Statistical analysis of two independent experiments for determining g

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In this article we present two different experimental approaches for determining the earth's gravitational acceleration g: a pendulum and a ball on an incline. Using error propagation and statistical analysis of the experimental data, we determined the gravitational acceleration and their respective uncertainties. For the pendulum we obtained $g = (9.807 \pm 0.002)$ m/s^2. With the ball on an incline, the measured gravity is $g = (8.91 \pm 0.19)$ m/s^2. Here, the determination of the acceleration on the incline a has the highest impact on the uncertainty, followed by the measurement of the rail angle θ. Compared to the gravity measured at the University of Copenhagen using a free-fall gravimeter of $g = 9.815$ m/s^2 [1], both the pendulum measurement deviates and the ball incline measurement differ by 5σ. This may be due systematic effects, such as large deflection angles for the pendulum experiment or friction effects for the ball on incline experiment.

INTRODUCTION

Obtaining the same results for a physical constant using two independent and different experiments is a good way of cross-checking and validating experimental data. Here, we use two different measurements to obtain the gravitational acceleration g. Due to their simplicity, the experiments used for determining g are a pendulum and a ball on an incline. Analysing the respective results of two experiments thoroughly with common statistical techniques, good agreement between experimental data and the well-known earth's acceleration g should be obtained.

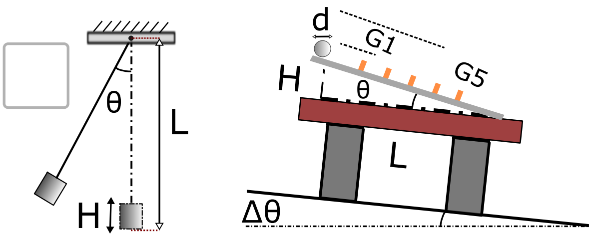

Figure 1: Experimental setups with respective indication what variables refer to for pendulum and ball on incline. For the ball on incline, the red-brown color represents the table top on which the incline stands. The gates are drawn in orange.

We use a pendulum in its simplest form. It consists of a nail to fix the string of the pendulum, a simple string and a cylindrical formed weight (Figure 1). The gravitational constant g can then be determined solely based on measuring only two things: the period time T of a pendulum swing and the length L of the pendulum from its suspension to its center of mass. The relation between g, T and L in the small angle approximation is given as

$$g = L\left(\frac{2\pi}{T}\right)^2 . \tag{1.1}$$

Measuring the period time and length of the pendulum multiple times in form of independent measurements allows to improve the result compared to a single measurement.

In a similar fashion, a ball on incline experiment is performed multiple times. Using gates that time the passage time of the ball rolling down the incline, the acceleration of the ball can be determined (Figure 1). The gravitational constant g follows then from the angle of the incline θ that a ball is rolling down on, the ball's acceleration a and diameter d_{ball}, and the diameter of the rail d_{rail}

$$g = \frac{a}{\sin(\theta + \Delta\theta)}\left(1 + \frac{2}{5}\cdot\frac{d_{\text{ball}}^2}{d_{\text{ball}}^2 - d_{\text{rail}}^2}\right) . \tag{1.2}$$

This measurement is repeated multiple times using different balls and orientations of the experimental setup. The angle θ is determined using trigonometry and measurements with a goniometer. For each orientation of the setup, the goniometer is used to measure the angle in both directions, allowing to determine $\Delta\theta$ and counteracting systematic errors on the device (Figure 1).

As a further cross-check, we compare our values for g with the officially measured value for $g_{\text{off.}} = 9.81546301(1)$ m/s^2 from [1].

METHOD

Pendulum

To get a higher precision on the period time of the pendulum, not a single pendulum swing is measured at a time, but 30 - 50 swings in a row are timed with lap times after each full swing. To further improve the measurements, three people are taking at the same time the period times. Each person starts the pendulum once. Therefore, the data consists of 9 data sets, plus 2 additional independent ones. In between the period time measurements, the length of the pendulum is measured by all three persons with respective estimates on the precision. Here, the length is the length from top to bottom of the pendulum. At the end of the experiment, each person determined the height of the weight once. The actual pendulum length is then given by

$$L = L_{\text{String}} - \frac{1}{2} \cdot H_{\text{Weight}} \, . \tag{1.3}$$

Since each measurement is done multiple times with respective uncertainties, the weighted means and errors are used to determine the actual values for L_{String} and H_{Weight}. To determine the uncertainty on the pendulum length, we have to propagate errors as

$$\sigma_L = \sqrt{\sigma^2_{L_{\text{String}}} + \frac{\sigma^2_{H_{\text{Weight}}}}{4}} \, . \tag{1.4}$$

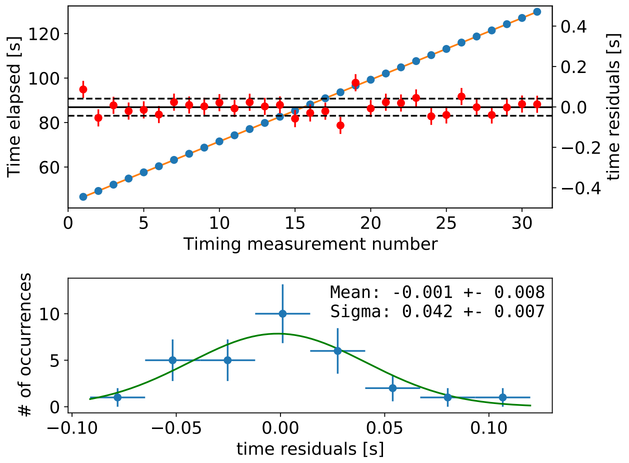

Figure 2: Exemplary data of number of periods versus the cumulative time including uncertainties, with respective initial linear fit. In addition, the resulting residuals are plotted with respective errors for each point. In the lower part, the residuals are plotted as histogram and fitted with a Gaussian function centered around zero using a binned likelihood fit.

For the period time of the pendulum, things are a bit more involved. Each data set contains pairs for the number of pendulum swings and respective cumulative time (Figure 2). This can be used to determine an averaged period time by fitting a linear function $t(N) = a \cdot N + c$ to this data. For fitting we use the fitting routine provided in the python package *iminuit*. The period time is then given as the change in value for an unit step. To estimate the initial uncertainty, residuals from the initial fit to the data points are determined. These should be normally distributed, therefore a Gaussian function is fitted towards the binned residuals using a binned likelihood fit. From the resulting Gaussian, we get the timing accuracy for each time in the data set through its half width. In Figure 2, this is shown exemplary using one of the data sets.

This timing accuracy is then used as uncertainty on each data point and the fit is repeated using this addition. By construction, the fit should fit the timing points nicely. Therefore, the well-ness of the fit is guaranteed and the Chi-squared probability should be by construction $P\left(\chi^2\right) \approx 0.5$. In addition, with the improved fit the uncertainty on the period time T is then simply given by the uncertainty on the fit constant. Repeating this for all data sets, the obtained mean period times from the fits with their respective uncertainties are used to calculate the weighted mean and error of the period time.

Having pendulum length and period time and associated uncertainties, the gravitational constant g can be deterumine using equation (1.1). Similar to before, we have to propagate uncertainties to get the true error on the gravitational constant:

$$\sigma_g = \sqrt{\left(\frac{2\pi}{T}\right)^4 \sigma_L^2 + 4L^2 \frac{(2\pi)^4}{T^6} \sigma_T^2} \tag{1.5}$$

Ball on an incline

For the ball on incline experiment, the height H and length L (Figure 1) are determined with a 90 degree angle between them (Table IX). Only then we can determine the angle θ using trigonometry. The gate positions are then measured with respect to a common starting point to reduce the uncertainty on the value (Figure 1). Each person measured them independently, the weighted averages are shown along with the original values in table XIV. The same applies for the measured ball diameters and rail diameter (Table X).

In addition, we measure the additional angle $\Delta\theta$ that arises from a slight tilting of the table the incline is positioned on. Every person measures the angle of a goniometer standing on the incline from both sides and both incline directions. Then, the difference between the angle of the two incline directions A and B on the table is $\theta_{\text{A}} - \theta_{\text{B}} = 2\Delta\theta$. Out of this, we extract the tilting angle $\Delta\theta$.

In the acceleration measurement, every person rolls each ball twice down the incline in both table directions, leading to 24 data sets in total. Each run, a software records the times at which the ball passes by the LED photo detectors of the gates. Assuming a constant acceleration, the movement of the ball through all gates will follow the function

$$s(t) = \frac{1}{2} a t^2 + v_0 t + s_0 \, , \tag{1.6}$$

where a is the acceleration, v_0 the start velocity, and s_0 the start position. By fitting this function on the photo detector recordings, and by combining this with the previously mentioned measurements, we can determine g.

Similar to the pendulum, the different parameters going into the expression of g are each associated with their respective uncertainties. To find the proper uncertainty, the errors are propagated using the standard error propagation formula:

$$\sigma_g = \left(\left(\frac{\delta g}{\delta a}\right)^2 \sigma_a^2 + \left(\frac{\delta g}{\delta d_{\text{ball}}}\right)^2 \sigma_{d_{\text{ball}}}^2 + \left(\frac{\delta g}{\delta d_{\text{rail}}}\right)^2 \sigma_{d_{\text{rail}}}^2 + \left(\frac{\delta g}{\delta\left(\theta+\Delta\theta\right)}\right)^2 \sigma_{\theta+\Delta\theta}^2 \right)^{1/2} \tag{1.7}$$

Where the individual contributions in equation (1.7) are given by:

$$\begin{aligned}
\left(\frac{\delta g}{\delta a}\right) &= \frac{1}{\sin\theta \pm \Delta\theta}\left(1+\frac{2}{5}\frac{d_{\text{ball}}^2}{d_{\text{ball}}^2 - d_{\text{rail}}^2}\right)\\
\left(\frac{\delta g}{\delta d_{\text{ball}}}\right) &= \frac{a}{\sin\theta \pm \Delta\theta}\left(\frac{-4d_{\text{rail}}^2 d_{\text{ball}}}{5\left(d_{\text{ball}}^2 - d_{\text{rail}}^2\right)^2}\right)\\
\left(\frac{\delta g}{\delta d_{\text{rail}}}\right) &= \frac{a}{\sin\theta \pm \Delta\theta}\left(\frac{4d_{\text{ball}}^2 d_{\text{rail}}}{5\left(d_{\text{ball}}^2 - d_{\text{rail}}^2\right)^2}\right)\\
\left(\frac{\delta g}{\delta\left(\theta+\Delta\theta\right)}\right) &= \frac{-a\cos\theta \pm \Delta\theta}{\sin^2\theta \pm \Delta\theta}\left(1+\frac{2}{5}\frac{d_{\text{ball}}^2}{d_{\text{ball}}^2 - d_{\text{rail}}^2}\right)\\
\sigma_{\theta+\Delta\theta} &= \sqrt{\theta^2\sigma_{\Delta\theta}^2 + \left(\Delta\theta\right)^2\sigma_\theta^2}
\end{aligned}$$

RESULTS

Pendulum

First, the actual length of the pendulum consisting of the length of the strength and half the height of the weight attached to it is determined based on the data presented in Table VIII. Using weighted means and propagating errors, the total length of the pendulum is determined to be 191.19 ± 0.02 cm.

As discussed in the methods, the residuals of an initial fit are used to determine the uncertainty on the timings by fitting a Gaussian to the binned residuals. This is shown in Figure 2 for one exemplary data set with $\mu = -0.001 \pm 0.008$ and $\sigma = 0.034 \pm 0.006$. The results for the other eleven data sets are presented in table VII. Using this constructed uncertainty on the timings for a second linear fit allows us to directly retrieve the uncertainty on the period time from the fitting uncertainty. This leads for the above data set to $T = 2.7763 \pm 0.0009$ s (table V. By construction, the chi square probability should then be 0.5. In table V we see that this is close for some of the data sets, but not for all. When comparing the residual histograms with respective Gaussian fits, then this coincides with cases where the fitting is not easy and the width and position of the Gaussian is overestimated. Because of this, the chi square probability is too big. Even with 30 to 50 data points, the heights of the bins in the histogram are sensitive to random fluctuations, compromising proper estimates of the Gaussian and its width.

Using these results, the combined period time of all eleven data sets amounts to $T = 2.7743 \pm 0.0002$ s. These data sets are partly independent, and partly dependent, since each person measured multiple data sets. The respective chi squared probability is hence 0, as the weighted average of the timings is directly connected to the individual values for the period time.

Finally, the gravitational constant can be determined using the combined values for the period time and pendulum length as stated in table I according to equation (1.1). For the error, we use the error propagation formula as stated in equation (1.5). Using this, g amounts to $g = 9.807 \pm 0.002\ \ \text{m/s}^2$.

Ball on Incline

When rolling down a ball on the incline with the light gates, we get a photo voltage time trace as shown in Figure 4. We assume that the signal is a square peak when the ball passes the detectors. Each peak is determined as subset of data above a reference voltage (Figure 4 red line). We chose a voltage of 3.4 V as level, to ensure to include the complete top of each peak. The passage time is then given as the mean of each subset.

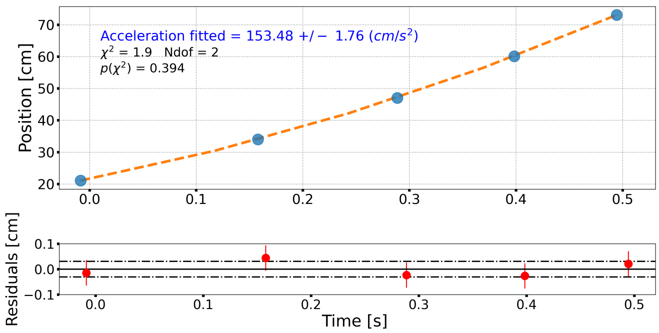

Figure 3: Exemplary data of detector voltage versus the gate passing time including uncertainties. The data is fitted with respective parabola function. At the bottom graph, the resulting residuals are also plotted with uncertainty for each point.

To determine the standard deviation of each square peak, we record the first (b) and last (c) time data point of each time subset. Then, the standard deviation of this square peak is then given as

$$\sigma_t = \frac{c-b}{2\sqrt{3}}\ . \tag{1.8}$$

Plotting these time values together with their measured gate positions results in scatter plots as shown in fig. 3 and fitted with a model based on equation (1.6).

To reduce the uncertainty on the acceleration, we fit each timing data set with the two measured gate

	Value	Statistical error	Systematic error	Impact on σ_g through $(\mathrm{d}g/\mathrm{d}i)^2\sigma_i^2$
Period time T	2.7743 s	0.0002 s	-2.0 ± 1.0 ms	2.00×10^{-6} m^2/s^4
Pendulum length L	1.9119 m	0.0002 m	unknown	1.05×10^{-6} m^2/s^4
Gravitational constant g	9.807 m/s^2	0.002 m/s^2	$+0.014 \pm 0.007$ m/s^2	
	Value	Statistical error	Systematic error	Impact on σ_g through $(\mathrm{d}g/\mathrm{d}i)^2\sigma_i^2$
Acceleration a_A for d_small	1.50 m/s^2	0.08 m/s^2	unknown	2×10^{-1} m^2/s^4
Acceleration a_A for d_big	1.53 m/s^2	0.06 m/s^2	unknown	1×10^{-1} m^2/s^4
Acceleration a_B for d_small	1.51 m/s^2	0.08 m/s^2	unknown	2×10^{-1} m^2/s^4
Acceleration a_B for d_big	1.52 m/s^2	0.02 m/s^2	unknown	2×10^{-2} m^2/s^4
Rail angle θ	14.58 deg	0.04 deg	unknown	6×10^{-4} m^2/s^4
Table tilting angle $\Delta\theta$	0.31 deg	0.03 deg	unknown	3×10^{-4} m^2/s^4
Diameter small ball d_small	12.03 mm	0.03 mm	unknown	2×10^{-5} m^2/s^4
Diameter big ball d_big	19.47 mm	0.03 mm	unknown	1×10^{-6} m^2/s^4
Rail diameter d_rail	5.80 mm	0.03 mm	unknown	1×10^{-4} m^2/s^4
Gravitational constant g	8.9 m/s^2	0.2 m/s^2		

Table I: Main results for the pendulum (top) and the ball on incline analysis (bottom). Errors on the acceleration are based on each subset's RMS value.

positions separately (Tables XII and XIII) leading to 12 acceleration values for each orientation and ball size of the experiment. These are then combined to the four weighted accelerations: 2 for each ball size, and 2 for each direction of the experiment as stated in Table I. Using these accelerations, along with weighted averages of the remaining parameters, gives us $g = 8.9 \pm 0.2$ m/s^2

DISCUSSION

To quantify how well our obtained sub- and main results are, we performed cross-checks using the chi square probability on individually calculated values and also with their weighted averages. In general, the combined values for the length L, height H, period time T and the gravitational constant g have a vanishing $P\left(\chi^2\right)$ when being compared with the individual data sets as shown in the Tables III and VII. This is since the data points and the combined values are not independent. A good way of cross-checking is also comparing the g values for individual data sets and doing a weighted average of these instead of averaging the period times. Comparing the values in Table VI and I we see, that both values agree within their uncertainties. However, the error of the latter is smaller.

For the pendulum, the formula stated in equation (1.1) holds only in the small angle approximation where $\theta \approx \sin\theta$. When performing the experiment, we did not pay attention to that, but rather used an angle of 7.5 ± 2.5 °. Formally, the small angle approximation does not hold here. Using [2], the first order correction to the period time is then $(0.2 \pm 0.2) \cdot 10^{-2}$ s. This means, that we are systematically overestimating the period time due to that. Using this, we expect to systematically underestimate our value for g, such that we estimate to have to account for a systematic error of $+0.014 \pm 0.007$ m/s^2. With this rough estimate for the systematic error on g, it is possible to explain the deviation from the official value as stated in [1]. However, we only achieve millimeter precision on our statistical errors.

In the case of the ball on incline we clearly see that both, accuracy and precision are significantly worse for the value of the gravitational acceleration g when comparing it to the official value as stated in [1]. The accuracy might be limited by systematic effects such as friction, slowing the ball down compared to the true value we would expect for rolling down the incline. However, we cannot provide an estimate of the systematic error due to this effect. The precision in our value for g is limited due to the various individual uncertainties going into it, but in particular the values and uncertainties in the acceleration a seem to have significant effect on the value of g and its uncertainty. This can be quantified through the impact factor as stated in Table I). This factor is two to three orders larger than any other contribution (Table I). Therefore, we seem to be mainly limited by the obtained acceleration values and uncertainties, leading to the discrepancy of more than five σ_g between theoretical and experimental value, followed by the measurement uncertainty of the angle θ.

CONCLUSION

We successfully performed two independent experiments to determine g using two different methods and analysed the results statistically. Here, the pendulum experiment showed a much higher precision compared to the ball on incline experiment. While we seem to be not limited by the statistical error on g for the pendulum experiment, the deviation from the value stated in [1] can be explained by a systematic error arising from the small angle approximation which did not hold in our case. Accounting for this, our value for g agrees with the theoretical value within its uncertainty.

For the ball on incline, the discrepancy between expected and obtained value for g is far larger. This must originate in systematic errors that we are unable to estimate properly, especially due to the acceleration. One possible way to reduce the impact on the acceleration would be, to increase θ to 90 degrees and performing the experiment as free fall. This should reduce the impact due to friction and the like.

Overall, the precision and accuracy in both experiments used are limited and lack in comparison with [1]. However, the accuracy for the pendulum is significantly better than for the ball on incline. The precision on the value with respect to statistical errors is 2 orders of magnitude better than for the ball on incline experiment. Considering the simplicity of the methods used here, our results are nevertheless appropriately fitting with the theoretical value.

APPENDIX

[1] G. Strykowski, L. Timmen, O. Gitlein, R. Forsberg, B. Madsen and C. J. Andersen, *Gravity measurements in Denmark in 2005* , National Space Center, Technical report No. 6, 2006 May 2006, Table 2, value: *University of Copenhagen*

[2] Webpage for determining contribution due to large angle: `http://hyperphysics.phy-astr.gsu.edu/hbase/pendl.html`, last accessed 07.12.2020

Pendulum data

Original experimental data for the pendulum along with intermediate results for fits and chi squared values and probabilities.

Length L [cm]	σ_L [cm]
192.80	0.05
193.0	0.1
193.20	0.05
192.90	0.05
193.0	0.1
193.0	0.05
192.85	0.05
193.3	0.1
193.30	0.05
193.20	0.05
193.00	0.05

Height H [cm]	σ_H [cm]
3.73	0.01
3.64	0.05
3.65	0.01

Table II: Measured pendulum lengths and heights of the weight of the pendulum with respective uncertainties.

Length L [cm]	σ_L [cm]	χ^2	$P\left(\chi^2\right)$
193.04	0.02	101.17	0.00

Height H [cm]	σ_H [cm]	χ^2	$P\left(\chi^2\right)$
3.689	0.007	32.98	3 e-7

g [m/s^2]	σ_g [m/s^2]	χ^2	$P\left(\chi^2\right)$
9.807	0.002	224.06	7 e-42

Table III: Weighted means for length of pendulum string and weight and respective chi squared value and probability with respect to values from table VIII.

period time T [s]	σ_T [s]	offset c [s]	σ_c [s]
2.776	0.015	209.6	0.4
2.78	0.02	43.7	0.4
2.77	0.02	72.3	0.4
2.78	0.02	7.1	0.4
2.77	0.02	13.1	0.4
2.78	0.02	-0.1	0.4
2.774	0.015	0.0	0.3
2.77	0.02	47.6	0.4
2.77	0.02	-0.7	0.3
2.771	0.010	18.5	0.3
2.772	0.010	99.7	0.3

Table IV: Resulting slope and offset parameters from initial linear fit of period times with respective fit uncertainties.

period time T [s]	σ_T [s]	χ^2	$P\left(\chi^2\right)$
2.7760	0.0005	33.51	0.54
2.7763	0.0009	29.54	0.44
2.7744	0.0007	29.14	0.66
2.7769	0.0005	26.38	0.82
2.7736	0.0011	17.36	0.98
2.7756	0.0007	13.68	1.00
2.7740	0.0008	27.18	0.82
2.7725	0.0009	29.76	0.68
2.7711	0.0007	27.15	0.75
2.7707	0.0008	37.82	0.83
2.7717	0.0005	45.73	0.57
2.7743	0.0002	92.17	0.00

Table V: Resulting average period times per data set with uncertainties, chi squared values and probabilities. Last line are the resulting combined values.

g [m/s^2]	σ_g [m/s^2]
9.783	0.004
9.791	0.006
9.815	0.005
9.782	0.003
9.810	0.008
9.796	0.005
9.800	0.006
9.833	0.006
9.843	0.005
9.841	0.006
9.823	0.004
9.811	0.005

Table VI: Individual calculations of g for comparison with the one calculated from first averaging T and L. Comparing individual values, weighted average of individual values and the calculated g from averaged values seem to be consistent.

μ	σ_μ	σ	σ_σ
0.000	0.006	0.034	0.006
-0.001	0.008	0.042	0.007
-0.001	0.008	0.040	0.008
0.006	0.008	0.029	0.006
0.00	0.02	0.06	0.02
0.02	0.03	0.04	0.02
0.02	0.02	0.054	0.011
-0.002	0.011	0.057	0.012
-0.001	0.008	0.042	0.008
0.000	0.014	0.08	0.02
-0.001	0.009	0.056	0.008

Table VII: Results for the fit of the histogram of residuals with a Gaussian function as visualized exemplary in Figure 2.

Period number n	T [s]
1	46.6092
2	49.2447
3	52.0831
4	54.8310
5	57.6133
6	60.3665
7	63.2040
8	65.9663
9	68.7360
10	71.5305
11	74.2783
12	77.0851
13	79.8409
14	82.6246
15	85.3332
16	88.1375
17	90.9226
18	93.6287
19	96.6162
20	99.2651
21	102.0722
22	104.8447
23	107.6462
24	110.3314
25	113.1146
26	115.9821
27	118.7053
28	121.4433
29	124.2569
30	127.0502
31	129.8253

Table VIII: Exemplary data set for number of oscillations and corresponding timings. This data set was used to produce Figure 2.

Ball on incline

Original experimental data for the Ball on incline experiment along with other relevant results from the analysis not presented in the main text of the report.

Length L [cm]	σ_L [cm]
89.20	0.05
89.15	0.05
88.85	0.05
89.07	0.03

Height H [cm]	σ_H [cm]
22.10	0.05
22.05	0.05
25.30	0.05
22.08	0.04

Table IX: Lengths and heights for the incline and their respective weighted mean. Height and length are measured such that they are perpendicular to each other. For the height the third value is not included in the weighted mean since it is multiple σ away and obviously an error.

Small ball d_{small} [mm]	$\sigma_{d_{\mathrm{small}}}$ [mm]
12.10	0.05
12.00	0.05
12.00	0.05
12.03	0.03

Big ball d_{big} [mm]	$\sigma_{d_{\mathrm{big}}}$ [mm]
20.30	0.05
19.10	0.05
19.00	0.05
19.47	0.03

Rail diameter d_{rail} [mm]	$\sigma_{d_{\mathrm{rail}}}$ [mm]
5.80	0.05
6.10	0.05
5.50	0.05
5.80	0.03

Table X: Measured diameters for the two balls used in the experiment and the diameter of the rail used as incline with respective weighted averages and uncertainties.

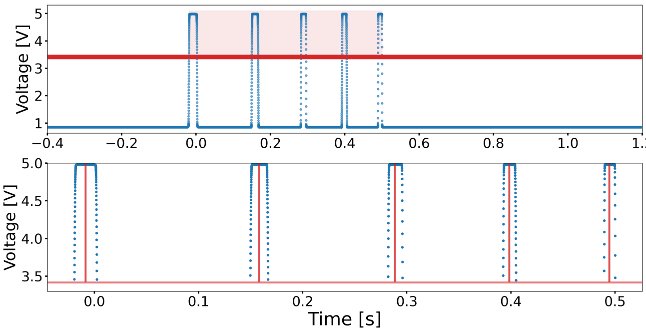

Figure 4: Example voltage peaks as the ball passes time gates. Gate timing data is shown in blue, and the mean on each subset is shown at bottom graph by the red lines. Each of the five peaks represents the passing through one of the five gates.

Direction	Side	Angle θ [deg]	σ_θ [deg]
A	n	14.2	0.1
		14.2	0.1
		14.2	0.1
	mean:	14.20	0.06
	an	13.8	0.1
		13.6	0.1
		16.1	0.1
	mean:	13.70	0.07
	$\mu_{\theta A}$	13.95	0.05
B	n	16.2	0.1
		15.2	0.1
		16.65	0.1
	mean:	16.02	0.06
	an	12.5	0.1
		13.0	0.1
		13.9	0.1
	mean:	13.13	0.06
	$\mu_{\theta B}$	14.58	0.04
$\Delta\theta$		0.31	0.03

Table XI: Measured angles of the ramps using two different goniometers to derive $\Delta\theta$. A and B stand for the two different directions of the ramp, i.e. the ramp turned by 180 deg in the horizontal plane, and a and an stand for the two different sides of the ramp. For the mean calculation of direction A and side an, we discard the 16.1 deg value, as it lays more than 20 σ away from the other two measurements.

Side A - Ball size small				
Gate meas.	acceleration a [cm/s^2]	σ_a [cm/s^2]	χ^2	$P\left(\chi^2\right)$
I	151.268	1.599	0.2	0.973
II	126.952	1.599	0.2	0.000
III	150.041	1.599	0.2	0.078
I	151.016	1.593	0.2	0.926
II	126.794	1.593	0.2	0.000
III	149.794	1.593	0.1	0.089
I	151.105	1.728	0.1	0.941
II	124.853	1.727	0.1	0.000
III	149.801	1.728	0.1	0.109
I	151.264	1.728	0.2	0.923
II	125.008	1.728	0.2	0.000
III	149.960	1.728	0.2	0.123
I	150.988	1.591	0.2	0.917
II	126.802	1.592	0.2	0.000
III	149.768	1.591	0.2	0.917
I	150.996	1.600	0.2	0.904
II	126.661	1.601	0.2	0.000
III	149.770	1.602	0.2	0.100

Side A - Ball size big				
Gate meas.	acceleration a [cm/s^2]	σ_a [cm/s^2]	χ^2	$P\left(\chi^2\right)$
I	150.418	1.598	0.1	0.955
II	126.944	1.693	2.9	0.000
III	149.193	1.598	0.1	0.087
I	151.418	1.500	0.1	0.955
II	127.113	1.715	2.2	0.000
III	150.253	1.500	0.1	0.096
I	151.504	1.505	0.1	0.944
II	126.750	1.759	1.8	0.000
III	150.338	1.505	0.1	0.145
I	151.663	1.557	0.1	0.973
II	127.200	1.740	1.8	0.000
III	150.463	1.557	0.1	0.119
I	151.297	1.608	0.2	0.914
II	128.404	1.687	1.5	0.000
III	150.066	1.607	0.2	0.109
I	151.297	1.608	0.2	0.914
II	128.665	1.660	1.9	0.000
III	150.066	1.607	0.2	0.109

Table XII: Fitted acceleration based on eq. 1.6, including the corresponding χ^2 value at probability $P\left(\chi^2\right)$. We discard the gate measurement II as it provides $P\left(\chi^2\right) < 0.01$ for all measurements.

Side B - Ball size small				
Gate meas.	acceleration a [cm/s^2]	σ_a [cm/s^2]	χ^2	$P\left(\chi^2\right)$
I	152.703	1.692	2.9	0.239
II	126.115	1.598	0.1	0.000
III	151.411	1.692	2.9	0.007
I	153.205	1.715	2.2	0.336
II	128.618	1.506	0.1	0.000
III	151.899	1.715	2.2	0.013
I	153.504	1.759	1.8	0.408
II	128.634	1.505	0.1	0.000
III	152.171	1.759	1.8	0.015
I	150.418	1.598	0.1	0.955
II	128.634	1.505	0.1	0.000
III	149.193	1.598	0.1	0.087
I	151.418	1.500	0.1	0.955
II	128.634	1.505	0.1	0.000
III	150.253	1.500	0.1	0.096
I	151.504	1.505	0.1	0.944
II	127.992	1.559	0.1	0.000
III	150.338	1.505	0.1	0.145

Side B - Ball size big				
Gate meas.	acceleration a [cm/s^2]	σ_a [cm/s^2]	χ^2	$P\left(\chi^2\right)$
I	152.428	1.708	3.6	0.166
II	126.445	1.708	3.6	0.000
III	151.149	1.706	3.6	0.005
I	151.967	1.699	4.4	0.110
II	126.044	1.989	4.4	0.000
III	150.669	1.692	4.4	0.004
I	153.137	1.741	2.7	0.256
II	126.798	1.651	2.7	0.000
III	151.821	1.747	2.7	0.009
I	152.398	1.836	3.6	0.168
II	124.499	1.796	3.6	0.000
III	151.021	1.840	3.6	0.006
I	154.512	1.625	1.9	0.390
II	129.764	1.626	1.9	0.000
III	153.259	1.627	1.9	0.017
I	153.643	1.588	3.0	0.223
II	129.468	1.588	3.0	0.000
III	152.414	1.588	3.0	0.009

Table XIII: Fitted acceleration based on eq. 1.6, including the corresponding χ^2 value at probability $P\left(\chi^2\right)$. We discard the gate measurement II as it provides $P\left(\chi^2\right) < 0.01$ for all measurements.

Meas.	Gate 1 [cm]	Gate 2 [cm]	Gate 3 [cm]	Gate 4 [cm]	Gate 5 [cm]	σ [cm]
I	21.05	34.00	47.05	60.10	73.10	0.05
II	21.50	35.30	48.80	60.90	74.10	0.05
III	77.95	64.95	51.85	38.95	25.85	0.05
	21.28	34.65	47.93	60.50	73.60	0.04

Table XIV: Measured gate positions for the ball on incline measurement. The third data set was measured with respect to a different reference point, therefore this set was excluded for the weighted average and further analysis.