MTHSTAT 564/564G/764–Time Series Analysis
Spring 2024
Homework Assignment 1: Due Wednesday, 14 February in Lecture (or by 5:00 P.M. in my office or mailbox

This homework consists of seven problems, none of which require R. You may feel free to work with classmates, but please be sure to turn in your own work.

Reading

Chapter 2

Problems

1. Let $\{e_t\}$ be a zero-mean white noise process. Suppose that the observed process is $Y_t = e_t + \theta e_{t-1}$, where θ is either 3 or $\frac{1}{3}$.

   (a) Find the autocorrelation function for $\{Y_t\}$ both when $\theta = 3$ and $\theta = \frac{1}{3}$.

   (b) You should have discovered that the time series is stationary regardless of the value of θ and that the autocorrelation functions are the same for $\theta = 3$ and $\theta = \frac{1}{3}$. For simplicity, suppose that the process mean is known to be zero and the variance of Y_t is known to be 1. You observe the series $\{Y_t\}$ for $t = 1, 2, \ldots, n$ and suppose that you can produce good estimates of the autocorrelations ρ_k. Do you think that you could determine which value of θ (3 or 1/3) is correct based on the estimate of ρ_k? Why or why not?

2. Suppose that $\{Y_t\}$ is stationary with autocovariance function γ_k. Show that for any fixed positive integer n and any constants $c_1, \ldots, c_n$, the process $\{W_t\}$ defined by $W_t = c_1 Y_t + c_2 Y_{t-1} + \ldots + c_n Y_{t-n+1}$ is stationary.

3. Suppose that $Y_t = e_t - e_{t-12}$, where $\{e_t\}$ is zero-mean white noise. Show that $\{Y_t\}$ is stationary, and that, for $k > 0$, its autocorrelation function is nonzero only for lags $k = 0$ and $k = 12$.

4. Let $Y_t = e_t - \theta(e_{t-1})^2$, where $\{e_t\}$ is zero-mean, normally distributed white noise.

   (a) Find the autocorrelation function for $\{Y_t\}$.

   (b) Is $\{Y_t\}$ stationary?

5. Consider the standard random walk model where $Y_t = Y_{t-1} + e_t$, with $Y_1 = e_1$ and $\{e_t\}$ is zero-mean white noise.

   (a) Use the representation of Y_t above to show that $\mu_t = \mu_{t-1}$ for all $t > 1$ with initial condition $\mu_1 = \mathbb{E}e_1 = 0$. Hence, show that $\mu_t = 0$ for all t.

   (b) Similarly, show that $\text{Var}(Y_t) = \text{Var}(Y_{t-1}) + \sigma_e^2$ for $t > 1$ with $\text{Var}(Y_1) = \sigma_e^2$. Hence, show that $\text{Var}(Y_t) = t\sigma_e^2$.

   (c) For $0 \leq t \leq s$, show that $Y_s = Y_t + e_{t+1} + e_{t+2} + \ldots + e_s$ to show that $\text{Cov}(Y_t, Y_s) = \text{Var}(Y_t)$, and hence, $\text{Cov}(Y_t, Y_s) = \min(t, s)\sigma_e^2$.

6. For a random walk with random starting value Y_0, let $Y_t = Y_0 + e_t + e_{t-1} + \ldots + e_1$ for $t > 0$, where Y_0 has mean μ_0 and variance σ_0^2 and is independent of the zero-mean white noise process $\{e_t\}$.

   (a) Show that $\mathbb{E}[Y_t] = \mu_0$ for all t.

   (b) Show that $\text{Var}(Y_t) = t\sigma_e^2 + \sigma_0^2$.

   (c) Show that $\text{Cov}(Y_t, Y_s) = \min(t, s)\sigma_e^2 + \sigma_0^2$.

   (d) Show that $\text{Corr}(Y_t, Y_s) = \sqrt{\frac{t\sigma_e^2 + \sigma_0^2}{s\sigma_e^2 + \sigma_0^2}}$ for $0 \leq t \leq s$.

7. Let $\{X_t\}$ be a time series in which we are interested. However, because the measurement process itself is imperfect, we actually observe $Y_t = X_t + e_t$, where $\{e_t\}$ is mean-zero white noise that is independent of $\{X_t\}$. We call X_t the signal and e_t the measurement noise or error process. If $\{X_t\}$ is stationary with variance σ_X^2 and autocorrelation ρ_k, show that $\{Y_t\}$ is also stationary with

$$\text{Corr}(Y_t, Y_{t-k}) = \frac{\rho_k}{1 + \frac{\sigma_e^2}{\sigma_X^2}}$$

   for $k >\geq 1$. We call $\frac{\sigma_X^2}{\sigma_e^2}$ the **signal-to-noise ratio** (SNR). The larger the SNR, the nearer the autocorrelation function of the observed process is to the autocorrelation function of the signal process $\{X_t\}$.