ and $\mathcal{I}(Y; \mathbf{u} + \mathbf{v})$ must go to 0 as $d \to \infty$.

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(S3) $\mathcal{I}(X; \mathbf{u} + \mathbf{v})$ and $\mathcal{I}(Y; \mathbf{u} + \mathbf{v})$ must go to 0 as $d \to \infty$.

- Observe that $X \to (\mathbf{u} + \mathbf{v}) \to (\mathbf{u} + \mathbf{v} + \mathbf{z})$ forms a Markov chain.
- Therefore, $\mathcal{I}(X; \mathbf{u} + \mathbf{v} + \mathbf{z}) \leq \mathcal{I}(X; \mathbf{u} + \mathbf{v})$.
- Hence, (S3) guarantees secrecy even in presence of noise.

A strongly secure coding scheme

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- Encoding: Given message Λ_j, encoder outputs a random point $\mathbf{u}$ from Λ_j according to

$$p_j(\mathbf{u}) = \begin{cases} \frac{g_{\sqrt{P}}(\mathbf{u})}{g_{\sqrt{P}}(\Lambda_j)}, & \text{if } \mathbf{u} \in \Lambda_j \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

A strongly secure coding scheme

- Decoding: The relay has $\mathbf{w} = \mathbf{u} + \mathbf{v} + \mathbf{z}$.
  1. Let $\tilde{\mathbf{w}}$ be the closest point in $\Lambda^{(d)}$ to $\mathbf{w}$.
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- Achievable power-rate pairs: $(\mathcal{P}, \mathcal{R})$ is achievable with strong secrecy if for every $\delta > 0$, there exists a sequence of $(\Lambda^{(d)}, \Lambda_0^{(d)})$ nested lattice codes such that for all sufficiently large d,
  - Avg. transmit power, $\frac{1}{d}\mathbb{E}\|\mathbf{u}\|^2 = \frac{1}{d}\mathbb{E}\|\mathbf{v}\|^2$, is less than $\mathcal{P} + \delta$.
  - Transmission rate, $\frac{1}{d}\log_2|\mathbb{G}^{(d)}|$, is greater than $\mathcal{R} - \delta$.
  - The average probability of decoding $X \oplus Y$ incorrectly from $\mathbf{u} + \mathbf{v} + \mathbf{z}$ is less than δ.
  - The mutual information, $\mathcal{I}(X; \mathbf{u} + \mathbf{v}) = \mathcal{I}(Y; \mathbf{u} + \mathbf{v})$ is less than δ.

Strong secrecy

Define the average variational distance between $U+V$ and X,

$$d_V := \sum_{\mathbf{x}} \sum_{\mathbf{w}\in\Lambda^{(d)}} |p_{U+V,X}(\mathbf{w},\mathbf{x}) - p_{U+V}(\mathbf{w})p_X(\mathbf{x})|$$

Lemma (Csiszár, Narayan (2004))

For $|\mathbb{G}^{(d)}| \geq 4$, we have

$$\mathcal{I}(X;\mathbf{u}+\mathbf{v}) \leq d_V \left(\log_2 |\mathbb{G}^{(d)}| - \log_2 d_V\right).$$

Strong secrecy

For any $\theta > 0$, the flatness factor, $\epsilon_\Lambda(\theta)$, is defined as

$$\epsilon_\Lambda(\theta) = \frac{\max_{\mathbf{x}\in\mathcal{V}(\Lambda)} \left| \left(\sum_{\lambda\in\Lambda} g_{\theta,\lambda}(\mathbf{x})\right) - 1/\mathrm{vol}(\Lambda)\right|}{1/\mathrm{vol}(\Lambda)}. \tag{3}$$

If $\mathbf{z}$ is a $\mathcal{N}(\mathbf{0}, \theta^2\mathbf{I}_d)$ random vector, then $\epsilon_\Lambda(\theta)$ is a measure of how far the distribution of $[\mathbf{z}] \bmod \Lambda$ is from the uniform distribution over $\mathcal{V}(\Lambda)$.

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Theorem

If the sequence of nested lattice pairs, $(\Lambda^{(d)}, \Lambda_0^{(d)})$ is such that the flatness factor, $\epsilon_{\Lambda_0^{(d)}}\left(\sqrt{\mathcal{P}/2}\right)$ is less than $1/2$, then the average variational distance,

$$d_V \;\leq\; 216\, \epsilon_{\Lambda_0^{(d)}}\left(\sqrt{\mathcal{P}/2}\right). \tag{4}$$

Achievable rate

Secrecy good lattices: $\Lambda^{(d)}$ is secrecy good if the flatness factor, $\epsilon_{\Lambda^{(d)}}(\theta)$ decays exponentially in d for all θ that satisfies

$$\frac{(\mathrm{vol}(\Lambda^{(d)}))^{2/d}}{2\pi\theta^2} < 1. \tag{5}$$

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We choose a sequence of nested lattices that satisfy the following "goodness" properties:

- The sequence of coarse lattices, $\Lambda_0^{(d)}$ is good for covering, MSE quantization, AWGN channel coding, and secrecy good.
- The sequence of fine lattices, $\Lambda^{(d)}$ is good for AWGN channel coding and secrecy good.

Nested lattice pairs that satisfy the above properties indeed exist.[1]

[1] Erez et al. (2004, 2005), Ling et al. (2012)

Achievable rate

- To have $\epsilon_{\Lambda_0^{(d)}}(\sqrt{\mathcal{P}/2}) \to 0$ exponentially, we need (5).

$$\frac{\left(\mathrm{vol}(\Lambda_0^{(d)})\right)^{2/d}}{2\pi(\mathcal{P}/2)} < 1.$$

  Scale the nested lattice pair so that $\left(\mathrm{vol}(\Lambda_0^{(d)})\right)^{2/d} = \pi\mathcal{P} - \delta$.
- For the average transmit power to converge to $\mathcal{P}$, we require $\epsilon_{\Lambda^{(d)}}(\sqrt{\mathcal{P}}/2) \to 0$ as $d \to \infty$, which imposes the following constraint:

$$\frac{1}{d}\log_2 |\mathbb{G}^{(d)}| > \frac{1}{2} - \frac{1}{2}\log_2\left(1 - \frac{\delta}{\pi\mathcal{P}}\right).$$

- To have the average probability of error decay to zero as $d \to \infty$, we need[2]

$$\frac{\left(\mathrm{vol}(\Lambda^{(d)})\right)^{2/d}}{2\pi e\sigma^2} > 1.$$

[2] Erez, Zamir (2004)

Achievable rate

Theorem

For any $\mathcal{P} \geq 4e\sigma^2$, a power-rate pair of

$$\left(\mathcal{P}, \frac{1}{2}\log_2\frac{\mathcal{P}}{\sigma^2} - \frac{1}{2}\log_2 2e\right)$$

can be achieved with strong secrecy.

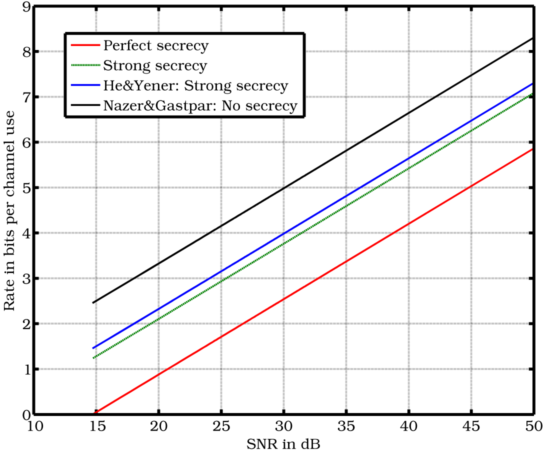
Using dithering techniques and MMSE equalization[3] at the relay, a power-rate pair of

$$\left(\mathcal{P}, \frac{1}{2}\log_2\left(\frac{1}{2} + \frac{\mathcal{P}}{\sigma^2}\right) - \frac{1}{2}\log_2 2e\right)$$

can be achieved with strong secrecy.

[3] Erez & Zamir (2004), Nazer & Gastpar(2011)

A comparison of achievable rates



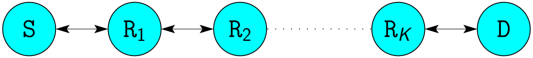
He and Yener (strong secrecy)

$$\mathcal{R} = \frac{1}{2}\log_2\left(\frac{1}{2} + \frac{\mathcal{P}}{\sigma^2}\right) - 1.$$

Kashyap et al. (perfect secrecy)

$$\mathcal{R} = \frac{1}{2}\log_2\frac{\mathcal{P}}{\sigma^2} - \log_2 2e.$$

Multi-hop line network with $K + 1$ hops



- All links are identical wireless AWGN links with unit gain.
- Source node $\mathtt{S}$ wants to send M independent messages (chosen uniformly at random) to destination.

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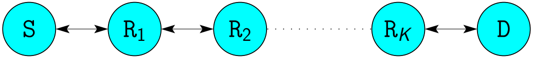
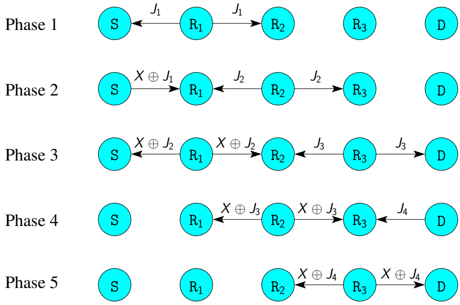


- All links are identical wireless AWGN links with unit gain.
- Source node $\mathtt{S}$ wants to send M independent messages (chosen uniformly at random) to destination.
- Relay nodes are independent passive eavesdroppers.
- Want strong secrecy at relay nodes.

Multi-hop line network with $K+1$ hops

- He and Yener (Allerton '08) proposed a weakly secure scheme using cooperative jamming.
- Each relay node independently generates a jamming signal J_i $(i = 0, 1, \ldots, K)$. Destination generates M jamming signals.



Multi-hop network

Any strongly secure scheme for the bidirectional relay can be used with the cooperative jamming protocol of He and Yener to achieve strong secrecy.[4]

[4] *"Secure compute-and-forward in a bidirectional relay,"* submitted to IEEE Trans. Inf. Theory, online: http://ece.iisc.ernet.in/∼shashank/publications.html

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Theorem

For $\mathcal{P} \geq 4e\sigma^2$, a power-rate pair of

$$\left(\mathcal{P}, \frac{1}{4}\log_2\left(\frac{\mathcal{P}}{\sigma^2}\right) - \frac{1}{4}\log_2 2e\right)$$

is achievable with strong secrecy in a multi-hop network.

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Conclusions

- Nested lattice based coding scheme satisfying an average power constraint that achieves strong secrecy over the bidirectional relay.
- Basic idea: Given the ith message (coset), transmit random point from Λ_i according to a pmf obtained by sampling Gaussian distributions.
- Possible that pmfs obtained by sampling different distributions may give interesting secrecy properties.
- Extension to the multi-hop line network.