Modified GUP in Extra Dimensions, update

Sven Köppel
koeppel@fias.uni-frankfurt.de

Generation date: August 29, 2014, 12:14

Abstract

This document is an addition for my proposal from July 6, 2014. It brings corrected prefactors, a more compact result for $\mathcal{T}^{00}$, numerical results for r_0 and M_*, corrected plots of the metric and the black hole temperatures. It ends with an outlook about the Heat capacity and stability.

This document is written in the context of the currently prepared paper *Self-Completeness and the Generalized Uncertainty Principle in Extra Dimensions* calculated by Maximiliano Isi and Marco Knipfer [1]. My proposal from July introduced a way to solve higher dimensional fourier transformations that occur in the computation, where Marcos approach (Schwinger Operator representation and identification as higher dimensional Gaussian integral) fails.

Internal working title: CALC18

Contents

Mini symbols key

- $\lfloor x \rfloor := \max\{k \in \mathbb{N}_0 : k \leq x\}$: Gaussian step function
- $z = x + iy,\ \bar{z} = x - iy$ the complex conjugate
- $\Omega_d = \frac{\pi^{(d-1)/2}}{\Gamma\left(\frac{d-1}{2}+1\right)}$: Surface factor of d-sphere, the surface is given by $A_d = \Omega_d r^{d-1}$ or something like that.
- $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \mathrm{d}t$ the Gamma function
- $\Gamma(a, z) = \int_z^\infty t^{a-1} e^{-t} \mathrm{d}t$ the upper Gamma function
- $\gamma(a, z) = \int_0^z t^{a-1} e^{-t} \mathrm{d}t$ the lower Gamma function.
- $M_{\text{4d Planck}} = V_n M_*^{n+2}$ as defined in [5]: M_* is the reduced Planck length and V_n the volume of the compactified dimensions, e.g. in a torus $V_n = (2\pi R_c)^n$ with R_c the compactification radius.
- $L_* = 1/M_*$ the reduced Planck length and $M_* = 1/(8\pi G)$ the link to newtonian Physics (the 8π must be preserved somewhere)

1 Framework

This document follows the reasoning of the 2013 JHEP paper [3], the work in progress [1] and my last proposal from July 2014. I won't repeat the details but write down the improved results.

1.1 The modified GUP

We want to modify the GUP relation discussed in [3] to higher dimensions in a way that without extra dimensions, it reduces to the ordinary case. Consider the "ordinary" GUP as modification of the canonical commutation relations ($p = |\mathbf{p}|$):

$$[x^i, p_j] = i\delta^i_j(1 + \beta p^2) \tag{1}$$

Now our improved versions in total $N + 1 = 4 + n$ space-time dimensions[1] looks like

$$[x^i, p_j] = i\delta^i_j(1 + L^{2+n}p^{2+n}) \tag{2}$$

with $L^2 = \beta$. The modified energy-momentum tensor is given by smearing the classical energy-momentum tensor (Schwarzschild static isotropic point-like matter source $T^0_0 = M\delta^N(\mathbf{x})$) with a bilocal function $\mathcal{A}^{-2}$.

$$\mathcal{T}^\mu_\nu = \mathcal{A}^{-2}(\Box)T^\mu_\nu = M\mathcal{A}^{-2}(\Box)\delta(\mathbf{x}) \tag{3}$$

Representing the Dirac in momentum space, the usual approach is

$$\mathcal{A}^{-2}(\Box)\delta(\mathbf{x}) = \frac{1}{(2\pi)^N}\int \mathrm{d}^N p\, \mathcal{A}^{-2}(\Box)\, e^{i\mathbf{x}\cdot\mathbf{p}} = \frac{1}{(2\pi)^N}\int \mathrm{d}^N p\, \mathcal{A}^{-2}(p^2)\, e^{i\mathbf{x}\cdot\mathbf{p}} := \mathcal{F}^{-1}_N\{\mathcal{A}^{-2}(p^2)\} \tag{4}$$

with $\mathcal{F}^{-1}_N$ the N-dimensional inverse fourier transformation. Using the modified momentum integration measure given by [2],

$$\int \frac{\mathrm{d}^3 p}{1 + L^{2+n}p^{2+n}} |p\rangle\langle p| = 1, \tag{5}$$

we end up determining the smeared matter density by

$$\mathcal{T}^0_0 = \frac{M}{(2\pi)^N}\int \frac{\mathrm{d}^3 p}{1 + L^{2+n}p^{2+n}} e^{i\mathbf{x}\cdot\mathbf{p}}. \tag{6}$$

In this text, we will solve $\mathcal{T}^0_0$, afterwards derive $g_{\mu\nu}$ and be happy.

1.2 The effective 1-dimensional FT

In order to solve (6), I already proposed an approach using Heaviside-step functions $\Theta(z) = \Theta(\mathrm{Re}\ z)$ on the complex plane.

Computing the Fourier transformation

$$\hat{V}(\mathbf{r}) = \frac{1}{(2\pi)^N}\int \mathrm{d}^N p\ e^{+i\mathbf{r}\cdot\mathbf{p}}\ V(p) \tag{7}$$

we integrate out all angles up to one, used to make the identification with the $\mathbf{x}\cdot\mathbf{p}$ scalar product:

$$\hat{V}(\mathbf{r}) = \frac{1}{(2\pi)^N}\frac{\Omega_{N-1}}{2}\int_0^\infty \mathrm{d}p\ p^{N-1}\ V(p)\ \frac{e^{+irp} - e^{-irp}}{irp} \tag{8}$$

Following the reasoning in the Jul 2014 proposal, we end up with an one dimensional integration

$$\hat{V}(\mathbf{r}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}p\ v(p)\ e^{+irp} \tag{9}$$

with the effective 1d Fourier kernel $v(r)$, given by

$$v(p) := \frac{1}{(2\pi)^{N-1}}\frac{\Omega_{N-1}}{2}\frac{1}{ir}p^{N-2}\left[V(p)\Theta(p) + (-1)^{N-1}V(-p)\Theta(-p)\right] \tag{10}$$

[1] Remember: N = Number of spatial dimensions, n = number of large extra dimensions

2 Properties of the modified GUP

2.1 Gaining the matter density

With $V(p) = 1/(1 + L^{2+n}p^{2+n})$, following (10) gives us the integral

$$\mathcal{T}_0^0 = \frac{M}{(2\pi)^N}\frac{\Omega_{N-1}}{2}\frac{1}{ir}\int_{-\infty}^{\infty} \mathrm{d}p\; p^{N-2}\left[\frac{\Theta(p)}{1+L^{N-1}p^{N-1}} + \frac{(-1)^{N-1}\Theta(-p)}{1+L^{N-1}(-p)^{N-1}}\right]e^{ipr}. \tag{11}$$

For easier evaluation of the poles, I prefer using dimensionless units $q = pL$ and $z = r/L$:

$$\mathcal{T}_0^0 = \underbrace{\frac{M}{(2\pi)^N}\frac{\Omega_{N-1}}{2L^N}\frac{1}{iz}}_{f_0}\int_{-\infty}^{\infty} \mathrm{d}q\;\left[\underbrace{\frac{q^{N-2}}{1+q^{N-1}}}_{f_+(q)}\Theta(q) + \underbrace{\frac{q^{N-2}(-1)^{N-1}}{1+(-q)^{N-1}}}_{f_-(q)}\Theta(-q)\right]e^{iqz} \tag{12}$$

The explicit determination of the poles of $f_\pm(q)$ was already given in the July proposal. Note that the solution set of the equation $1/f_+(q) = 0$ is just the negative of the solution set of $1/f_-(q) = 0$. The poles of $f_+(q)$ were already given in the July proposal by

$$1 + q^{n+2} = 0 \quad\Leftrightarrow\quad q = (-1)^{\frac{1}{n+2}} = \exp\left\{\frac{i\pi + 2\pi ik}{n+2}\right\} \quad \forall k \in \mathbb{N}_0, \tag{13}$$

and we also already summed up the residues $f_\pm(q_0)e^{iq_0z}$ of all eligible poles q_0. In the July proposal, the result was lacking all prefactors, called f_0 in (12), or to be more specific, I worked with the wrong $f_0 = 2\pi i/z$.

All our poles q_0, as given in (13), have the same value for $\mathrm{Res}_{q_0} f_\pm(q_0) = \frac{1}{2+n}$. Moreover, using the identity

$$e^{iqz} + e^{i\bar{q}z} = 2e^{-z\sin(\alpha)}\cos(z\cos(\alpha)), \quad q = e^{i\pi\alpha} \tag{14}$$

as introduced in the July proposal, we get the overall real result

$$\boxed{\mathcal{T}_0^0 = \frac{M}{(2+n)r}\frac{\Omega_{N-1}}{(2\pi L)^{N-1}}\sum_{\varphi\in\Phi_n} e^{-r/L\sin(\varphi)}\cos\left(r/L\cos(\varphi)\right),} \tag{15}$$

with Φ_n the phases of the poles taken into account for n extra dimensions, given as

$$\Phi_n = \left\{\varphi = \arg(q) \;:\; 1 + q^{n+2} = 0 \wedge \mathrm{Im}(q) \geq 0 \wedge \mathrm{Re}(q) \geq 0\right\} \tag{16}$$

$$= \left\{\varphi = \pi\frac{1+2k}{n+2} \;:\; k \in \mathbb{N}_0 \wedge k \leq \frac{n}{4}\right\}. \tag{17}$$

Since the number of angles $|\Phi_n| = \lfloor\frac{n}{4}\rfloor$ is a step-function, in this notation there is no way to write (15) in a more compact way.

For $n = 0$, our result reduces to [3], as can bee seen when computing $\Phi_0 = \{\pi/2\}$, $\Omega_2 = 2\pi$:

$$\mathcal{T}_0^0 = \frac{M}{2r}\frac{\Omega_2}{(2\pi L)^2}e^{-z} = \frac{M}{\beta r4\pi}e^{-r/\sqrt{\beta}}. \tag{18}$$

2.2 The metric

Following the $N+1$-dimensional solution of the Einstein Equations made by Rizzo 2005 [5], the correct line element is given by the solution of the first order differential equation

$$V'(r) + \frac{n+1}{r}V(r) = \frac{1}{M_*^{n+2}}\frac{2r\rho(r)}{n+2} \tag{19}$$

with $\rho(r) = \mathcal{T}_0^0$, in a way that the line element is then given by

$$\mathrm{d}s^2 = (1 - V(r))\mathrm{d}t^2 - (1 - V(r))^{-1}\mathrm{d}r^2 - \mathrm{d}\Omega^2. \tag{20}$$

The general solution of (19) is given by

$$V(r) = \frac{1}{r^{n+1}} \left(\frac{2}{(n+2)M_*^{n+2}} \int_{c_1}^{r} x^{n+2} \rho(x) \mathrm{d}x + c_2 \right) \quad \text{with} \quad c_1, c_2 = \text{const}, \tag{21}$$

and after inserting our density $\rho = \mathcal{T}_0^0$ given by (15), with $p_0 = e^{i\varphi}$,

$$V(r) = \frac{1}{r^{n+1}} \frac{2M}{(n+2)^2 M_*^{n+2}} \frac{\Omega_{n+2}}{(2\pi L)^{n+2}} \sum_{\varphi \in \Phi_n} \int_0^r \mathrm{d}x \; x^{n+1} \left(e^{ip_0 x} + e^{i\bar{p}_0 x} \right). \tag{22}$$

By substitution $x' = x/(-ip_0)$ or $x' = x/(-i\bar{p}_0)$, respectively, the occurring integrals can be rewritten to lower gamma functions. Note that $q_0^{2+n} \equiv \bar{q}_0^{2+n} \equiv -1$. Our final result for the metric is

$$\boxed{V(r) = \frac{1}{r^{n+1}} \frac{2M}{(n+2)^2 M_*^{n+2}} \frac{\Omega_{n+2}}{(2\pi L)^{n+2}} \frac{(-1)^{1+n}}{i^{2+n}} \sum_{\varphi \in \Phi_n} \gamma(2+n, -ip_0 r) + \gamma(2+n, -i\bar{p}_0 r).} \tag{23}$$

Let's check the result for $n \to 0$: The sum only contains two times the same $p_0 = \bar{p}_0 = i/L$ and $1/M_* = 8\pi G$, so we derived the 4d GUP metric of [3],

$$V(r) = \frac{2GM}{r\beta} \gamma(2, {}^{r}/\!\sqrt{\beta}). \tag{24}$$

Figure 1 shows the metric with the special properties which are basically the same as in $n = 0$, that is, three possible situations (no black hole, extremal horizon r_0 or two horizons $r_\pm$), the presence of a remnant for a special mass, etc.

2.3 Self-completeness

Casting $L = L_*$ the reduced Planck mass, the special mass $M = M_*$ exhibits the extremal configuration at $r = r_0$ (actually no self-encoding), see e.g. figure 1 and table 1 for numerical values. Actually, the extremal black hole masses get enormous values, compared to other models I know (e.g. holographic ones in LXDs, [4]).

2.4 Black Hole Temperature

Computing the temperatures $T_H = \frac{1}{4\pi} \partial_r g_{00}|_{r=r_H}$ is a straightforward process and generates a long expression, c.f. the length of the temperature expressions of the non-modified GUPs in [1]. In figure 2, the temperature is plotted for the GUP-modified Black Holes. Further computation may be done, but not in this document.

n	0	1	2	3	4	5	6	7
r_0	1.79328	1.27534	1.07714	0.993701	1.01592	0.981144	0.953649	0.932502
M_*	3.35092	53.0073	621.491	6536.7	35182.9	359680.	$3.69058 \cdot 10^6$	$3.8323 \cdot 10^7$

Table 1: Self-encoding horizon radius $r_0 = L_*$ and Remnant masses $M_* = 1/L_*$, in 4d Planck Units for the modified GUP

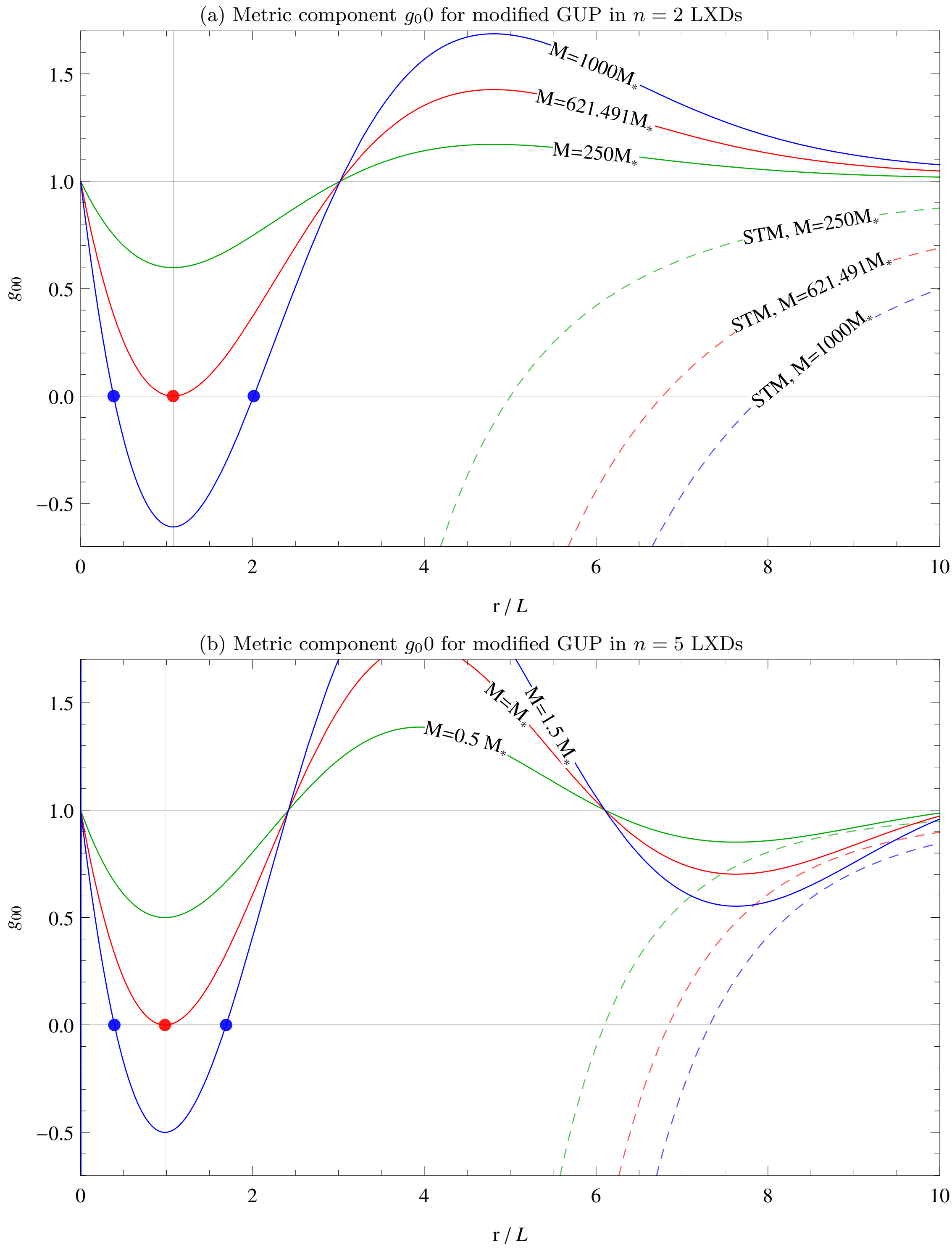

Figure 1: The metric behaviour for the modified GUP $[x^i, p_j] = i\delta^i_j(1 + L^{2+n}p^{2+n})$ in n large extra dimensions, compared to the Schwarzschild-Tangherlini metric in n large extradimensions (dashed lines). The red dot indicates the extremal radius r_C, for small r, this metric behaves always like the $n = 0$ GUP.

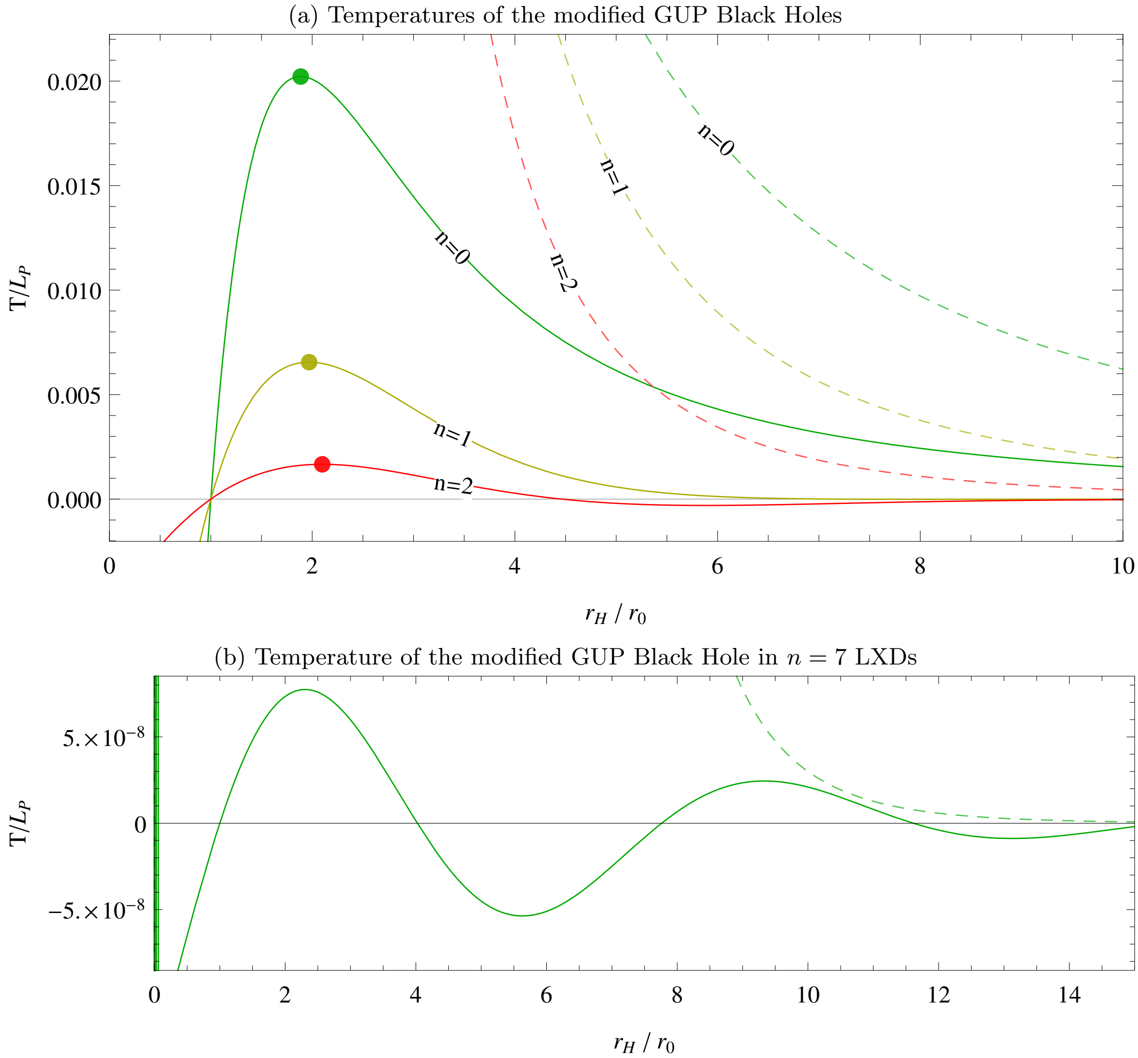

Figure 2: Temperatures of the modified GUP Black holes in n large extra dimensions, compared to Schwarzschild-Black holes (dashed lines). The x-axis is scaled by the extremal radius r_0. In figure (a), the circles indicate the critical radii r_C where the Heat Capacity diverges.
With increasing $n > 0$, the temperature fluctuates more and more around $T = 0$. The model therefore suffers negative temperatures which have actually the same order of magnitude as the remnant temperature for large n, as shown in figure (b). With each inflexion point, a diverging heat capacity is associated. Therefore the Black holes should exhibit a rich phase structure with alternating stable and unstable phases.

References

[1] M. Isi, M. Knipfer, J. Mureika and P. Nicolini, "Self-Completeness and the Generalized Uncertainty Principle in Extra Dimensions," *in progress* (my latest copy: April 26, 2014).

[2] A. Kempf, G. Mangano and R. B. Mann, "Hilbert space representation of the minimal length uncertainty relation," Phys. Rev. D **52** (1995) 1108 [hep-th/9412167].

[3] M. Isi, J. Mureika and P. Nicolini, "Self-Completeness and the Generalized Uncertainty Principle," JHEP **1311** (2013) 139 [arXiv:1310.8153 [hep-th]].

[4] P. Nicolini and E. Spallucci, "Holographic screens in ultraviolet self-complete quantum gravity," Adv. High Energy Phys. **2014** (2014) 805684 [arXiv:1210.0015 [hep-th]].

[5] T. G. Rizzo, "Noncommutative Inspired Black Holes in Extra Dimensions," JHEP **0609** (2006) 021 [arXiv:hep-ph/0606051].