764 Midterm Part 3 Helen Kafka

1. AR(1) : $Y_t = \varphi Y_{t-1} + e_t$ $\qquad e_t \overset{iid}{\sim} WN(0, \sigma_e^2)$

stationary condition: $|\varphi| < 1$

$$\mathbb{E}(Y_t) = \varphi\, \mathbb{E}(Y_{t-1}) + \underbrace{\mathbb{E}(e_t)}_{=0} = \varphi\, \mathbb{E}(Y_{t-1})$$

such that the process is stationary, it should hold:

$$\mathbb{E}(Y_t) = \mathbb{E}(Y_{t-1})$$

$$\Rightarrow \quad \mathbb{E}(Y_t) = \varphi\, \mathbb{E}(Y_{t-1}) = \varphi\, \mathbb{E}(Y_t) \quad \Leftrightarrow \quad \mathbb{E}(Y_t) - \varphi\, \mathbb{E}(Y_t) = 1$$

$$\Leftrightarrow \quad \mathbb{E}(Y_t) = \frac{1}{1-\varphi} \qquad \varphi \neq 1$$

$\mathbb{E}(Y_t) = \varphi\, \mathbb{E}(Y_{t-1})$ only holds if $\mathbb{E}(Y_t) = 0$

$$\mathrm{Var}(Y_t) = \mathrm{Var}(\varphi Y_{t-1} + e_t) = \varphi^2\, \mathrm{Var}(Y_{t-1}) + \mathrm{Var}(e_t) + 2\varphi\, \mathrm{Cov}(Y_{t-1}, e_t)$$

$$= \varphi^2\, \mathrm{Var}(Y_{t-1}) + \sigma_e^2$$

for the process to be stationary

① $\{ \mathrm{Var}(Y_t) = \mathrm{Var}(Y_{t-1})$ has to hold

$$\Rightarrow \quad \mathrm{Var}(Y_t) = \varphi^2\, \mathrm{Var}(Y_{t-1}) + \sigma_e^2 \overset{①}{=} \varphi^2\, \mathrm{Var}(Y_t) + \sigma_e^2$$

$$\Leftrightarrow \quad \mathrm{Var}(Y_t) = \frac{\sigma_e^2}{1-\varphi^2} \qquad \varphi^2 < 1 \Leftrightarrow |\varphi| < 1 \quad \text{because } \mathrm{Var}(\cdot) \geq 0$$

since $1 = (1-\varphi)\, \mathbb{E}Y_t$, we can center the process around $\mathbb{E}Y_t$

and get $Y_t - \mathbb{E}Y_t = \varphi(Y_{t-1} - \mathbb{E}Y_t) + e_t$

$$\gamma_k = \mathbb{E}((Y_t - \mathbb{E}Y_t)(Y_{t-k} - \mathbb{E}Y_t))$$

$$= \mathbb{E}((\varphi(Y_{t-1} - \mathbb{E}Y_t) + e_t)(Y_{t-k} - \mathbb{E}Y_t))$$

$$= \mathbb{E}(\varphi(Y_{t-1} - \mathbb{E}Y_t)(Y_{t-k} - \mathbb{E}Y_t) + e_t \cdot (Y_{t-k} - \mathbb{E}Y_t))$$

$$= \mathbb{E}(\varphi(Y_{t-1} - \mathbb{E}Y_t)(Y_{t-k} - \mathbb{E}Y_t)) + \underbrace{\mathbb{E}(e_t)}_{=0}\, \mathbb{E}(Y_{t-k} - \mathbb{E}Y_t)$$

$$= \varphi\, \mathbb{E}((Y_{t-1} - \mathbb{E}Y_t)(Y_{t-k} - \mathbb{E}Y_t))$$

$$= \varphi\, \gamma_{k-1}$$

with repeated substituition, we get

$$\gamma_k = \varphi \gamma_{k-1} = \dots = \varphi^k \gamma_0 = \varphi^k \frac{\sigma_e^2}{1-\varphi^2} < \infty \quad \text{iff} \quad |\varphi| < 1$$

→ if $|\varphi|<1$ the process does not depend on t and thus it is stationary

AR(2) : $Y_t = \varphi_1 Y_{t-1} + \varphi_2 Y_{t-2} + e_t \qquad e_t \overset{iid}{\sim} WN(0, \sigma_e^2)$

stationary condition:

roots of $1 - \varphi_1 x - \varphi_2 x^2 = 0$ exceed 1 in absolut value

This is true iff
1. $\varphi_1 + \varphi_2 < 1$
2. $\varphi_2 - \varphi_1 < 1$
3. $|\varphi_2| < 1$

it is assumed that e_t is independent of $Y_{t-i} \quad i = 1, 2, \dots$

AR characteristic polynomial: $\varphi(x) = 1 - \varphi_1 x - \varphi_2 x^2$

AR characteristic equation: $1 - \varphi_1 x - \varphi_2 x^2 = 0$

$$1 - \varphi_1 x - \varphi_2 x^2 = 0 \qquad | \cdot x^{-2}$$

$$\Leftrightarrow \quad x^{-2} - \varphi_1 x^{-1} - \varphi_2 = 0$$

$\lambda = x^{-1}$

$$\Leftrightarrow \quad \lambda^2 - \varphi_1 \lambda - \varphi_2 = 0$$

$\Rightarrow$ roots are given by $\lambda_{1,2} = \dfrac{\varphi_1 \pm \sqrt{\varphi_1^2 + 4 \cdot \varphi_2}}{2}$

the roots are real if $\varphi_1^2 + 4\varphi_2 > 0$:

AR(2) is stationary if $|\lambda| < 1$

$$-1 < \frac{\varphi_1 \pm \sqrt{\varphi_1^2 + 4\varphi_2}}{2} < 1$$

$$-2 < \varphi_1 \pm \sqrt{\varphi_1^2 + 4\varphi_2} < 2$$

$$\Rightarrow \quad \varphi_1 + \sqrt{\varphi_1^2 + 4\varphi_2} < 2$$

$$\Leftrightarrow \quad \varphi_1^2 + 4\varphi_2 < (2 - \varphi_1)^2 = \varphi_1^2 - 4\varphi_1 + 4$$

$$\Leftrightarrow \quad +4\varphi_2 + 4\varphi_1 < 4$$

$$\varphi_1 + \varphi_2 < 1$$

$$\Rightarrow \quad -2 < \varphi_1 - \sqrt{\varphi_1^2 + 4\varphi_2}$$

$$\Leftrightarrow \quad -2 - \varphi_1 < -\sqrt{\varphi_1^2 + 4\varphi_2}$$

$$\Leftrightarrow \quad \varphi_1^2 + 4\varphi_2 < (2 + \varphi_1)^2 = \varphi_1^2 + 4\varphi_1 + 4$$

$$\Leftrightarrow \quad \varphi_2 - \varphi_1 < 1$$

the roots are complex if $\varphi_1^2 + 4\varphi_2 < 0$

$$\lambda_{1,2} = \frac{\varphi_1}{2} \pm \frac{i}{2}\sqrt{-(\varphi_1^2 + 4\varphi_2)}$$

$$\Rightarrow \lambda^2 = \left(\frac{\varphi_1}{2}\right)^2 + \left(\frac{1}{2}\sqrt{-(\varphi_1^2 + 4\varphi_2)}\right)^2$$

$$= \frac{\varphi_1^2}{4} - \frac{1}{4}(\varphi_1^2 + 4\varphi_2)$$

$$= -\varphi_2$$

$\rightarrow$ this is stable if $|\lambda| < 1 \Leftrightarrow -\varphi_2 < 1$ or $\varphi_2 > -1$

$$\Leftrightarrow |\varphi_2| < 1$$

$\Rightarrow$ conditions
1. $\varphi_1 + \varphi_2 < 1$ fulfilled
2. $\varphi_2 - \varphi_1 < 1$
3. $|\varphi_2| < 1$

$\Rightarrow$ roots of $1 - \varphi_1 x - \varphi_2 x^2 = 0$ exceed 1 in absolut value

2. show $\gamma_0 = \dfrac{(1-2\varphi\vartheta+\vartheta^2)}{1-\varphi^2}\sigma_e^2$

and $\rho_k = \dfrac{(1-\vartheta\varphi)(\varphi-\vartheta)}{1-2\vartheta\varphi+\vartheta^2}\varphi^{k-1}$ for $k \geq 1$

for an ARMA (1,1) process $e_t \sim WN(0,\sigma_e^2)$

ARMA(1,1) can be written as: $Y_t = \varphi Y_{t-1} + e_t - \vartheta e_{t-1}$

1. $\mathbb{E}(e_t Y_t) = \mathbb{E}(e_t(\varphi Y_{t-1} + e_t - \vartheta e_{t-1}))$

$= \mathbb{E}(\varphi e_t Y_{t-1}) + \mathbb{E}(e_t^2) - \vartheta \underbrace{\mathbb{E}(e_t e_{t-1})}_{=0}$

$e_t \perp\!\!\!\perp Y_{t-1}$

$= \varphi \underbrace{\mathbb{E}(e_t)}_{=0} \cdot \mathbb{E}(Y_{t-1}) + \sigma_e^2$

$= \sigma_e^2$

2. $\mathbb{E}(e_{t-1}Y_t) = \mathbb{E}(e_{t-1}(\varphi Y_{t-1} + e_t - \vartheta e_{t-1}))$

$= \mathbb{E}(\varphi e_{t-1}Y_{t-1}) + \underbrace{\mathbb{E}(e_{t-1}\cdot e_t)}_{=0} - \vartheta\,\mathbb{E}(e_{t-1}^2)$ $e_t \perp\!\!\!\perp e_{t-1}$

e_{t-1} is independent with all terms of Y_{t-1} except its e_{t-1} term, for all other terms the expected value is zero

$= \varphi\,\mathbb{E}(e_{t-1}\cdot e_{t-1}) - \vartheta\sigma_e^2$

$= \varphi\sigma_e^2 - \vartheta\sigma_e^2 = (\varphi-\vartheta)\sigma_e^2$

$\mathbb{E}(e_{t-i}Y_t) = \varphi\sigma_e^2 \qquad i \geq 2$

$\gamma_k = COV(Y_t, Y_{t-k}) = \mathbb{E}(Y_t \cdot Y_{t-k}) - \underbrace{\mathbb{E}Y_t}_{=0} \cdot \underbrace{\mathbb{E}(Y_{t-k})}_{=0}$

$= \mathbb{E}(Y_t \cdot Y_{t-k})$

$= \mathbb{E}((\varphi Y_{t-1} + e_t - \vartheta e_{t-1})Y_{t-k})$

$= \varphi\,\mathbb{E}(Y_{t-1}Y_{t-k}) + \mathbb{E}(e_t Y_{t-k}) - \vartheta\,\mathbb{E}(e_{t-1}Y_{t-k})$

$k=0$

$$\gamma_0 = \varphi\,\mathbb{E}(Y_{t-1}\cdot Y_t) + \mathbb{E}(e_t Y_t) - \vartheta\,\mathbb{E}(e_{t-1}Y_t)$$

$$\overset{①②}{=} \varphi\,\mathbb{E}(Y_{t-1}\cdot Y_t) + \sigma^2 - \vartheta(\varphi-\vartheta)\sigma^2$$

$$\mathbb{E}(Y_{t-1}\cdot Y_t) = \mathbb{E}(Y_{t-1}(\varphi Y_{t-1} + e_t - \vartheta e_{t-1}))$$

$$= \mathbb{E}(\varphi Y_{t-1}^2) + \underbrace{\mathbb{E}(e_t Y_{t-1})}_{=0} - \vartheta\underbrace{\mathbb{E}(e_{t-1}Y_{t-1})}_{=\sigma^2}$$

$$= \varphi\underbrace{\mathbb{E}(Y_{t-1}^2)}_{=\mathrm{Var}(Y_{t-1}) = \gamma_0} - \vartheta\sigma^2$$

$$= \varphi\gamma_0 - \vartheta\sigma^2$$

$$\Rightarrow \gamma_0 = \varphi(\varphi\gamma_0 - \vartheta\sigma^2) + \sigma^2 - \vartheta(\varphi-\vartheta)\sigma^2$$

$$= \varphi^2\gamma_0 - \varphi\vartheta\sigma^2 + \sigma^2 - \vartheta\varphi\sigma^2 + \vartheta^2\sigma^2$$

$$= \varphi^2\gamma_0 + \sigma^2 - 2\vartheta\varphi\sigma^2 + \vartheta^2\sigma^2$$

$$= \varphi^2\gamma_0 + \sigma^2(1 - 2\vartheta\varphi + \vartheta^2)$$

$$\Leftrightarrow \gamma_0 = \frac{(1-2\varphi\vartheta+\vartheta^2)\sigma^2}{1-\varphi^2}$$

$k=1$

$$\gamma_1 = \varphi\underbrace{\mathbb{E}(Y_{t-1}^2)}_{=\mathrm{Var}(Y_{t-1}) = \gamma_0} + \mathbb{E}(e_t Y_{t-1}) - \vartheta\,\mathbb{E}(e_{t-1}Y_{t-1})$$

$$= \varphi\gamma_0 - \vartheta\sigma^2$$

$k \geq 2$

$$\gamma_k = \varphi \, \mathbb{E}(Y_{t-1} Y_{t-k}) + \underbrace{\mathbb{E}(e_t Y_{t-k})}_{=0} - \vartheta \underbrace{\mathbb{E}(e_{t-1} Y_{t-k})}_{=0}$$

$$= \varphi \, \gamma_{k-1}$$

$$= \dots = \varphi^{k-1} \gamma_{k-k+1} = \varphi^{k-1} \gamma_1$$

$$\rho_k = \frac{\gamma_k}{\gamma_0} = \frac{\varphi^{k-1} \gamma_1}{\gamma_0} = \varphi^{k-1} \frac{\gamma_1}{\gamma_0}$$

$$\frac{\gamma_1}{\gamma_0} = \frac{\varphi \gamma_0 - \vartheta \sigma^2}{\gamma_0} = \varphi - \frac{\vartheta \sigma^2}{\gamma_0}$$

$$= \varphi - \frac{\vartheta \sigma^2}{(1 - 2\varphi\vartheta + \vartheta^2)\sigma^2} \cdot (1 - \varphi^2)$$

$$= \varphi - \frac{\vartheta (1 - \varphi^2)}{1 - 2\varphi\vartheta + \vartheta^2}$$

$$= \frac{\varphi - 2\varphi^2\vartheta + \varphi\vartheta^2 - \vartheta + \vartheta\varphi^2}{1 - 2\varphi\vartheta + \vartheta^2}$$

$$= \frac{\varphi - \vartheta + \varphi\vartheta^2 - \varphi^2\vartheta}{1 - 2\varphi\vartheta + \vartheta^2} = \frac{(1 - \vartheta\varphi)(\varphi - \vartheta)}{1 - 2\varphi\vartheta + \vartheta^2}$$

$$\Rightarrow \rho_1 = \varphi^{k-1} \frac{(1 - \vartheta\varphi)(\varphi - \vartheta)}{1 - 2\varphi\vartheta + \vartheta^2}$$