Cloud Dynamics Lectures

Mathematical Background Material

Boring Math (Vector Calculus)

Coordinates:

Position — $(x, y, z) = (x_1, x_2, x_3)$.

Direction — $(\hat{x}, \hat{y}, \hat{z}) = (\hat{1}, \hat{2}, \hat{3})$.

Velocity — $(u, v, w) = (u_1, u_2, u_3)$.

Dot Product:

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta = a_x b_x + a_y b_y + a_z b_z.$$

Here θ is the angle between vectors $\vec{a}$ and $\vec{b}$. The dot product of two vectors that are 90° apart is zero. The dot product of two vector that are parallel is just the length of one vector times the length of the other.

Independent and Dependent Variables:

Consider a function $f(x, y)$ of variables x and y. f may represent the height of a hill at a location specified by the coordinates x and y. x and y are allowed to have arbitrary values. Furthermore, the value of x is not determined by the value of y, but is specified independently. The same is true for y. Therefore x and y are said to be independent variables. f is said to be a dependent variable, since its value depends on the independent variables.

In summary, independent variables are variables that are specified independently of each other before we begin any analysis. They are the coordinates. Dependent variables depend on the independent variables. They are what we solve for.

Linear and Non-linear Terms in an Equation:

A linear term is a term that contains only one dependent variable whose exponent is 1. The dependent variable may be differentiated any number of times, and may be multiplied by a constant or an independent variable. A non-linear term is one that contains at least one dependent variable, but is not a linear term. A term that contains no dependent terms is neither linear nor non-linear; it could be called a forcing term.

For instance, suppose f and g are functions of x, and a is a constant. In the equation

$$ax\frac{d^2 f}{dx^2} + x + f^2 + \frac{1}{g} + fg = 0,$$

the first term is linear, the second is a forcing term, and the rest are nonlinear.

Definition of Total Derivative:

$$\frac{df(x)}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \equiv \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$$

The derivative df/dx is simply the slope of f at x. We often approximate derivatives in meteorology by using the above formula, but not taking the limit to zero, i.e. leaving Δx non-zero.

Definition of a Partial Derivative:

The expression

$$\left(\frac{\partial f(x,y)}{\partial x}\right)_y$$

means the derivative of $f(x, y)$ with respect to x when y is held fixed. To find the derivative, f must be written out in terms of the variables x and y, and x and y only. Normally we omit the $()_y$ because it is assumed that that is what is meant:

$$\frac{\partial f(x,y)}{\partial x}.$$

This can be illustrated by drawing a hill whose elevation at x and y is given by f.

Gradient:

$$\vec{\nabla} f(x,y) = \left(\frac{\partial f(x,y)}{\partial x}\right)_y \hat{x} + \left(\frac{\partial f(x,y)}{\partial y}\right)_x \hat{y}$$

$\vec{\nabla} f$ is a vector field. The gradient vector lies in the (x, y) plane and points in the uphill direction. $\vec{\nabla} f$ is perpendicular to constant elevation contours.

Material Derivative:

Consider a function $f = f(x, y, t)$, where $x = x(t)$ and $y = y(t)$. This function might be useful if you are walking on a hill during an earthquake.

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial t}dt$$

This function might be of use to you if you are walking on a hill during an earthquake. Dividing by dt,

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial t}\frac{dt}{dt}.$$

$dt/dt = 1$. If we interpret x and y as spatial coordinates and t as time, then $dx/dt = u$ is the $\hat{x}$ component of velocity, $dy/dt = v$ is the $\hat{y}$ component of velocity. Then we have

$$\frac{df}{dt} = \frac{\partial f}{\partial x}u + \frac{\partial f}{\partial y}v + \frac{\partial f}{\partial t} = \frac{\partial f}{\partial t} + \vec{u} \cdot \vec{\nabla} f.$$

This is the so-called material derivative. It is important to understanding the Navier-Stokes equations, which are the fundamental governing equations of fluid flow. df/dt is the rate of change of f following a parcel. $\partial f/\partial t$ is the rate of change of f at a particular location in space. It is zero if the flow is steady. The formula above states that the difference between these two quantities is $\vec{u} \cdot \vec{\nabla} f$. (Plot and explain example of a smokestack emitting smoke.)

Divergence:

Consider a vector $\vec{u}(x, y) = u_1(x, y)\hat{x} + u_2(x, y)\hat{y}$. Then

$$\vec{\nabla} \cdot \vec{u} = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y}.$$

$\vec{\nabla} \cdot \vec{u} > 0$ means that the vector field diverges (expands) from a point. Although $\vec{u}$ is a vector, $\vec{\nabla} \cdot \vec{u}$ is a scalar.

Laplacian:

The Laplacian is the divergence of the gradient, $\nabla^2 = \vec{\nabla} \cdot \vec{\nabla}$:

$$\nabla^2 f(x, y) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}.$$

For a sinusoid, the Laplacian of f is opposite in sign to f itself.

The Product Rule:

$$\frac{d}{dt}(fg) = f\frac{dg}{dt} + g\frac{df}{dt}.$$

One can visualize this by considering a rectangle with sides of length f and g. If the rectangle grows by a small amount in each direction, the area increases by $d(fg)$. The area can be broken into 3 areas, a strip with area $f\ dg$, a strip with area $g\ df$, and a corner whose area is neglected. (Thanks for the example, Nori!)

Definition of a (Definite) Integral:

$$\int_a^b f(x)dx = \lim_{\Delta x \to 0} \sum_{n=0}^{N} f(x_n)\Delta x.$$

This means that an integral is the area under a curve. This formula is illustrated graphically by breaking up the area into rectangles. The convergence rate for integrating the area under a straight line is

$$\text{Error} \propto \frac{1}{N}$$

This is faster than some other methods of integration that we will discuss later.

Boring Math II (Index Notation)

Cotton and Anthes 1989, p. 16 (p. 19, 2nd ed.).

The first thing we will do in this course is write down equations of motion that are suitable for convection. Before doing so, we must discuss index notation, also called summation notation or subscript notation. This is an alternative to standard vector notation that is a compact and convenient way of writing down the equations.

Consider how one writes the velocity vector in vector notation. Let $(\hat{x}, \hat{y}, \hat{z}) = (\hat{1}, \hat{2}, \hat{3})$ be the unit vectors in the three directions (plot axes). Let $(u, v, w) = (u_1, u_2, u_3)$ be the components of the velocity vector. Then we may write

$$\begin{aligned} \vec{u} &= u\hat{x} + v\hat{y} + w\hat{z} && (1) \\ &= u_1\hat{1} + u_2\hat{2} + u_3\hat{3} && (2) \\ &= \sum_{i=1}^{3} u_i\hat{i} && (3) \\ &= u_i\hat{i}. && (4) \end{aligned}$$

These are all equivalent ways of writing the same thing. For the last step, the summation sign has been suppressed, although summation still occurs. The rule is that summation is implied whenever an index is repeated. This is known as the summation convention. Similarly, one can write a gradient as follows:

$$\begin{aligned}
\vec{\nabla} f &= \hat{x}\frac{\partial f}{\partial x} + \hat{y}\frac{\partial f}{\partial y} + \hat{z}\frac{\partial f}{\partial z} && (5)\\
&= \hat{1}\frac{\partial f}{\partial x_1} + \hat{2}\frac{\partial f}{\partial x_2} + \hat{3}\frac{\partial f}{\partial x_3} && (6)\\
&= \sum_{i=1}^{3} \hat{i}\frac{\partial f}{\partial x_i} && (7)\\
&= \hat{i}\frac{\partial f}{\partial x_i}. && (8)
\end{aligned}$$

Some authors also use the notation

$$\frac{\partial}{\partial x_i} \equiv \nabla_i,$$

so that $\vec{\nabla} f = \hat{i}\nabla_i f$. Note that the final expression is a vector expression, not a scalar expression. Often, one just is interested in an arbitrary component of the gradient; this is just $\nabla_i f$.

Now suppose that we want to use index notation to write a dot product between two vectors, $\vec{a}$ and $\vec{b}$.

$$\begin{aligned}
\vec{a}\cdot\vec{b} &= (a_x\hat{x} + a_y\hat{y} + a_z\hat{z}) \cdot (b_x\hat{x} + b_y\hat{y} + b_z\hat{z}) && (9)\\
&= \left(\sum_{i=1}^{3} a_i\hat{i}\right) \cdot \left(\sum_{j=1}^{3} b_j\hat{j}\right) && (10)\\
&= \sum_{i=1}^{3}\sum_{j=1}^{3} a_i b_j \hat{i}\cdot\hat{j} && (11)\\
&= a_i b_j \hat{i}\cdot\hat{j}. && (12)
\end{aligned}$$

It is very important to use different indices i and j for the different vectors. Now we let

$$\delta_{ij} = \hat{i}\cdot\hat{j}. \qquad (13)$$

δ_{ij} is called the Kronecker delta. Since the unit axis vectors are orthogonal and have magnitude unity,

$$\begin{aligned} \delta_{ij} &= 1 \qquad\qquad i = j \qquad (14)\\ &= 0 \qquad\qquad i \neq j. \qquad (15) \end{aligned}$$

δ_{ij} has two free (unsummed) indices; therefore it is a two-dimensional tensor (matrix). δ_{ij} represents the elements of the identity matrix:

$$\delta_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \qquad (16)$$

Because $\delta_{ij} = 0$ for $i \neq j$, we have

$$\delta_{ij} b_j = b_i,$$

just as for matrix multiplication. Substituting this into (12), we find

$$\vec{a} \cdot \vec{b} = a_i b_i. \qquad (17)$$

Similarly,

$$a_i \delta_{ij} = a_j,$$

which leads to

$$\vec{a} \cdot \vec{b} = a_j b_j. \qquad (18)$$

Note that a dot product is a scalar, so that there is no vector in the final expression. Also, there is no free index; rather the index is summed over. Therefore, the index is a dummy argument. That is why Eqs. (17) and (18) are equivalent. Combining the above rules, we see, for example, that

$$\vec{u} \cdot \vec{\nabla} f = u_i \frac{\partial f}{\partial x_i}.$$

Boring Math III: Taylor Series

Another important thing we must do if we are to get anywhere in life is to make approximations. A main approximation we will make is to expand functions in Taylor series.

$$f(x) = f(x)|_{x=a} + \frac{1}{1!}\left.\frac{\partial f}{\partial x}\right|_{x=a}(x-a) + \frac{1}{2!}\left.\frac{\partial^2 f}{\partial x^2}\right|_{x=a}(x-a)^2 + ...$$

Consider the first two terms on the right-hand side. They approximate the curved function $f(x)$ as a line that passes through $f(x)$ at $x = a$. As an example of a Taylor series approximation, consider the function $f(x) = (1+x)^n$. The Taylor series approximation about $x = 0$ is

$$(1+x)^n \cong 1 + nx + n(n-1)\frac{x^2}{2!} + ...$$

Taylor series are useful in approximating nonlinear terms as linear ones.

Equations of Motion

Reading: Cotton and Anthes, Sections 2.2 and 2.3 (Sections 2.2 — 2.11, 2nd ed.). Also Emanuel, Chapter 1, Sections 4.1–4.4, and Appendices 1 and 2.

The equations of motion are important because of the kind of science meteorology is. What meteorologists do does not conform to the usual definition of scientific method. Namely, we don't have laboratory experiments in meteorology. Hence we can't check theoretical results in detail. Hence the deductive aspects of the science are important.

First, we define some terms. "Dry" air is air that contains no water vapor or liquid water or ice. Water in any of its phases affects buoyancy. Dry convection is free from such effects. So we may speak of convecting water in a laboratory tank as undergoing dry convection! The opposite is "moist" air, which contains either water vapor, liquid, or ice. Non-cloudy air, i.e. clear air, may contain water vapor but does not contain liquid or ice. Cloudy air contains liquid or ice.

[Venn diagram.]

Ideal Gas Law

Section 2.2.1 Cotton and Anthes. (Section 2.2.1 in 2nd ed., too.)

The point of the ideal gas law tells us, given a gas at a particular temperature and pressure, the density of the gas.

[Sketch of cylindrical container with weight on the top, illustrating constant pressure.]

The gas law for dry air is probably familiar to you, but we write it for a gas that includes water vapor and suspended liquid droplets as well as dry air.

First, some definitions:

Define density.

[Plot a parcel containing dry air, water vapor, and liquid droplets.]

The density of the parcel is:

$$\rho \equiv \rho_d + \rho_v + \rho_l. \tag{19}$$

Here ρ_d is the mass of dry air divided by the volume of the parcel. Similarly for ρ_v. ρ_l is the mass of liquid droplets divided by the volume of the *parcel*, not the volume of the droplets.

This implicitly treats the dry air plus vapor and liquid as one "fluid." Why is it reasonable to think of the fluid as one heterogeneous system instead of two coupled fluids? The droplets are falling at approximately their terminal velocity.

[Picture of gravitational force, friction force acting on a droplet.]

In doing so, the droplets apply a force on the air equal to their weight. We are merely including this force in the weight of the air/water mixture.

Also useful is the quantity

$$\rho_m \equiv \rho_d + \rho_v. \tag{20}$$

Mixing ratio and specific humidity

The water vapor mixing ratio r is defined as

$$r \equiv \rho_v/\rho_d.$$

I.e. it is the mass of water vapor per mass of dry air. Similarly, the liquid water mixing ratio is

$$r_l \equiv \rho_l/\rho_d.$$

Finally, the total water mixing ratio is

$$r_t \equiv r + r_l.$$

The specific humidity is the mass of vapor per mass of moist air, including vapor and liquid:

$$q \equiv \rho_v/(\rho_d + \rho_v + \rho_l) = \frac{r}{1 + r_t}.$$

Likewise, the specific liquid water content is

$$q_l \equiv \rho_l/(\rho_d + \rho_v + \rho_l) = \frac{r_l}{1 + r_t}.$$

Finally, the total specific water content is

$$q_t \equiv (\rho_v + \rho_l)/(\rho_d + \rho_v + \rho_l) = \frac{r_t}{1 + r_t}.$$

The atmosphere obeys the ideal gas law. This law is slightly complicated because there are two major contributors to the pressure: "dry air" (i.e. nitrogen and oxygen) and water vapor. Suppose the partial pressure of dry air is denoted p_d and that of water vapor, e. Suppose further that ρ_d is the density of dry air (with units of mass per volume), ρ_v is the density of water vapor, and T is temperature (in Kelvins). Furthermore, let $R_d = 287$ J kg^{-1} K^{-1} and $R_v = 461.5$ J kg^{-1} K^{-1} be the gas constants for dry air and water vapor, respectively. Then the ideal gas law becomes:

$$p = p_d + e = R_d \rho_d T + R_v \rho_v T \tag{21}$$

It is convenient to write this in terms of $\rho_m = \rho_v + \rho_d$. Pull out a factor of $\rho_m R_d T$, which involves multiplying and dividing (21) by ρ_m:

$$p = \rho_m R_d \frac{\rho_d + (R_v/R_d)\rho_v}{\rho_m} T. \tag{22}$$

In order to write the expression in terms of mixing ratios, divide the numerator and denominator by ρ_d. Then we find

$$p = \rho_m R_d \frac{1 + (R_v/R_d)(\rho_v/\rho_d)}{1 + \rho_v/\rho_d} T. \tag{23}$$

If we let $\epsilon = R_d/R_v \cong 0.622$ and $r = \rho_v/\rho_d$, then we find

$$p = \rho_m R_d \frac{1 + r/\epsilon}{1 + r} T. \tag{24}$$

Now convert this expression to one written in terms of specific humidities. Assume that $q = r/(1+r)$, that is, that the mixing ratio of liquid is negligible. Add and subtract r to the numerator.

$$p = \rho_m R_d \frac{1 + r + r/\epsilon - r}{1+r} T = \rho_m R_d \left(1 + \frac{r}{1+r}\left(1/\epsilon - 1\right)\right) T \cong \rho_m R_d \left(1 + \left(1/\epsilon - 1\right) q\right) T. \tag{25}$$

Meteorologists like to measure everything in units of temperature. In particular, they measure density in units of temperature. If we define the virtual temperature $T_v \equiv (1 + (1/\epsilon - 1)q)T$ (Emanuel 1994, Eqn. 4.3.6), then

$$p \cong \rho_m R_d T_v. \tag{26}$$

The density temperature is defined as $T_\rho = (1 + (1/\epsilon - 1)q - q_l)T$. For non-cloudy air, $T_\rho \equiv T_v$. If two unsaturated parcels are at the same pressure, then the virtual temperature is inversely related to the density. Hence T_v is a measure of buoyancy: more buoyant parcels have greater T_v. T_v can be thought of as the temperature dry air would have to have to yield the same density as moist unsaturated air. (This may be illustrated with a sketch of two parcels, one dry, one moist but non-cloudy. Make them both the same virtual temperature, but make the dry one warmer.) (Is T_v greater than T? Yes. Why? Because water vapor is lighter than dry air. Why? Because water has only one (heavy, 16 g per mole) oxygen and two (light) hydrogens, whereas dry air has two (heavy, 14 g per mole) nitrogens or two (heavy) oxygens (remember that $PV = NkT$. Does density temperature always exceed temperature? No. Water loading can, in principle, exceed water vapor.)

Continuity Equation

Sections 2.2.3, 2.3.3 Cotton and Anthes. (Section 2.4, 2nd ed.)
Batchelor, p. 74. Tritton, p. 52.
The continuity equation may be written

$$\frac{1}{\rho_d}\frac{d\rho_d}{dt} = -\vec{\nabla} \cdot \vec{u} = -\frac{\partial u_i}{\partial x_i}. \tag{27}$$

(Is this equation linear or non-linear?) The material derivative is defined by

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \vec{u} \cdot \vec{\nabla} \equiv \frac{\partial}{\partial t} + u_i \frac{\partial}{\partial x_i}. \tag{28}$$

This material derivative may be interpreted as stating that ρ_d changes either if (1) there are not necessarily any spatial gradients but the density changes in time, or if (2) there is not necessarily any change in time but the parcel moves into a region of greater or lesser ρ_d (Tritton, p. 54).

Combining the continuity equation (27) with the material derivative (28), we find

$$\frac{1}{\rho_d}\left(\frac{\partial \rho_d}{\partial t} + \vec{u}\cdot\vec{\nabla}\rho_d\right) = -\vec{\nabla}\cdot\vec{u}. \tag{29}$$

In index notation,

$$\frac{1}{\rho_d}\left(\frac{\partial \rho_d}{\partial t} + u_i\frac{\partial \rho_d}{\partial x_i}\right) = -\frac{\partial u_i}{\partial x_i}. \tag{30}$$

We will not derive the continuity equation, but it expresses mass conservation. Consider a parcel of air whose mass is constant and which is advected by a flow. (Draw sketch.) Write $\rho_d = M/V$, where M is the parcel mass and V is the volume. Then keep M constant and let V vary. The continuity equation states that the fractional volume change of this parcel is equal to the divergence of the flow:

$$\frac{1}{V}\frac{dV}{dt} = \vec{\nabla}\cdot\vec{u} = \frac{\partial u_i}{\partial x_i}.$$

(Why not include liquid in the continuity equation? Or vapor? Because both have sources and sinks that would greatly complicate the equation.)

It is useful to approximate the continuity equation for two reasons. First, theoretically, it is very difficult to work with a non-linear equation. Secondly, the above continuity equation permits sound waves. To simulate these in a numerical model requires a very small time step, which is costly.

First we construct a reference state whose variables are denoted by $()_0$; deviations from this state are denoted ()'. We will linearize about this state. The reference state is assumed to be horizontally uniform; that is, x and y derivatives of basic state variables are zero. It is also time independent. Then $()_0 = ()_0(z)$. The reference state is dry (contains no vapor or liquid): $r_0 = r_{l,0} = 0$. Draw figure of horizontally uniform reference state.

Then

$$\rho_d(x,y,z,t) = \rho_0(z) + \rho'(x,y,z,t). \tag{31}$$

Although it is not the most realistic assumption, for simplicity, we assume

$$\vec{u}_0 = 0. \tag{32}$$

Now we ask whether the ρ' term is small. Let $\rho_d = \rho_0 + \rho'$, where $\rho_0 = \rho_{d0}$, i.e., the reference state density is the dry density, because the reference state contains no moisture. Subbing into (27), we find

$$\frac{1}{\rho_d}\frac{d(\rho_0+\rho')}{dt} = \frac{1}{\rho_d}\vec{u}\cdot\vec{\nabla}\rho_0 + \frac{1}{\rho_d}\frac{d\rho'}{dt} = -\vec{\nabla}\cdot\vec{u}. \tag{33}$$

How large is $1/\rho_d d\rho'/dt$ compared to $\vec{\nabla}\cdot\vec{u}$? The magnitude of $1/\rho_d d\rho'/dt$ may be estimated as

$$\frac{1}{\rho_d}\frac{d\rho'}{dt} \sim \frac{1}{\rho_d}\frac{\Delta\rho'}{\Delta t} \sim \frac{1}{\rho_d}\frac{\rho'}{\tau}. \tag{34}$$

Here we have set $\Delta\rho' \cong (\rho' - 0) = \rho'$ because ρ' often is zero throughout the flow. The time scale τ is, for the motions of interest to us (which do not include sound waves), in order of magnitude,

$$\frac{1}{\tau} \sim \left|\frac{\partial u}{\partial x}\right|, \left|\frac{\partial v}{\partial y}\right|, \left|\frac{\partial w}{\partial z}\right|. \tag{35}$$

Thus, the timescale we choose is an advective timescale, the time to advect a parcel a typical length. But $\rho'/\rho_d << 1$. (What is an estimate of ρ'/ρ_d?) Hence we can neglect $(1/\rho_d)(d\rho'/dt)$ in (33) relative to $|\partial u/\partial x|$, $|\partial v/\partial y|$, and $|\partial w/\partial z|$. Furthermore, we write ρ_d in terms of ρ_0. Re-writing (33), we find

$$\vec{u}\cdot\vec{\nabla}\rho_0 = -\rho_d\vec{\nabla}\cdot\vec{u} = -(\rho_0+\rho')\vec{\nabla}\cdot\vec{u}. \tag{36}$$

Now drop the term containing ρ' because it is of order ρ'/ρ_d compared to the term containing ρ_0. Then, using the product rule, we find that

$$\vec{\nabla}\cdot(\rho_0(z)\vec{u}) = \frac{\partial}{\partial x_i}(\rho_0(z)u_i) = 0. \tag{37}$$

This is the "anelastic equation." It is valid for deep convection. But sound waves are no longer permitted (i.e. the model is "soundproof.") This reflects a determination on our part to exclude sound waves from consideration. Our scaling assumption was not true for sound waves, by design. (Is the anelastic equation nonlinear? Not if one considers ρ_0 to be given.) For shallow convection, ρ_0 is approximately constant with altitude; but for deep convection, it varies considerably with height.

A plot of $\rho_0(z)$ may be constructed using the following standard atmosphere data:

Geopotential height [km]	Density [kg m^{-3}]
0	1.23
1	1.11
2	1.01
3	0.91
6	0.66
9	0.47
12	0.31
15	~ 0.2

Boundary layer clouds occur in the lowest 1 to 2 km of the troposphere. For these clouds, assuming constant density incurs only a 10 to 20 per cent error. But for deep convection, the error is large. For shallow convection ρ_0, we can further approximate to obtain the so-called "Boussinesq" approximation:

$$\vec{\nabla} \cdot \vec{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{\partial u_i}{\partial x_i} = 0. \tag{38}$$

Then the continuity reduces to the form used for an incompressible fluid (i.e., a fluid which does not change density upon a change in pressure, although this is not strictly true for an ideal gas).

Physically interpreted, a divergence-less flow is one whose streamlines cannot end in the interior of the fluid (Batchelor, p. 75). A streamline is a line that is everywhere parallel to a flow vector at a instant in time (Batchelor, p. 72).

In derivations of the Boussinesq form of the continuity equation, the argument often appears to go something like: "The left-hand side of (27) is much smaller than the right-hand side. Therefore we set the right-hand side to zero." When there are only two terms in an equation, it is difficult to use scaling arguments to get rid of one. I think of it as follows. $\vec{\nabla} \cdot \vec{u}$ consists of three terms: $\partial u/\partial x$, $\partial v/\partial y$, and $\partial w/\partial z$. Each of these is bigger than the terms on the left-hand side in a Boussinesq flow. So these three terms tend to cancel each other, and the density term has little contribution. (Is this true for the $\partial\rho/\partial z$ term?)

Momentum Equation

Follow Sections 2.2.2 and 2.3.2 of Cotton and Anthes.

The Navier-Stokes equation for a non-rotating fluid is (2.17 Cotton and Anthes 1989):

$$\rho_m \frac{d\vec{u}}{dt} = \rho_m \left(\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \vec{\nabla} \vec{u} \right) = -\vec{\nabla} p - (\rho_m + \rho_l) g \hat{z} + \rho_m \nu \nabla^2 \vec{u}. \tag{39}$$

In index notation, an arbitrary component is

$$\rho_m \frac{du_i}{dt} = \rho_m \left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right) = -\frac{\partial p}{\partial x_i} - (\rho_m + \rho_l) g \delta_{i3} + \rho_m \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} \tag{40}$$

Here u_i is the ith component of velocity, $\rho_m = \rho_d + \rho_v$, ρ_l is the "density" of condensate, p is pressure, ν is the kinematic viscosity (with magnitude 1.3×10^{-5} m^2 s^{-1}), and δ is the Kronecker delta. This really represents three scalar equations, that is, a vector equation.

Dividing by ρ_m and re-arranging,

$$\frac{du_i}{dt} = \left(-\frac{1}{\rho_m} \frac{\partial p}{\partial x_i} - g \delta_{i3} \right) - \frac{\rho_l}{\rho_m} g \delta_{i3} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} \tag{41}$$

Now we linearize the pressure and buoyancy terms. Specify a reference state that is 1) horizontally uniform, 2) free of liquid and ice and water vapor, and 3) hydrostatic. (Draw sketch.)

(What is the physical meaning of hydrostaticity? It's when pressure is given by the weight of a column of fluid above the location of interest. Then the buoyancy force balances the pressure gradient force exactly, and all other terms in the Navier-Stokes equation vanish.) Now expand pressure and density about the reference state:

$$\rho_m(x, y, z, t) = \rho_0(z) + \rho_m'(x, y, z, t)$$

$$p(x, y, z, t) = p_0(z) + p'(x, y, z, t).$$

Here $\rho_{m0} = \rho_{d0} = \rho_0$. Now ρ_0 and p_0 are known, specified functions, and the old dependent variables, ρ_m and p, have been replaced by new ones, ρ_m' and p'. Since we want the reference state to be hydrostatic, for simplicity, we choose the reference state such that

$$\frac{1}{\rho_0} \frac{\partial p_0}{\partial z} = -g \quad \text{and} \quad \frac{\partial p_0}{\partial x} = \frac{\partial p_0}{\partial y} = 0. \tag{42}$$

We shall use a subscript zero to denote the basic state. Substitute into the terms in parentheses in (41).

$$-\frac{1}{\rho_m} \frac{\partial p}{\partial z} - g = -\frac{1}{\rho_0} \left(1 + \rho_m'/\rho_0 \right)^{-1} \frac{\partial (p_0 + p')}{\partial z} - g. \tag{43}$$

We linearize this expression, using the assumption that $\rho_m'/\rho_0 << 1$. Then we obtain:

$$-\frac{1}{\rho_m}\frac{\partial p}{\partial z} - g \cong -\left(1 - \rho_m'/\rho_0\right)\frac{1}{\rho_0}\frac{\partial p_0}{\partial z} - \frac{1}{\rho_0}\frac{\partial p'}{\partial z} - g. \tag{44}$$

Using hydrostaticity, we obtain

$$-\frac{1}{\rho_m}\frac{\partial p}{\partial z} - g \cong -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} - g\frac{\rho_m'}{\rho_0} \tag{45}$$

Now we linearize the 3rd term on the right-hand side of (41). We assume that in the reference state, there is no condensed water (or vapor): $\rho_{l0} = 0$ and $\rho_l' = \rho_l$. Then

$$-\frac{1}{\rho_m}\rho_l g \cong -\frac{\rho_l'}{\rho_m}g \cong -\frac{\rho_l'}{\rho_d}g = -r_l' g, \tag{46}$$

where $r_l' = r_l = \rho_l/\rho_d \cong \rho_l'/\rho_0$.

Substituting (45) and (46) into (41), we find

$$\frac{du_i}{dt} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x_i} + g\left(-\frac{\rho_m'}{\rho_0} - r_l'\right)\delta_{i3} + \nu\frac{\partial^2 u_i}{\partial x_j \partial x_j}. \tag{47}$$

Now we want to write ρ_m' in terms of the more familiar quantities potential temperature and specific humidity. To do so, first we linearize the ideal gas law, (25), written in terms of ρ_m, not ρ (Eqn. 2.12, Cotton and Anthes 1989) ($p = \rho_m R_d(1 + (1/\epsilon - 1)q)T$). To linearize, we divide the equation by the gas law for the reference state, $p_0 = \rho_0 R_d(1 + (1/\epsilon - 1)q_0)T_0$:

$$\frac{p}{p_0} = \frac{\rho_m}{\rho_0}\frac{R_d}{R_d}\frac{(1 + (1/\epsilon - 1)q)}{(1 + (1/\epsilon - 1) \times 0)}\frac{T}{T_0}. \tag{48}$$

Then we Taylor-expand. We use the fact that for a dry reference state, $q_0 = 0$ and $q = q'$. We find

$$\rho_m'/\rho_0 = -T'/T_0 - (1/\epsilon - 1)q' + p'/p_0. \tag{49}$$

This form occurs because $q_0 = 0$ in a dry atmosphere. To write this in terms of potential temperature, we linearize the definition of potential temperature,

$$\theta = T(p_{\text{ref}}/p)^{R_d/c_{pd}}. \tag{50}$$

Here c_{pd} is the heat capacity of dry air per unit mass of dry air at constant pressure, and p_{ref} is a reference pressure, usually taken to be 1000 mb. This yields

$$-T'/T_0 = -\theta'/\theta_0 - (R_d/c_{pd})(p'/p_0). \tag{51}$$

Substituting (51) into (49) and using $R_d = c_{pd} - c_{vd}$, where c_{vd} is the heat capacity at constant volume for dry air, we find

$$\rho_m'/\rho_0 = -\theta'/\theta_0 - (1/\epsilon - 1)q' + (c_{vd}/c_{pd})(p'/p_0). \tag{52}$$

Finally, substituting (52) into (47), we find

$$\frac{du_i}{dt} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x_i} + g\left(\frac{\theta'}{\theta_0} + (1/\epsilon - 1)q' - \frac{c_{vd}}{c_{pd}}\frac{p'}{p_0} - r_l'\right)\delta_{i3} + \nu\frac{\partial^2 u_i}{\partial x_j \partial x_j}. \tag{53}$$

How should we physically interpret the buoyancy terms? A parcel is more buoyant if it is hotter, has more water vapor, has lower pressure, and has less condensate. (Why does a decrease in pressure lead to more buoyancy? Look at ideal gas law. If temperature stays fixed, then a decrease in pressure implies a decrease in density.)

Thermodynamic Equation

Follow Section 2.2.5 of Cotton and Anthes.

The heat equation is given by

$$\frac{d\ln\theta}{dt} = (R_m/c_{pm} - R_d/c_{pd})\frac{d\ln p}{dt} - \frac{L_{lv}}{c_{pm}T}\frac{dr}{dt} + \frac{L_{il}}{c_{pm}T}\frac{dr_i}{dt} + \frac{\dot{Q}(R)}{c_{pm}} + \frac{\dot{Q}(D)}{c_{pm}}. \tag{54}$$

Here $c_{pm} = c_{pd} + rc_{pv} + r_l c_l + r_i c_i$ and $R_m = R_d + rR_v$. r_l and r_i are the mixing ratios of water and ice respectively. c_{pm} is the heat capacity of a moist, cloudy mixture per unit mass of dry air at constant pressure. c_{pv} is the heat capacity at constant pressure for water vapor, per unit mass of water vapor (see Emanuel, Eqn. 4.2.2). L_{lv} and L_{il} are the latent heat of the liquid-to-vapor phase change (latent heat of vaporization), and the latent heat of ice-to-liquid phase change (latent heat of fusion). They vary slightly with temperature. (What do the latent heating terms on the right hand signify? That freezing releases latent heat and evaporative causes cooling.) $\dot{Q}(R)$ is the radiative heating hate, and $\dot{Q}(D)$ is molecular dissipation, i.e. frictional heating, which is the same effect that occurs when one rubs one's hand to warm them.

How important is the latent heat of fusion? (See Section 4.6.5 Cotton and Anthes 1989.) At 0 degrees C, $L_{lv} = 2.5 \times 10^6$ J kg^{-1} and $L_{il} = 0.33 \times 10^6$ J kg^{-1}. Latent heat of fusion is a factor of eight smaller. However, in upper levels of clouds, it can be important since relatively little condensation occurs there.

If we neglect the heat capacity of condensed water, set $R_m \cong R_d$, take the derivative of $\ln \theta$, and use the definition of potential temperature, we find

$$\frac{d\theta}{dt} = -\frac{L_{lv}}{c_{pd}} \left(\frac{p}{p_{ref}} \right)^{-R_d/c_{pd}} \frac{dr}{dt} + \frac{L_{il}}{c_{pd}} \left(\frac{p}{p_{ref}} \right)^{-R_d/c_{pd}} \frac{dr_i}{dt} + \theta \frac{\dot{Q}(R)}{c_{pd}} + \theta \frac{\dot{Q}(D)}{c_{pd}}. \tag{55}$$

The third term on the right-hand side governs latent heat of fusion. When liquid droplets freeze, they heat the parcel, and vice-versa. The second term on the RHS represents latent heat of vaporization. When a parcel condenses liquid, vapor is lost, leading to heating. Conversely, when a parcel evaporates liquid, there is evaporative cooling. When rain falls out of a parcel, there is no heating or cooling. The equation neglects the effect of diffusion of moisture from a humid parcel to a dry one.

Water advection-diffusion

Follow section 2.2.4 of Cotton and Anthes.

$$\frac{dr_T}{dt} = \frac{\partial r_T}{\partial t} + u_j \frac{\partial r_T}{\partial x_j} = \kappa_r \frac{\partial^2}{\partial x_j \partial x_j} r_T - P. \tag{56}$$

Here $r_T = r + r_l + r_i$, κ_r is the kinematic diffusivity of water vapor, and P represents precipitation.

We have r in the heat equation but r_T in the water conservation equation. How can we close the equations of motion? Take one simple case.

Suppose that

1. There is no ice. Then $r_T = r_t = r + r_l$. We have r if we can solve for r_l.
2. Any vapor in excess of saturation is immediately converted to liquid. This is usually the case for the atmosphere, since there is usually plenty of aerosol on which to condense liquid.
3. Likewise, any parcel that is subsaturated immediately evaporates its liquid until all is gone or the parcel saturates.

Then (Eqn. 2.22 Cotton and Anthes 1989):

$$r_l = (r_t - r_s)H(r_t - r_s).$$

Here r_s is the saturation mixing ratio and H is the step function. Draw plot of $r_t - r_s$, $H(r_t - r_s)$, and r_l all vs. r_t. Physically, this means that latent heating effects occur only in cloud.

How do we find r_s? First we must find the saturation vapor pressure, e_s. If there is no ice, we can approximate it using the Clausius-Clapeyron equation (Emanuel 1994, Eqn. 4.4.11):

$$\frac{de_s(T)}{dT} = \frac{L_{lv}e_s}{R_vT^2}. \tag{57}$$

This shows that the saturation vapor pressure, e_s, increases rapidly as the temperature increases. This is in part because $L_{lv} \cong 2.5 \times 10^6$ J kg^{-1} is such a large number. Integrating this expression is difficult if L_{lv} is allowed to vary with temperature. However, one can write down the following empirical approximation (Emanuel 1994, Eqn. 4.4.14, from Bolton):

$$e_s(T) \cong 6.112\text{m}b\ \exp\left(\frac{17.67(T - 273.15)}{T - 29.65}\right).$$

Here T is in Kelvins. To convert from vapor pressure to mixing ratio, we use the following formula (Emanuel 1994, Eqn. 4.1.2):

$$r \equiv \frac{\rho_v}{\rho_d} = \frac{e/(R_vT)}{p_d/(R_dT)} = \frac{R_d}{R_v}\frac{e}{p-e} \equiv \epsilon\frac{e}{p-e} \cong \epsilon\frac{e}{p}.$$

Thus, we have

$$r_s(T,p) \cong \epsilon\frac{e_s(T)}{p}.$$

Do we have enough equations to solve for all variables? Yes (if no ice or supersaturation):

Variable	Equation
u_i	momentum
θ	heat
r_t	water advection-diffusion
r	$r = r_t - r_l$ (r is in heat eqn.)
r_l	$r_l = (r_t - r_s)H(r_t - r_s)$
r_s	Clausius-Clapeyron
p'	continuity

Reynolds averaging

Cotton and Anthes, Sections 3.6, 3.8 (3.2, 2nd ed.)

Stull, Sections 2.4.2, 2.4.3, 3.4.3

Oftentimes we are only concerned with mean properties, averaged over a horizontal area. For instance, we might want to know the mean wind averaged over a 100 km × 100 km area, even if we don't care to know about every gust and fluctuation. We can derive an equation for the average velocity as follows.

Consider the Navier-Stokes equations, simplified such that there is no moisture and that p'/p_0 is small. Then the buoyancy term simplifies considerably. Furthermore, we neglect the viscosity term. Then we have

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho_{ref}} \frac{\partial p'}{\partial x_i} + g\left(\frac{\theta'}{\theta_{ref}}\right)\delta_{i3}. \tag{58}$$

Here ρ_{ref} and θ_{ref} are constants. Now we divide the velocity into a mean part, denoted by an overbar, and a perturbation part, denoted by a double prime:

$$u_i(x,y,z,t) = \overline{u}_i(x,y,z,t) + u_i''(x,y,z,t). \tag{59}$$

$$p(x,y,z,t) = p_0(z) + p'(x,y,z,t) = \overline{p}(x,y,z,t) + p''(x,y,z,t).$$

$$\theta(x,y,z,t) = \theta_0(z) + \theta'(x,y,z,t) = \overline{\theta}(x,y,z,t) + \theta''(x,y,z,t).$$

Then $p' = \overline{p} - p_0 + p''$ and $\theta' = \overline{\theta} - \theta_0 + \theta''$. Note that the mean $\overline{()}$ can vary with time and is distinct from the reference state $()_0$. The mean is a running spatial average over a horizontal area. It takes as input a noisy space series and returns a smoothly varying function in the horizontal directions. (Sketch example of noisy and smoothed curve in the shape of the error function. Show how θ' and θ'' differ.) Hence $\overline{u}$ is not merely a constant number, but rather $\overline{u} = \overline{u}(x,y,z,t)$.

Substituting into the Navier-Stokes equation, we obtain:

$$\frac{\partial(\overline{u}_i + u_i'')}{\partial t} + (\overline{u}_j + u_j'')\frac{\partial}{\partial x_j}(\overline{u}_i + u_i'') = -\frac{1}{\rho_{ref}}\frac{\partial}{\partial x_i}(\overline{p} - p_0 + p'') + g\left(\frac{\overline{\theta} - \theta_0 + \theta''}{\theta_{ref}}\right)\delta_{i3}. \tag{60}$$

Now we average this equation, using Reynolds averaging rules:

1. For instance, we assume that an average of an average equals a single average:

$$\overline{\overline{u_i}} \cong \overline{u_i}.$$

This is not strictly true; extra filtering produces a smoother function (sketch example of noisy and averaged series with a "cliff"). But it is approximately true.

2. Then, using (59), we see that averages of perturbations are zero:

$$\overline{u_i''} \cong 0 \qquad \overline{p''} \cong 0 \qquad \overline{\theta''} \cong 0$$

3. We also assume

$$\overline{\overline{u}_j u_i''} \cong 0.$$

4. Also

$$\overline{\overline{u}_j \overline{u}_i} \cong \overline{u}_j \overline{u}_i.$$

Averaging all terms in this equation, we find:

$$\frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j \frac{\partial}{\partial x_j} \overline{u}_i + \overline{u_j'' \frac{\partial}{\partial x_j} u_i''} = -\frac{1}{\rho_{ref}} \frac{\partial}{\partial x_i} (\overline{p} - p_0) + g \left(\frac{\overline{\theta} - \theta_0}{\theta_{ref}} \right) \delta_{i3}. \tag{61}$$

Turbulence Kinetic Energy Equation

Cotton and Anthes, Section 3.8
Emanuel, Section 10.3.2

We now write an equation for the turbulence kinetic energy (TKE),

$$\overline{e} \equiv \frac{1}{2} \overline{u_i'' u_i''} = \frac{1}{2} \left(\overline{u''^2} + \overline{v''^2} + \overline{w''^2} \right). \tag{62}$$

The TKE is the amount of kinetic energy in the turbulent (as opposed to mean) air motions. It is useful to study TKE because it is a measure of convective strength. The governing equation for TKE tells us what process generate or destroy TKE.

To find the TKE equation, we first need to write an equation for the perturbation velocity, $u_i'' = u_i - \overline{u_i}$. To do so, we subtract the mean velocity equation (61) from the full velocity equation, (60). We find

$$\frac{\partial u_i''}{\partial t} + u_j'' \frac{\partial \overline{u}_i}{\partial x_j} + \overline{u}_j \frac{\partial u_i''}{\partial x_j} + u_j'' \frac{\partial u_i''}{\partial x_j} - \overline{u_j'' \frac{\partial u_i''}{\partial x_j}} = -\frac{1}{\rho_{ref}} \frac{\partial p''}{\partial x_i} + g \left(\frac{\theta''}{\theta_{ref}} \right) \delta_{i3}. \tag{63}$$

Now we multiply this equation by u_i'', average, and re-arrange. We only give the final result:

$$\frac{\partial \overline{e}}{\partial t} + \overline{u}_j \frac{\partial \overline{e}}{\partial x_j} + \overline{u_j'' \frac{\partial e}{\partial x_j}} = -\overline{u_i'' u_j''} \frac{\partial \overline{u_i}}{\partial x_j} - \frac{1}{\rho_{ref}} \frac{\partial (\overline{u_i'' p''})}{\partial x_i} + \frac{g}{\theta_{ref}} \overline{u_3'' \theta''}. \tag{64}$$

Here we have assumed the Boussinesq approximation. These terms can be interpreted as follows. The time tendency term determines how $\overline{e}$ changes with time at a fixed point in space. The next two terms govern how the turbulence is advected by the mean flow and the turbulent fluctuations. TKE can be thought of as a dye that gets advected by the flow, except that it has flow-dependent source terms. The shear generation term determines how turbulent fluctuations generate TKE. The next term is the pressure fluctuation term. The final term shows that a positive heat flux generates TKE.

A “turbulent flux” is any term of the form

$$\overline{u_i'' a''}.$$

Consider, for example, the “vertical turbulent heat flux,”

$$\overline{w'' \theta''}.$$

This is positive if $w'' > 0$ when $\theta'' > 0$, and $w'' < 0$ when $\theta'' < 0$. In other words, the flux is positive if updrafts are warm and downdrafts are cold. In this case, TKE is generated, because, in the absence of other forces, updrafts will accelerate upwards and downdrafts will accelerate downward. When the heat flux is negative, TKE is consumed.

0.1 Filtering the equations of motion

Cotton et al., 2nd ed., Section 3.3.

Leonard (1974); Germano (1992) and Chapter 13 of Pope (2000).

What equations does a large-eddy simulation (LES) model solve? It does not solve the Navier-Stokes equations. Those equations contain small molecular diffusivities that would lead to numerical instability. Instead, LES models approximate resolved (coarse-grained) fields using equations that contain a much larger eddy diffusivity. The eddy diffusivity arises from a parameterization (reduced model) of subgrid turbulent fluxes. But what does that parameterized eddy diffusivity truly represent?

The unclosed subgrid fluxes that we wish to parameterize can be given a precise, mathematical definition through the use of the filtering approach. The filtering approach involves four steps:

1. smooth the Navier-Stokes equations using a low-pass filter — this yields unclosed, higher-order terms;
2. parameterize the unclosed terms;
3. discretize the smoothed, parameterized equations; and
4. solve them numerically.

Applying a low-pass filter is similar to Reynolds averaging, but Reynolds' rules of averaging are only strictly valid for stationary fields and large averaging regions. In contrast, the averages needed by numerical models of the atmosphere are averages of fields with trends over small regions. Nevertheless, in this situation, an analogue to Reynolds rules holds (Germano 1992). That analogue is what we will discuss now.

But before we start filtering the *equations*, let's discuss the outcome of filtering a *field*. A field (velocity, temperature, moisture) $f(\tilde{\mathbf{x}})$ can be used to create a coarse-grained (filtered) version, $\overline{f}(\mathbf{x})$, by the following operation:

$$\overline{f}(\mathbf{x}) = \int G(\mathbf{x} - \tilde{\mathbf{x}}; L) f(\tilde{\mathbf{x}}) d\tilde{\mathbf{x}}. \tag{65}$$

Here, G is a localized, spatial filter, with an averaging length of L. G is normalized:

$$\int G(\mathbf{x} - \tilde{\mathbf{x}}; L) d\tilde{\mathbf{x}} = 1. \tag{66}$$

G could also include an average over a time interval, in addition to a spatial average:

$$\overline{f}(\mathbf{x}, t) = \int \int G(\mathbf{x} - \tilde{\mathbf{x}}, \tilde{t}; L) f(\tilde{\mathbf{x}}, \tilde{t}) d\tilde{\mathbf{x}} d\tilde{t}. \tag{67}$$

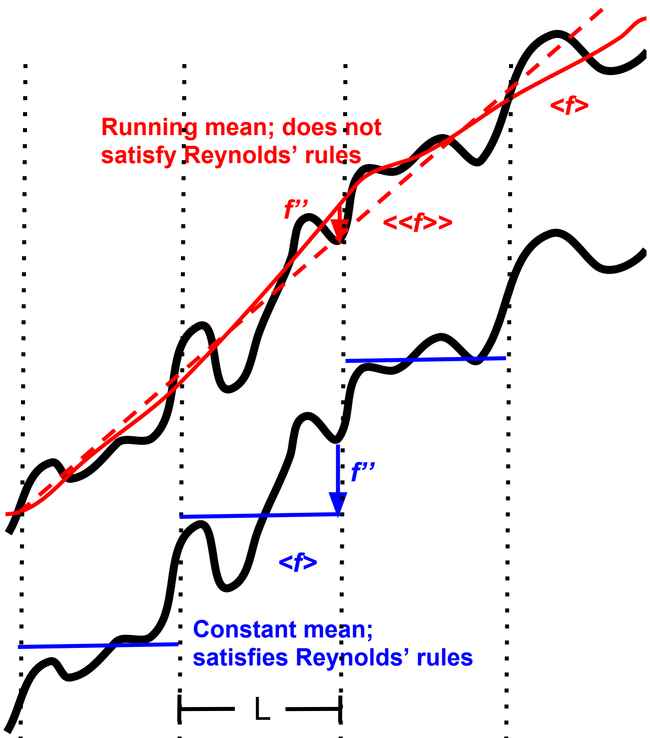

Figure 1: Two methods of filtering a field: using a running mean filter, as proposed by Germano (1992), and using a step-wise average, which is more akin to Reynolds averaging. Each black line corresponds to the unfiltered field, as might be provided by a direct numerical simulation. The perturbations for a running-mean filter are smaller than those from a step-wise filter. Adapted from Leonard (1974).

G can have a number of functional forms, such as a Gaussian. One of the simplest forms is a box filter, in which case the filtering operation reduces to

$$\overline{f}(\mathbf{x}) = \frac{1}{L} \int_{\tilde{x}=x-L/2}^{\tilde{x}=x+L/2} \int_{\tilde{y}=y-L/2}^{\tilde{y}=y+L/2} \int_{\tilde{z}=z-L/2}^{\tilde{z}=z+L/2} f(\tilde{\mathbf{x}})\, d\tilde{x} d\tilde{y} d\tilde{z}. \tag{68}$$

LES models are usually regarded as deterministic models, not stochastic models. LES models contain no random number generators. Furthermore, the Navier-Stokes equations are deterministic, and the filter is deterministic. However, filtering introduces terms that are unclosed. As such, those terms are uncertain: many realizations of underlying turbulent fields correspond to each value of the parameterized term (e.g., eddy diffusivity). This perhaps opens the door to a stochastic interpretation. One could imagine running an ensemble of LES, each of which has a different random seed corresponding to a different underlying turbulent field.

The filtering operation simply yields a running mean, which is smoother than the original field. If this coarse-grained field is filtered again, then the field becomes even smoother. Thus

$$\overline{\overline{f}} \neq \overline{f} \tag{69}$$

violating one of Reynolds' rules. Because $f'' = f - \overline{f}$, we also have,

$$\overline{f''} = \overline{f} - \overline{\overline{f}}. \tag{70}$$

In most cases,

$$\overline{f''} \neq 0. \tag{71}$$

The step-wise filter shown in Fig. 1 does obey Reynolds rules, but it introduces step discontinuities in the filtered field. That would seem to introduce complexities if the field were horizontally advected. Perhaps these complexities could be side-stepped by setting the grid spacing $dx = L$ and discretizing the horizontal advection. But the running-mean approach has the benefit of allowing dx and L to be kept separate. Thus the continuous equations and their discretized version are cleanly separated. For instance, one could imagine keeping L fixed at a large value, basing the parameterizations on that large value of L, but then testing the convergence as $dx \to 0$.

Nevertheless, we can still derive analogues to the Reynolds averaged equations, without assuming $\overline{\overline{f}} = \overline{f}$ or invoking f'' at all. Instead, the procedure is to form equations from the following "moments":

$$\begin{aligned}
\tau(f_i, f_j) &\equiv \overline{f_i f_j} - \overline{f_i}\,\overline{f_j} \\
\tau(f_i, f_j, f_k) &\equiv \overline{f_i f_j f_k} - \overline{f_i}\tau(f_j, f_k) - \overline{f_j}\tau(f_i, f_k) - \overline{f_k}\tau(f_i, f_j) - \overline{f_i}\,\overline{f_j}\,\overline{f_k} \\
&...
\end{aligned} \tag{72}$$

Here, the τ functions are analogous to Reynolds central moments:

$$\begin{aligned}
\tau(f_i, f_j) &\approx \overline{f_i' f_j'} \\
\tau(f_i, f_j, f_k) &\approx \overline{f_i' f_j' f_k'} \\
&...
\end{aligned} \tag{73}$$

Within the filtering framework, the parameterization (model reduction) problem boils down to approximating unclosed high-order quantities like $\overline{f_i f_j} - \overline{f_i}\,\overline{f_j}$ in terms of known low-order quantities like $\overline{f_i}$. To construct such a parameterization, one could obtain $\overline{f_i f_j} - \overline{f_i}\,\overline{f_j}$ by filtering a direct numerical (high-resolution) simulation and try to fit it in terms of quantities like $\overline{f_i}$ derived in the same way from the same simulation. The resulting parameterization will depend on the shape of the filter (e.g., box or Gaussian) and the size of the filtering scale, L. The best choices for these quantities is not obvious.

One can form equations that relate the τ-moments without ever using f''. To illustrate, consider the simple set of equations:

$$\text{Continuity}: \quad \frac{\partial u_i}{\partial x_i} = 0 \tag{74}$$

$$\text{Total water advection}: \quad \frac{\partial r_t}{\partial t} = -\frac{\partial}{\partial x_i}\left(u_i r_t\right) \tag{75}$$

The Reynolds averaged equation for total water variance is:

$$\frac{\partial}{\partial t}\overline{r_t''^2} = -\frac{\partial}{\partial x_i}\left(\overline{u_i}\overline{r_t''^2}\right) - \frac{\partial}{\partial x_i}\left(\overline{u_i'' r_t''^2}\right) - 2\overline{u_i'' r_t''}\frac{\partial \overline{r_t}}{\partial x_i} \tag{76}$$

We wish to create a similar equation, but involving the τ-moments, rather than perturbation quantities $()''$. In particular, we anticipate needing τ-moments that are similar to the moments in Eqn. (76):

$$\tau(r_t, r_t) = \overline{r_t^2} - \overline{r_t}^2 \tag{77}$$

$$\tau(u_i, r_t) = \overline{u_i r_t} - \overline{u_i}\;\overline{r_t} \tag{78}$$

$$\tau(u_i, r_t, r_t) = \overline{u_i r_t^2} - 2\overline{r_t}\tau(u_i, r_t) - \overline{u_i}\tau(r_t, r_t) - \overline{u_i}\;\overline{r_t}^2 \tag{79}$$

We want to form an equation for $\overline{r_t^2} - \overline{r_t}^2$. (When Reynolds' rules apply, $\overline{r_t''^2} = \overline{r_t^2} - \overline{r_t}^2$). First, let's form an equation for $\overline{r_t}^2$. Averaging the equation for r_t (75), and substituting in (78) yields:

$$\begin{aligned} \frac{\partial \overline{r_t}}{\partial t} &= -\frac{\partial}{\partial x_i}\left(\overline{u_i r_t}\right) \\ &= -\frac{\partial}{\partial x_i}\left(\overline{u_i}\;\overline{r_t}\right) - \frac{\partial}{\partial x_i}\tau(u_i, r_t) \end{aligned} \tag{80}$$

Multiplying this by $\overline{r_t}$ yields:

$$\frac{\partial \overline{r_t}^2}{\partial t} = -\frac{\partial}{\partial x_i}\left(\overline{u_i}\;\overline{r_t}^2\right) - 2\overline{r_t}\frac{\partial}{\partial x_i}\tau(u_i, r_t) \tag{81}$$

Now form an equation for $\overline{r_t^2}$. Multiplying the equation for r_t (75) by r_t yields:

$$\frac{\partial r_t^2}{\partial t} = -\frac{\partial}{\partial x_i}\left(u_i r_t^2\right) \tag{82}$$

Averaging this, and substituting in Eq. (79) yields:

$$\begin{aligned}\frac{\partial \overline{r_t^2}}{\partial t} &= -\frac{\partial}{\partial x_i}\left(\overline{u_i r_t^2}\right) \\ &= -\frac{\partial}{\partial x_i}\left(\tau(u_i, r_t, r_t) + 2\overline{r_t}\tau(u_i, r_t) + \overline{u_i}\tau(r_t, r_t) + \overline{u_i}\ \overline{r_t}^2\right)\end{aligned} \tag{83}$$

Subtract Eqn. (81) from Eqn. (83):

$$\begin{aligned}\frac{\partial(\overline{r_t^2} - \overline{r_t}^2)}{\partial t} &= \frac{\partial \tau(r_t, r_t)}{\partial t} \\ &= -\frac{\partial}{\partial x_i}\left(\overline{u_i}\tau(r_t, r_t)\right) - \frac{\partial}{\partial x_i}\tau(u_i, r_t, r_t) - \frac{\partial}{\partial x_i}\left(2\overline{r_t}\tau(u_i, r_t)\right) + 2\overline{r_t}\frac{\partial}{\partial x_i}\tau(u_i, r_t) \\ &= -\frac{\partial}{\partial x_i}\left(\overline{u_i}\tau(r_t, r_t)\right) - \frac{\partial}{\partial x_i}\tau(u_i, r_t, r_t) - 2\tau(u_i, r_t)\frac{\partial \overline{r_t}}{\partial x_i}\end{aligned} \tag{84}$$

This equation is structurally identical to the Reynolds averaged equation (76) but does not require any of the Reynolds assumptions and does not even require a calculation of r_t''. Note that the choices of filter shape and filter length (L) do not appear explicitly in the equations. However, they will appear, implicitly at least, in the parameterizations. For instance, the Smagorinsky diffusivity is proportional to L^2.

How are perturbations related to the τ-moments? A perturbation may be defined as

$$f''(x'') \equiv f(x'') - \overline{f}. \tag{85}$$

However, suppose we wish to filter a term involving f'', such as $\overline{f''^2}$: should $\overline{f}$ be regarded as $\overline{f}(x)$ or $\overline{f}(x'')$? If we set

$$f''(x'') \equiv f(x'') - \overline{f}(x). \tag{86}$$

then $\overline{f}$ is constant with respect to x'', and Reynolds rules, such as $\overline{\overline{f}} = \overline{f}$, hold. In fact, in this case,

$$\overline{f''^2} = \overline{f^2} - \overline{f}^2. \tag{87}$$

Alternatively, if we define

$$f''(x'') \equiv f(x'') - \overline{f}(x''). \tag{88}$$

then the mean is a running mean over the filtered region, and the perturbations from this running mean are (typically) much smaller. Reynolds rules are not valid.

To follow notational conventions in meteorology, the rest of the course will use the perturbation notation, but one may regard the perturbation moments as being shorthand for the corresponding τ-moments.