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Problem Set 4
Math modeling of the atmosphere

Problem 1: Conditional distribution of a 2D normal distribution

Consider a normal distribution with two variates, x_1 and x_2. Assume that the distribution has mean $\mu = 0$ and standard deviation $\sigma = 1$. Therefore, we can call the distribution $\mathcal{N}(x_1, x_2|0, 1)$.

(I) Assume that the covariance, Σ_{12}, between x_1 and x_2 is 0.

(a) Sketch contours of $\mathcal{N}(x_1, x_2|0, 1)$.

(b) What is the conditional average $\mathbb{E}(X_1|X_2 = 1)$? Is $\mathbb{E}(X_1|X_2 = 1) > \mu_1$, or is $\mathbb{E}(X_1|X_2 = 1) < \mu_1$, or is $\mathbb{E}(X_1|X_2 = 1) = \mu_1$? ($\mu_1$ is mean of marginal in x_1 direction.)

(c) What is the standard deviation σ of the conditional distribution? Is it less than or greater than the standard deviation of $\mathcal{N}$, which is 1? Sketch the conditional distribution.

(II) Now repeat (a), (b), and (c), but assume that $\Sigma_{12} = 0.5$.

(III) Repeat (a), (b), and (c), but assume that $\Sigma_{12} = 0.99$.

(IV) Which is larger, the conditional average in (II) or that in (III)? Which is larger, the standard deviation of the conditional distribution in (II) or that in (III)?

(V) Repeat (a), (b), and (c), but assume that $\Sigma_{12} = -0.5$.

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Problem Set 4, Answer Outline
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(I) Assume that the covariance, Σ_{12}, between x_1 and x_2 is 0.

(a) Sketch contours of $\mathcal{N}(x_1, x_2|0, 1)$.

(b) $\mathbb{E}(X_1|X_2 = 1) = \mu_1$, because the covariance is 0.

(c) The standard deviation σ of the conditional distribution is unchanged at 1, again because there is zero covariance.

(II) Now repeat (a), (b), and (c), but assume that $\Sigma_{12} = 0.5$.

In this case, the covariance is positive. So the conditional mean is positive
$(= 0 + 0.5(1^{-1})(1 - 0) = 0.5)$ and the conditional variance is less than 1 $(= 1 - 0.5(1^{-1})0.5)$.

(III) Repeat (a), (b), and (c), but assume that $\Sigma_{12} = 0.99$.

(IV) The conditional average in (III) is larger than that in (II) because the correlation concentrates the values at high values.

The standard deviation of the conditional distribution in (III) is less than that in (II) because the correlation concentrates the points along the diagonal.

(V) Repeat (a), (b), and (c), but assume that $\Sigma_{12} = -0.5$.

With negative covariance, the conditional mean is negative, but the conditional variance is still reduced.