Multivariate Statistical Analysis

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Exercise 1

Problem

Determine which of the following matrices admit a matrix square root, and for the ones that do, find it.

(i)

```
A_1 <- matrix(c(2, 3, -1, 3, 2, 1, -1, 1, 0), 3, 3)
print(A_1)
```

```
##      [,1] [,2] [,3]
## [1,]    2    3   -1
## [2,]    3    2    1
## [3,]   -1    1    0
```

Check if the eigenvalues of A_1 are positive:

```
e_1 <- eigen(A_1)$values
print(e_1)
```

```
## [1]  5  1 -2
```

Because one of the eigenvalues is negative, more precisely -2, according to the lecture it does not exist a square root matrix for A_1.

(i)

```
A_2 <- matrix(c(3, 2, -1, 2, 3, 1, -1, 1, 2), 3, 3)
print(A_2)
```

```
##      [,1] [,2] [,3]
## [1,]    3    2   -1
## [2,]    2    3    1
## [3,]   -1    1    2
```

Check if the eigenvalues of A_2 are positive:

```
e_2 <- eigen(A_2)$values
print(e_2)
```

```
## [1] 5.000000e+00 3.000000e+00 8.881784e-16
```

Because the eigenvalues of A_2 are positive and the matrix is symmetric, A_2 is positive definite, and it has a matrix square root given by the following matrix:

```
L <- diag(eigen(A_2)$values)
V <- eigen(A_2)$vectors
L_sq <- sqrt(L)
B <- V %*% L_sq %*% solve(V)
print(B)
```

```
##            [,1]      [,2]       [,3]
## [1,]  1.4067091 0.8293588 -0.5773503
## [2,]  0.8293588 1.4067091  0.5773503
## [3,] -0.5773503 0.5773503  1.1547005
```

We can verify this by Multiplication:

```
print(B %*% B)
```

```
##      [,1] [,2] [,3]
## [1,]    3    2   -1
## [2,]    2    3    1
## [3,]   -1    1    2
```

Exercise 2

Problem

Let H be the subspace of $\mathbb{R}^3$ with basis v_1 and v_2 given by

```
v_1 <- c(1, 1, 1)
print(v_1)
```

```
## [1] 1 1 1
```

```
v_2 <- c(1, -1, 0)
print(v_2)
```

```
## [1]  1 -1  0
```

a)

Find the projection matrix P associated with $\mathcal{H}$:

```r
A <- matrix(c(v_1, v_2), 3, 2)
print(A)
```

```
##      [,1] [,2]
## [1,]    1    1
## [2,]    1   -1
## [3,]    1    0
```

```r
A_t <- t(A)
A_i <- solve(A_t %*% A)
P <- A %*% A_i %*% A_t
print(P)
```

```
##            [,1]       [,2]      [,3]
## [1,]  0.8333333 -0.1666667 0.3333333
## [2,] -0.1666667  0.8333333 0.3333333
## [3,]  0.3333333  0.3333333 0.3333333
```

b)

i)

```r
x1 <- c(0, 1, 1)
x1_best <- P %*% x1
print(x1_best)
```

```
##           [,1]
## [1,] 0.1666667
## [2,] 1.1666667
## [3,] 0.6666667
```

ii)

```r
x2 <- c(2, 0, 1)
x2_best <- P %*% x2
print(x2_best)
```

```
##               [,1]
## [1,]  2.000000e+00
## [2,] -5.551115e-17
## [3,]  1.000000e+00
```

```r
print(x2 %*% A)
```

```
##      [,1] [,2]
## [1,]    3    2
```

The result tells us, that x_2 is part of the subspace. We can confirm this because x_2 is a linear combination of v_1 and v_2.

iii)

```
x3 <- c(-1, -1, 2)
x3_best <- P %*% x3
print(x3_best)
```

```
##              [,1]
## [1,] 1.110223e-16
## [2,] 1.110223e-16
## [3,] 0.000000e+00
```

```
print(v_1 %*% x3)
```

```
##      [,1]
## [1,]    0
```

```
print(v_2 %*% x3)
```

```
##      [,1]
## [1,]    0
```

The result tells us, x_3 is perpendicular to v_1 and v_2.

Exercise 3

Problem

Consider the matrix

```
B <- matrix(c(2 / 3, 1 / 3, 1 / 3, 1 / 3, 2 / 3, -1 / 3, 1 / 3, -1 / 3, 2 / 3), 3, 3)
print(B)
```

```
##           [,1]       [,2]       [,3]
## [1,] 0.6666667  0.3333333  0.3333333
## [2,] 0.3333333  0.6666667 -0.3333333
## [3,] 0.3333333 -0.3333333  0.6666667
```

Is B a projection matrix for some space $\mathcal{H}$? If so, find a basis for $\mathcal{H}$.

According to lecture, such that a matrix is a projection matrix it has to be symmetric and idempotent.

```
print(B)
```

```
##           [,1]       [,2]       [,3]
## [1,] 0.6666667  0.3333333  0.3333333
## [2,] 0.3333333  0.6666667 -0.3333333
## [3,] 0.3333333 -0.3333333  0.6666667
```

```r
print(t(B))
```

```
##           [,1]       [,2]       [,3]
## [1,] 0.6666667  0.3333333  0.3333333
## [2,] 0.3333333  0.6666667 -0.3333333
## [3,] 0.3333333 -0.3333333  0.6666667
```

```r
print(B %*% B)
```

```
##           [,1]       [,2]       [,3]
## [1,] 0.6666667  0.3333333  0.3333333
## [2,] 0.3333333  0.6666667 -0.3333333
## [3,] 0.3333333 -0.3333333  0.6666667
```

Because the matrices are the same, B is symmetric and idempotent. Hence, B is a projection matrix.

To find a basis for $\mathcal{H}$, we have to find the eigenvectors:

```r
eigen(B)$vector
```

```
##           [,1]       [,2]       [,3]
## [1,] 0.8164966  0.0000000  0.5773503
## [2,] 0.4082483 -0.7071068 -0.5773503
## [3,] 0.4082483  0.7071068 -0.5773503
```