Time series analysis Homework 2

1.
$$Y_1 = \mu + e_1 - e_0$$
$$Y_2 = \mu + e_2 - e_1$$
$$Y_3 = \mu + e_3 - e_2$$
$$\vdots$$
$$Y_n = \mu + e_n - e_{n-1}$$

$$\bar{Y}_1 = \frac{1}{n}\sum_{i=1}^{n} Y_i = \frac{1}{n}\sum_{i=1}^{n} \mu + e_i - e_{i-1}$$

$$= \frac{1}{n}\left(n\mu + \sum_{i=1}^{n} e_i - e_{i-1}\right)$$

$$\overset{\text{telescopic sum}}{=} \frac{1}{n}\left(n\mu + e_n - e_0\right)$$

$$Var(\bar{Y}_1) = Var\left(\frac{1}{n}(n\mu + e_n - e_0)\right)$$

$$= \frac{1}{n^2} Var(\underbrace{n\mu}_{\text{constant}} + e_n - e_0)$$

$$= \frac{1}{n^2} Var(e_n - e_0)$$

$$\overset{e_i \perp\!\!\!\perp e_j \text{ for } i \neq j}{=} \frac{1}{n^2}\left(Var(e_n) + Var(e_0)\right)$$

$$= \frac{2\sigma_e^2}{n^2}$$

$$Y_1 = \mu + e_1$$
$$Y_2 = \mu + e_2$$
$$\vdots$$
$$Y_n = \mu + e_n$$

$$\bar{Y}_2 = \frac{1}{n}\sum_{i=1}^{n} Y_i = \frac{1}{n}\sum_{i=1}^{n} \mu + e_i = \frac{1}{n}\left(n\mu + \sum_{i=1}^{n} e_i\right)$$

$$Var(\bar{Y}_2) = Var\left(\frac{1}{n}\sum_{i=1}^{n}\mu + e_i\right)$$

$$= \frac{1}{n^2} Var(\underbrace{n\mu}_{=\text{const.}} + e_1 + \dots + e_n)$$

$$= \frac{1}{n^2} Var(e_1 + \dots + e_n)$$

$$\overset{e_i \perp\!\!\!\perp e_j \text{ for } i \neq j}{=} \frac{1}{n^2}\left(Var(e_1) + \dots + Var(e_n)\right)$$

$$= \frac{n\sigma_e^2}{n^2} = \frac{\sigma_e^2}{n}$$

$$\Rightarrow \quad Var(\bar{Y}_1) = \frac{2\sigma_e^2}{n^2} \geq \frac{\sigma_e^2}{n} = Var(\bar{Y}_2) \quad \text{for } n = 1,2$$

$$Var(\bar{Y}_1) = \frac{2\sigma_e^2}{n^2} < \frac{\sigma_e^2}{n} = Var(\bar{Y}_2) \quad \text{for } n \geq 3$$

3. $\rho_k = \varphi^k$

a) show $\quad Var(\bar{Y}) = \frac{\gamma_0}{n}\left(\frac{1+\varphi}{1-\varphi} - \frac{2\varphi}{n}\frac{1-\varphi^n}{(1-\varphi)^2}\right)$

use 3.2.3 $\quad Var(\bar{Y}) = \frac{\gamma_0}{n}\left(1 + 2\sum_{k=1}^{n-1}\left(1-\frac{k}{n}\right)\rho_k\right)$ (1)

$$\sum_{k=0}^{n}\varphi^k = \frac{1-\varphi^{n+1}}{1-\varphi} \qquad (2)$$

$$\sum_{k=0}^{n}k\varphi^{k-1} = \frac{d}{d\varphi}\sum_{k=0}^{n}\varphi^k \qquad (3)$$

$$Var(\bar{Y}) \overset{(1)}{=} \frac{\gamma_0}{n}\left(1 + 2\sum_{k=1}^{n-1}\left(1-\frac{k}{n}\right)\rho_k\right)$$

$$= \frac{\gamma_0}{n}\left(1 + 2\underbrace{\sum_{k=1}^{n-1}\underbrace{\rho_k}_{\varphi^k}}_{= -1 + \sum_{k=0}^{n-1}\varphi^k} - \frac{2}{n}\underbrace{\sum_{k=1}^{n-1}k\underbrace{\rho_k}_{\varphi^k}}_{= \sum_{k=0}^{n-1}k\varphi^k \,=\, \varphi\sum_{k=0}^{n-1}k\varphi^{k-1}}\right)$$

$$= \frac{\gamma_0}{n}\left(1 - 2 + 2\sum_{k=0}^{n-1}\varphi^k - \frac{2\varphi}{n}\sum_{k=0}^{n-1}k\varphi^{k-1}\right)$$

$$\overset{(2),(3)}{=} \frac{\gamma_0}{n}\left(-1 + 2\frac{1-\varphi^n}{1-\varphi} - \frac{2\varphi}{n}\frac{d}{d\varphi}\left(\sum_{k=0}^{n-1}\varphi^k\right)\right)$$

$$\overset{(2)}{=} \frac{\gamma_0}{n}\left(-1 + 2\frac{1-\varphi^n}{1-\varphi} - \frac{2\varphi}{n}\underbrace{\frac{d}{d\varphi}\frac{1-\varphi^n}{1-\varphi}}\right)$$

$$= \frac{(1-\varphi)(-n\varphi^{n-1}) + 1-\varphi^n}{(1-\varphi)^2}$$

$$= \frac{n\varphi^n - n\varphi^{n-1}}{(1-\varphi)^2} + \frac{1-\varphi^n}{(1-\varphi)^2}$$

$$= \frac{\gamma_0}{n}\left(-1 + 2\frac{1-\varphi^n}{1-\varphi} - \frac{2\varphi}{n}\cdot\frac{n\varphi^n - n\varphi^{n-1}}{(1-\varphi)^2} - \frac{2\varphi}{n}\frac{1-\varphi^n}{(1-\varphi)^2}\right)$$

$$= \frac{\gamma_0}{n}\left(-1 + \underbrace{\frac{(2-2\varphi^n)(1-\varphi)}{(1-\varphi)^2} - \frac{2\varphi^{n+1} - 2\varphi^n}{(1-\varphi)^2}} - \frac{2\varphi}{n}\frac{1-\varphi^n}{(1-\varphi)^2}\right)$$

$$= \frac{2 - 2\varphi - 2\varphi^n + 2\varphi^{n+1} - 2\varphi^{n+1} + 2\varphi^n}{(1-\varphi)^2}$$

$$= \frac{2(1-\varphi)}{(1-\varphi)^2} = \frac{2}{1-\varphi}$$

$$= \frac{\gamma_0}{n}\left(\frac{2}{1-\varphi} - \frac{1-\varphi}{1-\varphi} - \frac{2\varphi}{n}\frac{1-\varphi^n}{(1-\varphi)^2}\right)$$

$$= \frac{\gamma_0}{n}\left(\frac{1+\varphi}{1-\varphi} - \frac{2\varphi}{n}\frac{1-\varphi^n}{(1-\varphi)^2}\right)$$

b) $$\mathrm{Var}(\bar{Y}) = \frac{\gamma_0}{n}\left(\frac{1+\varphi}{1-\varphi} - \underbrace{\underbrace{\frac{2\varphi}{n}}_{\xrightarrow{n\to\infty} 0} \cdot \underbrace{\frac{1-\varphi^n}{(1-\varphi)^2}}_{\xrightarrow{n\to\infty} \frac{1}{(1-\varphi)^2}}}_{\xrightarrow{n\to\infty} 0}\right)$$

$\varphi^n \xrightarrow{n\to\infty} 0$ because $|\varphi| < 1$

$\Rightarrow$ for n large enough: $\mathrm{Var}(\bar{Y}) \approx \frac{\gamma_0}{n} \frac{1+\varphi}{1-\varphi}$