Cost models and advanced Futhark programming

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Parallel cost models

Prefix sums (scans)

Using scans

Auxiliary

The need for cost models

Which is better?

```python
import numpy as np

def inc_scalar(x):
  for i in range(len(x)):
    x[i] = x[i] + 1

def inc_par(x):
  return x + np.ones(x.shape)
```

The need for cost models

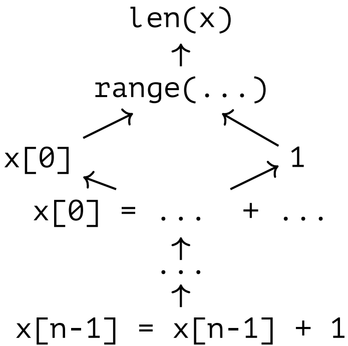
Which is better?

```
import numpy as np

def inc_scalar(x):
  for i in range(len(x)):
    x[i] = x[i] + 1

def inc_par(x):
  return x + np.ones(x.shape)
```

Intuitively, `inc_par` is better because it is “more parallel”.

Parallel cost models make this notion precise.

Computation DAG for `inc_scalar`



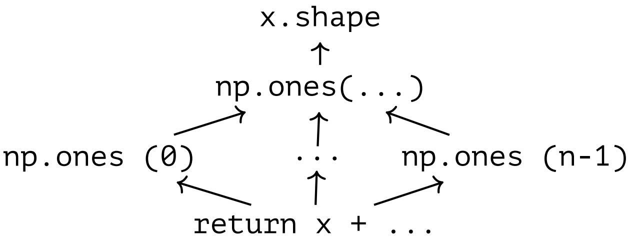
- Total count of nodes is the *work*, $W(p)$.
- Length of longest path from a leaf to the root is the *span*.
- **With an infinite number of processors, if a program p has span k, written $S(p) = k$, the program can execute in $O(k)$ time.**
- Here, $W(p) = O(n), S(p) = O(n)$.

Computation DAG for `inc_par`

```
                 x.shape
                    ↑
               np.ones(...)
          ↗         ↑         ↖
np.ones (0)        ...       np.ones (n-1)
          ↖         ↑         ↗
              return x + ...
```

What is the work and span complexity?

Computation DAG for `inc_par`



What is the work and span complexity?

- $W(p) = O(n)$
- $S(p) = O(1)$

Parallel cost model based on work and span

Instead of giving just a simple cost-model based on the total notion of work carried out by a program, we give instead a *refined* cost model, which aims at providing both:

- a notion of how much total work (W) the program does;
- a notion of the *span*[1] (S) of the program, specifying the maximum depth required by the computation.

Notice:

- The span is the length of the longest sequence of operations that must be performed sequentially due to data dependencies.
- With an infinite number of processors, if a program p has span k, written $S(p) = k$, the program can execute in $O(k)$ time.

[1] Sometimes also called *depth*.

Brent's Theorem (1974) (or Lemma, or Law...)

Writing T_i for the time taken to execute an algorithm on i processors, Brent's Theorem states that

$$\frac{T_1}{p} \leqslant T_p \leqslant T_\infty + \frac{T_1}{p}$$

Proof sketch: At level j of the DAG there are M_j independent operations, which can clearly be executed by p processors in time

$$\left\lceil \frac{M_j}{p} \right\rceil$$

Sum these for each level of the DAG. $\square$

Ramification

We can simulate an "infinitely parallel" machine on a real machine at an overhead proportional to the amount of "missing" hardware parallelism.

Language-based cost models

- Tallying up levels in an infinite DAG is impractical for real programs. Instead we prefer a *language-based cost model*
- E.g. $W(x + y)$ is defined as $W(x) + W(y)$.
- The following slides define work and span cost for a small subset of Futhark.
- Write $[\![e]\!]$ for the result of evaluating expression e (we are being intuitive about scopes and such).

Language-based cost models

- Tallying up levels in an infinite DAG is impractical for real programs. Instead we prefer a *language-based cost model*
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Cost model must be implementable

A provable time and space efficient implementation of NESL—Guy Blelloch and John Greiner, 1996

Simple cases

$$
\begin{aligned}
W(v) &= \\
S(v) &= \\
W(e_1 \oplus e_2) &= \\
S(e_1 \oplus e_2) &= \\
W(\texttt{\textbackslash}x \texttt{ -> } e) &= \\
S(\texttt{\textbackslash}x \texttt{ -> } e) &=
\end{aligned}
$$

Simple cases

$$
\begin{aligned}
W(v) &= 1 \\
S(v) &= 1 \\
W(e_1 \oplus e_2) &= \\
S(e_1 \oplus e_2) &= \\
W(\backslash x \texttt{ -> } e) &= \\
S(\backslash x \texttt{ -> } e) &=
\end{aligned}
$$

Simple cases

$$
\begin{aligned}
W(v) &= 1 \\
S(v) &= 1 \\
W(e_1 \;\oplus\; e_2) &= W(e_1) + W(e_2) + 1 \\
S(e_1 \;\oplus\; e_2) &= S(e_1) + S(e_2) + 1 \\
W(\backslash x \texttt{ -> } e) &= \\
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\end{aligned}$$

$$\begin{aligned}
W([e_1, \cdots, e_n]) &=\\
S([e_1, \cdots, e_n]) &=\\
W((e_1, \cdots, e_n)) &=\\
S((e_1, \cdots, e_n)) &=
\end{aligned}$$

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S(e_1 \ \oplus \ e_2) &= S(e_1) + S(e_2) + 1 \\
W(\backslash x \texttt{ -> } e) &= 1 \\
S(\backslash x \texttt{ -> } e) &= 1
\end{aligned}
$$

$$
\begin{aligned}
W([e_1, \cdots, e_n]) &= W(e_1) + \ldots + W(e_n) + 1 \\
S([e_1, \cdots, e_n]) &= S(e_1) + \ldots + S(e_n) + 1 \\
W((e_1, \cdots, e_n)) &= W(e_1) + \ldots + W(e_n) + 1 \\
S((e_1, \cdots, e_n)) &= S(e_1) + \ldots + S(e_n) + 1
\end{aligned}
$$

Interesting cases

$$
\begin{aligned}
W(\texttt{iota}\ e) &= \\
S(\texttt{iota}\ e) &=
\end{aligned}
$$

Interesting cases

$$W(\texttt{iota}\ e) = W(e) + [\![e]\!]$$
$$S(\texttt{iota}\ e) = S(e) + 1$$

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$$S(\texttt{iota}\ e) = S(e) + 1$$

$$W(\textbf{let}\ x = e\ \textbf{in}\ e') =$$
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Interesting cases

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$$W(\texttt{let}\ x = e\ \texttt{in}\ e') = W(e) + W(e'[x \mapsto [\![e]\!]]) + 1$$
$$S(\texttt{let}\ x = e\ \texttt{in}\ e') = S(e) + S(e'[x \mapsto [\![e]\!]]) + 1$$

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$$W(e_1\ e_2) = W(e_1) + W(e_2) + W(e'[x \mapsto [\![e_2]\!]]) + 1$$
$$\text{where } [\![e_1]\!] = \backslash x \texttt{ -> } e'$$
$$S(e_1\ e_2) = S(e_1) + S(e_2) + S(e'[x \mapsto [\![e_2]\!]]) + 1$$
$$\text{where } [\![e_1]\!] = \backslash x \texttt{ -> } e'$$

Work and span of map

$$W(\texttt{map}\ e_1\ e_2) =$$

$$S(\texttt{map}\ e_1\ e_2) =$$

Work and span of map

$$
\begin{aligned}
&W(\texttt{map}\ e_1\ e_2) = \\
&\quad W(e_1) + W(e_2) + W(e'[x \mapsto v_1]) + \ldots + W(e'[x \mapsto v_n]) \\
&\quad \text{where } [\![\, e_1 \,]\!] = \backslash x\ \texttt{->}\ e' \\
&\quad \text{where } [\![\, e_2 \,]\!] = [v_1, \cdots, v_n]
\end{aligned}
$$

$$
\begin{aligned}
&S(\texttt{map}\ e_1\ e_2) = \\
&\quad S(e_1) + S(e_2) + max(S(e'[x \mapsto v_1]), \ldots, S(e'[x \mapsto v_n])) + 1 \\
&\quad \text{where } [\![\, e_1 \,]\!] = \backslash x\ \texttt{->}\ e' \\
&\quad \text{where } [\![\, e_2 \,]\!] = [v_1, \cdots, v_n]
\end{aligned}
$$

Reduction by contraction

```
def npow2 (n:i64) : i64 =
  loop a = 2 while a < n do 2*a

-- Pad a vector to make its size a power of two
def padpow2 [n] (ne: i32) (v:[n]i32) : []i32 =
  concat v (replicate (npow2 n - n) ne)

-- Reduce by contraction
def red (xs : []i32) : i32 =
  let xs =
    loop xs = padpow2 0 xs
    while length xs > 1 do
      let n = length xs / 2
      in map2 (+) xs[0:n] xs[n:2*n]
  in xs[0]
```

Work and span of `loop`

$$W(\texttt{loop}\ x = e_1\ \texttt{while}\ e_2\ \texttt{do}\ e_3) =$$

$$S(\texttt{loop}\ x = e_1\ \texttt{while}\ e_2\ \texttt{do}\ e_3) =$$

Work and span of `loop`

$$
\begin{aligned}
W(\texttt{\textbf{loop}}\ x = e_1\ \texttt{\textbf{while}}\ e_2\ \texttt{\textbf{do}}\ e_3) = {} & W(e_1) + W(e_2[x \mapsto \llbracket e_1 \rrbracket]) \ + \\
& \text{if } \llbracket e_2[x \mapsto \llbracket e_1 \rrbracket] \rrbracket = \texttt{\textbf{false}} \\
& \text{then } 0 \\
& \text{else } W(e_3[x \mapsto \llbracket e_1 \rrbracket]) \ + \\
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S(\texttt{\textbf{loop}}\ x = e_1\ \texttt{\textbf{while}}\ e_2\ \texttt{\textbf{do}}\ e_3) = {} & S(e_1) + S(e_2[x \mapsto \llbracket e_1 \rrbracket]) \ + \\
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$$

Work and Span for npow2 n

Work and Span for `npow2 n`

By inspection, we have

$$W(\texttt{npow2 n}) = S(\texttt{npow2 n}) = O(\log \texttt{n})$$

Work and Span for `padpow2 ne v`

Work and Span for `npow2 n`

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Work and Span for `padpow2 ne v`

Because $\texttt{npow2 n} \leqslant 2\,\texttt{n}$, we have (where $\texttt{n} = \texttt{length v}$)

$$\begin{aligned}
W(\texttt{padpow2 ne v}) &= W(\texttt{concat}\ v\ (\texttt{replicate}\ (\texttt{npow2 n - n})\ \texttt{ne}))\\
&= O(\texttt{n})
\end{aligned}$$

$$S(\texttt{padpow2 ne v}) = O(\log\ \texttt{n})$$

Work and Span for `red`

Work and Span for npow2 n

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Because $\texttt{npow2 n} \leqslant 2\,\texttt{n}$, we have (where $\texttt{n} = \texttt{length v}$)

$$\begin{aligned} W(\texttt{padpow2 ne v}) &= W(\texttt{concat}\ v\ (\texttt{replicate (npow2 n - n) ne})) \\ &= O(\texttt{n}) \end{aligned}$$

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Work and Span for red

Each loop iteration in red has span $O(1)$. Because the loop is iterated at-most $\log(2\,\texttt{n})$ times, we have (where $\texttt{n} = \texttt{length v}$)

$$W(\texttt{red v}) = O(\texttt{n}) + O(\texttt{n}/2) + O(\texttt{n}/4) + \cdots + O(1) =$$

Work and Span for `npow2 n`

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Work and Span for `padpow2 ne v`

Because $\texttt{npow2 n} \leqslant 2\,\texttt{n}$, we have (where $\texttt{n} = \texttt{length v}$)

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Work and Span for `red`

Each loop iteration in `red` has span $O(1)$. Because the loop is iterated at-most $\log(2\,\texttt{n})$ times, we have (where $\texttt{n} = \texttt{length v}$)

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Work and Span for `npow2 n`

By inspection, we have

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Work efficiency

A parallel algorithm is said to be *work efficient* if it has at most the same work as the best sequential algorithm.

Is `red` work efficient?

Work efficiency

A parallel algorithm is said to be *work efficient* if it has at most the same work as the best sequential algorithm.

Is `red` work efficient?

Yes, because it does $O(n)$ work, which is as good as a sequential summation.

Is it also *efficient*?

Performance Compared to the Built-in Reduction SOAC

```
-- ==
-- entry: test_red test_reduce
-- random input { [10000000]i32 }
entry test_red = red
entry test_reduce = reduce (+) 0
```

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```
$ futhark bench --backend=opencl reduce.fut
Compiling reduce.fut...
Results for reduce.fut:test_red:
dataset [10000000]i32:    4675.40μs
Results for reduce.fut:test_reduce:
dataset [10000000]i32:     273.80μs
```

Performance Compared to the Built-in Reduction SOAC

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```

If you are not using `futhark bench`, then you are probably doing it wrong.

Parallel cost models

Prefix sums (scans)

Using scans

Auxiliary

Inclusive and exclusive prefix sum

Exclusive prefix sum (“prescan”)

Given

$$[1, 2, 3, 4]$$

produce

$$[0, 1, 3, 6]$$

Inclusive prefix sum

Given

$$[1, 2, 3, 4]$$

produce

$$[1, 3, 6, 10]$$

Prefix sums are scans

Generalising the addition and zero used by a prefix sum to an arbitrary associative operator $\oplus$ and neutral element $0_{\oplus}$, we get *scan*.

```
-- The scan in Futhark is inclusive.
> scan (+) 0 [1,2,3,4]
[1, 3, 6, 10]
```

Prefix sums are scans

Generalising the addition and zero used by a prefix sum to an arbitrary associative operator $\oplus$ and neutral element $0_{\oplus}$, we get *scan*.

```
-- The scan in Futhark is inclusive.
> scan (+) 0 [1,2,3,4]
[1, 3, 6, 10]
```

- Scans are a fundamental tool for parallelising seemingly-sequential algorithms.
- Let us see how scans can be computed in parallel.

Sequential prefix sum

```
acc = 0
for i < n:
  acc = acc + input[i]
  scanned[i] = acc
```

Sequential prefix sum

```
acc = 0
for i < n:
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```

Work: $O(n)$

Span: $O(n)$

Brute force

To calculate the prefix sum of $[x_0, \ldots, x_{n-1}]$, compute

$$\begin{aligned}
&[sum([x_0]) \\
&\ sum([x_0, x_1]) \\
&\ \vdots \\
&\ sum([x_0, x_1, \ldots, x_{n-1}])]
\end{aligned}$$

Assume $S(sum([x_0, \ldots, x_{n-1}])) = log_2(n)$.

Brute force

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$$
\begin{aligned}
[&sum([x_0]) \\
&sum([x_0, x_1]) \\
&\vdots \\
&sum([x_0, x_1, \ldots, x_{n-1}])]
\end{aligned}
$$

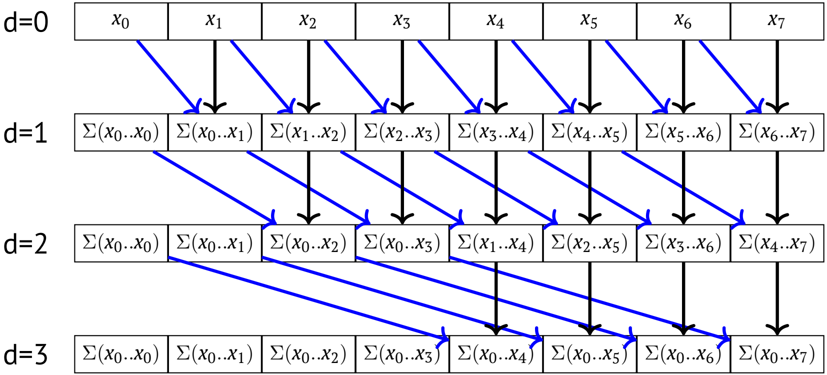
Assume $S(sum([x_0, \ldots, x_{n-1}])) = log_2(n)$.

Work: $O(\sum_{i<n} i) = O(n^2)$

Span: $O(max(S(sum([x_0])), \ldots S(sum([x_0, \ldots, x_{n-1}])))) = O(log_2(n))$

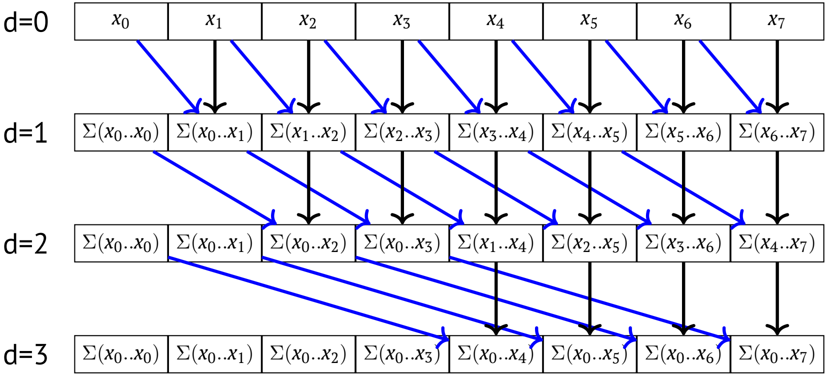
Terrible. The sequential implementation is faster for large n!

Hillis—Steele scan (1986)



For each d, element x_i^d is updated by $x_{i-2^d}^{d-1} + x_i^{d-1}$.

Hillis–Steele scan (1986)



For each d, element x_i^d is updated by $x_{i-2^d}^{d-1} + x_i^{d-1}$.

Work: For $n = 2^m, O(\sum_{i<m} 2^m - 2^i) = O(n \log(n))$

Span: $\log(n)$

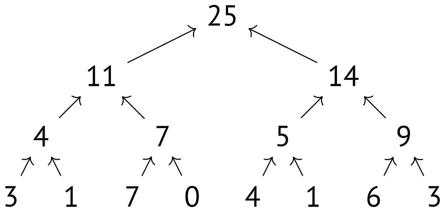
Work-efficient exclusive scan by Guy Blelloch

Two passes

Upsweep Build a balanced binary tree of partial sums.

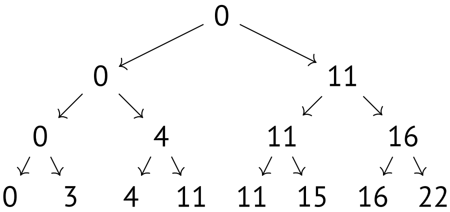
Downsweep Set root to 0, use partial sums to propagate.

Upsweep



$$\text{sum}[v] = \text{sum}[L[v]] + \text{sum}[R[v]]$$

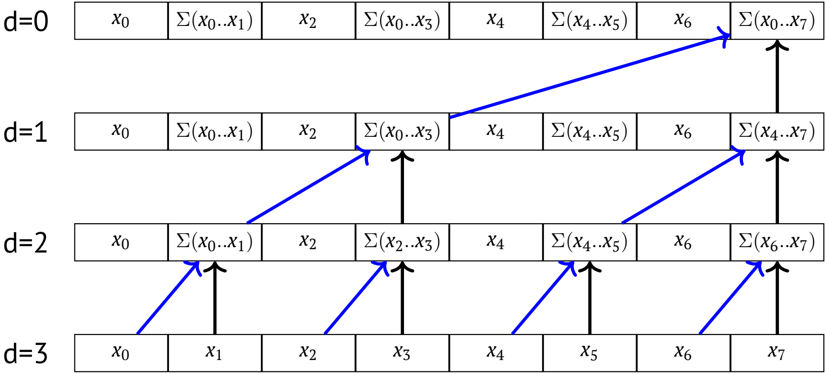
Downsweep



$$\text{scan}[L[v]] = \text{scan}[v]$$

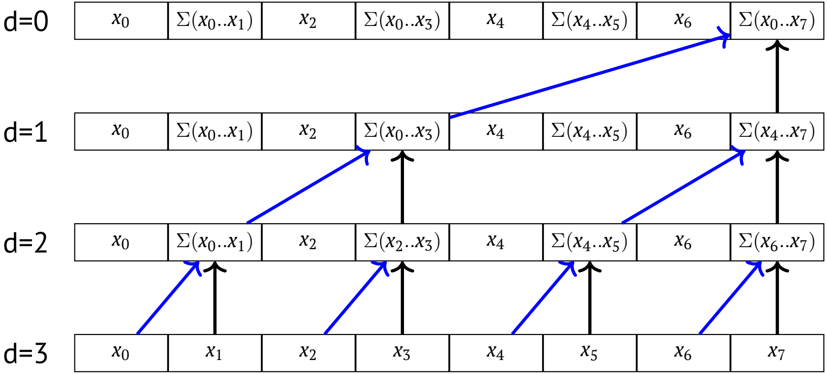
$$\text{scan}[R[v]] = \text{sum}[L[v]] + \text{scan}[v]$$

On arrays: upsweep ("reduction phase")



$$x_i^d = x_{i-2^{m-d-1}}^{d+1} + x_i^{d+1}$$

On arrays: upsweep ("reduction phase")

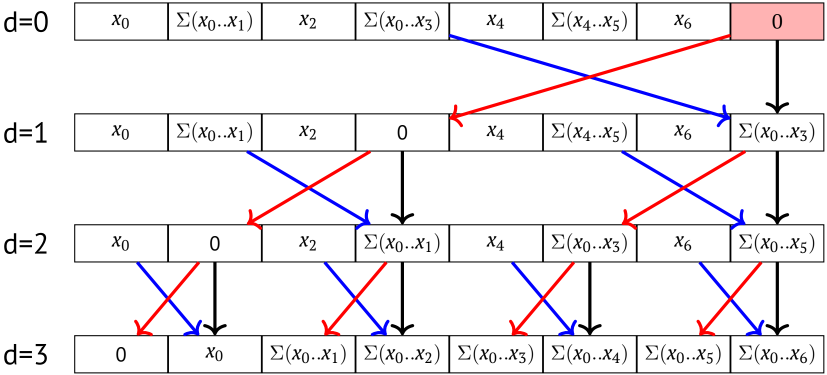


$$x_i^d = x_{i-2^{m-d-1}}^{d+1} + x_i^{d+1}$$

Work: For $n = 2^m, O(\sum_{i<m} 2^i) = O(n)$

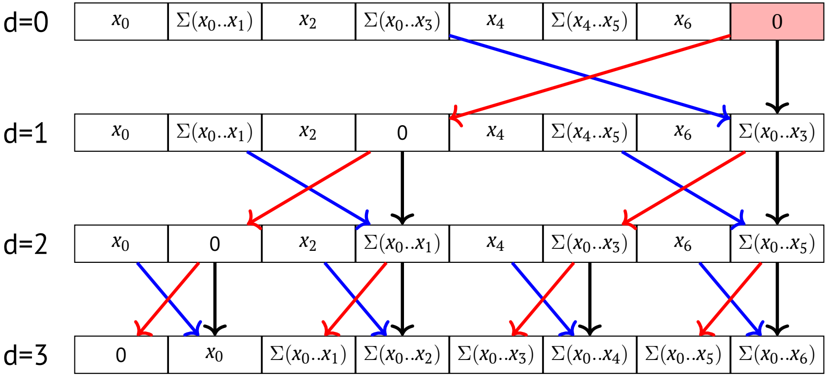
Span: $\log(n)$

On arrays: downsweep



Inverse indexing of the upsweep phase.

On arrays: downsweep



Inverse indexing of the upsweep phase.

Work: For $n = 2^m, O(\sum_{i \leqslant m} 2^i) = O(n)$

Span: $\log(n)$

Work efficient scan

Complexity of *scan* on size-n input

Work: $O(n)$

Span: $\log(n)$

- Optimal, as *reduce* is the same.
- Can now depend on scan as a relatively cheap building block.

Real-world scan implementations are often very different for technical reasons, but we can depend on these asymptotics when analysing and designing parallel algorithms.

Parallel cost models

Prefix sums (scans)

Using scans

Auxiliary

Filtering

Suppose we wish to remove negative elements from the list

```
let as = [-1, 2, -3, 4, 5, -6]
```

Filtering

Suppose we wish to remove negative elements from the list

```
let as = [-1, 2, -3, 4, 5, -6]
```

For each element, see if we want to keep it:

```
let keep = map (\a -> if a >= 0 then 1 else 0) as
-- [ 0, 1, 0, 1, 1, 0]
```

Filtering

Suppose we wish to remove negative elements from the list

```
let as = [-1, 2, -3, 4, 5, -6]
```

For each element, see if we want to keep it:

```
let keep = map (\a -> if a >= 0 then 1 else 0) as
-- [ 0, 1, 0, 1, 1, 0]

let offsets1 = scan (+) 0 keep
-- [ 0, 1, 1, 2, 3, 3]
```

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let offsets = map (\x -> x - 1) offsets1
-- [-1, 0, 0, 1, 2, 2]
```

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-- [-1, 0, 0, 1, 2, 2]
```

`offsets[i]` now indicates position in filtered list if `keep[i] == 1`.

`scatter`

`scatter xs is vs` computes equivalent of the imperative pseudocode

```
for j < n:
  xs[is[j]] = vs[j]
```

- Out-of-bound writes are ignored.
- Writing different values to same index is *undefined*.[2]
- Work $O(n)$, span $O(1)$.

Just what we need for filtering!

[2] `reduce_by_index` handles conflicts with provided operator.

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Just what we need for filtering!

```
scatter (replicate (last offsets1) 0)
        (map2 (\i k -> if k == 1 then i else -1)
              offsets keep)
        as
```

[2] `reduce_by_index` handles conflicts with provided operator.

Implementing filter

```
def filter [n] 'a (p: a -> bool) (as: [n]a): []a =
  let keep = map (\a -> if p a then 1 else 0) as
  let offsets1 = scan (+) 0 keep
  let num_to_keep = n == 0 || last offsets1 == 0
  in if num_to_keep == 0
     then []
     else scatter (replicate num_to_keep as[0])
                  (map2 (\i k -> if k == 1
                                 then i-1
                                 else -1)
                        offsets1 keep)
                  as
```

Radix sort

- Many classical sorting algorithms are a poor fit for data parallelism, but *radix sort* works well.
- Radix-2 sort works by repeatedly partitioning elements according to one bit at a time, while preserving the ordering of the previous steps.
  - *Stable* partitioning.

Example with radix-10

3 2 6		6 9 0		7 0 4		3 2 4
4 5 3		7 5 1		6 0 8		4 3 8
6 0 8		4 5 3		3 2 6		4 5 6
8 3 5		7 0 4		8 3 5		6 0 5
7 5 1	$\Rightarrow$	8 3 5	$\Rightarrow$	4 3 5	$\Rightarrow$	6 9 5
4 3 5		4 3 5		7 5 1		7 0 1
7 0 4		3 2 6		4 5 3		7 5 3
6 9 0		6 0 8		6 9 0		8 3 0

Example with radix-10

3 2 6		6 9 0		7 0 4		3 2 4
4 5 3		7 5 1		6 0 8		4 3 8
6 0 8		4 5 3		3 2 6		4 5 6
8 3 5		7 0 4		8 3 5		6 0 5
7 5 1	$\Rightarrow$	8 3 5	$\Rightarrow$	4 3 5	$\Rightarrow$	6 9 5
4 3 5		4 3 5		7 5 1		7 0 1
7 0 4		3 2 6		4 5 3		7 5 3
6 9 0		6 0 8		6 9 0		8 3 0

- **Radix sort is not as general as a comparison-based sort.**
- Assumes sorting key can be decomposed into “digits”.

Sorting xs : [n]u32 by bit b

```
let bits = map (u32.get_bit b) xs
let bits_neg = map (1-) bits
let offs = reduce (+) 0 bits_neg
```

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```
let bits = map (u32.get_bit b) xs
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let offs = reduce (+) 0 bits_neg
```

Example

```
b        = 0
xs       = [0, 1, 2, 3, 4]
bits     = [0, 1, 0, 1, 0]
bits_neg = [1, 0, 1, 0, 1]
offs     = 3
```

```
let idxs0 = map2 (*)
                 bits_neg
                 (scan (+) 0 bits_neg)
let idxs1 = map2 (*)
                 bits
                 (map (+offs) (scan (+) 0 bits))
```

```
let idxs0 = map2 (*)
                 bits_neg
                 (scan (+) 0 bits_neg)
let idxs1 = map2 (*)
                 bits
                 (map (+offs) (scan (+) 0 bits))
```

Example

```
bits                    = [0, 1, 0, 1, 0]
bits_neg                = [1, 0, 1, 0, 1]
offs                    = 3
idxs0                   = [1, 0, 2, 0, 3]
idxs1                   = [0, 4, 0, 5, 0]
map2 (+) idxs0 idxs1 = [1, 4, 2, 5, 3]
```

Then `scatter` as when filtering.

The whole step

```
def radix_sort_step [n] (xs: [n]u32) (b: i32): [n]u32 =
  let bits = map (u32.get_bit b) xs
  let bits_neg = map (1-) bits
  let offs = reduce (+) 0 bits_neg
  let idxs0 = map2 (*) bits_neg
                   (scan (+) 0 bits_neg)
  let idxs1 = map2 (*) bits
                   (map (+offs) (scan (+) 0 bits))
  let idxs2 = map2 (+) idxs0 idxs1
  let idxs  = map (\x->x-1) idxs2
  let xs' = scatter (copy xs) idxs xs
  in xs'
```

Radix sort in Futhark

```
def radix_sort [n] (xs: [n]u32): [n]u32 =
  loop xs for i < 32 do radix_sort_step xs i
```

See worked example at
`https://futhark-lang.org/examples/radix-sort.html`

Segmented scan

```
val segmented_scan [n] 't
    :  (op: t -> t -> t) -> (ne: t)
    -> (flags: [n]bool) -> (as: [n]t)
    -> [n]t
```

`true` starts a segment and `false` continues a segment.

Example

```
segmented_scan (+) 0
  [true, false, true, false, false, true]
  [0, 1, 2, 3, 4, 5]
== scan (+) 0 [0,1] ++
   scan (+) 0 [2,3,4] ++
   scan (+) 0 [5]
== [0, 1, 2, 5, 9, 5]
```

Segmented reduction

```
val segmented_reduce [n] 't
    :  (op: t -> t -> t) -> (ne: t)
    -> (flags: [n]bool) -> (as: [n]t)
    -> []t
```

Example

```
segmented_reduce (+) 0
  [true, false, true, false, false, true]
  [0, 1, 2, 3, 4, 5]
== [reduce (+) 0 [0,1],
    reduce (+) 0 [2,3,4],
    reduce (+) 0 [5]]
== [1, 9, 5]
```

Generalised histograms

Like `scatter`, but uses a provided `reduce`-like operator to handle multiple writes to same index.

Type

```
val reduce_by_index [k] [n] 'a :
     (dest: *[k]a)
  -> (f: a -> a -> a) -> (ne: a)
  -> (is: [n]i64) -> (vs: [n]a) -> *[k]a
```

Semantics

```
for index in 0..k-1:
  i = is[index]
  v = vs[index]
  dest[i] = f(dest[i], v)
```

Futhark uses parallel implementation with GPU *atomics*.

Proving associativity and neutral elements

```
let op (x, i) (y, j) : (i32, i32) =
  if x < y then (y, j) else (x, i)

let argmax [n] (xs: [n]i32) =
  reduce op
         (i32.lowest, -1)
         (zip xs (iota n))
```

- Is op associative?
- Is `(i32.lowest, -1)` a neutral element?

argmax: associativity

First, inline definitions:

```
   (a `op` b) `op` c
== ((ax, ai) `op` (bx, bi)) `op` (cx, ci)
== let (x, i) = if ax < bx then (bx, bi)
                            else (ax, ai)
   in if x < cx then (cx, ci)
                else (x, i)
```

```
   a `op` (b `op` c)
== (ax, ai) `op` ((bx, bi) `op` (cx, ci))
== let (x, i) = if bx < cx then (cx, ci)
                            else (bx, bi)
   in if ax < x then (x, i)
                else (ax, ai)
```

Then enumerate all possible comparisons between `ax`, `bx`, and `cx` and show that these two expressions are equivalent.

E.g. for `!(ax < bx) && bx < cx && cx < ax`

```
    let (x, i) = if ax < bx then (bx, bi)
                             else (ax, ai)
    in if x < cx then (cx, ci)
                 else (x, i)
 == if ax < cx then (cx, ci)
               else (ax, ai)
 == (ax, ai)
```

```
    let (x, i) = if bx < cx then (cx, ci)
                             else (bx, bi)
    in if ax < x then (x, i)
                 else (ax, ai)
 == if ax < cx then (cx, ci)
               else (ax, ai)
 == (ax, ai)
```

□

argmax: neutral element

Similarly, by equational reasoning.

```
   a `op` (i32.lowest, -1)
== (x, i) `op` (i32.lowest, -1)
== if x < i32.lowest then (i32.lowest, -1)
                     else (x, i)
== (x, i)
```

```
   (i32.lowest, -1) `op` a
== (i32.lowest, -1) `op` (x, i)
== if i32.lowest < x then (x, i)
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== (x, i)
```

□

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== (i32.lowest, -1) `op` (x, i)
== if i32.lowest < x then (x, i)
                     else (i32.lowest, -1)
== (x, i)
```

□

(Actually, the second case is wrong—see if you can figure out why, and try to fix it by modifying the operator.)

A more calculational approach

`https://byorgey.wordpress.com/2020/02/23/what-would-dijkstra-do-proving-the-associativity-of-min/`

- Worth a read!
- More elegant and concise, but requires more creative thinking to characterise a useful property of the operator.

Commutativity?

Exercise for home: The `argmax` operator is not commutative. Try to come up with a counterexample, and see if you can change its definition such that it becomes commutative.

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Commutative reductions

Futhark has a `reduce_comm` function that can be used for commutative operators. This runs faster than normal `reduce`. Not necessary for built-in operators.

Summary

- *Work* measures the total number of operations, *span* measures the longest chain of dependencies.
- Language-based cost models let us reason about program performance in a hardware-agnostic and composable way.
- Scans are a useful building block in advanced data parallel algorithms, but an efficient implementation is not straightforward.