

# Multivariate Statistical Analysis

## Homework 2

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### Problem 1

(a)

We have  $X \in \mathbb{R}^p$ ,  $\mathbb{E}[X] = \mu$ ,  $\text{Cov}(X) = \Sigma$ ,  $A$  is a  $p \times p$  constant matrix and  $\text{tr}(Avv^\top) = v^\top Av$ . Because  $A$  is a constant  $p \times p$  matrix,  $A$  is symmetric. Because of this symmetry, it follows that it has a Cholesky decomposition as  $A = C^\top C$ .

Let  $y = CX$ .

Then

$$\begin{aligned}\mathbb{E}[X^\top AX] &= \mathbb{E}[X^\top C^\top CX] \\ &= \mathbb{E}[(CX)^\top CX] \\ &= \mathbb{E}[y^\top y] \\ &= \sum_i \mathbb{E}[y_i^2] \\ &= \sum_i \text{Var}(y_i) + \mathbb{E}[y_i]^2 \\ &= \text{tr}(\Sigma_y) + \mu_y^\top \mu_y\end{aligned}$$

where  $\Sigma_y = \mathbb{E}[(y - \mathbb{E}[y])(y - \mathbb{E}[y])^\top] = C\Sigma C^\top$  and  $\mu_y = C \cdot \mu$ .

$$\begin{aligned}\implies \mathbb{E}[X^\top AX] &= \text{tr}(C\Sigma C^\top) + \mu^\top \underbrace{C^\top C}_{=A} \mu \\ &= \text{tr}(\Sigma C^\top C) + \mu^\top A \mu \\ &= \text{tr}(\Sigma A) + \mu^\top A \mu\end{aligned}$$

(b)

$X_1, \dots, X_n$  uncorrelated:

$\text{Cov}(X_i, X_j) = 0$  for  $i \neq j$  and  $\text{Cov}(X_i, X_i) = \text{Var}(X_i) = \sigma^2$

$$\Rightarrow \Sigma = \begin{pmatrix} \sigma^2 & 0 & 0 & \dots & 0 \\ 0 & \sigma^2 & 0 & \dots & 0 \\ 0 & 0 & \ddots & \dots & \vdots \\ \vdots & \vdots & \dots & \ddots & \vdots \\ 0 & 0 & \dots & \dots & \sigma^2 \end{pmatrix}, J = \begin{pmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{pmatrix} \in \mathbb{R}^{p \times p}.$$

$$A = I - \frac{1}{p}J = \begin{pmatrix} 1 - \frac{1}{p} & -\frac{1}{p} & -\frac{1}{p} & \dots & -\frac{1}{p} \\ -\frac{1}{p} & 1 - \frac{1}{p} & -\frac{1}{p} & \dots & -\frac{1}{p} \\ -\frac{1}{p} & -\frac{1}{p} & \ddots & \dots & \vdots \\ \vdots & \vdots & \dots & \ddots & \vdots \\ -\frac{1}{p} & -\frac{1}{p} & \dots & \dots & 1 - \frac{1}{p} \end{pmatrix} \in \mathbb{R}^{p \times p},$$

$$A\Sigma = \begin{pmatrix} (1 - \frac{1}{p})\sigma^2 & -\frac{\sigma^2}{p} & -\frac{\sigma^2}{p} & \dots & -\frac{\sigma^2}{p} \\ -\frac{\sigma^2}{p} & (1 - \frac{1}{p})\sigma^2 & -\frac{1}{p} & \dots & -\frac{1}{p} \\ -\frac{\sigma^2}{p} & -\frac{\sigma^2}{p} & \ddots & \dots & \vdots \\ \vdots & \vdots & \dots & \ddots & \vdots \\ -\frac{\sigma^2}{p} & -\frac{\sigma^2}{p} & \dots & \dots & (1 - \frac{1}{p})\sigma^2 \end{pmatrix} \in \mathbb{R}^{p \times p}.$$

$$\mathbb{E}[X^\top AX] \stackrel{a)}{=} \text{tr}(A\Sigma) + \mu^\top A\mu$$

$$\begin{aligned} &= \sum_{i=1}^p (1 - \frac{1}{p})\sigma^2 + (\mu \dots \mu)^\top \begin{pmatrix} 1 - \frac{1}{p} & -\frac{1}{p} & -\frac{1}{p} & \dots & -\frac{1}{p} \\ -\frac{1}{p} & 1 - \frac{1}{p} & -\frac{1}{p} & \dots & -\frac{1}{p} \\ -\frac{1}{p} & -\frac{1}{p} & \ddots & \dots & \vdots \\ \vdots & \vdots & \dots & \ddots & \vdots \\ -\frac{1}{p} & -\frac{1}{p} & \dots & \dots & 1 - \frac{1}{p} \end{pmatrix} \begin{pmatrix} \mu \\ \vdots \\ \mu \end{pmatrix} \\ &= p \cdot (1 - \frac{1}{p}) \cdot \sigma^2 + \underbrace{(\mu \cdot (1 - \frac{1}{p}) + (p-1) \cdot (-\frac{1}{p}) \cdot \mu \dots \mu \cdot (1 - \frac{1}{p}) + (p-1) \cdot (-\frac{1}{p}) \cdot \mu)}_{\substack{= \frac{p-1}{p} \cdot \mu + \frac{1-p}{p} \cdot \mu \\ = 0}} \begin{pmatrix} \mu \\ \vdots \\ \mu \end{pmatrix} \\ &= (p-1) \cdot \sigma^2 + (0 \dots 0) \cdot \begin{pmatrix} \mu \\ \vdots \\ \mu \end{pmatrix} \\ &= (p-1) \cdot \sigma^2 \end{aligned}$$

(c)

$$\text{Cov}(X_i, X_j) = \rho\sigma^2, i \neq j, \text{Cov}(X_i, X_i) = \sigma^2.$$

$$\Sigma = \text{Cov}(X) = \begin{pmatrix} \sigma^2 & \rho\sigma^2 & \rho\sigma^2 & \dots & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 & \rho\sigma^2 & \dots & \rho\sigma^2 \\ \rho\sigma^2 & \rho\sigma^2 & \ddots & \dots & \vdots \\ \vdots & \vdots & \dots & \ddots & \rho\sigma^2 \\ \rho\sigma^2 & \rho\sigma^2 & \dots & \rho\sigma^2 & \sigma^2 \end{pmatrix}$$

$$A = I - \frac{1}{p}J = \begin{pmatrix} 1 - \frac{1}{p} & -\frac{1}{p} & -\frac{1}{p} & \dots & -\frac{1}{p} \\ -\frac{1}{p} & 1 - \frac{1}{p} & -\frac{1}{p} & \dots & -\frac{1}{p} \\ -\frac{1}{p} & -\frac{1}{p} & \ddots & \dots & \vdots \\ \vdots & \vdots & \dots & \ddots & \vdots \\ -\frac{1}{p} & -\frac{1}{p} & \dots & \dots & 1 - \frac{1}{p} \end{pmatrix} \in \mathbb{R}^{p \times p}.$$

$$\begin{aligned} \text{tr}(A\Sigma) &= \sum_{i=1}^p \left(1 - \frac{1}{p}\right) \cdot \sigma^2 + (p-1) \cdot \left(-\frac{1}{p}\right) \cdot \rho\sigma^2 \\ &= p \cdot \left(1 - \frac{1}{p}\right) \cdot \sigma^2 + (p-1) \cdot \left(-\frac{1}{p}\right) \cdot \rho\sigma^2 \\ &= (p-1) \cdot \sigma^2 + (1-p)\rho \cdot \sigma^2 \\ &= (p-1) \cdot \sigma^2 + (\rho - p \cdot \rho) \cdot \sigma^2 \\ &= (p-1 + \rho - p \cdot \rho) \cdot \sigma^2 \\ \mathbb{E}[X^\top AX] &= \text{tr}(A\Sigma) + \underbrace{\mu^\top A \mu}_{=0 \text{ (as shown above)}} \\ &= (p-1 + \rho - p \cdot \rho) \cdot \sigma^2 \end{aligned}$$

## Problem 2

(a)

$X \sim \mathcal{N}(\mu, \Sigma)$ . We have to prove that  $Z = \Sigma^{-\frac{1}{2}}(X - \mu) \sim \mathcal{N}(0, I)$ .

$$\begin{aligned} \mathbb{E}[Z] &= \mathbb{E}[\Sigma^{-\frac{1}{2}}(X - \mu)] = \Sigma^{-\frac{1}{2}}(\mathbb{E}[X] - \mu) = \Sigma^{-\frac{1}{2}}(\mu - \mu) = 0, \\ \text{Var}(Z) &= \text{Var}(\Sigma^{-\frac{1}{2}}(X - \mu)) = (\Sigma^{-\frac{1}{2}})^2 \text{Var}(X - \mu) \\ &= \Sigma^{-1} \cdot \text{Var}(X) = \Sigma^{-1} \cdot \Sigma = I. \\ \implies Z &\sim \mathcal{N}(\mu, \Sigma). \end{aligned}$$

(b)

$$\begin{aligned} Z &= \Sigma^{-\frac{1}{2}}(X - \mu) \\ &= \Sigma^{-\frac{1}{2}}(X - 0) \\ &= \Sigma^{-\frac{1}{2}} \cdot X. \end{aligned}$$

```
Sigma <- matrix(data = c(1, -2, 0, -2, 5, 0, 0, 0, 2), nrow = 3, ncol = 3)
print(Sigma)
```

```
##      [,1] [,2] [,3]
## [1,]    1  -2    0
## [2,]   -2    5    0
## [3,]    0    0    2
```

Calculate the matrix square root:

```
Sigma_sqrt <- expm::sqrtm(Sigma)
print(Sigma_sqrt)
```

```
##      [,1]      [,2]      [,3]
## [1,] 0.7071068 -0.7071068 0.0000000
## [2,] -0.7071068  2.1213203 0.0000000
## [3,] 0.0000000  0.0000000 1.414214
```

Check if the matrix square root times the matrix square root equals A:

```
print(Sigma_sqrt %*% Sigma_sqrt)
```

```
##      [,1] [,2] [,3]
## [1,]    1  -2    0
## [2,]   -2    5    0
## [3,]    0    0    2
```

```
Sigma_sqrt_inv <- solve(Sigma_sqrt)
print(Sigma_sqrt_inv)
```

```
##      [,1]      [,2]      [,3]
## [1,] 2.1213203 0.7071068 0.0000000
## [2,] 0.7071068 0.7071068 0.0000000
## [3,] 0.0000000 0.0000000 0.7071068
```

So it follows:

$$Z = \begin{pmatrix} 2.1213203 \cdot x_1 + 0.7071068 \cdot x_2 \\ 0.7071068 \cdot x_1 + 0.7071068 \cdot x_2 \\ 0.7071068 \cdot x_3 \end{pmatrix}$$

### Problem 3

$X = (X_1, X_2)$  with joint pdf:

$$f(x_1, x_2) = \begin{cases} 2\varphi(X_1)\varphi(X_2), & X_1 \cdot X_2 > 0, \\ 0, & \text{otherwise} \end{cases}$$

where

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x}{2}\right)$$

$$\implies f(x_1, x_2) = 2 \cdot \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x_1}{2}\right) \cdot \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x_2}{2}\right) \cdot I_{\{x_1 \cdot x_2 > 0\}}(x_1, x_2)$$

Case 1 ( $x_1, x_2 > 0$ ):

$$\begin{aligned} f_{X_1}(x_1) &= \int_0^\infty 2 \cdot \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x_1}{2}\right) \cdot \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x_2}{2}\right) dx_2 \\ &= 2 \cdot \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x_1}{2}\right) \int_0^\infty \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x_2}{2}\right) dx_2 \\ &= 2 \cdot \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x_1}{2}\right) \cdot \frac{1}{2} \cdot \underbrace{\int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x_2}{2}\right) dx_2}_{=1 \text{ (pdf of } \mathcal{N}(0,1) \text{ over its whole support)}} \\ &= \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x_1}{2}\right) \cdot I_{\{0, \infty\}}(x_1). \\ f_{X_2}(x_2) &= \int_0^\infty 2 \cdot \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x_2}{2}\right) \cdot \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x_1}{2}\right) dx_1 \\ &= 2 \cdot \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x_2}{2}\right) \int_0^\infty \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x_1}{2}\right) dx_1 \\ &= 2 \cdot \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x_2}{2}\right) \cdot \frac{1}{2} \cdot \underbrace{\int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x_1}{2}\right) dx_1}_{=1 \text{ (pdf of } \mathcal{N}(0,1) \text{ over its whole support)}} \\ &= \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x_2}{2}\right) \cdot I_{\{0, \infty\}}(x_2). \end{aligned}$$

Case 2 ( $x_1, x_2 < 0$ ):

$$\begin{aligned}
f_{X_1}(x_1) &= \int_0^\infty 2 \cdot \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x_1}{2}\right) \cdot \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x_2}{2}\right) dx_2 \\
&= 2 \cdot \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x_1}{2}\right) \int_{-\infty}^0 \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x_2}{2}\right) dx_2 \\
&= 2 \cdot \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x_1}{2}\right) \cdot \frac{1}{2} \cdot \underbrace{\int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x_2}{2}\right) dx_2}_{=1 \text{ (pdf of } \mathcal{N}(0,1) \text{ over its whole support)}} \\
&= \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x_1}{2}\right) \cdot I_{\{-\infty, 0\}}(x_1). \\
f_{X_2}(x_2) &= \int_0^\infty 2 \cdot \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x_2}{2}\right) \cdot \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x_1}{2}\right) dx_1 \\
&= 2 \cdot \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x_2}{2}\right) \int_{-\infty}^0 \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x_1}{2}\right) dx_1 \\
&= 2 \cdot \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x_2}{2}\right) \cdot \frac{1}{2} \cdot \underbrace{\int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x_1}{2}\right) dx_1}_{=1 \text{ (pdf of } \mathcal{N}(0,1) \text{ over its whole support)}} \\
&= \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x_2}{2}\right) \cdot I_{\{-\infty, 0\}}(x_2).
\end{aligned}$$

Then

$$\begin{aligned}
f_{X_1}(x_1) &= \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x_1}{2}\right) \cdot I_{\{-\infty, \infty\}}(x_1), \\
f_{X_2}(x_2) &= \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x_2}{2}\right) \cdot I_{\{-\infty, \infty\}}(x_2).
\end{aligned}$$

Thus,  $f_{X_1}(x_1) \sim \mathcal{N}(0, 1)$  and  $f_{X_2}(x_2) \sim \mathcal{N}(0, 1)$ , but  $X$  is not multivariate Normal since the support of  $f(X_1, X_2)$  is not  $\mathbb{R}^2$ .