PROJECT: APPLIED STATISTICS 2020

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ABSTRACT

Two experiments were conducted to obtain two independent measurements of the gravitational acceleration g in the Copenhagen area, which has a standard value of $g = 9.80665$ m s^{-2}. The first experiment, a simple pendulum, yielded a result of $g = 9.876 \pm 0.003$ m s^{-2}, while the second experiment, a ball on an incline, yielded a result of $g = 9 \pm 1$ m s^{-2}. The former had higher precision, while the latter had higher accuracy. We show the formulas used as well as the error propagation using the *Law of Combination of Errors* [2]. We describe our methodology and fitting, and discuss our results and the impact of the different parameters and their associated errors.

INTRODUCTION

As part of this project, two experiments will be conducted in order to determine the value of the gravitational acceleration in the Copenhagen area, namely $g = 9.80665$ m s^{-2} [1]. The two experiments will be described in the following, along with the used formulas and error propagation, needed to statistically determine g through repeated experiments.

Pendulum

In the first experiment, we will measure the period of a pendulum with the length L and calculate g by using the pendulum equation:

$$T = 2\pi\sqrt{\frac{L}{g}} \tag{1.1}$$

This means that g can be determined as:

$$g = L\left(\frac{2\pi}{T}\right)^2 \tag{1.2}$$

We will thus measure both the length L and the period T of the pendulum, along with their respective uncertainties σ_L and σ_T. This means that we can propagate the errors using the *Law of Combination of Errors* [2], as we are measuring L and T independently:

$$\begin{aligned}\sigma_g^2 &= \left(\frac{\mathrm{d}g}{\mathrm{d}L}\right)^2\sigma_L^2 + \left(\frac{\mathrm{d}g}{\mathrm{d}T}\right)^2\sigma_T^2 \\ &= \left(\frac{4\pi^2}{T^2}\right)^2\sigma_L^2 - \left(\frac{8L\pi^2}{T^3}\right)^2\sigma_T^2\end{aligned}$$

Such that the standard deviation of g is:

$$\sigma_g = 4\pi^2\sqrt{\frac{1}{T^4}\sigma_L^2 - \frac{4L^2}{T^6}\sigma_T^2} \tag{1.3}$$

The setup is illustrated in figure 1.

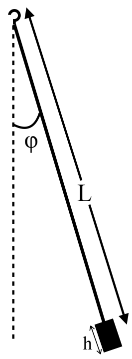

Figure 1: Diagram of pendulum experiment with L, the pendulum length, ϕ the displacement angle and h the height of the bob.

Ball on Incline

In the second experiment, we will consider a ball rolling down an inclined track. Through the laws of energy conservation and the assumption that the acceleration is constant, g can be found to be:

$$g = \frac{a}{\sin(\theta + \Delta\theta)}\left[1 + \frac{2}{5}\frac{R^2}{R^2 - \left(\frac{d}{2}\right)^2}\right] \tag{1.4}$$

where a is the measured acceleration of the center of mass (measured using the distance and time between the light gates), θ is the angle of the incline with respect to the table, $\Delta\theta$ is the angle of the table, R is the radius of the ball and d is the width of the track it is rolling on.
We can thus propagate the error on g as, again using the *Law of Combination of Errors* [2]:

$$\begin{aligned}\sigma_g^2 = &\left(\frac{\mathrm{d}g}{\mathrm{d}\theta}\right)^2\sigma_\theta^2 + \left(\frac{\mathrm{d}g}{\mathrm{d}\Delta\theta}\right)^2\sigma_{\Delta\theta}^2 + \left(\frac{\mathrm{d}g}{\mathrm{d}a}\right)^2\sigma_a^2 \\ &+ \left(\frac{\mathrm{d}g}{\mathrm{d}R}\right)^2\sigma_R^2 + \left(\frac{\mathrm{d}g}{\mathrm{d}d}\right)^2\sigma_d^2\end{aligned}$$

The calculations are somewhat cumbersome, so we will simply state the result on the variance:

$$\begin{aligned}\sigma_g^2 = &\left(1+\frac{2R^2}{5\left(R^2-\frac{d^2}{4}\right)}\right)^2 \frac{\sigma_a^2}{\sin^2(\theta+\Delta\theta)} \\ &+\left(1+\frac{2R^2}{5\left(R^2-\frac{d^2}{4}\right)}\right)^2 \frac{a^2\sigma_\theta^2\cos^2(\theta+\Delta\theta)}{\sin^4(\theta+\Delta\theta)} \\ &+\left(1+\frac{2R^2}{5\left(R^2-\frac{d^2}{4}\right)}\right)^2 \frac{a^2\sigma_{\Delta\theta}^2\cos^2(\theta+\Delta\theta)}{\sin^4(\theta+\Delta\theta)} \\ &+\frac{a^2R^4d^2\sigma_d^2}{25\sin^2(\theta+\Delta\theta)\left(R^2-\frac{d^2}{4}\right)^4} \\ &+\left(\frac{4R}{5\left(R^2-\frac{d^2}{4}\right)}-\frac{4R^3}{5\left(R^2-\frac{d^2}{4}\right)^2}\right)^2 \frac{a^2\sigma_R^2}{\sin^2(\theta+\Delta\theta)}\end{aligned} \tag{1.5}$$

The setup is illustrated in figure 2

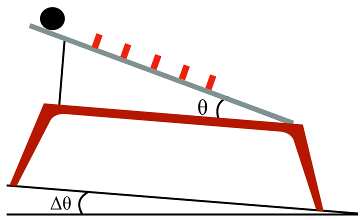

Figure 2: Diagram of the ball on incline experiment with θ the angle between the table and the incline, and $\Delta\theta$ the angle between the table and horizontal ground.

METHOD

All of the quantities used in the calculations in Equations (1.3) and (1.5) were measured in the lab as described in this section. The measurements were taken each by 3 experimenters, independently. The readings were taken blind to reduce bias.

Pendulum

1. *The pendulum period*: The period T of the pendulum was measured by timing its swing during simple linear motion. The pendulum was set into motion by raising the bob to one side or the other and then released. The pendulum equation (1.1) uses the small angle approximation which holds for roughly $\phi < |20°|$, where ϕ is the angle of displacement of the bob from equilibrium, so the bob was not released above this angle. The period timing was performed using the `stopwatch_py3.py` script, where an experimenter pressed a button each time they judged the bob to have passed a specific point at the end of a period. This was performed for 25 periods. The pendulum was not reset between measurements.

2. *The pendulum length*: The length L of the pendulum was taken to be the length of string from the bottom of the hook to the middle of the bob (we assume this is where the centre of mass is). A length measurement was taken of the distance from the bottom of the ceiling hook to the bottom of the bob, and then the height of the bob. From the first measurement was subtracted 1/2 the height of the bob to find the pendulum length.

3. *The length of the string*: The length of the string was measured using a plastic 2m folding rule, held flush with the ceiling and left to hang downwards. A reading was taken at bottom of the hook and at the bottom of the bob (not counting the hook on the bottom of the bob). The vertical thickness of the bob was measured using a manual vernier caliper with error 0.05mm.

Ball on an incline

To find the acceleration of a ball rolling down an incline, the timing of the ball passing fixed points on the incline was measured, along with several other parameters of the experimental setup.

1. *The separation of the time gates*: This was measured first using the folding white 2m rule, and secondly a rigid wooden 2m rule. The rule was rested on the ramp, with the 0mm mark flush with the top of the ramp. It was found with the folding rule that it rested on different parts of the ramp along the incline ie. it was difficult to rest it on the elevated part of the ramp (where the ball rolls) along the entire length of the rule. For this reason the measurements taken with the rigid 2m rule were used in analysis. Measurements were then read off, corresponding to the top (higher up) edge of each gate. This method is preferable to the alternative where the ruler is moved for each measurement, as this would correlate measurements.

2. *The radius of the ball, R*: Two balls of different diameters and the same material were used for the experiment; the diameter of the balls was measured using the manual vernier caliper.

3. *The width of the ramp, d*: The width was measured in between the upper two time gates using a manual vernier caliper.

4. *The angle between the incline and the table, θ*: A large analogue goniometer was used to measure the angle of the ramp. It was set upon the ramp, and the angle read off from where the needle fell on the scale. The angle was measured first in the configuration where the ramp sloped upwards from left to right, and the goniometer faced the back of the lab room. Then the goniometer was turned around and the angle measured again. Then the ramp was rotated 180°and the angle was measured with the goniometer facing the back of the room, and when the goniometer faced the front of the room.

5. *The angle between the table and horizontal ground, $\Delta\theta$*: As the table holding the apparatus may not be perpendicular to the direction of g, we wish to correct for this offset angle $\Delta\theta$. We include two methods of finding this angle:

$$2\Delta\theta = \theta_L - \theta_R \tag{1.6}$$

and:

$$\Delta\theta = \frac{(a_L - a_R)\cos\theta}{(a_L + a_R)\sin\theta} \tag{1.7}$$

In equation 1.6 we use the averaged θ measurements when the incline is oriented to the left and the right. In equation 1.7 we use the calculated acceleration of the ball when the incline is oriented to the left and the right.

6. *Ball descent timings*: The timing was measured by releasing the ball at the top of the track, below the trigger (as to ensure the ball rolls). The Waveforms software records the timing of the ball crossing each gate as a continuous signal. The timing of the decent of each ball was recorded with the ramp inclined to the left, and then also with the ramp rotated 180°.

RESULTS

Pendulum

In order to use our measurements from this experiment to determine g, we need to convert our time series from the `stopwatch_py3.py` script to periods for the pendulum with some associated errors. Assuming that the period of the pendulum is constant over the measured interval, we gain this from doing a linear fit on the time series, which can be found in link [3]. As it can be seen on figure 3, we observe that the residuals from this linear fit are normally distributed around 0, and hence we use the residuals as the statistical error on the measured time series, when we conduct the fit to find the period. This is illustrated in figure 4, whereas all results and their associated errors can be found in tables 2-7 in the appendix.

This gives us 3 different periods with associated, independent errors. We perform a χ^2-test on them to ensure these measurements agree with each other, and given the result $\chi^2 = 0.492$ and $p = 0.782$ we conclude that they do, which allows us to do a weighted mean and gain $T = 2.7677 \pm 0.0002$ s. In order to gain the length of the pendulum, we combine our measurements of the string and the bob as described in the method section. For the three independent measurements of the pendulum bob height, we get $\chi^2 = 2.960$ and $p = 0.228$, which again suggests that the datapoints agree on the weighted average $H = 15.28 \pm 0.02$ mm. However, for the string length we get $\chi^2 = 10.09$ and $p = 0.005$ and hence no agreement between our three measurements. This suggests that we have underestimated the uncertainties on these measurements, which makes us shift away from the weighted mean and use a non-weighted average algorithm to gain $L = 190.87 \pm 0.08$ cm with $\chi^2 = 5.36$ and $p = 0.069$. Even though this is significant within a 5% significance level, it is still a high χ^2-value and a low probability. The reason might be due to how we measured the length, which as described in the method section was quite cumbersome. From equations (1.2) and (1.3), we find $g = 9.876 \pm 0.003$ m s^{-2}.

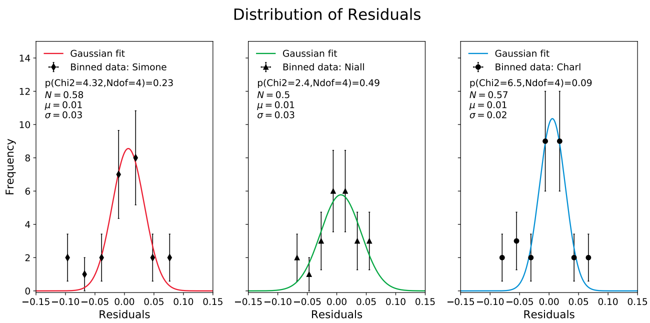

Figure 3: Distribution of time residuals of the pendulum measurements.

Ball on an incline

In order to use our measurements from this experiment to determine g, we need to detect the peaks in the time series from the light gates on the incline connected to `Waveform` to gain the time intervals between these gates. Unfortunately, we didn't save our time series properly and as a consequence our files only contained noise from the electrical setup. Another group, who did measurements on the same setup, was kind enough to send us their measurements, so we have used those in the following. We locate the peaks using the `find_peaks` function from the `scipy.signal` library. This function needs the voltage series as well as two positional arguments to return the indices of the peaks. These two additional arguments are needed, because the function operates on simple comparison of neighbouring values, and the voltage series includes electrical noise, which induces a lot of small peaks in the series. We are not interested in these peaks, and thus we set a minimum height of the peaks `height` = 4 V. Further, the peaks will have finite widths and 'flat' tops due to the high resolution of the equipment. Hence, we would also expect the electrical noise to produce small peaks on top of the peaks caused by the ball rolling past the light gates. We don't want these small peaks to be counted as well, and thus we set a minimum distance between the peaks of interest `distance` = 100 indices. The relevant script can be found in link [4]. All other measurements needed to determine g can be found in table 5-13 in the appendix, except for $\Delta\theta$ which we will describe further in this section. The measurements, errors, the fit and the residuals of the fit are illustrated in figure 5. From equation (1.6), we get $\Delta\theta = 0.06 \pm 0.04^o$ and from equation (1.7) we get $\Delta\theta = 0.013 \pm 0.004^o$. When we do a χ^2-test on these, we find $\chi^2 > 200$ and $p \sim 0$ meaning that these two values don't agree, and hence we deduce that we have underestimated the errors on these values. As a consequence, we increase the errors on each of the measurements by $\pm 0.03^o$ to gain a weighted average $\Delta\theta = 0.05 \pm 0.04$ $\chi^2 = 1.45$ and $p = 0.23$. Using Eq. (1.4) and Eq. (1.5), we get $g = 9 \pm 1$ m s^{-2} for both balls.

DISCUSSION

The objective of this project was to obtain two independent measurements of the gravitational acceleration g, and then compare it to an official value of high accuracy and precision, which we took to be the standard gravity value, $g = 9.80665$ m s^{-2}. For the first experiment, g was calculated using a simple pendulum setup, and a result of $g = 9.876 \pm 0.003$ m s^{-2} was obtained. For the second experiment, a ball on an incline, the result was $g = 9 \pm 1$ m s^{-2}. The pendulum experiment was the closest to the standard value, with the obtained value having high precision, but is not very accurate, since the standard value of g falls quite far outside the calculated error. For the ball-on-an-incline experiment, however, the precision was very low, but the standard value for g falls within the obtained uncertainty, making it more accurate than the pendulum experiment.

Table 1 shows the values obtained for the measured variables contributing to the uncertainties. The pendulum experiment had only two variables, with the uncertainty on the length L the highest. For the ball on an incline, there were 5 measured variables, making it a lot more difficult to get a precise estimate of g, since it leads to an error propagation formula with 5 terms (see equations 1.3 and 1.5). The parameters with the largest impact on error propagation are, for the pendulum experiment, the length L, and for the ball on an incline, the incline angle θ, and the angle of the table, $\Delta\theta$. Two different methods were used to calculate $\Delta\theta$, but the results did not agree, and the errors on these results were therefore increased to give a lower χ^2 value, and a higher probability. This resulted in the large error seen in table 1, and the large impact it has on the final error of calculated g value. For the pendulum, the estimate of L can be improved by taking more independent measurements, since only three measurements were used in the calculations. For the ball on incline, more precise equipment can be used to measure the angles, and this can be combined with more independent measurements.

Variable	Value	Error	Impact [m^2 s^{-4}]
Pendulum			
Period T	2.7677 s	$\pm$0.0002 s	3.7×10^{-9}
Length L	190.87 cm	$\pm$0.04 cm	1.0×10^{-8}
Calculated g	9.876 m s^{-2}	$\pm$0.003 m s^{-2}	
Ball on incline (large ball, R)			
Acceleration a	1.42 m s^{-2}	$\pm$0.01 m s^{-2}	6.0×10^{-4}
Inclination angle θ	14.32°	$\pm$0.03°	3.4×10^{-1}
Table angle $\Delta\theta$	0.05°	$\pm$0.04°	4.8×10^{-1}
Rail diameter d	3.87 mm	$\pm$0.06 mm	1.5×10^{-4}
Ball diameter D	13.00 mm	$\pm$0.03 mm	5.1×10^{-4}
Calculated g	9 m s^{-2}	$\pm$1 m s^{-2}	

Table 1: Measured variables for the two experiments, together with the impact it has on the uncertainty of g. The values in the Impact column are the terms of the error propagation formulae (equations 1.3 and 1.5) that each variable is associated with. The values shown for the ball-on-incline experiment are for the large ball, with the incline to the right.

CONCLUSION

We have conducted two experiments to obtain two independent measurements of the gravitational acceleration g. For the first experiment, a simple pendulum, a result of $g = 9.876 \pm 0.003$ m s^{-2} was obtained. This value is the closest to the standard value for g, which is $g = 9.80665$ m s^{-2}, but given the the high precision of the result, it is not very accurate. The second experiment was a ball on an incline, with the obtained value being $g = 9 \pm 1$ m s^{-2}. This value had low precision, a result of 5 different variables that were measured, and then combined in the error propagation equations. The accuracy for this experiment was therefore better, even though it was further away from the standard value.

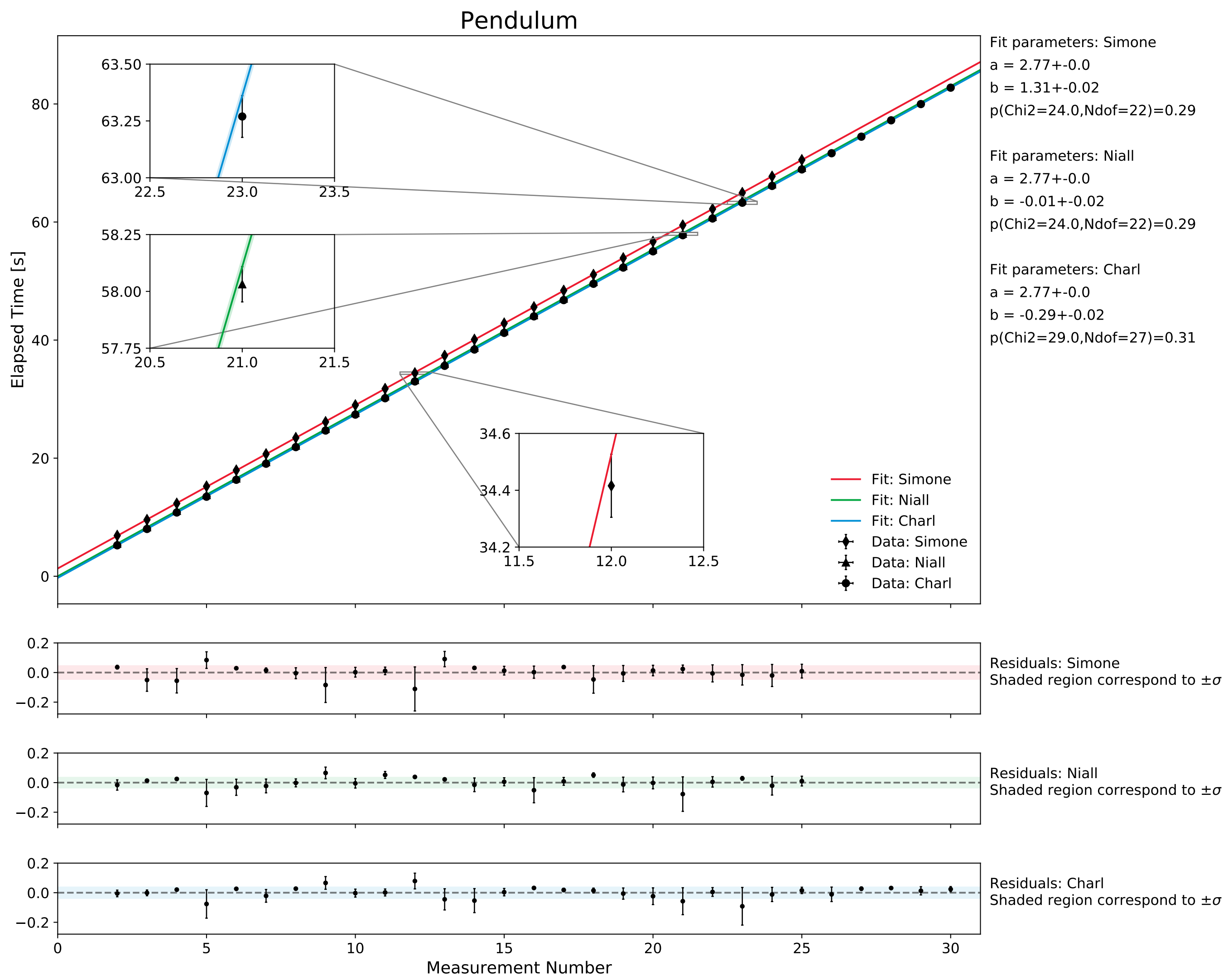

Figure 4: Results from the fitting of the pendulum measurements are shown in the main figure, including residuals in the lower three figures. The errors on the measurements are the residuals from the resulting fit. The fits, shown as solid lines, are straight lines on the form $y = ax + b$. The shaded area surrounding the solid lines are the errors on the fit. between each The fit parameters as well as χ^2, number of degrees of freedom and the resulting p-value between the fit and measurements are listed on the right. The insert figures show the measurements with the largest residuals for each respective experiment.

REFERENCES

[1] Standard gravity as defined on Wikipedia: `https://en.wikipedia.org/wiki/Standard_gravity`, (2020)

[2] R. J. Barlow, *Statistics: A Guide to the Use of Statistical Methods in the Physical Sciences*, (John Wiley And Sons Ltd, 1989)

[3] Our script for determining the residuals: `https://github.com/svejlgaard/AppStat2020Project/blob/main/Scripts/Residuals.py`, (2020)

[4] Our script for determining the time intervals between the light gates: `https://github.com/svejlgaard/AppStat2020Project/blob/main/Scripts/Ball.py`, (2020)

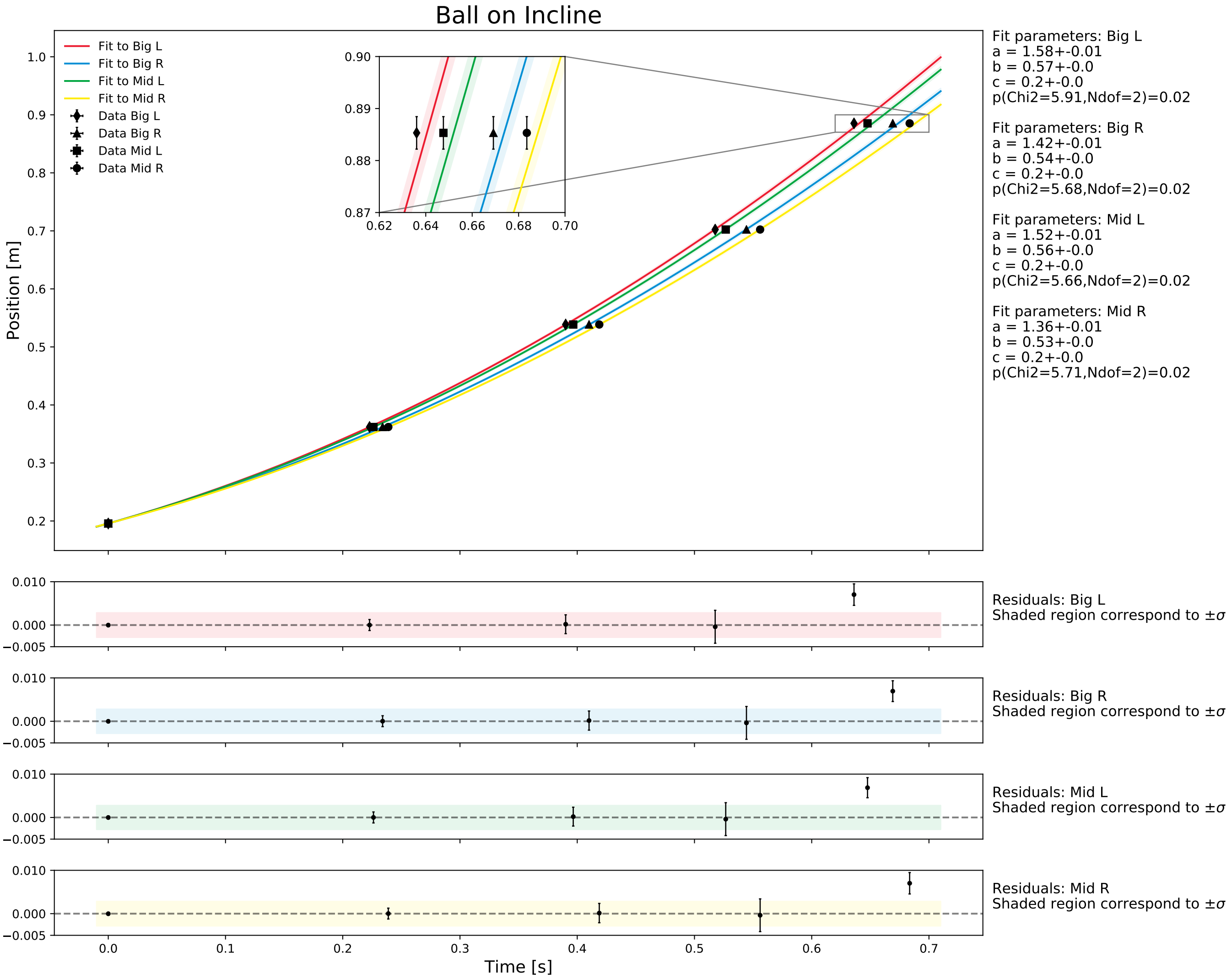

Figure 5: Results from the fitting of the pendulum measurements are shown in the main figure, including residuals in the lower four figures. The fit, shown as solid lines, are second order polynomials on the form $y = ax^2 + bx + c$. The shaded area surrounding the solid lines are the errors on the fit. The fit parameters as well as χ^2, number of degrees of freedom and the resulting p-value between the fit and measurements are listed on the right. The insert figure shows the measurements that fall the farthest away from the fit.

APPENDIX

All of the data from the pendulum and ball on incline experiments can be found in table 2-4 and 5-13, respectively.

Pendulum

Height of the bob [mm]		
	Value	Error
Simone	15.28	± 0.05
Niall	15.32	± 0.05
Charl	15.25	± 0.05
Mean:	**15.28**	**± 0.02**

Table 2: The height of the pendulum bob as measured with a caliper by Simone, Niall, and Charl. The error on each independent measurement is the stated error on the given caliper. The assumption of independent measurements is also included in the weighted mean and its error.

Length of the string [cm]		
	Value	Error
Simone	190.98	± 0.05
Niall	190.84	± 0.05
Charl	190.80	± 0.05
Mean:	**190.87**	**± 0.08**

Table 3: The length of the pendulum string as measured with a folding ruler by Simone, Niall, and Charl. The error on each independent measurement is half of the smallest measure on the folding ruler. The assumption of independent measurements is also included in the weighted mean and its error.

Period [s]		
	Value	Error
Simone	2.7681	± 0.0013
Niall	2.7673	± 0.0010
Charl	2.7676	± 0.0008
Mean:	**2.7677**	**± 0.0002**

Table 4: The period of the pendulum as measured by Simone, Niall, and Charl with the `stopwatch_py3.py` script. The error on each independent measurement is the residuals from fitting to a straight line. The assumption of independent measurements is also included in the weighted mean and its error.

Ball on an incline

Diameter of the small ball [mm]		
	Value	Error
Simone	10.00	± 0.05
Niall	10.01	± 0.05
Charl	9.95	± 0.05
Mean:	**9.99**	**± 0.02**

Table 5: The diameter of the small ball as measured with a caliper by Simone, Niall, and Charl. The error on each independent measurement is the stated error on the given caliper. The assumption of independent measurements is also included in the weighted mean and its error.

Diameter of the large ball [mm]		
	Value	Error
Simone	13.05	± 0.05
Niall	13.01	± 0.05
Charl	12.95	± 0.05
Mean:	**13.00**	**± 0.03**

Table 6: The diameter of the large ball as measured with a caliper by Simone, Niall, and Charl. The error on each independent measurement is the stated error on the given caliper. The assumption of independent measurements is also included in the weighted mean and its error.

Diameter of the rail [mm]		
	Value	Error
Simone	3.75	± 0.05
Niall	3.89	± 0.05
Charl	3.98	± 0.05
Mean:	**3.87**	**± 0.06**

Table 7: The diameter of the large ball as measured with a caliper by Simone, Niall, and Charl. The error on each independent measurement is the stated error on the given caliper. The assumption of independent measurements is also included in the weighted mean and its error.

Inclination angle, R, F [^{o}]		
	Value	Error
Simone	14.0	± 0.5
Niall	13.9	± 0.5
Charl	13.9	± 0.5
Mean:	**13.93**	**± 0.03**

Table 8: The inclination angle as measured by Simone, Niall, and Charl when the setup is oriented right with a goniometer facing the front. The error on each independent measurement is half of the smallest measure on the goniometer. The assumption of independent measurements is also included in the weighted mean and its error.

Inclination angle, R, B [^{o}]		
	Value	Error
Simone	14.8	± 0.5
Niall	14.6	± 0.5
Charl	14.7	± 0.5
Mean:	**14.70**	**± 0.05**

Table 9: The inclination angle as measured by Simone, Niall, and Charl when the setup is oriented right with a goniometer facing the back. The error on each independent measurement is half of the smallest measure on the goniometer. The assumption of independent measurements is also included in the weighted mean and its error.

Inclination angle, L, F [^{o}]		
	Value	Error
Simone	13.1	± 0.5
Niall	13.2	± 0.5
Charl	13.2	± 0.5
Mean:	**13.17**	**± 0.03**

Table 10: The inclination angle as measured by Simone, Niall, and Charl when the setup is oriented left with a goniometer facing the front. The error on each independent measurement is half of the smallest measure on the goniometer. The assumption of independent measurements is also included in the weighted mean and its error.

Inclination angle, L, B [^{o}]		
	Value	Error
Simone	15.3	± 0.5
Niall	15.2	± 0.5
Charl	15.1	± 0.5
Mean:	**15.20**	**± 0.05**

Table 11: The inclination angle as measured by Simone, Niall, and Charl when the setup is oriented left with a goniometer facing the back. The error on each independent measurement is half of the smallest measure on the goniometer. The assumption of independent measurements is also included in the weighted mean and its error.

Gate positions [cm]		
	Value	Error
Gate 1		
Simone	19.62	± 0.05
Niall	19.58	± 0.05
Charl	19.56	± 0.05
Mean:	**19.587**	**± 0.017**
Gate 2		
Simone	36.62	± 0.05
Niall	36.17	± 0.05
Charl	36.15	± 0.05
Mean:	**36.177**	**± 0.018**
Gate 3		
Simone	53.91	± 0.05
Niall	53.85	± 0.05
Charl	53.85	± 0.05
Mean:	**53.87**	**± 0.02**
Gate 4		
Simone	70.25	± 0.05
Niall	70.35	± 0.05
Charl	70.30	± 0.05
Mean:	**70.3**	**± 0.03**
Gate 5		
Simone	88.97	± 0.05
Niall	88.15	± 0.05
Charl	89.89	± 0.05
Mean:	**89.0**	**± 0.5**

Table 12: The light gate positions as measured with a folding ruler by Simone, Niall, and Charl. The error on each independent measurement is half of the smallest measure on the folding ruler. The assumption of independent measurements is also included in the weighted mean and its error.

Gate positions [cm]		
	Value	Error
	Gate 1	
Simone	19.65	± 0.05
Niall	19.61	± 0.05
Charl	19.42	± 0.05
Mean:	**19.56**	**± 0.07**
	Gate 2	
Simone	36.31	± 0.05
Niall	36.22	± 0.05
Charl	36.12	± 0.05
Mean:	**36.22**	**± 0.05**
	Gate 3	
Simone	54.01	± 0.05
Niall	53.85	± 0.05
Charl	53.68	± 0.05
Mean:	**53.85**	**± 0.09**
	Gate 4	
Simone	70.03	± 0.05
Niall	70.30	± 0.05
Charl	70.15	± 0.05
Mean:	**70.16**	**± 0.08**
	Gate 5	
Simone	88.23	± 0.05
Niall	88.10	± 0.05
Charl	87.85	± 0.05
Mean:	**88.1**	**± 0.1**

Table 13: The light gate positions as measured with a ruler by Simone, Niall, and Charl. The error on each independent measurement is half of the smallest measure on the folding ruler. The assumption of independent measurements is also included in the weighted mean and its error.