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Problem Set 6
Math modeling of the atmosphere

Problem 1: Unbiasedness of sample variance

An estimate of the variance is given by

$$s^2 = \frac{1}{N-1} \sum_{i=1}^{N} (X_i - \hat{\mu})^2, \tag{1}$$

where

$$\hat{\mu} = \frac{1}{N} \sum_{i=1}^{N} X_i. \tag{2}$$

Show that s^2 is an unbiased estimate of the variance of the distribution, σ^2. That is, show that $\mathbb{E}(s^2) = \sigma^2$. Assume that the samples are independent and identically distributed. The samples come from a distribution with population mean μ and population variance σ.

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Problem Set 6, Answer Outline
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$$
\begin{aligned}
\mathbb{E}(s^2) &= \mathbb{E}\left[\frac{1}{N-1}\sum_{i=1}^{N}(X_i-\hat{\mu})^2\right] \\
&= \frac{1}{N-1}\sum_{i=1}^{N}\mathbb{E}\left[\left(X_i-\frac{1}{N}\sum_{j=1}^{N}X_j\right)^2\right] \\
&= \frac{1}{N-1}\sum_{i=1}^{N}\mathbb{E}\left[X_i^2-\frac{2}{N}X_i\sum_{j=1}^{N}X_j+\left(\frac{1}{N}\sum_{j=1}^{N}X_j\right)^2\right] \\
&= \frac{1}{N-1}\sum_{i=1}^{N}\mathbb{E}\left[X_i^2-\frac{2}{N}X_i\sum_{j=1}^{N}X_j+\frac{1}{N^2}\sum_{j=1}^{N}X_j\sum_{k=1}^{N}X_k\right] \\
&= \frac{1}{N-1}\sum_{i=1}^{N}\mathbb{E}\left[X_i^2-\frac{2}{N}X_i\sum_{j=1}^{N}X_j+\frac{1}{N^2}\sum_{j=1}^{N}X_j^2+\sum_{j=1}^{N}\sum_{k\neq j}^{N-1}X_jX_k\right] \\
&= \frac{1}{N-1}\sum_{i=1}^{N}\left[\frac{N-2}{N}\mathbb{E}(X_i^2)-\frac{2}{N}\sum_{j\neq i}^{N-1}\mathbb{E}(X_iX_j)+\frac{1}{N^2}\sum_{j=1}^{N}\mathbb{E}(X_j^2)+\sum_{j=1}^{N}\sum_{k\neq j}^{N-1}\mathbb{E}(X_jX_k)\right]
\end{aligned} \tag{3}
$$

Here we let $X_i \equiv X_i' + \mu$. We note that $\mathbb{E}(X_i'^2) = \sigma^2$ and $\mathbb{E}(X_i') = 0$. So $\mathbb{E}(X_i^2) = \sigma^2 + \mu^2$ and $\mathbb{E}(X_iX_j) = \mu^2$ if $i \neq j$.

$$
\begin{aligned}
&= \frac{1}{N-1}\sum_{i=1}^{N}\left[\frac{N-2}{N}(\sigma^2+\mu^2)-\frac{2}{N}(N-1)\mu^2+\frac{1}{N^2}N(\sigma^2+\mu^2)+\frac{1}{N^2}N(N-1)\mu^2\right] \\
&= \frac{1}{N-1}\sum_{i=1}^{N}\left[\frac{N-1}{N}\sigma^2\right] \\
&= \frac{1}{N}\sum_{i=1}^{N}\sigma^2 \\
&= \sigma^2.
\end{aligned} \tag{4}
$$