The static potential in momentum space

Performing the Fourier transform of a generic *position space* potential $V(|\vec{r}|)$ in $d=3$ dimensions to momentum space, we obtain

$$\begin{aligned}\hat{V}(p) &= \int \mathrm{d}^3r\, \mathrm{e}^{-i\vec{r}\cdot\vec{p}}\, V(r) = 2\pi \int_{-1}^{1} \mathrm{d}\cos\theta \int_0^\infty \mathrm{d}r\, r^2\, \mathrm{e}^{-irp\cos\theta}\, V(r) \\ &= \frac{2\pi i}{p} \int_0^\infty \mathrm{d}r\, r\, V(r)\, \left(\mathrm{e}^{-irp} - \mathrm{e}^{+irp}\right) = \frac{2\pi i}{p} \int_{-\infty}^{\infty} \mathrm{d}r\, \mathrm{e}^{-irp}\, r\, [V(r)\,\Theta(r) + V(-r)\,\Theta(-r)]\,, \end{aligned} \tag{1}$$

with $r=|\vec{r}|$ and $p=|\vec{p}|$. Note that this effectively amounts to a one dimensional Fourier transform

$$\hat{v}(p) = \int_{-\infty}^{\infty} \mathrm{d}r\, \mathrm{e}^{-irp}\, v(r)\,, \tag{2}$$

with

$$v(r) = r\,[V(r)\,\Theta(r) + V(-r)\,\Theta(-r)] \quad \text{and} \quad \hat{V}(p) = \frac{2\pi i}{p}\, \hat{v}(p)\,. \tag{3}$$

A simple effective potential

Consider the following *effective* potential in position space,

$$V(r) = -\frac{\pi}{12}\frac{1}{r} + \sigma r + V_0\,, \tag{4}$$

with the force defined as $F(r) = |\frac{\mathrm{d}V(r)}{\mathrm{d}r}|$. Given the explicit values $F(r_0)r_0^2 = 1.65$ and $r_0 = 0.5\,\mathrm{fm}$ we infer $\sigma = \frac{r_0^2 F(r_0) - \pi/12}{r_0^2} \approx 5.55\,\mathrm{fm}^{-2}$. The potential $v(r)$ associated with Eq. (4) is given by [cf. Eq. (3)]

$$v(r) = \left(\frac{\pi}{12} - \sigma r^2\right) [\Theta(-r) - \Theta(r)] + rV_0\,, \tag{5}$$

such that

$$\begin{aligned}\hat{v}(p) &= \left(\frac{\pi}{12} + \sigma\partial_p^2\right) \int_{-\infty}^{\infty} \mathrm{d}r\, \mathrm{e}^{-irp}\, [\Theta(-r) - \Theta(r)] + V_0\, i\partial_p \int_{-\infty}^{\infty} \mathrm{d}r\, \mathrm{e}^{-irp} \\ &= 2i\left(\frac{\pi}{12} + \sigma\partial_p^2\right) \int_0^\infty \mathrm{d}r\, \mathrm{e}^{-rp} + 2\pi i\, V_0\, \partial_p\delta(p) = \frac{i\pi}{6}\frac{1}{p} + 4i\sigma\frac{1}{p^3} + 2\pi i\, V_0\, \partial_p\delta(p)\,. \end{aligned} \tag{6}$$

The potential in momentum space thus reads [cf. Eq. (3)]

$$\hat{V}(p) = -\frac{\pi^2}{3}\frac{1}{p^2} - 8\pi\sigma\frac{1}{p^4} - 4\pi^2\, V_0\, \frac{1}{p}\partial_p\delta(p))\,. \tag{7}$$

Obviously, for $p\to\infty$, i.e., in the realm of perturbation theory, Eq. (7) is well approximated by

$$\hat{V}(p) \approx -\frac{\pi^2}{3}\frac{1}{p^2} \quad \leftrightarrow \quad V(r) \approx -\frac{\pi}{12}\frac{1}{r}\,, \tag{8}$$

whereas for $0 < p \ll 1$, i.e., in the nonperturbative regime,

$$\hat{V}(p) \approx -8\pi\,\frac{\sigma}{p^4} \quad \leftrightarrow \quad V(r) \approx \sigma r \tag{9}$$

constitutes a good approximation.