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Review Article

Nonlinearity and solution techniques in reservoir simulation: A review

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A B S T R A C T

Reservoir simulation is used to demonstrate the dynamic physical processes of rocks and fluid properties with high-order nonlinear equations. Currently, different types of simulation models are used in the petroleum industry. These models are solved by numerical techniques to get a solution considering lots of inherent assumption. In this paper, an extensive review is offered on the state-of-the-art literature with a focus on the nonlinearity in partial differential equations related to the petroleum reservoir simulation. A critical analysis is done on the different techniques for solving nonlinear governing equations in a petroleum reservoir. It also addresses the inherent assumptions and properties, the significance of nonlinear solvers and their technical challenges. Finally, the article discusses the impact of the solution of these nonlinearity problems by following the different numerical techniques.

Contents

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1. Introduction

The petroleum industry is the primary key to the global economy, and technological advancement moves forward based on this sector. The energy demand is increasing and currently, the crude oil production is some 90 million barrels per day (EIA, 2016). Extracting more oil and gas out of existing reservoirs is therefore of paramount importance if the industry is to meet the future growth of energy consumption. Therefore, there is a need to improve reservoir performance and enhance the hydrocarbon recovery mechanism, which is mostly influenced by proper reservoir simulation models. Crichlow (1977) presented a general overview of the simulation approach for the petroleum industry. However, the physical dimension of a reservoir is always an uncertain issue because every reservoir has different geometrical structure and unique geological characteristics (Mustafiz and Islam, 2008). The prediction of reserves based on the theory behind the fluid flow through porous media, and existing mathematical models for the oil displacement process misleading the petroleum industry's existence.

Simulation models capture complex physical phenomena related to the inherent geological complexity of earth models. Due to the nonlinearity complexity in governing equations, the dynamic simulation and solution processes in a petroleum field remains a major challenge and an ongoing research topic. The governing equations for the fluid flow in porous media based on conservation of mass, momentum, and energy equations. Reservoir simulation models categorised into two groups, based on standard black-oil models and compositional models (Aziz and Settari, 1979). In compositional models, conservation equation had written for individual components (Young and Stephenson, 1983). Despite the increase in the use of compositional models, the high computational cost associated with nonlinearity complexities remains a major drawback. On the other hand, the black oil models are more attractive candidates for most reservoir simulation studies in the industry due to their simplifying assumptions regarding realistic field-scale simulations (Lee et al., 2008).

For solving the simulation models, several analytical and numerical methods were applied to handle the nonlinear problems. However, the solutions are not exact due to their possibilities of linearization and various assumptions and failed to provide multiple solutions instead of a single solution for a set of governing equations. Therefore, an advanced numerical tool is needed to predict the exact solution for a multivariable problem and the solutions are realistic rather than impractical.

Based on above issues, the focus of this paper is to review the solution techniques for various nonlinear equations in petroleum reservoir engineering and simulation, technical challenges in solving those governing equations, and future facilitation for the reservoir simulation. Here, we reviewed the related literature of various researcher, summarized their solution techniques and model along with assumptions and properties. Finally, we provide some guidelines for future research and development (R&D).

1.1. Background of the research

In reservoir engineering, most of the equations expressed the nonlinear behaviours due to effects of the time interval, fluid and formation properties (e.g. porosity, permeability, water saturation, viscosity, etc.) variation, distribution of pressure responses, simplification of the governing equations at formulation stage or the feasibility of multiple solutions. Islam and Nandakumar (1986,1990) showed the nonlinear behaviour of the governing equations in petroleum reservoir engineering and simulation. To avoid the nonlinearity, the previous researchers solved the governing equations using linearized methods (i.e. Taylor series expansion, Optimal linearization method, Global linearization method, Perturbation theory, Euler's method, Runge-Kutta method, Newton's Iteration, etc.) along with some assumptions (Jordan, 2006). These procedures help the researcher to understand the simulation model and control the system design methods quickly. However, the linearization effects were significant, and the results were not accurate due to the wrong prediction of the parameters distribution and number of errors. The interpretation of the simulation models also affected by neglecting higher order roots and taking the assumption (Islam et al., 2016). The material balance equation, Navier-Stokes equation, Buckley-Leverett's equation, etc. are the typical example of nonlinear behaviour in petroleum reservoir engineering (Table 1).

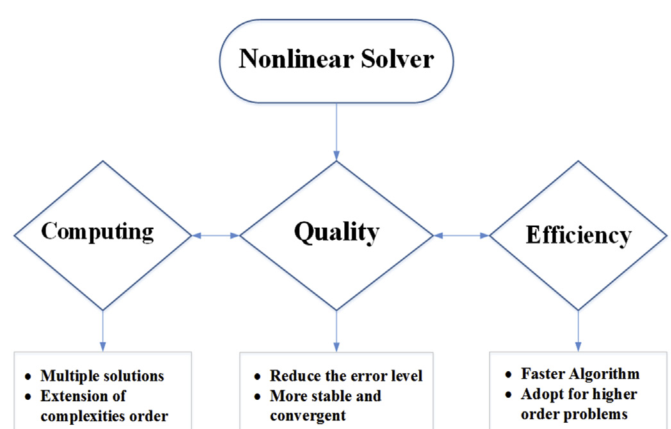

Figure 1. Importance of nonlinear solver.

Table 1
A few examples of nonlinear equations

Sl. No.	Equations	Reason's for nonlinearity	References
01	Material Balance equation	Nonlinear nature of pressure decline with time or distance	Islam et al. (2016)
02	Navier-Stokes equation	Nonlinear stress-rate of strain relationship	Islam et al. (2016)
03	Buckley-Leverett's equation	Nonlinear behaviour due to the inclusion of capillary pressure	Mustafiz et al. (2008b); Islam et al. (2016)
04	Darcy's law: Fluid flow through porous medium	Nonlinear nature of pressure-dependent properties	Abou-Kassem et al. (2006); Islam et al. (2016)

1.2. Technical challenges towards the research

All current commercial computer simulators (e.g. Eclipse, CMG Suite, Tempest MORE, ExcSim, Nexus, FlowSim, etc.) in the petroleum industry solve the set of governing equations (including all algebraic, PDEs and ODEs equations) by linearizing nonlinear governing equations and considering several assumptions. Majority of cases, the solutions are not ideal due to the nonlinear behaviours of the equations. Islam et al. (2010, 2016) showed that these solutions are varied with the realistic range of most of the petroleum parameters in a single-phase flow. The researcher also found significant errors in the prediction time of petroleum reservoir performance using advanced fuzzy logic. The scenarios are worse for multiphase flow when linearization occurs in governing equations (Islam et al., 2016). Thus, to predict the reservoir behaviour and its future performance, we need more accurate solutions from nonlinear solvers, which develop a consistent solution scheme by solving nonlinear algebraic equations and helps us to optimize the hydrocarbons recovery.

1.3. Objectives of the research

Nonlinearity increases the complexity of reservoir operations and reduces the performance while applying various simulation technologies. It also raised the computational cost and took more time to complete. Only the numerical solutions of the nonlinear governing equations in the reservoir engineering are helping us to understand, forecast, and manage subsurface fluid migration in a reservoir, especially in complex geometry and highly nonlinear multiphase fluid flow system. An effective numerical technique can handle the nonlinear parameters and enhance the productivity. As such it has drawn a lot of interest by the diverse group of scientists.

Most of the petroleum properties were not justified, and approximately 30% deviation occurred due to the linearization of nonlinear algebraic equations regarding pressure values for single phase flow (Hossain and Islam, 2010a; Islam et al., 2010). That type of error restricts the ability of petroleum reservoir model. The performance further deteriorates when the governing equations for multiphase flow are linearized. The residual equations of the governing equations are also nonlinear due to various natures of nonlinearity including saturation-dependent nonlinear terms (e.g. relative permeability, and capillary pressure functions), or pressure-dependent nonlinear terms such as viscosities and densities (Shahvali, 2012; Nooruddin et al., 2014). Thus, it is necessary to solve the nonlinear algebraic equations in time and space dimensions. A nonlinear solver reduces the time step and error level with its better algorithm for the equations using the engineering approach. The solver also maintains the stability and consistency of a solution. The stability and consistency of a solution are also maintained by a nonlinear solver. However, limited work has been done using this technique to represent the importance of nonlinear solver (Fig. 1) and resolve the nonlinearity difficulties. Nonlinear solver tasks are more fruitful for enhanced oil recovery (EOR) scheme to optimizing the oil recovery and improve the thermal flooding operations (Hossain et al., 2009; Islam et al., 2010; Al-Mutairi et al., 2014).

2. Critical literature analysis

The equations used in reservoir engineering and simulation are inherently nonlinear due to the interaction and inclusion of various parameters. These equations may be algebraic, differential, integral, partial differential equations (PDE), ordinary differential equations (ODE), or integro differential equations. These equations consist of higher-order roots and indicates the nonlinear behaviours. Depending on the number of parameters, the nonlinear algebraic equations obtain multiple solutions instead of a single solution. We need to solve such type of equations to find an exact solution using

Table 2
A Critical analysis of different model equations from different researchers.

Researcher	Parameters of the model equations																Solution scheme
	ϕ	k	P	T	S_w	f_w	μ	P_c	x	σ	g	q	α	τ	Reservoir dimension	No. of phases	
Buckley-Leverett (1942)	√	√	-	-	√	√	√	√	√	-	√	√	-	-	1-D	2	Theoretical
Holmgren and Morse (1951a,b)	√	√	-	-	√	-	√	√	√	-	-	√	-	-	1-D	2	Analytical
Welge (1952)	√	√	-	-	√	√	√	√	√	-	-	√	-	-	1-D	2	Analytical
Fayers and Sheldon (1959)	√	√	-	-	√	-	√	√	√	-	√	√	-	-	1-D	2	Analytical
Hovanessian and Fayers (1961)	√	√	-	-	√	√	√	√	√	-	√	√	-	-	1-D	2	Theoretical
Slattery (1967)	-	√	√	-	-	-	√	-	-	√	-	-	√	√	-	1	Theoretical
Bentsen (1978)	√	√	-	-	-	√	√	√	√	-	√	√	-	-	1-D	2	Numerical
Mifflin and Schowalter (1986)	-	√	√	-	-	-	√	-	-	-	-	-	√	√	3-D	1	Analytical
Eringen (1991)	-	-	√	√	-	-	√	-	√	√	-	√	√	√	1-D	1	Theoretical
Nibbi (1994)	-	-	√	√	-	-	√	-	√	-	-	√	√	√	1-D	1	Theoretical
Broszeit (1997)	-	-	√	√	-	-	-	-	-	-	-	-	√	√	1-D	1	Numerical
Caputo (1999)	-	√	√	-	-	-	√	-	√	-	-	√	√	-	-	-	Theoretical
Shin et al. (2003)	-	-	√	√	-	-	√	-	-	-	-	-	√	-	-	1	Theoretical
Liu et al. (2003)	√	√	√	√	-	-	√	-	-	-	√	-	-	√	1-D	1	Numerical
Chen et al. (2005)	√	√	√	√	-	-	√	-	√	-	-	-	√	√	1-D	2	Theoretical
Hossain and Islam (2006)	√	-	√	√	-	-	-	-	-	√	-	-	√	√	-	-	Theoretical
Abou-Kassem (2007)	√	√	√	√	-	-	√	-	√	-	-	√	-	-	1-D	1	Analytical
Hossain et al. (2007)	√	√	-	√	-	-	√	-	-	-	-	-	√	√	1-D	1	Analytical
Mustafiz et al. (2008b)	√	√	-	-	√	√	√	√	√	-	√	√	-	-	1-D	2	Numerical
Hossain et al. (2008)	√	√	√	-	-	-	√	-	√	-	-	√	√	-	1-D	2	Numerical
Hossain et al. (2009b)	-	√	√	√	-	-	√	-	-	-	-	√	√	√	1-D	2	Numerical
Younis et al. (2010)	√	√	-	-	√	√	√	√	√	-	√	√	√	-	1-D	2	Numerical
Hossain (2012)	√	-	√	√	-	-	√	-	√	-	-	√	√	-	1-D	2	Analytical
Wang and Tchelepi (2013)	√	√	-	-	√	-	√	√	√	-	√	-	√	-	1-D	2	Numerical
Li and Tchelepi (2014)	√	√	√	√	-	-	√	√	√	-	-	√	-	-	1-D	2	Numerical
Hossain (Unpublished results)	√	√	√	√	-	-	√	-	√	-	-	√	√	-	1-D	2	Analytical
Hossain (2016)	√	√	√	√	-	-	√	-	√	-	-	√	√	-	1-D	2	Numerical
Obembe et al. (2016b)	√	√	√	-	-	-	√	-	√	-	-	√	√	-	1-D	2	Numerical

Table 3
A comparative and critical study on assumptions and limitations of different model equations.

Investigator	Assumptions	Limitations	Applications
Ciarletta and Scarpetta (1989)	• Homogeneous, incompressible and viscous fluid • Aware of symmetric velocity gradient • Defined boundary conditions	• Not suitable for complex simulation model equations • Depends only on fluid viscosity	• In mathematics field for minimizing viscous-fluids related problems
Eringen (1991)	• Identified local convergences • Applicable only for spherical problems	• Undefined global convergences • Incompetent for thermally active fluid flow systems	• Developed the suspension mechanism for colloidal problems
Nibbi (1994)	• Homogeneous, incompressible and viscous fluid • Linear and isotropic problems	• Imprudent for practical problems	• Fluid related quasi-static problem in Mathematics field
Caputo (1999)	• Linear isotropic, homogeneous, viscous and incompressible fluid • Predicts fluid pressure with time	• Permeability decreases with time • Undefined fluid properties role in a porous media	• In geothermal areas for studying the pore size of the minerals
Shin et al. (2003)	• Memory is dependent on the diffusion time scale • Homogeneous model • Used Taylor series expansion	• Did not explain the memory impact fully in the media • More dependency on time scale	• Explained particle deposition facts and Non-equilibrium mechanism for fully developed turbulent channel flows
Hossain and Islam (2006)	• Permeability and fluid compressibility unchanged with time • Fluid and media properties have been considered	• No explanation about nonlinear trends of stress-strain model	• Visco-elastic fluid flow behaviour in a reservoir.
Hossain et al. (2007)	• No slip condition • No lower contact surface velocity • Consider dimensionless parameters • Ignored the roles of surface tension on the fluid viscosity, and memory	• No explanation about matrix heterogeneity, anisotropy, and inelasticity of a reservoir	• Used in reservoir simulation, rheological study, well test analysis, and EOR surfactant selection
Hossain et al. (2008)	• Used time-dependent fluid and media properties • Used equation of motion and continuity equation • Followed fractional order of differentiation	• Generates only singular form of equations • Need to remove singularities using different numerical techniques	• Used for crude oil flow in porous media, EOR process, and few special case (like decline pressure reservoir condition)
Hossain et al. (2009b)	• Non-Newtonian and shear thinning fluids • Ignored scale-up problems • Trapezoidal method used for numerical simulation • Porosity, permeability, shape factor and flow velocity considered	• Captured physical phenomena of a model only • Limited works at low shear rates • Not showed the effect of apparent viscosity in their model • Analytical solution was absent	• Used in EOR and polymer manufacturing applications
Hossain (2012)	• Compressible fluids • Single phase flow • 1-D horizontal reservoir with irregular block size • Rectangular coordinates	• Analytical and numerical solutions was absent	• Used for the development of petroleum reservoir simulator
Wang and Tchelepi (2013)	• Newton-based nonlinear solver • Large time steps • Analysed entire capillary, viscous, and buoyancy parameter spaces of porous media • Nonlinear immiscible, incompressible, two-phase flow in porous media	• Failed to explain time-step based truncation errors • Only for single cell problems	• Developed for coupled multiphase flow and transport in porous media
Li and Tchelepi (2014)	• Interface of discretized flux function between two cells • Guided only Newton iterations for various saturation regions • Considered time-truncation errors • Oscillation of convergence failure avoided • Two-phase flow	• Undefined many unknown variables in multiple dimensions • Unable to explain sharp kinks of relative permeability curves	• Used for oil/gas recovery, groundwater remediation, and CO_2 geological sequestration
Hossain (Unpublished results)	• Fluid and rock properties are time-dependent continuous functions • Followed engineering approach • 1-D Cartesian reservoir considered	• Have no realistic estimation regarding fractional derivatives • Need experimental data set for validation of memory-based diffusivity equation	• Used in well testing analysis, petroleum reservoir simulation and history matching
Hossain (2016)	• Permeability is constant with respect to time and space • 1-D model of an oil reservoir • Only considered transient phase • Asymptotic value is set equal to zero	• Not explain about non-local aspects of fluid transport • Singularity of the integral formulation with the memory was inherent for integrodifferential equations	• Determined pressure response in the formation rock and fluid properties for a petroleum reservoir • Captured memory in reservoir fluid flow in porous media
Obembe et al. (2016b)	• Extended the diffusivity equation for multiphase flow system • Considered wellbore geometries • Defined volumetric flux with time • Material balance check per time step	• R-L fractional operator failed to interpret the initial conditions of a system • Time-step must be known for stability check	• Handled fractional flow problem in porous media

Newton's method, Finite Difference Method (FDM), Adomian Decomposition Method (ADM), etc. However, these conventional techniques do not provide multiple solutions for simultaneous nonlinear equations, which has grown the interest of diverse groups of scientists to work more on this section. There are several effects (i.e., pressure-dependent properties, capillary pressure, viscous fingering, memory, gas flow modelling, mixing, phase exchange, absorption and desorption etc.) involved which increases the nonlinearity in the governing equations. A critical analysis of different models is shown in Table 2 based on those effects. Here, a summary of the model equations, assumptions, parameters behind these equations, and the limitations for the developed equations are presented in Tables 2 and 3. These tables represented the last few decades advancement in reservoir simulation models and provided a scope for the future research. Furthermore, the identified key features will help the researcher for improving their ideology and philosophy to develop a reservoir emulator considering the time and space. In this section, we are trying to review the researcher works and sort out the problems from their solution technique and finally looking for a research scope in the current area of interest.

2.1. Effect of capillary pressure

Capillary pressure is the pressure differential between the wetting and non-wetting phase in porous media due to the effects of capillary forces across the fluids interface. The behaviour of capillary pressure in reservoir engineering and enhanced oil recovery problems is a challenging task. Using the capillary pressure, the scientist and petroleum engineers evaluates the quality of reservoir rock, the depth of reservoir fluid contacts, seal capacity, pay versus nonpay zone, estimation of recovery efficiency, etc. from the petroleum fields (Morrow, 1970; Melrose and Brandner, 1974; Wardlaw et al., 1988).

Most researchers tried to solve the governing equations of these areas by considering capillary effects and found significant effects. Based on Buckley-Leverett's equation (1942); Holmgren and Morse (1951a,b) explained the capillary effects for removing nonlinearities through the average saturation of water calculation. After that, Welge (1952) found the mean water saturation at a breakthrough point in an oil reservoir and shock effects, that described at Buckley-Leverett's equation. Fayers and Sheldon (1959) solved the displacement equation with gravity and capillary pressure using Lagrangian approach. However, they did not obtain saturation values for a required time. Hovanessian and Fayers (1961) also presented the capillary pressure effects by avoiding the multiple-valued of saturation profiles. Hence, Bentsen (1978) explained that fact with separate equations for a particular distance travelled by zero saturation with the numerical investigation. He also showed the incorrect formulation of Fayers and Sheldon (1959) for slow rates of injection at constant normalized saturation condition. However, Craft and Hawkins (1991) showed the fluid distributions of a single homogeneous formation for different days and detected multiple water saturation values along the formation bed. For simplicity, the authors assumed that the forces generated by gravity and the capillary pressures are negligible. The capillary pressure is also significant during fluid injection into confined aquifers. Nordbotten and Celia (2006) predicted the formation zone behaviour with semi-analytical solutions for the case of CO_2 injecting into deep saline aquifers, considering capillary pressure effects. The authors also attempted to reduce the nonlinearity of PDEs in their modeled system by presenting a couple of solutions through iterative solution techniques and finally compared with numerical solutions depicted by Nordbotten et al. (2005). However, to avoid the complexity, the capillary pressure was neglected in their solution. Later, Schmid et al. (2011) derived semi-analytical solutions for two-phase flow and concerned with the effects of viscous and capillary forces in counter-current and co-current imbibition processes. Due to the highly nonlinear nature of the problem, they only considered the capillary-free Buckley-Leverett problem.

To minimize the effects of nonlinearity, Mustafiz et al. (2008a, and b) solved nonlinear Buckley-Leverett's equation (1942), considering capillary pressure effects and validated their solution for 1-D, two-phase flow using ADM. Recently, Islam et al. (2016) argued that the multiple solutions might come using advanced fuzzy logic and numerical techniques. Therefore, the finding of consistent multiple solutions for multiphase flow from Buckley and Leverett's equation using advanced numerical techniques will be a new scope of research.

From Buckley-Leverett's theory (1942), the equation can be written as:

$$\frac{\partial S_w}{\partial t}+\frac{q}{A\varphi}\frac{\partial f_w}{\partial S_w}\frac{\partial S_w}{\partial x}=0 \tag{1}$$

where,

$$f_w=\left(\frac{1}{1+\frac{k_{ro}\mu_w}{k_{rw}\mu_o}}\right)\left(1+\frac{Akk_{ro}}{q\mu_o}\left[\frac{\partial P_c}{\partial x}-(\rho_w-\rho_o)gsin\alpha\right]\right) \tag{2}$$

The equations (1, 2) are nonlinear PDEs. Taking few assumptions, Craft and Hawkins (1991) presented the following equation (3) by calculating water saturation along the formation bed:

$$x(t,S_{w0})=\frac{qt}{A\varphi}\frac{\partial}{\partial S_{w0}}\left(\frac{1}{1+\frac{k_{ro}\mu_w}{k_{rw}\mu_o}}\right)_t \tag{3}$$

The above equation (3) neglects the gravity and capillary forces in the formation, and the flow is horizontal. The results showed multiple saturation values at any points along the bed, which is fully unrealistic in the reservoir simulation. Therefore, Mustafiz et al. (2008b) presented the following equation (4) for the case of horizontal flow, considering the capillary pressure effects:

$$S_{w0}(x,t)=S_w(0,x)-\int_0^t\left[\frac{q}{A\varphi}\frac{\partial}{\partial S_{w0}}\left(\frac{1}{1+\frac{k_{ro}\mu_w}{k_{rw}\mu_o}}\right)\frac{\partial S_{w0}}{\partial x}\right]dt \tag{4}$$

$$\begin{aligned}S_{w1}(x,t)=-\int_0^t\Bigg[&\frac{kk_{ro}}{\mu_0\varphi}\frac{\partial}{\partial S_{w0}}\left(\frac{1}{1+\frac{k_{ro}\mu_w}{k_{rw}\mu_o}}\right)\frac{\partial P_c}{\partial S_{w0}}\\&+\frac{k}{\mu_0\varphi}\left(\frac{1}{1+\frac{k_{ro}\mu_w}{k_{rw}\mu_o}}\right)\frac{\partial k_{ro}}{\partial S_{w0}}\frac{\partial P_c}{\partial S_{w0}}\\&+\frac{kk_{ro}}{\mu_0\varphi}\left(\frac{1}{1+\frac{k_{ro}\mu_w}{k_{rw}\mu_o}}\right)\frac{\partial^2P_c}{\partial {S_{w0}}^2}\Bigg]\left(\frac{\partial S_{w0}}{\partial x}\right)^2dt\end{aligned} \tag{5}$$

Mustafiz et al. (2008b) solved these equations (4 and 5) using ADM, which is a powerful technique to solve nonlinear equations. The researcher maintained the capillary pressure and water saturation behaviour and obtained one saturation value for a given time at one point along the formation bed. In fact, the solution converged quickly but failed to produce multiple solutions for the governing Buckley-Leverett's equation. Hence, there is a scope for the future researcher to find out the new scheme and check the feasibility of generating multiple solutions.

On the contrary, the capillary effects could be explained by bundle tube model and network model concepts. Some researchers had focused only on the bundle tube models (Yuster, 1951; Scheidegger, 1953; Bartley and Ruth, 1999; Dong et al., 1998), and other researchers discussed 2-D and 3-D network models (Chatzis and Dullien, 1977, 1982; Chandler et al., 1982; Lapidus et al., 1985; Diaz et al., 1987; Blunt and King, 1991). However, the researchers failed to simulate the fluid dynamics in porous media through network modelling. In addition, Dong et al. (2005) analyzed immiscible displacement processes using interacting capillary bundle model concept. In this model, the capillary pressure had considered, and the fluids used in different capillary pressure were independent of each other. The saturation profiles were calculated using the backward difference approximation which is second order nonlinear equations and solved through Newton's iteration method for each time and space steps. The stability and convergence were also in good approximation based on experimental, numerical and analytical results. However, this is a linearized and time-consuming method. Therefore, the researchers are looking forward advanced nonlinear solution schemes which will be less time consuming and will provide stable and accurate solutions.

2.2. Effect of memory

Memory is a function of fluid and media properties, and the pressure changes with respect to time in a reservoir, which influences the other rock and fluid properties (Caputo, 1998a, 1999, 2000; Caputo and Cametti, 2009; Caputo and Fabrizio, 2015). During the flow process, the rock and fluid properties alter continuously within the reservoir, and that alteration is directly or indirectly related to time function (Hossain et al., 2009; Hossain and Islam, 2009). Research shows that these properties are inherently nonlinear. Later, Hossain and Abu-Khamsin (2012a, 2012b) presented the notion of memory, which stated the continuous time function or history dependency and leads the nonlinearity and multiple solutions.

Getting memory effects in the petroleum reservoir, Slattery (1967) defined the memory, which represents the deformation tensor rate as a function of extra stresses. He made it through studying the viscoelastic fluid behaviour with the Buckingham-pi theorem and found the nonlinear behaviour in his theorem. Using the memory function, Mifflin and Schowalter (1986) presented a relationship between the fluid viscosity and stress tensor in the non-Newtonian fluid. That research was conducted for solving 3-D steady flows in closed or open flow systems. They divided the memory into velocity gradients and continued the calculation until the fluid memory decomposed adequately while the rest of the integral could either be neglected or set to a small constant value. Besides, Ciarletta and Scarpetta (1989) focused on the linearized progress of an incompressible fluid flow equation, ignoring the nonlinear convective term of the equation. They were concerned with the symmetric velocity gradient, which helps to observe the immediate stress effects based on memory.

Apart from the above fluids, Eringen (1991) developed a nonlocal theory of micro-polar fluids by considering the orientation and memory effects along with stress and fluid viscosity. The researcher showed the significant memory effects by measuring the small characteristic length compared with the average gyration radius of the fluid molecular elements. This type of situation arises in the case of thin film lubrications which exhibits a nonlinear behaviour. However, none of the above researchers did explain and solve the nonlinearity. Nibbi (1994); Broszeit (1997); Li et al. (2001); Shin et al. (2003); Chen et al. (2005); and Hossain and Islam (2006) used memory function based on the characteristic and intermediate diffusion time scale for different fluids and parameters. These authors reviewed the memory effects comprehensively, but unsuccessful at the stage of experimental validation.

After few years, Hossain et al. (2009a) applied fluid memory into the non-Newtonian fluids during an EOR process. Here, the rock compressibility had affected by the pressure decline during the production life of a reservoir. The fluid and formation properties were investigated using space and time derivatives in fractional order, which made it a rigorous model. A significant pressure response was observed due to the increasing production time of the reservoir, which made the fluid memory effects dominant. Recently, a model was solved by Obembe et al. (2017) for heterogeneous media and explained the memory as a time derivative fractional order in a diffusion model.

Furthermore, Hossain et al. (2009b) described the characteristics of polymer flooding, reservoir simulation, and the characterization of complex reservoirs with the help of memory effects which could characterize the fluid movement. The researchers were concerned with the characterization of the rheological behaviour along with memory effects for the shear-thinning fluids. But, that model did not explain the delaying of fluid movement in a viscoelastic fluid. Therefore, Hossain et al. (2009c) presented a mathematical stress-strain model for a complex reservoir considering the effects of memory (α). For different values of α, they identified the fluid memory effects, considering time and space over the stress-strain relationship. The memory mechanism helps in interpreting the reservoir phenomenology with matrix heterogeneity, anisotropy, and inelasticity. Moreover, the variation of distance and time defines a chaotic behaviour with non-monotonous trends of the stress-strain relationship, which was a strong indication of the memory effect. The trapezoidal method used to solve their proposed model, ignoring the complexity of nonlinear second degree PDEs. The higher order roots PDEs unresolved for the absence of a nonlinear solver. Recently, memory has been applied appropriately by Hossain (Unpublished results) and Obembe et al. (2016b) to present the modified diffusivity equation where the rock and fluid properties as a continuous time-dependent function. According to Hossain (Unpublished results), the model equation was highly nonlinear due to the application of memory and validated by numerical experiment. Here, the solution of the equation was dependent on memory (α), which was more realistic compared to conventional Darcy's model. The memory was predominant surroundings the wellbore and diminished toward the outer boundary of the reservoir. However, that model could not provide better matching prediction due to the inconsistent rate of diffusion and incomplete solution.

Here, we show an example where memory incorporated into stress-strain relation and made the governing equation nonlinear. The relationship between stress and strain (Hossain and Islam, 2006; Hossain et al., 2007) becomes (Equation (6)):

$$\tau_T = \frac{k^2 \Delta p A_{xz} \Gamma(1-\alpha)}{\mu_0^2 \eta \rho_0 \phi y c I} \times \left[\left(\frac{\partial \sigma}{\partial T} \frac{\Delta T}{\alpha_D M_a} \right) \times e^{\left(\frac{E}{RT}\right)} \right] \tag{6}$$

where, $I = \int_0^t (t-\xi)^{-\alpha} \left(\frac{\partial^2 p}{\partial \xi^2} \right) d\xi$ and the permeability of media and fluid compressibility are unchanged with time.

The above equation is the simplified form of a mathematical model (Lu and Hanyga, 2005; Hossain et al., 2007) which consists of all fluid and media properties, and the equation is the following:

$$\tau_T = \frac{k^2 \Delta p A_{xz} \Gamma(1-\alpha)}{\mu_0^2 \eta \rho_0 \phi y c \int_0^t (t-\xi)^{-\alpha} \left(\frac{\partial c}{\partial \xi}\frac{\partial p}{\partial \xi} - \frac{c}{k}\frac{\partial k}{\partial \xi}\frac{\partial p}{\partial \xi} + c\frac{\partial^2 p}{\partial \xi^2} \right) d\xi} \times \left[\left(\frac{\partial \sigma}{\partial T} \frac{\Delta T}{\alpha_D M_a} \right) \times e^{\left(\frac{E}{RT}\right)} \right] \frac{du_x}{dy} \tag{7}$$

In equation (7), the memory is a function of all fluid and media properties over time. When we considered the memory and other properties simultaneously, the equation behaves nonlinear rather than a linear function. A proper nonlinear solver or a new numerical scheme could handle that type of equation and minimize the challenges of memory induced rock and fluid properties in porous media.

2.3. Effect of viscous fingering for miscible and immiscible displacement

To enhance the secondary and tertiary oil recovery, modelling viscous fingering is important in both miscible and immiscible displacement. Though this task is formidable, few researchers attempted to established a suitable model by following some of the chaos theory. The biggest problem was generating of higher-order root equations with respect to time and space, and researchers faced complexity to solve those types of governing equations. Naami et al. (1999) conducted an experiment on modelling viscous fingering in a 2-D system and interpreted the model using a numerical method. The similar approach was taken by Saghir et al. (2000) where the governing equations of viscous fingering model was solved using finite difference scheme. Both research groups explained PDEs of the model by propagation of fingers in the system. Some advanced numerical techniques also applied on modelling viscous fingering by other researchers (Aboudheir et al., 1999; and Bokhari and Islam, 2005). Bokhari and Islam (2005) showed their advancements in the order of Δt^4 in time and Δx^2 in space and presented some reasonable agreement through Barakat-Clark scheme and experimental works. Till to date, the solution is undefined due to lack of nonlinear solvers which will work on this higher order roots and will solve the viscous fingering model accurately.

A 2-D convective-diffusive equation is written as (Equation (8)):

$$\frac{\partial C}{\partial t} = D_x \frac{\partial^2 C}{\partial x^2} + D_y \frac{\partial^2 C}{\partial x^2} - u\frac{\partial C}{\partial x} - v\frac{\partial C}{\partial x} \tag{8}$$

The researchers used modified Barakat-Clark (CTD) scheme with the central time difference (Islam et al., 2016), and the initial and boundary conditions considered from Aboudheir et al. (1999). The formulation of the above equation (8) is as follows (Equation (9)):

$$\frac{Ca_{i,j}^{n+1} - Ca_{i,j}^{n-1}}{2\Delta t} = Dx\frac{Ca_{i+1,j}^{n} - Ca_{i,j}^{n} - Ca_{i,j}^{n+1} + Ca_{i-1,j}^{n+1}}{\Delta x^2} + Dy\frac{Ca_{i,j+1}^{n} - Ca_{i,j}^{n} - Ca_{i,j}^{n+1} + Ca_{i,j-1}^{n+1}}{\Delta y^2} - U\frac{Ca_{i+1,j}^{n} - Ca_{i-1,j}^{n}}{2\Delta x} - V\frac{Ca_{i,j+1}^{n} - Ca_{i,j-1}^{n}}{2\Delta y} \tag{9}$$

Here, the central time difference scheme created artificial oscillations, and the solution was only accurate for the first order and introduced a second order truncation error (Mathews, 1992). The Barakat-Clark (CTD) scheme presented the stability for time term and showed the accuracy for the time function in the order of Δt^4, while the Barakat-Clark (FTD) scheme showed approximately of the order of Δt^2. If the memory term is added into the equations, it leads higher order nonlinearity which is a very hard task for the linear solver to solve. One should take the initiative to continue research for better time and space accuracy in the convection-diffusion equation. Finally, an advanced nonlinear solver can tackle this situation, which is another research scope in the area.

2.4. Effect of fluid and media properties

In petroleum reservoir, the most basic fluid and media properties include permeability, density, viscosity, temperature, pressure, diffusivity, compressibility, surface tension, specific gravity and so on. These parameters are interconnected with each other, and for a smaller change, the reservoir estimation and performance would significantly change. For any governing equations, the equation must show nonlinear behaviours without considering assumption and follow the linearization techniques. From that point of view, a nonlinear solver would be able to handle higher complexities and higher roots nonlinear governing equations.

The memory is the only term in reservoir simulation which is well described by Islam et al. (2010, 2016), Hossain and Islam (2006, 2010a), Hossain et al. (2007, 2008, 2009c), and few other research groups (Slattery, 1967; Caputo, 1999, 2000, Caputo and fabrizio, 2015; Mustafiz et al., 2005, 2008a,b; Obembe et al., 2016a,b; and Obembe et al., 2017). Earlier, the present authors tried to highlight the memory effects concisely and showed its independency. On the other hand, several parameters including but not limited (i.e., permeability, density, viscosity, temperature, pressure, diffusivity, compressibility, surface tension, specific gravity, etc.) are discussed here. These parameters are interconnected with each other, but for a smaller change, the reservoir estimation and performance would significantly change. The basic difference is that sometimes memory will influence the reservoir simulation application independently and rest of the time it will act with other parameters.

From last few decades, several researchers tried to deal with nonlinear equations along with those properties. Slattery (1967) explained the viscoelastic behaviour of Newtonian and incompressible fluid and presented a fluid behaviour model, which was unrealistic due to the permeability effect. Based on previous research findings, Nibbi (1994) discussed the relationship between viscous fluids and free energies of linear viscoelastic fluid extensively for the quasi-static problem. Later, Broszeit (1997) showed the fluid deformation activities into simulation behaviour and attempted to describe and solve the stress related fluid problems. The researcher applied a single integral constitutive law, where the fluid kinematics were known and dealt with the numerical simulation of steady state isothermal flow for Newtonian fluids. However, the exact solution was not clearly present in their numerical simulation approach. Liu et al. (2003) also verified the analytical solutions of Navier-Stokes equation for fractured reservoirs by numerical simulation but failed to define the boundary conditions between the porous medium and the fracture. Later, Hossain et al. (2007) explained the viscoelastic fluid flow behaviour in porous media, which is an important variable for predicting oil flow and helps to understand the reservoir performances. The researchers worked on existing fluid flow models considering time and other fluid (e.g. viscosity, density, diffusivity, compressibility) and media (e.g. surface tension, porosity, permeability) properties (Hossain et al., 2009c). Further, some researchers showed intangible problems of memory and recognized the need for considering memory and other rock/fluid properties (e.g. stress, viscosity, surface tension, temperature, etc.) (Hossain et al., 2009c). After that, Hossain and co-authors were able to describe the effects of fluid variation and formation properties over time and proposed a model for describing the flow of the fluid in porous media (Hossain et al., 2008; Hossain, 2016a). Besides, Hassan and Hossain (2016), and Obembe et al. (2016a) reviewed the thermal displacement processes for an oil reservoir and presented the rock and fluid properties alteration with time. The authors (Hossain et al., 2008; Hossain, Unpublished results) modified the equation of motion by applying the memory concept in mathematical and computational models. The new form of the equation referred to the memory based diffusivity equation, which is a fully nonlinear integrodifferential equation. Here, an implicit-explicit finite difference method was used to obtain a mathematical formulation, where the researchers tried to avoid the singular form of the equation. The only significant featured was the properties of rock and fluid as a function of time. The solutions of the model equations are still inherent due to lack of nonlinear solver.

In addition, the modified momentum balance equation has presented by Hossain (2016) for 1-D oil reservoir, which helps to established the memory contribution in reservoir fluid flow through porous media. This equation was solved numerically using spline function and applied the trapezoidal rule to crosscheck the stability and accuracy of the solution. The time-domain and space-domain fractional order derivatives solutions were obtained from that numerical solution. Yet the spline function and trapezoidal rule are not suitable for such type of equation and the researcher is looking for a nonlinear solver to solve the equation with better accuracy.

The general form of the Darcy diffusivity equation (Hossain et al., 2008) is expressed as (Equation (10):

$$\frac{\partial^2 p}{\partial x^2} = \frac{\phi \mu_o c_t}{k}\frac{\partial p}{\partial t} \tag{10}$$

With the inclusion of rock and fluid memory, Hossain et al. (2008) showed modified Darcy's law in 1-D reservoir (Equation (11)):

$$u_x = -\beta_c \eta \left\{ \left[\frac{1}{\Gamma(1-\alpha)}\right] \int_0^t (t-\xi)^{-\alpha} \frac{\partial}{\partial \xi}\left(\frac{\partial \phi}{\partial x}\right) \partial \xi \right\} \tag{11}$$

The researcher took the range of α from 0 to 1, and equation (11) looks a nonlinear form. Later, Hossain (Unpublished results) applied engineering approach for the development of memory-based diffusivity equation and expressed the rock and fluid properties as a function of time as shown in equation (12).

$$\begin{aligned} &T_{x_{i+\frac{1}{2}}} p_{i+1}^{n+1} - \left\{ T_{x_{i-\frac{1}{2}}} + T_{x_{i-\frac{1}{2}}} + \frac{V_{b_i}}{\alpha_c \Delta t}\left(\frac{\phi c_t}{B}\right)_i \right\} p_i^{n+1} + T_{x_{i-\frac{1}{2}}} p_{i-1}^{n+1} \\ &= T_{x_{i+\frac{1}{2}}} p_{i+1}^{n} - \left\{ T_{x_{i+\frac{1}{2}}} + T_{x_{i-\frac{1}{2}}} + \frac{V_{b_i}}{\alpha_c \Delta t}\left(\frac{\phi c_t}{B}\right)_i \right\} p_i^{n} + T_{x_{i-\frac{1}{2}}} p_{i-1}^{n} - q_{sc_i} \\ &\quad - T_{x_{i+\frac{1}{2}}} \sum_{k=1}^{n} b_k^{(1-\alpha)} \left\{ \left(p_{i+1}^{n+1-k} - p_{i+1}^{n-k}\right) - \left(p_i^{n+1-k} - p_i^{n-k}\right) \right\} \\ &\quad - T_{x_{i-\frac{1}{2}}} \sum_{k=1}^{n} b_k^{(1-\alpha)} \left\{ \left(p_{i-1}^{n+1-k} - p_{i-1}^{n-k}\right) - \left(p_i^{n+1-k} - p_i^{n-k}\right) \right\} \end{aligned} \tag{12}$$

which is the final form of the following equation (13):

$$\begin{aligned} &\left[\frac{\beta_c \eta A_x}{B}\left\{\frac{\partial^\alpha}{\partial t^\alpha}\left(\frac{\partial P}{\partial x}\right)\right\}\right]_{x_{i+\frac{1}{2}}} - \left[\frac{\beta_c \eta A_x}{B}\left\{\frac{\partial^\alpha}{\partial t^\alpha}\left(\frac{\partial P}{\partial x}\right)\right\}\right]_{x_{i-\frac{1}{2}}} + q_{sc_i} \\ &= \frac{V_{b_i}}{\alpha_c \Delta t}\left(\frac{\phi c_t}{B}\right)_i \left(p_i^{n+1} - p_i^{n}\right) \end{aligned} \tag{13}$$

The equation (12) is the discretized form of diffusivity equation, where the boundary conditions applied to reduce nonlinearity and performed a numerical experiment to validated the model equation. Here, the author provides only the outline of the solution of model equation instead of an exact solution. Yet, the solution is invisible which might be solved by a nonlinear solver.

2.5. Effect of time step size

The effects of time step to computational accuracy are generated due to the use of different time steps on the simulation system. The calculation of time step effects started from formulation level to solution level. The results of relative percentage error are one way to express the time interval implications in the reservoir simulation, where the relative error calculated based on pressure, temperature, or water saturation values at different time steps. To validate the models, time steps must consider for long-term simulation (Islam, 2008). In addition, the Newton's method is not always guaranteed to converge for large time steps, due to the nonlinearity of conservation equations (Aziz and Settari, 1979).

Ertekin et al. (2001) found significant changes in the percentage of relative error due to the implementation of time step in the simulation system. The gradual reduction of error observed during each iteration steps and made the convergence rate slower. After that observation, Mustafiz et al. (2008a) investigated the time effects for compressible fluid in a single-phase flow problem. They conducted their experiment on Darcy's law which is described earlier by Abou-Kassem et al. (2006). The researchers collected pressure responses for different time intervals using the interpolation of cubic spline function and continuous function. The time effect was more sensitive for the results of continuous function rather than spline function, and the pressure drop also increased with increasing the time steps. In fact, for short time interval, the relative error percentage accuracy was less in a single-phase flow system.

Based on Ertekin et al. (2001) iteration process, Younis et al. (2010) presented a nonlinear iteration process with a converging time step and developed ideas to address nonlinear solver issues in reservoir simulation. The researchers demonstrated the robustness and computational efficiency of their proposed iteration method and the solutions were more attainable than the standard Newton's method. Also, they interpreted an algorithm for the examples of single-phase implicit residual system at different time steps by following the Newton's method. Later, Li (2014) solved the nonlinear equations using the standard Newton's method which arise from the fully implicit discretization of fluid flow in porous media. The researcher formulated, verified and analyzed the computational efficiency of a new nonlinear solution technique for a given time step size, compared with Sequential-Implicit Method and Newton-based iterative method. That solution scheme had a lower computational cost without compromising the allowable time step size in each iteration. However, the proposed algorithm was full of rigorous analysis, and only nonlinear solvers could solve the order arbitrarily.

After that, Wang and Tchelepi (2013) described the time steps effects for immiscible two-phase transport in porous media by a nonlinear solver (called a trust-region solver), where viscous, buoyancy and capillary forces were deemed significant. They highlighted the flux function (F) as a nonlinear function of saturation, which was the primary source of complexity for nonlinear solvers of coupled multiphase flow and transport in porous media. The authors also described a modified Newton algorithm method and showed that two successive iterations cannot cross any trust region boundary, and demonstrated significant extension of the inflection-point strategy for viscous dominated flows. They analyzed the discrete nonlinear transport equation using finite-volume discretization with a phase-based upstream weighting system. Later, the convergence was numerically proved for the trust region Newton method irrespective of the time step size for single-cell problems. However, they tried to overcome the limitations by analyzing the larger heterogeneous reservoir models with the proper time step size and developed the performance by reducing overall computational cost.

Consequently, Li and Tchelepi (2014) developed a convergent nonlinear solution technique for immiscible multiphase flow and transport in heterogeneous porous media where numerical performance was much superior to previous nonlinear solution methods and bypassed the convergence failure effectively. They also highlighted about convergences difficulties (i.e. erratic time stepping, greater number of Newton's iteration steps and time steps interval), and achieved solutions for arbitrary time step sizes followed by Newton's iterations method. The researchers studied their proposed solution technique for oil/gas recovery, groundwater remediation, and carbon-di-oxide geological sequestrations related to large-scale problems in the presence of viscous, buoyancy, and capillary forces. In addition, Li and Tchelepi (2015) developed another convergent nonlinear solution technique to analyze the discrete nonlinear transport (i.e. mass conservation) equations for immiscible, incompressible, multiphase flow transport in porous media, considered with the time step size and improved the computational speed of numerical simulations. They analyzed the nonlinearities in heterogeneous domains across the viscous, buoyancy, and capillary forces in detail, which dominate the transport dynamics. The authors also pointed out heterogeneities in the capillary pressure–saturation relationship extensively using their proposed numerical solution scheme. Recently, Hamon and Tchelepi (2016) presented nonlinear ordering based solving techniques to reduce the number of iterations significantly, which lead to avails in the entire computational cost and promoted the vigour of the potential based ordering method in the presence of gravity. They introduced the Fully Implicit Method (FIM) for the temporal discretization for multiphase flow in porous media and solving large coupled systems of nonlinear algebraic equations instead of Newton-based iterations method (Ortega and Rheinboldt, 1970; Deuflhard, 2004). The researchers also extended their nonlinear approach to interphase (i.e. between liquid and gas) mass transfer with the function of pressure and composition, where the algorithm deals accurately. The detailed comparisons of the robustness and efficiency of the potential nonlinear and linear solvers for immiscible two-phase (dead oil), black oil, and compositional problems also presented. Still the researchers are working on efficient solution techniques where they utilize the time step size for each grid blocks in a microscopic way and saves the computational time in the modern reservoir simulation systems.

2.6. Nonlinearity from the modelling of gas flow in reservoir

In gas reservoirs, modelling of pore size distribution and understanding the pore structure of the formation are necessary for fluid flow measurement and hydrocarbon estimation. The recovery of most reservoirs is highly dependent on the pore structure characteristics (Clarkson et al., 2013; Hu et al., 2012; Josh et al., 2012; Kuila and Prasad, 2013; Wang et al., 2014; Lin et al., 2015). To explain this issue, many researchers investigated the pore characteristics of the formation but failed due to the lack of proper mathematical models (Clarkson et al., 2013, 2013; Hu et al., 2012; Josh et al., 2012; Kuila and Prasad, 2013; Wang et al., 2014). Lin et al. (2015) proposed a numerical fitting model method for estimating the pore volume of the shale formation under reservoir conditions and analyzed the samples in qualitative and quantitative ways. Yet, the researchers are working continuously to represent a complete scenario of pore volume structure and its suitability in gas flow modelling.

The residual natural gas saturation also affected the multiphase flow behaviour in gas reservoirs, and it would be more challenging during the modelling of multiphase flow characteristics (Caudle et al., 1951; Holmgren and Morse, 1951a,b; Gonzalez et al., 2007; Idem and Ibrahim, 2002; Roman et al., 2008, 2009). Sometimes, the pressure responses in fractured (i.e., symmetrical, asymmetrical, longitudinal, transverse, vertical, incline, etc.) gas reservoir affect the fluid flow models (i.e., analytical, semi-analytical or numerical) and exhibit the nonlinear behaviour (Crosby et al., 2002; Wan and Aziz, 2002; Al-Kobaisi et al., 2006; Zhu et al., 2007; Lin and Zhu, 2010; Rbeawi and Tiab, 2013; Huang et al., 2015). The situation would be more complicated while the shale gas moved from the tight gas reservoirs to using various fracture network models (Bustin et al., 2008; Gong et al., 2011; Firoozabadi, 2012; Swami et al., 2013; Kudapa et al., 2017). Therefore, the researchers and engineers need to solve the highly nonlinear models. They need to obtain the fluid flow rate from the matrix to the wellbore before proceeding the productions.

2.7. Nonlinearity in EOR applications

The exact flow regime in EOR applications should be determined through the accurate mathematical or numerical models. A well designed EOR method such as miscible gas injection also provides the exact phase behaviour for multi-components fluid systems (Islam et al., 2016). The solving of nonlinear governing equations for an unstable flow regime will carry significant value. The chances of numerical errors would be increased without any rigorous solution techniques. Moreover, governing equations could become more and more nonlinear because of the carbon-dioxide (CO_2) sequestration, the presence of non-Newtonian fluid flow, injecting gas and fluids for recovery, thermal-flooding, fractal permeability and porosity, phase changes during the measurement of rock and fluid

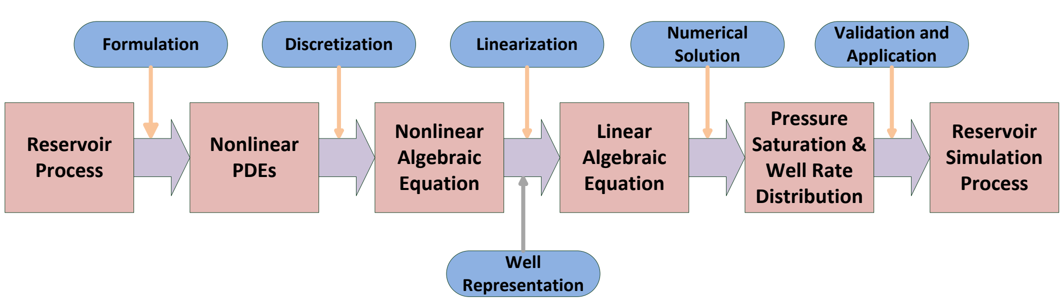

Figure 2. Major steps of the present reservoir simulator (modified after Hossain, 2012).

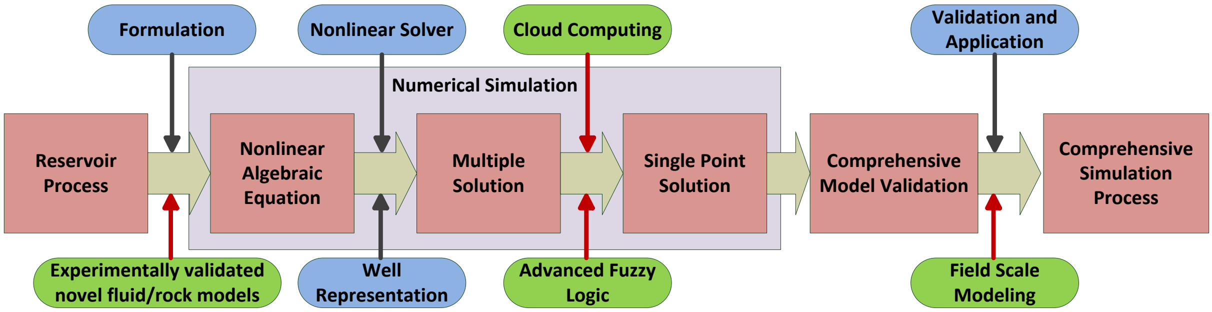

Figure 3. Future simulator steps for the reservoir simulation.

properties, absorption and desorption during flooding mechanism etc (Mungan, 1992; Özkılıç, and Gumrah, 2009; Farajzadeh et al., 2009; Gogoi, 2011; Ju et al., 2012; Sheng, 2010, 2015; Jang et al., 2014; Patacchini et al., 2014; Wang et al., 2015; Sun et al., 2017). Sometimes, the researcher found semi-analytical solutions for those governing equations which may be solved by a nonlinear solver using advanced numerical simulations (Wentao et al., 2012; Cossio et al., 2013; Wang et al., 2015).

3. Outline of present simulator steps, challenges and solution techniques

3.1. *Present simulator steps*

In the petroleum reservoir simulation, several numerical techniques used to find the exact and approximate solutions of the governing equations. Most of them ended up with unique solutions. However, due to the computational advancement, it is the time to looking forward for multiple solutions and/or a cloud of solutions with high computational efficiency. This target may be achieved through an efficient nonlinear solver or a new scheme. According to Islam et al. (2016), the ADM was successfully used to solve nonlinear PDEs (e.g. Buckley-Leverett's equation) and generated solutions in single-phase flow system.

Based on Ertekin et al. (2001) and Hossain (2012), the outlook of the current simulator steps is shown in Fig. 2. In formulation stage, the simulator describes the governing nonlinear PDEs along with underlying assumption and mathematical terms. The equations are discretized by following time discretization and generates a set of nonlinear algebraic equations for the choices of different time steps (i.e. old-time step, intermediate time step or new time step). Using the Taylor series expansion, the nonlinear algebraic equations turned into linear algebraic equations and defined the production and injection well for a petroleum field. To get the solutions, different numerical techniques is applied to these equations. Finally, an experiment needs to run for the validation of the model equations.

3.2. *Challenges of the present simulator steps*

Most of the governing equations are in nonlinear form and need to be solved to predict reservoir performance. The current commercial computer simulators (e.g. Eclipse, CMG Suite, Tempest MORE, ExcSim, Nexus, FlowSim, etc.) use linear solvers that produce solutions by linearizing nonlinear governing equations. Using mathematical approach, mathematicians provided solutions of the nonlinear equations by linearizing the governing equations and simplifying it with initial and boundary conditions (Ertekin et al., 2001).

Apart from the previous approach, Abou-Kassem et al. (2006); and Abou-Kassem (2007) presented an outline of engineering approach for solving the governing nonlinear equations. This approach showed the nonlinear equations in an integral form instead of the nonlinear algebraic equation using time and space discretization, which made the solution simpler than what mathematicians had done earlier (Fig. 3). Besides, Mousavizadegan et al. (2006) proposed ADM for solving nonlinear equations through computing the governing equations via engineering approach formulation. Hence, several researchers claimed that multiple solutions are achievable for the nonlinear governing equations without linearization using ADM technique (Mustafiz and Islam, 2008; Mustafiz et al., 2008a; Islam et al., 2010; 2016). In addition, Islam et al. (2016) raised concern about the single point solution because cloud computing solution for reservoir can be achieved even by advanced fuzzy logic theory. The approximate solutions are close to exact solutions but not achievable due to the lack of nonlinear solvers.

As further elaborated by Islam et al. (2016), these solutions without linearization revealed a number of key observations, such as (i) a broad range of operating parameters for which the nonlinear solvers predict results remarkably different from those predicted by linear solvers; (ii) the possibility of multiple solutions inherent to the reservoir simulation problems; and (iii) linearization of governing equations likely to divert subsequent results, hence biasing the decision-making process irreversibly.

3.3. *Solution techniques for nonlinear algebraic equations*

From the beginning of the simulation, the researchers used different numerical methods for solving nonlinear algebraic equations in a fluid flow system. Most of those techniques consist of various assumptions, which influence the governing equations directly and make it simple linear equations. Some of them are iterative based by linearizing the governing equations, and the remainders are the phase-based solution. Only few researchers tried to apply advanced numerical techniques (e.g. ADM, Implicit Pressure Explicit Saturation (IMPES), Multilevel Nonlinear Method (MNM), Nested Iteration, etc.) without eliminating assumptions, which gives reliable results rather than conventional solution techniques (e.g. Newton's Method, Secant Method, Finite element method (FEM), Jacobi method, Relaxation method, Gauss-Seidel method, Alternating-Direction Implicit Procedure (ADIP), Iterative ADIP, Linearized-Implicit method, etc.) (Crichlow, 1977). Here, we briefly narrate some standard techniques to solve nonlinear equations in reservoir simulation.

3.3.1. *Techniques for linear equations*

From the very beginning of petroleum reservoir simulation, several numerical methods have been applied to get an accurate solution, which influenced the operations, forecasting

Table 4
Summary of the linear equations solution techniques.

Methods	Features	Limitations	References
Matrix inversion	• Straight forward process • Better elimination quality • Real-time simulation method	• Very cumbersome method • Usable for finding unique solution	Crichlow (1977); Nash (1990); Lipschutz (1991)
Cramer's rule	• Explicit type solution method • Systemic and rapid solution • Determinants of the matrices will never zero value • Only applicable when coefficient of the matrices is square	• Numerically unstable • Computationally disable • Concern with trivial parameters	Hoffman and Frankel (2001); Higham (2002); Shores (2007); Habgood and Arel (2012)
Gaussian elimination	• Efficient algorithm • Two-step calculations and unknown parameters obtain from second step algorithm • Determine coefficient of the matrices sequentially	• Handle only band matrices problem • Do not find approximate solution • Numerically unstable for large number of equations and unknowns	Gentle (1998); Marc and Seymour (2001); Higham (2002); Grcar (2011)
Gauss-Jordan method	• Directly obtain identity matrix • Reduced order elimination method • Results obtain from the matrix without back substitution	• Increases the chance of round-off errors	Crichlow (1977);Shores (2007); Higham (2002); Grcar (2011)
Matrix decomposition	• Transform the larger matrices problem into smaller matrices • Helpful for singularities in a problem • Better solution comes from linear algebraic equations	• Only follow the back-substitution rules • Generate erroneous solutions	Choudhury and Horn (1987); Meyer (2000); Townsend and Trefethen (2015)
Jacobi method	• Determine the solutions from diagonal matrices system of linear equations • Iteration is continuing until it converges.	• Required well-conditioned linear system	Bronshtein and Semendyayev (1997); Saad (2003); Yang and Mittal (2014)
Relaxation method	• Introduce a relaxation parameter to accelerate the convergence • Improve the quality of solution • Solve PDEs by splitting and iterating until solution is found	• Slow convergence • High computational cost	Jeffreys and Jeffreys (1988); Ortega and Rheinboldt (2000); Goffin (1980); Richard (2002)
Gauss-Seidel method	• Stable method • Faster converges than Jacobi method • Require current approximation for unknown vector	• Find only one solution • High computational cost	Crichlow (1977); Jeffreys and Jeffreys (1988); Richard (2002)

performances and computational costs of petroleum fields. These numerical methods applied to the nonlinear algebraic equations by following two ways: (i) Direct processes (i.e.: Matrix inversion, Cramer's rule, Gaussian Elimination, Gauss-Jordan method, Matrix decomposition, etc.); and (ii) Iterative Processes (i.e.: Jacobi method, Relaxation method, and Gauss-Seidel method). In direct processes, the solution of the system of equations is obtained after the completion of a fixed number of operations. The methods are also easily compatible with pressure equations in simulation, and the algorithms are reasonably efficient. However, it has the higher possibility of round-off error and required a significant amount of computational labour (Crichlow, 1977). In contrary, the iterative processes are more efficient, faster convergence, and provide greater levels of accuracy. Here, the solution generated after a systematic computation of solution approximations at each iteration steps. The important features of both processes are critically analyzed and reported in Table 4. However, both processes increase the ease of complexity and number of errors, time-steps and computational speed for all types of reservoir simulation nowadays (Ertekin et al., 2001). Due to technological advancements and higher demands of petroleum products, the simulation models are more complex higher order equations, considering with all rock and fluid properties (i.e., porosity, permeability, water saturation, pressure-temperature distribution, etc.) and the linearized solving techniques are not usable here. Therefore, the researcher is looking for more accurate, stable and convergent numerical solution methods or new scheme to solve these higher order equations.

3.3.2. Techniques for nonlinear equations

Researchers studies some techniques (i.e., IMPES, Sequential Implicit method (SIM), Standard Newton's method, Newton's method with heuristic solution, Continuation Newton, Ordering-based methods, ADM, MNM, Deflation-Nested Iteration (NI) Method, etc.) to avoid the higher order complexity, rapid convergence and saving of the computational time by nonlinear solvers. These nonlinear solvers are more reliable, accurate and stable than previous linearized solution techniques (Nordbotten et al., 2005b; Yavneh and Dardyk, 2006; Kwok and Tchelepi, 2007; Mustafiz et al., 2008b; Younis et al., 2010; Younis, 2011; Adler et al., 2017). Sometimes, the analytical solutions are failed to match with exact solutions. This challenge can be addressed by the numerical techniques along with field data interpretations and validation. The approximate solutions have been improved and reduce the problems of previously linearized solution techniques, considering with local and global convergence features, solution accuracy and stability. Here, we summarised the numerical solution techniques concerning the basic principles, advantages, limitations and significant features related to the reservoir simulation.

3.3.2.1. IMPES and SIM methods. Aziz and Settari (1979) used SIM to solve the black oil model equations for old time step by solving pressure and computed a new total velocity field based on the pressure. The solution of pressure equations is conveniently separated in sequential solution, which permits approximation of saturation equations (Peaceman and Rachford, 1955). After that, they solved the transport problem implicitly considering the total velocity field. Then the FIM made both the saturation and pressure variables implicit with respect to time and generated a solution of the nonlinear algebraic equations. Later, Coats (2000) introduced the IMPES method where the saturation variables are explicit in time, and the pressure variables are implicit in time. IMPES is inversely proportional to the largest fluid velocity in the reservoir due to its stability limit, and this limit is often too restrictive in practice. Later, the IMPES stability was derived by Coats (2001) for multidimensional black oil and compositional models for three-phase flow. The truncation error was smaller in IMPES (Coats, 2001; Coats, 1968; Bansal et al., 1979). On the other side, the SIM did not suffer stability problems like IMPES due to the absence of capillarity. The General Purpose Reservoir Simulator (GPSIM) used SIM (Spillette et al., 1973) for more computational efficiency (Duuglas et al., 1959).

3.3.2.2. Standard Newton's method. This is an iterative scheme where a sequence of iteration generated by solving a nonlinear system of equations, obtained from the linearization of the original nonlinear system. At each Newton iteration, this method solves:

$$J^{\nu}\delta x^{\nu} = -R^{\nu}$$

where, J denotes the Jacobean matrix, representing the derivatives of the residual to unknowns, and δx^{ν} is the vector of Newton updates. Thus, the sequence of iteration generated starting from the beginning can be written as:

$$x^{n+1,0} = x^{n} \tag{14}$$

$$x^{n+1,\nu+1} = x^{n+1,\nu} - J^{-1}R\left(x^{n+1,\nu};\Delta t, x^{n}\right), \nu = 0, 1 \tag{15}$$

This method is an explicit first-order time stepping scheme and the dynamical system is defined by Deuflhard (2004) and Younis (2011) as follows:

$$x^{n+1} = x^{n}, \quad \mathrm{v} = 0 \tag{16}$$

$$\frac{dx^{n+1}}{d\nu} = -J^{-1}R\left(x^{n+1,\nu};\Delta t, x^{n}\right), \quad \mathrm{v} > 0 \tag{17}$$

where, the Newton iteration index (v) is considering to be a continuous quantity. A first-order explicit discretization of equation (13) results in the following discrete form representing the dynamical system:

$$x^{n+1,\nu+1} - x^{n+1,\nu} = -\Delta\nu J^{-1}R\left(x^{n+1,\nu};\Delta t, x^{n}\right) \tag{18}$$

Comparison of Equation (15 and 17) exhibits that Newton's method approximates the derivative of the new state concerning the embedded time (ν) using a first-order finite-difference scheme with a unit step size, $\Delta\nu = 1$ (Deuflhard, 2004; Younis, 2011). Since explicit first-order time stepping may be unstable due to time step restriction, thus, Newton iterations may not converge even though the continuous Newton flows are well-behaved (Shahvali, 2012).

3.3.2.3. Newton's method with heuristic safeguards. In commercial simulators (e.g. Eclipse, CMG Suite, Tempest MORE, ExcSim, Nexus, FlowSim, etc.), the convergence behaviour of standard Newton's method has been improved by using heuristic solution methods; especially the buoyancy forces are influential in the fluid flow (Naccache, 1997; Nordbotten et al., 2005). In porous media, modified Appleyard chop algorithm (e.g. used in Eclipse™) is one of the most common heuristic approaches. It is a cell-based approach, and for any grid cell, Newton saturation iterates from old iteration levels to new iteration levels using that algorithm (Geoquest, 2005; Younis, 2011). The Buckley-Leverett's (1942) 1-D displacement problem is one of the examples, who followed modified Appleyard chop algorithm. The convergences behaviour improved and the saturation is changed from immobile to mobile for the ranges between 0 and 1. Here, the problem is discretized implicitly and separated it into 100 grid cells. The generated nonlinear problem is solved for one time-step size using standard Newton's method and Modified Appleyard chop algorithm. The results showed that standard Newton's method converged for time-steps less than 10^{-2} PVI (pore volumes injected), where the modified Appleyard chop converged from the beginning of the pore volumes injection time-step sizes (Younis, 2011; Shahvali, 2012).

3.3.2.4. Continuation Newton method. This method formulated by Younis et al. (2010) and Younis (2011), where the time step size considered for each iteration sequence and the result of the continuation-based solution process. At the initial iteration step, the time step was zero and convergence neighbourhood would have considered for increasing computational efficiency and avoiding the iteration solution path (Shahvali, 2012). The researcher developed the existing nonlinear methods by this approach and handled the current and upcoming challenges in physical nonlinearity in a reservoir simulator. The significant finding was removing converging difficulties by practicing time-step sizes, which increased the computational effort and performed the iteration for small time-steps (Younis, 2011).

For Buckley-Leverett Problem, the researcher calculated time-steps when the fractional flow was horizontal and when the fractional flow had affected due to gravity effects. In both cases, time-steps influenced the governing equations, and here, the continuation Newton method converges the solution considering with small time-step sizes. The results showed that the better approximation of solution generated for smaller time steps instead of larger time steps. In fact, the saturation changed sharply with distances and provided the worst approximation of governing equation for large time steps (Younis et al., 2010; Younis, 2011).

3.3.2.5. Ordering-based methods. Appleyard and Cheshire (1982); and Natvig et al. (2006) proposed an ordering based method to solve flow and transport equations. The approaches of the solution may cell-based (Cascade method and Natvig's method) or phase-based ordering method. Cell-based approaches considered when cells are rearranged along the direction of fluid flow and solve transport equations on a cell-by-cell basis sequentially. On the other hand, when ordering is performed on phase potentials due to capillarity effects, the method is known as the phase-based method.

Kwok and Tchelepi (2007); and, Natvig and Lie (2008) implemented this ordering based method for a set of multiphase flow equations in porous media, which is in nonlinear form due to gravity and capillarity effects. They also concerned with the linear solution step and investigated the advantages of the potential ordering methods for saturation variables on large heterogeneous problems. The researcher simplified the nonlinear algebraic equations for pressure dependence using the fully implicit method (FIM). Here, the saturation as a function of pressure and solved it for small time steps. This FIM method speeds up the solution of the nonlinear systems of algebraic equations implicitly. They also achieved real convergences for larger time steps using reduced-order Newton's method instead of standard Newton's method. Shahvali (2012) used the ordering-based methods for solving flow and transport equations and faced the gravitational challenges of the counter-current flow equations efficiently.

3.3.2.5.1. Cascade method. Appleyard and Cheshire (1982) proposed a cell-based ordering method, known as the cascade method, which accelerates the standard Newton's method and solves the conservation equations simultaneously for pressure and saturation in a two-phase flow. Kwok (2007) and Shahvali (2012) also proved the cascade method for an incompressible 1-D modelling problem when the counter-current flow was absent.

For N number of cells in the grid block, the researcher applied cascade method for evaluating the pressure values without saturation from the higher order cells to the lower order cells. This approach performed sweep activity, and for each cell phase, this method runs sequentially within each phase. Depending on the phases, the nonlinear equations would be generated for pressure and saturations. This method converges the pressure and saturations solutions in the presence of the local minima and counter-current flow in multiple dimensions. Due to the poor initial guess, the cascade method solution might not be converged in real fields application and failed to provide the guarantee of local minima existence (Kwok, 2007; Shahvali, 2012).

3.3.2.5.2. Natvig's method. Natvig et al. (2006), Natvig and Lie (2008), and Shahvali (2012) presented a solver method based on discontinuous Galerkin spatial discretization to solve hyperbolic transport equations when gravity and capillarity were absent. They solved the equations by applying an optimal reordering of grid cells on a cell-by-cell basis from upstream to downstream, using a standard Newton algorithm.

This cell-based ordering method used to solve the multiphase advection problem, when the gravity and capillarity have no influences. The equations generated from a discontinuous Galerkin discretization could be solved by applying this method. Sometimes, this approach is eligible to apply equally with standard FVM approach (Shahvali, 2012). The cell problems are consisting of multiple nonlinear systems and solved from upstream to downstream order by decoupling system. The current reservoir simulator also developed based on this sequential solution scheme and the computational efficiency is achieved for its unified cell-based applications. However, the convergence of the solution is no longer linear due to the presence of countercurrent fluid flow, and the robust implementation would be needed at that time. In addition, for higher order nonlinear problems, the solution procedure solely responsible for time-dependent problems and the sequential solution procedure is still undiscovered (Natvig et al., 2006; Natvig and Lie, 2008).

3.3.2.5.3. Phase-based potential ordering. Kwok and Tchelepi (2007) performed a rigorous mathematical analysis of 1-D problems, where the algorithm was always convergent due to the absence of counter-current flow and derived a reduced Newton algorithm for multiphase flow in porous media based on the new phase-based potential ordering. Before that, they presented an order of equations and unknowns based on phase potentials and showed that saturation dependence in the Jacobian method takes a lower-triangular form and solves one unknown at a time. They also extended this approach for counter-current flow due to gravity, using different orderings corresponding to different phases. On the other hand, reduced Newton algorithm converges only for the presence of counter-current flow and satisfaction of backward CFL conditions. This algorithm may cycle or diverge when the backward CFL number is greater than one (Kwok, 2007; Shahvali, 2012). However, this phase-based potential ordering might not have necessity in simulation for computing time-step when the fluid flow directions are unchanged. At that time, this approach used only for beginning time-steps counts and the researcher could validate that ordering easily (Kwok and Tchelepi, 2007).

For Buckley-Leverett equation, the phase based potential ordering method only working in the presence of capillarity due to the downstream water saturation. From that governing nonlinear equation, the pressure of one phase is evaluated concerning another pressure phase. Therefore, the researcher considered some precaution steps to avoiding the downstream dependency of water saturation. Also, the gas equations are solved in the last step of phase ordering due to the presence of countercurrent fluid flow and the gas components have coexisted in oil and gas phases (Kwok, 2007; Shahvali, 2012).

3.3.2.6. Adomian Decomposition Method. The ADM is one of the potential methods to solve nonlinear equations without considering any linearization steps or inherent assumptions and provides a way for generating multiple solutions (Adomian, 1984, 1986, 1991; Wazwaz, 2001; Wazwaz and El-Sayed, 2001; Gu and Li, 2007). For reservoir engineering, ADM is a very robust solution method for solving various equations including algebraic, integral, differential, integrodifferential, higher order PDEs and ODEs, etc. This technique determines the roots of a nonlinear parameter in the governing equations and solves the nonlinearity without considering linearization or unjustified assumptions of a problem. This method provides more analytic, verifiable and rapid convergent approximations than other numerical methods (Wazwaz, 2001; Wazwaz and El-Sayed, 2001). ADM handles the nonlinear problems with full strength with saving the computational time (Biazar and Ebrahimi, 2005). Following this technique, Mustafiz et al. (2008b) could demonstrate the real scenario of the 1-D governing equations for single-phase flow reservoir problems. The ADM solution profile is also validated through numerous experimental studies in petroleum reservoir simulation (Whitaker, 1986a,b; Mustafiz et al., 2005, 2008b; Mustafiz and Islam, 2005; Islam et al., 2010).

In the field of physics, mathematics, and medical research, ADM attracted the scientists and researchers for its deterministic and stochastic problem-solving capability, and for less computational time (Adomian, 1986). Using ADM, the governing equations replaced by a recursive relationship and the solution comes in the power series form. For a smaller region or grid block, the solutions converge rapidly and are more exact than any approximate solution. The accuracy of ADM solutions also depends on the boundary conditions of the problem and intervals lengths of the governing equation variables (Islam et al., 2016). However, in Buckley-Leverett (1942) analysis, ADM solved nonlinear part of this governing equation by using Adomian polynomial and obtained saturation values in terms of distance. Still, there is a scope to find out the multiple solutions from Buckley-Leverett's nonlinear governing equation, and ADM is not more feasible in that case (Mustafiz et al., 2008b; Islam et al., 2016).

3.3.2.7. Multilevel nonlinear method. The MNM is another technique for solving nonlinear problems in the applications of engineering, mathematics, physics or other branches of science (Yavneh and Dardyk, 2006). MNM technique is less dependent on the variables in a nonlinear problem and provides a good initial approximation when Newton's methods are limited. The procedure successfully implemented for 2-D phase complex problems and avoided the distinct behaviours which were found earlier from Newton's method and the Full Approximation Scheme (FAS). That technique eliminates the differences between global and local linearization, provides consistency through the scaling analysis, and increases the computational efficiency. Sometime coefficients are non-smooth in coarse grid approximations, and only MNM solve this by direct discretization (Hackbusch, 1985; Brandt, 1977, 1982; Yavneh and Dardyk, 2006).

MNM is capable to solve nonlinear equation by taking smaller nonlinear parts of the governing equation and provides sufficient convergence than Newton's iteration method and FAS (Yavneh and Dardyk, 2006). This technique is less effective for nonlinear problems in the presence of discontinues coefficients and irregular boundary values, which indicates a significant disadvantage of this approach. By examine the nature of the problem, the MNM is more capable to represents important conceptual and statistical limitations related to the area of the problems (Brandt, 1982; Yavneh and Dardyk, 2006).

3.3.2.8. Deflation-nested iteration method. To solve nonlinear governing equations in an efficient way, Nested Iteration (NI) is an excellent choice for nonlinear solvers. The solver used this technique for improving the computational efficiency and adopt exact solutions for specific problems by using deflation methodology (Farrell et al., 2015). Deflation method is an unsystematic approach, and the algorithm employed here for providing multiple solutions by modifying the nonlinear problems (Adler et al., 2015a, 2015b; 2017). Adler et al. (2017) presented the combined deflation NI method for computing the multiple solutions of nonlinear PDEs efficiently. The researcher performed few numerical simulations with the combined deflation NI algorithm on liquid crystals and found multiple solutions for nonlinear governing equations. They also investigated and demonstrated the algorithm performance and its accuracy for different physical phenomena (i.e., bifurcation and disclination behaviours). That powerful technique reduced the overall computational cost regarding the solution of nonlinear PDEs.

This technique works sequentially into a nonlinear problem and obtained distinct solutions from previous stages of the problem by eliminating the deflation iterations. After satisfying the local and global minima, the deflation approach provides highly effective solutions. Here, the nonlinear solvers integrate the deflation technique with an NI approach and improve the efficiency of the solution. This technique is mostly applicable in the branch of physics especially in liquid crystal problems where NI approach solved coarse and fine grid problems even in mild complexity problems. Using the Dirichlet boundary conditions, the multiple solutions are revealed sequentially considering with stable and unstable local minima and increasing the scope of solution for global minima. By blending the deflation and NI approach, the solutions are easily detectable for both coarser and fine mesh grids, and the set of initial guesses for deflation method are closer for each grid. This combination technique is still working on smaller grids problem and is challenging for higher order grid problems.

3.3.2.9. Legendre method. This method is used to solve nonlinear PDE problems for the first and second kind of order using the separation of variables. The singularities of a problem are defined using Legendre polynomials, and the solutions will generate for each domain points of that issue through symmetric or antisymmetric ways. This technique is mostly used for fractional derivative calculations, multiple expansions in physics, and trigonometry from where we get many eigenvalues (Beylkin et al., 1991). Especially, Legendre Wavelet method has presented a solution for the linear and nonlinear fractional and partial differential equations which is more accurate, stable, reliable and less computational efforts. The convergences and boundary conditions also verified through this technique. This method rapidly adopted with diverse fields of science and engineering, and transformed the boundary value problems into algebraic equations system (Razzaghi and Yousefi, 2001). The researcher considered only for calculus variations concerning mathematics fields and showed the validation of their proposed technique for Brachistochrone problem. The results

Table 5
Comparisons of different numerical techniques.

Schemes	Why scheme used?	How scheme used?	Strength of the scheme	Accuracy level of the scheme	Limitations
FEM (Schnipke, 1986; Chaskalovic, 2008; Zienkiewicz et al., 2005)	Extensively applicable for computational fluid dynamics, structural mechanics problems	Require boundary conditions naturally	Extremely powerful and useful for more complex problems	Quickly evaluate the PDEs solution at any point with high accuracy	Hard to follow calculation steps
	Smooth representation of the solution	Inclusion of dissimilar material properties	Local effects adoption	Accurate representation of complex geometry	Longer time needed
	Approximate solutions of PDEs are achievable	Subdivide large problem into smaller problems	Faster and less expensive scheme	Visualisation is possible in detail	Less physical significance
FDM (Schnipke, 1986; Chaskalovic, 2008; Zienkiewicz et al., 2005)	Used to find the values and the problem derivatives at discrete points	Boundary conditions required	Intuitive and easy to implement into PDEs	Accuracy is better than FEM due to short time applications	Solves simple PDEs and failed to interpolate the solution
	Very easy to implement on the problems	Problem discretizes into large number of cell/grid points	Lower order approximation within each cell	High accuracy	Increases round-off errors
	Solutions are more accurate due to multiple parameters	Solutions are achievable by Taylor series expansion	Less expensive	Observe better formulation	Only for rectangular geometry shapes
FVM (Schnipke, 1986; Versteeg and Malalasekera, 1995; Toro, 1999; LeVeque, 2002)	Calculate the average values of the conserved variables across the volume	Boundary conditions applied to PDEs	Structured mesh is not required	Especially influential on abrasive non-uniform grids	Need more effort to solve irregular shapes
	Represent PDEs into algebraic equations	Values calculated at discrete point on each small volume mesh	Simply formulated for unstructured meshes	Strongly applicable for discrete places on a meshed geometry	Need to aware of flux calculation
	Piece-wise linear variation may helpful for accuracy	Need to balance the fluxes across the boundaries of individual volume	Flux calculation at neighbouring domain provides accuracy		
EA (Islam et al., 2010, 2016;Mustafiz et al., 2008a,b)	New technique to solve PDEs with boundary conditions	Avoid formulations step	Solve nonlinear PDEs comprehensively	Provide accurate multiple solutions	Lack of nonlinear solver
	Provides physical interpretation of forward, central and backward differences for time derivatives	Nonlinear algebraic equations come from discretized nonlinear PDEs	Reduce the time steps and inherent assumptions of a model	Solutions are convergent, stable, and consistent	Selection of efficient numerical methods
	Algebraic equations are easily attainable	Solve PDEs through bypassing linearization	Solve higher order complex nonlinearities equation	Enhance the production and recovery of oil and gas fields	Field applications

Table 6
Pros and cons of different numerical methods used for solving nonlinear algebraic equations.

Techniques		Opportunities	Limitations
Newton–Raphson Method (Kelley, 2003; Lu, 2008)		• Used in heterogeneous reservoir properties simulation • Count small time steps • Explain nonlinearities in linearizing and iterative ways • Produces local convergence • Stable than IMPES method	• Time-consuming • Less accurate solver • Less robust technique • Not applicable for large time step simulation • Fails to obtain global convergence
Runge-Kutta Method (Mathews, 1992; Kumar and Unny, 2002)		• Easy implementation for ODEs • Very stable solution • Self-motivated method	• Limited solving capacity for PDEs • Larger computation time • Uncertain truncation errors • High computational cost
IMPES (Snyder and Ramey, 1967; Chen et al., 2006)		• Better stability • Require less computation time • Best fitted only for two-phase incompressible flow problems • Efficient for countercurrent problems	• Inefficient for larger nonlinearities problem • Applicable up to two-phase fluid flows
ADM (Adomian, 1991; Holmquist, 2007; Wang and Bajaj, 2006; Waewcharoen et al., 2008)		• Provide rapid convergent series solutions • Solutions are more realistic, which are comes from differential, delay-differential, integrodifferential, and partial differential equations. • A powerful method to obtain explicit and numerical solutions at a time. • Analytical solutions are achievable from deterministic and stochastic ODEs and PDEs. • Linearization is not required	• Exhibits small region of convergence • Uncertain to apply boundary value problems • Accuracy is dependent on the interval of the independent variable lengths (For example time)
Ordering-based methods (Appleyard and Cheshire, 1982; Shahvali, 2012; Natvig et al., 2006; Natvig and Lie, 2008; Kwok and Tchelepi, 2007; Kwok, 2007)	Cascade method	• Pressure and saturation equations are solving for two-phase flow • Used individually for each phase	• Not applicable in the presence of counter-current flow
	Natvig's method	• Solve equations when gravity and capillary pressure are absent	• Only solve hyperbolic equations • Time-consuming
	Phase-based potential ordering	• Applicable for counter-current flow problems • Reduce order Newton's method follows	• Only solved convergent problems
MNM (Yavneh and Dardyk, 2006)		• Suitable for 1D and 2D PDE's problem • Problem has been broken into steps or grids • Helpful for discontinuous and noisy problems	• Operation cost is more expensive than other methods • Need another simplify model for calculation
Deflation-NI (Adler et al., 2017; Tang and Vuik, 2008; Gambolati et al., 1995)		• Useful iterative technique • Efficient only for linear systems • Exhibit more inner properties and behaviour • Applicable for Symmetric Eigenproblems • Used for regular and irregular finite element grids	• Used for singular matrices only • Less stability • Limitations of lower-order scheme is undefined
Explicit methods (Chen et al., 2006)	Forward Euler method	• Self-adaptive time stepping techniques • Satisfied time step constraint • Computationally efficient • Solve equations in a linearized way	• Extensive computations and longer time needed to address a field scale model • Not applicable for strong nonlinearities problem • Can be unstable sometimes
Implicit methods (Duffy, 2004; Chen et al., 2006; Lu, 2008)	FIM	• Accurate method for linear solver • Stable method • Used for fractured reservoirs	• Implementation is hard • Higher computational cost
	Backward Euler method	• Reasonably stable • Applicable for complex quadratic equations	• Solution require for large nonlinear equations at each time step
	Crank-Nicolson method	• Better stability properties • Faster convergence • Time discretization is possible	• Need to choose the grid dimensions for removing the nonlinearity error's • Space discretization is not possible
	SIM	• Used for complex reservoir simulation problems • Solving equations implicitly without coupling • Suitable for the compositional and chemical compositional flow problems includes chemical components	• Less stable but more computationally effective than the SS scheme, and more stable but less efficient than the IMPES scheme • Can't solve the rapid change of capillary pressure problems
	Adaptive Implicit Scheme	• Efficient at the mid-level of IMPES and SS schemes • The resulting equations are more efficient and stable	• Used linearize and iterative equations • Time-consuming
Simultaneous Solution Method (SS) (Chen et al., 2006)		• Solves all coupled nonlinear equations simultaneously and implicitly • Stable technique and can handle enormous time steps with excellent stability • Suitable for the black oil and thermal models	• Analyse only a few components in the black oil and thermal models • Not appropriate for the compositional and chemical compositional flow problems
Legendre Wavelet Method (Beylkin et al., 1991; Razzaghi and Yousefi, 2001)		• Analyse orthogonality properties of a reservoir • Exact polynomials representation up to certain degree • Represents the functions with different resolution levels • Build fast connection with numerical algorithms	• Fail to provide analytical Solutions • Time sensitive with flow rate changes

were excellent due to the smooth implementation of that technique and reduced the solving system of nonlinear algebraic equations (Razzaghi and Yousefi, 2001). Later, Ablaoui-Lahmar et al. (2014) solved PDEs using Legendre wavelet decomposition method by considering spatial and temporal variables and reduced the time-dependent solutions into a set of ODE. They also highlighted the resolution of differential, linear integrodifferential and fractional differential equations; and optimal control problems.

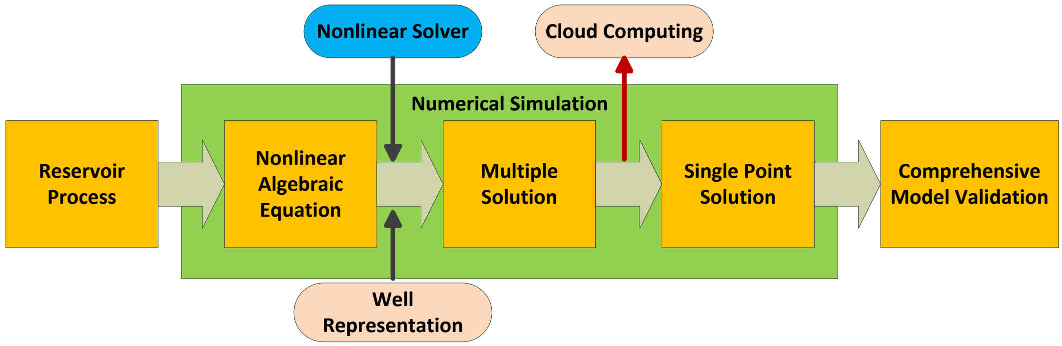

Figure 4. Future research scope in the reservoir simulation.

3.4. *Fundamental differences among widely used numerical methods*

In reservoir simulation, the governing equations coming from mathematical models that are solvable by various numerical and analytical methods. The solutions came from numerical methods are understandable, and the models are used to optimize the complex reservoir fluid flow processes. These applications have greatly expanded due to computational capabilities and provide a set of solutions for large problems (Chen et al., 2006). There are several methods incorporated into reservoir simulation to predict the multiple solutions of nonlinear governing equations of a complex reservoir (Peaceman, 1977; Aziz and Settari, 1979). Here, we summarised the fundamental methods (such as FDM, Finite Element Method (FEM), Finite Volume Method (FVM), Engineering Approach (EA), etc.) which are using to solve the nonlinear governing equations of the petroleum reservoirs (Table 5). Besides, the pros and cons of different numerical methods used for solving nonlinear algebraic equations are presented in tabular form (Table 6). Both of these tables influence the researcher for investigating more and more regarding numerical techniques selection and solve the nonlinear algebraic equations in the reservoir simulation.

4. Discussion and future directions

Nonlinearity is a challenging issue for the science and engineering fields. Most researchers were targeted to reduce the computational cost and increase the accuracy by reducing nonlinearity behaviours. To mitigate these problems, they utilize several numerical techniques by considering the different parameters and inherent assumptions. Based on the previous review on numerical techniques and problem descriptions, the researcher worked on this issue till now and tried to find more accurate and efficient solutions.

Among all the numerical techniques, the engineering approach and finite volume method will be adapted to solve the nonlinear algebraic equations, following by time and space discretization and the eventual solution. Specifically, ordering-based methods, deflation-nested iteration, and multilevel nonlinear methods provided a clear solution for a set of nonlinear algebraic equations. The ADM technique also tried to solve the nonlinear equations, but after some certain steps, that method failed to provide the multiple solutions. Thus, the researcher is looking for a stable, consistent and accurate solution, which is raising the task for the nonlinear solver. The challenges for the future reservoir simulation steps is shown in Fig. 4. From this figure, the researcher will pick the nonlinear algebraic equations, which is discretized for time function. The nonlinearity is starting from that point and using an efficient numerical scheme; the solver will solve the equations and provides a set of multiple solutions. The principle focus of this research is to bypass the linearization of nonlinear algebraic equations and reduces assumptions from the governing equations. Later, the solutions will be stored in the reservoir cloud system and validate comprehensively by the future researcher. Moreover, the solver will check the stability, convergence, and accuracy of the solutions, and will find out the single solution for governing equations. Finally, this solution will work for developing a reservoir emulator and predict the future reservoir performance in the petroleum industry.

5. Conclusions

This article presents critical reviews on nonlinearity problems and solution techniques in petroleum reservoir simulation. This study also accumulates the significance of nonlinearity by discussing stepwise nonlinear equation development, their solution strategy and the pros and cons for various solution schemes. The authors also point out the scope of the research in terms of governing equation parameters and solution scheme variables. Finally, the author shows the future research directions by sequential analysis of current reservoir simulator steps and addressing their challenges.

Declarations of interest

All authors have seen and approved the final version of the manuscript being submitted. We warrant that the article is original work, has not received prior publication and is not under consideration for publication elsewhere. Finally, the authors have no conflict of interest.

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NOMENCLATURE

1-D	One-dimensional
2-D	Two-dimensional
3-D	Three-dimensional
A_{xz}	Cross-sectional area of rock perpendicular to the flow of flowing fluid, m^2

A Area of the cross-section for a sample volume
a Parameter in Carreau–Yasuda model, dimensionless
B_o Oil formation volume factor, $m^3(std.m)^{-3}$
C Total compressibility of the system, $1/Pa$
c_f $c_0 + c_w =$ total fluid compressibility of the system, $1/Pa$
c_t $c_f + c_s =$ total compressibility of the system, $1/Pa$
c_s Formation rock compressibility of the system, $1/Pa$
c_w Formation water compressibility of the system, $1/Pa$
$\frac{du_x}{dy}$ Velocity gradient along y-direction, $1/sec$
$D_t^{1-\gamma}$ Grunwald-Letnikov (G-L) operator
E Activation energy for viscous flow, KJ/mol
F Wetting phase flux
f Fractional flow rate
g Gravitational acceleration, ft/sec^2
k Initial reservoir permeability, m^2
M_a Marangoni number
n Power-law exponent for Carreau–Yasuda model, dimensionless
p Pressure of the system, N/m^2
p_{i-1} Pressure of gridblock i-1, psia [kPa]
p_i Pressure of grid block i, psia [kPa]
p_{i+1} Pressure of grid block i+1, psia [kPa]
p_i^n Pressure of grid block i at time t^n, psi [kPa]
p_i^{n+1} Pressure of grid block i at time t^{n+1}, psi [kPa]
P_e Peclet number
p_c Capillary force: pressure difference between oil phase and water phase
q Total flow
R Universal gas constant, KJ/mole − K
S Wetting phase saturation
T Temperature, K
T Time, s
t Time, sec
$T_{x_{i+\frac{1}{2}}}$ Transmissibility between block i and i + 1
$T_{x_{i-\frac{1}{2}}}$ Transmissibility between block i and i - 1
u_x Fluid velocity in porous media in the direction of x-axis, m/s
V_{b_i} Bulk volume of grid block i, ft^3
y Distance from the boundary plan, m

Greek alphabets
α Fractional order of differentiation, known as Memory
α_c Volume conversion factor = 5.614583 for customary units or 1 for SPE preferred SI units
α_d Thermal diffusivity, m^2/sec
α_{SF} Shape factor which is medium-dependent
Γ Euler gamma function
ΔT $T_T - T_0 =$ Temperature difference, K
Δt Time step, s
Δx Grid block size, m
η Ratio of the pseudo-permeability of the medium with memory to fluid viscosity, $m^3 s^{1+\alpha}/kg$
λ Time constant in Carreau–Yasuda model, s
μ_0 Fluid dynamic viscosity at reference temperature, $T_0,\ Pa-sec$
μ_∞ Fluid dynamic viscosity at infinite shear rate, pa-s
ξ A dummy variable for time, i.e., real part in the plane of the integral, s
$\Delta\xi$ Dummy time step, s
ρ_0 Density of the fluid at reference temperature T_0, kg/m^3
σ Surface tension, N/m
$\left|\frac{d\sigma}{dT}\right|$ The derivative of surface tension α with temperature; can be positive or negative, depending on the substance, $N/m-K$
τ_T Shear stress at temperature T, Pa
ϕ Porosity of fluid media, m^3/m^3

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