TFY4240 - Computational assignment

Svein Åmdal

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1 Setting up the equation

Given the boundary conditions, we introduce the reduced coordinates $\xi = x/L$ and $\eta = y/L$. We infer that there is no free charge on the square cylinder given in the problem. Therefore, we can find the potential V from solving Laplace's equation

$$\nabla^2 V(\xi, \eta) = 0, \tag{1}$$

for some reduced (i.e. dimensionless) potential $V(\xi, \eta)$. The boundary conditions become

$$V(\xi = 0, \eta) = 0 \tag{2}$$
$$V(\xi = 1, \eta) = 0 \tag{3}$$
$$V(\xi, \eta = 0) = 0 \tag{4}$$
$$V(\xi, \eta = 1) = V_0(\xi). \tag{5}$$

There is no dependence on z, so we view the problem as two-dimensional. Assuming the potential has a solution of the form $V(\xi, \eta) = X(\xi)Y(\eta)$, we can separate the equation into the form

$$\frac{\partial_\xi^2 X(\xi)}{X(\xi)} + \frac{\partial_\eta^2 Y(\eta)}{Y(\eta)} = 0. \tag{6}$$

Because ξ and η are independent, each of the terms in eq. (6) must individually be constant, say $-k^2$ and k^2, respectively. We may assume $k > 0$. Solving the indivicual equations, we get that

$$X(\xi) = Acos(k\xi) + Bsin(k\xi) \tag{7}$$
$$Y(\eta) = Ce^{k\eta} + De^{-k\eta}. \tag{8}$$

Using eq. (2), we find that $A = 0$. Using eq. (4) we get that $C = -D$. Thus $X(\xi) \to Bsin(k\xi)$ and $Y(\eta) \to 2Csinh(k\eta)$. Using eq. (3), we find that either $B = 0$, or $k = n\pi$. To ensure the solution is nontrivial, we choose to demand the latter. Finally eq. (5) implies that the equation

$$2Bsin(n\pi\xi)Csinh(n\pi) = V_0(\xi) \tag{9}$$

holds $\forall n \in \mathbb{N}$. The general solution is a linear combination of these. Letting $B \to B_n$ and $C \to C_n$ and combining all constants into $F_n = B_n C_n$, we get

$$V(\xi, \eta) = 2\sum_n F_n sin(n\pi\xi) sinh(n\pi\eta). \tag{10}$$

Inserting $\eta = 1$ we re-obtain eq. (9). Mulitplying both sides of eq. (9) with $sin(m\pi\xi)$ and integrating over the interval of $\xi \in (0, 1)$, we obtain

$$2\int_0^1 d\xi V_0(\xi) sin(m\pi\xi) = 2\sum_n F_n sinh(n\pi) \int_0^1 d\xi sin(n\pi\xi) sin(n\pi\xi) = 2\sum_n F_n sinh(n\pi)\frac{\delta_{nm}}{2} \tag{11}$$

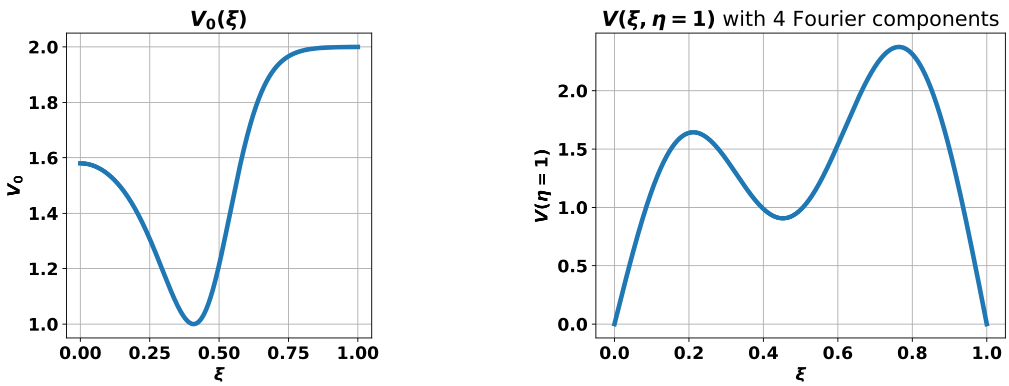
$$= F_m sinh(m\pi) \implies F_n = \frac{1}{sinh(n\pi)}\int_0^1 dq V_0(q) sin(n\pi q). \tag{12}$$

Finally, the equation that yields V is then given by

$$V(\xi, \eta) = 2 \sum_{n=1}^{\infty} F_n sin(n\pi\xi) sinh(n\pi\eta), \tag{13}$$

where F_n is the n-th *Fourier coefficient*, given by eq. (12).

2 Plots



(a) The target function.

(b) Reconstruction of the target function by four Fourier components.

Figure 1: Plot of $V(\xi, \eta = 1)$.

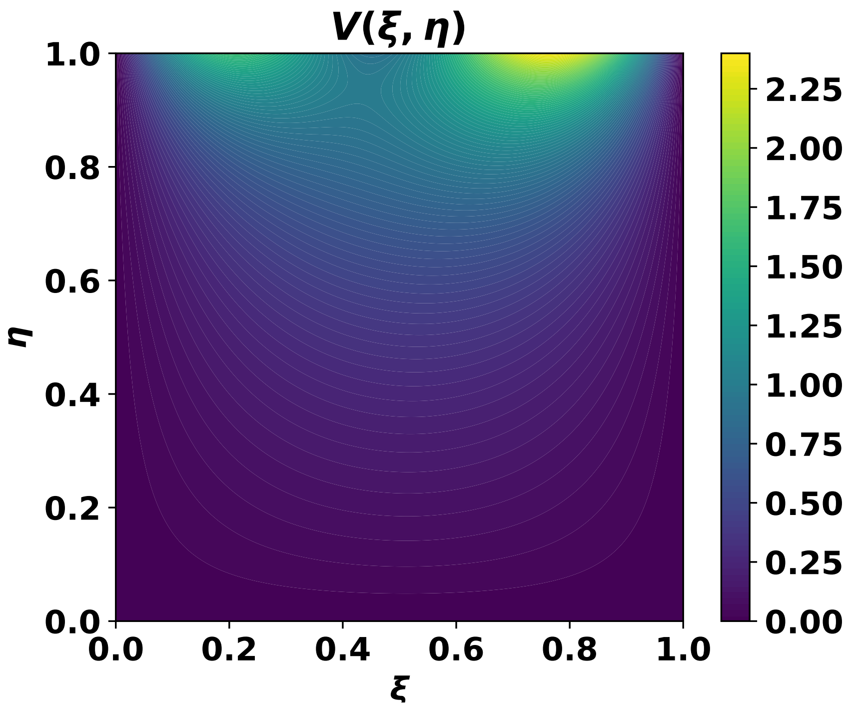


Figure 2: Contour plot of $V(\xi, \eta)$.

Comparison between V_0 and $V(\xi, \eta = 1)$ is given in fig. 1. This will be commented further later on. Using 4 Fourier components, the full solution for V is given in fig. 2. We use the fact that $\vec{E}(\xi, \eta) = -\nabla_{\xi,\eta} V(\xi, \eta)$ to obtain the solution of $\vec{E}$ (also reduced) in fig. 3.

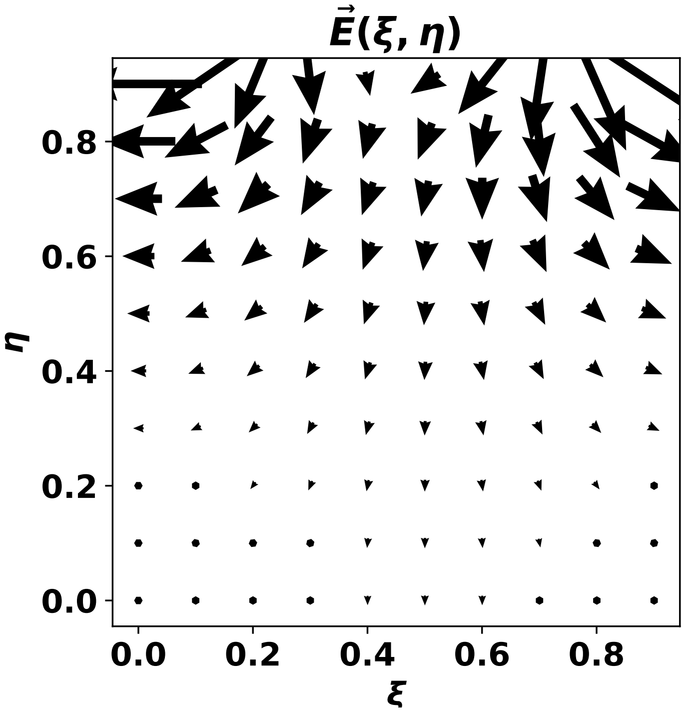
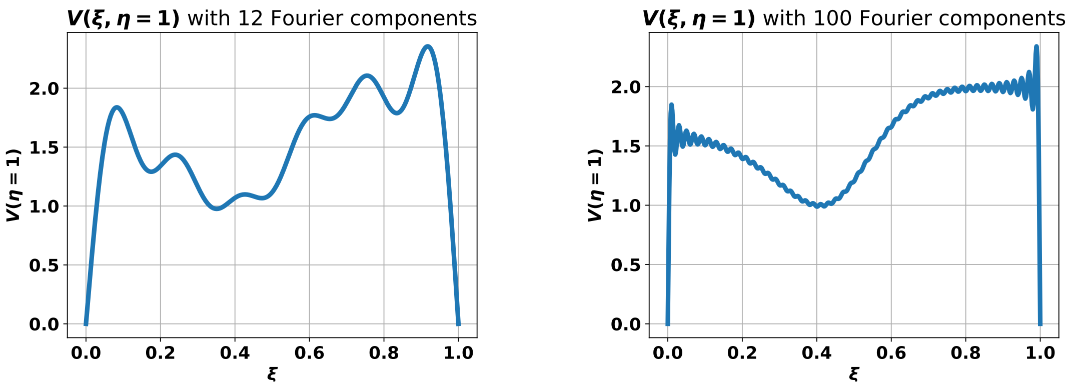


Figure 3: Quiver plot of $\vec{E}(\xi, \eta)$.

3 Discussion of errors

We have used the function $V_0(\xi) = 1 + tanh^2(1 - 6\xi^2)$. This does not vanish in the end points. The boundary conditions given by eq. (2) and eq. (3), *do* however require V_0 be 0 here. The function we reconstruct in fig. 1b is in fact discontinious, so any finite Fourier series experiences the Gibbs' phenomena close to the discontinuity. Therefore, we opt to only use terms up to $n = 4$. Using higher order terms yields better accuracy close to the boundaries, but also results in some wild behaviour slightly further away. This is shown in fig. 4. Ultimately, we determined that $n = 4$ provided a nice-looking potential, without being computationally intensive. With fig. 4 we also show that increasing n increases the accuracy, in case we want to consider a different V_0.



(a) n=12.

(b) n=100.

Figure 4: Plot of $V(\xi, \eta = 1)$ with differing amounts of Fourier components.