LASTNOSTI DETERMINANT

① Če iz matrike A dobimo matriko B tako, da neko vrstico pomnožimo s številom λ, potem je $\det(B) = \lambda \cdot \det(A)$

$$A: \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ \lambda a_{i1} & \lambda a_{i2} & \cdots & \lambda a_{in} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = B = \sum_{\pi} \operatorname{sign}(\pi)\, a_{1\pi(1)} \cdots \lambda a_{i\pi(i)} \cdots a_{n\pi(n)}$$

Primer: $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ $\quad \det(A) = 2\cdot 5 - 3\cdot 4 = 10 - 12 = -2$

$B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \cdot 5 \sim \begin{bmatrix} 10 & 15 \\ 4 & 5 \end{bmatrix}$ $\quad \det(B) = 10\cdot 5 - 4\cdot 15 = 50 - 60 = -10 = 5\cdot(-2)$
$= 5 \cdot \det(A)$

$B' = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{matrix} \cdot 5 \\ \cdot 3 \end{matrix} \sim \begin{bmatrix} 10 & 15 \\ 12 & 15 \end{bmatrix}$ $\quad \det(B') = 10\cdot 15 - 12\cdot 15 = 150 - 180 = -30$
$= 5\cdot 3\cdot(-2)$
$= 5\cdot 3\cdot \det(A)$

② Če iz matrike A dobimo matriko B tako, da v matriki A zamenjamo 2 vrstici, potem je $\det(B) = -\det(A)$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & & & \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & & & \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix} \qquad B = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$\det(A) = \sum_{\pi} \operatorname{sign}(\pi)\, a_{1\pi(1)} \cdots a_{i\pi(i)} \cdots a_{j\pi(j)} \cdots a_{n\pi(n)}$$

$\Rightarrow$ v vsaki permutaciji (v vsakem členu vsote) imamo 1 inverzijo več

$\Rightarrow$ vsaki permutaciji se spremeni predznak

$\Rightarrow$ determinanti se spremeni predznak

Primer: $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ $\quad \det(A) = -2$

$B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \sim \begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix}$ $\quad \det(B) = 12 - 10 = 2 = -(-2) = -\det(A)$

③ Če iz matrike A dobimo matriko B tako, da poljubni vrstici matrike A prištejemo (ali odštejemo) nek večkratnik poljubne druge vrstice, potem je $\det(B) = \det(A)$.

Primer: $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ $\quad \det(A) = -2$

$B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} v_2 - 2v_1 \sim \begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix}$ $\quad \det(B) = 2(-1) - 3\cdot 0 = -2 = \det(A)$

④ Če je matrika A zgornje (ali spodnje) trikotna, potem je determinanta enaka produktu koeficientov na diagonali.

Primer: $B = \begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix}$ $\quad \det(B) = 2 \cdot (-1) = -2$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ & \emptyset & \ddots & \vdots \\ & & & a_{nn} \end{bmatrix}$$

z razvojem po 1. stolpcu:

$$\det(A) = a_{11} \begin{vmatrix} a_{22} & \cdots & & a_{2n} \\ 0 & a_{33} & \cdots & a_{3n} \\ \emptyset & & \ddots & \vdots \\ & & & a_{nn} \end{vmatrix} = a_{11} \cdot a_{22} \begin{vmatrix} a_{33} & \cdots & & a_{3n} \\ 0 & a_{44} & \cdots & a_{4n} \\ \emptyset & & \ddots & \vdots \\ & & & a_{nn} \end{vmatrix}$$

$$= \cdots = a_{11}\, a_{22}\, a_{33} \cdots a_{nn}$$

⑤ Če v matriki A vrstice NISO LINEARNO NEODVISNE, potem je $\det(A) = 0$.

Primer: $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$ vrstice niso lin. neodvisne, saj je $v_3 = v_1 + v_2$

SARRUSOVO PRAVILO (ker je 3×3 matrika)

$$\det(A) = \left|\begin{matrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{matrix}\right| \begin{matrix} 1 & 2 \\ 2 & 3 \\ 3 & 5 \end{matrix} = \underline{1\cdot3\cdot7 + 2\cdot4\cdot3 + 3\cdot2\cdot5} - \underline{3\cdot3\cdot3 - 1\cdot4\cdot5 - 2\cdot2\cdot7}$$

$$= 21 + 24 + 30 - 27 - 20 - 28$$

$$= 75 - 75 = 0$$

Posledica 1: Če ima matrika A dve vrstici enaki, potem je $\det(A) = 0$.

Primer: $A = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}$ $\quad \det(A) = 6 - 6 = 0$

Posledica 2: Če ima matrika A ničelno vrstico, potem je $\det(A) = 0$

Primer: $A = \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix}$ $\quad \det(A) = -0 \cdot 3 + 0 \cdot 2 = 0$

$\quad = 2 \cdot 0 - 3 \cdot 0 = 0$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix} \begin{matrix} \\ v_2 - 2v_1 \\ v_3 - 3v_1 \end{matrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & -1 & -2 \end{bmatrix} \begin{matrix} \\ \\ v_3 - v_2 \end{matrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Downarrow$ 2 enaki vrstici $\Rightarrow \det(A) = 0$

$\Downarrow$ ničelna vrstica $\Downarrow$ $\det(A) = 0$.

OPOMBA: Vse zgornje lastnosti veljajo tudi, če besedo "vrstice" zamenjamo z besedo "stolpci".

Primer: $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 1 & 2 & 5 \end{bmatrix}$ $\quad s_2 = 2s_1 \quad \overset{?}{\Longrightarrow} \quad \det(A) = 0$

I) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 1 & 2 & 5 \end{bmatrix} \begin{matrix} \\ v_2 - v_1 \\ v_3 - v_1 \end{matrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{matrix} \\ \\ v_3 - 2v_2 \end{matrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow$ ničelna vrstica

$\Rightarrow \det(A) = 0$

II) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 1 & 2 & 5 \end{bmatrix} s_2 - 2s_1 \sim \begin{bmatrix} 1 & 0 & 3 \\ 1 & 0 & 4 \\ 1 & 0 & 5 \end{bmatrix} \Rightarrow \det(A) = 0$

(6) $\det(A^T) = \det(A)$

Primer: $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \qquad \det(A) = -2$

$A^T = \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix} \qquad \det(A^T) = 2 \cdot 5 - 3 \cdot 4 = 10 - 12 = -2 = \det(A)$

(7) Naj bo B $n \times n$ matrika, potem je

$$\det(AB) = \det(A) \cdot \det(B)$$

Primer: $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -1 \\ 0 & -3 \end{bmatrix}$

$\det(A) = -2 \qquad \det(B) = 1 \cdot (-3) - 0(-1) = -3$

$\det(A) \cdot \det(B) = -2 \cdot (-3) = 6$

$AB = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} 2+0 & -2-9 \\ 4+0 & -4-15 \end{bmatrix} = \begin{bmatrix} 2 & -11 \\ 4 & -19 \end{bmatrix}$

$\det(AB) = 2 \cdot (-19) - 4 \cdot (-11) = -38 + 44 = 6 = \det(A) \cdot \det(B)$

PRIMER: Izračunajte determinanto matrike $A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ -2 & -2 & 3 & 3 \\ 1 & -2 & -2 & -3 \end{bmatrix}$.

I) z razvojem po 1. vrstici

$$\begin{array}{cccc} + & - & + & - \end{array}$$
$$\begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ -2 & -2 & 3 & 3 \\ 1 & -2 & -2 & -3 \end{vmatrix} = 0 - 1 \cdot \left|\begin{array}{ccc|cc} 1 & 1 & 1 & 1 & 1 \\ -2 & 3 & 3 & -2 & 3 \\ 1 & -2 & -3 & 1 & -2 \end{array}\right| + 2 \cdot \left|\begin{array}{ccc|cc} 1 & 1 & 1 & 1 & 1 \\ -2 & -2 & 3 & -2 & -2 \\ 1 & -2 & -3 & 1 & -2 \end{array}\right| - 3 \cdot \left|\begin{array}{ccc|cc} 1 & 1 & 1 & 1 & 1 \\ -2 & -2 & 3 & -2 & -2 \\ 1 & -2 & -2 & 1 & -2 \end{array}\right|$$

$= -1 \cdot (-9 + \cancel{3} + 4 - \cancel{3} + \cancel{6} - \cancel{6}) + 2 \cdot (6 + 3 + 4 + 2 + \cancel{6} - \cancel{6})$

$-3 \cdot (4 + 3 + \cancel{4} + 2 + 6 - \cancel{4})$

$= -1 \cdot (-5) + 2 \cdot 15 - 3 \cdot 15$

$= 5 + 30 - 45 = \underline{\underline{-10}}$

II) $\begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ -2 & -2 & 3 & 3 \\ 1 & -2 & -2 & -3 \end{bmatrix}$ $-$

$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ -2 & -2 & 3 & 3 \\ 1 & -2 & -2 & -3 \end{bmatrix} \begin{matrix} \\ \\ V_3+2V_1 \\ V_4-V_1 \end{matrix}$

$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 5 & 5 \\ 0 & -3 & -3 & -4 \end{bmatrix} \begin{matrix} \\ \\ :5 \\ V_4+3V_2 \end{matrix}$ $\cdot 5$

razvoj determinante po 1. stolpcu

$-1 \cdot \begin{vmatrix} 1 & 2 & 3 \\ 0 & 5 & 5 \\ -3 & -3 & -4 \end{vmatrix} \begin{matrix} 1 & 2 \\ 0 & 5 \\ -3 & -3 \end{matrix}$

$= -1 \cdot (-20 - 30 + 0 + 45 + 15 + 0)$

$= -1 \cdot (60 - 50) = \underline{\underline{-10}}$

$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 3 & 5 \end{bmatrix} \begin{matrix} \\ \\ \\ V_4-3V_3 \end{matrix}$

$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

$\det(A) = -5 \cdot (1 \cdot 1 \cdot 1 \cdot 2) = \underline{\underline{-10}}$

Kje vse nam je lahko determinanta v pomoč?

1) pri računanju vektorskega in mešanega produkta

$\vec{a} = (1,0,1)$, $\vec{b} = (0,-1,2)$, $\vec{c} = (2,1,3)$

Vektorski produkt $\begin{matrix} 1 & 0 & 1 \\ 0 & -1 & 2 \end{matrix}$

I) $\vec{a} \times \vec{b} = \left(\begin{vmatrix} 0 & 1 \\ -1 & 2 \end{vmatrix}, -\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} \right)$

$= (0+1, -(2-0), -1-0)$

$= (1, -2, -1)$

II) $\vec{a} \times \vec{b} = \begin{vmatrix} \overset{+}{\vec{i}} & \overset{-}{\vec{j}} & \overset{+}{\vec{k}} \\ 1 & 0 & 1 \\ 0 & -1 & 2 \end{vmatrix} \begin{matrix} \vec{i} & \vec{j} \\ 1 & 0 \\ 0 & -1 \end{matrix}$

$\vec{i} = (1,0,0)$
$\vec{j} = (0,1,0)$
$\vec{k} = (0,0,1)$

$\Rightarrow$ z razvojem po 1. vrstici

$= \vec{i} \begin{vmatrix} 0 & 1 \\ -1 & 2 \end{vmatrix} - \vec{j} \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}$

$= \vec{i} \cdot 1 - \vec{j} \cdot 2 + \vec{k} \cdot (-1) = 1\vec{i} - 2\vec{j} - 1\vec{k} = (1,-2,-1)$

$\Rightarrow$ po Sarrusu

$= 0\vec{i} + 0\vec{j} - 1\vec{k} - 0\vec{k} + 1\vec{i} - 2\vec{j}$

$= 1\vec{i} - 2\vec{j} - 1\vec{k} = (1,-2,-1)$

mešani produkt

$$(\vec{a},\vec{b},\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c} = (1,-2,-1)\cdot(2,1,3) = 2-2-3 = -3$$

$$(\vec{a},\vec{b},\vec{c}) = \left|\begin{array}{ccc} 1 & 0 & 1 \\ 0 & -1 & 2 \\ 2 & 1 & 3 \end{array}\right| \begin{array}{cc} 1 & 0 \\ 0 & -1 \\ 2 & 1 \end{array} = -3+0+0+2-2-0 = -3$$

$$(\vec{a},\vec{b},\vec{b}) = \begin{vmatrix} 1 & 0 & 1 \\ 0 & -1 & 2 \\ 0 & -1 & 2 \end{vmatrix} \quad \text{enaki vrstici} \Rightarrow \det = 0$$

2 obrnljivost matrike in računanje inverza

IZREK: $n\times n$ matrika A je obrnljiva, če in samo če je $\det(A)\neq 0$.

TRDITEV: Če je A obrnljiva matrika, potem je $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

Dokaz:

$$1 = \det(I) = \det(A\cdot A^{-1}) = \det(A)\cdot\det(A^{-1})$$

$$1 = \det(A)\cdot\det(A^{-1}) \quad \Big/ : \det(A)$$

$$\frac{1}{\det(A)} = \det(A^{-1})$$

Definicija: Naj bo $A = [a_{ij}]_{n\times n}$. Potem je ADJUNGIRANKA, $\operatorname{adj}(A)$ matrike A matrika velikosti $n\times n$ v kateri je ij-ti element enak kofaktorju A_{ji} matrike A.

Primer: Izračunajte $\operatorname{adj}(A)$, če je $A = \begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 3 \\ 2 & -3 & 4 \end{bmatrix}$.

$\underset{a_{11}}{A_{11}} = (-1)^2\cdot\begin{vmatrix} -1 & 3 \\ -3 & 4 \end{vmatrix} = -4+9 = 5$

$\underset{a_{12}}{A_{21}} = (-1)^3\cdot\begin{vmatrix} 2 & 1 \\ -3 & 4 \end{vmatrix} = -(8+3) = -11$

$\underset{a_{13}}{A_{31}} = (-1)^4\begin{vmatrix} 2 & 1 \\ -1 & 3 \end{vmatrix} = 6+1 = 7$

$\underset{a_{21}}{A_{12}} = (-1)^3\cdot\begin{vmatrix} 6 & 3 \\ 2 & 4 \end{vmatrix} = -(24-6) = -18$

$\underset{a_{22}}{A_{22}} = (-1)^4\cdot\begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix} = 4-2 = 2$

$\underset{a_{23}}{A_{32}} = (-1)^5\cdot\begin{vmatrix} 1 & 1 \\ 6 & 3 \end{vmatrix} = -(3-6) = 3$

$\underset{a_{31}}{A_{13}} = (-1)^4\cdot\begin{vmatrix} 6 & -1 \\ 2 & -3 \end{vmatrix} = -18+2 = -16$

$\underset{a_{32}}{A_{23}} = (-1)^5\begin{vmatrix} 1 & 2 \\ 2 & -3 \end{vmatrix} = -(-3-4) = 7$

$\underset{a_{33}}{A_{33}} = (-1)^6\begin{vmatrix} 1 & 2 \\ 6 & -1 \end{vmatrix} = -1-12 = -13$

$$\operatorname{adj}(A) = \begin{bmatrix} 5 & -11 & 7 \\ -18 & 2 & 3 \\ -16 & 7 & -13 \end{bmatrix}$$

IZREK: Če je A matrika velikosti $n \times n$, potem velja

$$A \cdot \mathrm{adj}(A) = \det(A) \cdot I_n = \mathrm{adj}(A) \cdot A$$

IZREK: Če je A obrnljiva matrika, potem je

$$A^{-1} = \frac{1}{\det(A)} \mathrm{adj}(A)$$

Dokaz: $A \cdot \mathrm{adj}(A) = \det(A) \cdot I$

$$A \cdot A^{-1} = A \cdot \left(\frac{1}{\det(A)} \mathrm{adj}(A) \right) = \frac{1}{\det(A)} A \cdot \mathrm{adj}(A) = \frac{1}{\det(A)} \cdot \det(A) \cdot I = I$$

Primer: Izračunajte inverz matrike $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

I) Gaussova metoda

$$A|I = \left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right] V_2 + \tfrac{1}{2} V_1 \sim \left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & -1 & 1/2 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \cdot 2 \sim \left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 3 & -2 & 1 & 2 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right] V_3 + \tfrac{1}{3} V_2$$

$$\sim \left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 3 & -2 & 1 & 2 & 0 \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{array}\right] \begin{array}{l} :2 \\ :3 \\ \cdot \frac{3}{4} \end{array} \sim \left[\begin{array}{ccc|ccc} 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{array}\right] V_1 + \tfrac{1}{2} V_2 \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 1 & -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{array}\right] \begin{array}{l} V_1 + \frac{1}{3} V_3 \\ V_2 + \frac{2}{3} V_3 \\ \end{array}$$

$$\frac{1}{2} + \frac{1}{2} \cdot \frac{1}{3} = \frac{3+1}{6} = \frac{4}{6} = \frac{2}{3}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{array}\right] = I \,|\, A^{-1}$$

$$\frac{2}{3} + \frac{1}{3} \cdot \frac{1}{4} = \frac{8+1}{12} = \frac{9}{12} = \frac{3}{4}$$

$$\frac{1}{3} + \frac{1}{3} \cdot \cancel{\frac{2}{3}}\, \frac{1}{2} = \cancel{\frac{3+2}{9}} = \cancel{\frac{5}{9}} = \frac{2+1}{6} = \frac{3}{6} = \frac{1}{2}$$

$$\frac{1}{3} + \frac{\cancel{2}}{3} \cdot \frac{1}{\cancel{4}_2} = \frac{2+1}{6} = \frac{3}{6} = \frac{1}{2}$$

$$\frac{2}{3} + \frac{\cancel{2}}{3} \cdot \frac{1}{\cancel{2}} = 1$$

II) $\det(A) = \left|\begin{array}{ccc} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{array}\right| \begin{array}{cc} 2 & -1 \\ -1 & 2 \\ 0 & -1 \end{array} = 8 + 0 + 0 - 0 - 2 - 2 = \underline{\underline{4}}$

$A_{11} = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4 - 1 = 3$ $\quad A_{21} = -\begin{vmatrix} -1 & 0 \\ -1 & 2 \end{vmatrix} = -(-2+0) = 2$ $\quad A_{31} = \begin{vmatrix} -1 & 0 \\ 2 & -1 \end{vmatrix} = 1 - 0 = 1$

$A_{12} = -\begin{vmatrix} -1 & -1 \\ 0 & 2 \end{vmatrix} = -(-2+0) = 2$ $\quad A_{22} = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 - 0 = 4$ $\quad A_{32} = -\begin{vmatrix} 2 & 0 \\ -1 & -1 \end{vmatrix} = -(-2+0) = 2$

$A_{13} = \begin{vmatrix} -1 & 2 \\ 0 & -1 \end{vmatrix} = 1 - 0 = 1$ $\quad A_{23} = -\begin{vmatrix} 2 & -1 \\ 0 & -1 \end{vmatrix} = -(-2+0) = 2$ $\quad A_{33} = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4 - 1 = 3$

$$\mathrm{adj}(A) = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix} \qquad A^{-1} = \frac{1}{4} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{bmatrix}$$