V. Larson
November 15, 2022

Problem Set 5
Math modeling of the atmosphere

Problem 1: Transformation of samples by use of the Cholesky decomposition

Suppose that a covariance matrix is given by

$$\mathbf{\Sigma} = \begin{bmatrix} 1 & r_{12} \\ r_{12} & 1 \end{bmatrix}. \tag{1}$$

Then the Cholesky matrix $\mathbf{L}$ is given by

$$\boldsymbol{L} = \begin{bmatrix} 1 & 0 \\ r_{12} & \sqrt{1 - r_{12}^2} \end{bmatrix}. \tag{2}$$

1) Verify, by direct computation, that $\mathbf{L}\mathbf{L}^T = \mathbf{\Sigma}$.

2) Now suppose that the desired PDF is positively correlated, with $r_{12} = 0.99$, and it has zero mean: $\boldsymbol{\mu} = 0$. Use the formula

$$\mathbf{X} = \mathbf{L} \cdot \mathbf{Y} + \boldsymbol{\mu} \tag{3}$$

to sketch how the Cholesky decomposition $\boldsymbol{L}$ maps the following four points, $\mathbf{Y}$ — $[1\ \ 1]^T$, $[1\ \ -1]^T$, $[-1\ \ -1]^T$, $[-1\ \ 1]^T$ — into $\mathbf{X}$. Does the distribution of points $\mathbf{X}$ look positively correlated? Does it matter whether $\mathbf{X}$, $\mathbf{Y}$, and μ are treated as row or column vectors?

3) Repeat part 2), but let $r_{12} = -0.99$. Does the resulting distribution look negatively correlated?

4) What is the Cholesky decomposition $\boldsymbol{L}$ for the covariance matrix

$$\mathbf{\Sigma} = \begin{bmatrix} \sigma_x^2 & r_{12}\sigma_x\sigma_y \\ r_{12}\sigma_x\sigma_y & \sigma_y^2 \end{bmatrix}? \tag{4}$$