

Calc7

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Generation date: Thursday 3rd April, 2014, 13:07

Calc7 contains properties of $h(r)$ and $h_\alpha(r)$. **Contents**

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1 Remnant Masses M_*

In this writeup I collect the properties of two models $h(r)$ and $h_\alpha(r)$ in $D = n + 4$ dimensions.

$$h(r) = \frac{r^2}{r^2 + L^2} \tag{1}$$

$$h'(r) = \frac{2rL^2}{(r^2 + L^2)^2} \tag{2}$$

$$h_\alpha(r) = \frac{r^{3+n}}{(r^\alpha + L^\alpha/2)^{\frac{3+n}{\alpha}}} \tag{3}$$

$$h'_\alpha(r) = \frac{(n+3)L^\alpha r^{n+2} \left(\frac{L^\alpha}{2} + r^\alpha\right)^{-\frac{n+3}{\alpha}}}{L^\alpha + 2r^\alpha} \tag{4}$$

Let now $h(r)$ be a generic profile. I frequently derived the metric $g_{00} = 1 - V(r)$ for these profiles,

$$\rho(r) = \frac{M}{\Omega_{n+2}} \frac{dh(r)}{dr} \Rightarrow V(r) = \frac{2}{n+2} \frac{M}{M_*^{n+2}} \frac{1}{\Omega_{n+2}} \frac{h(r)}{r^{n+1}}. \tag{5}$$

Until now, L is just a constant. The Holography (1) and Self-Regular (3) profiles can be expressed as $h(r) = h(r/L)$, so L here is just a scaling factor for r .

1.1 Extremal radius

The remnant is the smallest possible Black Hole solution and considered as a stable particle that can no more evaporate. Self encoding solutions encode the remnant radius by it's degrees of freedom. Typically the (reduced) Planck Length is supposed to be equal to the remnant's size. Considering figure 3, the remnant equations require

$$\begin{cases} \partial_r|_{r=r_0} g_{00}(r) &= 0 \\ g_{00}(r_0) &= 0 \end{cases} \tag{6}$$

Finding the remnant : $g_{00}=1-V(r)$

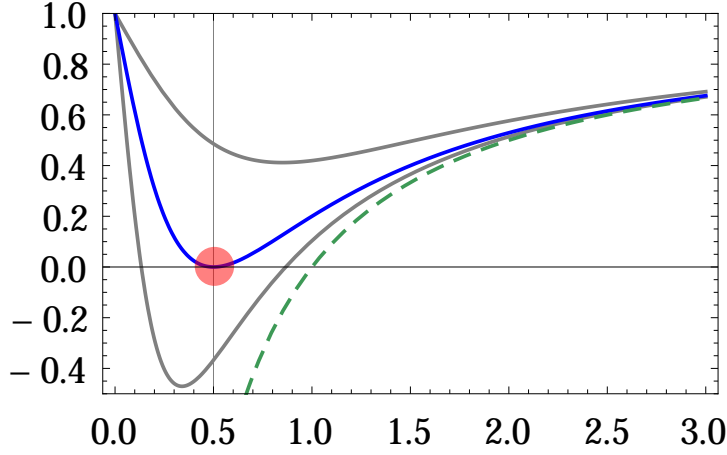


Figure 1: Extremal Configuration in regularized Schwarzschild metrics. Since $g_{00}(r) \rightarrow 0$ at $r \in \{0, \infty\}$, there must be an extremal r_0 . In the extremal configuration (blue), $r_0 = r_H$. There are also solutions with two r_H and $r_0 < 0$ and no $r_H, r_0 > 0$. The dashed line is the Schwarzschild behaviour.

For our models (eq 5), we can in general rewrite this set of equations in terms of $h(r)$. Especially, for finding the extremal value,

$$\begin{aligned} \partial_r V(r) = 0 &\Leftrightarrow 0 = \partial_r \frac{h(r)}{r^{n+1}} \Leftrightarrow 0 = -(n+1) \frac{h(r)}{r} + h'(r) \\ &\Leftrightarrow 0 = \left[L h'(z) - \frac{n+1}{z} h(z) \right]_{z=r_0/L} \end{aligned} \quad (7)$$

a substitution $z = r/L$ makes it easier to find a solution.

Finding the extremal radius r_0 (let $r_{0,\alpha}$ be for the self-regular model, r_0 for the holography model) is a bit of calculation. In the end, I got

$$r_0 = L \frac{1+n}{1-n} \quad (8)$$

$$r_{0,\alpha} = \frac{L}{(1+n)^{1/\alpha}} \quad (9)$$

1.2 Extremal value for the holographic model $h(r) = \frac{r^2}{r^2+L^2}$

Equation 8 tells us that there is *no extremum in higher dimensions*. At $n = 0$ the result coincides with [NS2012], at $n > 1$ all $r_0 < 0$. Actually this profile exhibits always a single horizon r_H as soon as $n > 1$. There is no more regularization for $n > 1$ which can be read off the metric:

$$V(r) \propto \frac{1}{r^{n+1}} \frac{r^2}{r^2+L^2} \propto \frac{1}{r^{n-1}} \frac{1}{r^2+L^2} \quad (10)$$

As soon as $n > 1$, there is a singularity at the origin. There is a special case in five dimensions ($n = 1$). A naked black hole at origin can be made with $L = 1$. etc.

1.3 Self-Encoding for $h_\alpha(r)$

The self regular solution looks like figure 3 in all dimensions. We can therefore always find a remnant and therefor fix α for any number of dimensions. Inserting r_0 to

$$1 \stackrel{!}{=} V(r_0) = \frac{2}{n+2} \frac{M(r_0)}{M_*^{n+2}} \frac{1}{\Omega_{n+2}} \frac{1}{r_0^{n+1}} \frac{h_\alpha(r_0)}{1} \quad (11)$$

After Calc6 (The surface issue), I think this equation needs some fixing. When $\frac{2\Omega_2}{\Omega_{n+2}}$ replaces the Ω fraction in equation 11, in the $n \rightarrow 0$ limit this gives a $\frac{2 \cdot 4\pi}{4\pi}$ which is needed to reproduce SMM correctly. Furthermore, let's remember that

$$M_P^2 = V_n M_*^{n+2}, \quad V_n = (2\pi R_c)^n, \quad \frac{M_P = 1/\sqrt{G}}{L_P = \sqrt{G}} = M_P^2 = \frac{1}{G}, \quad \Rightarrow \frac{M_*}{L_*^{n+1}} = M_*^{n+2} \quad (12)$$

Inserting $h_\alpha \left(L/(1+n)^{1/\alpha} \right) = \left(\frac{3+n}{2} \right)^{\frac{3+n}{\alpha}}$, I finally get

$$M(r_0) = \frac{1}{n+2} \left(\frac{3+n}{2} \right)^{\frac{3+n}{\alpha}} \left(\frac{r_0^{n+1}}{L_*^{n+1}} \right) \frac{\Omega_{n+2}}{\Omega_2} M_* \quad (13)$$

This looks very much like eq. 3 from [NS2012], expect the surface terms. The remnant's mass is equal to the reduced Planck mass if all coefficients get 1. This has multiple implications: We can determine α as well as identify $r_0 = L_*$, so the remnants radius is the reduced planck length. Eventually this means $L_P = (1+n)^{1/\alpha} L_*$. If I ignore the surface terms,

$$\alpha_0 = \frac{3+n}{\ln(2+n)} \ln \frac{3+n}{2} \quad (14)$$

which reduces exactly to $\alpha_0 = \frac{3}{\ln 2} \ln \frac{3}{2}$ found by [NS2012] in $n = 0$. Presumably they have to be taken into account, so

$$\alpha_0 = \frac{3+n}{\ln(2+n) - \ln \omega_n} \ln \frac{3+n}{2}, \quad \omega_n = \frac{\Omega_{n+2}}{\Omega_2} = \frac{\pi^{(n+3)/2}}{2\pi\Gamma((n+3)/2)} \quad (\text{Calc6, eq.5}) \quad (15)$$

This is fine, since $\omega_0 = 1$. I will use eq 15 for further calculations.

n	0	1	2	3	4	5	6	7
α	1.755	4.285	7.081	9.333	10.642	11.128	11.098	10.805
L_*	1.	0.851	0.856	0.862	0.86	0.851	0.839	0.825
M_*	1.	1.176	1.168	1.16	1.163	1.175	1.192	1.212
G_*	1.	0.724	0.733	0.743	0.739	0.725	0.704	0.681

Figure 2: Self-encoding horizon radius $r_0 = L_*$ and Remnant masses $M_* = 1/L_*$, in 4d Planck Units.

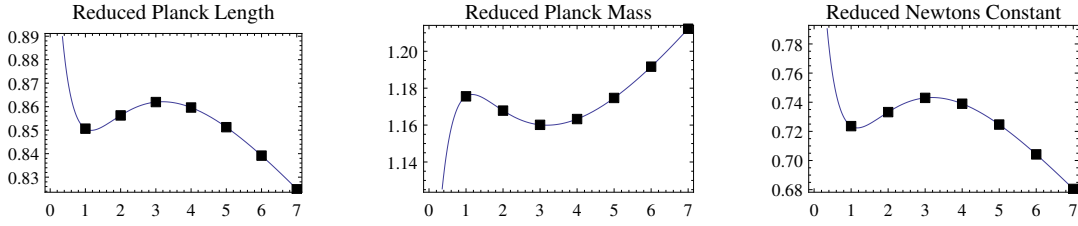


Figure 3: Table 2 as plot, from left to right: $L_*(n)$, $M_*(n) = 1/L_*$, $G_*(n) = L_*^2$

2 Hawking Temperature

Considering the h_α model with bigger masses $M > M_*$, the resulting Black Holes have multiple event horizons r_H . Solving $0 = g_{00}$ gives (compare eqn 11ff)

$$1 \stackrel{!}{=} V(r_H) = \frac{2}{n+2} \frac{M(r_H)}{M_*^{n+2}} \frac{1}{\Omega_{n+2}} \frac{1}{r_H^{n+1}} \frac{h_\alpha(r_H)}{1} \quad (16)$$

Eq 16 gives us the mass $M(r_H)$ of a BH with event horizon r_H in the same way as eq 13 told that for $M(r_0)$. Then I determine the Hawking Temperature $T_H = \frac{1}{4\pi} \left(\frac{dg_{00}}{dr} \right)_{r=r_H}$, plugging in $\alpha = \alpha_0$, $M = M(r_H)$. Let

$$V(r) = AM(r) \frac{1}{r^{n+1}} h_\alpha(r) \quad (17)$$

$$\Leftrightarrow M(r_H) = \frac{r_H^{n+1}}{A h_\alpha(r)} \quad \text{like in eq 13} \quad (18)$$

$$\frac{dg_{00}}{dr} = (n+1) \frac{M}{r^{n+2}} h_\alpha(r) - A M \frac{h'_\alpha(r)}{r^{n+1}} \quad (19)$$

$$\left. \frac{dg_{00}}{dr} \right|_{r=r_H} = (n+1) \frac{1}{r_H} - \frac{h'_\alpha(r_H)}{h_\alpha(r_H)} \quad (20)$$

$$= \frac{1}{r_H} \left((n+1) - (n+3) \frac{1}{1 + \left(\frac{r_H}{L} \right)^\alpha} \right) \quad (21)$$

$$= \frac{1}{L z_H} \left((n+1) - (n+3) \frac{1}{1 + z_H^\alpha} \right) \quad \text{with } z_H = r_H/L \quad (22)$$

Equations 20, 21 and 22 are wrong. Eq 17, 18 and 19 are correct.

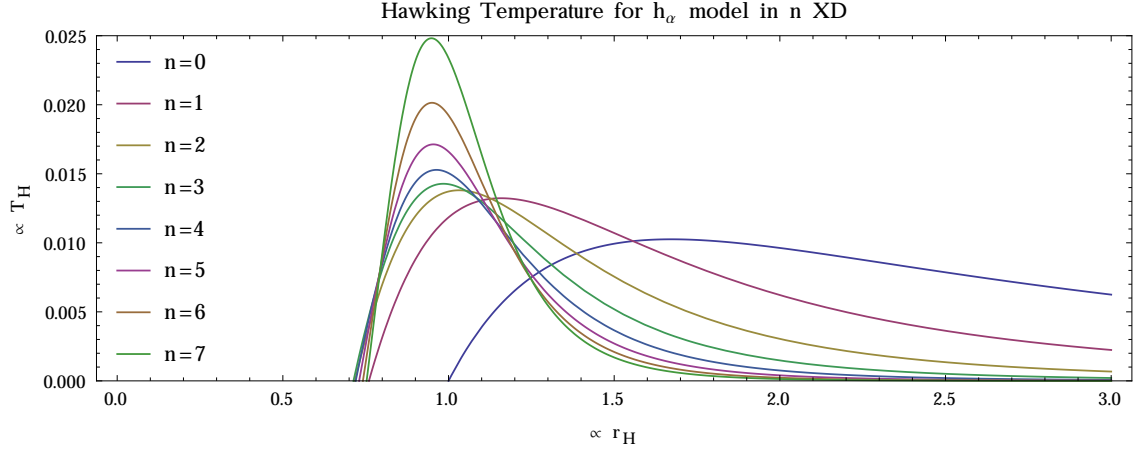


Figure 4: Hawking temperature, plot of eqn. 19 with $L = M = M_* = 1$ and $\alpha = \alpha_0$.

The Hawking Temperature vanishes¹ at r_0 , that is, $T_H(r_{0,\alpha_0}) = 0$. Again, this confirms the self-encoding property of h_α .

3 Heat capacity

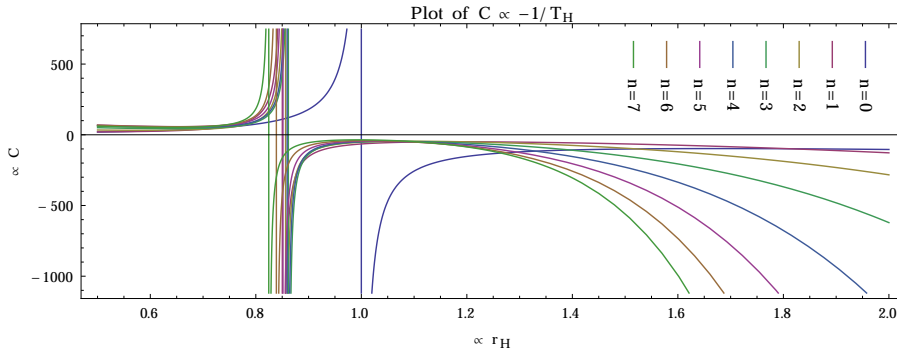
The Heat Capacity C is defined as $C = \partial U / \partial T_H$ with the internal energy $U = M$, so [NS 06.11.2013]

$$C = \frac{\partial M}{\partial T_H} \quad (23)$$

I wonder how this works. In SMM ($G = 1$), $T_H = 2M/r^2$, so $M = T_H r^2 / 2$ and $\partial M / \partial T_H = r^2 / 2$. This is not $C \propto -r^2$ as stated in [NS 06.11.2013]. Is it correct that

$$\frac{\partial M}{\partial T} = \left(\frac{\partial T}{\partial M} \right)^{-1} = \left(\frac{T}{M} \right)^{-1} \propto T^{-1} \quad (24)$$

because there is also the simple relationship $T_H = M f(r)$? This shows $-1/T_H$:



¹I made a numeric check for $L = 1$ like in figure 4. I got a table (like figure 2) for the r which solve the equation $T_H(r) = 0$, the values match $r = L_*$ for all n .