Multivariate Statistical Analysis

Homework 2

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02/15/24

```r
library(expm)
```

Problem 1

(a)

We have $x \in \mathbb{R}^p$, $\mathbb{E}[x] = \mu$, $\text{Cov}(x) = \Sigma$, A is a $p \times p$ constant matrix and $\text{tr}(Avv^\top) = v^\top Av$. Because A is a constant $p \times p$ matrix, A is symmetric. Because of this symmetry, it follows that it has a Cholesky decomposition as $A = C^\top C$.

Let $y = Cx$.

Then

$$
\begin{aligned}
\mathbb{E}[x^\top Ax] &= \mathbb{E}[x^\top C^\top Cx] &(1)\\
&= \mathbb{E}[(Cx)^\top Cx] &(2)\\
&= \mathbb{E}[y^\top y] &(3)\\
&= \sum_i \mathbb{E}[y_i^2] &(4)\\
&= \sum_i \text{Var}(y_i) + \mathbb{E}[y_i]^2 &(5)\\
&= \text{tr}(\Sigma_y) + \mu_y^\top \mu_y &(6)
\end{aligned}
$$

where $\Sigma_y = \mathbb{E}[(y - \mathbb{E}[y])(y - \mathbb{E}[y])^\top) = C\Sigma C^\top$ and $\mu_y = C_\mu$.

$$
\begin{aligned}
\implies \mathbb{E}[x^\top Ax] &= \text{tr}(C\Sigma C^\top) + \mu^\top \underbrace{C^\top C}_{=A} \mu &(7)\\
&= \text{tr}(\Sigma C^\top C) + \mu^\top A\mu &(8)\\
&= \text{tr}(\Sigma A) + \mu^\top A\mu &(9)
\end{aligned}
$$

(b)

(c)

Problem 2

(a)

(b)

```r
Sigma <- matrix(data = c(1, -2, 0, -2, 5, 0, 0, 0, 2), nrow = 3, ncol = 3)
print(Sigma)
```

```
##      [,1] [,2] [,3]
## [1,]    1   -2    0
## [2,]   -2    5    0
## [3,]    0    0    2
```

```r
Sigma_sqrt <- sqrtm(Sigma)
print(Sigma_sqrt)
```

```
##            [,1]       [,2]     [,3]
## [1,]  0.7071068 -0.7071068 0.000000
## [2,] -0.7071068  2.1213203 0.000000
## [3,]  0.0000000  0.0000000 1.414214
```

Problem 3