Multivariate Statistical Analysis

Homework 2

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Problem 1

(a)

We have $x \in \mathbb{R}^p$, $\mathbb{E}[x] = \mu$, $\mathrm{Cov}(x) = \Sigma$, A is a $p \times p$ constant matrix and $\mathrm{tr}(Avv^\top) = v^\top Av$. Because A is a constant $p \times p$ matrix, A is symmetric. Because of this symmetry, it follows that it has a Cholesky decomposition as $A = C^\top C$.

Let $y = Cx$.

Then

$$\begin{aligned}
\mathbb{E}[x^\top Ax] &= \mathbb{E}[x^\top C^\top Cx] \\
&= \mathbb{E}[(Cx)^\top Cx] \\
&= \mathbb{E}[y^\top y] \\
&= \sum_i \mathbb{E}[y_i^2] \\
&= \sum_i \mathrm{Var}(y_i) + \mathbb{E}[y_i]^2 \\
&= \mathrm{tr}(\Sigma_y) + \mu_y^\top \mu_y
\end{aligned}$$

where $\Sigma_y = \mathbb{E}[(y - \mathbb{E}[y])(y - \mathbb{E}[y])^\top) = C\Sigma C^\top$ and $\mu_y = C_\mu$.

$$\begin{aligned}
\implies \mathbb{E}[x^\top Ax] &= \mathrm{tr}(C\Sigma C^\top) + \mu^\top \underbrace{C^\top C}_{=A} \mu \\
&= \mathrm{tr}(\Sigma C^\top C) + \mu^\top A\mu \\
&= \mathrm{tr}(\Sigma A) + \mu^\top A\mu
\end{aligned}$$

(b)

$X_1, \ldots, X_n$ uncorrelated:

$\mathrm{Cov}(X_i, X_j) = 0$ for $i \neq j$ and $\mathrm{Cov}(X_i, X_i) = \mathrm{Var}(X_i) = \sigma^2$

$$\implies \Sigma = \begin{pmatrix} \sigma^2 & 0 & 0 & \dots & 0 \\ 0 & \sigma^2 & 0 & \dots & 0 \\ 0 & 0 & \ddots & \dots & \vdots \\ \vdots & \vdots & \dots & \ddots & \vdots \\ 0 & 0 & \dots & \dots & \sigma^2 \end{pmatrix}$$

$$J = \begin{pmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{pmatrix} \in \mathbb{R}^{p \times p}.$$

$$A = T - \frac{1}{p}J = \begin{pmatrix} 1-\frac{1}{p} & -\frac{1}{p} & -\frac{1}{p} & \dots & -\frac{1}{p} \\ -\frac{1}{p} & 1-\frac{1}{p} & -\frac{1}{p} & \dots & -\frac{1}{p} \\ -\frac{1}{p} & -\frac{1}{p} & \ddots & \dots & \vdots \\ \vdots & \vdots & \dots & \ddots & \vdots \\ -\frac{1}{p} & -\frac{1}{p} & \dots & \dots & 1-\frac{1}{p} \end{pmatrix} \in \mathbb{R}^{p \times p}.$$

$$A\Sigma = \begin{pmatrix} (1-\frac{1}{p})\sigma^2 & -\frac{\sigma^2}{p} & -\frac{\sigma^2}{p} & \dots & -\frac{\sigma^2}{p} \\ -\frac{\sigma^2}{p} & (1-\frac{1}{p})\sigma^2 & -\frac{1}{p} & \dots & -\frac{1}{p} \\ -\frac{\sigma^2}{p} & -\frac{\sigma^2}{p} & \ddots & \dots & \vdots \\ \vdots & \vdots & \dots & \ddots & \vdots \\ -\frac{\sigma^2}{p} & -\frac{\sigma^2}{p} & \dots & \dots & (1-\frac{1}{p})\sigma^2 \end{pmatrix} \in \mathbb{R}^{p \times p}.$$

$$\begin{aligned}
\mathbb{E}[x^\top A x] &\overset{a)}{=} \operatorname{tr}(A\Sigma) + \mu^\top A \mu \\
&= \sum_{i=1}^{p} (1-\frac{1}{p})\sigma^2 + (\mu \dots \mu)^\top \begin{pmatrix} 1-\frac{1}{p} & -\frac{1}{p} & -\frac{1}{p} & \dots & -\frac{1}{p} \\ -\frac{1}{p} & 1-\frac{1}{p} & -\frac{1}{p} & \dots & -\frac{1}{p} \\ -\frac{1}{p} & -\frac{1}{p} & \ddots & \dots & \vdots \\ \vdots & \vdots & \dots & \ddots & \vdots \\ -\frac{1}{p} & -\frac{1}{p} & \dots & \dots & 1-\frac{1}{p} \end{pmatrix} \begin{pmatrix} \mu \\ \vdots \\ \mu \end{pmatrix} \\
&= p \cdot (1-\frac{1}{p}) \cdot \sigma^2 + (\underbrace{\mu \cdot (1-\frac{1}{p}) + (p-1) \cdot (-\frac{1}{p}) \cdot \mu}_{\underbrace{= \frac{p-1}{p} \cdot \mu + \frac{1-p}{p} \cdot \mu}_{=0}} \dots \mu \cdot (1-\frac{1}{p}) + (p-1) \cdot (-\frac{1}{p}) \cdot \mu) \begin{pmatrix} \mu \\ \vdots \\ \mu \end{pmatrix} \\
&= (p-1) \cdot \sigma^2 + (0 \dots 0) \cdot \begin{pmatrix} \mu \\ \vdots \\ \mu \end{pmatrix} \\
&= (p-1) \cdot \sigma^2
\end{aligned}$$

(c)

Problem 2

(a)

(b)

```
Sigma <- matrix(data = c(1, -2, 0, -2, 5, 0, 0, 0, 2), nrow = 3, ncol = 3)
print(Sigma)
```

```
##      [,1] [,2] [,3]
## [1,]    1   -2    0
## [2,]   -2    5    0
## [3,]    0    0    2
```

```
library(expm)
```

```
Sigma_sqrt <- sqrtm(Sigma)
print(Sigma_sqrt)
```

```
##            [,1]       [,2]     [,3]
## [1,]  0.7071068 -0.7071068 0.000000
## [2,] -0.7071068  2.1213203 0.000000
## [3,]  0.0000000  0.0000000 1.414214
```

Problem 3