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Problem Set 2
Math modeling of the atmosphere

Problem 1: Cloud fraction for a "double delta function" PDF

Consider a volume of air that is half occupied by air with $r_t = r_{t1}$ and half occupied by air with $r_t = r_{t2}$. (Let $r_{t1} > r_{t2}$, and let $\Delta r_t \equiv r_{t1} - r_{t2}$.) Further suppose that the volume has uniform temperature everywhere and that the saturation mixing ratio is r_s. Assume that any vapor in excess of saturation is immediately converted to liquid, as we assumed earlier in the semester for non-precipitating, ice-free clouds.

(i) Write an expression for cloud fraction, C, in terms of r_{t1}, r_{t2}, r_s, and the Heaviside step function.

(ii) Write an expression for average liquid water mixing ratio, $\langle r_l \rangle$, in terms of r_{t1}, r_{t2}, r_s, and the Heaviside step function.

Problem 2: Cloud fraction for a triangular PDF

Consider now a volume of air that has a symmetric PDF of r_t, $P(r_t)$, with the shape of an isosceles triangle. The PDF is a piecewise linear function of r_t that becomes non-zero at $r_t = \overline{r_t} - \Delta r_t$, increases linearly to a peak at $r_t = \overline{r_t}$, and then decreases linearly to zero at $r_t = \overline{r_t} + \Delta r_t$. Assume that temperature is constant and that the saturation mixing ratio is r_s. You don't need to write in terms of Heaviside step functions if you don't wish to do so. Instead you can just leave the answer in terms of discrete cases.

(i) Write the functional form of $P(r_t)$ in terms of $\overline{r_t}$, Δr_t, and r_t. Recall that the area under $P(r_t)$ must be 1.

(ii) Find the cloud fraction, C, in terms of $\overline{r_t}$, Δr_t, and r_s.

Problem 3: Sketch cloud fractions for the double delta and triangular PDFs

Overplot the two cloud fraction formulas that you have found in problems 1 and 2 as a function of $\overline{r_t}$ for fixed r_s and Δr_t. You may sketch by hand, if you wish.

Problem Set 2 Solution Outline
Math modeling of the atmosphere

Problem 1: Cloud fraction and liquid water for a "double delta function" PDF

Consider a volume that is half occupied by air with $r_t = r_{t1}$ and half occupied by air with $r_t = r_{t2}$. (Let $r_{t1} > r_{t2}$, and let $\Delta r_t \equiv r_{t1} - r_{t2}$.) Further suppose that the volume has uniform temperature everywhere and that the saturation mixing ratio is r_s.

(i) Write an expression for cloud fraction, C, in terms of r_{t1}, r_{t2}, r_s, and the Heaviside step function.

$$C = \frac{1}{2}H(r_{t1} - r_s) + \frac{1}{2}H(r_{t2} - r_s) \tag{1}$$

(ii) Write an expression for liquid water mixing ratio, r_l, in terms of r_{t1}, r_{t2}, r_s, and the Heaviside step function.

$$\langle r_l \rangle = \frac{1}{2}(r_{t1} - r_s)H(r_{t1} - r_s) + \frac{1}{2}(r_{t2} - r_s)H(r_{t2} - r_s) \tag{2}$$

Problem 2: Cloud fraction for a triangular PDF

Consider now a volume of air that has a symmetric PDF of r_t, $P(r_t)$, with the shape of an isosceles triangle. The PDF is piecewise linear function of r_t that becomes non-zero at $r_t = \overline{r_t} - \Delta r_t$, increases linearly to a peak at $r_t = \overline{r_t}$, and then decreases linearly to zero at $r_t = \overline{r_t} + \Delta r_t$. Assume that temperature is constant and that the saturation mixing ratio is r_s.

(i) Write the functional form of $P(r_t)$. Recall that the area under $P(r_t)$ must be 1.

$$\begin{aligned}
P(r_t) = \quad & 0 & r_t < \overline{r_t} - \Delta r_t \quad & (3)\\
& \tfrac{r_t - \overline{r_t}}{\Delta r_t^2} + \tfrac{1}{\Delta r_t} & \overline{r_t} - \Delta r_t \le r_t < \overline{r_t} \quad & (4)\\
& -\tfrac{r_t - \overline{r_t}}{\Delta r_t^2} + \tfrac{1}{\Delta r_t} & \overline{r_t} \le r_t < \overline{r_t} + \Delta r_t \quad & (5)\\
& 0 & r_t \ge \overline{r_t} + \Delta r_t \quad & (6)
\end{aligned}$$

$$ \tag{7} $$

(ii) Write an expression for cloud fraction, C, in terms of $\overline{r_t}$, Δr_t, and r_s.

In general,

$$C = \int P(r_t) H(r_t - r_s)\, dr_t \tag{8}$$

For the triangular PDF,

$$
\begin{aligned}
C = \quad & 0 & \overline{r_t} < r_s - \Delta r_t \quad & (9)\\
& \frac{1}{2}\left[\frac{r_s - (\overline{r_t} + \Delta r_t)}{\Delta r_t}\right]^2 & r_s - \Delta r_t \le \overline{r_t} < r_s \quad & (10)\\
& 1 - \frac{1}{2}\left[\frac{-r_s + (\overline{r_t} - \Delta r_t)}{\Delta r_t}\right]^2 & r_s \le \overline{r_t} < r_s + \Delta r_t \quad & (11)\\
& 1 & \overline{r_t} \ge r_s + \Delta r_t \quad & (12)\\
& & & (13)
\end{aligned}
$$

Problem 3: Sketch cloud fractions for the double delta and triangular PDFs

Overplot the two cloud fraction formulas that you have found as a function of $\overline{r_t}$ for fixed r_s and Δr_t. You may sketch by hand, if you wish.

Both formulas for C must asymptote to 0 for low values of $\overline{r_t}$ and asymptote to 1 for high values of $\overline{r_t}$. However, the triangle PDF yields a smooth function of C, whereas the double delta formula yields discrete steps from $C = 0, 0.5, 1$. These steps are rather unrealistic, which is a drawback of the double delta PDF.