Calc9

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Generation date: Thursday 27th March, 2014, 16:14

Calc9 ties up to Calc7, making more calculations with the holographic models. I clean up the syntax for $H \in \{h, h_\alpha\}$ and work with dimensionless quantities $z = r/L$. Furthermore I derive formulas from the general to the specific.

Contents

1 Heat capacity and Entropic corrections

This is the first document where I propose a $D = n + 4$ dimensional extension to the holographic model $h(r)$. The self-regular model $h_\alpha(r)$ was already discussed in Calc7.

$$h(r) = \frac{r^{2+n}}{r^{2+n} + L^{2+n}} \tag{1}$$

$$h'(r) = \frac{(2+n)\, r^{1+n} L^{2+n}}{(r^{2+n} + L^{2+n})^2} \tag{2}$$

$$h_\alpha(r) = \frac{r^{3+n}}{(r^\alpha + L^\alpha/2)^{\frac{3+n}{\alpha}}} \tag{3}$$

$$h'_\alpha(r) = \frac{(n+3) L^\alpha r^{n+2} \left(\frac{L^\alpha}{2} + r^\alpha\right)^{-\frac{n+3}{\alpha}}}{L^\alpha + 2r^\alpha} \tag{4}$$

Let $H \in \{h, h_\alpha\}$ be a generic profile as approximation of the theta function $\Theta(r)$ in $\rho(r)$, a class of densities for which I frequently derived the metric $g_{00} = 1 - V(r)$:

$$\rho(r) = \frac{M}{\Omega_{n+2}} \frac{\mathrm{d}H(r)}{\mathrm{d}r} \quad \Rightarrow \quad V(r) = \frac{2}{n+2} \frac{M}{M_*^{n+2}} \frac{1}{\Omega_{n+2}} \frac{H(r)}{r^{n+1}}. \tag{5}$$

1.1 The Mass

We can argue that M is just a constant, responsible for fulfilling the horizon equation $V(r_H) = 1$. If we set (»arbitrarily«)

$$M = \frac{n+2}{2} M_*^{n+2} \Omega_{n+2} \frac{r_H^{n+1}}{H(r_H)} = \frac{n+2}{2}\, \Omega_{n+2}\, \frac{1}{H(r_H)} \left(\frac{r_H}{L_*}\right)^{n+1} M_* \tag{6}$$

then the horizon equation $V(r) = 0$ is fulfilled at $r = r_H$. In general, equation 6 gives us a relationship $M = M(r)$. It can be used for given r to get the mass neccessary to create an event horizon at that r. In Calc7, I used it already for determining the remnant mass at $r = r_0$, in such a way that $M = M_*$ was obtained and $r_0 = L_*$ could be identified. Eventually, in models with (further) degrees of freedom (like $H = h_\alpha$), that equation also fixed α.

In particular, for $n = 0$, eq. 6 reduces to $M = r_H/2L^2 h(r_H)$. For $H = h$, we end with the well known $M = (r^2 + L^2)/2L^2 r$ from [NS 06.11.2013].

1.2 Dimensionless notation

My models $H(r)$ can be expressed in units of the dimensionless variable $z = r/L$ (which may be interpreted as »Multiples of the Planck unit«):

$$h(z) = 1/(1 + (1/z)^{2+n}) \tag{7}$$

$$h_\alpha(z) = 1/(1 + (1/z)^\alpha/2)^{(n+3)/\alpha} \tag{8}$$

The derivative $\frac{\mathrm{d}}{\mathrm{d}r}$ can be replaced by $\propto \frac{\mathrm{d}}{\mathrm{d}z}$ by determining $\frac{\mathrm{d}f}{\mathrm{d}r} = \frac{\mathrm{d}f}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}r} = \frac{1}{L}\frac{\mathrm{d}f}{\mathrm{d}z}$. We write $f'(z)$ for $\partial_z f(z)$:

$$h'(z) = (2 + n) h^2(z)/z^{3+n} \tag{9}$$

All quantities $Q \in \{g_{00}, V, M, T_H, C, S, \dots\}$ can be written in units of z. If $[Q] = L^k$ is the unit of Q (that is, the kth power of length which equals the $-k$ths power of energy), a seperation

$$Q(r) = L^k \tilde{Q}(z) \tag{10}$$

is always possible, with $[\tilde{Q}] = 1$. This can be checked, let's write some already derived expressions in terms of z:

$$V(z) = \frac{2}{n+2}\frac{M}{M_*}\frac{L^{n+1}}{M_*^{n+1}}\frac{1}{\Omega_{n+2}}\frac{H(z)}{z^{n+1}}. \qquad [V] = 1 \qquad V(r) = V(z) \tag{11}$$

$$M(z) = \frac{n+2}{2}\,\Omega_{n+2}\,\frac{z_H^{n+1}}{H(r_H)}\left(\frac{M_*}{L}\right)^{n+1} M_* \qquad [M] = 1/L \qquad M(r) = M(z)\dots \tag{12}$$

1.3 Extremal Radius and Remnants

For h_α, this section was discussed in Calc7. For h it is new.

The extremal radius equation $\partial_r g_{00} = {}^1\!/_L\, \partial_z g_{00} = 0$ can be written as

$$0 = \frac{\mathrm{d}H(z)}{\mathrm{d}z} - (n+1)\frac{H(z)}{z}, \tag{13}$$

an expression which looks like the one derived in Calc7, only by replacing $r \to z$. After inserting $H(z) = h(z)$, the expression $0 = (n+2)\frac{h^2}{z^{3+n}} - (n+1)\frac{h}{z}$ can be easily solved, giving

$$\boxed{r_0 = L\, z_0 = L\left(\frac{1}{1+n}\right)^{\frac{1}{2+n}}} \tag{14}$$

We can enforce the holographic metric to have the event horizon at $r_H = r_0$. Using (6), this gives us

$$M(r_0) = \frac{n+2}{2} \underbrace{\Omega_{n+2}}_{\text{ignored}} \underbrace{(n+2)}_{1/h(r_0)} \left(\frac{r_0^{n+1}}{L_*^{n+1}} \right) M_* \tag{15}$$

So unlike for h_α, no self encoding $M(r_0) = M_*$ can take place since $\frac{(n+2)^2}{2} \neq 1$.

1.4 The Heat Capacity

Equation 6 is important for determining the heat capacity, when using the expression

$$C = \frac{\partial M}{\partial T_H} = \frac{\partial M}{\partial r_H} \left(\frac{\partial T_H}{\partial r_H} \right)^{-1} \tag{16}$$

Actually it would be nice to have a closed form expression $T_H = T_H(M)$ but it is hard to become, sagt Nicolini. For calculating C in terms of z, we simply write

$$C = \frac{\partial M}{\partial T_H} = \frac{\partial M}{\partial z_H} \left(\frac{\partial T_H}{\partial z_H} \right)^{-1} \tag{17}$$

Expressions could also be mixed in r and z. Nothing special about that:

$$C = \frac{\partial M}{\partial r_H} \frac{\partial r_H}{\partial z_H} \frac{\partial z_H}{\partial T_H} = L \frac{\partial M}{\partial r_H} \left(\frac{\partial T_H}{\partial z_H} \right)^{-1} \tag{18}$$

1.5 The Entropy

The Black hole entropy integral can be rewritten in the same way like the Heat Capacity was rewritten in equation 16:

$$S(r) = \int_{M_1}^{M_2} \frac{\mathrm{d}M}{T} = \int_{r_1}^{r_2} \frac{\mathrm{d}M}{\mathrm{d}r_H} \frac{\mathrm{d}r_H}{T} = \int \mathrm{d}r_H \frac{1}{T} \left(\frac{\mathrm{d}M(r_H)}{\mathrm{d}r_H} \right) \tag{19}$$

This allows me to reproduce the NS2011 result, using $H = h$, $n = 0$:

$$\frac{\mathrm{d}M}{\mathrm{d}r_H} = \frac{\mathrm{d}}{\mathrm{d}r} \left(\frac{1}{2L^2 r} (r^2 + L^2) \right) = \frac{1}{2} \left(\frac{1}{L^2} + \frac{1}{r^2} \right) \tag{20}$$

$$T = \frac{1}{4\pi r_H} \left(1 - \frac{2L^2}{r_H^2 + L^2} \right) \tag{21}$$

$$S = 4\pi \int_L^{r_H} r \left(\frac{r}{2L^2} + \frac{1}{2r} \right) \frac{1}{1 - \frac{2L^2}{r^2+L^2}} = \pi \left(\frac{r^2}{L^2} + 2\log(r) \right)_L^{r_H} \tag{22}$$

Like always, $S(r) = S(z)$ since $[S] = 1$ in natural units:

$$S(z) = \int_{z_1}^{z_2} \frac{\mathrm{d}M}{T} = \int \mathrm{d}z_H \frac{1}{T} \left(\frac{\mathrm{d}M(z_H)}{\mathrm{d}z_H} \right) \tag{23}$$

1.6 A generic approach to T_H, C and S

By merging all constant (non-r dependent) terms in the metric (5) and mass term (6), generic calculations with any H and n can be performed in a very simple way.

To do these calculations, let's shortly forget about H and just seperate $V(r)$ in a suggestive way:

$$
\begin{aligned}
V(r) &= M(r_H)\cdot Y(r) && (24)\\
M(r_H) &= Y^{-1}(r_H) && (25)\\
T &= \frac{1}{4\pi}\,\partial_r g_{00}|_{r=r_H} = -\frac{1}{4\pi}V'(r_H) && (26)\\
&= -\frac{1}{4\pi}M(r_H)\cdot Y'(r_H) = -\frac{1}{4\pi}\frac{1}{L}M(z_H)Y'(z_H) && (27)\\
S(z) &= \int^z \mathrm{d}z_H \frac{M'(z_H)}{T} = -4\pi L\int^z \mathrm{d}z_H \frac{M'(z_H)}{M(z_H)}\frac{1}{Y'(z_H)} && (28)
\end{aligned}
$$

It is important to note that (25) is only valid for r_H, so Y is not the inverse of M and the inverse derivative law cannot be applied (in general, $M(r) \neq Y^{-1}(r)$). In terms of r, M is constant: $M'(r) = 0$.

We can now introduce the holographic approach

$$
Y(r) = A\frac{H}{r^{n+1}} \tag{29}
$$

A can be eliminated quickly in T due to the property of (25).

$$
\begin{aligned}
V &= A\,M(r_H)\,\frac{H(r)}{r^{n+1}} && (30)\\
M(r_H) &= \frac{1}{A}\frac{r_H^{n+1}}{H(r_H)} && (31)\\
T &= \frac{1}{4\pi r_H}\left(1+n-r_H\frac{H'(r_H)}{H(r_H)}\right) = \frac{1}{L}\frac{1}{4\pi z_H}\left(1+n-z_H\frac{H'(z_H)}{H(z_H)}\right) && (32)\\
C &= \frac{4\pi r_H^{n+2}}{A}\frac{r_H H'(r_H)-(n+1)H(r_H)}{r_H^2 H(r_H)\,H''(r_H)-r_H^2 H'(r_H)^2+(n+1)H(r_H)^2} && (33)\\
S(z) &= -4\pi L\int^z \mathrm{d}x\left(\frac{n+1}{x}-\frac{H'(x)}{H(x)}\right)\frac{1}{Y'(x)} && (34)\\
&= -4\pi LA\int^z \mathrm{d}x\left(\frac{n+1}{x}-\frac{H'(x)}{H(x)}\right)\frac{x^{n+2}}{xH'(x)-(n+1)H(x)} && (35)
\end{aligned}
$$

The simple relation (32) was already asserted in Calc7, eq. 20, but not believed yet. Writing terms in z is handy because the resulting terms are dimensionless. Attached powers of L entirely indicate the physical units of quantities, like $[T] = [1/L]$.

1.6.1 Check with $n = 0$

I checked that, with $n = 0, H = h$, (32) and (33) gives the [NS 06.11.2013] result

$$T = \frac{1}{4\pi r}\left(1 - \frac{2L^2}{L^2 + r_H^2}\right) \tag{36}$$

$$C = -4\pi \frac{(L - r_H)(r_H + L)\left(r_H^2 + L^2\right)^2}{2L^2\left(4L^2 r_H^2 - r_H^4 + L^4\right)} \tag{37}$$

Therefore I claim (32) and (33) to be true.

1.6.2 Values for $h_\alpha(r)$

This section was already done in Calc7.

1.6.3 Values for $h(r)$

Inserting $H(r) = h(r)$ in (32) and (33) gives

$$T = \frac{1}{4\pi r_H}\left(1 + n - \frac{(2+n)\left(\frac{L}{r_H}\right)^n}{\left(\frac{L}{r_H}\right)^n + \left(\frac{r_H}{L}\right)^2}\right) = \frac{1}{4\pi z_H}\frac{1}{L}\left(1 + n - \frac{2+n}{1 + z_H^{2+n}}\right) \tag{38}$$

$$C = -\frac{r_H^{n+2}}{A} \cdot \text{langes zeug} \tag{39}$$

$$S_h(z) = 4\pi A L\left(\frac{x^{n+2}}{n+2} + \log(x)\right)_1^z \tag{40}$$

2 Questions

- (Minor) Integral boundaries for S
- (Major) Propagator calculations: Eq (20) in [N Feb2012] is at least $\propto \frac{1}{r}\partial_r h_{n=0}(r)$. How to extend?