Calc10

Sven Köppel
koeppel@fias.uni-frankfurt.de

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Calc10 ties up Calc8, calculating the Propagator and modified Einstein Equations.

Contents

1 The modified Action

The aim of this section is to motivate potential $V(r) \propto \frac{h(r)}{r^{1+n}}$ in the metric $g_{00} = 1 - V(r)$ not only as the result of a modified density $\rho = \frac{M}{\Omega}\delta(r) \to \frac{M}{\Omega}\frac{\mathrm{d}h(r)}{\mathrm{d}r}$ but also by modified Einstein equations, a modified Action, a modified mass or gravitational constant term. That is, the step $\delta(r) \to \frac{\mathrm{d}h(r)}{\mathrm{d}r}$ shall be performed by introducing a more fundamental concept.

This concept looks like a modified delta distribution again: A bilocal distribution

$$\mathcal{A}^2(x-y) = \mathcal{A}^2(\Box_x)\delta^4(x-y) \tag{1}$$

One way to introduce it is smearing the Ricci scalar $R(x)$ [N 02.2012] by

$$\mathcal{R}(x) = \int \mathrm{d}^4 y \sqrt{-g}\mathcal{A}^2(x-y)R(y) \tag{2}$$

This modifies the Action and the Einstein equations immediately:

$$S = \frac{1}{16\pi G}\int \mathrm{d}^4 x \sqrt{-g}\mathcal{R}(x) \qquad \mathcal{A}^2(\Box)\left(R_{\mu\nu} - \frac{g_{\mu\nu}}{2}R\right) = 8\pi G\, T_{\mu\nu} \tag{3}$$

Having $\mathcal{T}_{\mu\nu} = \mathcal{A}^{-2}(\Box)T_{\mu\nu}$ and the Schwarzschild density $\rho_0 = M\delta(\vec{x})$, we end up with

$$\mathcal{T}_0^0 = -M\mathcal{A}^{-2}(\Box)\delta(\vec{x}) \tag{4}$$

Finally, this yields our requirement for matching this formalism to the holographic approach in [Calc 1-10]:

$$M\mathcal{A}^{-2}(\Box)\delta(\vec{x}) \stackrel{!}{=} \frac{M}{\Omega_{n+2}}\frac{\mathrm{d}H(r)}{\mathrm{d}r} \tag{5}$$

I want to find profiles $\mathcal{A}$ that corresponds to my functions $H \in \{h, h_\alpha\}$ in n dimensions.

1.1 Operator rewrites and Fourier Transformation

In [N 02.2012], there is a special choice of the D'Alambert operator, using a length scale ℓ. This gives the momentum operator $\hat{P}$ another form than usual:

$$
\begin{aligned}
&\hat{P} = -i\hbar\nabla && (6)\\
&\Box = \ell^2\nabla^2 && (7)\\
\Rightarrow\quad &\hat{P}^2 = -\Box/\ell^2 && (8)
\end{aligned}
$$

1.1.1 Spherical Fourier transformation in $3+n$ dimensions

I use the Fourier transformation $\mathcal{F}$ which is defined in d dimensions ($\vec{x} \in \mathbb{R}^d$) as

$$
\mathcal{F}\left\{f\right\}(\vec{p}) = \tilde{f}(\vec{p}) = \frac{1}{(2\pi)^d}\int \mathrm{d}^d x\, e^{-i\vec{p}\cdot\vec{x}} f(\vec{x}) \tag{9}
$$

$$
\mathcal{F}^{-1}\left\{\tilde{f}\right\}(\vec{x}) = f(\vec{x}) = \int d^d p\, e^{+i\vec{p}\cdot\vec{x}}\tilde{f}(\vec{p}) \tag{10}
$$

The subsequent use is in $d = 3+n$ dimensions. If the function f only depends on the radius, $f(\vec{x}) = f(|\vec{x}|)$, then the integrals 9 and 10 can be transformed to one dimensional integrals. In $d = 3$, we obtain

$$
\begin{aligned}
\hat{V}(p) &= \int \mathrm{d}^3 r\, e^{-i\vec{r}\cdot\vec{p}}\, V(r) = 2\pi\int_{-1}^{+1}\mathrm{d}\cos\theta\int_0^\infty \mathrm{d}r\, r^2\, e^{-irp\cos\theta}V(r) && (11)\\
&= \frac{2\pi i}{p}\int_0^\infty \mathrm{d}r\, r\, V(r)\left(e^{-irp} - e^{+irp}\right) && (12)\\
&= \frac{2\pi i}{p}\left[\int_0^\infty e^{-ipr}\, r\, V(r) - \int_{-\infty}^0 e^{-ipr}\,(-r)\, V(-r)\right] && (13)\\
&= \frac{2\pi i}{p}\int_{-\infty}^\infty \mathrm{d}r\, e^{-irp}\, r\,\left[V(r)\Theta(r) + V(-r)\Theta(-r)\right] && (14)
\end{aligned}
$$

with $r = |\vec{r}|$, $p = |\vec{p}|$.

The rewrite made the trick of substituting the $\theta \in [0,\pi]$ angle by the angle between $\vec{r}$ and $\vec{p}$ in the scalar product $\vec{r}\cdot\vec{p} = |\vec{r}|\,|\vec{p}|\cos\varphi$, with $\varphi \in [0,\pi]$. This is also possible in higher dimensions. Consider in the spherical spacial coordinates $\vec{r} = (r, \phi, \theta_1, \ldots, \theta_{d-2})$:

$$
\begin{aligned}
\int \mathrm{d}^d r &= \int_0^\infty \mathrm{d}r\, r^{d-1}\int_0^{2\pi}\mathrm{d}\phi\prod_{i=1}^{d-2}\int_0^\pi \mathrm{d}\theta_i \sin^i(\theta_i) := \int_0^\infty \mathrm{d}r\,\Omega_{d-1}r^{d-1} && (15)\\
&= \frac{\Omega_{d-1}}{2}\underbrace{\int_0^\pi \mathrm{d}\theta_1\sin(\theta_1)}_{=2}\int_0^\infty \mathrm{d}r\, r^{d-1} \qquad \text{with}\quad \Omega_{n+2} = 2\frac{\pi^{\frac{n+3}{2}}}{\Gamma\left(\frac{n+3}{2}\right)} && (16)
\end{aligned}
$$

Making the scalar product substitution with only θ_1, one gets

$$
\begin{aligned}
\tilde{V}(p) &= \frac{\Omega_{d-1}}{2}\frac{i}{p}\int_0^\infty \mathrm{d}r\, r^{d-2}\, V(r)\,\left(e^{-irp} - e^{+irp}\right) && (17)\\
&= \frac{\Omega_{n+2}}{2}\frac{i}{p}\int_0^\infty \mathrm{d}r\, r^{n+1}\, V(r)\,\left(e^{-irp} - e^{+irp}\right) && (18)
\end{aligned}
$$

When writing the integral in form of (14), the Residual theorem

$$\frac{1}{2\pi\mathrm{i}} \int_\Gamma f = \sum_{a \in D_f} \mathrm{ind}_\Gamma(a)\, \mathrm{Res}_a\, f \tag{19}$$

can be used to solve these integrals.

1.1.2 Operator Eigenvalues

We also use the so called Schwinger-Representation ([IMN Nov 2013], eq. 21) for operators $\mathcal{O}$ (actually, I don't know why):

$$(\mathcal{O})^\alpha = \frac{1}{\Gamma(-\alpha)} \int_0^\infty \mathrm{d}s\; s^{\alpha-1} e^{-s\mathcal{O}} \tag{20}$$

Furthermore, power series are used to reason why functions of operators can be replaced by functions of the applied operators. E.g.

$$\exp x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots \tag{21}$$

$$\frac{1}{1+x} \approx 1 - x + x^2 - x^3 + \ldots \tag{22}$$

Consider linear operators, especially differential operators $\mathcal{O} = \frac{\mathrm{d}}{\mathrm{d}x}$. Then consider any infinetly differentiable function f of that differential operator $\frac{\mathrm{d}}{\mathrm{d}x}$. I will switch between the representations $f(\frac{\mathrm{d}}{\mathrm{d}x})$ and $f(p)$, which does *not* denote momentum space, but an evaluation of differentiation. The eigenvalue equation with eigenfunction $e^{\lambda x}$

$$f\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)\, e^{\lambda x} = f(\lambda)\, e^{\lambda x} \tag{23}$$

makes that replacement feasible. In the next section, this will be used with a semi-Fourier-transform:

$$f\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)\, \delta(x) = f(p^2)\, \delta(x) \tag{24}$$

1.2 How to get the $\mathcal{A}$

We want to solve eq. (5). This can be done exploiting one dimensional Fourier Transformationations. Consider the Fourier transformation of the Dirac Delta $\delta(x)$,

$$\tilde{\delta}(p) = \frac{1}{2\pi} \int \mathrm{d}x\, e^{-ipx} \delta(x) = 1 \qquad\qquad \delta(x) = \int \mathrm{d}p\, e^{ipx} \tag{25}$$

We write both sides of (5) as the reverse Fourier transformation $\mathcal{F}^{-1}$ of their Fourier transformations. The integrands, which are basically the Fourier transformations, then can be compared directly (comparison of equation coefficients):

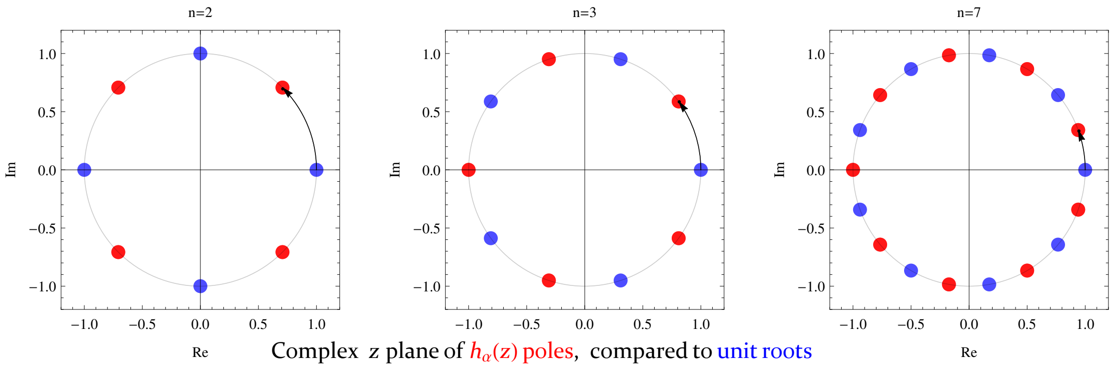

Figure 1: Unit roots in comparison to the poles of $h(z)/(2+n) = h(r)/2l$. Each pole occures two times, so the number of poles $|\{z_0\}|$ for given n is $|\{z_0\}| = 2(n+2)$, while the number of unit roots $|\{x_0\}|$ is only $|\{x_0\}| = n+2$. The black arrow indicates the $e^{i\pi/(2+n)}$ rotation.

$$\mathcal{A}^{-2}(\Box)\delta(\vec{x}) = \frac{\mathrm{d}H(x)}{\mathrm{d}x} \tag{26}$$

$$\Leftrightarrow \int \mathrm{d}p\mathcal{A}^{-2}(\Box)\, e^{ipx} = \frac{\mathrm{d}H(x)}{\mathrm{d}x} \tag{27}$$

$$\Leftrightarrow \int \mathrm{d}p\mathcal{A}^{-2}(\Box)\, e^{ipx} = \int \mathrm{d}p\ \mathcal{F}\left\{\frac{\mathrm{d}H(x)}{\mathrm{d}x}\right\} e^{ipx} \tag{28}$$

$$\Leftrightarrow \mathcal{A}^{-2}(p^2) = \mathcal{F}\left\{\frac{\mathrm{d}H(x)}{\mathrm{d}x}\right\} \tag{29}$$

Determining $\mathcal{A}$ (in position space) therefore means just calculating the Fourier transform of the derivative of the holographic function.

As told in the section before, $\mathcal{A}^{-2}(p^2)$ must not be confused with the fourier transformed $\tilde{\mathcal{A}}^{-2}(p^2) = \int \mathrm{d}x\mathcal{A}^{-2}(\Box)e^{-ipx}$. Actually, the latter is never used in the present calculations.

1.3 $\mathcal{A}$ for $h(r)$ in n dimensions

Consider the holographic function in dimensionless coordinates $z = rL$ in $3+n$ dimensions:

$$h(z) = \frac{1}{1+\left(\frac{1}{z}\right)^{2+n}} \qquad\qquad h'(z) = \frac{(2+n)\left(\frac{1}{z}\right)^{3+n}}{\left(1+\left(\frac{1}{z}\right)^{2+n}\right)^2} \tag{30}$$

Calculating the Fourier transform of (30) can be done with Residue theorem, which requires knowledge of the poles, which are given when the demoninator of (30) equals 0. This problem can be reduced to determination of the *roots of unity* (Einheitswurzeln):

$$z^{3+n}\left(1+\left(\frac{1}{z}\right)^{2+n}\right)^2 = 0 \quad \Leftrightarrow \quad 1 = -z^{2+n} \tag{31}$$

Given the nth root of unity, $x^n = 1$, and $-1 = e^{i\pi}$, the relation of the solution set $\{z_0\}$ and $\{x_0\}$ is given by $x_0\, e^{i\pi/(2+n)} = z_0$. Due to the power of 2 in the denominator, all poles of $h'(z)$ are doubled. See also figure 1.

Knowing the poles of the integrand, the integral can be performed by summing the residues.

1.3.1 Results

The results, done correctly, can be expressed with the Meijer G-function $G^{m,n}_{p,q}\left(\begin{matrix} a_1,\dots,a_p \\ b_1,\dots,b_q \end{matrix} \middle| z\right)$. Table 1 lists the results of the integral

$$\mathcal{A}^{-2}(p) = \int \mathrm{d}r\,(-1)^{n+1}\frac{i}{p}\,r^{n+1}\left(\Theta(-r)h'(-r)+\Theta(r)h'(r)\right)e^{-ipr} \tag{32}$$

Constants like $\Omega_{n+2}/2$ as given in eq. (18) are omitted.

Table 1: Closed form expressions for $\mathcal{A}^{-2}(p)$ for $h(r)$ in n dimensions

n	$p\cdot\mathcal{A}^{-2}(p)$
0	$-2\sqrt{\pi}lG^{2,1}_{1,3}\left(\frac{l^2p^2}{4}\middle\| \begin{matrix} -\frac{1}{2} \\ \frac{1}{2},\frac{1}{2},0 \end{matrix}\right)$
1	$2i\sqrt{\frac{3}{\pi}}l^2G^{5,1}_{1,7}\left(\frac{l^6p^6}{46656}\middle\| \begin{matrix} -\frac{1}{3} \\ 0,\frac{1}{6},\frac{1}{3},\frac{2}{3},\frac{2}{3},\frac{1}{2},\frac{5}{6} \end{matrix}\right)$
2	$-2\sqrt{2\pi}l^3G^{3,1}_{1,5}\left(\frac{l^4p^4}{256}\middle\| \begin{matrix} -\frac{3}{4} \\ \frac{1}{4},\frac{1}{4},\frac{3}{4},0,\frac{1}{2} \end{matrix}\right)$
3	$2i\sqrt{\frac{5}{\pi}}l^4G^{7,1}_{1,11}\left(\frac{l^{10}p^{10}}{10000000000}\middle\| \begin{matrix} -\frac{2}{5} \\ 0,\frac{1}{10},\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{3}{5},\frac{4}{5},\frac{3}{10},\frac{1}{2},\frac{7}{10},\frac{9}{10} \end{matrix}\right)$
4	$-2\sqrt{3\pi}l^5G^{4,1}_{1,7}\left(\frac{l^6p^6}{46656}\middle\| \begin{matrix} -\frac{5}{6} \\ \frac{1}{6},\frac{1}{6},\frac{1}{2},\frac{5}{6},0,\frac{1}{3},\frac{2}{3} \end{matrix}\right)$
5	$2i\sqrt{\frac{7}{\pi}}l^6G^{9,1}_{1,15}\left(\frac{l^{14}p^{14}}{11112006825558016}\middle\| \begin{matrix} -\frac{3}{7} \\ 0,\frac{1}{14},\frac{1}{7},\frac{2}{7},\frac{3}{7},\frac{4}{7},\frac{4}{7},\frac{5}{7},\frac{6}{7},\frac{3}{14},\frac{5}{14},\frac{1}{2},\frac{9}{14},\frac{11}{14},\frac{13}{14} \end{matrix}\right)$

1.4 $\mathcal{A}$ for $h_\alpha(r)$ in n dimensions

When studying the self-regular profile in dimensionless coordinates $z = rL$ in $3+n$ dimensions:

$$h_\alpha(z) = \frac{1}{\left(1+\left(\frac{1}{z}\right)^\alpha/2\right)^{\frac{3+n}{\alpha}}} \qquad h'_\alpha(z) = \frac{\frac{3+n}{2}\left(\frac{1}{z}\right)^{\alpha+1}}{\left(1+\left(\frac{1}{z}\right)^\alpha/2\right)^{\frac{3+n}{\alpha}+1}} \tag{33}$$

It is hard to find a closed form expression for the Fourier transform $\mathcal{F}\{h'_\alpha(z)\}$ even for $n=0$, but in principle it should be possible, since the roots should be clearly determinable.