Calc3

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1 Schwarzschild modifications in D dimensions

Consider D dimensional spacetime. This is an $n = D - 4$ dimensional extension to the 4-dimensional spacetime. We commonly define the greek indices $\mu, \nu, \cdots = [1..4]$ for classical 4d-coordinates, big latin indices $A, B, \ldots K, L, \cdots = [1..D]$ for all coordinates and small latin indices $i, j, \cdots = [1..n+2]$ for the angles. So a vector may be noted as $x_K = (x_0, \ldots, X_D)$. In radial coordinates it can be written as $x_K = (t, r, \phi, \theta_1, \ldots, \theta_{D-3})$.

We start with arbitrary $\rho(r)$, with r beging the radial value of x_K. We derive the metric g_{AB} and require SS behaviour $g_{AB} = 0$ when $r \to \infty$. The Ansatz done by Rizzo is

$$\mathrm{d}s^2 = e^\nu \mathrm{d}x_0^2 - e^\mu \mathrm{d}r^2 - r^2 \mathrm{d}\Omega_{D-2}^2 \tag{1}$$

SS requires $e^{\nu,\mu} \to 1 \Leftrightarrow \mu = -\nu$ when $r \to \infty$. We write $e^\nu = 1 - f(r)$ and examine the D dimensional conservation of energy equation, $\nabla_B T^{AB} = 0$. *Now skipping all Ricci deriving stuff.*

R_i^i Einstein equations yield this first order ODE in $f(r)$:

$$f'(r) + \frac{n+1}{r} f(r) = \frac{1}{M_\star} \frac{2r\rho(r)}{n+2} \tag{2}$$

with $M_\star = M_*^{n+2}$ the reduced fundamental mass scale of the theory. This can be solved for any $\rho(r)$ to

$$f(r) = r^{-n-1} \left(\frac{2}{(n+2)M_\star} \int_{c_1}^r (r')^{n+2} \rho(r')\, \mathrm{d}r' + c_2 \right) \quad \text{with } c_1, c_2 = \text{const} \tag{3}$$

Setting c_1 arbitrary, like $c_1 = L_P$ or $c_1 = 0$, and $c_2 = 0$ to match the boundary conditions $g_{00} \xrightarrow{r \to \infty} 0$, a solution is

$$f(r) = \frac{2}{(n+2)} \frac{m(r)}{M_\star} \frac{1}{r^{n+1}} := \frac{\mu(r)}{r^{D-3}} \quad \text{with} \quad m(r) = \int_{L_P}^r (r')^{n+2} \rho(r') \mathrm{d}r' \tag{4}$$

This looks like the general Schwarzschild-Tangherlini-Solution $f(r) = \mu / r^{D-3}$ which is the D-dimensional SSM $f(r) = 2M/r$ generalization.

1.1 Noncommutation in D dim

I can insert the NSS 2006 density $\rho(r)$ into solution (3):

$$\rho(r) = \frac{M}{(4\pi\theta)^{(n+3)/2}} e^{-r^2/4\theta} \tag{5}$$

$$f(r) = r^{-1-n} \left(c_1 - \frac{1}{M_\star} \frac{M}{(2+n)\pi^{(n+3)/2}} \Gamma\left(\frac{3+n}{2}; \frac{r^2}{4\theta} \right) \right) \quad \text{with } c_1 = \text{const} \tag{6}$$

Since $\Gamma(a, r) \xrightarrow{r \to \infty} 0$, boundary conditions are met, but our $f(r) < 0$ if $c_1 = 0$, so we arbitrary set $c_1 = \frac{1}{M_\star} \frac{M}{(2+n)\pi^{(n+3)/2}} \Gamma\left((3+n)/2\right)$. This enables us writing $f(r)$ in a compact way, following

Rizzo 2006 and using the identity $\gamma(a,x) + \Gamma(a,x) = \Gamma(a)$, exploiting the incomplete Gamma functions

$$\gamma(a,x) = \int_0^x t^{a-1} e^{-t} \mathrm{d}t, \quad \Gamma(a,x) = \int_x^\infty t^{a-1} e^{-t} \mathrm{d}t \tag{7}$$

Finally I derived Rizzo 2006:

$$f(r) = \frac{1}{M_\star} \frac{M}{(n+2)\pi^{(n+3)/2}} \frac{1}{r^{n+1}} \gamma\left(\frac{n+3}{2}; \frac{r^2}{4\theta}\right) \tag{8}$$

For $\theta \to 0$ we have $\gamma(\frac{n+3}{2}, x) \xrightarrow{x\to\infty} \Gamma(\frac{n+3}{2})$ which is just a constant factor.
For $n \to 0$ (leaving θ as is) we end up with the not so nice

$$f_{\theta=0}(r) = \frac{1}{M_\star} \frac{M}{2\pi^{3/2}} \frac{1}{r} \gamma\left(\frac{3}{2}, \frac{r^2}{4\theta}\right) \tag{9}$$

1.2 Holography in D dim

With the NS 2011 generalized density $\rho(r)$ to D dimensions,

$$\rho(r) = \frac{M}{\Omega} \frac{\mathrm{d}h(r)}{\mathrm{d}r}, \quad \Omega = \Omega_{D-2} \tag{10}$$

using the differential equation solution (3) we have

$$f(r) = r^{-n-1} \left(\frac{2M}{M_\star(n+2)\Omega} \int_{c_1}^r (r')^{n+2} \frac{\mathrm{d}h(r')}{\mathrm{d}r'} \mathrm{d}r' + c_2 \right) \quad \text{with } c_1, c_2 = \text{const} \tag{11}$$

It seems that there can be made requirements for the shape of $h(r)$ based upon eq.(11). I explored a partial integration series in n which probably could tell me a maximal leading power, above which the integral no more converges. It looks like

$$\begin{aligned} f(r) \propto \frac{1}{r^{n-1}} \Bigg\{ & \left[x^{n+2} \int_\infty^x \mathrm{d}y_1\, h'(y_1) \right]_0^r \\ & - \left[x^{n+1} \int_\infty^x \int_\infty^{y_1} \mathrm{d}y_1 \mathrm{d}y_2\, h'(y_2) \right]_0^r \\ & + \left[x^{n} \quad \int_\infty^x \int_\infty^{y_1} \int_\infty^{y_2} \mathrm{d}y_1 \mathrm{d}y_2 \mathrm{d}y_3\, h'(y_3) \right]_0^r \\ & - \left[x^{n-1} \int_\infty^x \int_\infty^{y_1} \int_\infty^{y_2} \int_\infty^{y_3} \mathrm{d}y_1 \mathrm{d}y_2 \mathrm{d}y_3 \mathrm{d}y_4\, h'(y_4) \right]_0^r \\ & \ldots \\ & + (-1)^{(m+1)} \left[x^{n-(m+1)} \prod_{i=1}^m \int_\infty^{y_{i-1}} \quad h'(y_m) \right]_0^r \quad \text{in the } m\text{. line, with } y_0 := x \\ & \ldots \Bigg\} \end{aligned} \tag{12}$$

Eq (12) tells me that $h(r)$ must be at least $n+2$ times integrable, and, unfortunately, one cannot state that the first line $[x^{n+2} \ldots]_0^r = r^{n+2} h(r)$ contributes most.

1.2.1 Using $h(r) = r^2/(r^2+L^2)$

If we insert the approach $h(r) = r^2/(r^2+L^2)$, we have

$$
\begin{aligned}
f(r) =& \frac{c_1}{r^{n+1}} + \frac{1}{r^{n+1}} \frac{2M}{M_\star(n+2)\Omega} \left[L^2 \left(\frac{1}{1+L^2} - \frac{r^{2+n}}{L^2+r^2} \right) \right. \\
& \left. - {}_2F_1\left(1, \frac{n}{2}+1; \frac{n}{2}+2; -\frac{1}{L^2}\right) + r^{2+n}\, {}_2F_1\left(1, \frac{n}{2}+1; \frac{n}{2}+2; -\frac{r^2}{L^2}\right) \right]
\end{aligned}
\tag{13}
$$

with ${}_2F_1$ the hypergeometric function ${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$ (with Pochhammer Symbol $(x)_n = n!\binom{x}{n}$).

A check for $n \to 0$ gives

$$
f_{n=0}(r) = \frac{L^2 M}{M_\star \Omega r} \left(\frac{1}{1+L^2} - \frac{r^2}{L^2+r^2} - \log\left(1+\frac{1}{L^2}\right) + \log\left(1+\frac{r^2}{L^2}\right) \right) \tag{14}
$$

Notice the bad units e.g. in $1+1/L^2$ (so the calculation needs to be checked). Expected was something roughly like

$$
f_{n=0}(r) = \frac{2M}{r}\rho(r) \approx \frac{2M^2 \left(-\frac{10r^2}{(L^2+r^2)^2} + \frac{2}{L^2+r^2} + \frac{8r^4}{(L^2+r^2)^3} \right)}{r\Omega} \tag{15}
$$

1.2.2 Using $h(r) = h_\alpha(r)$

The approach

$$
h_\alpha(r) = \frac{r^3}{(r^\alpha + (\tilde{r}_0)^\alpha/2)^{3/\alpha}}, \quad \text{Call} \quad r_0 := \tilde{r}_0 := \tilde{r} \tag{16}
$$

yields something like

$$
f(r) = c_1 r^{-n-1} + \frac{2r^5 \left(2\left(\frac{r}{\tilde{r}}\right)^\alpha + 1\right)^{3/\alpha} \left(r^\alpha + \frac{1}{2}\tilde{r}^\alpha\right)^{-3/\alpha} {}_2F_1\left(\frac{3}{\alpha}, \frac{n+6}{\alpha}; \frac{n+6}{\alpha}+1; -2\left(\frac{r}{\tilde{r}}\right)^\alpha\right)}{M_\star(n+2)(n+6)} \tag{17}
$$

The very present number 3 seems to be motivated by 3 spatial dimensions, so if we change that to $3+n$, thus considering a modified density

$$
h_\alpha(r) = \frac{r^{(n+3)}}{(r^\alpha + r_0^\alpha/2)^{(n+3)/\alpha}} \tag{18}
$$

This has the solution

$$
f(r) = c_1 r^{-n-1} + \frac{r^{n+5}\left(2(r/r_0)^\alpha + 1\right)\left(r^\alpha + \frac{r_0^\alpha}{2}\right)^{-\frac{n+3}{\alpha}} {}_2F_1\left(1, \frac{n+\alpha+3}{\alpha}; \frac{2n+\alpha+6}{\alpha}; -2(r/r_0)^\alpha\right)}{M_\star(n+2)(n+3)} \tag{19}
$$