MthStat 568/768 – Multivariate Statistical Analysis – Spring 2024

Homework 2

Due Wednesday, February 21

1. (a) Prove that if a random vector $\mathbf{X} \in \mathbb{R}^p$ has mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$, and $\mathbf{A}$ is a $p \times p$ constant matrix, then $E(\mathbf{X}^T \mathbf{A} \mathbf{X}) = \mathrm{tr}(\mathbf{A}\boldsymbol{\Sigma}) + \boldsymbol{\mu}^T \mathbf{A} \boldsymbol{\mu}$, where tr() denotes the trace of a matrix. (*Hint:* note that $\mathrm{tr}(\mathbf{A}\mathbf{v}\mathbf{v}^T) = \mathbf{v}^T \mathbf{A} \mathbf{v}$ for any vector $\mathbf{v}$.)

   (b) Apply the previous result to the specific case of $X_1, \ldots, X_p$ being uncorrelated random variables with mean μ and variance σ^2, and $\mathbf{A} = \mathbf{I} - \frac{1}{p}\mathbf{J}$, where $\mathbf{J}$ denotes the $p \times p$ matrix of ones. What does $\mathbf{X}^T \mathbf{A} \mathbf{X}$ come down to in this case?

   (c) Repeat part (b) but now assuming that the X_js have constant correlations, that is, $\mathrm{cov}\,(X_i, X_j) = \rho\sigma^2$ for some $\rho \in (-1, 1)$ when $i \neq j$.

2. (a) Prove that if $\mathbf{X} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ then $\mathbf{Z} = \boldsymbol{\Sigma}^{-1/2}(\mathbf{X} - \boldsymbol{\mu}) \sim N(\mathbf{0}, \mathbf{I})$ (where $\boldsymbol{\Sigma}^{-1/2}$ denotes $\left(\boldsymbol{\Sigma}^{1/2}\right)^{-1}$ and $\boldsymbol{\Sigma}^{1/2}$ is the matrix square root of $\boldsymbol{\Sigma}$).

   (b) Find $\mathbf{Z}$ explicitly when $\boldsymbol{\mu} = \mathbf{0}$ and

$$\boldsymbol{\Sigma} = \begin{pmatrix} 1 & -2 & 0 \\ -2 & 5 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

3. Prove that if $\mathbf{X} = (X_1, X_2)$ has joint density

$$f(x_1, x_2) = \begin{cases} 2\varphi(x_1)\varphi(x_2), & \text{if } x_1 \text{ and } x_2 \text{ are both negative or both positive,} \\ 0, & \text{otherwise,} \end{cases}$$

   where $\varphi(x)$ is the $N(0,1)$ density, then the marginal densities of X_1 and X_2 are $N(0,1)$. But is $\mathbf{X}$ multivariate Normal?