Time Series Analysis

Homework 3

1. $Y_t = e_t + e_{t-1} - 0.5\, e_{t-2} + 0.25\, e_{t-3} - 0.125\, e_{t-4} + 0.0625\, e_{t-5}$
$- 0.03125\, e_{t-6} + 0.0151625\, e_{t-7}$

$\rightarrow$ MA(7) process

Goal: simplify

series alternates: $\varphi < 0$

$\vartheta_j = (-0.5)^{j-1}$

$Y_t = \varphi Y_{t-1} + e_t - \vartheta e_{t-1} \rightarrow$ ARMA-process

the general linear process form of ARMA(1,1) model can be obtained written like

$$Y_t = e_t + (\varphi - \vartheta) \Sigma_{j=1}^{\infty} \varphi^{j-1} e_{t-j}$$

$$\psi_j = (\varphi - \vartheta)\varphi^{j-1} \qquad j \geq 1$$

By above equation we see the same type of behavior for an ARMA(1,1) as we saw in our MA(7) model

$\Rightarrow \psi_1 = \vartheta_1 \qquad \psi_2 = \vartheta_2$

$\Rightarrow \varphi - \vartheta = 1 \qquad (\varphi - \vartheta)\varphi = -0.5$

$\Leftrightarrow \varphi = -0.5 \qquad \vartheta = 0.5$

So a simpler model for the MA(7) model is given by an ARMA(1,1) with $\varphi = -1/2$, $\vartheta = 1/2$

$$\Rightarrow Y_t = -\tfrac{1}{2} Y_{t-1} + e_t - \tfrac{1}{2} e_{t-1}$$

2. $Y_t = e_{t-1} - e_{t-2} + 0.5\, e_{t-3}$

a) $\text{Cov}(Y_t, Y_{t-k}) = \text{cov}(e_{t-1} - e_{t-2} + 0.5\, e_{t-3},\ e_{t-1-k} - e_{t-2-k} + 0.5\, e_{t-3-k})$

$= \text{cov}(e_{t-1}, e_{t-1-k}) - \text{cov}(e_{t-1}, e_{t-2-k}) + \tfrac{1}{2}\text{cov}(e_{t-1}, e_{t-3-k})$

$- \text{cov}(e_{t-2}, e_{t-1-k}) + \text{cov}(e_{t-2}, e_{t-2-k}) - \tfrac{1}{2}\text{cov}(e_{t-2}, e_{t-3-k})$

$+ \tfrac{1}{2}\text{cov}(e_{t-3}, e_{t-1-k}) - \tfrac{1}{2}\text{cov}(e_{t-3}, e_{t-2-k}) + \tfrac{1}{4}\text{cov}(e_{t-3}, e_{t-3-k})$

$|k| = 0$: $\quad Cov(Y_t, Y_t) = Var(Y_t) = \sigma^2 + \sigma^2 + 1/4\, \sigma^2 = \frac{9}{4}\sigma^2$

$|k| = 1$: $\quad Cov(Y_t, Y_{t-1}) = 0 - 0 + 0 - \sigma^2 + 0 - 0 + 0 - \frac{1}{2}\sigma^2 + 0$

$$= -3/2\, \sigma^2$$

$|k| = 2$: $\quad Cov(Y_t, Y_{t-2}) = 0 - 0 + 0 - 0 + 0 - 0 + 1/2\, \sigma^2 - 0 + 0 = 1/2\, \sigma^2$

$|k| \geq 3$: $\quad Cov(Y_t, Y_{t-k}) = 0$

$$Cov(Y_t, Y_{t-k}) = \begin{cases} 9/4\, \sigma^2 & |k| = 0 \\ -3/2\, \sigma^2 & |k| = 1 \\ 1/2\, \sigma^2 & |k| = 2 \\ 0 & |k| \geq 3 \end{cases}$$

b)

3. $\Rightarrow$ suppose that $|f|>1$ is the root of

$1-\varphi_1 x-\varphi_2 x^2-\ldots-\varphi_p x^p$

i.e. $1-\varphi_1\cdot f-\varphi_2\cdot f^2-\ldots-\varphi_p\cdot f^p=0$

$\Leftrightarrow f^p\left(\left(\frac{1}{f}\right)^p-\varphi_1\left(\frac{1}{f}\right)^{p-1}-\varphi_2\left(\frac{1}{f}\right)^{p-2}-\ldots-\varphi_p\right)=0$

$\Leftrightarrow \left(\frac{1}{f}\right)^p-\varphi_1\left(\frac{1}{f}\right)^{p-1}-\varphi_2\left(\frac{1}{f}\right)^{p-2}-\ldots-\varphi_p=0$

$\Rightarrow \frac{1}{f}$ is a root for the second statement

$$x^p-\varphi_1 x^{p-1}-\varphi_2 x^{p-2}-\ldots-\varphi_p=0$$

Because of $|m|>1$, it holds $1>\left|\frac{1}{m}\right|$

$\Leftarrow$ suppose $\frac{1}{f}$ is the root of $x^p-\varphi_1 x^{p-1}-\ldots-\varphi_p=0$ (1)

with $\left|\frac{1}{f}\right|<1$

multiplying (1) by f^p yields

$1-\varphi_1 f-\varphi_2 f^2-\ldots-\varphi_p f^p=0$

$\Rightarrow$ so the root of the second equation is given by f

with $|f|>1$ $\left(\Leftrightarrow \left|\frac{1}{f}\right|<1\right)$