Prof. Larson
January 17, 2023

Problem Set 1
Math modeling of the atmosphere

Reading: Germano, M., 1992: Turbulence: The filtering approach. *J. Fluid Mech.*, **238**, 325–336..

Problem 1: Spatial filtering

Consider a 1D field

$$f(x'') = \frac{1}{2}(1 - \cos(x'')) \tag{1}$$

that extends indefinitely to the left and the right. In this problem, we'll see how running-mean filtering of f affects the perturbations of f, namely f'', from the filtered version of f, namely $\overline{f}$.

Assume that the filter used is a simple box (uniform) filter, extending from $x'' = x - \pi/2$ to $x'' = x + \pi/2$.

Throughout this problem, when you are asked to plot functions, you may sketch the functions by hand or plot them using a computer.

i) Calculate $\overline{f}(x)$. Show your work.

ii) Overplot the original function $f(x'')$ and the filtered version $\overline{f}$.

iii) Suppose we define $f'' \equiv f(x'') - \overline{f}(x)$. (Note that the definition here uses $\overline{f}(x)$ rather than $\overline{f}(x'')$.) Calculate $\overline{f''}(x)$, showing your work.

iv) Now suppose we define $f'' \equiv f(x'') - \overline{f}(x'')$. (Note that this time the definition uses $\overline{f}(x'')$.) Calculate $\overline{f''}(x'')$ using this new definition, showing your work. Plot $\overline{f''}(x)$.

v) Explain in physical terms why $\overline{f''}(x)$ from part iii) is different from $\overline{f''}(x'')$ from part iv). Which part obeys Reynolds' rules?

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Problem Set 1, Solution Outline
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Problem 1, Solution Outline

i)

$$\begin{aligned}
\overline{f}(x) &= \frac{1}{\pi}\int_{x''=x-\pi/2}^{x''=x+\pi/2} f(x'')\, dx'' && (2)\\
&= \frac{1}{2}\left(1-\frac{2}{\pi}\cos(x)\right) && (3)\\
&= \frac{1}{2}-\frac{\cos(x)}{\pi} && (4)\\
& && (5)
\end{aligned}$$

ii)

A plot shows that $f(x'')$ is a negative cosine function that extends from 0 to 1 in the vertical with a mean of 1/2. $\overline{f}(x'')$ is a smoothed version of $f(x'')$.

iii)

For the purpose of filtering in x'', $f(x)$ is a constant.

$$\begin{aligned}
\overline{f''}(x) &= \frac{1}{\pi}\int_{x''=x-\pi/2}^{x''=x+\pi/2} f''(x'')\, dx'' && (6)\\
&= \frac{1}{\pi}\int_{x''=x-\pi/2}^{x''=x+\pi/2} \left(f(x'')-\overline{f}(x)\right)\, dx'' && (7)\\
&= \overline{f}(x)-\overline{f}(x) && (8)\\
&= 0 && (9)\\
& && (10)
\end{aligned}$$

iv)

Here f'' is the difference between f and $\overline{f}(x'')$, both of which are shown in the plot above.

$$\overline{f''}(x) = \frac{1}{\pi}\int_{x''=x-\pi/2}^{x''=x+\pi/2}\left(f(x'') - \overline{f}(x'')\right)\,dx'' \tag{11}$$
$$= \overline{f}(x) - \overline{\overline{f}}(x) \tag{12}$$

Here

$$\overline{\overline{f}}(x) = \frac{1}{\pi}\int_{x''=x-\pi/2}^{x''=x+\pi/2}\left(\frac{1}{2} - \frac{\cos(x'')}{\pi}\right)\,dx'' \tag{13}$$
$$= \frac{1}{2}\left(1 - \left(\frac{2}{\pi}\right)^2 \cos(x)\right) \tag{14}$$

Therefore,

$$\overline{f''}(x) = -\frac{\pi - 2}{\pi^2}\cos(x) \tag{15}$$

A plot of $\overline{f''}(x)$ shows a damped, inverted cosine. It is non-zero, unlike for the previous definition of f''.

v) Part iv) uses a running mean filter, and hence the average deviation from the running mean varies in space. In contrast, part iii) uses a constant (w.r.t. x'') filter, and hence the local deviations are larger, but they average to zero. Because the filter in part iii) is a constant, it obeys Reynolds' rules.