MTHSTAT 564/564G/764–Time Series Analysis
Spring 2024
Problem Solving Set 3

Please think about the following problems from the textbook in advance of our problem solving sessions on them:

Problem Solving 3

1. Suppose $Y_t = \beta_0 + \beta_1 t + X_t$, where $\{X_t\}$ is a zero-mean stationary series with autocovariance function γ_k and β_0 and β_1 are constants.

   (a) Show that $\{Y_t\}$ is not stationary, but that $W_t = \nabla Y_t = Y_t - Y_{t-1}$ is stationary.

   (b) In general, show that if $Y_t = \mu_t + X_t$, where $\{X_t\}$ is a zero-mean stationary series, and μ_t is a polynomial in t of degree d, then $\nabla^m Y_t = \nabla(\nabla^{m-1} Y_t)$ is stationary for $m \geq d$ and nonstationary for $0 \leq m < d$.

2. Let $\{X_t\}$ be a zero-mean, unit-variance stationary process with autocorrelation function ρ_k. Suppose that μ_t is a nonconstant function, and that σ_t is a positive-valued, nonconstant function. The observed series is formed as $Y_t = \mu_t + \sigma_t X_t$.

   (a) Find the mean and covariance function for the $\{Y_t\}$ process.

   (b) Show that the autocorrelation function for the $\{Y_t\}$ process depends only on the time lag. Is the $\{Y_t\}$ process stationary?

   (c) Is it possible to have a time series with a constant mean and with $\mathrm{Corr}(Y_t, Y_{t-k})$ free of t, but with $\{Y_t\}$ not stationary?

3. Suppose $\mathrm{Cov}(X_t, X_{t-k}) = \gamma_k$ is free of t, but that $\mathbb{E}[X_t] = 3t$.

   (a) Is $\{X_t\}$ stationary?

   (b) Let $Y_t = 7 - 3t + X_t$. Is $\{Y_t\}$ stationary?

4. Evaluate the mean and covariance function for each of the following processes. In each case, determine whether or not the process is stationary.

   (a) $Y_t = \theta_0 + t e_t$.

   (b) $W_t = \nabla Y_t$, where Y_t is given in part (a).

   (c) $Y_t = e_t e_{t-1}$, where $\{e_t\}$ is normally distributed white noise.

5. Let $\{Y_t\}$ be stationary with autocovariance function γ_k. Define the sample variance as

$$S^2 = \frac{1}{n-1}\sum_{t=1}^{n-1}(Y_t - \bar{Y})^2.$$

   (a) First show that

$$\sum_{t=1}^{n}(Y_t - \mu)^2 = \sum_{t=1}^{n}(Y_t - \bar{Y})^2 + n(\bar{Y} - \mu)^2.$$

   (b) Use part (a) to show that

$$\mathbb{E}[S^2] = \frac{n}{n-1}\gamma_0 - \frac{n}{n-1}\mathrm{Var}(\bar{Y}) = \gamma_0 - \frac{2}{n-1}\sum_{k=1}^{n-1}\left(1 - \frac{k}{n}\right)\gamma_k$$

   using the results of Exercise 2.17 for the last expression.

   (c) If $\{Y_t\}$ is a white noise process with variance γ_0, show that $\mathbb{E}[S^2] = \gamma_0$.

6. Let $Y_1 = \theta_0 + e_1$, and then for $t > 1$, define Y_t recursively by $Y_t = \theta_0 + Y_{t-1} + e_t$. Here, θ_0 is a constant. The process $\{Y_t\}$ is called a random walk with drift.

   (a) Show that Y_t may be rewritten as $Y_t = t\theta_0 + e_t + e_{t-1} + \ldots + e_1$.

   (b) Find the mean function for Y_t.

   (c) Find the autocovariance function for Y_t.

7. Two processes $\{Z_t\}$ and $\{Y_t\}$ are said to be independent if for any time points $t_1, t_2, \ldots, t_m$ and $s_1, s_2, \ldots, s_n$, the random variables $\{Z_{t_1}, \ldots, Z_{t_m}\}$ are independent of the random variables $\{Y_{s_1}, \ldots, Y_{s_n}\}$. Show that if $\{Z_t\}$ and $\{Y_t\}$ are independent stationary processes, then $W_t = Z_t + Y_t$ is stationary.

8. Suppose that $Y_t = R\cos(2\pi(ft + \Phi))$ for $t = 0, \pm1, \pm2, \ldots$, where $0 < f < \frac{1}{2}$ is a fixed frequency and R and Φ are uncorrelated random variables and with Φ uniformly distributed on (0,1).

   (a) Show that $\mathbb{E}[Y_t] = 0$ all t.

   (b) Show that the process is stationary with $\gamma_k = \frac{1}{2}\mathbb{E}[R^2]\cos(2\pi fk)$. Use the calculations leading up to Eq 2.3.4 on Page 19.

9. Suppose that $Y_t = R\cos[2\pi(ft+\Phi)]$ for $t = 0, \pm1, \pm2, \ldots$, where R and Φ are independent random variables and f is a fixed frequency. The phase Φ is assumed to be uniformly distributed on (0,1), and the amplitude R has a Rayleigh distribution with pdf $f(r) = re^{-\frac{r}{2}}$ for $r > 0$. Show that for each time point t, Y_t has a normal distribution. (Hint: Let $Y = R\cos[2\pi(ft + \Phi)]$ and $X = R\sin[2\pi(ft + \Phi)]$. Now find the joint distribution of X and Y. It can also be shown that all of the finite-dimensional distributions are multivariate normal, and hence the process is stationary.)