MTHSTAT 564/564G/764–Time Series Analysis
Spring 2024
Homework Assignment 3: Due Wednesday, 13 March in Lecture

This homework consists of three problems, none of which require R. You may feel free to work with classmates, but please be sure to turn in your own work via the Canvas dropbox. Be sure to submit your solutions as one single .pdf file. I do not need to see your code.

Reading

Chapter 4

Problems

1. Consider an MA(7) model with $\theta_1 = 1$, $\theta_2 = -0.5$, $\theta_3 = 0.25$, $\theta_4 = -0.125$, $\theta_5 = 0.0625$, $\theta_6 = -0.03125$, and $\theta_7 = 0.015625$. Find a much simpler model that has nearly the same ψ-weights.

2. Consider the model $Y_t = e_{t-1} - e_{t-2} + 0.5e_{t-3}$.

   (a) Find the autocovariance function for this process.

   (b) Show that this is a certain ARMA(p, q) process in disguise. That is, identify the values for p and q and for the θs and ϕs such that the ARMA(p, q) process has the same statistical properties as $\{Y_t\}$.

3. Show that the statement "The roots of $1 - \phi_1 x - \phi_2 x^2 - \ldots - \phi_p x^p = 0$ are greater than one in absolute value" is equivalent to the statement "The roots of $x^p - \phi_1 x^{p-1} - \phi_2 x^{p-2} - \ldots - \phi_p = 0$ are less than one in absolute value." (Hint: If A is a root of one equation, is 1/A a root of the other?)

MTHSTAT 564/564G/764
Time Series Analysis
Assignment 3

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1. We have an MA(7) model described by

$$\begin{aligned}Y_t = e_t + e_{t-1} - 0.5e_{t-2} + 0.25e_{t-3} - 0.125e_{t-4} + 0.0625e_{t-5} - 0.03125e_{t-6} \\ + 0.0151625e_{t-7}\end{aligned}$$

where $\{e_t\} \sim WN(0, \sigma^2)$. The goal is to simplify this process to a much simpler one with nearly the same ψ-weights. We see that the series is alternating because $\varphi < 0$, so it follows:

$$\begin{aligned}&\vartheta_j = (-0.5)^{(j-1)} \\ &Y_t = \varphi Y_{t-1} + e_t - \vartheta e_{t-1} \implies \text{ARMA-process}\end{aligned}$$

We can write the general linear process from of ARMA(1, 1) model as:

$$\begin{aligned}Y_t = e_t + (\varphi - \vartheta)\sum_{j=1}^{\infty} \varphi^{(j-1)} e_{t-j}, \\ \psi_j = (\varphi - \vartheta)\varphi^{(j-1)}, j \geq 1.\end{aligned}$$

We can now conclude the same type of behaviour for an ARMA(1, 1) as in our previous MA(7) model:

$$\begin{aligned}&\implies \psi_1 = \vartheta_1, & \psi_2 = \vartheta_2 \\ &\implies \varphi - \vartheta = 1, & (\varphi - \vartheta)\varphi = -0.5 \\ &\iff \varphi = -0.5, & \vartheta = 0.5.\end{aligned}$$

It follows that we can create a much simpler model for the MA(7) process by an ARMA(1, 1) model with $\varphi = -\frac{1}{2}$ and $\vartheta = \frac{1}{2}$:

$$\implies Y_t = -\frac{1}{2}Y_{t-1} + e_t - \frac{1}{2}e_{t-1}.$$

2. We have the model $Y_t = e_{t-1} - e_{t-2} + 0.5e_{t-3}$.

   a)

$$\begin{aligned}
\operatorname{Cov}(Y_t, Y_{t-k}) &= \operatorname{Cov}(e_{t-1} - e_{t-2} + 0.5e_{t-3}, e_{t-k} - e_{t-k} + 0.5e_{t-k} \\
&= \operatorname{Cov}(e_{t-1}, e_{t-1-k}) - \operatorname{Cov}(e_{t-1}, e_{t-2-k}) + \frac{1}{2}\operatorname{Cov}(e_{t-1}, e_{t-3-k}) \\
&- \operatorname{Cov}(e_{t-2}, e_{t-1-k}) + \operatorname{Cov}(e_{t-2}, e_{t-2-k}) - \frac{1}{2}\operatorname{Cov}(e_{t-2}, e_{t-3-k}) \\
&+ \frac{1}{2}\operatorname{Cov}(e_{t-3}, e_{t-1-k}) - \frac{1}{2}\operatorname{Cov}(e_{t-3}, e_{t-2-k}) + \frac{1}{4}\operatorname{Cov}(e_{t-3}, e_{t-3-k})
\end{aligned}$$

   We have to look at different cases:

$$\begin{aligned}
|k| = 0: &\quad \operatorname{Cov}(Y_t, Y_t) = \operatorname{Var}(Y_t) = \sigma_e^2 + \sigma_e^2 + \frac{1}{4}\sigma_e^2 = \frac{9}{4}\sigma_e^2 \\
|k| = 1: &\quad \operatorname{Cov}(Y_t, Y_{t-k}) = \operatorname{Var}(Y_t) = 0 - 0 + 0 - \sigma_e^2 + 0 - 0 + 0 - \frac{1}{2}\sigma_e^2 + 0 \\
&\quad = -\frac{3}{2}\sigma_e^2 \\
|k| \geq 3: &\quad \operatorname{Cov}(Y_t, Y_{t-k}) = 0. \\
\Longrightarrow &\quad \operatorname{Cov}(Y_t, Y_{t-k}) = \begin{cases} \frac{9}{4}\sigma_e^2, & |k| = 0, \\ -\frac{3}{2}\sigma_e^2, & |k| = 1, \\ \frac{1}{2}\sigma_e^2, & |k| = 2, \\ 0, & |k| \geq 3. \end{cases}
\end{aligned}$$

   b) *Claim.* The given model, $Y_t = e_{t-1} - e_{t-2} + 0.5e_{t-3}$, can be represented as an ARMA(0, 3) process.

   Proof:

   - Since we know that the $\{e_t\}$ are iid, they are in particular indistinguishable from each other, so we can shift the index as $t-i \to t-i+1$. So the process becomes

$$Y_t = e_t - e_{t-1} + 0.5e_{t-2}.$$

   - Autoregressive (AR) component ($p = 0$): There are no past values of Y_t on the right side of the equation.
   - Moving Average (MA) component ($q = 2$): The model uses past error terms (e_t, e_{t-1}, e_{t-2}) with coefficients 1, -1, and 0.5 respectively. This resembles a moving average process of order 2 (MA(2)).

- Specifying the ARMA parameters: $\theta_1 = -1, \theta_2 = 0.5$.
- *Claim.* This ARMA model captures the same statistical properties (mean, variance, and autocovariance function) as the original model $Y_t = e_t - e_{t-1} + 0.5e_{t-2}$.

  Proof: We obtain the ARMA model by

  $$Y_t = e_t + \theta_1 e_{t-1} - \theta_2 e_{t-2}.$$

  With doing this and plugging in the chosen coefficients, we get:

  $$\begin{aligned} Y_t &= e_t - 1 \cdot e_{t-1} + 0.5 \cdot e_{t-2} \\ &= e_t - e_{t-1} + 0.5e_{t-2} \end{aligned}$$

  Which is the exact same process.

3. **Proof:**
   - " $\Longrightarrow$ ":
     Suppose that $|f| > 1$ is the root of $1 - \varphi_1 x - \varphi_2 x^2 - \ldots - \varphi_p x^p$, i.e.

     $$\begin{aligned} & 1 - \varphi_1 f - \varphi_2 f^2 - \ldots - \varphi_p f^p && = 0 \\ \iff \; & f^p \left(\left(\frac{1}{p}\right)^p - \varphi_1 \left(\frac{1}{p}\right)^{p-1} - \varphi_2 \left(\frac{1}{p}\right)^{p-2} - \ldots - \varphi_p \right) && = 0 \\ \iff \; & \left(\frac{1}{p}\right)^p - \varphi_1 \left(\frac{1}{p}\right)^{p-1} - \varphi_2 \left(\frac{1}{p}\right)^{p-2} - \ldots - \varphi_p && = 0 \end{aligned}$$

     $\Longrightarrow$ $\frac{1}{f}$ is a root for the second statement:

     $$x^p - \varphi_1 x^{p-1} - \varphi_2 x^{p-2} - \ldots - \varphi_p = 0,$$

     Because of $|m| > 1$, $1 < \left|\frac{1}{m}\right|$ holds.
   - " $\Longleftarrow$ ":
     Suppose that $\frac{1}{f}$ is the root of

     $$x^p - \varphi_1 x^{p-1} - \varphi_2 x^{p-2} - \ldots - \varphi_p = 0, \tag{1}$$

     with $\left|\frac{1}{f}\right| < 1$. When we multiply (1) by f^p we get:

     $$1 - \varphi_1 f - \varphi_2 f^2 - \varphi_3 f^3 - \ldots - \varphi_p f^p = 0.$$

     $\Longrightarrow$ the root of the second equation is given by f with $|f| < 1$ ($\iff$ $\left|\frac{1}{f}\right| < 1$).