Calc11

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This is a summary of all things I calculated or determined so far, that is, everything from the papers *Calc1* to *Calc10*, with corrections (like better graphs). It will not repeat every detailed calculation.

Contents

1 Setup

I investigate Schwarzschild-like spherical symmetric Black Holes at short scales. The modification is expressed as a smeared Dirac mass/density term. Calculations are done with n Large spatial extra dimensions in total $D = n + 4$ dimensions.

Let $H(r)$ be an approximation of the Heaviside step function $\Theta(r)$, then I frequently derived the metric $g_{00} = 1 - V(r)$ starting from the energy density $\rho(r)$

$$\rho(r) = \frac{M}{\Omega_{n+2}} \frac{\mathrm{d}H(r)}{\mathrm{d}r} \quad \Rightarrow \quad V(r) = \frac{2}{n+2} \frac{M}{M_*^{n+2}} \frac{1}{\Omega_{n+2}} \frac{H(r)}{r^{n+1}}. \tag{1}$$

with the surface term $\Omega_{n+2} = 2\frac{\pi^{\frac{n+3}{2}}}{\Gamma\left(\frac{n+3}{2}\right)}$ and the reduced d-dimensional Planck mass M_*.

I examined two special choices of $H \in \{h, h_\alpha\}$, however most relations can be derived for general (infinetly differentiable) profiles $H(r)$. These two choices each exhibit special features that will be discussed. Because Θ is dimensionless, they can be expressed in the dimensionless variable $z = r/L$ (for details see Calc9) with $H(r) = H(z)$:

$$h(r) = \frac{r^{2+n}}{r^{2+n} + L^{2+n}} \qquad h(z) = \frac{1}{1 + \left(\frac{1}{z}\right)^{2+n}} \tag{2a}$$

$$h_\alpha(r) = \frac{r^{3+n}}{(r^\alpha + L^\alpha/2)^{\frac{3+n}{\alpha}}} \qquad h_\alpha(z) = \frac{1}{\left(1 + \left(\frac{1}{z}\right)^\alpha/2\right)^{\frac{3+n}{\alpha}}} \tag{2b}$$

The derivatives appear in many places and is therefore denoted here. Since $\frac{\mathrm{d}f}{\mathrm{d}r} = \frac{\mathrm{d}f}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}r} = \frac{1}{L}\frac{\mathrm{d}f}{\mathrm{d}z}$, we can always substitute $H'(r) = H'(z)/L$.

$$h'(r) = \frac{(2+n)\, r^{1+n} L^{2+n}}{(r^{2+n} + L^{2+n})^2} \qquad h'(z) = \frac{(2+n)\left(\frac{1}{z}\right)^{3+n}}{\left(1 + \left(\frac{1}{z}\right)^{2+n}\right)^2} = (2+n)\frac{h^2(z)}{z^{3+n}} \tag{3a}$$

$$h'_\alpha(r) = \frac{(n+3)L^\alpha r^{n+2}\left(\frac{L^\alpha}{2} + r^\alpha\right)^{-\frac{n+3}{\alpha}}}{L^\alpha + 2r^\alpha} \qquad h'_\alpha(z) = \frac{\frac{3+n}{2}\left(\frac{1}{z}\right)^{\alpha+1}}{\left(1 + \left(\frac{1}{z}\right)^\alpha/2\right)^{\frac{3+n}{\alpha}+1}} \tag{3b}$$

1.1 The Mass

We can argue that M is just a constant, responsible for fulfilling the horizon equation $V(r_H) = 1$. Therefore we set (»arbitrarily«)

$$M = \frac{n+2}{2} M_*^{n+2} \Omega_{n+2} \frac{r_H^{n+1}}{H(r_H)} = \frac{n+2}{2}\, \Omega_{n+2}\, \frac{1}{H(r_H)} \left(\frac{r_H}{L_*}\right)^{n+1} M_* \tag{4}$$

then the horizon equation $V(r) = 0$ is fulfilled at $r = r_H$. In general, equation 4 gives us a relationship $M = M(r)$. See Calc9, Section 1.1 for details.

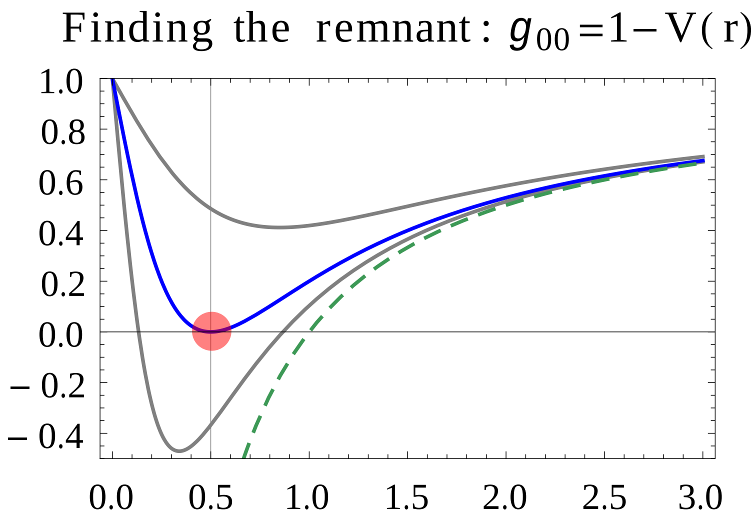
1.2 Event Horizons

(The explicit determination of $g_00(r_H) = 0$ is straightforward and was skipped in the Calc$X$ series until now)

2 Remnants

The remnant is the smallest possible Black Hole solution and considered as a stable particle that can no more evaporate. Self encoding solutions (h_α) encode the remnant radius by it's degrees of freedom. Typically the (reduced) Planck Length is supposed to be equal to the remnant's size. See Calc7, Section 1.1 for details. See figure 1 for a picture.

Figure 1: Extremal Configuration in regularized Schwarzschild metrics. Since $g_{00}(r) \rightarrow 0$ at $r \in \{0, \infty\}$, there must be an extremal r_0. In the extremal configuration (blue), $r_0 = r_H$. There are also solutions with two r_H and $r_0 < 0$ and no r_H, $r_0 > 0$. The dashed line is the Schwarzschild behaviour. See Calc7, Section 1.1 for details.



2.1 Extremal radius and Minimal Length

The remnant equations require

$$\begin{cases} \partial_r|_{r=r_0}\, g_{00}(r) & = 0 \\ g_{00}(r_0) & = 0 \end{cases} \tag{5}$$

For our generic solution (1), we rewrite the first equation to (see Calc7 for derivation)

$$\partial_r V(r) = 0 = -(n+1)\frac{G(r)}{r} + H'(r) = L\, H'(z) - \frac{n+1}{z}\, H(z)\bigg|_{z=\frac{r_0}{L}} \tag{6}$$

This equation can be solved for both $h(r_0)$ and $h_\alpha(r_{0,\alpha})$. In Calc7 and Calc9 I derived

$$r_0 = L\left(\frac{1}{1+n}\right)^{\frac{1}{2+n}} \tag{7a}$$

$$r_{0,\alpha} = L\left(\frac{1}{1+n}\right)^{\frac{1}{\alpha}} \tag{7b}$$

2.2 Self-Encoding

Self-Encoding means that $M(r_0) = M*$, so at the minimal length, the remnant has the Planck Mass. Self-Encoding only occurs in the self-regular metric, as derived in Calc7. It allows relating $\alpha = \alpha(n)$. I found that

$$\alpha_0 = \frac{3+n}{\ln(2+n)} \ln\frac{3+n}{2} \tag{8}$$

3 Thermodynamical properties

All calculations in this section start with a generic $V(r)$, so we forget (1) for a short moment, but keep (4). That is, we have:

$$V(r) \equiv M(r_H) \cdot Y(r) \tag{9a}$$

$$M(r_H) \equiv Y^{-1}(r_H) \tag{9b}$$

The smearing solution (10a) reconstructs equation (1), with an appropriate dimensionful constant A:

$$Y(r) = A \frac{H}{r^{n+1}} \tag{10a}$$

$$V(r) = A\ M(r_H)\ \frac{H(r)}{r^{n+1}} \tag{10b}$$

$$M(r_H) = \frac{1}{A} \frac{r_H^{n+1}}{H(r_H)} \tag{10c}$$

3.1 Hawking Temperature

Let's start with a generic $V(r)$ from which we don't know the inner structure yet:

$$T \equiv \frac{1}{4\pi} \partial_r g_{00}|_{r=r_H} = -\frac{1}{4\pi} M(r_H) \cdot Y'(r_H) = -\frac{1}{4\pi} \frac{1}{L} M(z_H) Y'(z_H) \tag{11}$$

Details can be found in Calc9, Section 1.6. With the holographic approach these equations read

$$T = \frac{1}{4\pi r_H} \left(1 + n - r_H \frac{H'(r_H)}{H(r_H)} \right) = \frac{1}{L} \frac{1}{4\pi z_H} \left(1 + n - z_H \frac{H'(z_H)}{H(z_H)} \right) \tag{12}$$

Now is simple to insert our $H \in \{h, h_\alpha\}$. Figure (2) shows T_H for both of them.

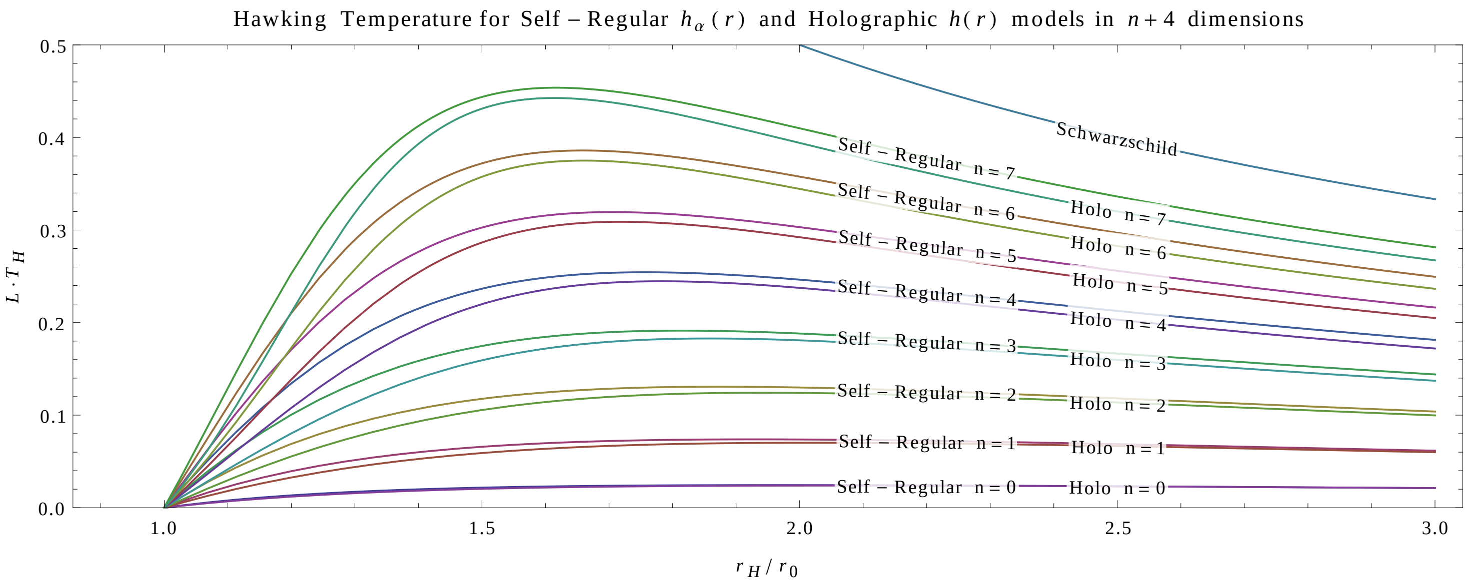


Figure 2: Hawking temperature for the holographic model h and the Self-Regular mode h_α. The functions are $L \cdot T_H(r_H/r_0)$. The Schwarzschild-Tangherlini solution is shown for comparison.

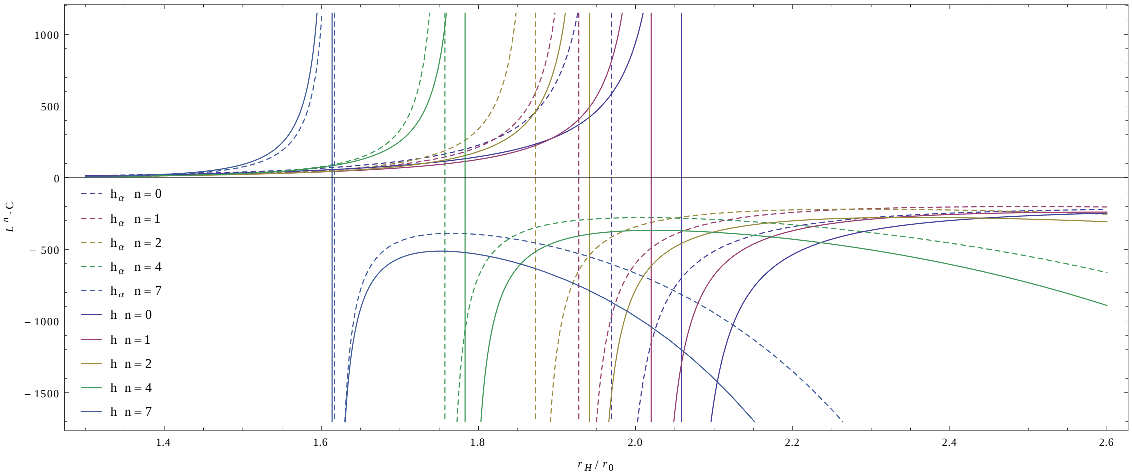


Figure 3: Heat Capacity for $h(r)$ and $h_\alpha(r)$ in various dimensions with critical z_C clearly visible (c.f. figure 2). Figure 4 shows the same functions shifted with r_C

3.2 Heat Capacity

The determiniation of the Heat Capacity is done by variable substitution:

$$C = \frac{\partial M}{\partial T_H} = \frac{\partial M}{\partial r_H} \left(\frac{\partial T_H}{\partial r_H} \right)^{-1} = \frac{\partial M}{\partial z_H} \left(\frac{\partial T_H}{\partial z_H} \right)^{-1} \tag{13}$$

Inserting (4) and (12), we get

$$C = \frac{4\pi r_H^{n+2}}{A} \frac{r_H H'(r_H) - (n+1) H(r_H)}{r_H^2 H(r_H) H''(r_H) - r_H^2 H'(r_H)^2 + (n+1) H(r_H)^2} \tag{14}$$

Especially $C(r) = L^n \, C(z)$.

At the critical radius r_C a phase transition takes place. It is $C(r_C) = 0$ and $T_H(r_C)$ is extremal (so $\partial_{r_H} T_H|_{r_H = r_C} = 0$). The critical radius for $h(r_C)$ and $h(r_{C,\alpha})$ is given by

$$r_C = 2^{\frac{1}{n+2}} \left(-n^2 + (n+2)\sqrt{n^2 + 2n + 5} - 3n - 4 \right)^{-\frac{1}{n+2}} \tag{15}$$

$$r_{C,\alpha} = \textit{no closed expression for general n, but possible for fixed n} \tag{16}$$

Figure 3 shows the well-known curve. In figure 4 the abscissa is rescaled by r_C, so $(r_H - r_C) r_0$ is displayed (numerical evaluation of r_C for convenience).

3.3 Entropy

The entropy defining integral can also be subsituted like in the Heat Capacity in the section before:

$$S(r) = \int_{M_1}^{M_2} \frac{\mathrm{d}M}{T} = \int_{r_1}^{r_2} \frac{\mathrm{d}M}{\mathrm{d}r_H} \frac{\mathrm{d}r_H}{T} = \int \mathrm{d}r_H \frac{1}{T} \left(\frac{\mathrm{d}M(r_H)}{\mathrm{d}r_H} \right) \tag{17a}$$

$$S(z) = \int \mathrm{d}z_H \frac{1}{T} \left(\frac{\mathrm{d}M(z_H)}{\mathrm{d}z_H} \right) = -4\pi L \int^z \mathrm{d}z_H \frac{M'(z_H)}{M(z_H)} \frac{1}{Y'(z_H)} \tag{17b}$$

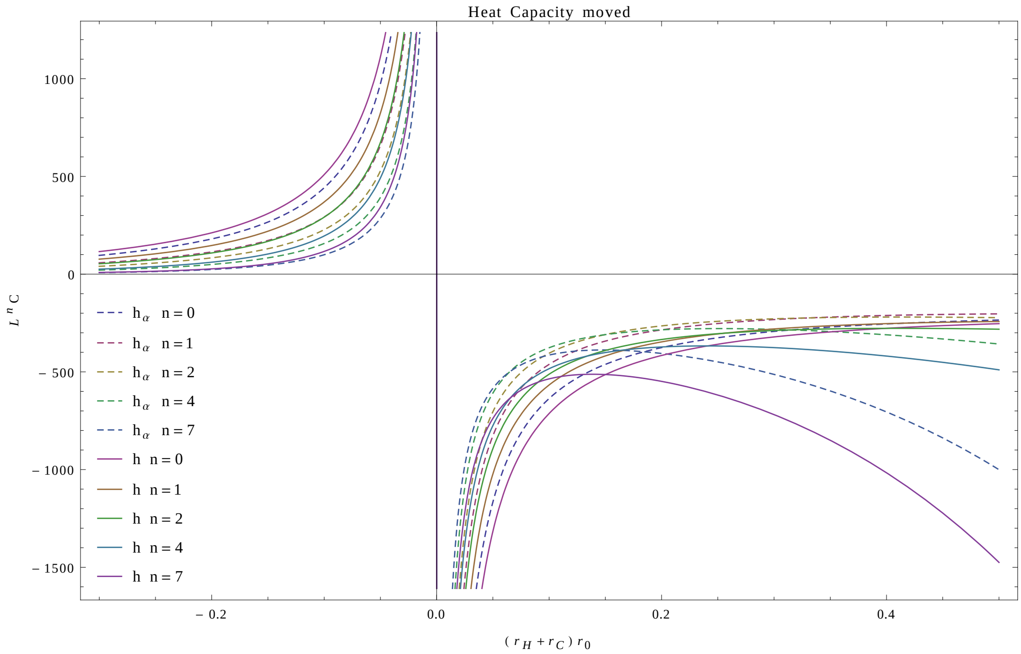


Figure 4: Heat Capacity shifted around r_C (c.f. figure 3)

Inserting (4) and (12) yields (I label $z_H = x$)

$$S(z) = -4\pi L \int^z \mathrm{d}x \left(\frac{n+1}{x} - \frac{H'(x)}{H(x)} \right) \frac{1}{Y'(x)} \tag{18a}$$

$$= -4\pi L A \int^z \mathrm{d}x \left(\frac{n+1}{x} - \frac{H'(x)}{H(x)} \right) \frac{x^{n+2}}{xH'(x) - (n+1)H(x)} \tag{18b}$$

This integral can be computed, at least for the hologrpahic model $h(r)$. This allows us to see logarithmic corrections in any dimension:

$$S_h(z) = 4\pi A L \left(\frac{x^{n+2}}{n+2} + \log(x) \right)_1^z \tag{19}$$

See Calc9 Section 1.6.3 for details.

4 Modified Einstein Equations

It is reasonable to find a deeper concept to justify the smearing of the Schwarzschild source. This can be smearing the Ricci scalar with a bilocal distribution $\mathcal{A}^2(x-y) = \mathcal{A}^2(\Box_x)\delta^D(x-y)$ and was done in Calc8 and Calc10.

Based on the density (1), one tries to find $\mathcal{A}$:

$$\mathcal{T}_0^0 = -M\mathcal{A}^{-2}(\Box)\delta(\vec{x}) \stackrel{!}{=} -\frac{M}{\Omega_{n+2}}\frac{\mathrm{d}H(r)}{\mathrm{d}r} \tag{20}$$

A Fourier transform helps to find the solution for the operator (for Details see Calc10, Section 1.1)

$$\mathcal{A}^{-2}(p^2) = \mathcal{F}\left\{\frac{\mathrm{d}H(x)}{\mathrm{d}x}\right\} = \int_{-\infty}^{\infty} \mathrm{d}^{3+n}r\,\frac{\mathrm{d}H(r)}{\mathrm{d}r}\,e^{-ipr} \tag{21}$$

For $h(r)$, I found the Meijer G-function as a closed algebraic solution,

$$p\,\mathcal{A}^{-2}(p) \propto G_{p,q}^{m,n}\left(\begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}\,\middle|\, z\right) \tag{22}$$

The p-dependence enters into the z part while the lists are only l and n-dependent.