Generalized Uncertainty Principle, Extra-dimensions and Holography

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Abstract

We consider Uncertainty Principles which take into account the role of gravity and the possible existence of extra spatial dimensions. Explicit expressions for such Generalized Uncertainty Principles in $4+n$ dimensions are given and their holographic properties investigated. In particular, we show that the predicted number of degrees of freedom enclosed in a given spatial volume matches the holographic counting only for one of the available generalizations and without extra dimensions.

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1 Introduction

During the last years many efforts have been devoted to clarifying the role played by the existence of extra spatial dimensions in the theory of gravity [1, 2]. One of the most interesting predictions drawn from the theory is that there should be measurable deviations from the $1/r^2$ law of Newtonian gravity at short (and perhaps also at large) distances. Such new laws of gravity would imply modifications of those Generalized Uncertainty Principles (GUP's) designed to account for gravitational effects in the measure of positions and energies.

On the other hand, the holographic principle is claimed to apply to all of the gravitational systems. The existence of GUP's satisfying the holography in four dimensions (one of the main examples is due to Ng and Van Dam [3]) led us to explore the holographic properties of the GUP's extended to the brane-world scenarios. The results, at least for the examples we considered, are quite surprising. The expected holographic scaling indeed seems to hold only in four dimensions, and only for the Ng and van Dam's GUP. When extra spatial dimensions are admitted, the holography is destroyed. This fact allows two different interpretations: either the holographic principle

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is not universal and does not apply when extra dimensions are present; or, on the contrary, we take seriously the holographic claim in any number of dimensions, and our results are therefore evidence against the existence of extra dimensions.

In Section 2 we analyze GUP's obtained by linearly combining quantum mechanical expressions with general relativistic bounds [4]; in Section 3 we repeat the same analysis for the type of GUP's discussed in Refs. [3] and in Section 4 we comment on our results. The four-dimensional Newton constant is denoted by $G_{\rm N}$ throughout the paper.

2 Linear GUP's from micro black holes

In this Section we derive GUP's via a micro black hole gedanken experiment, following closely the content of Ref. [4] which we then generalize to space-times with extra dimensions.

2.1 GUP in four dimensions

When we measure a position with precision of order Δx, we expect quantum fluctuations of the metric field around the measured position with energy amplitude

$$\Delta E \sim \frac{\hbar\, c}{2\,\Delta x}\ . \tag{2.1}$$

The Schwarzschild radius associated with the energy ΔE,

$$R_{\rm S} = \frac{2\,G_{\rm N}\,\Delta E}{c^4}\ , \tag{2.2}$$

falls well inside the interval Δx for practical cases. However, if we wanted to improve the precision indefinitely, the fluctuation ΔE would grow up and the corresponding $R_{\rm S}$ would become larger and larger, until it reaches the same size as Δx. As it is well known, the critical length is the Planck length,

$$\Delta x = R_{\rm S} \quad \Rightarrow \quad \Delta x = \left(\frac{G_{\rm N}\,\hbar}{c^3}\right)^{1/2} \equiv \ell_{\rm p}\ , \tag{2.3}$$

and the associated energy is the Planck energy

$$\epsilon_{\rm p} \equiv \frac{\hbar\, c}{2\,\ell_{\rm p}} = \frac{1}{2}\left(\frac{\hbar\, c^5}{G_{\rm N}}\right)^{1/2}\ . \tag{2.4}$$

If we tried to further decrease Δx, we should concentrate in that region an energy greater than the Planck energy, and this would enlarge further the Schwarzschild radius $R_{\rm S}$, hiding more and more details of the region beyond the event horizon of the micro hole. The situation can be summarized by the inequalities

$$\Delta x \gtrsim \begin{cases} \frac{\hbar\, c}{2\,\Delta E} & \text{for} \quad \Delta E < \epsilon_{\rm p} \\[2ex] \frac{2\,G_{\rm N}\,\Delta E}{c^4} & \text{for} \quad \Delta E > \epsilon_{\rm p}\ . \end{cases} \tag{2.5}$$

which, if combined linearly, yield

$$\Delta x \gtrsim \frac{\hbar\, c}{2\,\Delta E} + \frac{2\,G_{\rm N}\,\Delta E}{c^4}\ . \tag{2.6}$$

This is a generalization of the uncertainty principle to cases in which gravity is important, i.e. to energies of the order of ϵ_{p}. We note that the minimum value of Δx is reached for $(\Delta E)_{\mathrm{min}} = \epsilon_{\mathrm{p}}$ and is given by $(\Delta x)_{\mathrm{min}} = 2\,\ell_{\mathrm{p}}$.

2.2 GUP with n extra dimensions

We shall now generalize the procedure outlined in the previous Subsection to a space-time with $4+n$ dimensions, where n is the number of space-like extra dimensions. The first problem we should address is how to relate the gravitational constant G_{N} in four dimensions with the one in $4+n$, henceforth denoted by $G_{(4+n)}$.

This of course depends on the model of space-time with extra dimensions we consider. Models appeared in the literature in recent years belong mostly to two scenarios:

- the Arkani-Hamed–Dimopoulos–Dvali (ADD) model [1], where the extra dimensions are compact and of size L;
- the Randall–Sundrum (RS) model [2], where the extra dimensions have an infinite extension but are warped by a non-vanishing cosmological constant.

A feature shared by (the original formulations of) both scenarios is that only gravity propagates along the n extra dimensions, while Standard Model fields are confined on a four-dimensional sub-manifold usually referred to as the *brane-world*.

In the ADD case the link between G_{N} and $G_{(4+n)}$ can be fixed by comparing the gravitational action in four dimensions with the one in $4+n$ dimensions. The space-time topology in such models is $\mathcal{M} = \mathcal{M}^4 \otimes \Re^n$, where $\mathcal{M}^4$ is the usual four-dimensional space-time and $\Re^n$ represents the extra dimensions of finite size L. The space-time brane has no tension and therefore the action $S_{(4+n)}$ can be written as

$$S_{(4+n)} = \frac{c^4}{16\,\pi\,G_{(4+n)}} \int_{\mathcal{M}^4\otimes\Re^n} d^{4+n}x\,\sqrt{-g}\,R \sim \frac{c^4}{16\,\pi\,G_{(4+n)}} \int_{\mathcal{M}^4} d^4x\,\sqrt{-\tilde{g}}\,L^n\,\tilde{R}\ , \tag{2.7}$$

where $\tilde{R}$, $\tilde{g}$ are the projections on $\mathcal{M}^4$ of R and g. Here L^n is the "volume" of the extra dimensions and we omitted unimportant numerical factors. On comparing the above expression with the purely four-dimensional action

$$S_{(4)} = \frac{c^4}{16\,\pi\,G_{\mathrm{N}}} \int_{\mathcal{M}^4} d^4x\,\sqrt{-\tilde{g}}\,\tilde{R}\ , \tag{2.8}$$

we obtain

$$G_{(4+n)} \sim G_{\mathrm{N}}\,L^n\ . \tag{2.9}$$

The RS models are more complicated. It can be shown [2] that for $n=1$ extra dimension we have $G_{(4+n)} = \sigma^{-1}\,G_{\mathrm{N}}$, where σ is the brane tension with dimensions of $length^{-1}$ in suitable units. The gravitational force between two point-like masses m and M on the brane is now given by

$$F = G_{\mathrm{N}}\,\frac{m\,M}{r^2}\left(1+\frac{e^{-\sigma r}}{\sigma^2 r^2}\right)\ , \tag{2.10}$$

where the correction to Newton law comes from summing over the extra dimensional graviton modes in the graviton propagator [2]. However, since Eq. (2.10) is obtained by perturbative calculations, not immediately applicable to a non-perturbative structure such as a black hole, we shall consider only the ADD scenario in this paper. To be more precise, from table-top tests of the gravitational force one finds that $n \geq 2$ in ADD [1, 5]. On the other hand, black holes with mass $M \ll \sigma^{-1}$ are likely to behave as pure five-dimensional in RS [6], therefore our results for $n = 1$ should apply to such a case.

In order to proceed as in the previous Section, we now need a formula for the Schwarzschild radius in $4+n$ dimensions. This can be obtained, heuristically, from the gravitational force law in $4+n$ dimensions [7] as determined by Gauss theorem,

$$F = G_{(4+n)} \frac{m\,M}{r^{2+n}} \ . \tag{2.11}$$

Therefore, the total energy of a particle of mass m in the gravitational field of the source M is given by

$$E_{\rm tot} = \frac{1}{2}\, m\, v^2 - G_{(4+n)} \frac{m\,M}{r^{1+n}} \ , \tag{2.12}$$

and the escape velocity is

$$v_f^2 = 2\, G_{(4+n)} \frac{M}{r^{1+n}} \ . \tag{2.13}$$

Requiring $v_f = c$, we obtain for the Schwarzschild radius

$$R_{(4+n)} = \left(\frac{2\, G_{(4+n)}\, M}{c^2}\right)^{\frac{1}{1+n}} \ . \tag{2.14}$$

We shall show in the next Section that an exact calculations based on the higher dimensional Schwarzschild solution [12] just modifies this result by numerical factors.

In the following, we shall just consider micro black holes with $R_{(4+n)} \ll L$, so as to avoid the complications that are expected when the Schwarzschild radius approaches the compactification length [8]. Moreover, the opposite case ($R_{(4+n)} \gg L$) would imply the complete non-observability of extra dimensions, hidden beyond the event horizon.

The radius $R_{(4+n)}$ is related to $R_{(4)} \equiv R_{\rm S}$ as given in Eq. (2.2) according to

$$R_{(4+n)} = \left(\frac{2\, G_{(4+n)}\, M}{c^2}\right)^{\frac{1}{1+n}} = \left(\frac{2\, G_{\rm N}\, M\, L^n}{c^2}\right)^{\frac{1}{1+n}} = R_{\rm S}^{\frac{1}{1+n}}\, L^{\frac{n}{1+n}} \ , \tag{2.15}$$

and, from $R_{(4+n)} \ll L$, we can infer the following inequalities:

$$R_{(4+n)} \ll L \quad \Rightarrow \quad R_{(4+n)}^{1+n} \ll L^{1+n} \quad \Rightarrow \quad R_{\rm S}\, L^n \ll L^{1+n} \quad \Rightarrow \quad R_{\rm S} \ll L \ ; \tag{2.16}$$

$$R_{\rm S} \ll L \quad \Rightarrow \quad R_{\rm S}^{\frac{n}{n+1}} \ll L^{\frac{n}{n+1}} \quad \Rightarrow \quad R_{\rm S} \ll L^{\frac{n}{n+1}}\, R_{\rm S}^{\frac{1}{n+1}} \quad \Rightarrow \quad R_{\rm S} \ll R_{(4+n)} \ . \tag{2.17}$$

Therefore,

$$R_{\rm S} \ll R_{(4+n)} \ll L \tag{2.18}$$

and the Schwarzschild radius of a black hole in a space-time with extra dimensions is greater than in four dimensions [7, 8].

Since measurements can be performed only on the brane, to the uncertainty Δx in position we can still associate an energy given by Eq (2.1). The corresponding Schwarzschild radius is now given by Eq. (2.14) with $M = \Delta E/c^2$ and the critical length such that $\Delta x = R_{(4+n)}$ is the Planck length in $4+n$ dimensions,

$$\Delta x = \left(\frac{G_{(4+n)}\,\hbar}{c^3}\right)^{\frac{1}{2+n}} = \left(\frac{G_{\rm N}\,\hbar}{c^3}\,L^n\right)^{\frac{1}{2+n}} = \left(\ell_{\rm p}^2\,L^n\right)^{\frac{1}{n+2}} \equiv \ell_{(4+n)} \ . \tag{2.19}$$

The energy associated with $\ell_{(4+n)}$ is analogously the Planck energy in $4+n$ dimensions,

$$\epsilon_{(4+n)} = \frac{1}{2}\left(\frac{\hbar\,c^5}{G_{\rm N}}\frac{\hbar^n\,c^n}{L^n}\right)^{\frac{1}{n+2}} = \frac{1}{2}\left[4\,\epsilon_{\rm p}^2\left(\frac{\hbar\,c}{L}\right)^n\right]^{\frac{1}{n+2}} \ , \tag{2.20}$$

where $\epsilon_{\rm p}$ is the Planck energy in 4 dimensions given in Eq. (2.4).

It is reasonable to assume that $\ell_{\rm p} \ll L$, otherwise the extra dimensions would not have a classical space-time structure. We then have

$$\ell_{\rm p} \ll L \quad \Rightarrow \quad \ell_{\rm p}^n \ll L^n \quad \Rightarrow \quad \ell_{\rm p}^{2+n} \ll \ell_{\rm p}^2\,L^n = \ell_{(4+n)}^{2+n} \quad \Rightarrow \quad \ell_{\rm p} \ll \ell_{(4+n)} \ . \tag{2.21}$$

Further, we can also prove that

$$\ell_{\rm p} \ll L \quad \Rightarrow \quad \ell_{\rm p}^2 \ll L^2 \quad \Rightarrow \quad \ell_{(4+n)}^{2+n} = \ell_{\rm p}^2\,L^n \ll L^{2+n} \quad \Rightarrow \quad \ell_{(4+n)} \ll L \ . \tag{2.22}$$

Summarizing, from $\ell_{\rm p} \ll L$ we obtain

$$\ell_{\rm p} \ll \ell_{(4+n)} \ll L \ , \tag{2.23}$$

and correspondingly

$$\frac{\hbar\,c}{2\,L} \ll \epsilon_{(4+n)} \ll \epsilon_{\rm p} \ , \tag{2.24}$$

so that the Planck energy threshold, where quantum gravity phenomena become important, is lowered by the existence of extra dimensions [1]. Finally, we can check the inequalities among $\ell_{\rm p}$, $R_{\rm S}$, $R_{(4+n)}$, and $\ell_{(4+n)}$. We can easily prove that

$$\ell_{(4+n)} < R_{(4+n)} \ , \tag{2.25}$$

since

$$\ell_{(4+n)} < R_{(4+n)} \quad \Leftrightarrow \quad \left(\ell_{\rm p}^2\,L^n\right)^{\frac{1}{n+2}} < \left(R_{\rm S}\,L^n\right)^{\frac{1}{n+1}} \quad \Leftrightarrow \quad \ell_{\rm p}^{n+2}\,\ell_{\rm p}^n < R_{\rm S}^{n+2}\,L^n \ , \tag{2.26}$$

and the last inequality holds by virtue of $\ell_{\rm p} < R_{\rm S}$ and $\ell_{\rm p} < L$. We are therefore left with two possible chains of inequalities,

$$\ell_{\rm p} < R_{\rm S} < \ell_{(4+n)} < R_{(4+n)} < L \tag{2.27}$$

$$\ell_{\rm p} < \ell_{(4+n)} < R_{\rm S} < R_{(4+n)} < L \ , \tag{2.28}$$

and, in general, it is not possible to tell if $R_{\rm S} < \ell_{(4+n)}$ or $R_{\rm S} > \ell_{(4+n)}$.

Now, let us come back to the GUP. The argument goes precisely as in four dimensions and one therefore obtains the following inequalities

$$\Delta x \gtrsim \begin{cases} \frac{\hbar c}{2\,\Delta E} & \text{for} \quad \Delta E < \epsilon_{(4+n)} \\ \left(\frac{2\,G_{(4+n)}\,\Delta E}{c^4}\right)^{\frac{1}{n+1}} & \text{for} \quad \Delta E > \epsilon_{(4+n)} \ . \end{cases} \tag{2.29}$$

Combining linearly the previous inequalities, we obtain

$$\Delta x \gtrsim \frac{\hbar\, c}{2\,\Delta E} + \left(\frac{2\,G_{(4+n)}\Delta E}{c^4}\right)^{\frac{1}{n+1}} \ , \tag{2.30}$$

which is a straightforward generalization of Eq. (2.6) to the case with n extra dimensions. The minimum value for Δx is now reached when $(\Delta E)_{\rm min} = (1+n)^{\frac{1+n}{2+n}}\,\epsilon_{\rm p}$ and we then have

$$(\Delta x)_{\rm min} = \left[(1+n)^{-\frac{1+n}{2+n}} + (1+n)^{\frac{1}{2+n}}\right]\,\ell_{(4+n)} \ . \tag{2.31}$$

2.3 Holographic properties

In this Subsection, we investigate the holographic properties of the GUP's which we have proposed this far. We shall estimate the number of degrees of freedom $n(V)$ contained in a spatial volume (cube or "hypercube") of size l. The holographic principle claims that $n(V)$ scales as the area of the (hyper-)surface enclosing the given volume, that is $(l/\ell_{\rm p})^{2+n}$ in $4+n$ dimensions.

For the GUP's considered in the previous Subsections we find:

a) for the four-dimensional GUP in Eq. (2.6), this scaling does not occur. In fact, $(\Delta x)_{\rm min} \sim \ell_{\rm p}$ and a cube of side l contains a number of degrees of freedom equal to

$$n(V) \sim \left(\frac{l}{\ell_{\rm p}}\right)^3 \ . \tag{2.32}$$

b) for the GUP in $4+n$ dimensions of Eq. (2.30), the minimum value for Δx is given in Eq. (2.31) and, a part from numerical factors, we see that the holographic scaling again does not hold,

$$n(V) \sim \left(\frac{l}{(\Delta x)_{\rm min}}\right)^{3+n} \sim \left(\frac{l}{\ell_{(4+n)}}\right)^{3+n} \ . \tag{2.33}$$

We then conclude that GUP's obtained by linearly combining the quantum mechanical expression with gravitational bounds do not imply the holographic counting of degrees of freedom.

3 Ng and Van Dam GUP's

An interesting GUP that satisfies the holographic principle in four dimensions has been proposed by Ng and van Dam in various papers [3]. They start from the Wigner inequalities about distance measurements with clocks and light signals.

3.1 GUP in four dimensions

Suppose we wish to measure a distance l. Our measuring device is composed of a clock, a photon detector and a photon gun. A mirror is put at the distance l we want to measure and m is the mass of the system "clock + photon detector + photon gun". We call "detector" the whole system and let a be its size. Obviously, we suppose

$$a > r_{\rm g} \equiv \frac{2\,G_{\rm N}\,m}{c^2} = R_{\rm S}(m) \ , \tag{3.1}$$

which means that we are not using a black hole as a clock. Be Δx_1 the uncertainty in the position of the detector. Then the uncertainty on the detector velocity is

$$\Delta v = \frac{\hbar}{2\,m\,\Delta x_1} \ . \tag{3.2}$$

After a time $T = 2\,l/c$, elapsed during the light trip detector–mirror–detector, the uncertainty on the detector position (i.e. the uncertainty on the actual length of the segment l) has become

$$\Delta x_{\rm tot} = \Delta x_1 + T\,\Delta v = \Delta x_1 + \frac{\hbar\,T}{2\,m\,\Delta x_1} \ . \tag{3.3}$$

We can minimize $\Delta x_{\rm tot}$ by suitably choosing Δx_1,

$$\frac{\partial \Delta x_{\rm tot}}{\partial \Delta x_1} = 0 \quad \Rightarrow \quad (\Delta x_1)_{\rm min} = \left(\frac{\hbar\,T}{2\,m}\right)^{1/2} \ . \tag{3.4}$$

Hence

$$(\Delta x_{\rm tot})_{\rm min} = (\Delta x_1)_{\rm min} + \frac{\hbar T}{2\,m\,(\Delta x_1)_{\rm min}} = 2\left(\frac{\hbar\,T}{2\,m}\right)^{1/2} \ . \tag{3.5}$$

Since $T = 2\,l/c$, we have

$$(\Delta x_{\rm tot})_{\rm min} = 2\left(\frac{\hbar\,l}{m\,c}\right)^{1/2} \equiv \delta l_{\rm QM} \ . \tag{3.6}$$

This is a purely quantum mechanical result obtained, for the first time, by Wigner in 1958 [9]. From Eq. (3.6), it seems that we can reduce the error $(\Delta x_{\rm tot})_{\rm min}$ as much as we want by choosing m very large, since $(\Delta x_{\rm tot})_{\rm min} \to 0$ for $m \to \infty$. But, obviously, here gravity enters the game.

A first remark is that the length l must be greater than the Schwarzschild radius of the detector with mass m,

$$l > r_{\rm g} \quad \Rightarrow \quad \frac{1}{m} > \frac{2\,G_{\rm N}}{l\,c^2} \quad \Rightarrow \quad (\Delta x_{\rm tot})^2_{\rm min} \gtrsim 8\,\ell_{\rm p}^2 \ . \tag{3.7}$$

A second consideration (due to Amelino-Camelia [10]) is that the measuring device must not be a black hole,

$$a > r_{\rm g} \quad \Rightarrow \quad (\Delta x_{\rm tot})^2_{\rm min} \gtrsim 8\,\ell_{\rm p}^2\,\frac{l}{a} \ . \tag{3.8}$$

The typical scenario is $\ell_{\rm p} \ll a \leq l$. Also in the ideal case, when $a \sim \ell_{\rm p}$, we have

$$(\Delta x_{\rm tot})_{\rm min} \gtrsim 2\sqrt{2\, l\, \ell_{\rm p}}\ . \tag{3.9}$$

Ng and van Dam have also considered a further source of error, a purely gravitational error, besides the purely quantum mechanical one already addressed. Suppose the clock has spherical symmetry, with $a > r_{\rm g}$. Then the error due to curvature can be computed from the Schwarzschild metric surrounding the clock. The optical path from $r_0 > r_{\rm g}$ to a generic point $r > r_0$ is given by (see, for example, Ref. [11])

$$c\,\Delta t = \int_{r_0}^{r} \frac{d\rho}{1-\frac{r_{\rm g}}{\rho}} = (r - r_0) + r_{\rm g}\,\log\frac{r - r_{\rm g}}{r_0 - r_{\rm g}}\ , \tag{3.10}$$

and differs from the "true" (spatial) length $(r - r_0)$. If we put $a = r_0$, $l = r$, the gravitational error on the measure of $(l - a)$ is thus

$$\delta l_{\rm C} = r_{\rm g}\,\log\frac{l - r_{\rm g}}{a - r_{\rm g}} \sim r_{\rm g}\,\log\frac{l}{a}\ , \tag{3.11}$$

where the last estimate holds for $l > a \gg r_{\rm g}$.

If we measure a distance $l \geq 2a$, then the error due to curvature is

$$\delta l_{\rm C} \geq r_{\rm g} \log 2 \simeq \frac{G_{\rm N} m}{c^2}. \tag{3.12}$$

Thus, according to Ng and van Dam the total error is

$$\delta l_{\rm tot} = \delta l_{\rm QM} + \delta l_{\rm C} = 2\left(\frac{\hbar\, l}{m\, c}\right)^{1/2} + \frac{G_{\rm N}\, m}{c^2}\ . \tag{3.13}$$

This error can be minimized again by choosing a suitable value for the mass of the clock,

$$\frac{\partial l_{\rm tot}(m)}{\partial m} = 0 \quad \Rightarrow \quad m_{\rm min} = c\left(\frac{\hbar\, l}{G_{\rm N}^2}\right)^{1/2} \tag{3.14}$$

and we then have

$$(\delta l_{\rm tot})_{\rm min} = 2\left(\frac{\hbar\, G_{\rm N}\, l}{c^3}\right)^{1/3} + \left(\frac{\hbar\, G_{\rm N}\, l}{c^3}\right)^{1/3} = 3\,\left(\ell_{\rm p}^2\, l\right)^{1/3}\ . \tag{3.15}$$

The global uncertainty on l contains therefore a term proportional to $l^{1/3}$.

3.2 GUP with n extra dimensions

Ng and van Dam's derivation can be generalized to the case with n extra dimensions. The Wigner relation (3.6) for the quantum mechanical error is not modified by the presence of extra dimensions and we just need to estimate the error $\delta l_{\rm C}$ due to curvature.

We are not considering now micro black holes created by the fluctuations ΔE in energy, as in the previous Section. Instead, we have to deal with (more or less) macroscopic clocks and distances and this implies that we have to distinguish four different cases:

1. $0 < L < r_g < a < l$;
2. $0 < r_{(4+n)} < L < a < l$;
3. $0 < r_{(4+n)} < a < L < l$;
4. $0 < r_{(4+n)} < a < l < L$;

where $r_{(4+n)} = R_{(4+n)}(m)$, and $r_{\rm g} = r_{(4)}$ as before. The curvature error will be calculated (as in the previous Subsection) by computing the optical path from $a \equiv r_o$ to $l \equiv r$. Of course, we will use a metric which depends on the relative size of L with respect to a and l, that is the usual four-dimensional Schwarzschild metric in the region $L < r$, and the $4+n$ dimensional Schwarzschild solution in the region $r < L$ (where the extra dimensions play an actual role).

In cases 1. and 2. the optical path from a to l can be computed using just the four-dimensional Schwarzschild solution and the result is given by Eq. (3.15) in the previous Subsection.

In cases 3. and 4. we instead have to use the Schwarzschild solution in $4+n$ dimensions [12],

$$ds^2 = -\left(1 - \frac{C}{r^{n+1}}\right) c^2 dt^2 + \left(1 - \frac{C}{r^{n+1}}\right)^{-1} dr^2 + r^2 d\Omega_{n+2}^2 \ , \tag{3.16}$$

at least for a part of the optical path. In the above,

$$C = \frac{16\,\pi\, G_{(4+n)}\, m}{(n+2)\, A_{n+2}\, c^2} \ , \tag{3.17}$$

and A_{n+2} is the area of the unit $(n+2)$-sphere, that is

$$A_{n+2} = \frac{2\,\pi^{\frac{n+3}{2}}}{\Gamma\left(\frac{n+3}{2}\right)} \ . \tag{3.18}$$

Besides, we note that, for $n = 0$,

$$C = \frac{2\, G_{\rm N}\, m}{c^2} = r_{\rm g} \ , \tag{3.19}$$

that is, C coincides in four dimensions with the Schwarzschild radius of the detector. The Schwarzschild horizon is located where $(1 - C/r^{n+1}) = 0$, that is at $r = C^{1/(n+1)} \equiv r_{(4+n)}$, or

$$r_{(4+n)} = \left[\frac{16\,\pi\, G_{(4+n)}\, m}{(n+2)\, A_{n+2}\, c^2}\right]^{\frac{1}{n+1}} \ , \tag{3.20}$$

in qualitative agreement with the expression obtained in Subsection 2.2.

In case 3. we obtain the optical path from a to l by adding up the optical path from a to L and that from L to l. We have to use the solution in $4+n$ dimensions for the first part, and the four-dimensional solution for the second part of the path,

$$\begin{aligned} c\,\Delta t &= \int_a^L \left(1 + \frac{C}{r^{n+1} - C}\right) dr + \int_L^l \left(1 + \frac{r_{\rm g}}{r - r_{\rm g}}\right) dr \\ &= (L - a) + (l - L) + C \int_a^L \frac{dr}{r^{n+1} - C} + r_{\rm g} \int_L^l \frac{dr}{r - r_{\rm g}} \ . \end{aligned} \tag{3.21}$$

We have shown before that from $r_{(4+n)} < L$ (that holds in cases 3. and 4.) we can infer

$$r_g < r_{(4+n)} < L \ . \tag{3.22}$$

Now, suppose $a^{n+1} \gg C = r_{(4+n)}^{n+1}$, that is $a \gg r_{(4+n)}$, so that we are not doing measures inside a black hole. Then $r_g < r_{(4+n)} \ll a < L < l$ and

$$\begin{aligned} c\,\Delta t &\simeq (l-a) + C \int_a^L \frac{dr}{r^{n+1}} + r_{\rm g} \int_L^l \frac{dr}{r} = (l-a) + \frac{C}{n}\left(\frac{1}{a^n} - \frac{1}{L^n}\right) + r_{\rm g}\, \log\frac{l}{L} \\ &= (l-a) + \frac{1}{n}\left(\frac{1}{a^n} - \frac{1}{L^n}\right) \frac{16\,\pi\, G_{(4+n)}}{(n+2) A_{n+2}\, c^2}\, m + \left(\frac{2\,G_{\rm N}}{c^2}\, \log\frac{l}{L}\right) m \ . \end{aligned} \tag{3.23}$$

The error caused by the curvature (when $a < L < l$) is therefore linear in m,

$$\delta l_{\rm C} = \left[\frac{1}{n}\left(\frac{1}{a^n} - \frac{1}{L^n}\right) \frac{16\,\pi\, G_{\rm N} L^n}{(n+2)\, A_{n+2}\, c^2} + \frac{2\,G_{\rm N}}{c^2} \log\frac{l}{L}\right] m \equiv K\, m \ . \tag{3.24}$$

We recall that the curvature error in four dimensions does not contain the size of the clock. On the contrary, this error in $4+n$ dimensions depends explicitly on the size a of the clock and on the size L of the extra dimensions. Hence the total error is given by

$$\delta l_{\rm tot} = \delta l_{\rm QM} + \delta l_C = 2\left(\frac{\hbar\, l}{m\, c}\right)^{1/2} + K\, m = J\, m^{-1/2} + K\, m \ , \tag{3.25}$$

where $J = 2\,(\hbar\, l/c)^{1/2}$ and K was defined before. This error can be minimized with respect to m,

$$\frac{\partial \delta l_{\rm tot}}{\partial m} = 0 \quad \Rightarrow \quad m_{\rm min} = \left(\frac{J}{2\,K}\right)^{2/3} \ . \tag{3.26}$$

Finally,

$$\begin{aligned} (\delta l_{\rm tot})_{\rm min} &= \left(2^{1/3} + 2^{-2/3}\right) \left(K\, J^2\right)^{1/3} \\ &= 2\left(2^{1/3} + 2^{-2/3}\right) \left[\frac{1}{n}\left(\frac{1}{a^n} - \frac{1}{L^n}\right) \frac{8\,\pi}{(n+2)\, A_{n+2}}\, \ell_{(4+n)}^{2+n}\, l + \ell_{\rm p}^2\, l\, \log\frac{l}{L}\right]^{1/3} \end{aligned} \tag{3.27}$$

where we used the definition of J and K.

In case 4., the optical path from a to l can be obtained by using simply the Schwarzschild solution in $4+n$ dimensions. We get

$$c\,\Delta t = \int_a^l \left(1 + \frac{C}{r^{n+1} - C}\right) dr = (l-a) + C \int_a^l \frac{dr}{r^{n+1} - C} \ . \tag{3.28}$$

Suppose now, as before, that $a^{n+1} \gg C = r_{(4+n)}^{n+1}$, that is $a \gg r_{(4+n)}$ (i.e. our clock is not a black hole). We then have

$$c\,\Delta t \simeq (l-a) + C \int_a^l \frac{dr}{r^{n+1}} = (l-a) + \frac{C}{n}\left(\frac{1}{a^n} - \frac{1}{l^n}\right) \ . \tag{3.29}$$

If the distance we are measuring is, at least, of the size of the clock ($l \geq 2\,a$), we can write

$$c\,\Delta t \gtrsim (l-a) + \frac{C}{n}\left(\frac{2^n-1}{2^n\,a^n}\right) . \tag{3.30}$$

The error caused by the curvature is therefore (when $a < l < L$)

$$\delta l_{\mathrm{C}} = \frac{C}{n}\left(\frac{2^n-1}{2^n\,a^n}\right) . \tag{3.31}$$

Here we again note that the curvature error in $4+n$ dimensions explicitly contains the size of the clock. The global error can be computed as before

$$\delta l_{\mathrm{tot}} = \delta l_{\mathrm{QM}} + \delta l_C = 2\left(\frac{\hbar l}{m\,c}\right)^{1/2} + \frac{C}{n}\left(\frac{2^n-1}{2^n\,a^n}\right) , \tag{3.32}$$

where C is linear in m. Minimizing δl_{tot} with respect to m can be done in perfect analogy with the previous calculation. The result is

$$(\delta l_{\mathrm{tot}})_{\mathrm{min}} = \left(2^{1/3} + 2^{-2/3}\right)\left(\frac{2^n-1}{2^n\,n}\,\frac{64\,\pi}{(n+2)\,A_{n+2}}\right)^{1/3}\left(\frac{\ell_{(4+n)}^{n+2}\,l}{a^n}\right)^{1/3} . \tag{3.33}$$

We note that the expression (3.27) coincides in the limit $L \to a$ with Eq. (3.15) (taking $l \geq 2\,a$), while, in the limit $L \to l$, we recover from Eq. (3.27) the expression (3.33) (of course, supposing also that $l \geq 2\,a$).

3.3 Holographic properties

We finally examine the holographic properties of Eq. (3.33) for the GUP of Ng and van Dam type in $4+n$ dimensions. We just consider the expression in Eq. (3.33) because it also represents the limit of Eq. (3.27) for $L \to l$ and $l \geq 2\,a$. Moreover, for $n = 0$, Eq. (3.33) yields the four-dimensional error given in Eq. (3.15).

Since we are just interested in the dependence of $n(V)$ on l and the basic constants, we can write

$$(\delta l_{\mathrm{tot}})_{\mathrm{min}} \sim \left(\frac{\ell_{(4+n)}^{n+2}\,l}{a^n}\right)^{1/3} . \tag{3.34}$$

We then have that the number of degrees of freedom in the volume of size l is

$$n(V) = \left(\frac{l}{(\delta l_{\mathrm{tot}})_{\mathrm{min}}}\right)^{3+n} = \left(\frac{l^2\,a^n}{\ell_{\mathrm{p}}^2\,L^n}\right)^{1+\frac{n}{3}} , \tag{3.35}$$

and the holographic counting holds in four-dimensions ($n = 0$) but is lost when $n > 0$. Even if we take the ideal case $a \sim \ell_{(4+n)}$ we get

$$n(V) = \left(\frac{l}{\ell_{(4+n)}}\right)^{2\left(1+\frac{n}{3}\right)} , \tag{3.36}$$

and the holographic principle does not hold for $n > 0$.

4 Concluding remarks

In the previous Sections, we have shown that the holographic principle seems to be satisfied only by uncertainty relations in the version of Ng and van Dam and for $n = 0$. That is, only in four dimensions we are able to formulate uncertainty principles which predict the same number of degrees of freedom per spatial volume as the holographic counting. This could be an evidence for questioning the existence of extra dimensions. Moreover, such an argument based on the holography could also be used to support the compactification of string theory down to four dimensions, given that there seems to be no firm argument which forces the low energy limit of string theory to be four-dimensional (except from the obvious observation of our world). In this respect, we should also say that the cases 3. and 4. of Subsection 3.2 do not seem to have a good probability to be realized in nature since, if there are extra spatial dimensions, their size must be shorter than 10^{-1} mm [5]. Therefore, cases 1. and 2. of Subsection 3.2 are more likely to survive the test of future experiments.

A number of general remarks are however in order. First of all, we cannot claim that our list of possible GUP's is complete and other relations might be derived in different contexts which accommodate for both the holography and extra dimensions. Further, one might find hard to accept that quantum mechanics and general relativity enter the construction of GUP's on the same footing, since the former is supposed to be a fundamental framework for all theories while the latter can be just regarded as a theory of the gravitational interaction. We might agree on the point of view that GUP's must be considered as "effective" (phenomenological) bounds valid at low energy (below the Planck scale) rather than "fundamental" relations. This would in fact reconcile our result that four dimensions are preferred with the fact that string theory (as a consistent theory of quantum gravity) requires more dimensions through the compactification which must occur at low energy, as we mentioned above. Let us also note that general relativity (contrary to usual field theories) determines the space-time including the causality structure, and the latter is an essential ingredient in all actual measurements. It is therefore (at least) equally hard to conceive uncertainty relations which neglect general relativity at all. This conclusion would become even stronger in the presence of extra dimensions, since the fundamental energy scale of gravity is then lowered [1, 2] (possibly) within the scope of present or near-future experiments and the gravitational radius of matter sources is correspondingly enlarged [7].

A final remark regards cases with less than four dimensions. Since Einstein gravity does not propagate in such space-times and no direct analogue of the Schwarzschild solution exists, one expects a qualitative difference with respect to the cases that we have considered here. For instance, a point-like source in three dimensions would generate a flat space-time with a conical singularity and no horizon [1]. Consequently, one does expect that the usual Heisenberg uncertainty relations hold with no corrections for gravity.

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