MTHSTAT 564/564G/764–Time Series Analysis
Spring 2024
Midterm Project: Due Wednesday, 27 March at 5:00 PM

This project consists of two parts that everyone needs to do and one part that is for the students enrolled in MS 764. This is a midterm, so you are expected to work independently, but you may feel free to ask me any questions that might arise during your completion of this work. Please submit your entire project as one .pdf file. Include any plots you make. I do not need to see your code.

Part 1

Using the “retail” data set in the TSA package, which lists total UK retail sales in billions of pounds from January, 1986 through March, 2007, please do the following: (2000 is the base year and is coded as 100).

1. Prior to making any plots or doing any analysis, describe any types of trends you might expect in the retail sales. Give practical reasons why these might be present. Then provide a time series plot of retail sales and compare it to your expectations. Include plotting symbols in the plot.

2. State and fit a model that captures any trend you might see. Estimate and interpret the coefficients and comment on their significance or lack thereof, and comment on any quantities that might indicate the quality of your model fit.

3. Carry out a full analysis of the standardized residuals, including examination of the autocorrelation. Comment on the validity of your model assumptions. If you see issues with the model you fit, transform the data in some way so that your model assumptions are approximately satisfied. Your descriptions of residual plots for each model should be detailed. You need only present those for the original model and the one that satisfied the assumptions on the innovation terms, if a transformation and new fit was needed. If you need to fit a new model, state that model.

Part 2

Using the series “MidtermPt2.txt”, examine this series for stationarity and comment on the results. If there is evidence that the series is not stationary, consider transformations to stationarity to remedy the situation. Include all plots from the original series and the final model in your report, along with the requisite explanations/interpretation.

Part 3–MS 764 Students Only

1. Derive the stationarity conditions for the AR(1) and AR(2) processes.
2. Derive the autocovariance and autocorrelation functions for the ARMA(p, q) model that are given in the textbook by Equation (4.4.7).

MTHSTAT 564/564G/764
Time Series Analysis
Midterm Project – Part 3

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March 3, 2024

1. Derive the stationarity conditions for the AR(1) and AR(2) processes.
   - For an AR(1) process we have: $Y_t = \varphi Y_{t-1} + e_t$, where $e_t \sim WN(0, \sigma_e^2)$. The stationarity is given by $|\varphi| < 1$. For the mean we have that

$$\mathbb{E}[Y_t] = \varphi \mathbb{E}[Y_{t-1}] + \underbrace{\mathbb{E}[e_t]}_{=0} = \varphi \mathbb{E}[Y_{t-1}].$$

For a process to be stationary, we want to have that

$$\mathbb{E}[Y_t] = \mathbb{E}[X_{t-1}]. \tag{1}$$

Plugging this in yields:

$$\begin{aligned} \mathbb{E}[Y_t] = \varphi \mathbb{E}[Y_{t-1}] \overset{1}{=} \varphi \mathbb{E}[Y_t] &\iff \mathbb{E}[Y_t] - \varphi \mathbb{E}[Y_t] = 1 \\ &\iff \mathbb{E}[Y_t] = \frac{1}{1-\varphi}, \varphi \neq 1. \end{aligned}$$

$\mathbb{E}[Y_t] = \varphi \mathbb{E}[Yt-1]$ only holds if $\mathbb{E}[Y_t] = 0$.

$$\begin{aligned} \mathrm{Var}(Y_t) = \mathrm{Var}(\varphi Y_{t-1} + e_t) &= \varphi^2 \mathrm{Var}(Y_{t-1}) + \mathrm{Var}(Y_t) + 2\varphi \mathrm{Cov}(Y_{t-1}, e_t) \\ &= \varphi^2 \mathrm{Var}(Y_{t-1}) + \sigma_e^2 \end{aligned}$$

We also have the condition

$$\mathrm{Var}(Y_t) = \mathrm{Var}(Y_{t-1}), \tag{2}$$

which needs to be fulfilled.

$$\begin{aligned}\implies \operatorname{Var}(Y_t) &= \varphi^2 \operatorname{Var}(Y_{t-1}) + \sigma_e^2 \\ &\overset{Equation\ 2}{=} \varphi^2 \operatorname{Var}(Y_t) + \sigma_e^2 \\ \iff \operatorname{Var}(Y_t) &= \frac{\sigma_e^2}{1-\varphi^2},\end{aligned}$$

$\varphi^2 < 1 \iff |\varphi| < 1$ because $\operatorname{Var}(\cdot) \geq 0$. Since $1 = (1-\varphi)\mathbb{E}[Y_t]$, we can center the process around $\mathbb{E}[Y_t]$ and get

$$Y_t - \mathbb{E}[Y_t] = \varphi(Y_{t-1} - \mathbb{E}[Y_t]) + e_t.$$

$$\begin{aligned}\gamma_k &= \mathbb{E}[(Y_t - \mathbb{E}[Y_t])(Y_{t-k} - \mathbb{E}[Y_t])] \\ &= \mathbb{E}[(\varphi(Y_{t-1} - \mathbb{E}[Y_t]) + e_t)(Y_{t-k} - \mathbb{E}[Y_t])] \\ &= \mathbb{E}[\varphi(Y_{t-1} - \mathbb{E}[Y_t])(Y_{t-k} - \mathbb{E}[Y_t]) + e_t \cdot (Y_{t-k} - \mathbb{E}[Y_t])] \\ &= \mathbb{E}[\varphi(Y_{t-1} - \mathbb{E}[Y_t])(Y_{t-k} - \mathbb{E}[Y_t])] + \underbrace{\mathbb{E}[e_t]}_{=0} \cdot \mathbb{E}[(Y_{t-k} - \mathbb{E}[Y_t])] \\ &= \varphi \mathbb{E}[\varphi(Y_{t-1} - \mathbb{E}[Y_t])(Y_{t-k} - \mathbb{E}[Y_t])] \\ &= \varphi\gamma_{k-1}.\end{aligned}$$

with repeated substitution, we get

$$\gamma_k = \varphi\gamma_{k-1} = \ldots = \varphi^k \gamma_0 = \varphi^k \frac{\sigma_e^2}{1-\varphi^2} < \infty, \iff |\varphi| < 1$$

$\implies$ if $|\varphi| < 1$ the process does not depend on t and thus it is stationary.

- For an AR(2) process we have: $Y_t = \varphi_1 Y_{t-1} + \varphi_2 Y_{t-2} + e_t$, where $e_t \sim WN(0, \sigma_e^2)$. The stationarity condition here is that the roots of $1 - \varphi_1 x - \varphi_2 x^2 = 0$ exceed 1 in absolute value. This happens if and only if
  a) $\varphi_1 + \varphi_2 < 1$
  b) $\varphi_2 - \varphi_1 < 1$
  c) $|\varphi_2| < 1$

  We assume that $e_t \perp\!\!\!\perp Y_{t-1}, i = 1, 2, \ldots$. For an AR we have the ...
  - ... characterisitc polynomiae: $\varphi(x) = 1 - \varphi_1 x - \varphi_2 x^2$

– ... characterisitc equation: $1 - \varphi_1 x - \varphi_2 x^2 = 0$,
therefore,

$$\begin{aligned} & 1 - \varphi_1 x - \varphi_2 x^2 = 0 \\ \Longleftrightarrow\ & x^{-2} - \varphi_1 x^{-1} - \varphi_2 = 0 \\ \overset{\lambda = x^{-1}}{\Longleftrightarrow}\ & \lambda^2 - \varphi_1 \lambda - \varphi_2 = 0. \end{aligned}$$

$\Longrightarrow$ the roots are given by $\lambda_{1,2} = \frac{\varphi_1 \pm \sqrt{\varphi_1^2 - 4 \cdot 1 \cdot (-\varphi_2)}}{2}$. Those are real if the discriminant is $> 0 \iff \varphi_1^2 + 4\varphi_2 > 0$: Since AR(2) is stationary if $|\lambda| < 1$:

$$\begin{aligned} -1 < \frac{\varphi_1 \pm \sqrt{\varphi_1^2 + 4\varphi_2}}{2} < 1 \\ -2 < \varphi_1 \pm \sqrt{\varphi_1^2 + 4\varphi_2} < 2 \end{aligned}$$

– Case 1:

$$\begin{aligned} \Longrightarrow\quad & \varphi_1 + \sqrt{\varphi_1^2 + 4\varphi_2} < 2 \\ \Longleftrightarrow\quad & \sqrt{\varphi_1^2 + 4\varphi_2} < 2 - \varphi_1 \\ \Longleftrightarrow\quad & \varphi_1^2 + 4\varphi_2 < (2 - \varphi_1)^2 \\ \Longleftrightarrow\quad & \varphi_1^2 + 4\varphi_2 < 4 - 4\varphi_1 + \varphi_1^2 \\ \Longleftrightarrow\quad & 4\varphi_1 + 4\varphi_2 < 4 \\ \Longleftrightarrow\quad & \varphi_1 + \varphi_2 < 1 \end{aligned}$$

– Case 2:

$$\begin{aligned} \Longrightarrow\quad & \varphi_1 - \sqrt{\varphi_1^2 + 4\varphi_2} > -2 \\ \Longleftrightarrow\quad & -\sqrt{\varphi_1^2 + 4\varphi_2} > -2 - \varphi_1 \\ \Longleftrightarrow\quad & \sqrt{\varphi_1^2 + 4\varphi_2} < 2 + \varphi_1 \\ \Longleftrightarrow\quad & \varphi_1^2 + 4\varphi_2 < (2 + \varphi_1)^2 \\ \Longleftrightarrow\quad & \varphi_1^2 + 4\varphi_2 < 4 + 4\varphi_1 + \varphi_1^2 \\ \Longleftrightarrow\quad & -4\varphi_1 + 4\varphi_2 < 4 \\ \Longleftrightarrow\quad & -\varphi_1 + \varphi_2 < 1 \end{aligned}$$

We have complex roots if the discriminant is $< 0 \iff \varphi_1^2 + 4\varphi_2 < 0$:

$$\begin{aligned}\lambda_{1,2} &= \frac{\varphi_1 \pm \sqrt{-(\varphi_1^2 + 4\varphi_2)}}{2} \\ &= \frac{\varphi_1 \pm i\sqrt{-(\varphi_1^2 + 4\varphi_2)}}{2}\end{aligned}$$

Since the squared modulus of a complex number is the square of the real plus the square of the imaginary part. Therefore,

$$\begin{aligned}\lambda^2 &= \left(\frac{\varphi_1}{2}\right)^2 + \left(\frac{\sqrt{-(\varphi_1^2 + 4\varphi_2)}}{2}\right)^2 \\ &= \frac{\varphi_1^2}{4} - \frac{\varphi_1^2 + 4\varphi_2}{4} \\ &= -\varphi_2\end{aligned}$$

$\implies$ this is stable if $|\lambda| < 1 \iff -\varphi_2 < 1$ or $\varphi_2 > -1 \iff |\varphi_2| < 1$. Therefore, the following conditions are fulfilled:

a) $\varphi_1 + \varphi_2 < 1$

b) $\varphi_2 - \varphi_1 < 1$

c) $|\varphi_2| < 1$

It follows, that the roots of $1 - \varphi_1 x - \varphi_2 x^2 = 0$ exceed 1 in absolute value.

2. For an ARMA(1, 1) process, we want to show

$$\gamma_0 = \frac{(1 - 2\varphi\vartheta + \vartheta^2)}{1 - \varphi^2}\sigma_e^2, \tag{3}$$

$$\rho_k = \frac{(1 - \vartheta\varphi)(\varphi - \vartheta)}{1 - 2\vartheta\varphi + \vartheta^2}\varphi^{k-1}, k \geq 1. \tag{4}$$

This process can be written as

$$Y_t = \varphi Y_{t-1} + e_t - \vartheta e_{t-1}, e_t \sim WN(0, \sigma_e^2).$$

$$\begin{aligned}\mathbb{E}[e_t Y_t] &= \mathbb{E}[e_t(\varphi Y_{t-1} + e_t - \vartheta e_{t-1})] \\ &= \mathbb{E}[e_t \varphi Y_{t-1}] + \mathbb{E}[e_t^2] - \vartheta \underbrace{\mathbb{E}[e_t e_{t-1}]}_{=0} \\ &\overset{e_t \perp\!\!\!\perp Y_{t-1}}{=} \varphi \underbrace{\mathbb{E}[e_t]}_{=0} \cdot \mathbb{E}[Y_{t-1}] + \sigma_e^2 \\ &= \sigma_e^2.\end{aligned} \tag{5}$$

$$
\begin{aligned}
\mathbb{E}[e_{t-1}Y_t] &= \mathbb{E}[e_{t-1}(\varphi Y_{t-1} + e_t - \vartheta e_{t-1})] && (6)\\
&= \mathbb{E}[e_{t-1}\varphi Y_{t-1}] + \underbrace{\mathbb{E}[e_{t-1}e_t]}_{=0(e_t \perp\!\!\!\perp e_{t-1})} - \vartheta \mathbb{E}[e_{t-1}^2] && (7)\\
&= \varphi \mathbb{E}[e_{t-1}\cdot e_t] - \vartheta\sigma_e^2 && (8)\\
&= \varphi\sigma_e^2 - \vartheta\sigma_e^2\\
&= (\varphi - \vartheta)\sigma_e^2.
\end{aligned}
$$

Equation 7 → Equation 8 works because e_{t-1} is independent to all terms of $Y_{t-1} = \varphi Y_{t-2} + e_{t-1} - \vartheta e_{t-2}$ except from the middle term, e_{t-1}.

$$
\mathbb{E}[e_{t-1}Y_t] = \varphi\sigma_e^2, i \geq 2. \tag{9}
$$

$$
\begin{aligned}
\gamma_k = \operatorname{Cov}(Y_t, Y_{t-k}) &= \mathbb{E}[Y_tY_{t-k}] - \underbrace{\mathbb{E}[Y_t]}_{=0}\cdot\underbrace{\mathbb{E}[Y_{t-k}]}_{=0}\\
&= \mathbb{E}[Y_tY_{t-k}]\\
&= \mathbb{E}[(\varphi Y_{t-1} + e_t - \vartheta e_{t-1})Y_{t-k}]\\
&= \varphi\mathbb{E}[Y_{t-1}Y_{t-k}] + \mathbb{E}[e_tY_{t-k}] - \vartheta\mathbb{E}[e_{t-1}Y_{t-k}]
\end{aligned}
$$

- Case $k = 0$:

$$
\begin{aligned}
\gamma_0 &= \varphi\mathbb{E}[Y_{t-1}Y_t] + \mathbb{E}[e_tY_t] - \vartheta\mathbb{E}[e_{t-1}Y_t]\\
&\overset{(5),(6)}{=} \varphi\mathbb{E}[Y_{t-1}Y_t] + \sigma_e^2 - \vartheta(\varphi - \vartheta)\sigma_e^2\\
\mathbb{E}[Y_{t-1}Y_t] &= \mathbb{E}[Y_{t-1}(Y_{t-1} + e_t - \vartheta e_{t-1})]\\
&= \mathbb{E}[\varphi Y_{t-1}^2] + \underbrace{\mathbb{E}(e_tY_{t-1}}_{=0} - \vartheta\underbrace{\mathbb{E}[e_{t-1}Y_{t-1}]}_{=\sigma_e^2}\\
&= \varphi\underbrace{\mathbb{E}[Y_{t-1}^2]}_{=\operatorname{Var}(Y_{t-1})=\gamma_0} - \vartheta\sigma_e^2\\
&= \varphi\gamma_0 - \vartheta\sigma_e^2\\
\implies \gamma_0 &= \varphi(\varphi\gamma_0 - \vartheta\sigma_e^2) + \sigma_e^2 - \vartheta(\varphi - \vartheta)\sigma_e^2\\
&= \varphi^2\gamma_0 - \varphi\vartheta\sigma_e^2 + \sigma_e^2 - \vartheta\varphi\sigma_e^2 + \vartheta^2\sigma_e^2\\
&= \varphi^2\gamma_0 - 2\varphi\vartheta\sigma_e^2 + \sigma_e^2 + \vartheta^2\sigma_e^2\\
&= \varphi^2\gamma_0 + \sigma_e^2(1 - 2\varphi\vartheta + \vartheta^2)\\
\implies \gamma_0 &= \frac{\sigma_e^2(1 - 2\varphi\vartheta + \vartheta^2)}{1 - \varphi^2}.
\end{aligned}
$$

- Case $k = 1$:

$$\begin{aligned}
\gamma_1 &= \varphi \underbrace{\mathbb{E}[Y_{t-1}^2]}_{=\mathrm{Var}(Y_{t-1})=\gamma_0} + \underbrace{\mathbb{E}(e_t Y_{t-1}}_{=0} - \vartheta \underbrace{\mathbb{E}[e_{t-1} Y_{t-1}]}_{=\sigma_e^2} \\
&= \varphi\gamma_0 - \vartheta\sigma_e^2
\end{aligned}$$

- Case $k \geq 2$:

$$\begin{aligned}
\gamma_k &= \varphi\mathbb{E}[Y_{t-1}Y_{t-k}] + \underbrace{\mathbb{E}(e_t Y_{t-1}}_{=0} - \vartheta \underbrace{\mathbb{E}[e_{t-1} Y_{t-k}]}_{=0} \\
&= \varphi\gamma_{k-1} \\
&= \dots \\
&= \varphi^{k-1}\gamma_{k-k+1} \\
&= \varphi^{k-1}\gamma_1.
\end{aligned}$$

$$\rho_k = \frac{\gamma_k}{\gamma_0} = \frac{\varphi^{k-1}\gamma_1}{\gamma_0} = \varphi^{k-1}\frac{\gamma_1}{\gamma_0}$$

$$\begin{aligned}
\frac{\gamma_1}{\gamma_0} &= \frac{\varphi\gamma_0 - \vartheta\sigma_e^2}{\gamma_0} = \varphi - \frac{\vartheta\sigma_e^2}{\gamma_0} \\
&= \varphi - \frac{\vartheta\sigma_e^2}{(1-2\varphi\vartheta+\vartheta^2)\sigma_e^2} \cdot (1-\varphi^2) \\
&= \varphi - \frac{\vartheta(1-\varphi^2)}{1-2\varphi\vartheta+\vartheta^2} \\
&= \varphi - \frac{\vartheta(1-\varphi^2)}{1-2\varphi\vartheta+\vartheta^2} \\
&= \frac{\varphi(1-2\varphi\vartheta+\vartheta^2)}{1-2\varphi\vartheta+\vartheta^2} - \frac{\vartheta(1-\varphi^2)}{1-2\varphi\vartheta+\vartheta^2} \\
&= \frac{\varphi - 2\varphi^2\vartheta + \varphi\vartheta^2}{1-2\varphi\vartheta+\vartheta^2} - \frac{\vartheta - \vartheta\varphi^2}{1-2\varphi\vartheta+\vartheta^2} \\
&= \frac{\varphi + \varphi\vartheta^2 - \vartheta + \vartheta\varphi^2}{1-2\varphi\vartheta+\vartheta^2} \\
&= \frac{(1-\vartheta\varphi)(\varphi-\vartheta)}{1-2\varphi\vartheta+\vartheta^2} \\
\implies \gamma_1 &= \varphi^{k-1}\frac{(1-\vartheta\varphi)(\varphi-\vartheta)}{1-2\varphi\vartheta+\vartheta^2}.
\end{aligned}$$