TFY4345 Computational Assignments 2 & 3: The Coriolis effect

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1 Introduction

Using the same premise as in the first computational assignment, the discussion of the Coriolis effect follows naturally. This document aims to give an answer to computational assignments 2 and 3, and can be viewed as a sequel to the first one.

Previously, we established the range of the Paris gun to be about 148 km (in lab coordinates) in our numerical model, using a firing angle of about 51.5°. This range is formidable, so we will now redo the calculations while taking the Coriolis effect into account.

Historically, the Germans fired the gun at Paris (48°36′18″N, 3°30′53″E) from Crépy-en-Valois (49°36′18″N, 3°30′53″E), which equates to an intermediate angle (with base in the Earth centre) of $\alpha = \sqrt{(\Delta\text{Latitude})^2 + (\Delta\text{Longitude})^2} = 1.384°$, and a corresponding arc length of $R_\oplus \cdot \alpha = 153.8\,\text{km}$ across the Earth's surface. In lab coordinates, this would be within the range we calculated.

2 Assumptions and simplifications

Like the last time, we solve the differential equations of motion using a Runge-Kutta algorithm of order 4. We make two assumptions to avoid coordinate transformation in each iteration. These simplify the code and reduce computational time.

1. We assume the altitude in the air drag model is equal to the z-component of the Crépy lab coordinate system.[1]

2. We assume the gravitational acceleration has constant magnitude and direction, in the negative z-direction of the lab coordinates.

We use the adiabatic air drag model from computational assignment 1, but this model is flawed at large altitudes.[2] Therefore, we infer that the error made by approximation 1 is less significant. Similarly, the change in gravitational acceleration is really slight. Anyway, these factors are not really the point of the exercise.

We also assume the Earth to be a perfect sphere with radius $R_\oplus = 6371\,\text{km}$, rotating without precession about the north-pole to south-pole axis with a constant angular frequency of $\omega_\oplus = 7.29 \times 10^{-5}\,\text{s}^{-1}$. In the following sections, we will completely neglect the centrifugal force, but we may sometimes take into account the Coriolis effect.

[1] The Crépy lab coordinate system is centered in Crépy with $\hat{x}_C = \hat{\theta}_\oplus$, $\hat{y}_C = \hat{\phi}_\oplus$, $\hat{z}_C = \hat{r}_\oplus$, and the referred Earth coordinate system is in standard spherical coordinates with $\phi_\oplus = 0$ on the Greenwich meridian.

[2] The model squares a value that becomes complex at large altitudes, giving a speed boost, rather than a speed decrease. This is ”fixed” by just neglecting the air drag in the regions where it behaves like a speed boost.

3 The Coriolis effect

The effective force on a particle in a rotating coordinate system is

$$\vec{F}_{\text{eff}} = \vec{F} + 2m\vec{v}_{\text{rot}} \times \vec{\omega} - m\vec{\omega} \times (\vec{\omega} \times \vec{r}_{\text{rot}}), \tag{1}$$

where we will omit the centrifugal force (the second additional term). The Coriolis force[3] (the first additional term) will give an apparent westward deflection on the northern hemisphere. Due to the high velocity component of the projectile perpendicular to the Earth's rotation, this term is significant.

With a projectile mass of 106 kg, the force of gravity is on the order of 1000 N. Utilizing the maximum speed of the trajectory, and a polar angle (which will turn out to be approximately 45° with respect to the Earth's rotation), we have that the Coriolis force is approximately 20 N. Evidently, the Coriolis effect must be taken into account if we desire accuracy better than 2%.

In conclusion, we then need to solve the following set of differential equations:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = v_x, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = v_y, \quad \frac{\mathrm{d}z}{\mathrm{d}t} = v_z,$$

$$\frac{\mathrm{d}v_x}{\mathrm{d}t} = -Dvv_x + 2(\vec{v} \times \vec{\omega})_x, \quad \frac{\mathrm{d}v_y}{\mathrm{d}t} = -Dvv_y + 2(\vec{v} \times \vec{\omega})_y, \quad \frac{\mathrm{d}v_z}{\mathrm{d}t} = -Dvv_z + 2(\vec{v} \times \vec{\omega})_z - g, \tag{2}$$

for $D = D(z)$ given by the adiabatic air drag model, and $v = \sqrt{v_x^2 + v_y^2 + v_z^2}$.

4 Finding a starting state to reach Paris

We first want to find the firing direction to hit Paris without taking the Coriolis effect into account. We pretend we are giving directions to the crew firing the cannon, and we want to provide them with a polar and azimuth angle in their local coordinate system. By transforming the earth coordinates of Paris into lab coordinates,[4] we find the coordinates of Paris the ground crew has to aim for. Thus, the lab polar angle is

$$\phi_C = \arctan\left(\frac{y_{\text{Paris}}}{x_{\text{Paris}}}\right) = -45.886^\circ. \tag{3}$$

The azimuth angle has no analytic solution, so we must try different angles and see which yields the desired arc length. Shooting with ϕ_C (C for Crépy-coordinates), we use the bisection algorithm to solve the equation

$$\text{range}(\theta_C) - R_{C,P} = 0, \tag{4}$$

where $R_{C,P}$ is the distance between Crépy and Paris. Within accuracy of 1 mm,[5] we have $\theta_C = 35.609^\circ$.

A plot of the trajectory in this case is included in fig. 1.

5 Trajectory, influenced by Coriolis

Using the same initial conditions as above, we study the trajectory of a projectile under the influence of the Coriolis effect. A plot of this trajectory is included in fig. 2. The deflection is small compared to the shooting range, so it is more useful to look at the East-West deflection during the course of the trajectory. This is included in fig. 3.

We note some of the other differences in the trajectory. Using the landing coordinates in both cases, we can calculate the deflection angle to be (positive) 0.470°, and the landing point to be 589 m *closer* to Crépy. Also, accounting for Coriolis *decreases* the maximum altitude by 146 m.

[3]Not a force in the Newtonian sense. There is no equal and opposite force to fulfil Newton's third law. This is sometimes known as a *fictitious* force.

[4]the "lab" is in Crépy, so we will also refer to this as Crépy-coordinates.

[5]This is definitely overkill. The error in i.e. the air drag model is much larger. However, we had a good initial estimate so we decided to push the accuracy.

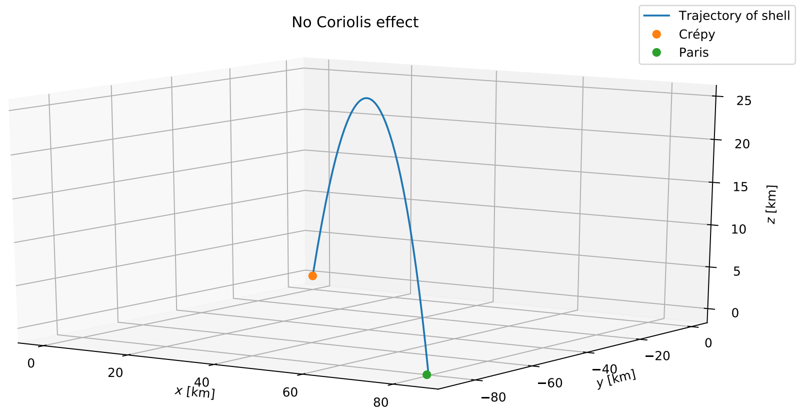

Figure 1: Trajectory of cannon projectile fired at Paris, not taking the Coriolis effect into account.

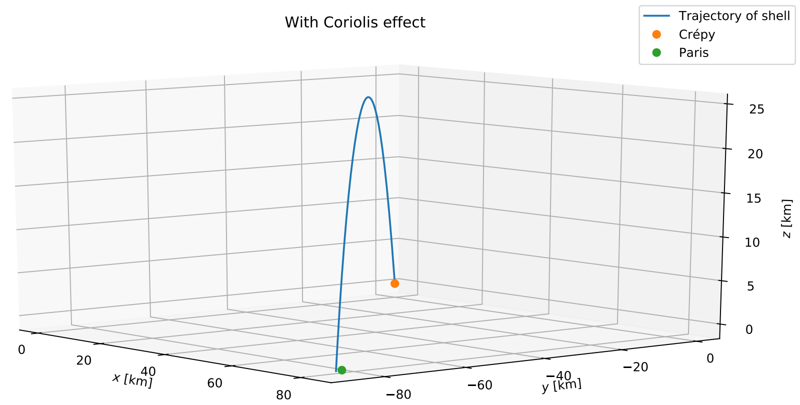

Figure 2: Trajectory of cannon projectile fired at Paris, taking the Coriolis effect into account.

6 Firing angles for arbitrary target

We now implement some code to search for the firing angles given arbitrary landing coordinates (within range). We discuss two methods for doing this:

1. Interpolating the landing coordinates for a selection of firing angles.
2. Alternately bisecting in polar and azimuth firing angles to approach the desired landing coordinates.

Interpolation turned out to be futile, as the domain (of the scale 10^0) is far smaller than the image (of the scale 10^6); the required number of interpolation nodes would be too large.

The alternating bisection algorithm works like this: Start with the analytical[6] firing angles to hit the target. Fire a shot with these angles, and check if the range is larger or smaller than the target. Bisect azimuth angle and fire again. Repeat until the fired range is close to the target range.

[6] Ignoring air drag and Coriolis. This gives a fine starting point to begin the search.

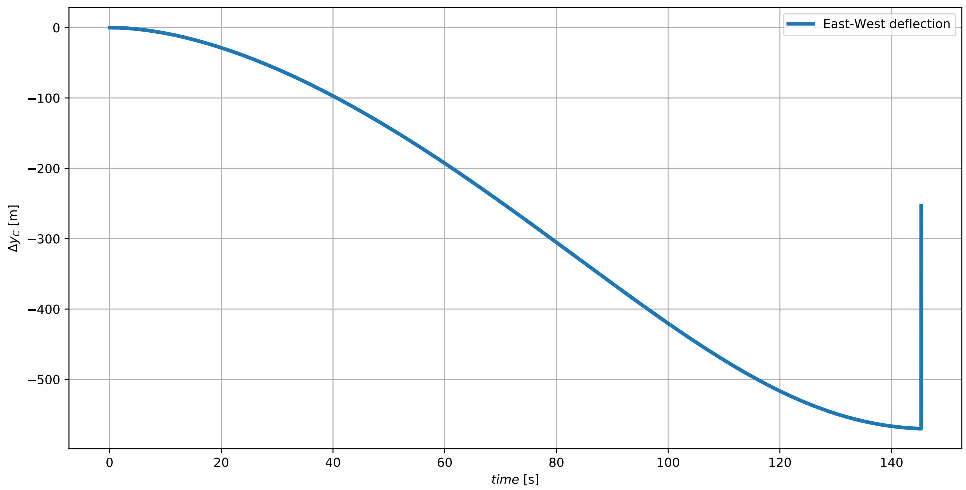

Figure 3: East-West deflection (Δy_C) during the course of the trajectory. Negative values means deflection in negative y_C (westward).

Then do the same process for azimuth angle. Repeat both processes until we land fairly close to the target.

As an example, we tried to find the firing angles corresponding to the (Crépy) coordinates $x_C = -81\,\text{km}$, $y_C = 109\,\text{km}$. The appropriate firing angles turned out to be $\theta \approx 62°$, $\phi \approx 127°$.

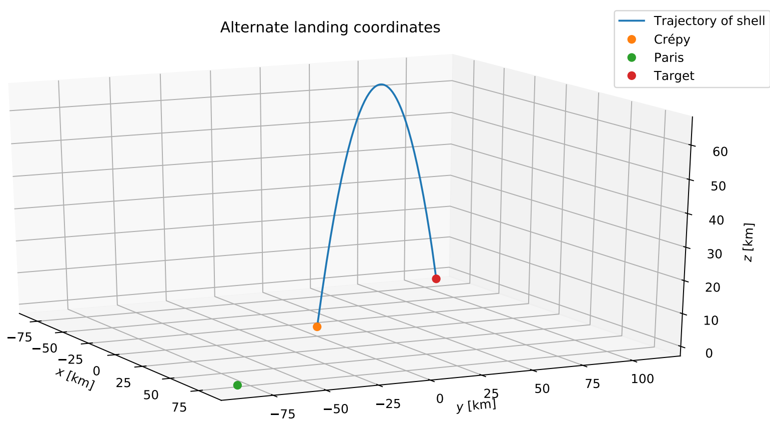

Figure 4: Example shot to hit the coordinates $(-81\,\text{km}, 109\,\text{km})$. The firing angles are $\theta \approx 62°$, $\phi \approx 127°$.

7 Closing words

Similar to the previous project, we have numerically solved the equations of motion for a projectile with the Runge-Kutta algortihm while taking into account the Coriolis effect due to Earth's rotation. Due to the relatively high initial velocity of the projectile, it turned out that the interpolation method for an arbitrary target was futile and too time consuming, leaving us no choice but using the robust bisection method to calculate the appropriate firing angles to hit the desired target.