Calc1

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1 From Density to Metric

1.1 Problem: Mass distribution is wrong!?

Ansatz, mit Plancklänge $L_P = L$ und Masse M, Density ρ:

$$\rho(r) = \frac{M}{2\pi r} \frac{L_P}{(r^2 + L_P^2)^2} \tag{1}$$

This density has two limes

$$\frac{M}{8\pi L_P^3} \xleftarrow{r \simeq L_P} \rho(r) \xrightarrow{r \gg L_P} 0 \tag{2}$$

Man kann daraus eine Massenverteilung ableiten: $\rho = -T_0^0$:

$$\mu(r) = 4\pi \int_0^r \mathrm{d}x\, x^2 \rho(x) \tag{3}$$

$$m(r) = 4\pi \int_r^\infty \mathrm{d}x\, x^2 (-\rho(x)) \tag{4}$$

Das unbestimmte Integral ist $4\pi \int x^2 \rho(x) \mathrm{d}x = -\frac{LM}{L^2+x^2} + C$, aber die bestimmten Integrale haben die eindeutige Lösung

$$\mu(r) = \frac{Mr^2}{L^2 + r^2} \tag{5}$$

$$m(r) = -\frac{L^2 M}{L^2 + r^2} \tag{6}$$

$$\frac{M}{L} = \mu(r) + m(r) \tag{7}$$

BUT the paper states

$$m(r) = \frac{Mr^2}{L^2 + r^2} = M - \frac{LM}{L^2 + r^2} \tag{8}$$

Which one is right? Equation 8 and 6 only differ by constant: $M + m_6(r) = m_8(r)$.

1.2 Problem: Source of EOS?

On some way this yields to a stress energy tensor

$$T_{\mu\nu} = \mathrm{diag}(-\rho, p_r, p_\perp, p_\perp) \tag{9}$$

Conservation of stress-energy tensor

$$\nabla_\nu T^{\mu\nu} = 0 \tag{10}$$

yields $p_r = -\rho$ and $p_\perp = p_r + \frac{r}{2}\partial_r p_r$, as statet in the paper. Wie kommt man dazu? This determines $T_{\mu\nu}$ components to

$$T_{00} = T_{11} = -\rho \tag{11}$$

$$T_{11} = T_{22} = -\frac{3}{2}\rho - \frac{2\rho}{r^3 + rL_P} = -\frac{L^2 M\left(L^2 - 3r^2\right)}{4\pi r\left(L^2 + r^2\right)^3} \tag{12}$$

Siehe weiter unten warum das ein Problem ist.

1.3 Notes

Ist die folgende Formel eine Ab-Initio-Annahme?

$$g_{00} = -\left(1 - \rho\frac{M^2 G}{\pi L_P^2}\right) = -g_{11}^{-1} \tag{13}$$

Griffith-Podolsky leiten mit der inneren Schwarzschildgleichung her:

$$\mu(r) = \int_0^r 4\pi x^2 \rho(x) \mathrm{d}x \tag{14}$$

Außerdem ist die Schwarzschild-Lösung SSM

$$\mathrm{d}s^2 = \left(1 - \frac{2m}{r}\right)\mathrm{d}t^2 \left(1 - \frac{2m}{r}\right)^{-1} \mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2 \tag{15}$$

Vgl A.S Vaidya Radiating SSM wo $m = m(r)$. Inner Schwarzschild:

$$\mathrm{d}s^2 = -\exp\left(2\Phi(r)\right)\mathrm{d}t^2 + \left(1 - \frac{2\mu(r)}{r}\right)^{-1} \mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2 \tag{16}$$

Aus Einsteingleichungen folgert man da typischerweise Constraints an $\frac{\mathrm{d}\Phi}{\mathrm{d}r}$. Ist $\mu(r) = m(r)$? Außerdem haben wir offensichtlich kein isotropes p, wovon inner Schwarzschild immer ausgeht.

Dazu: Adler Razin Seite 262ff

Sowie Inner Schwarzschild S. 290

2 From metric to density and pressure

Isotropic static metric, Standardform (Achtung, Signatur $+---$!)

$$ds^2 = B(r)\mathrm{d}t^2 - A(r)\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2 \tag{17}$$

Hier: $B(r) = 1 - \frac{2m(r)G}{r} = 1/A(r)$ mit $m(r) = Mr^2/(L^2+r^2)$, $L = L_P$.
Berechne Inverse Metrik:

$$g_{\mu\nu} = \mathrm{diag}(B, -A, -r^2, -r^2\sin^2\theta) \tag{18}$$

$$g^{\mu\nu} = \mathrm{diag}\left(\frac{1}{B}, \frac{-1}{A}, \frac{-1}{r^2}, \frac{-1}{r^2\sin^2\theta}\right) = g_{\mu\nu}^{-1} \tag{19}$$

2.1 Kristoffels und Kovariante Ableitung

Berechne damit Kristoffelsymbole:

$$\Gamma^\sigma_{\lambda\mu} = \frac{g^{\sigma\nu}}{2}\left(\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\lambda}}{\partial x^\nu}\right) \tag{20}$$

Berechne Komponenten (Fließbach ART, S. 134)

$$\Gamma^0_{01} = \Gamma^0_{10} = \frac{B'}{2B} = \frac{GM\left(r^2-L^2\right)}{\left(L^2+r^2\right)\left(r(r-2GM)+L^2\right)} \tag{21}$$

$$\Gamma^1_{00} = \frac{B'}{2A} = \frac{GM\left(r^2-L^2\right)}{A\left(L^2+r^2\right)^2} \tag{22}$$

$$\Gamma^1_{11} = \frac{A'}{2A} = \frac{GM(L-r)(L+r)}{\left(L^2+r^2\right)\left(r(r-2GM)+L^2\right)} \tag{23}$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r} \tag{24}$$

$$\Gamma^1_{22} = -\frac{1}{A} \tag{25}$$

$$\Gamma^1_{33} = -\frac{r\sin^2\theta}{A} \tag{26}$$

$$\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r} \tag{27}$$

$$\Gamma^3_{23} = \Gamma^3_{32} = \cot\theta \tag{28}$$

$$\Gamma^2_{33} = -\sin\theta\cos\theta \tag{29}$$

Die anderen Komponenten sind alle 0.

Dies erlaubt nun das Berechnen der kovarianten Ableitung:

$$\nabla_a t^\nu = \partial_a t^\nu + \Gamma^\mu_{ac} t^c \tag{30}$$
$$\nabla_a T^{\mu\nu} = \partial_a T^{\mu\nu} + \Gamma^\mu_{sa} T^{s\nu} + \Gamma^\nu_{as} T^{\mu s} \tag{31}$$

Nun können wir für die Metrik ausrechnen, was die Kontraktion

$$0 = \nabla_\mu T^{\mu\nu} = \partial_\mu T^{\mu\nu} + \Gamma^\mu_{s\mu} T^{s\nu} + \Gamma^\nu_{\mu s} T^{\mu s} \tag{32}$$

bedeutet. Dies ist ein Gleichungssystem mit vier Gleichungen ($0 = 0^\nu$). Mit Ansatz $T^{\mu\nu} = \operatorname{diag}(p_1 := \rho, p_2 := -\rho, p_3, p_4)$ kann man nun mit 4 Gleichungen die zwei Unbekannte $p_3 = p_3(p_1, p_2)$, $p_4 = p_4(p_1, p_2)$ ausdrücken:

$$p_3 = \frac{\frac{GMr\left(r^2-L^2\right)\rho(r)}{\left(L^2+r^2\right)^2} + \frac{p_r(r)\left(L^2r(4r-3GM)+r^3(2r-5GM)+2L^4\right)}{\left(r(r-2GM)+L^2\right)^2}}{2r^2} \tag{33}$$

$$p_4 = \frac{\csc^2(\theta)\left(\frac{GMr\left(r^2-L^2\right)\rho(r)}{\left(L^2+r^2\right)^2} + \frac{p_r(r)\left(L^2r(4r-3GM)+r^3(2r-5GM)+2L^4\right)}{\left(r(r-2GM)+L^2\right)^2}\right)}{2r^2} \tag{34}$$

BUT Gleichungen 33 und 34 sind unvereinbar mit 12 aus dem Paper, also

$$p_3 = p_4 \neq -\rho(r) - \frac{r}{2}\partial_r \rho(r) \quad !!! \tag{35}$$

Alternativ kann man auch erst mal die Energieerhaltungsgleichungen außen vor lassen und die Einsteingleichungen lösen.

2.2 Ricci-Tensor und Einstein-Gleichungen

Berechne nun Ricci-Tensor $R_{\mu\nu}$

$$R_{\mu\nu} = R^\rho_{\mu\rho\nu} = g^{\kappa\rho} R_{\kappa\mu\rho\nu} \tag{36}$$
$$= \frac{\partial \Gamma^\rho_{\mu\rho}}{\partial x^\nu} - \frac{\partial \Gamma^\rho_{\mu\nu}}{\partial x^\rho} + \Gamma^\sigma_{\mu\rho}\Gamma^\rho_{\sigma\nu} - \Gamma^\sigma_{\mu\nu}\Gamma^\rho_{\sigma\rho} \tag{37}$$

Erhalte hierbei

$$R_{00} = \frac{2GL^2M\left(L^2-3r^2\right)\left(r(r-2GM)+L^2\right)}{r\left(L^2+r^2\right)^4} \tag{38}$$

$$R_{11} = -\frac{2GM\left(L^4-3L^2r^2\right)}{r\left(L^2+r^2\right)^2\left(r(r-2GM)+L^2\right)} \tag{39}$$

$$R_{22} = -\frac{4GL^2Mr}{\left(L^2+r^2\right)^2} \tag{40}$$

$$R_{33} = R_{22}\sin^2\theta \tag{41}$$

$$R_{\mu\nu} = 0 \quad \text{für} \quad \mu \neq \nu \tag{42}$$

Der Ricci-Skalar $R = R^{\mu}_{\mu} = g^{\mu\nu}R_{\mu\nu}$ lautet

$$R = -\frac{40GL^2Mr}{\left(L^2+r^2\right)^3} + \frac{24GL^4M}{r\left(L^2+r^2\right)^3} + \frac{18L^2r^2}{\left(L^2+r^2\right)^3} + \frac{6r^4}{\left(L^2+r^2\right)^3} + \frac{6L^6}{r^2\left(L^2+r^2\right)^3} + \frac{18L^4}{\left(L^2+r^2\right)^3} \tag{43}$$

und sieht damit irgendwie falsch aus. Er geht in die Einstein-Gleichungen ein:

$$R_{\mu\nu} - \frac{R}{2}g_{\mu\nu} = -\frac{8\pi G}{c^4}T_{\mu\nu} \tag{44}$$

Wenn wir in gleicher Weise die Energiespur $T = T^{\mu}_{\mu}$ definieren, ergibt Kontraktion der Einsteingleichung $-R = -\frac{8\pi G}{c^4}T$ und damit eine andere Schreibweise von Einstein:

$$R_{\mu\nu} = -\frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{T}{2}g_{\mu\nu}\right) \tag{45}$$

Wenn ich den Argumentationsweg *Metrik* $\rightarrow$ *Christoffel* $\rightarrow$ *Ricci* $\rightarrow$ *Energieimpuls* wähle, stelle ich 44 nach $T_{\mu\nu}$ um ($c = 1$):

$$T_{\mu\nu} = -\frac{R_{\mu\nu} - \frac{R}{2}g_{\mu\nu}}{8\pi G} \tag{46}$$

Das Ergebnis von 46 ist die Monster-Diagonalmatrix

$$T_{00} = -\frac{5GL^2M^2r^2}{2\pi\left(L^2+r^2\right)^4} + \frac{9GM^2r^4}{4\pi\left(L^2+r^2\right)^4} + \frac{3r^6}{8\pi G\left(L^2+r^2\right)^4} + \frac{3L^2r^4}{2\pi G\left(L^2+r^2\right)^4} + \frac{3L^8}{8\pi Gr^2\left(L^2+r^2\right)^4} + \frac{3L^6}{2\pi G\left(L^2+r^2\right)^4} - \frac{7GL^4M^2}{4\pi\left(L^2+r^2\right)^4} + \frac{9L^4r^2}{4\pi G\left(L^2+r^2\right)^4} - \frac{3Mr^5}{2\pi\left(L^2+r^2\right)^4} - \frac{5L^2Mr^3}{2\pi\left(L^2+r^2\right)^4} +$$

$$T_{11} = \frac{11GL^2M^2r^2}{2\pi\left(L^2+r^2\right)^2\left(r(r-2GM)+L^2\right)^2} + \frac{9GM^2r^4}{4\pi\left(L^2+r^2\right)^2\left(r(r-2GM)+L^2\right)^2} + \frac{3r^6}{8\pi G\left(L^2+r^2\right)^2\left(r(r-2GM)+L^2\right)^2} - \frac{3Mr^5}{2\pi\left(L^2+r^2\right)^2\left(r(r-2GM)+L^2\right)^2} + \frac{3L^2r^4}{2\pi G\left(L^2+r^2\right)^2\left(r(r-2GM)+L^2\right)^2}$$

$$T_{22} = \frac{3r^6\csc^2(\theta)}{8\pi G\left(L^2+r^2\right)^3} - \frac{3r^6}{4\pi G\left(L^2+r^2\right)^3} + \frac{9L^2r^4\csc^2(\theta)}{8\pi G\left(L^2+r^2\right)^3} - \frac{9L^2r^4}{4\pi G\left(L^2+r^2\right)^3} + \frac{3L^6\csc^2(\theta)}{8\pi G\left(L^2+r^2\right)^3} - \frac{3L^6}{4\pi G\left(L^2+r^2\right)^3} + \frac{9L^4r^2\csc^2(\theta)}{8\pi G\left(L^2+r^2\right)^3} - \frac{9L^4r^2}{4\pi G\left(L^2+r^2\right)^3} + \frac{9L^2Mr^3}{2\pi\left(L^2+r^2\right)^3} + \frac{L^4Mr}{2\pi\left(L^2+r^2\right)^3}$$

$$T_{33} = -\frac{3r^6}{8\pi G\left(L^2+r^2\right)^3} - \frac{9L^2r^4}{8\pi G\left(L^2+r^2\right)^3} - \frac{3L^6}{8\pi G\left(L^2+r^2\right)^3} - \frac{9L^4r^2}{8\pi G\left(L^2+r^2\right)^3} - \frac{9L^2Mr^3\cos(2\theta)}{4\pi\left(L^2+r^2\right)^3} + \frac{9L^2Mr^3}{4\pi\left(L^2+r^2\right)^3} - \frac{L^4Mr\cos(2\theta)}{4\pi\left(L^2+r^2\right)^3} + \frac{L^4Mr}{4\pi\left(L^2+r^2\right)^3}$$

Das ist garantiert falsch. Warum?

Zum Crosscheck ließe sich mit dem neuen Tensor nochmal $0 = \nabla_{\mu}T^{\mu\nu}$ berechnen, aber das führt zu nichts neuem brauchbaren (ähnlich wie oben).

3 Hawking Temperature

Normales Vorgehen ist wohl bei SMM-artigen Metriken:

$$\kappa = \frac{1}{2}\frac{\partial C}{\partial r}\bigg|_{r=r_+} \rightarrow T_H = \frac{\kappa}{2\pi k_B} = \frac{\kappa}{2\pi} \tag{47}$$

Dabei ist $C(r)$ eigentlich durch den Übergang in Interior/Exterior-Koordinaten gegeben (vgl BH2-Präsentation). Mein erster Ansatz würde einfach einsetzen:

$$C(r) := 1 - \frac{2MLr}{r^2 + L^2} \tag{48}$$

Gemäß dem Paper gilt

$$r_+ = r_h = L^2 \left(M \pm + \sqrt{M^2 - M_p^2} \right) \tag{49}$$

Also ist

$$\kappa = \frac{1}{2}\left(\frac{4LMr^3}{(L^2+r^2)^2} - \frac{4LMr}{L^2+r^2} \right)_{r=r_h} \tag{50}$$

Setzt man hier

$$M = \frac{1}{2L^2 r_h}(r_h^2 + L^2) \tag{51}$$

aus dem Paper ein, dann bekommt man eine seltsame Hawkin-Temperatur von

$$T_H = -\frac{1}{2\pi}\left(\frac{L}{L^2 + r_+^2} \right) \tag{52}$$

Vergleich mit dem gesuchten Ergebnis:

$$T_{H,\text{Paper}} = \frac{1}{4\pi r_+}\left(1 - \frac{2L^2}{r_+^2 + L^2} \right) \tag{53}$$

WHY does this calculation not lead the correct result?
Back-calculation from $T_{H,\text{Paper}}$:

$$C_{\text{Paper}}(r) = 4\pi \int T_{H,\text{Paper}} \mathrm{d}r = \log\left(abc\right) \tag{54}$$

Looks like an entry in the SMM inner metric 16, $\mathrm{d}s^2 = -\exp\left(2\Phi(r)\right)\mathrm{d}t^2 + \dots$. True?

Calc2

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Calc2 is the second writeup of notices in my Master thesis research. This document lists up some formulas and expands some with that higher dimensional things.

Definition of frequently used formulas:

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + f^{-1}(r)\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2$$

$$T_H = \left(\frac{1}{4\pi}\frac{\mathrm{d}g_{00}}{\mathrm{d}r}\right)_{r=r_H}$$

$$\gamma(s;x) = \int_0^x t^{s-1}e^{-t}\mathrm{d}t$$

Contents

1 Summary of papers

1.1 NSS 2006

Title Noncommutative geometry inspired Schwarzschild black hole

Keywords NC

Genutzt von Rizzo 2006

Auch eines der wichtigsten, die ich gelesen habe.

$$\rho_\theta(r) = \frac{M}{(4\pi\theta)^{3/2}}e^{-r^2/4\theta} \tag{1}$$

$$f(r) = 1 - \frac{4M}{r\sqrt{\pi}}\gamma(3/2, r^2/4\theta) \tag{2}$$

$$r_H = \frac{4M}{\sqrt{\pi}}\gamma(3/2; r_H^2/4\theta) \tag{3}$$

$$T_H = \frac{1}{4\pi r_H}\left(1 - \frac{r_H^3}{4\theta^{3/2}}\frac{e^{-r_H^2/4\theta}}{\gamma(3/2, r_H^2/4\theta)}\right) \tag{4}$$

1.2 N Aug 2010

Title Entropic force, noncommutative gravity and ungravity

Keywords Emergent gravity, Verlinde

Basic Ideas Newton $F(r)$ herleiten aus $S = k_B \ln N$. Später mit n Raumdimensionen und in $\mathcal{G}$ einige Effekte.

$$f(r) = 1 - \frac{2M}{r^{n-2}c^2}\mathcal{G}(r) \tag{5}$$

$$F = \frac{GMm}{r^2}\left(1 + 4L^2\frac{\partial S}{\partial A}\right) \tag{6}$$

Nicht so passend zum Thema $f(r)$.

1.3 N Feb 2012

Title Nonlocal and generalized uncertainity principle black holes

Keywords EH-Action

Basic Ideas Operator $\mathcal{A}(x-y)$, running $\mathcal{G}(r)$, Length scale l of theory

Nicht passend zum Thema.

$$
\begin{aligned}
S &= \frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{-g}\mathcal{R}(x) \\
\mathcal{R}(x) &= \int \mathrm{d}^4y\sqrt{-g}\mathcal{A}^2(x-y)R(y) \\
\mathcal{A}^2(x-y) &= \mathcal{A}^2(\Box_x)\delta^4(x-y) \\
\Box_x &= l^2 g_{\mu\nu}\nabla^\mu\nabla^\nu \\
\mathcal{A}(p^2) &= \exp(l^2p^2/2)\ldots \\
\mathcal{T}_{\mu\nu} &= \mathcal{A}^{-2}(\Box)T_{\mu\nu} \\
f(r) &= 1 - \frac{GM\gamma(2;r/\sqrt{\beta})}{r}
\end{aligned}
$$

1.4 NS Okt 2012

Title Holographic screens in ultraviolet self-complete quantum gravity

Keywords Holography

Das Hauptpaper, was ich als erstes las, darüber geht auch Calc1.

Das Paper umfasst zwei Ansätze, $h_\alpha(r)$ und $h(r)$.

Im ersten Ansatz setzt Bedingung $M_P = M_0$, $M_0 = M(r_0)$ das $\alpha = \alpha_0$, $r_0 = L_P$. Im zweiten Ansatz wird eine der drei Bedigungen an eine Metrik verworfen.

$$f(r) = 1 - \frac{2MG}{r}h_{\alpha,\ldots}(r) \tag{7}$$

$$h_\alpha(r) = \frac{r^3}{(r^\alpha + (\tilde{r}_0)^\alpha/2)^{3/\alpha}} \tag{8}$$

$$h(r) = \frac{r^2}{r^2+L^2} = 1 - \frac{L^2}{r^2+L^2} \tag{9}$$

$$\rho(r) = \frac{M}{2\pi r}\frac{L^2}{(r^2+L^2)^2} \tag{10}$$

$$m(r) = \frac{Mr^2}{L^2+r^2} = M - \frac{LM}{L^2+r^2} \tag{11}$$

1.5 NS 06.11.2013

Title Holographic screens in ultraviolet self-complete quantum gravity

Keywords Holography

Source Elsevier Preprint by Mail am 12.11.13

$$\rho(r) = \frac{M}{4\pi r^2}\delta(r) \tag{12}$$

$$\delta(r) = \frac{\mathrm{d}}{\mathrm{d}r}\Theta(r) \tag{13}$$

$$\Theta(r) \to h(r) \tag{14}$$

$$\rho(r) = \frac{M}{4\pi r^2}\frac{\mathrm{d}}{\mathrm{d}r}h(r) = T^0_0 \tag{15}$$

$$h(r) = 1 - L^2/(r^2+L^2) \tag{16}$$

$$\sigma_h = M/(4\pi r_h^2) \tag{17}$$

1.6 NIM 07.11.2013

Title Self-Completeness and the Generalized Uncertainity Principle

Keywords -

Ein neues veröffentlichtes Paper auf ArXiv, parallel zum Preprint. Erstmals hübsche Bilder. Herleitung von $f(r)$ aus Operator $\mathcal{A}$:

$$f(r) = 1 - 2\frac{GM}{c^2 r}\gamma(2; \frac{r}{\sqrt{\beta}}) \tag{18}$$

1.7 Rizzo 2006

Title Noncommutative inspired black holes in extra dimensions

Basiert auf NSS 2006 NC Ansatz (Section 1.1)

$$\rho_\theta(r) = \frac{M}{(4\pi\theta)^{3/2}} e^{-r^2/4\theta} \tag{19}$$

$$\rightarrow \frac{M}{(4\pi\theta)^{(n+3)/2}} e^{-r^2/4\theta} \tag{20}$$

2 Extension von [1.4 NS2012] analog zu [1.7 Rizzo 2006]

In Paper [1.5 NS2013], in 4D, it was like (using $\Sigma := (r^2 + L^2)^2$)

$$\rho(r) = \frac{M}{A_2} h'(r) \propto \frac{1}{r^2}\frac{r}{\Sigma} \quad \Rightarrow \quad \mu(r) = \int_0^r \mathrm{d}r r^2 \rho(r) \quad \Rightarrow \quad \mu(r) = \int_0^r \mathrm{d}r \frac{r}{\Sigma} \propto \left[\frac{1}{\Sigma}\right]_0^r \tag{21}$$

In the holography picture, only the $A_{(n-2)}$-Sphere, which is the surface of an $V_{(n-1)}$ dimensional matter ball in $(n-1)$ spacial dimensions (+1 time dimension makes n space-time dimensions) seems to enter $\rho(r)$. So in n dim, combining [1.5 NS2013] + [1.7 Rizzo 2006]:

$$\rho(r) = \frac{M}{A_{(n-2)}} \frac{\mathrm{d}h(r)}{\mathrm{d}r} \quad \text{Units: } [\rho] = \frac{[M]}{[A_{n-2}]}\left[\frac{\mathrm{d}}{\mathrm{d}r}\right][h] = \frac{E}{L^{n-2}}\frac{1}{L}\cdot 1 = \frac{E}{L^3} = \frac{1}{L^4} = E^4 \tag{22}$$

Formulas to remember for the volume of an n-Ball and its corresponding $(n-1)$-Sphere:

$$V_n = r^n \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)} \quad \text{and} \quad A_{(n-1)} = \frac{\mathrm{d}V_n}{\mathrm{d}r} = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)} n r^{n-1} = 2\frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} r^{n-1} \tag{23}$$

$$\begin{matrix} \Gamma(x) = (x-1)! \\ \Gamma(x+1) = x\Gamma(x) \end{matrix} \quad \text{Prefactors:} \quad \begin{matrix} V_n = v_n r^n \\ A_n = a_n r^n \end{matrix} \quad \text{Recursion:} \quad \begin{matrix} v_0 = 1, & v_{n+1} = a_n/(n+1) \\ a_0 = 2, & a_{n+1} = 2\pi v_n \end{matrix} \tag{24}$$

Now we evaluate the $(n-1)$ dimensional integral measure in spherical coordinates $k_\mu = (k_0, \vec{k}) = (k_0, r, \phi, \theta_1, \ldots, \theta_{n-3})$, integrating only the spacial components:

$$\int \mathrm{d}^{(n-1)} r = \int_0^\infty \mathrm{d}r \; r^{n-2} \underbrace{\int_0^{2\pi} \mathrm{d}\phi \prod_{j=1}^{n-3} \int_0^\pi \mathrm{d}\theta_j \; \sin^j(\theta_j)}_{=a_{n-2},\text{ since }(n-2)-\text{Surface}} = \frac{2\pi^{(n-1)/2}}{\Gamma\left(\frac{n-1}{2}\right)} \int_0^r \mathrm{d}r \; r^{n-2} \tag{25}$$

The factor r^{n-2} is given by $(n-2)$ angles in n dimensions.
The derivation of eqn. 25 follows Wagner QFT2, not important here:

$$B(x,y) = \int_0^1 \mathrm{d}t t^{x-1}(1-t)^{y-1} = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{26}$$

$$\Gamma(x) = \int_0^\infty \mathrm{d}t\, t^{x-1}e^{-t} \quad \text{und} \quad \gamma(s,x) = \int_0^x t^{s-1}e^t \mathrm{d}t, \quad \Gamma(1/2) = \sqrt{\pi} \tag{27}$$

$$\int_0^\pi \theta_j \, \sin^j(\theta_j) = \frac{\sqrt{\pi}\,\Gamma\left(\frac{j+1}{2}\right)}{\Gamma\left(\frac{j+2}{2}\right)} \tag{28}$$

So in the end, it's straightforward for general $h(r)$:

$$\mu(r) = a_{n-2}\int_0^r \mathrm{d}r r^{n-2}\rho(r) = a_{n-2}\int_0^r \frac{M}{A_{n-2}} h'(r) r^{n-2}\mathrm{d}r = M\int_0^r \mathrm{d}r \frac{\mathrm{d}h(r)}{\mathrm{d}r} = M\left[h(r) - h(0)\right] \tag{29}$$

By construction of $\rho(r)$, it just kills the $(n-2)$ dimensional sphere. $\mu(r)$ only diverges if $h(0)$ diverges. The $h(r)$ Ansatz from [1.4 NS2012] yields

$$h(r) = \frac{r^2}{r^2+L^2}, \quad \frac{\mathrm{d}h(r)}{\mathrm{d}r} = \frac{2rL^2}{(r^2+L^2)^2} \quad \Rightarrow \quad \mu(r) = \frac{2r\,ML^2}{(r^2+L^2)^2} \tag{30}$$

for *any* dimension n.

Calc3

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1 Schwarzschild modifications in D dimensions

Consider D dimensional spacetime. This is an $n = D-4$ dimensional extension to the 4-dimensional spacetime. We commonly define the greek indices $\mu, \nu, \dots = [1..4]$ for classical 4d-coordinates, big latin indices $A, B, \dots K, L, \dots = [1..D]$ for all coordinates and small latin indices $i, j, \dots = [1..n+2]$ for the angles. So a vector may be noted as $x_K = (x_0, \dots, X_D)$. In radial coordinates it can be written as $x_K = (t, r, \phi, \theta_1, \dots, \theta_{D-3})$.

We start with arbitrary $\rho(r)$, with r beging the radial value of x_K. We derive the metric g_{AB} and require SS behaviour $g_{AB} = 0$ when $r \to \infty$. The Ansatz done by Rizzo is

$$ds^2 = e^{\nu} dx_0^2 - e^{\mu} dr^2 - r^2 d\Omega_{D-2}^2 \tag{1}$$

SS requires $e^{\nu,\mu} \to 1 \Leftrightarrow \mu = -\nu$ when $r \to \infty$. We write $e^{\nu} = 1 - f(r)$ and examine the D dimensional conservation of energy equation, $\nabla_B T^{AB} = 0$. *Now skipping all Ricci deriving stuff.*

R_i^i Einstein equations yield this first order ODE in $f(r)$:

$$f'(r) + \frac{n+1}{r} f(r) = \frac{1}{M_\star} \frac{2r\rho(r)}{n+2} \tag{2}$$

with $M_\star = M_*^{n+2}$ the reduced fundamental mass scale of the theory. This can be solved for any $\rho(r)$ to

$$f(r) = r^{-n-1} \left(\frac{2}{(n+2)M_\star} \int_{c_1}^{r} (r')^{n+2} \rho(r') \, dr' + c_2 \right) \quad \text{with } c_1, c_2 = \text{const} \tag{3}$$

Setting c_1 arbitrary, like $c_1 = L_P$ or $c_1 = 0$, and $c_2 = 0$ to match the boundary conditions $g_{00} \xrightarrow{r\to\infty} 0$, a solution is

$$f(r) = \frac{2}{(n+2)} \frac{m(r)}{M_\star} \frac{1}{r^{n+1}} := \frac{\mu(r)}{r^{D-3}} \quad \text{with} \quad m(r) = \int_{L_P}^{r} (r')^{n+2} \rho(r') dr' \tag{4}$$

This looks like the general Schwarzschild-Tangherlini-Solution $f(r) = \mu/r^{D-3}$ which is the D-dimensional SSM $f(r) = 2M/r$ generalization.

1.1 Noncommutation in D dim

I can insert the NSS 2006 density $\rho(r)$ into solution (3):

$$\rho(r) = \frac{M}{(4\pi\theta)^{(n+3)/2}} e^{-r^2/4\theta} \tag{5}$$

$$f(r) = r^{-1-n} \left(c_1 - \frac{1}{M_\star} \frac{M}{(2+n)\pi^{(n+3)/2}} \Gamma\left(\frac{3+n}{2}; \frac{r^2}{4\theta} \right) \right) \quad \text{with } c_1 = \text{const} \tag{6}$$

Since $\Gamma(a, r) \xrightarrow{r\to\infty} 0$, boundary conditions are met, but our $f(r) < 0$ if $c_1 = 0$, so we arbitrary set $c_1 = \frac{1}{M_\star} \frac{M}{(2+n)\pi^{(n+3)/2}} \Gamma\left((3+n)/2\right)$. This enables us writing $f(r)$ in a compact way, following

Rizzo 2006 and using the identity $\gamma(a,x) + \Gamma(a,x) = \Gamma(a)$, exploiting the incomplete Gamma functions

$$\gamma(a,x) = \int_0^x t^{a-1} e^{-t} \mathrm{d}t, \quad \Gamma(a,x) = \int_x^\infty t^{a-1} e^{-t} \mathrm{d}t \tag{7}$$

Finally I derived Rizzo 2006:

$$f(r) = \frac{1}{M_\star} \frac{M}{(n+2)\pi^{(n+3)/2}} \frac{1}{r^{n+1}} \gamma\left(\frac{n+3}{2}; \frac{r^2}{4\theta}\right) \tag{8}$$

For $\theta \to 0$ we have $\gamma(\frac{n+3}{2}, x) \xrightarrow{x\to\infty} \Gamma(\frac{n+3}{2})$ which is just a constant factor.
For $n \to 0$ (leaving θ as is) we end up with the not so nice

$$f_{\theta=0}(r) = \frac{1}{M_\star} \frac{M}{2\pi^{3/2}} \frac{1}{r} \gamma\left(\frac{3}{2}, \frac{r^2}{4\theta}\right) \tag{9}$$

1.2 Holography in D dim

With the NS 2011 generalized density $\rho(r)$ to D dimensions,

$$\rho(r) = \frac{M}{\Omega} \frac{\mathrm{d}h(r)}{\mathrm{d}r}, \quad \Omega = \Omega_{D-2} \tag{10}$$

using the differential equation solution (3) we have

$$f(r) = r^{-n-1} \left(\frac{2M}{M_\star(n+2)\Omega} \int_{c_1}^r (r')^{n+2} \frac{\mathrm{d}h(r')}{\mathrm{d}r'} \mathrm{d}r' + c_2 \right) \quad \text{with } c_1, c_2 = \text{const} \tag{11}$$

It seems that there can be made requirements for the shape of $h(r)$ based upon eq.(11). I explored a partial integration series in n which probably could tell me a maximal leading power, above which the integral no more converges. It looks like

$$\begin{aligned} f(r) \propto \frac{1}{r^{n-1}} \Bigg\{ & \left[x^{n+2} \int_\infty^x \mathrm{d}y_1\, h'(y_1) \right]_0^r \\ & - \left[x^{n+1} \int_\infty^x \int_\infty^{y_1} \mathrm{d}y_1 \mathrm{d}y_2\, h'(y_2) \right]_0^r \\ & + \left[x^{n} \quad \int_\infty^x \int_\infty^{y_1} \int_\infty^{y_2} \mathrm{d}y_1 \mathrm{d}y_2 \mathrm{d}y_3\, h'(y_3) \right]_0^r \\ & - \left[x^{n-1} \int_\infty^x \int_\infty^{y_1} \int_\infty^{y_2} \int_\infty^{y_3} \mathrm{d}y_1 \mathrm{d}y_2 \mathrm{d}y_3 \mathrm{d}y_4\, h'(y_4) \right]_0^r \\ & \quad \ldots \\ & + (-1)^{(m+1)} \left[x^{n-(m+1)} \prod_{i=1}^m \int_\infty^{y_{i-1}} \quad h'(y_m) \right]_0^r \quad \text{in the } m\text{. line, with } y_0 := x \\ & \quad \ldots \Bigg\} \end{aligned} \tag{12}$$

Eq (12) tells me that $h(r)$ must be at least $n+2$ times integrable, and, unfortunately, one cannot state that the first line $[x^{n+2} \ldots]_0^r = r^{n+2} h(r)$ contributes most.

1.2.1 Using $h(r) = r^2/(r^2 + L^2)$

If we insert the approach $h(r) = r^2/(r^2 + L^2)$, we have

$$\begin{aligned} f(r) =& \frac{c_1}{r^{n+1}} + \frac{1}{r^{n+1}} \frac{2M}{M_\star (n+2)\Omega} \left[L^2 \left(\frac{1}{1+L^2} - \frac{r^{2+n}}{L^2 + r^2} \right) \right. \\ & \left. - {}_2F_1\left(1, \frac{n}{2}+1; \frac{n}{2}+2; -\frac{1}{L^2}\right) + r^{2+n}\, {}_2F_1\left(1, \frac{n}{2}+1; \frac{n}{2}+2; -\frac{r^2}{L^2}\right) \right] \end{aligned} \tag{13}$$

with ${}_2F_1$ the hypergeometric function ${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$ (with Pochhammer Symbol $(x)_n = n!\binom{x}{n}$).

A check for $n \to 0$ gives

$$f_{n=0}(r) = \frac{L^2 M}{M_\star \Omega r} \left(\frac{1}{1+L^2} - \frac{r^2}{L^2 + r^2} - \log\left(1 + \frac{1}{L^2}\right) + \log\left(1 + \frac{r^2}{L^2}\right) \right) \tag{14}$$

Notice the bad units e.g. in $1 + 1/L^2$ (so the calculation needs to be checked). Expected was something roughly like

$$f_{n=0}(r) = \frac{2M}{r}\rho(r) \approx \frac{2M^2 \left(-\frac{10r^2}{(L^2+r^2)^2} + \frac{2}{L^2+r^2} + \frac{8r^4}{(L^2+r^2)^3} \right)}{r\Omega} \tag{15}$$

1.2.2 Using $h(r) = h_\alpha(r)$

The approach

$$h_\alpha(r) = \frac{r^3}{(r^\alpha + (\tilde{r}_0)^\alpha/2)^{3/\alpha}}, \quad \text{Call} \quad r_0 := \tilde{r}_0 := \tilde{r} \tag{16}$$

yields something like

$$f(r) = c_1 r^{-n-1} + \frac{2r^5 \left(2\left(\frac{r}{\tilde{r}}\right)^\alpha + 1\right)^{3/\alpha} \left(r^\alpha + \frac{1}{2}\tilde{r}^\alpha\right)^{-3/\alpha} {}_2F_1\left(\frac{3}{\alpha}, \frac{n+6}{\alpha}; \frac{n+6}{\alpha} + 1; -2\left(\frac{r}{\tilde{r}}\right)^\alpha\right)}{M_\star (n+2)(n+6)} \tag{17}$$

The very present number 3 seems to be motivated by 3 spatial dimensions, so if we change that to $3 + n$, thus considering a modified density

$$h_\alpha(r) = \frac{r^{(n+3)}}{(r^\alpha + r_0^\alpha/2)^{(n+3)/\alpha}} \tag{18}$$

This has the solution

$$f(r) = c_1 r^{-n-1} + \frac{r^{n+5}\left(2(r/r_0)^\alpha + 1\right)\left(r^\alpha + \frac{r_0^\alpha}{2}\right)^{-\frac{n+3}{\alpha}} {}_2F_1\left(1, \frac{n+\alpha+3}{\alpha}; \frac{2n+\alpha+6}{\alpha}; -2(r/r_0)^\alpha\right)}{M_\star (n+2)(n+3)} \tag{19}$$

Calc4

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1 Holography in D dim, corrected

This writeup contains the corrected calculations from Calc3 and then follows more strictly the Rizzo approach for calculation Black Hole properties.

1.1 Framework: Rizzo

From Rizzo2006 we have a generic solution for a Schwarzschild-like Metric in D dimensions (so $n = D - 4$ extra dimensions),

$$\mathrm{d}s^2 = (1 - f(r))\,\mathrm{d}t^2\,(1 - f(r))^{-1}\,\mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2_{D-2} \tag{1}$$

It is the ODE

$$f'(r) + \frac{n+1}{r} f(r) = \frac{1}{M^\star} \frac{2r\rho(r)}{n+2} \tag{2}$$

with M_* the reduced fundamental mass scale of the theory (shortcut $M^\star = M_*^{n+2}$). It is easy to solve this for any $\rho(r)$ to

$$f(r) = \frac{1}{r^{n+1}} \left(\frac{2}{(n+2)M^\star} \int_{c_1}^r (r')^{n+2} \rho(r')\,\mathrm{d}r' + c_2 \right) \quad \text{with } c_1, c_2 = \text{const} \tag{3}$$

like already done in Calc3.

1.2 Holography in D dim

With the NS 2011 generalized density $\rho(r)$ to D dimensions,

$$\rho(r) = \frac{M}{\Omega(r)} \frac{\mathrm{d}h(r)}{\mathrm{d}r}, \quad \Omega(r) = \Omega_{D-2} r^{D-2} = \Omega_{n+2} r^{n+2} \tag{4}$$

the integral in $f(r)$ is evaluated in a trivial manner (this was done wrong in Calc3). That is, it reads

$$\begin{aligned} f(r) &= \frac{1}{r^{n+1}} \left(\frac{2}{(n+2)M^\star} \int^r \frac{M}{\Omega_{D-2}} h'(r')\mathrm{d}r' + \text{const} \right) & (5) \\ &= \frac{2}{n+2} \frac{M}{M_*^{n+2}} \frac{1}{\Omega_{n+2}} \boxed{\frac{h(r)}{r^{n+1}}} & (6) \end{aligned}$$

We note that the units are correct. With $h(r) = \theta(r)$ (in 5, $h(r) = 1$ in 6), the result gets the proper Schwarzschild-Tangherlini result $f(r) \propto 1/r^{D-3} = 1/r^{n+1}$.

1.3 Getting r_H

Rizzo already made a lot of effort to calculate r_H at $g_{00} = 0$, that is, $f(r_H) = 1$. The bottom line is that there are no more closed form solutions in the models he explored (NC, Lorentzian). Rizzo writes the horizon equation $f(r_H) = 1$ as

$$\begin{aligned} m &= M/M_* \qquad & y &= M_*\sqrt{\theta} \qquad & c_n &\approx (n+2)\Omega_{n+2} \\ x &= M_* R_H \qquad & z &= x/y = R_H/\sqrt{\theta} \qquad & x^{n+1} &= \frac{m}{c_n} F_n(z) \end{aligned} \tag{7}$$

He lists possible $\delta(r)$ modeling expressions $\rho(r)$ and the functions $F_n(z)$ to be discussed. I added the two holography ones.

Label	$\rho(r)$	$F_n(z)$
D dim NC (Rizzo2006)	$\rho = \frac{M}{(4\pi\theta)^{(n+3)/2}} e^{-r^2/4\theta}$	$F_n(z) = \frac{1}{\Gamma\left(\frac{n+3}{2}\right)} \gamma\left(\frac{n+3}{2}; \frac{z^2}{4}\right)$
Lorentzian (Rizzo2006)	$\rho \sim \frac{1}{(r^2+L^2)^{\frac{n+4}{2}}}$	$G_n(z) = \frac{2}{\pi} \frac{(n+2)!!}{(n+1)!!} \int_0^z \mathrm{d}t \frac{t^{n+2}}{(1+t^2)^{(n/2+2)}}$
D dim Holography	$\rho = \frac{M}{\Omega_{n+2} r^{n+2}} h'(r)$	$H_n(r) = h(r)$
D dim NS2011 $h = \frac{r^2}{r^2+L^2}$	$\rho = \frac{M}{\Omega_{n+2}} \frac{1}{r^{n+1}} \frac{L^2}{(L^2+r^2)^2}$	$H_n(z) = \frac{z^2}{z^2+1}$ with $\sqrt{\theta} = L$

Rizzo claims that all ρ models behave quite similary. I wonder if his Lorentzian approach would be the right D dimensional extension to NS2011. All my holography functions lack a dependence of n.

1.4 Open Questions

- How to choose $\rho(r)$? Toy model or physical motivation? Where is the motivation?
- How much degrees of freedom in $\rho(r)$ choice?
  How to match $r_0 = l_*$, $M_0 = M_*$, $G = M_*^{1-m}$?
- *What* to do with the calculated quantities *Horizons*, *Hawking Temperature*, *surface energy density*, *heat capacity*?

Calc5

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1 A Master Thesis Intro

Calc5 is supposed to collect introductory things for an upcoming Master thesis. I especially want to handle literature in a proper way. Propriate Commenting is done in Calc3, anyway.

1.1 A common Introducion

- Idea of extra spacial dimensions goes back to 1920s work of Theodor Kaluza and Oskar Klein. They proposed 4+1 dimensions, 5th dimension shall be microscopic curled up (compactified).
- Famous 1998 paper of Arkani-Hamed, Dimopoulos, Dvali about large extra dimensions
- Paper of Randall and Sundrum of infinite extra dimensions

1.2 Topics and Issues handled by the Intro

- Warum reicht es, Schwarzschild zu betrachten? (BH phases, Spin-Down, etc.)
- Welches Ziel haben meine QGR-Modelle?
- Welche Vorarbeiten gibt es zu analytischen Lösungen der Einstein-Gleichung? - Siehe auch das gleichlautende Buch
- Approaches: NonCommutavity, Holography

Calc6

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1 The surface issue

These calculations point out a missing surface term when doing the plausibility check for the holographic approach done in [NS11.2013].

I work in $D = 4 + n$ dimensions, but any equation must hold for $n = 0$, too. I follow the Rizzo2006 deviation (also performed in Calc1) for deriving an ODE for the potential $V(r)$ in $g_{00} = 1 - V(r)$ SMM like metrics. Having given only $\rho(r)$, the continuity equation $T^{AB}_{;B} = 0$ gives

$$T^i_i = \rho + \frac{r}{n+2}\partial_r \rho. \tag{1}$$

Which then lead to the first order differential equation

$$V' + \frac{n+1}{r} V = \frac{1}{M_*^{n+2}} \frac{2r\rho}{n+2}. \tag{2}$$

The Ansatz $V(r) = r^{-(n+1)} \left(C \int^r x^2 \rho(x) \mathrm{d}x + D \right)$ solves equation 2. It is simple to derive, as done in Calc3 and Calc4,

$$V(r) = \frac{1}{r^{n+1}} \left(\frac{2}{(n+2)M_*^{n+2}} \int_{c_1}^r x^{n+2} \rho(x) \mathrm{d}x + c_2 \right). \tag{3}$$

It is important to remark that the integral in 3 only looks like the radial part of an partially performed spherical integration, but *there is no surface term*, as there would be if the integral really would be $m(r) = \int \mathrm{d}^{n+3}\, \vec{r}\, \rho(\vec{r})$. That is,

$$\int \mathrm{d}^{n+3}\vec{r}\, \rho(\vec{r}) = \int \mathrm{d}r \, \left(\Omega_{n+2} r^{n+2} \right) \, \rho(r), \tag{4}$$

With $\Omega_{n+2} r^{n+2}$ being the $(n+2)$ dimensional surface (of an $n+3$ dimensionall sphere)

$$\Omega_{n+2} = 2 \frac{\pi^{\frac{n+3}{2}}}{\Gamma\left(\frac{n+3}{2}\right)} \tag{5}$$

The missing Ω_{n+2} in eq. 3 compared to 4 stands out. This is important, because the holographic approach depends on that property of 4.

1.1 NC in D dim

Rizzo introduces the reduced Planck scale M_* by $M_P^2 = V_n M_*^{n+2}$, with $v_n = (2\pi R_c)^n$ the volume of the compacted dimensions as tori with radius R_c. Thus the $n \to 0$ limit gives $M_*^2 = M_P^2 = 1/G$.

Using the gaussian $\rho(r)$, Rizzo (and I in Calc3) got

$$V(r) = \frac{M}{M_*^{n+2}} \frac{1}{(n+2)\pi^{(n+3)/2}} \frac{1}{r} \Gamma\left(\frac{3+n}{2} ; \frac{r^2}{4\theta} \right) \tag{6}$$

In the $\theta, n \to 0$ limit, $\Gamma(\frac{3}{2}; \infty) = \sqrt{\pi}/2$ and therefore we end with

$$V(r) = \frac{GM}{4\pi r} \tag{7}$$

1.2 $h(r)$ Profile

In [NS 07.11.2013], the $\theta \to h(r)$ smearing function is introduced, so $\partial_r\theta = \delta \to \partial_r h$ enters a smeared density:

$$\rho(r) = \frac{M}{4\pi r^2}\frac{\mathrm{d}h}{\mathrm{d}r} \quad \xrightarrow{\text{D=n+2 dimensions}} \quad \rho(r) = \frac{M}{\Omega_{n+2} r^{n+2}}\frac{\mathrm{d}h}{\mathrm{d}r} \tag{8}$$

Since $\Omega_2 = 4\pi$, this seems to be true. I showed already in Calc2 that an integration (like in eq 4) over that class of $\rho(r)$ gets trivial in *any* dimension.

Lets apply the solution for $V(r)$ at this density. Since that integration is not a *full* one, it allows the surface constant Ω_{n+2} to enter the metric. We end up with (already showed in Calc4)

$$V(r) = \frac{2}{n+2}\frac{M}{M_*^{n+2}}\frac{1}{\Omega_{n+2}}\frac{h(r)}{r^{n+1}}. \tag{9}$$

This equation cannot produce the SMM value $V(r) = \frac{2GM}{r}$ any more, because nothing kills the $\Omega_2 = 4\pi$. Indeed, if we use the Schwarzschild-Tangherlini density $\rho(r) = M/(\Omega_{n+2})\delta(r)$ and apply it to eq. 3, [Reall-Review Section 3.2]

$$V(r) = \frac{1}{r^{n+1}}\left(\frac{2}{(n+2)M_*^{n+2}}\frac{M}{\Omega_{n+2}}\int \mathrm{d}r\delta(r)\right) = \frac{\mu}{r^{n+2}}, \quad \mu = \frac{16\pi GM}{(n+2)\Omega_{n+2}} \tag{10}$$

TODO: Why μ?. If we now send $n \to 0$, this does not reproduce SMM at all.

The Planck length $M_P^2 = V_n M_*^{n+2}$ is equal to M_* in $n = 0$ dimensions. Since $M_P = 1/\sqrt{G}$ Newtons constant $G = 1/M_*^2$ is restored. Thus, from eq. we get for $n = 0$

$$V(r) = \frac{GM}{r}\frac{1}{4\pi} \tag{11}$$

Conclusion: There seems always the factor 8π to be missing. There seems to be some $G \leftrightarrow 8\pi G$ issue.

Calc7

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Calc7 contains properties of $h(r)$ and $h_\alpha(r)$.

Contents

1 Remnant Masses M_*

In this writeup I collect the properties of two models $h(r)$ and $h_\alpha(r)$ in $D = n + 4$ dimensions.

$$h(r) = \frac{r^2}{r^2 + L^2} \tag{1}$$

$$h'(r) = \frac{2rL^2}{(r^2 + L^2)^2} \tag{2}$$

$$h_\alpha(r) = \frac{r^{3+n}}{(r^\alpha + L^\alpha/2)^{\frac{3+n}{\alpha}}} \tag{3}$$

$$h'_\alpha(r) = \frac{(n+3)L^\alpha r^{n+2} \left(\frac{L^\alpha}{2} + r^\alpha\right)^{-\frac{n+3}{\alpha}}}{L^\alpha + 2r^\alpha} \tag{4}$$

Let now $h(r)$ be a generic profile. I frequently derived the metric $g_{00} = 1 - V(r)$ for these profiles,

$$\rho(r) = \frac{M}{\Omega_{n+2}} \frac{\mathrm{d}h(r)}{\mathrm{d}r} \quad \Rightarrow \quad V(r) = \frac{2}{n+2} \frac{M}{M_*^{n+2}} \frac{1}{\Omega_{n+2}} \frac{h(r)}{r^{n+1}}. \tag{5}$$

Until now, L is just a constant. The Holography (1) and Self-Regular (3) profiles can be expressed as $h(r) = h(r/L)$, so L here is just a scaling factor for r.

1.1 Extremal radius

The remnant is the smallest possible Black Hole solution and considered as a stable particle that can no more evaporate. Self encoding solutions encode the remnant radius by it's degrees of freedom. Typically the (reduced) Planck Length is supposed to be equal to the remnant's size. Considering figure 3, the remnant equations require

$$\begin{cases} \partial_r|_{r=r_0}\, g_{00}(r) &= 0 \\ g_{00}(r_0) &= 0 \end{cases} \tag{6}$$

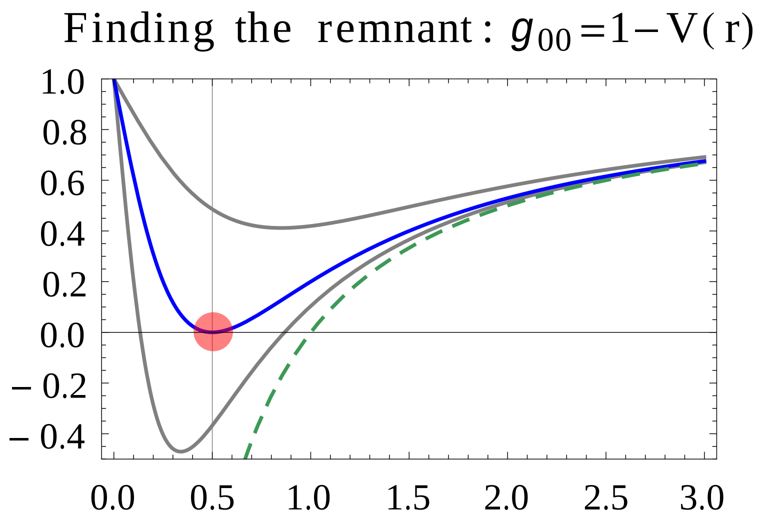


Figure 1: Extremal Configuration in regularized Schwarzschild metrics. Since $g_{00}(r) \to 0$ at $r \in \{0, \infty\}$, there must be an extremal r_0. In the extremal configuration (blue), $r_0 = r_H$. There are also solutions with two r_H and $r_0 < 0$ and no r_H, $r_0 > 0$. The dashed line is the Schwarzschild behaviour.

For our models (eq 5), we can in general rewrite this set of equations in terms of $h(r)$. Especially, for finding the extremal value,

$$\partial_r V(r) = 0 \Leftrightarrow 0 = \partial_r \frac{h(r)}{r^{n+1}} \Leftrightarrow 0 = -(n+1)\frac{h(r)}{r} + h'(r)$$
$$\Leftrightarrow 0 = \left[L\, h'(z) - \frac{n+1}{z}\, h(z) \right]_{z=\frac{r_0}{L}} \tag{7}$$

a substitution $z = r/L$ makes it easier to find a solution.

Finding the extremal radius r_0 (let $r_{0,\alpha}$ be for the self-regular model, r_0 for the holography model) is a bit of calculation. In the end, I got

$$r_0 = L\,\frac{1+n}{1-n} \tag{8}$$
$$r_{0,\alpha} = \frac{L}{(1+n)^{1/\alpha}} \tag{9}$$

1.2 Extremal value for the holographic model $h(r) = \frac{r^2}{r^2+L^2}$

Equation 8 tells us that there is *no extremum in higher dimensions*. At $n = 0$ the result coincides with [NS2012], at $n > 1$ all $r_0 < 0$. Actually this profile exhibits always a single horizon r_H as soon as $n > 1$. There is no more regularization for $n > 1$ which can be read off the metric:

$$V(r) \propto \frac{1}{r^{n+1}} \frac{r^2}{r^2+L^2} \propto \frac{1}{r^{n-1}} \frac{1}{r^2+L^2} \tag{10}$$

As soon as $n > 1$, there is a singularity at the origin. There is a special case in five dimensions ($n = 1$). A naked black hole at origin can be made with $L = 1$. etc.

1.3 Self-Encoding for $h_\alpha(r)$

The self regular solution looks like figure 3 in all dimensions. We can therefore always find a remnant and therefor fix α for any number of dimensions. Inserting r_0 to

$$1 \overset{!}{=} V(r_0) = \frac{2}{n+2} \frac{M(r_0)}{M_*^{n+2}} \frac{1}{\Omega_{n+2}} \frac{1}{r_0^{n+1}} \frac{h_\alpha(r_0)}{1} \tag{11}$$

After Calc6 (The surface issue), I think this equation needs some fixing. When $\frac{2\Omega_2}{\Omega_{n+2}}$ replaces the Ω fraction in equation 11, in the $n \to 0$ limit this gives a $\frac{2\cdot 4\pi}{4\pi}$ which is needed to reproduce SMM correctly. Furthermore, let's remember that

$$M_P^2 = V_n M_*^{n+2}, \quad V_n = (2\pi R_c)^n, \quad \frac{M_P = 1/\sqrt{G}}{L_P = \sqrt{G}} = M_P^2 = \frac{1}{G}, \quad \Rightarrow \frac{M_*}{L_*^{n+1}} = M_*^{n+2} \tag{12}$$

Inserting $h_\alpha\left(L/(1+n)^{1/\alpha}\right) = \left(\frac{3+n}{2}\right)^{\frac{3+n}{\alpha}}$, I finally get

$$M(r_0) = \frac{1}{n+2} \left(\frac{3+n}{2}\right)^{\frac{3+n}{\alpha}} \left(\frac{r_0^{n+1}}{L_*^{n+1}}\right) \frac{\Omega_{n+2}}{\Omega_2} M_* \tag{13}$$

This looks very much like eq. 3 from [NS2012], expect the surface terms. The remnant's mass is equal to the reduced Planck mass if all coefficients get 1. This has multiple implications: We can determine α as well as identify $r_0 = L_*$, so the remnants radius is the reduced planck length. Eventually this means $L_P = (1+n)^{1/\alpha} L_*$. If I ignore the surface terms,

$$\alpha_0 = \frac{3+n}{\ln(2+n)} \ln \frac{3+n}{2} \tag{14}$$

which reduces exactly to $\alpha_0 = \frac{3}{\ln 2} \ln \frac{3}{2}$ found by [NS2012] in $n = 0$. Presumably they have to be taken into account, so

$$\boxed{\alpha_0 = \frac{3+n}{\ln(2+n) - \ln \omega_n} \ln \frac{3+n}{2}}, \quad \omega_n = \frac{\Omega_{n+2}}{\Omega_2} = \frac{\pi^{(n+3)/2}}{2\pi\Gamma\left((n+3)/2\right)} \quad \text{(Calc6, eq.5)} \tag{15}$$

This is fine, since $\omega_0 = 1$. I will use eq 15 for further calculations.

n	0	1	2	3	4	5	6	7
α	1.755	4.285	7.081	9.333	10.642	11.128	11.098	10.805
L_*	1.	0.851	0.856	0.862	0.86	0.851	0.839	0.825
M_*	1.	1.176	1.168	1.16	1.163	1.175	1.192	1.212
G_*	1.	0.724	0.733	0.743	0.739	0.725	0.704	0.681

Figure 2: Self-encoding horizon radius $r_0 = L_*$ and Remnant masses $M_* = 1/L_*$, in 4d Planck Units.

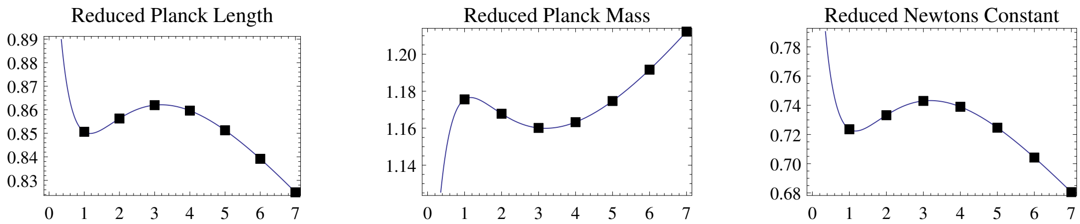


Figure 3: Table 2 as plot, from left to right: $L_*(n)$, $M_*(n) = 1/L_*$, $G_*(n) = L_*^2$

2 Hawking Temperature

Considering the h_α model with bigger masses $M > M_*$, the resulting Black Holes have multiple event horizons r_H. Solving $0 = g_{00}$ gives (compare eqn 11ff)

$$1 \stackrel{!}{=} V(r_H) = \frac{2}{n+2}\frac{M(r_H)}{M_*^{n+2}}\frac{1}{\Omega_{n+2}}\frac{1}{r_H^{n+1}}\frac{h_\alpha(r_H)}{1} \tag{16}$$

Eq 16 gives us the mass $M(r_H)$ of a BH with event horizon r_H in the same was as eq 13 told that for $M(r_0)$. Then I determine the Hawking Temperature $T_H = \frac{1}{4\pi}\left(\frac{\mathrm{d}g_{00}}{\mathrm{d}r}\right)_{r=r_H}$, plugging in $\alpha = \alpha_0$, $M = M(r_H)$. Let

$$V(r) = AM(r)\frac{1}{r^{n+1}}h_\alpha(r) \tag{17}$$

$$\Leftrightarrow M(r_H) = \frac{r^{n+1}}{A\,h_\alpha(r)} \quad \text{like in eq 13} \tag{18}$$

$$\frac{\mathrm{d}g_{00}}{\mathrm{d}r} = (n+1)\frac{M\,A}{r^{n+2}}h_\alpha(r) - A\,M\frac{h'_\alpha(r)}{r^{n+1}} \tag{19}$$

$$\left.\frac{\mathrm{d}g_{00}}{\mathrm{d}r}\right|_{r=r_H} = (n+1)\frac{1}{r_H} - \frac{h'_\alpha(r_H)}{h_\alpha(r_H)} \tag{20}$$

$$= \frac{1}{r_H}\left((n+1) - (n+3)\frac{1}{1+\left(\frac{r_H}{L}\right)^\alpha}\right) \tag{21}$$

$$= \frac{1}{L\,z_H}\left((n+1) - (n+3)\frac{1}{1+z_H^\alpha}\right) \quad \text{with } z_H = r_H/L \tag{22}$$

Equations 20, 21 and 22 are wrong. Eq 17, 18 and 19 are correct.

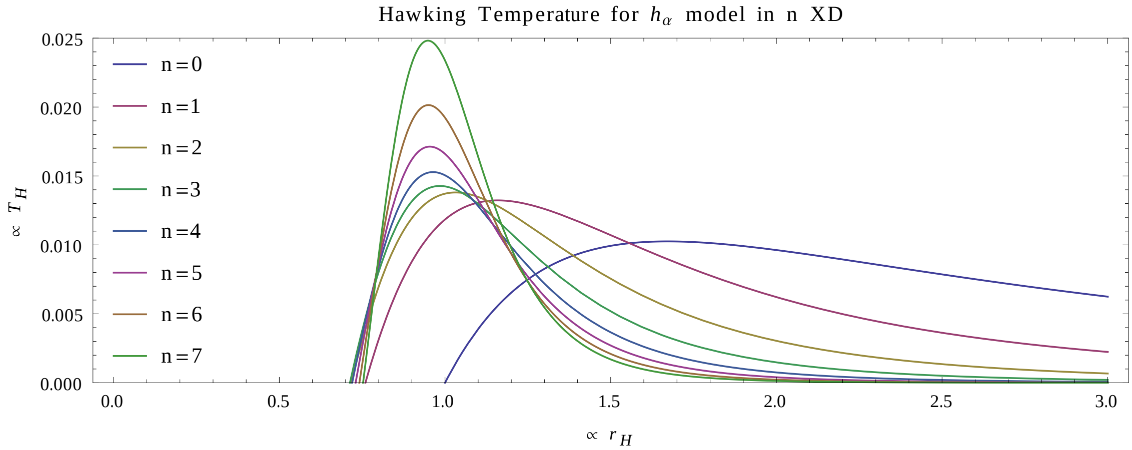


Figure 4: Hawking temperature, plot of eqn. 19 with $L = M = M_* = 1$ and $\alpha = \alpha_0$.

The Hawking Temperature vanishes[1] at r_0, that is, $T_H(r_{0,\alpha_0}) = 0$. Again, this confirms the self-encoding property of h_α.

3 Heat capacity

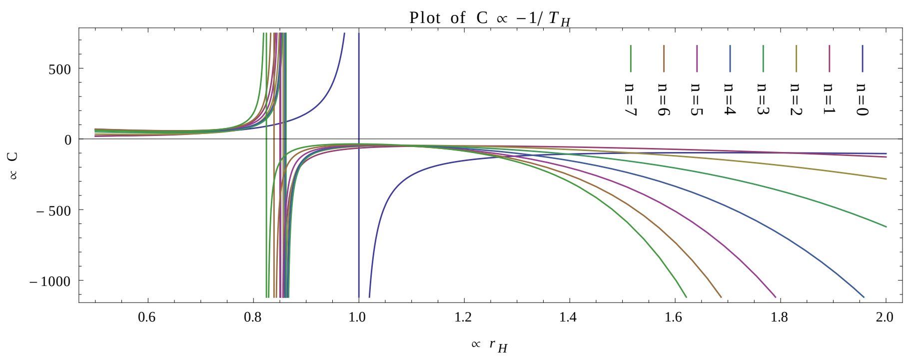
The Heat Capacaity C is defined as $C = \partial U/\partial T_H$ with the internal energy $U = M$, so [NS 06.11.2013]

$$C = \frac{\partial M}{\partial T_H} \tag{23}$$

I wonder how this works. In SMM ($G = 1$), $T_H = 2M/r^2$, so $M = T_H r^2/2$ and $\partial M/\partial T_H = r^2/2$. This is not $C \propto -r^2$ as stated in [NS 06.11.2013]. Is it correct that

$$\frac{\partial M}{\partial T} = \left(\frac{\partial T}{\partial M}\right)^{-1} = \left(\frac{T}{M}\right)^{-1} \propto T^{-1} \tag{24}$$

because is there is also the simple relationship $T_H = M\, f(r)$? This shows $-1/T_H$:



[1] I made a numeric check for $L = 1$ like in figure 4. I got a table (like figure 2) for the r which solve the equation $T_H(r) = 0$, the values match $r = L_*$ for all n.

Calc8

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1 Propagator and modified Einstein Equations

1.1 Einstein modifications with NCBHs

I got plenty of papers following basically modifications $G \to \mathcal{G}$ or $T_{\mu\nu} \to \mathcal{S}_{\mu\nu}$ or $R \to \mathcal{R}$, e.g.

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G_N \mathcal{S}_{\mu\nu} \quad \mathcal{S}_{\mu\nu} = \mathcal{F}^2(\Box(x)/\Lambda_G^2) T_{\mu\nu} \quad \text{[MMN 2010]} \tag{1}$$

See also

- [N Feb 2012]: *Nonlocal and generalized uncertainty principle black holes*
  Introduces the bi-local distribution $\mathcal{A}^2(x-y)$ that modifies R, the Einstein Equations, $T_{\mu\nu}$ and $\delta(\vec{x})$
- [Modesto Moffat N Dec 2010]: *Black holes in an ultraviolet complete quantum gravity*
  Introduces the entire (=holomorphic) function $\mathcal{F}$ that scales like $\mathcal{A}$ before.
- [Isi Nov2013]: *Self-Completeness and the Generalized Uncertainty Principle*
  Follows straightforward the [N Feb 2012] formalism.
- [Isi Feb2014]: *Self-Completeness in Alternative Theories of Gracity*

1.2 A Roadmap

With the approach $\mathcal{T}_{00} \propto M\,\mathcal{A}^{-2}(\Box)\delta(\vec{x})$, it's all about finding a differential operator that modifies the Dirac Delta to the smeared functions $\partial_r h$ or $\partial_r h_\alpha$. Since our approach always was $T_{00} \propto M/\Omega\,\delta(r) \to M/\Omega \frac{\mathrm{d}h}{\mathrm{d}r}$, with

$$h(r) = \frac{r^2}{r^2 + L^2} \tag{2}$$

$$h'(r) = \frac{2rL^2}{(r^2+L^2)^2} \tag{3}$$

$$h_\alpha(r) = \frac{r^{3+n}}{(r^\alpha + L^\alpha/2)^{\frac{3+n}{\alpha}}} \tag{4}$$

$$h'_\alpha(r) = \frac{(n+3)L^\alpha r^{n+2}\left(\frac{L^\alpha}{2} + r^\alpha\right)^{-\frac{n+3}{\alpha}}}{L^\alpha + 2r^\alpha} \tag{5}$$

the propagator $\mathcal{A}^{-2}(\Box)$ really must be a complex one to get $\mathcal{A}^{-2}\delta \to h'_\alpha$ (eq. 5)!

1.3 Note: Spherical Fourier transformation in $3 + n$ dimensions

From Felix Karbstein: Performing the Fourier transform of a generic position space potential $V(|\vec{r}|)$ in $d = 3$ dimensions to momentum space, we obtain

$$\hat{V}(p) = \int \mathrm{d}^3 r\, e^{-i\vec{r}\vec{p}}\; V(r) = 2\pi \int_{-1}^{+1} \mathrm{d}\cos\theta \int_0^\infty \mathrm{d}r\, r^2\, e^{-irp\cos\theta} V(r) \tag{6}$$

$$= \frac{2\pi i}{p} \int_0^\infty \mathrm{d}r\, r\, V(r) \left(e^{-irp} - e^{+irp}\right) = \frac{2\pi i}{p} \int_{-\infty}^\infty \mathrm{d}r\, e^{-irp}\, r\; [V(r)\Theta(r) + V(-r)\Theta(-r)] \tag{7}$$

with $r = |\vec{r}|$, $p = |\vec{p}|$. Note that this effectively amounts to an one dimensional Fourier transform

$$\hat{v}(p) = \int_{-\infty}^\infty \mathrm{d}r\, e^{-irp}\; v(r) \tag{8}$$

with

$$v(r) = r\; [V(r)\Theta(r) + V(-r)\Theta(-r)] \quad \text{and} \quad \hat{V}(p) = \frac{2\pi i}{p} \hat{v}(p) \tag{9}$$

Now proceed with going from $\mathrm{d}^3 r \to \mathrm{d}^{3+n} r$

Calc9

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Calc9 ties up to Calc7, making more calculations with the holographic models. I clean up the syntax for $H \in \{h, h_\alpha\}$ and work with dimensionless quantities $z = r/L$. Furthermore I derive formulas from the general to the specific.

Contents

1 Heat capacity and Entropic corrections

This is the first document where I propose a $D = n+4$ dimensional extension to the holographic model $h(r)$. The self-regular model $h_\alpha(r)$ was already discussed in Calc7.

$$h(r) = \frac{r^{2+n}}{r^{2+n} + L^{2+n}} \tag{1}$$

$$h'(r) = \frac{(2+n)\, r^{1+n} L^{2+n}}{(r^{2+n} + L^{2+n})^2} \tag{2}$$

$$h_\alpha(r) = \frac{r^{3+n}}{(r^\alpha + L^\alpha/2)^{\frac{3+n}{\alpha}}} \tag{3}$$

$$h'_\alpha(r) = \frac{(n+3) L^\alpha r^{n+2} \left(\frac{L^\alpha}{2} + r^\alpha\right)^{-\frac{n+3}{\alpha}}}{L^\alpha + 2r^\alpha} \tag{4}$$

Let $H \in \{h, h_\alpha\}$ be a generic profile as approximation of the theta function $\Theta(r)$ in $\rho(r)$, a class of densities for which I frequently derived the metric $g_{00} = 1 - V(r)$:

$$\rho(r) = \frac{M}{\Omega_{n+2}} \frac{\mathrm{d}H(r)}{\mathrm{d}r} \quad \Rightarrow \quad V(r) = \frac{2}{n+2} \frac{M}{M_*^{n+2}} \frac{1}{\Omega_{n+2}} \frac{H(r)}{r^{n+1}}. \tag{5}$$

1.1 The Mass

We can argue that M is just a constant, responsible for fulfilling the horizon equation $V(r_H) = 1$. If we set (»arbitrarily«)

$$M = \frac{n+2}{2} M_*^{n+2} \Omega_{n+2} \frac{r_H^{n+1}}{H(r_H)} = \frac{n+2}{2}\, \Omega_{n+2}\, \frac{1}{H(r_H)} \left(\frac{r_H}{L_*}\right)^{n+1} M_* \tag{6}$$

then the horizon equation $V(r) = 0$ is fulfilled at $r = r_H$. In general, equation 6 gives us a relationship $M = M(r)$. It can be used for given r to get the mass neccessary to create an event horizon at that r. In Calc7, I used it already for determining the remnant mass at $r = r_0$, in such a way that $M = M_*$ was obtained and $r_0 = L_*$ could be identified. Eventually, in models with (further) degrees of freedom (like $H = h_\alpha$), that equation also fixed α.

In particular, for $n = 0$, eq. 6 reduces to $M = r_H/2L^2 h(r_H)$. For $H = h$, we end with the well known $M = (r^2 + L^2)/2L^2 r$ from [NS 06.11.2013].

1.2 Dimensionless notation

My models $H(r)$ can be expressed in units of the dimensionless variable $z = r/L$ (which may be interpreted as »Multiples of the Planck unit«):

$$h(z) = 1/(1 + (1/z)^{2+n}) \tag{7}$$

$$h_\alpha(z) = 1/(1 + (1/z)^\alpha/2)^{(n+3)/\alpha} \tag{8}$$

The derivative $\frac{\mathrm{d}}{\mathrm{d}r}$ can be replaced by $\propto \frac{\mathrm{d}}{\mathrm{d}z}$ by determining $\frac{\mathrm{d}f}{\mathrm{d}r} = \frac{\mathrm{d}f}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}r} = \frac{1}{L}\frac{\mathrm{d}f}{\mathrm{d}z}$. We write $f'(z)$ for $\partial_z f(z)$:

$$h'(z) = (2+n)h^2(z)/z^{3+n} \tag{9}$$

All quantities $Q \in \{g_{00}, V, M, T_H, C, S, \dots\}$ can be written in units of z. If $[Q] = L^k$ is the unit of Q (that is, the kth power of length which equals the $-k$ths power of energy), a seperation

$$Q(r) = L^k \tilde{Q}(z) \tag{10}$$

is always possible, with $[\tilde{Q}] = 1$. This can be checked, let's write some already derived expressions in terms of z:

$$V(z) = \frac{2}{n+2}\frac{M}{M_*}\frac{L^{n+1}}{M_*^{n+1}}\frac{1}{\Omega_{n+2}}\frac{H(z)}{z^{n+1}}. \qquad [V] = 1 \qquad V(r) = V(z) \tag{11}$$

$$M(z) = \frac{n+2}{2}\,\Omega_{n+2}\,\frac{z_H^{n+1}}{H(r_H)}\left(\frac{M_*}{L}\right)^{n+1} M_* \qquad [M] = 1/L \qquad M(r) = M(z)\dots \tag{12}$$

1.3 Extremal Radius and Remnants

For h_α, this section was discussed in Calc7. For h it is new.

The extremal radius equation $\partial_r g_{00} = {}^1/_L\, \partial_z g_{00} = 0$ can be written as

$$0 = \frac{\mathrm{d}H(z)}{\mathrm{d}z} - (n+1)\frac{H(z)}{z}, \tag{13}$$

an expression which looks like the one derived in Calc7, only by replacing $r \to z$. After inserting $H(z) = h(z)$, the expression $0 = (n+2)\frac{h^2}{z^{3+n}} - (n+1)\frac{h}{z}$ can be easily solved, giving

$$\boxed{r_0 = L\, z_0 = L\left(\frac{1}{1+n}\right)^{\frac{1}{2+n}}} \tag{14}$$

We can enforce the holographic metric to have the event horizon at $r_H = r_0$. Using (6), this gives us

$$M(r_0) = \frac{n+2}{2} \underbrace{\Omega_{n+2}}_{\text{ignored}} \underbrace{(n+2)}_{1/h(r_0)} \left(\frac{r_0^{n+1}}{L_*^{n+1}} \right) M_* \tag{15}$$

So unlike for h_α, no self encoding $M(r_0) = M_*$ can take place since $\frac{(n+2)^2}{2} \neq 1$.

1.4 The Heat Capacity

Equation 6 is important for determining the heat capacity, when using the expression

$$C = \frac{\partial M}{\partial T_H} = \frac{\partial M}{\partial r_H} \left(\frac{\partial T_H}{\partial r_H} \right)^{-1} \tag{16}$$

Actually it would be nice to have a closed form expression $T_H = T_H(M)$ but it is hard to become, sagt Nicolini. For calculating C in terms of z, we simply write

$$C = \frac{\partial M}{\partial T_H} = \frac{\partial M}{\partial z_H} \left(\frac{\partial T_H}{\partial z_H} \right)^{-1} \tag{17}$$

Expressions could also be mixed in r and z. Nothing special about that:

$$C = \frac{\partial M}{\partial r_H} \frac{\partial r_H}{\partial z_H} \frac{\partial z_H}{\partial T_H} = L \frac{\partial M}{\partial r_H} \left(\frac{\partial T_H}{\partial z_H} \right)^{-1} \tag{18}$$

1.5 The Entropy

The Black hole entropy integral can be rewritten in the same way like the Heat Capacity was rewritten in equation 16:

$$S(r) = \int_{M_1}^{M_2} \frac{\mathrm{d}M}{T} = \int_{r_1}^{r_2} \frac{\mathrm{d}M}{\mathrm{d}r_H} \frac{\mathrm{d}r_H}{T} = \int \mathrm{d}r_H \frac{1}{T} \left(\frac{\mathrm{d}M(r_H)}{\mathrm{d}r_H} \right) \tag{19}$$

This allows me to reproduce the NS2011 result, using $H = h$, $n = 0$:

$$\frac{\mathrm{d}M}{\mathrm{d}r_H} = \frac{\mathrm{d}}{\mathrm{d}r} \left(\frac{1}{2L^2 r} (r^2 + L^2) \right) = \frac{1}{2} \left(\frac{1}{L^2} + \frac{1}{r^2} \right) \tag{20}$$

$$T = \frac{1}{4\pi r_H} \left(1 - \frac{2L^2}{r_H^2 + L^2} \right) \tag{21}$$

$$S = 4\pi \int_L^{r_H} r \left(\frac{r}{2L^2} + \frac{1}{2r} \right) \frac{1}{1 - \frac{2L^2}{r^2+L^2}} = \pi \left(\frac{r^2}{L^2} + 2\log(r) \right)_L^{r_H} \tag{22}$$

Like always, $S(r) = S(z)$ since $[S] = 1$ in natural units:

$$S(z) = \int_{z_1}^{z_2} \frac{\mathrm{d}M}{T} = \int \mathrm{d}z_H \frac{1}{T} \left(\frac{\mathrm{d}M(z_H)}{\mathrm{d}z_H} \right) \tag{23}$$

1.6 A generic approach to T_H, C and S

By merging all constant (non-r dependent) terms in the metric (5) and mass term (6), generic calculations with any H and n can be performed in a very simple way.

To do these calculations, let's shortly forget about H and just seperate $V(r)$ in a suggestive way:

$$V(r) = M(r_H) \cdot Y(r) \tag{24}$$

$$M(r_H) = Y^{-1}(r_H) \tag{25}$$

$$T = \frac{1}{4\pi}\, \partial_r g_{00}|_{r=r_H} = -\frac{1}{4\pi} V'(r_H) \tag{26}$$

$$= -\frac{1}{4\pi} M(r_H) \cdot Y'(r_H) = -\frac{1}{4\pi}\frac{1}{L} M(z_H) Y'(z_H) \tag{27}$$

$$S(z) = \int^z \mathrm{d}z_H \frac{M'(z_H)}{T} = -4\pi L \int^z \mathrm{d}z_H \frac{M'(z_H)}{M(z_H)} \frac{1}{Y'(z_H)} \tag{28}$$

It is important to note that (25) is only valid for r_H, so Y is not the inverse of M and the inverse derivative law cannot be applied (in general, $M(r) \neq Y^{-1}(r)$). In terms of r, M is constant: $M'(r) = 0$.

We can now introduce the holographic approach

$$Y(r) = A \frac{H}{r^{n+1}} \tag{29}$$

A can be eliminated quickly in T due to the property of (25).

$$V = A\, M(r_H)\, \frac{H(r)}{r^{n+1}} \tag{30}$$

$$M(r_H) = \frac{1}{A} \frac{r_H^{n+1}}{H(r_H)} \tag{31}$$

$$T = \frac{1}{4\pi r_H} \left(1 + n - r_H \frac{H'(r_H)}{H(r_H)} \right) = \frac{1}{L} \frac{1}{4\pi z_H} \left(1 + n - z_H \frac{H'(z_H)}{H(z_H)} \right) \tag{32}$$

$$C = \frac{4\pi r_H^{n+2}}{A} \frac{r_H H'(r_H) - (n+1) H(r_H)}{r_H^2 H(r_H) H''(r_H) - r_H^2 H'(r_H)^2 + (n+1) H(r_H)^2} \tag{33}$$

$$S(z) = -4\pi L \int^z \mathrm{d}x \left(\frac{n+1}{x} - \frac{H'(x)}{H(x)} \right) \frac{1}{Y'(x)} \tag{34}$$

$$= -4\pi L A \int^z \mathrm{d}x \left(\frac{n+1}{x} - \frac{H'(x)}{H(x)} \right) \frac{x^{n+2}}{x H'(x) - (n+1) H(x)} \tag{35}$$

The simple relation (32) was already asserted in Calc7, eq. 20, but not believed yet. Writing terms in z is handy because the resulting terms are dimensionless. Attached powers of L entirely indicate the physical units of quantities, like $[T] = [1/L]$.

1.6.1 Check with $n = 0$

I checked that, with $n = 0, H = h$, (32) and (33) gives the [NS 06.11.2013] result

$$T = \frac{1}{4\pi r}\left(1 - \frac{2L^2}{L^2 + r_H^2}\right) \tag{36}$$

$$C = -4\pi \frac{(L - r_H)(r_H + L)\left(r_H^2 + L^2\right)^2}{2L^2\left(4L^2 r_H^2 - r_H^4 + L^4\right)} \tag{37}$$

Therefore I claim (32) and (33) to be true.

1.6.2 Values for $h_\alpha(r)$

This section was already done in Calc7.

1.6.3 Values for $h(r)$

Inserting $H(r) = h(r)$ in (32) and (33) gives

$$T = \frac{1}{4\pi r_H}\left(1 + n - \frac{(2+n)\left(\frac{L}{r_H}\right)^n}{\left(\frac{L}{r_H}\right)^n + \left(\frac{r_H}{L}\right)^2}\right) = \frac{1}{4\pi z_H}\frac{1}{L}\left(1 + n - \frac{2+n}{1 + z_H^{2+n}}\right) \tag{38}$$

$$C = -\frac{r_H^{n+2}}{A} \cdot \text{langes zeug} \tag{39}$$

$$S_h(z) = 4\pi A L \left(\frac{x^{n+2}}{n+2} + \log(x)\right)_1^z \tag{40}$$

2 Questions

- (Minor) Integral boundaries for S
- (Major) Propagator calculations: Eq (20) in [N Feb2012] is at least $\propto \frac{1}{r}\partial_r h_{n=0}(r)$. How to extend?

Calc10

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Calc10 ties up Calc8, calculating the Propagator and modified Einstein Equations.

Contents

1 The modified Action

The aim of this section is to motivate potential $V(r) \propto \frac{h(r)}{r^{1+n}}$ in the metric $g_{00} = 1 - V(r)$ not only as the result of a modified density $\rho = \frac{M}{\Omega}\delta(r) \to \frac{M}{\Omega}\frac{\mathrm{d}h(r)}{\mathrm{d}r}$ but also by modified Einstein equations, a modified Action, a modified mass or gravitational constant term. That is, the step $\delta(r) \to \frac{\mathrm{d}h(r)}{\mathrm{d}r}$ shall be performed by introducing a more fundamental concept.

This concept looks like a modified delta distribution again: A bilocal distribution

$$\mathcal{A}^2(x-y) = \mathcal{A}^2(\Box_x)\delta^4(x-y) \tag{1}$$

One way to introduce it is smearing the Ricci scalar $R(x)$ [N 02.2012] by

$$\mathcal{R}(x) = \int \mathrm{d}^4 y \sqrt{-g}\mathcal{A}^2(x-y)R(y) \tag{2}$$

This modifies the Action and the Einstein equations immediately:

$$S = \frac{1}{16\pi G}\int \mathrm{d}^4 x \sqrt{-g}\mathcal{R}(x) \qquad\qquad \mathcal{A}^2(\Box)\left(R_{\mu\nu} - \frac{g_{\mu\nu}}{2}R\right) = 8\pi G\, T_{\mu\nu} \tag{3}$$

Having $\mathcal{T}_{\mu\nu} = \mathcal{A}^{-2}(\Box)T_{\mu\nu}$ and the Schwarzschild density $\rho_0 = M\delta(\vec{x})$, we end up with

$$\mathcal{T}^0_0 = -M\mathcal{A}^{-2}(\Box)\delta(\vec{x}) \tag{4}$$

Finally, this yields our requirement for matching this formalism to the holographic approach in [Calc 1-10]:

$$M\mathcal{A}^{-2}(\Box)\delta(\vec{x}) \overset{!}{=} \frac{M}{\Omega_{n+2}}\frac{\mathrm{d}H(r)}{\mathrm{d}r} \tag{5}$$

I want to find profiles $\mathcal{A}$ that corresponds to my functions $H \in \{h, h_\alpha\}$ in n dimensions.

1.1 Operator rewrites and Fourier Transformation

In [N 02.2012], there is a special choice of the D'Alambert operator, using a length scale ℓ. This gives the momentum operator $\hat{P}$ another form than usual:

$$\hat{P} = -i\hbar\nabla \tag{6}$$
$$\Box = \ell^2\nabla^2 \tag{7}$$
$$\Rightarrow \quad \hat{P}^2 = -\Box/\ell^2 \tag{8}$$

1.1.1 Spherical Fourier transformation in $3+n$ dimensions

I use the Fourier transformation $\mathcal{F}$ which is defined in d dimensions ($\vec{x} \in \mathbb{R}^d$) as

$$\mathcal{F}\{f\}(\vec{p}) = \tilde{f}(\vec{p}) = \frac{1}{(2\pi)^d}\int \mathrm{d}^d x\, e^{-i\vec{p}\cdot\vec{x}} f(\vec{x}) \tag{9}$$
$$\mathcal{F}^{-1}\{\tilde{f}\}(\vec{x}) = f(\vec{x}) = \int d^d p\, e^{+i\vec{p}\cdot\vec{x}}\tilde{f}(\vec{p}) \tag{10}$$

The subsequent use is in $d = 3+n$ dimensions. If the function f only depends on the radius, $f(\vec{x}) = f(|\vec{x}|)$, then the integrals 9 and 10 can be transformed to one dimensional integrals. In $d = 3$, we obtain

$$\hat{V}(p) = \int \mathrm{d}^3 r\, e^{-i\vec{r}\cdot\vec{p}}\, V(r) = 2\pi\int_{-1}^{+1}\mathrm{d}\cos\theta \int_0^\infty \mathrm{d}r\, r^2\, e^{-irp\cos\theta} V(r) \tag{11}$$
$$= \frac{2\pi i}{p}\int_0^\infty \mathrm{d}r\, r\, V(r)\left(e^{-irp} - e^{+irp}\right) \tag{12}$$
$$= \frac{2\pi i}{p}\left[\int_0^\infty e^{-ipr}\, r\, V(r) - \int_{-\infty}^0 e^{-ipr}\,(-r)\,V(-r)\right] \tag{13}$$
$$= \frac{2\pi i}{p}\int_{-\infty}^\infty \mathrm{d}r\, e^{-irp}\, r\,\left[V(r)\Theta(r) + V(-r)\Theta(-r)\right] \tag{14}$$

with $r = |\vec{r}|$, $p = |\vec{p}|$.

The rewrite made the trick of substituting the $\theta \in [0,\pi]$ angle by the angle between $\vec{r}$ and $\vec{p}$ in the scalar product $\vec{r}\cdot\vec{p} = |\vec{r}|\,|\vec{p}|\cos\varphi$, with $\varphi \in [0,\pi]$. This is also possible in higher dimensions. Consider in the spherical spacial coordinates $\vec{r} = (r, \phi, \theta_1, \ldots, \theta_{d-2})$:

$$\int \mathrm{d}^d r = \int_0^\infty \mathrm{d}r\, r^{d-1}\int_0^{2\pi}\mathrm{d}\phi \prod_{i=1}^{d-2}\int_0^\pi \mathrm{d}\theta_i \sin^i(\theta_i) := \int_0^\infty \mathrm{d}r\,\Omega_{d-1} r^{d-1} \tag{15}$$
$$= \frac{\Omega_{d-1}}{2}\underbrace{\int_0^\pi \mathrm{d}\theta_1\sin(\theta_1)}_{=2}\int_0^\infty \mathrm{d}r\, r^{d-1} \qquad \text{with} \quad \Omega_{n+2} = 2\frac{\pi^{\frac{n+3}{2}}}{\Gamma\left(\frac{n+3}{2}\right)} \tag{16}$$

Making the scalar product substitution with only θ_1, one gets

$$\tilde{V}(p) = \frac{\Omega_{d-1}}{2}\frac{i}{p}\int_0^\infty \mathrm{d}r\, r^{d-2}\, V(r)\,\left(e^{-irp} - e^{+irp}\right) \tag{17}$$
$$= \frac{\Omega_{n+2}}{2}\frac{i}{p}\int_0^\infty \mathrm{d}r\, r^{n+1}\, V(r)\,\left(e^{-irp} - e^{+irp}\right) \tag{18}$$

When writing the integral in form of (14), the Residual theorem

$$\frac{1}{2\pi i}\int_\Gamma f = \sum_{a\in D_f} \mathrm{ind}_\Gamma(a)\,\mathrm{Res}_a f \tag{19}$$

can be used to solve these integrals.

1.1.2 Operator Eigenvalues

We also use the so called Schwinger-Representation ([IMN Nov 2013], eq. 21) for operators $\mathcal{O}$ (actually, I don't know why):

$$(\mathcal{O})^\alpha = \frac{1}{\Gamma(-\alpha)}\int_0^\infty \mathrm{d}s\; s^{\alpha-1} e^{-s\mathcal{O}} \tag{20}$$

Furthermore, power series are used to reason why functions of operators can be replaced by functions of the applied operators. E.g.

$$\exp x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots \tag{21}$$

$$\frac{1}{1+x} \approx 1 - x + x^2 - x^3 + \dots \tag{22}$$

Consider linear operators, especially differential operators $\mathcal{O} = \frac{\mathrm{d}}{\mathrm{d}x}$. Then consider any infinetly differentiable function f of that differential operator $\frac{\mathrm{d}}{\mathrm{d}x}$. I will switch between the representations $f(\frac{\mathrm{d}}{\mathrm{d}x})$ and $f(p)$, which does *not* denote momentum space, but an evaluation of differentiation. The eigenvalue equation with eigenfunction $e^{\lambda x}$

$$f\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)\, e^{\lambda x} = f(\lambda)\, e^{\lambda x} \tag{23}$$

makes that replacement feasible. In the next section, this will be used with a semi-Fourier-transform:

$$f\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)\, \delta(x) = f(p^2)\, \delta(x) \tag{24}$$

1.2 How to get the $\mathcal{A}$

We want to solve eq. (5). This can be done exploiting one dimensional Fourier Transformationations. Consider the Fourier transformation of the Dirac Delta $\delta(x)$,

$$\tilde{\delta}(p) = \frac{1}{2\pi}\int \mathrm{d}x\, e^{-ipx}\delta(x) = 1 \qquad\qquad \delta(x) = \int \mathrm{d}p\, e^{ipx} \tag{25}$$

We write both sides of (5) as the reverse Fourier transformation $\mathcal{F}^{-1}$ of their Fourier transformations. The integrands, which are basically the Fourier transformations, then can be compared directly (comparison of equation coefficients):

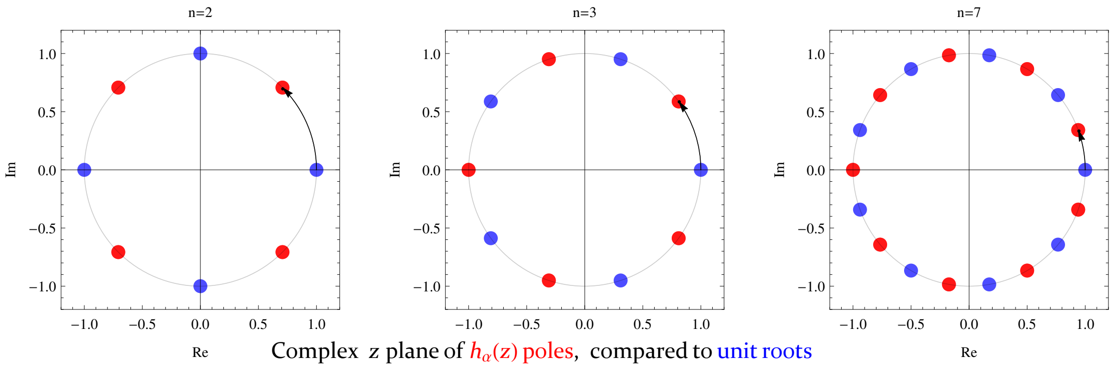

Figure 1: Unit roots in comparison to the poles of $h(z)/(2+n) = h(r)/2l$. Each pole occures two times, so the number of poles $|\{z_0\}|$ for given n is $|\{z_0\}| = 2(n+2)$, while the number of unit roots $|\{x_0\}|$ is only $|\{x_0\}| = n+2$. The black arrow indicates the $e^{i\pi/(2+n)}$ rotation.

$$\mathcal{A}^{-2}(\Box)\delta(\vec{x}) = \frac{\mathrm{d}H(x)}{\mathrm{d}x} \tag{26}$$

$$\Leftrightarrow \int \mathrm{d}p \mathcal{A}^{-2}(\Box)\, e^{ipx} = \frac{\mathrm{d}H(x)}{\mathrm{d}x} \tag{27}$$

$$\Leftrightarrow \int \mathrm{d}p \mathcal{A}^{-2}(\Box)\, e^{ipx} = \int \mathrm{d}p\; \mathcal{F}\left\{\frac{\mathrm{d}H(x)}{\mathrm{d}x}\right\} e^{ipx} \tag{28}$$

$$\Leftrightarrow \mathcal{A}^{-2}(p^2) = \mathcal{F}\left\{\frac{\mathrm{d}H(x)}{\mathrm{d}x}\right\} \tag{29}$$

Determining $\mathcal{A}$ (in position space) therefore means just calculating the Fourier transform of the derivative of the holographic function.

As told in the section before, $\mathcal{A}^{-2}(p^2)$ must not be confused with the fourier transformed $\tilde{\mathcal{A}}^{-2}(p^2) = \int \mathrm{d}x \mathcal{A}^{-2}(\Box)e^{-ipx}$. Actually, the latter is never used in the present calculations.

1.3 $\mathcal{A}$ for $h(r)$ in n dimensions

Consider the holographic function in dimensionless coordinates $z = rL$ in $3+n$ dimensions:

$$h(z) = \frac{1}{1+\left(\frac{1}{z}\right)^{2+n}} \qquad\qquad h'(z) = \frac{(2+n)\left(\frac{1}{z}\right)^{3+n}}{\left(1+\left(\frac{1}{z}\right)^{2+n}\right)^2} \tag{30}$$

Calculating the Fourier transform of (30) can be done with Residue theorem, which requires knowledge of the poles, which are given when the demoninator of (30) equals 0. This problem can be reduced to determination of the *roots of unity* (Einheitswurzeln):

$$z^{3+n}\left(1+\left(\frac{1}{z}\right)^{2+n}\right)^2 = 0 \quad \Leftrightarrow \quad 1 = -z^{2+n} \tag{31}$$

Given the nth root of unity, $x^n = 1$, and $-1 = e^{i\pi}$, the relation of the solution set $\{z_0\}$ and $\{x_0\}$ is given by $x_0\, e^{i\pi/(2+n)} = z_0$. Due to the power of 2 in the denominator, all poles of $h'(z)$ are doubled. See also figure 1.

Knowing the poles of the integrand, the integral can be performed by summing the residues.

1.3.1 Results

The results, done correctly, can be expressed with the Meijer G-function $G^{m,n}_{p,q}\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} \middle| \, z\right)$. Table 1 lists the results of the integral

$$\mathcal{A}^{-2}(p) = \int \mathrm{d}r \, (-1)^{n+1} \frac{i}{p} \, r^{n+1} \left(\Theta(-r)h'(-r) + \Theta(r)h'(r)\right) e^{-ipr} \tag{32}$$

Constants like $\Omega_{n+2}/2$ as given in eq. (18) are omitted.

Table 1: Closed form expressions for $\mathcal{A}^{-2}(p)$ for $h(r)$ in n dimensions

n	$p \cdot \mathcal{A}^{-2}(p)$
0	$-2\sqrt{\pi} l G^{2,1}_{1,3}\left(\frac{l^2p^2}{4} \middle| \begin{matrix} -\frac{1}{2} \\ \frac{1}{2}, \frac{1}{2}, 0 \end{matrix}\right)$
1	$2i\sqrt{\frac{3}{\pi}} l^2 G^{5,1}_{1,7}\left(\frac{l^6p^6}{46656} \middle| \begin{matrix} -\frac{1}{3} \\ 0, \frac{1}{6}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}, \frac{1}{2}, \frac{5}{6} \end{matrix}\right)$
2	$-2\sqrt{2\pi} l^3 G^{3,1}_{1,5}\left(\frac{l^4p^4}{256} \middle| \begin{matrix} -\frac{3}{4} \\ \frac{1}{4}, \frac{1}{4}, \frac{3}{4}, 0, \frac{1}{2} \end{matrix}\right)$
3	$2i\sqrt{\frac{5}{\pi}} l^4 G^{7,1}_{1,11}\left(\frac{l^{10}p^{10}}{10000000000} \middle| \begin{matrix} -\frac{2}{5} \\ 0, \frac{1}{10}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{3}{5}, \frac{4}{5}, \frac{3}{10}, \frac{1}{2}, \frac{7}{10}, \frac{9}{10} \end{matrix}\right)$
4	$-2\sqrt{3\pi} l^5 G^{4,1}_{1,7}\left(\frac{l^6p^6}{46656} \middle| \begin{matrix} -\frac{5}{6} \\ \frac{1}{6}, \frac{1}{6}, \frac{1}{2}, \frac{5}{6}, 0, \frac{1}{3}, \frac{2}{3} \end{matrix}\right)$
5	$2i\sqrt{\frac{7}{\pi}} l^6 G^{9,1}_{1,15}\left(\frac{l^{14}p^{14}}{11112006825558016} \middle| \begin{matrix} -\frac{3}{7} \\ 0, \frac{1}{14}, \frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}, \frac{3}{14}, \frac{5}{14}, \frac{1}{2}, \frac{9}{14}, \frac{11}{14}, \frac{13}{14} \end{matrix}\right)$

1.4 $\mathcal{A}$ for $h_\alpha(r)$ in n dimensions

When studying the self-regular profile in dimensionless coordinates $z = rL$ in $3+n$ dimensions:

$$h_\alpha(z) = \frac{1}{\left(1 + \left(\frac{1}{z}\right)^\alpha /2\right)^{\frac{3+n}{\alpha}}} \qquad h'_\alpha(z) = \frac{\frac{3+n}{2}\left(\frac{1}{z}\right)^{\alpha+1}}{\left(1 + \left(\frac{1}{z}\right)^\alpha /2\right)^{\frac{3+n}{\alpha}+1}} \tag{33}$$

It is hard to find a closed form expression for the Fourier transform $\mathcal{F}\{h'_\alpha(z)\}$ even for $n = 0$, but in principle it should be possible, since the roots should be clearly determinable.

Calc11

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This is a summary of all things I calculated or determined so far, that is, everything from the papers *Calc1* to *Calc10*, with corrections (like better graphs). It will not repeat every detailed calculation.

Contents

1 Setup

I investigate Schwarzschild-like spherical symmetric Black Holes at short scales. The modification is expressed as a smeared Dirac mass/density term. Calculations are done with n Large spatial extra dimensions in total $D = n + 4$ dimensions.

Let $H(r)$ be an approximation of the Heaviside step function $\Theta(r)$, then I frequently derived the metric $g_{00} = 1 - V(r)$ starting from the energy density $\rho(r)$

$$\rho(r) = \frac{M}{\Omega_{n+2}} \frac{\mathrm{d}H(r)}{\mathrm{d}r} \quad \Rightarrow \quad V(r) = \frac{2}{n+2} \frac{M}{M_*^{n+2}} \frac{1}{\Omega_{n+2}} \frac{H(r)}{r^{n+1}}. \tag{1}$$

with the surface term $\Omega_{n+2} = 2\frac{\pi^{\frac{n+3}{2}}}{\Gamma\left(\frac{n+3}{2}\right)}$ and the reduced d-dimensional Planck mass M_*.

I examined two special choices of $H \in \{h, h_\alpha\}$, however most relations can be derived for general (infinetly differentiable) profiles $H(r)$. These two choices each exhibit special features that will be discussed. Because Θ is dimensionless, they can be expressed in the dimensionless variable $z = r/L$ (for details see Calc9) with $H(r) = H(z)$:

$$h(r) = \frac{r^{2+n}}{r^{2+n} + L^{2+n}} \qquad h(z) = \frac{1}{1 + \left(\frac{1}{z}\right)^{2+n}} \tag{2a}$$

$$h_\alpha(r) = \frac{r^{3+n}}{(r^\alpha + L^\alpha/2)^{\frac{3+n}{\alpha}}} \qquad h_\alpha(z) = \frac{1}{\left(1 + \left(\frac{1}{z}\right)^\alpha/2\right)^{\frac{3+n}{\alpha}}} \tag{2b}$$

The derivatives appear in many places and is therefore denoted here. Since $\frac{\mathrm{d}f}{\mathrm{d}r} = \frac{\mathrm{d}f}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}r} = \frac{1}{L}\frac{\mathrm{d}f}{\mathrm{d}z}$, we can always substitute $H'(r) = H'(z)/L$.

$$h'(r) = \frac{(2+n)\, r^{1+n} L^{2+n}}{(r^{2+n} + L^{2+n})^2} \qquad h'(z) = \frac{(2+n)\left(\frac{1}{z}\right)^{3+n}}{\left(1 + \left(\frac{1}{z}\right)^{2+n}\right)^2} = (2+n)\frac{h^2(z)}{z^{3+n}} \tag{3a}$$

$$h'_\alpha(r) = \frac{(n+3)L^\alpha r^{n+2}\left(\frac{L^\alpha}{2} + r^\alpha\right)^{-\frac{n+3}{\alpha}}}{L^\alpha + 2r^\alpha} \qquad h'_\alpha(z) = \frac{\frac{3+n}{2}\left(\frac{1}{z}\right)^{\alpha+1}}{\left(1 + \left(\frac{1}{z}\right)^\alpha/2\right)^{\frac{3+n}{\alpha}+1}} \tag{3b}$$

1.1 The Mass

We can argue that M is just a constant, responsible for fulfilling the horizon equation $V(r_H) = 1$. Therefore we set (»arbitrarily«)

$$M = \frac{n+2}{2} M_*^{n+2} \Omega_{n+2} \frac{r_H^{n+1}}{H(r_H)} = \frac{n+2}{2}\, \Omega_{n+2}\, \frac{1}{H(r_H)} \left(\frac{r_H}{L_*}\right)^{n+1} M_* \tag{4}$$

then the horizon equation $V(r) = 0$ is fulfilled at $r = r_H$. In general, equation 4 gives us a relationship $M = M(r)$. See Calc9, Section 1.1 for details.

1.2 Event Horizons

(The explicit determination of $g_00(r_H) = 0$ is straightforward and was skipped in the Calc$X$ series until now)

2 Remnants

The remnant is the smallest possible Black Hole solution and considered as a stable particle that can no more evaporate. Self encoding solutions (h_α) encode the remnant radius by it's degrees of freedom. Typically the (reduced) Planck Length is supposed to be equal to the remnant's size. See Calc7, Section 1.1 for details. See figure 1 for a picture.

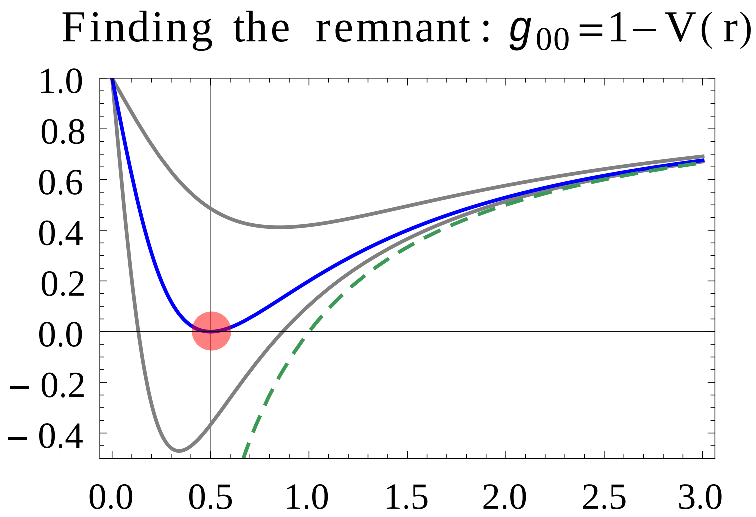

Figure 1: Extremal Configuration in regularized Schwarzschild metrics. Since $g_{00}(r) \to 0$ at $r \in \{0, \infty\}$, there must be an extremal r_0. In the extremal configuration (blue), $r_0 = r_H$. There are also solutions with two r_H and $r_0 < 0$ and no r_H, $r_0 > 0$. The dashed line is the Schwarzschild behaviour. See Calc7, Section 1.1 for details.

2.1 Extremal radius and Minimal Length

The remnant equations require

$$\begin{cases} \partial_r|_{r=r_0}\, g_{00}(r) & = 0 \\ g_{00}(r_0) & = 0 \end{cases} \tag{5}$$

For our generic solution (1), we rewrite the first equation to (see Calc7 for derivation)

$$\partial_r V(r) = 0 = -(n+1)\frac{G(r)}{r} + H'(r) = L\, H'(z) - \frac{n+1}{z}\, H(z)\Big|_{z=\frac{r_0}{L}} \tag{6}$$

This equation can be solved for both $h(r_0)$ and $h_\alpha(r_{0,\alpha})$. In Calc7 and Calc9 I derived

$$r_0 = L\left(\frac{1}{1+n}\right)^{\frac{1}{2+n}} \tag{7a}$$

$$r_{0,\alpha} = L\left(\frac{1}{1+n}\right)^{\frac{1}{\alpha}} \tag{7b}$$

2.2 Self-Encoding

Self-Encoding means that $M(r_0) = M*$, so at the minimal length, the remnant has the Planck Mass. Self-Encoding only occurs in the self-regular metric, as derived in Calc7. It allows relating $\alpha = \alpha(n)$. I found that

$$\alpha_0 = \frac{3+n}{\ln(2+n)} \ln\frac{3+n}{2} \tag{8}$$

3 Thermodynamical properties

All calculations in this section start with a generic $V(r)$, so we forget (1) for a short moment, but keep (4). That is, we have:

$$V(r) \equiv M(r_H) \cdot Y(r) \tag{9a}$$

$$M(r_H) \equiv Y^{-1}(r_H) \tag{9b}$$

The smearing solution (10a) reconstructs equation (1), with an appropriate dimensionful constant A:

$$Y(r) = A \frac{H}{r^{n+1}} \tag{10a}$$

$$V(r) = A\ M(r_H)\ \frac{H(r)}{r^{n+1}} \tag{10b}$$

$$M(r_H) = \frac{1}{A} \frac{r_H^{n+1}}{H(r_H)} \tag{10c}$$

3.1 Hawking Temperature

Let's start with a generic $V(r)$ from which we don't know the inner structure yet:

$$T \equiv \frac{1}{4\pi} \partial_r g_{00}|_{r=r_H} = -\frac{1}{4\pi} M(r_H) \cdot Y'(r_H) = -\frac{1}{4\pi} \frac{1}{L} M(z_H) Y'(z_H) \tag{11}$$

Details can be found in Calc9, Section 1.6. With the holographic approach these equations read

$$T = \frac{1}{4\pi r_H} \left(1 + n - r_H \frac{H'(r_H)}{H(r_H)} \right) = \frac{1}{L} \frac{1}{4\pi z_H} \left(1 + n - z_H \frac{H'(z_H)}{H(z_H)} \right) \tag{12}$$

Now is simple to insert our $H \in \{h, h_\alpha\}$. Figure (2) shows T_H for both of them.

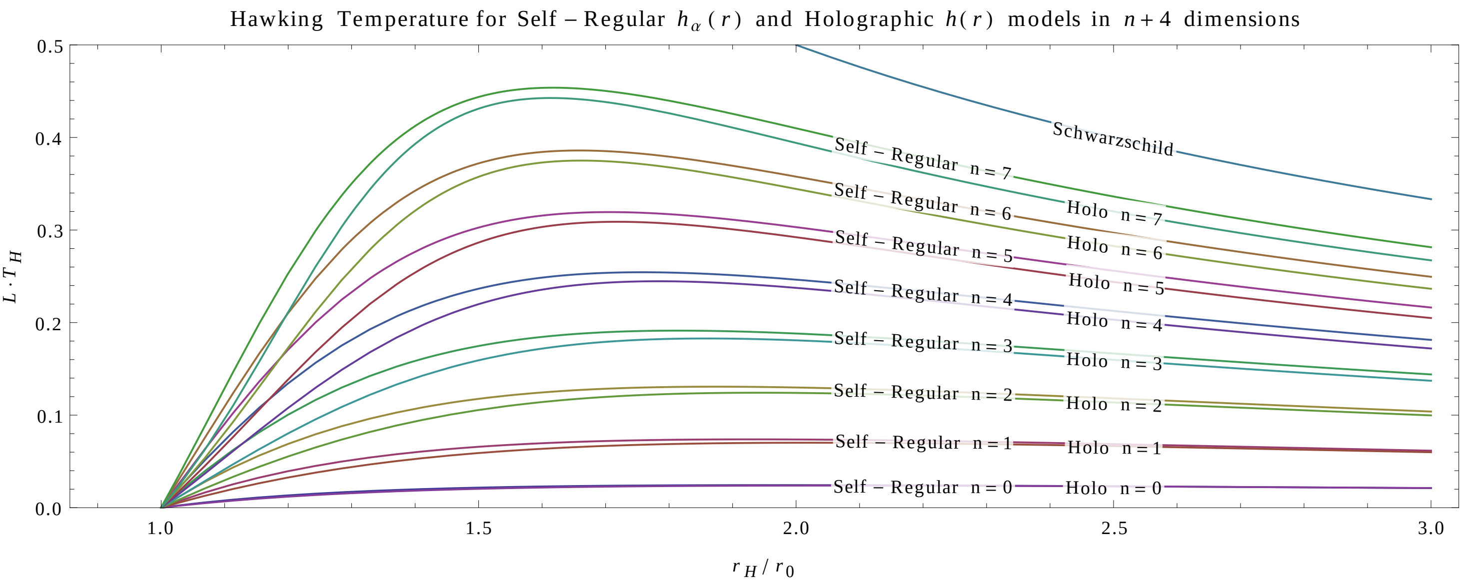

Figure 2: Hawking temperature for the holographic model h and the Self-Regular mode h_α. The functions are $L \cdot T_H(r_H/r_0)$. The Schwarzschild-Tangherlini solution is shown for comparison.

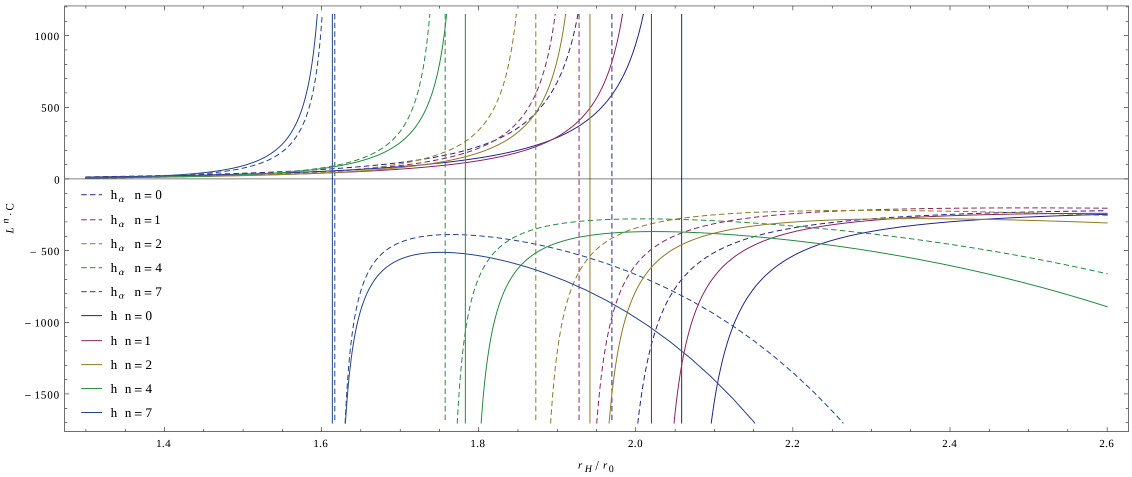

Figure 3: Heat Capacity for $h(r)$ and $h_\alpha(r)$ in various dimensions with critical z_C clearly visible (c.f. figure 2). Figure 4 shows the same functions shifted with r_C

3.2 Heat Capacity

The determiniation of the Heat Capacity is done by variable substitution:

$$C = \frac{\partial M}{\partial T_H} = \frac{\partial M}{\partial r_H}\left(\frac{\partial T_H}{\partial r_H}\right)^{-1} = \frac{\partial M}{\partial z_H}\left(\frac{\partial T_H}{\partial z_H}\right)^{-1} \tag{13}$$

Inserting (4) and (12), we get

$$C = \frac{4\pi r_H^{n+2}}{A}\frac{r_H H'(r_H) - (n+1)H(r_H)}{r_H^2 H(r_H)H''(r_H) - r_H^2 H'(r_H)^2 + (n+1)H(r_H)^2} \tag{14}$$

Especially $C(r) = L^n\, C(z)$.

At the critical radius r_C a phase transition takes place. It is $C(r_C) = 0$ and $T_H(r_C)$ is extremal (so $\partial_{r_H} T_H|_{r_H = r_C} = 0$). The critical radius for $h(r_C)$ and $h(r_{C,\alpha})$ is given by

$$r_C = 2^{\frac{1}{n+2}}\left(-n^2 + (n+2)\sqrt{n^2+2n+5} - 3n - 4\right)^{-\frac{1}{n+2}} \tag{15}$$

$$r_{C,\alpha} = \textit{no closed expression for general n, but possible for fixed n} \tag{16}$$

Figure 3 shows the well-known curve. In figure 4 the abscissa is rescaled by r_C, so $(r_H - r_C)r_0$ is displayed (numerical evaluation of r_C for convenience).

3.3 Entropy

The entropy defining integral can also be subsituted like in the Heat Capacity in the section before:

$$S(r) = \int_{M_1}^{M_2} \frac{\mathrm{d}M}{T} = \int_{r_1}^{r_2} \frac{\mathrm{d}M}{\mathrm{d}r_H}\frac{\mathrm{d}r_H}{T} = \int \mathrm{d}r_H \frac{1}{T}\left(\frac{\mathrm{d}M(r_H)}{\mathrm{d}r_H}\right) \tag{17a}$$

$$S(z) = \int \mathrm{d}z_H \frac{1}{T}\left(\frac{\mathrm{d}M(z_H)}{\mathrm{d}z_H}\right) = -4\pi L \int^z \mathrm{d}z_H \frac{M'(z_H)}{M(z_H)}\frac{1}{Y'(z_H)} \tag{17b}$$

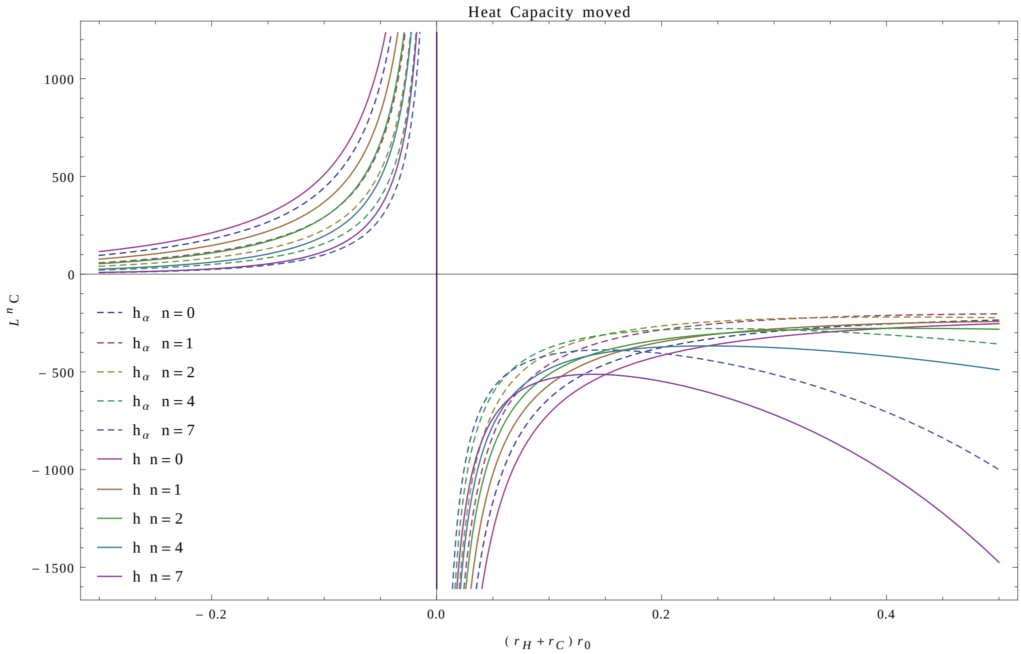

Figure 4: Heat Capacity shifted around r_C (c.f. figure 3)

Inserting (4) and (12) yields (I label $z_H = x$)

$$S(z) = -4\pi L \int^z \mathrm{d}x \left(\frac{n+1}{x} - \frac{H'(x)}{H(x)} \right) \frac{1}{Y'(x)} \tag{18a}$$

$$= -4\pi L A \int^z \mathrm{d}x \left(\frac{n+1}{x} - \frac{H'(x)}{H(x)} \right) \frac{x^{n+2}}{xH'(x) - (n+1)H(x)} \tag{18b}$$

This integral can be computed, at least for the hologrpahic model $h(r)$. This allows us to see logarithmic corrections in any dimension:

$$S_h(z) = 4\pi A L \left(\frac{x^{n+2}}{n+2} + \log(x) \right)_1^z \tag{19}$$

See Calc9 Section 1.6.3 for details.

4 Modified Einstein Equations

It is reasonable to find a deeper concept to justify the smearing of the Schwarzschild source. This can be smearing the Ricci scalar with a bilocal distribution $\mathcal{A}^2(x-y) = \mathcal{A}^2(\Box_x)\delta^D(x-y)$ and was done in Calc8 and Calc10.

Based on the density (1), one tries to find $\mathcal{A}$:

$$\mathcal{T}_0^0 = -M\mathcal{A}^{-2}(\Box)\delta(\vec{x}) \stackrel{!}{=} -\frac{M}{\Omega_{n+2}}\frac{\mathrm{d}H(r)}{\mathrm{d}r} \tag{20}$$

A Fourier transform helps to find the solution for the operator (for Details see Calc10, Section 1.1)

$$\mathcal{A}^{-2}(p^2) = \mathcal{F}\left\{\frac{\mathrm{d}H(x)}{\mathrm{d}x}\right\} = \int_{-\infty}^{\infty} \mathrm{d}^{3+n}r\ \frac{\mathrm{d}H(r)}{\mathrm{d}r}\ e^{-ipr} \tag{21}$$

For $h(r)$, I found the Meijer G-function as a closed algebraic solution,

$$p\ \mathcal{A}^{-2}(p) \propto G_{p,q}^{m,n}\left(\begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}\ \middle|\ z\right) \tag{22}$$

The p-dependence enters into the z part while the lists are only l and n-dependent.

Calc12

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Abstract

This is a writeup of one month research activity about one type of integral. This is a d dimensional spherical Fourier Transform over the Dirac Delta smearing function $H'(r)$.

This document reviews the Holographic Einstein modification term $\mathcal{A}^2(\Box)$ which was first derived in Calc10. Certainly the main focus is on a new chapter in my current research subjects: The Bardeen solution, which is a special case of the h_α model which is also capable to produce the self-regular solution h_{α_0} (For the overview about these terms see Calc11).

My main objective is to understand if the Integral can be done and why the results are so strange. When simply typing it into *Mathematica* and waiting some minutes, it returnes G-functions by Cornelius Simon Meijer which I just don't trust. The legitime question arises in this document wether Cauchy theorem is applicable in this case and how the results shall be interpreted.

Contents

1 The Scheme to get $\mathcal{A}$

At April 04. I presented my derivation of the modified Action by the bilocal distribution $\mathcal{A}^2(x-y)$ and it's derivation in momentum space with a $n+3$ dimensional Fourier transform (Calc10). By expressing the Ricci Scalar R as Integral identity $R(x) = \int \mathrm{d}y\, \delta(x-y)R(x)$ one can replace δ by it's smeared version $\delta \rightarrow \mathcal{A}^2(x-y)\ \delta$ and produces a smeared Ricci scalar $\mathcal{R}(x)$, smeared Einstein Equations and finally the smeared Energy-density tensor $\mathcal{T}_\mu^\nu$:

$$\mathcal{T}_0^0 = -M\mathcal{A}^{-2}(\Box)\delta(\vec{x}) \tag{1}$$

I identified the formalism with the generic holgraphic approach [NS March 2014] extended to $3+n$ dimensions

$$T_0^0 = -\frac{M}{\Omega_{n+2}\, r^{n+2}} \frac{\mathrm{d}H(r)}{\mathrm{d}r} \tag{2}$$

Actually I missed the r^{n+2} factor in my calculations. Irrespective of whether there is an r^{n+2} or not, I already reproduced the (correct?) way of obtaining an expression for $\mathcal{A}^{-2}$ with a Fourier Transformation in Calc10 (4th of April, 2014):

$$\mathcal{A}^{-2}(\Box)\delta(\vec{x}) = \frac{1}{r^{n+2}}\frac{\mathrm{d}H(x)}{\mathrm{d}x} \quad \text{(neglecting the term } \Omega_{n+2} \text{ for shortness)} \tag{3a}$$

$$\Leftrightarrow \int \mathrm{d}p\mathcal{A}^{-2}(\Box)\, e^{ipx} = \frac{1}{r^{n+2}}\frac{\mathrm{d}H(x)}{\mathrm{d}x} \tag{3b}$$

$$\Leftrightarrow \int \mathrm{d}p\mathcal{A}^{-2}(\Box)\, e^{ipx} = \int \mathrm{d}p\ \mathcal{F}\left\{\frac{1}{r^{n+2}}\frac{\mathrm{d}H(x)}{\mathrm{d}x}\right\} e^{ipx} \tag{3c}$$

$$\Leftrightarrow \mathcal{A}^{-2}(p^2) = \mathcal{F}\left\{\frac{1}{r^{n+2}}\frac{\mathrm{d}H(x)}{\mathrm{d}x}\right\} \tag{3d}$$

In line (3c), the *one dimensional* reverse FT was introduced on the right hand side, in order to archieve an equation for the integrands in (3d). So $\mathcal{F}$ must denote a *one dimensional* integration. If this is true, all calculations on the following pages are wrong.

1.1 Higher dimensional FT

What I did was thinking of $\mathcal{F}$ as an higher dimensional fourier transformation. This seems reasonable, because $\vec{r}$ is also an higher dimensional object **and** in the [Isi Paper Nov. 2013], they also make higher dimensional integrals. So my interpretation of (3d) was exactly

$$\mathcal{A}^{-2}(p^2) = \int \mathrm{d}^{3+n}r \frac{1}{r^{n+2}} H'(r) e^{-i\vec{p}\cdot\vec{r}} \tag{4a}$$

$$= \frac{2\pi i}{p}\int_{-\infty}^{+\infty} \mathrm{d}r\ r^{1+n} \frac{1}{|r|^{n+2}} H'(|r|) e^{-ipr} \tag{4b}$$

In Appendix B I derived how reducing the $(3+n)$ dimensional spherical integral (4a) to the effective one dimensional integral (4b).

It is very reasonable to ask if Cauchy Theorem may be applied to solve (4b). My argumentation was that by representing $v(|r|) = v(r)\Theta(r) + v(-r)\Theta(-r)$ and making again a smearing $\Theta \rightarrow H$ with an appropriate smearing function H *which has no poles*, the integrand *can be considered to be holomorphic*.

2 The Bardeen solution

The Bardeen solution is the spherical symmetric solution $g_{00} = 1 - V(r)$ with an electrical charge e:

$$V(r) = \frac{2mr^2}{(r^2+e^2)^{3/2}} = \frac{2m}{e}\frac{z^2}{(1+z^2)^{3/2}} \tag{5}$$

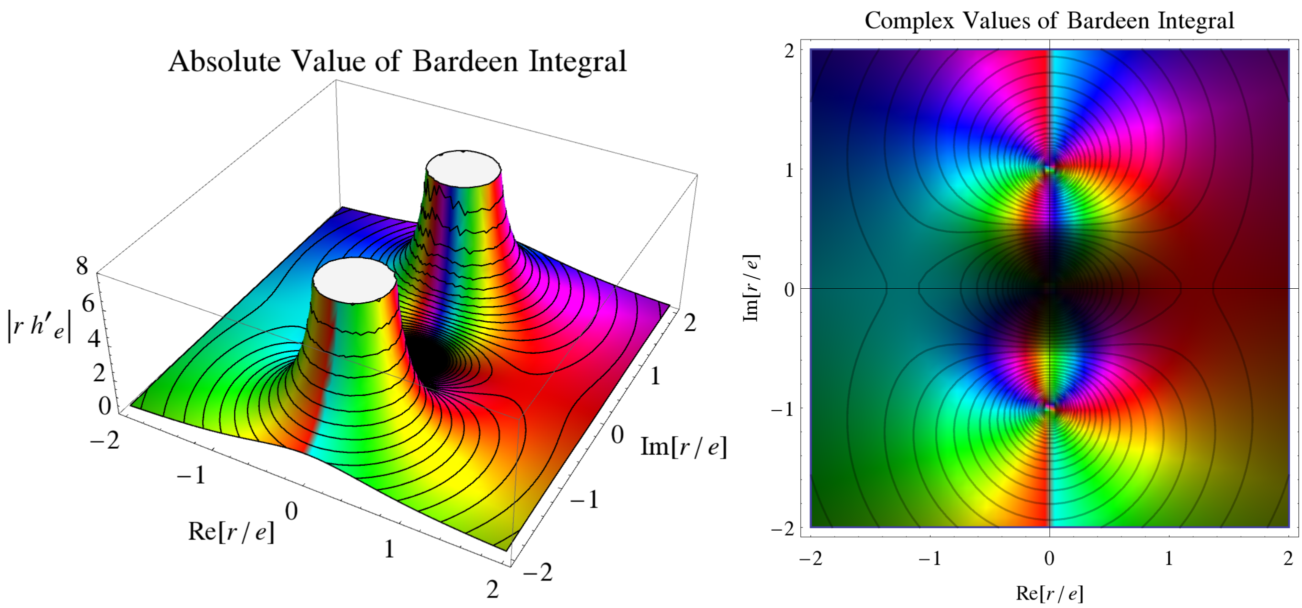
I introduced the dimensionless $z = \frac{r}{e}$, due to $[e] = L$ in Planck units. In the limit $e \to 0$, eq (5) gives 4-dimensional Schwarzschild. In terms of the $V(r) = 2m/r \cdot H(r)$ profiles, the Bardeen profile could be written as

$$h_e(r) = \frac{r^3}{(r^2+e^2)^{3/2}} \tag{6}$$

The Bardeen profile (6) equals to the self-regular h_α with $\alpha = 2$, $L = \sqrt[\alpha]{2}\, e$ and of course $n = 0$:

$$h_\alpha(r) = \frac{r^3}{(r^\alpha + L^\alpha/2)^{3/\alpha}} \tag{7}$$

Therefore, the Bardeen solution and the self-regular metric can be handled in one go.



(a) Absolute value on z-axis, argument as hue

(b) Absolute value as brightness, argument as hue

Figure 1: The two essential singularities of the Bardeen integrand (8) are severe (wesentliche Singularität, kein Pol da $\nexists k \in \mathbb{N} : (z - z_0)^k v(z)$ hebbar *mit* k minimal. Defakto $\exists k = 6/2$).

2.1 Bardeen Poles

We evaluate Bardeen only in 4 dimensions, so (20j) is applied. The effective function

$$v(r) = \frac{r}{|r|^2} \frac{\mathrm{d}h_e(|r|)}{\mathrm{d}r} = \frac{1}{|r|^2} \frac{3e^2 r |r|^2}{(e^2 + r^2)^{5/2}} \tag{8}$$

is odd and has two purely imaginary poles $r_0 = \pm ie$ with multiplicity $\frac{5}{2}$.

The Residue at that pole is really bad, c.f. figure 1. Without extra dimensions, this cannot be handled, and I think Cauchy's integral formula cannot be applied.

3 The more general h_α

As said in the section before, the Bardeen solution is a special case of the self-regular Metric. In general it is given by (7). When setting (as derived in Calc7, Calc11 Section 2.2)

$$\alpha := \alpha_0 = \frac{3+n}{\ln(2+n)} \ln \frac{3+n}{2} \tag{9}$$

then h_{α_0} shows the self-encoding radius property and L is no more an arbitrary additional degree of freedom. Actually, the Bardeen solution is *not* self-regular, but shares the same form of equation.

I didn't consider the poles of $r^{1+n}h'_\alpha$ yet in any CalcX paper, so this will be caught up here.

3.1 The h_α Poles

The effective Function $v(r)$ to fourier-transform is

$$v(r) = r^{1+n}\frac{\mathrm{d}h_\alpha(|r|)}{\mathrm{d}r} = \frac{3+n}{2}\frac{r^{1+n}}{|r|^{2+n}}\frac{L^\alpha |r|^{2+n}}{\left(\frac{L^\alpha}{2}+r^\alpha\right)^{\frac{3+n}{\alpha}+1}} \tag{10}$$

The obvious pole of eq (10) is

$$r_0 = \left(-\frac{L^\alpha}{2}\right)^{1/\alpha} = 2^{-1/\alpha}e^{i\pi/\alpha}L, \tag{11}$$

but it is not the only one, due to the freedom of α, there are plenty of others. It is uncomfortable (but possible, c.f. *Symmetrie für Bardeen.nb*, Section 1.1. *Die tatsächlichen Pole für beliebige α und n*) to write down all possible poles, dependent on α and n.

Symmetry discussions about $v(r)$ lead to a bunch of plots, c.f. figure 3.

3.2 The self-encoding Poles

When we fix $\alpha := \alpha_0$ for the self-encoding property, it is more reasonable to really denote all poles. Working in dimensionless $z = r/L$, the number of poles $|P_{v(r)}|$ can now be expressed merely in terms of n:

$$|P_{r^{1+n}h'_{\alpha_0}(r)}| = (2,2,4,4,4,6,6,8) \quad \text{for } n = 0..7 \tag{12}$$

The numeration and including is given by the shape of $v_\pm(r) = r^{1+n}h'(\pm r)$. That is, when writing in the spirit of (20j)

$$\mathcal{A}^{-2}(p) = \frac{2\pi i}{p}\int_{-\infty}^{\infty}\mathrm{d}r\,\frac{r^{1+n}}{|r|^{2+n}}\left(h'_{\alpha_0}(-r)\,\Theta(-r) + h'_{\alpha_0}(r)\,\Theta(r)\right)\,e^{-ipr} \tag{13}$$

$$= \frac{2\pi i}{p}\int_{-\infty}^{\infty}\mathrm{d}r\,\left(v_-(r)\,\Theta(-r) + v_+(r)\,\Theta(+r)\right)\,e^{-ipr} \tag{14}$$

Then the Cauchys Theorem can be applied for the roots of $v_-(r)$ and $v_+(r)$ (or $-$ for **L** and $+$ for **R** for *left* and *right* side of the complex plane). See figure 3.

3.3 The evaluated Integral (α_0)

It is pure coincidence that at least for $n \in \{0, 1, \ldots, 7\}$ no pole of $v_{\pm}(r)$ is on $\mathrm{Im}(r) = 0$ or on $\mathrm{Re}(r) = 0$. So the whole procedure should be well-defined. See figure 3 for a plot of all singularities.

Nevertheless, the Residuum is not well defined. This is the same situation as for Bardeen (figure 1). The Laurent series at the pole positions do not converge, therefore a Residuum cannot be given. Only for $n = 0$, the result is trivially 0 because there is no pole. This special case highlights another problem, because the Fourier Transform of the 4d Holographic Model is most likely not vanishing. Does this indicate that the complete calculation is not applicable?

For example, consider $n = 1$. As plotted in figure 3, there are two poles, $z_{\pm} = \pm 3^{-\frac{1}{4} + \frac{i\pi}{4\log(2)}}$. When trying to taylor v_R at z_+, we encounter diverging terms. This is because (for illustration)

$$v_R'(r/L) = -\frac{4r^4 \left(r^\alpha + \frac{1}{2}\right)^{-4/\alpha} (2\alpha r^\alpha - 2r^\alpha - 5)}{(2r^\alpha + 1)^2} \tag{15}$$

has even higher powers, so $v_R'(z_+)$ diverges. This is even more serious at higher orders.

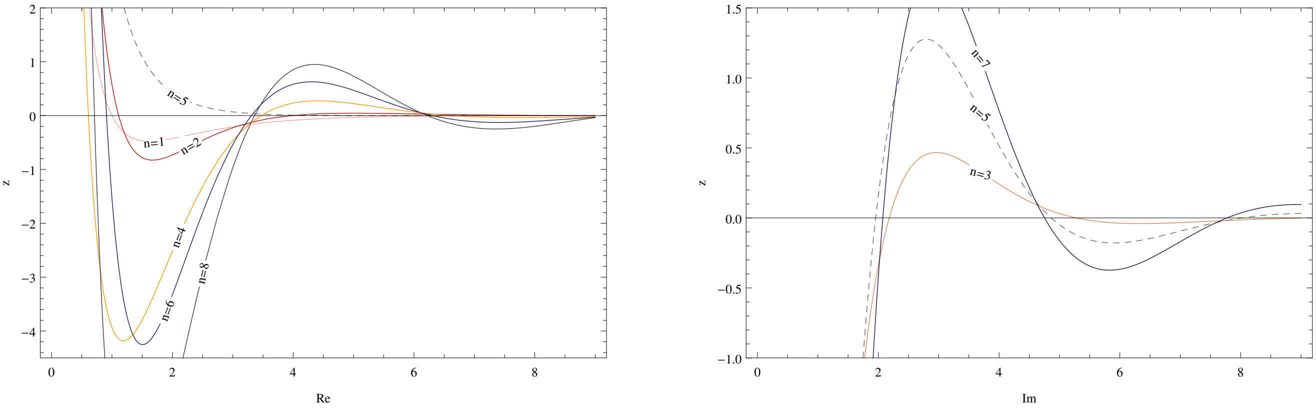


Figure 2: $\mathcal{A}^{-2}(p)$ for Holographic Function. $n = 0$ is Real and to small to be visible, but behaves like $n = 1$. $n = 5$ is special because it has both Re and Im. Defacto these plots lack another imaginary i prefactor and perhaps an axis scaling (c.f. Appendix B.2), but this doesnt change the nature of the curves.

4 The holography model, revisited

The holographic model

$$h(r) = \frac{r^{2+n}}{r^{2+n} + L^{2+n}} \tag{16}$$

was the first one which I tried to Fourier transform. While my first try was rather CAS assisted, I know performed pole summing, that is, regular Cauchy Theorem. Actually, figure 4 shows the poles of

$$v_{\pm}(r) = \frac{r^{1+n}}{(\pm r)^{2+n}} \frac{L^{2+n}(2+n)(\pm r)^{1+n}}{\left(L^{2+n} + (\pm r)^{2+n}\right)^2} \tag{17}$$

Solutions get quickly very complex. Only $n = 0$ one can write it in one line:

$$\mathcal{A}^{-2}(p) \propto -2\pi e^{-p} + \frac{2\pi e^{-p}}{p} \tag{18}$$

Figure 2 plots the Real and Imaginary parts for $n = 0..7$. The curves look somewhat promising, but this probably changes dramatically if one computes $\left(\mathcal{A}^{-2}(p)\right)^{-1/2}$.

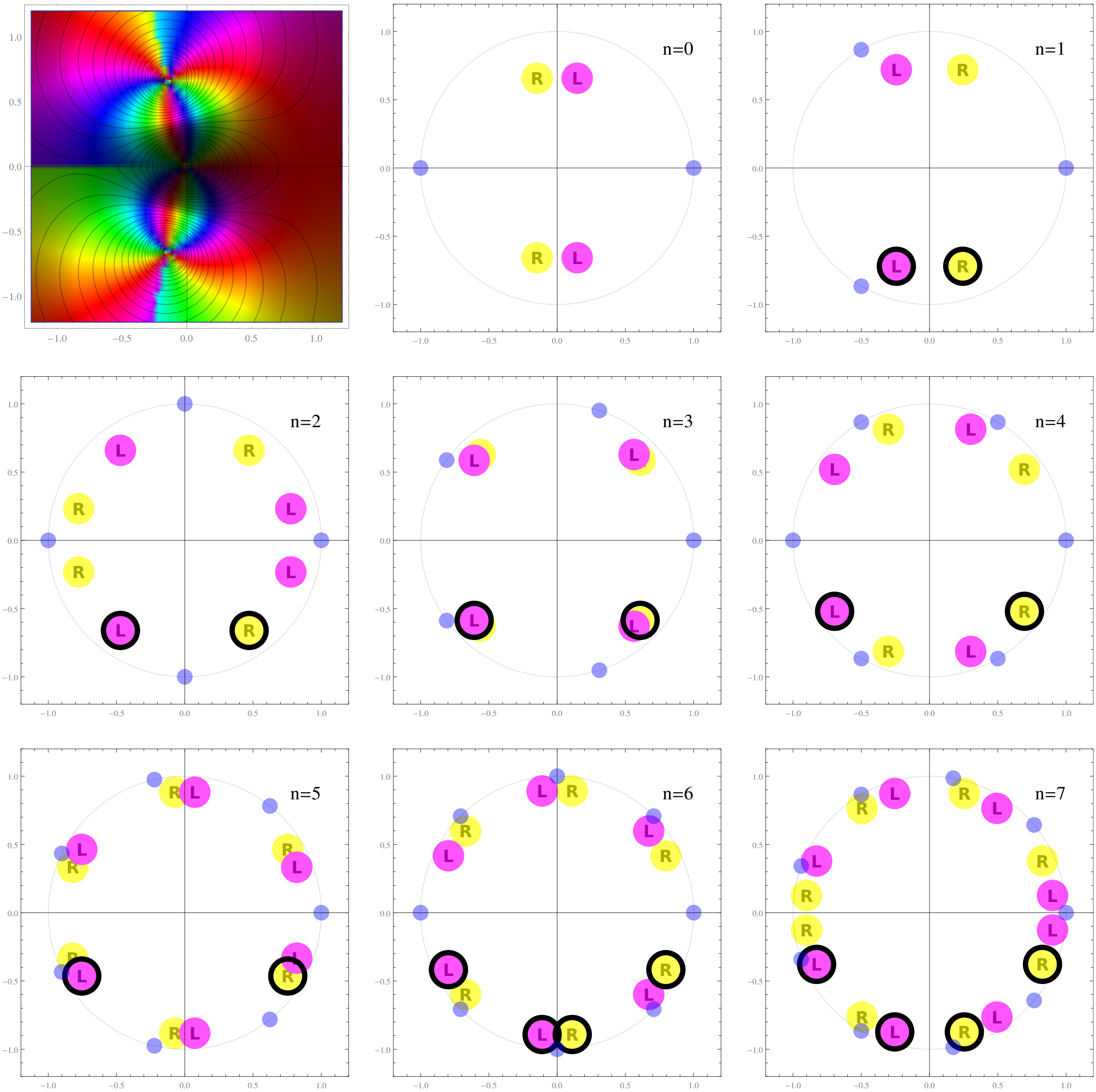


Figure 3: The poles of $v_{\mathbf{L}}(r) = r^{1+n}h'_{\alpha_0}(-r)$ and $v_{\mathbf{R}}(r) = r^{1+n}h'_{\alpha_0}(+r)$ for different dimensions $d = 3+n$ in the complex plane. Encircled poles can be take into account for Cauchys theorem. The small blue circles on the unit sphere indicate the $n+2$. root of unity. The upper left panel shows a complex plot of $v_{\mathbf{L}}$ for $n = 0$ (c.f. figure 1b). Therefore the first panel shows a subset of the second panel. As one can see, $n = 0$ is a special case where there is no pole in the integration area.

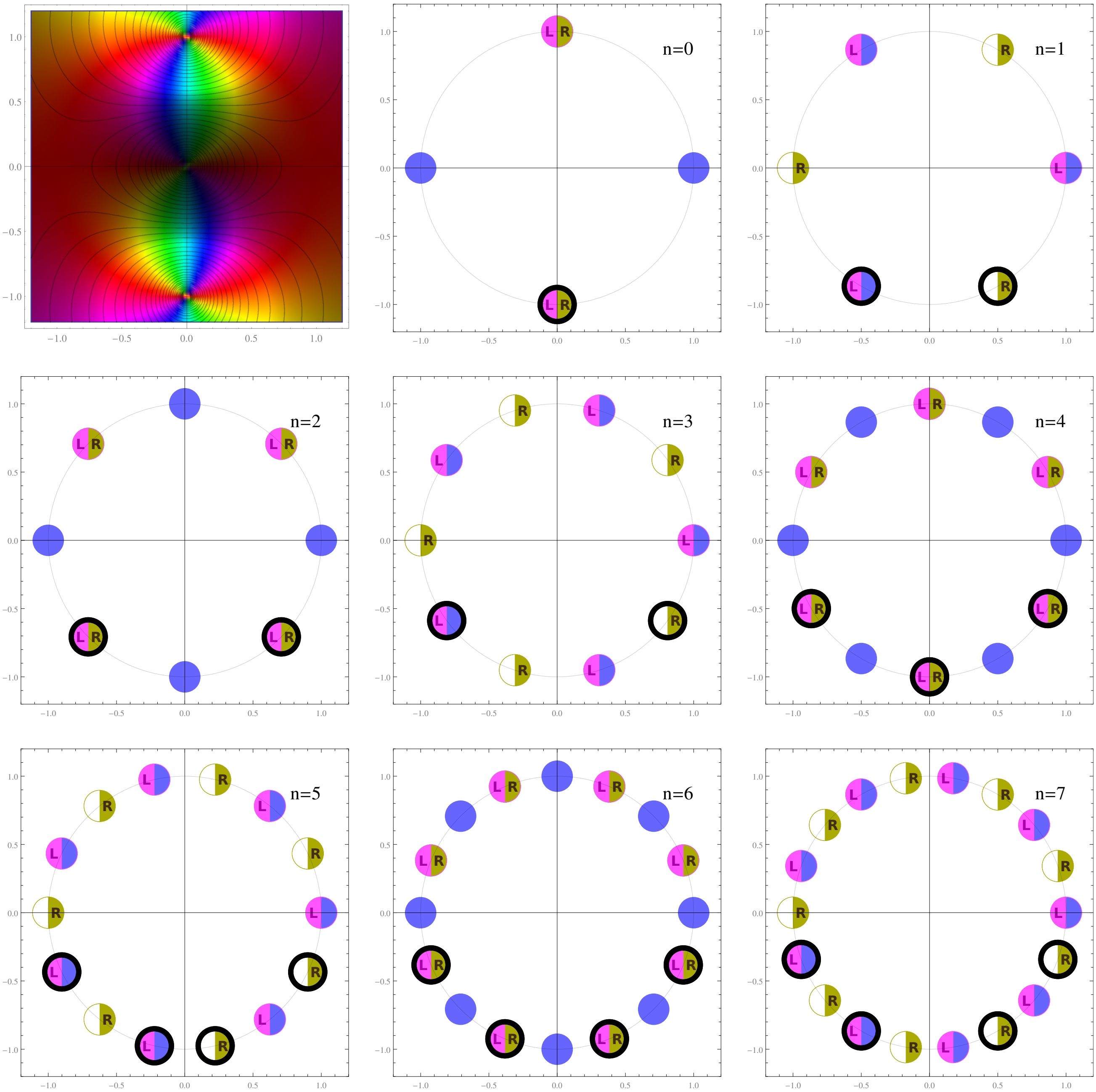


Figure 4: The poles of Holographic Function Integration kernel $v_{\pm}(r) = r^{1+n}h'(\pm r)$ in n dimensions. Same conventions used as in figure 3.
Note the poles on $r = \pm 1$ for $n = 1, 3, 5, 7$. They are straight on the integration contour, but do not count because in all cases, R and L are on the wrong side. That was close!

Appendices

A Verification Strategies

- **Searching the Poles**: Analytic result (Solve/Reduce) *vs.* 3d Plots like figure 1.
- **Performing the FT**: Cauchy Theorem (Summation of Poles) *vs.* Numerical evaluation of Integral, see Appendix C.

B Review of the radial symmetric d-dimensional FT

This calculation was already performed in Calc10, section 1.1.1, but not in detail. Actually I always thought, a minus was missing there. Let's review it.

In the end I want to use the the Fourier transformation $\mathcal{F}$ which is defined in d dimensions ($\vec{x} \in \mathbb{R}^d$) as

$$\mathcal{F}\left\{f\right\}(\vec{p}) = \tilde{f}(\vec{p}) = \frac{1}{(2\pi)^d} \int \mathrm{d}^d x \, e^{-i\vec{p}\cdot\vec{x}} f(\vec{x}) \tag{19a}$$

$$\mathcal{F}^{-1}\left\{\tilde{f}\right\}(\vec{x}) = f(\vec{x}) = \int d^d p \, e^{+i\vec{p}\cdot\vec{x}} \tilde{f}(\vec{p}) \tag{19b}$$

For shortness of notation, I will supress the leading $(2\pi)^{-d}$ in the following equations.

We begin with $V = V(|\vec{r}|)$, a radially symmetric potential, and work at first in $d = 3$ total spacial dimensions:

$$\hat{V}(p) = \int \mathrm{d}^3 r \, e^{-i\vec{r}\cdot\vec{p}} \, V(r) \tag{20a}$$

$$= \int_0^\infty \mathrm{d}r \int_0^\pi r^2 \sin\theta \, \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\varphi \, V(r) \, e^{-ipr\cos\theta_2} \tag{20b}$$

In line (20b) we already wrote the scalar product with an inner angle θ_2. We now substitute the radial angle θ (the θ which is part of $\vec{r} = (r, \theta, \varphi)$) integration with a $\cos\theta$ integration. This can be done because $\frac{\mathrm{d}\cos\theta}{\mathrm{d}\theta} = -\sin\theta$ and so $\int_0^\pi \sin\theta \mathrm{d}\theta = -\int_{-1}^{1} \mathrm{d}\cos\theta = \int_{-1}^{1} \mathrm{d}\cos\theta := \int_{-1}^{1} \mathrm{d}x$. We now identify $\cos\theta := x$ with $\cos\theta_1$ because they share the same domain, actually $\theta, \theta_1 \in \{0, \pi\}$. We continue (naturally, $\int_0^{2\pi} \mathrm{d}\varphi = 2\pi$ was already integrated out in the next line):

$$= 2\pi \int_{-1}^{+1} \mathrm{d}x \int_0^\infty \mathrm{d}r \, r^2 \, e^{-irpx} V(r) \tag{20c}$$

$$= 2\pi \int_0^\infty r^2 \, \mathrm{d}r \, V(r) \left[\frac{1}{-ipr} e^{-iprx}\right]_{-1}^{+1} \tag{20d}$$

$$= \frac{2\pi i}{p} \int_0^\infty r \, \mathrm{d}r \, V(r) \left\{e^{-ipr} - e^{+ipr}\right\} \tag{20e}$$

$$= \frac{2\pi i}{p} \left\{\int_0^\infty r \, \mathrm{d}r \, V(r) e^{-ipr} - \int_0^\infty r \, \mathrm{d}r \, V(r) e^{+ipr}\right\} \tag{20f}$$

In line (20f), we splitted the integral, and we now make two recastings: At first, switching the integral borders, which inserts one minus: $\int_a^b = -\int_b^a$ in (20g). Second, another substitution of the integration parameter $r := -r'$ and therefore $\mathrm{d}r = -\mathrm{d}r'$. The two minus signs kill each other in (20h), so $r\mathrm{d}r = r'\mathrm{d}r'$. After substitution, we will call r' again r, which is totally valid.

$$= \frac{2\pi i}{p} \left\{ \int_0^\infty r \, \mathrm{d}r \; V(r) e^{-ipr} + \int_\infty^0 r \, \mathrm{d}r \; V(r) e^{+ipr} \right\} \tag{20g}$$

$$= \frac{2\pi i}{p} \left\{ \int_0^\infty r \, \mathrm{d}r \; V(r) e^{-ipr} + \int_{-\infty}^0 r' \, \mathrm{d}r' \; V(-r') e^{-ipr'} \right\} \tag{20h}$$

$$= \frac{2\pi i}{p} \int_{-\infty}^\infty r \, \mathrm{d}r \; e^{-ipr} \; \{ V(r)\Theta(r) + V(-r)\Theta(-r) \} \tag{20i}$$

$$= \frac{2\pi i}{p} \int_{-\infty}^\infty \mathrm{d}r \; \{ rV(|r|) \} \; e^{-ipr} \tag{20j}$$

Basically (20j) is our final result. The more extensive eq. (20i) may be better to argue why I think the new *effective* one dimensional function $v(r) \sim r \; V(|r|)$ can still be treated as a holomorphic function, because all this work is about smearing the Θ, so it is no more a special distribution. Probabily, since our functions $V(r)$ don't have poles at $r = 0$, the discontinuity issue at $r = 0$ may be silently ignored, reasoning that it is always possible to let $V(-r)$ and $V(r)$ blend into each other in a continous way.

Whats about the real and complex parts of this fourier transformation? By construction, $V(|r|)$ is an even function (definition: $f(x) = f(-x)$ is even, $-f(x) = f(-x)$ is odd). Therefore $r \; V(|r|)$ is an odd function. By eulers formula $e^{i\varphi} = \cos\varphi + i \sin\varphi$, one quickly finds that the Fourier Transform of an even function includes only (also even) cos terms and the complex part vanishes, while the FT of an odd function only contains sin terms and the real part vanishes. The integral in (20j) is therefore ony complex, $\int \mathrm{d}r \; rV(|r|)e^{-ipr} \in \mathbb{C} \setminus \mathbb{R}$. But the prefactor makes the final result in (20j) completely real again. This can help as a quick check wether the computed result of the integral is correct.

B.1 From 3 to (3+n) dimensions

The same calculation can be performed in $d = 3 + n$ spatial dimensions. It is easy to derive then the pendant of (20j) as (TODO not sure about Ω, to be checked)

$$\hat{V}(p) = \Omega_{2+n} \frac{i}{p} \int_{-\infty}^\infty \mathrm{d}r \; r^{1+n} V(|r|) \; e^{-ipr} \tag{21}$$

The symmetry of the effective function $v(r) \sim r^{1+n} \; V(|r|)$ now dramatically depends on n. Actually, for even $n = 0, 2, 4, \dots$, $v(r)$ is odd, where for odd $n = 1, 3, 5, \dots$, $v(r)$ is even. Figure 5 shows this on an arbitrary function $V(|r|)$.

In consequence, this means for the Fourier Transformation in $(n + 3)$ dimensions,

$$\mathcal{F}_{n+3} \{ V(|r|) \} \in \begin{cases} \mathbb{C} \setminus \mathbb{R} & \text{if } n \in \{1, 3, 5, \dots\} \\ \mathbb{R} & \text{if } n \in \{0, 2, 4, \dots\} \end{cases} \tag{22}$$

Furthermore, (21) allows us to use Cauchy theorem to solve all $\mathcal{F}_{n+3}$. Since $p = |\vec{p}| > 0$, we always close the integration path at $r = -i\infty$ because then $e^{-ipr} \to e^{-i(+\#)(-i\infty)} = e^{-\#\infty} \to 0$. We therefore sum over all poles r_0 in the region $\mathrm{Im} r_0 < 0$. Poles in the region $(\mathrm{Re}\, r_0 > 0 \wedge \mathrm{Im}\, r_0 < 0)$ (IV. quadrant in two dimensional complex plane) can be determined by solving the equation $r^{-(1+n)}/V(r) = 0$ (and ignoring all poles outside the target region), where poles in the region $(\mathrm{Re}\, r_0 < 0 \wedge \mathrm{Im}\, r_0 < 0)$ can be determined with $r^{-(1+n)}/V(-r) = 0$ (same applies here). I'm not sure yet if mirroring at the y axis, which means effectively two count every pole *two (more) times* is also valid.

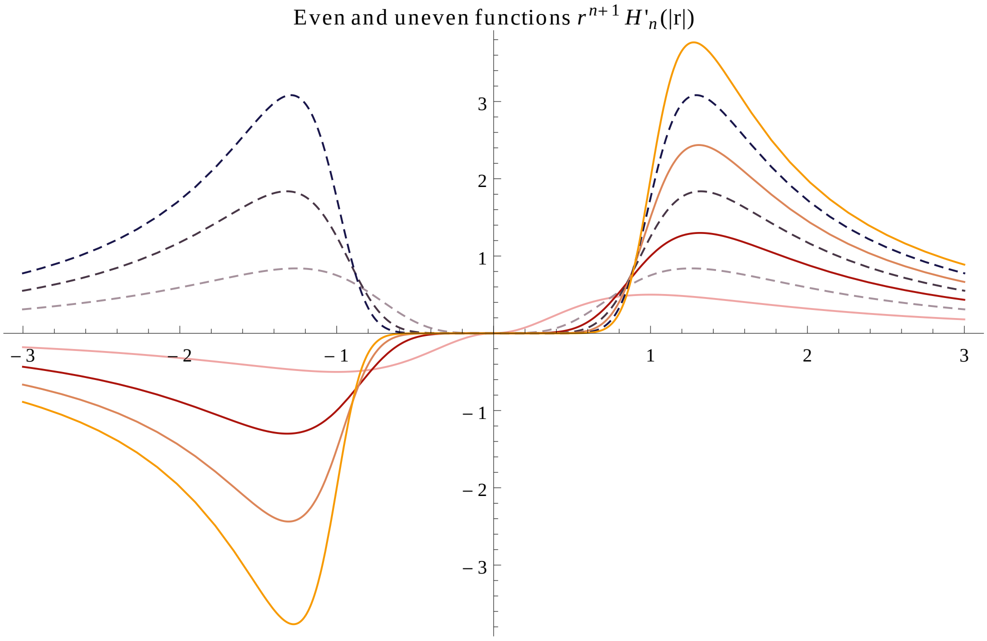


Figure 5: Dashed lines represent symmetric (even) functions $v(r)$ and thus belong to $n = 1, 3, 5$. Solid lines belong to $n = 0, 2, 4$. Here, $V(|r|) = h'_n(|r|/L) = \partial_r \left(r^2/(r^2+1)\right)_{r\to|r|}$

B.2 Dimensionless calculation

To simplify calculation, I frequently use dimensionless $z = r/L$ ($L > 0$) when working with functions $V(r) = V(z)$. Having $\mathrm{d}r = L\mathrm{d}z$ and $\frac{\mathrm{d}f(r)}{\mathrm{d}r} = \frac{\mathrm{d}f(r)}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}r} = \frac{1}{L}\frac{\mathrm{d}f(z)}{\mathrm{d}r}$, the integral

$$\int_{-\infty}^{\infty} \mathrm{d}r \frac{\mathrm{d}H(r)}{\mathrm{d}r} = \int_{-L\infty}^{L\infty} \mathrm{d}z \frac{\mathrm{d}H(z)}{\mathrm{d}z} \tag{23}$$

does not change under transformation. But there are other factors. Consider again (21). By inserting $q = Lp$, the transformation does not change the shape of the integral:

$$\hat{V}(p) = \Omega_{2+n}\frac{i}{p}\int_{-\infty}^{\infty} \mathrm{d}r\, r^{1+n} V(|r|)\, e^{-ipr} \tag{24a}$$

$$= \Omega_{2+n}\frac{i}{p}L^{1+n}\int_{-\infty}^{\infty} \mathrm{d}z\, z^{1+n} V(|z|) e^{-iqz} \tag{24b}$$

$$= L^n \left(\Omega_{2+n}\frac{i}{q}\int_{-\infty}^{\infty} \mathrm{d}z\, z^{1+n} V(|z|) e^{-iqz}\right) := L^n \hat{V}_z(q) \tag{24c}$$

So for comparing the graphs of the dimensionless quantity $\hat{V}_z(q)$ with the dimensionful, the mapping is given by

$$\left(p, \hat{V}(p)\right) = \left(q/L, L^n \hat{V}_z(q/L)\right) \tag{24d}$$

C d-dimensional discretized FT

These calculations are inspired by numerical physics computations, as they are done e.g. in Lattice QCD. We start with a simple one dimensional example.

Let $\vec{r} = (r)$ be the one dimensional position space vector. Working in a box (or on a finit one dimensional number line) with $N \in \mathbb{N}$ distinct elements (spacial extend of the box), we discretize $r = n\,a$ with $n \in [0, N-1] \setminus \mathbb{N}$ and the lattice spacing a which has the physical unit $[a] = L$.

On such lattice with a spacing a, the minimum wave length (and therefore the maximal momenta) is $\frac{2\pi}{a}$. The momentum is therefore discretized following $p = \frac{2\pi k}{a\,N}$ with $k \in [0, \frac{2\pi}{a}] \setminus \mathbb{N}$.

With these conventions, the transition from the analytic Fourier Transformation to the Discrete Fourier Transform (DFT, not to be confused with the Fourier Series) can be archived. Lets start with the one dimensional

$$\hat{f}(p) = \int_{-\infty}^{\infty} f(r)\, e^{-ipr}\, \mathrm{d}r \tag{25a}$$

At first, we make the infinities finite, so $\int \mathrm{d}r \to \sum \Delta r$. Due to the boxing, $\sum^{\infty} \to \sum^{N}$. In the limes, this process is well-defined.

$$= \lim_{\Delta t \to 0} \sum \Delta t\, f(r)\, e^{-ipr} \tag{25b}$$

$$= \lim_{\Delta t \to 0} \Delta t \sum_{n=0}^{N} f(an)\, e^{-i\left(\frac{2\pi k}{aN}\right) a\, n} \tag{25c}$$

$$= \lim_{\Delta t \to 0} \Delta t \sum_{n=0}^{N} f(an)\, e^{-\frac{2\pi i}{N} kn} \tag{25d}$$

We end up with the definition of the DFT. When we treat functions as lists, $f(an) \to f_n$ and $\hat{f}(p) \to f_k$, we can compute Fourier Transformations of lists.

Caution has to be taken when comparing DFT and ordinary FT results, both in respect to the prefactors and the p scaling. E.g. *Mathematica* defines these operations as

$$\texttt{FourierTransform}[f(t), t, \omega] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{+i\omega t}\, \mathrm{d}t \tag{26a}$$

$$\texttt{Fourier}[\{u_r\}] = \{v_s\} = \frac{1}{\sqrt{N}} \sum_{r=1}^{N} u_r\, e^{+\frac{2\pi i}{N}(r-1)(s-1)} \quad \text{with } N = |\{u_r\}| = |\{v_s\}| \tag{26b}$$

which means identifying the graphs $(x, f(x))$ of these equations requires (using again $n, k = r, s$)

$$\left(\frac{2\pi k}{aN}, \sqrt{\frac{2\pi}{N}} \hat{f}_k \right) = (p, \hat{f}(p)) \tag{27}$$

Actually, althought I investigated two or three days of work to examine numerical evaluation of the Transformation, in the hope of a simple comparison between analytical and numerical results from (27), this was not possible yet. Perhaps my lattice was too small – the results were only suitable as a very bad approximation.

Calc13

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Abstract

Calc13, a correction of Calc12.

Contents

1 The Scheme to get $\mathcal{A}$

At April 04. I presented my derivation of the modified Action by the bilocal distribution $\mathcal{A}^2(x-y)$ and it's derivation in momentum space with a $n+3$ dimensional Fourier transform (Calc10). By expressing the Ricci Scalar R as Integral identity $R(x) = \int \mathrm{d}y\, \delta(x-y)R(x)$ one can replace δ by it's smeared version $\delta \to \mathcal{A}^2(x-y)\ \delta$ and produces a smeared Ricci scalar $\mathcal{R}(x)$, smeared Einstein Equations and finally the smeared Energy-density tensor $\mathcal{T}_\mu^\nu$: Todo: ist der Abschnitt hierdrüber alt oder neu? Also Section 1.

2 Holography, closed form result

Consider again the holographic integral (ignoring $r \to z$ transformation)

$$\mathcal{A}^{-2}(p^2) = \frac{1}{\Omega_{n+2}} \int \mathrm{d}^{3+n}\vec{z} z^{-(2+n)} \frac{\mathrm{d}h}{\mathrm{d}z} e^{-i\vec{p}\vec{z}} \tag{1}$$

Using Appendix A which contains the improved and with signs corrected version of the effective 1d Fourier integral,

$$\mathcal{I} = \int \mathrm{d}^{3+n}\vec{z}\ V(|z|)e^{-i\vec{p}\vec{z}} = \frac{1}{2\Omega_{n+2}} \frac{2\pi i}{p} \int_{-\infty}^{\infty} \mathrm{d}z\ v(z) e^{-ipz} \tag{2a}$$

with the Fourier kernel

$$v(z) = z^{1+n} \left(V(z)\Theta(z) + (-1)^n V(-z)\Theta(-z)\right) \tag{2b}$$

Here, $V(z) := z^{-(2+n)} \frac{\mathrm{d}h}{\mathrm{d}z}$ and therefore

$$v(z) = z^{1+n} \left(\frac{1}{z^{n+2}} \frac{\mathrm{d}h}{\mathrm{d}z} \Theta(z) + \frac{(-1)^n}{(-z)^{n+2}} \frac{\mathrm{d}h}{\mathrm{d}z}(-z)\Theta(-z) \right) \tag{3}$$

As told in the appendix, by construction $v(z)$ is always odd.

The integral value is now given by

$$\begin{aligned}
\mathcal{A}^{-2}(p^2) &= \frac{\Omega_{n+2}}{2\Omega_{n+2}} \frac{2\pi i}{p} \int_{-\infty}^{\infty} \mathrm{d}z\ v(r) e^{-ipr} &\text{(4a)}\\
&= \frac{\pi i}{2p} \left[(2\pi i)(-1) \sum_{z_0} \left(\operatorname*{Res}_{z \to z_0} v(z) + \operatorname*{Res}_{z \to -\overline{z_0}} v(z) \right) \right] &\text{(4b)}
\end{aligned}$$

Where z_0 is given by the poles of $v(z)$, which means basically of the poles of $\frac{\mathrm{d}h}{\mathrm{d}z}$ or where the it's denominator is zero: $(-1) = z^{2+n}$, as proposed in eq. (31) in Calc10. So

$$z_0 = \exp \frac{2\pi i (1/2 + k)}{2+n}, \quad k \in \mathbb{N} \tag{5}$$

Actually there are $|z_0| = 2+n$ solutions, so basically $k = 0, 1, \ldots, 2+n$. In my pole summation rule I proposed in Calc10 and more verbosely in Calc12, I only consider poles where $\operatorname{Re} z_0 > 0 \wedge \operatorname{Im} z_0 < 0$ for reasons of the integration path choice. This corresponds to a special range of k that can be derived when performing Eulers formula $e^{i\varphi} = \cos\varphi + i \sin\varphi$ and searching where $\cos > 0 \wedge \sin < 0 \Leftrightarrow 3/2\pi \leq \varphi \leq 2\pi \Leftrightarrow$

$$\begin{aligned}
k_{\min} &= \lceil 1 + 3/4n \rceil \quad \text{(ceiling, aufrunden)} &\text{(6a)}\\
k_{\max} &= \lfloor 3/2 + n \rfloor \quad \text{(floor, abrunden)} &\text{(6b)}
\end{aligned}$$

Caution must also be made that one does not count poles two times. I did those summations Semi-automated since two month and the problem is, there are always alternating complex results which should not be there according to the Appendix.

n	# poles	Wert für A(p)
0	2	$\frac{2i\pi e^{-p}(p+1)}{p}$
1	2	$\frac{i\left(-\frac{2}{3}\pi e^{(-1)^{5/6}p}\left(p+\sqrt[6]{-1}\right)-\frac{2}{3}(-1)^{5/6}\pi e^{-\sqrt[6]{-1}p}\left(\sqrt[6]{-1}p+1\right)\right)}{p}$
2	2	$\frac{i\left(\frac{1}{2}i\pi e^{(-1)^{3/4}p}\left((-1)^{3/4}+ip\right)-\frac{1}{2}i\pi e^{-\sqrt[4]{-1}p}\left(\sqrt[4]{-1}+ip\right)\right)}{p}$
3	2	$\frac{i\left(-\frac{2}{5}\pi e^{(-1)^{7/10}p}\left(p+(-1)^{3/10}\right)-\frac{2}{5}(-1)^{7/10}\pi e^{-(-1)^{3/10}p}\left((-1)^{3/10}p+1\right)\right)}{p}$
4	4	$\frac{i\left(\frac{2}{3}\pi e^{-p}(p+1)+\frac{1}{3}(-1)^{5/6}\pi e^{(-1)^{2/3}p}\left(\sqrt[6]{-1}p+i\right)-\frac{1}{3}(-1)^{2/3}\pi e^{-\sqrt[3]{-1}p}\left(\sqrt[3]{-1}p+1\right)\right)}{p}$
5	4	$\frac{i\Big(-\frac{2}{7}\pi e^{(-1)^{13/14}p}\big(p+\sqrt[14]{-1}\big)-\frac{2}{7}\pi e^{(-1)^{9/14}p}\big(p+(-1)^{5/14}\big)-\frac{2}{7}(-1)^{13/14}\pi e^{-\sqrt[14]{-1}p}\big(\sqrt[14]{-1}p+1\big)-\frac{2}{7}(-1)^{9/14}\pi e^{-(-1)^{5/14}p}\big((-1)^{5/14}}{p}$
6	4	$\frac{i\left(-\frac{1}{4}\pi e^{(-1)^{7/8}p}\left(p+\sqrt[8]{-1}\right)-\frac{1}{4}\pi e^{(-1)^{5/8}p}\left(p+(-1)^{3/8}\right)-\frac{1}{4}(-1)^{7/8}\pi e^{-\sqrt[8]{-1}p}\left(\sqrt[8]{-1}p+1\right)-\frac{1}{4}(-1)^{5/8}\pi e^{-(-1)^{3/8}p}\left((-1)^{3/8}p+1\right)\right)}{p}$
7	4	$\frac{i\Big(-\frac{2}{9}\pi e^{(-1)^{5/6}p}\big(p+\sqrt[6]{-1}\big)-\frac{2}{9}\pi e^{(-1)^{11/18}p}\big(p+(-1)^{7/18}\big)-\frac{2}{9}(-1)^{5/6}\pi e^{-\sqrt[6]{-1}p}\big(\sqrt[6]{-1}p+1\big)-\frac{2}{9}(-1)^{11/18}\pi e^{-(-1)^{7/18}p}\big((-1)^{7/18}p+1}{p}$

(7)

Appendices

A Improvement of the radial symmetric $(n+3)$-dimensional FT

These calculations were performed already in

- Calc10, Section 1.1.1: No detailed derivation, just the use of the 3-dimensional Karbstein approach
- Calc12, Appendix A: Derivation only for 3 dimensions, the generalization from $3 \to (3+n)$ was just wrong.

This section will review the calculation of Calc12, but insert the $d = 3 + n$ in every step *and* take care of the prefactors.

Let's define the Fourier transform $\mathcal{F}_d$ in d dimensions ($\vec{x} \in \mathbb{R}^d$) once again as

$$\mathcal{F}_d \{f\}(\vec{p}) = \tilde{f}(\vec{p}) = \frac{1}{(2\pi)^d} \int \mathrm{d}^d x \, e^{-i\vec{p}\cdot\vec{x}} f(\vec{x}) \tag{8a}$$

$$\mathcal{F}_d^{-1} \{\tilde{f}\}(\vec{x}) = f(\vec{x}) = \int d^d p \, e^{+i\vec{p}\cdot\vec{x}} \tilde{f}(\vec{p}) \tag{8b}$$

As this section wants to discuss how the integrals in (8a,8b) are computed, lets consider only the integral of the forward transformation $\mathcal{F}_d$ in the next part (That is, supressing the leading $(2\pi)^{-d}$ in the following equations).

Consider the $d = 3 + n$ dimensional spherical integral measure, like introduced in Calc10, eqs (15,16):

$$\int \mathrm{d}^d r = \int_0^\infty \mathrm{d}r \, r^{d-1} \int_0^{2\pi} \mathrm{d}\phi \prod_{i=1}^{d-2} \int_0^\pi \mathrm{d}\theta_i \sin^i(\theta_i) := \int_0^\infty \mathrm{d}r \, \Omega_{d-1} r^{d-1} \tag{9a}$$

$$= \frac{\Omega_{d-1}}{2} \underbrace{\int_0^\pi \mathrm{d}\theta_1 \sin(\theta_1)}_{=2} \int_0^\infty \mathrm{d}r \, r^{d-1} \qquad \text{with} \quad \Omega_{n+2} = 2\frac{\pi^{\frac{n+3}{2}}}{\Gamma\left(\frac{n+3}{2}\right)} \tag{9b}$$

The θ_1 integral in (9b) can only be evaluated to $\int \mathrm{d}\theta_1 \cdots = 2$ if the integrand (which is ommitted in these equations) is not dependend of θ_1. In our calculation, this is not the case.

I won't retrace the full Calc12 calculations here. We end up with

$$\hat{V}(p) = \frac{\Omega_{2+n}}{2} \frac{2\pi i}{p} \int_{-\infty}^{\infty} \mathrm{d}r \, r^{1+n} \Big(V(r)\Theta(r) + (-1)^{2+n} V(-r)\Theta(-r) \Big) \tag{10}$$

Pay attention the toggling minus $(-1)^{2+n}$, this does **not** allow writing the effective integrand function $v(r) \neq r^{1+n} V(|r|)$ as supposed in Calc12. Why is it not $(-1)^{1+n}$? Because when substituting $r \to -r'$ and $\mathrm{d}r \to -\mathrm{d}r'$, it is

$$r^{1+n}\mathrm{d}r = (-1)^{1+n}(r')^{1+n}(-1)\mathrm{d}r' = (-1)^n r' \mathrm{d}r' = (-1)^{2+n} r' \mathrm{d}r' \tag{11}$$

So opposed as stated in Calc12, $v(r)$ is **always odd** for all n, therefore $\forall n$:

$$\int \mathrm{d}r \, v(r) \in \mathbb{C} \setminus \mathbb{R} \tag{12}$$

$$\mathcal{F}_{n+3} \{V(|\vec{r}|)\} \in \mathbb{R} \tag{13}$$

Calc14

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Abstract

In this document, I consider an alternative non vanishing $[x_i, p_j] = \Theta_{ij}$ for GUP in n spatial extra dimensions. I will show the connection to my previous holographic and self-encoding calculations with the Heaviside step function Θ *toH* smearing.

We will see, that in order to give self-regular solutions, the GUP function $f(\beta\vec{p})$ must scale with the number of extra dimensions. Otherwise, integral divergences cannot be cured.

This work closes up with the Knipfer2014 paper (in progress).

Contents

1 GUP in Large Extra Dimensions

In the Isi 2013 paper, the GUP

$$[x^i, p_j] = i\delta^i_j(1 + \beta\vec{p}^2) \tag{1}$$

was explored. Isi and Knipfer currently work on exploiting this equations in n total spatial dimensions, by using the identity

$$1 = \int_{-\infty}^{+\infty} \frac{\mathrm{d}^n p}{1 + \beta\vec{p}^2} \mid p\rangle\langle p \mid \tag{2}$$

proposed by Kempf1995. It feels like this smearing is not »strong enough« for fighting against nominator p powers, as Nicolini outlined:

$$\int \frac{p^{n-1}\mathrm{d}^n p}{1 + \beta\vec{p}^2} \sim \int p^{n-3}\mathrm{d}p \tag{3}$$

An appealing approach is switching to another GUP. Indeed, Kempf showed that for the general case, (4a) and (4b) can be proposed. Nicolini deduced (4c) at the 03.Jun Group meeting (yet possibly without proof):

$$[x_i, p_j] = i\hbar\delta_{ij}\left(1 + f(\vec{p}^2)\right) \tag{4a}$$

$$[x_i, x_j] = -2i\hbar f'(\vec{p}^2)\left(x_i p_j - x_j p_i\right) \tag{4b}$$

$$1 = \int \frac{\mathrm{d}^n p}{1 + f(\vec{p}^2)} \tag{4c}$$

The important question of the freedom of arbitrary choice of $f(\vec{p}^2)$ arises. It is this if (2) depends on the eikonal approximation made by Amati, Ciafaloni, Vereziano at string scattering computations (...).

Simpleminded, I will ignore this question and perform a calculation with $f(\vec{p}^2) = L_p^{n-1}p^{n-1}$, where $p = \sqrt{\vec{p}^2}$.

1.1 H-model theory

Diesen Abschnitt hab ich nur eingefügt, um einen Zusammenhang zu meinen bisherigen Untersuchungen zu beschreiben. Mir ist aufgefallen, dass der gar nicht so wirklich vorhanden ist. Die Integrale sind allenfalls ähnlich, aber das GUP-Zeug kann man nur *unbequem* als $H(r)$-Funktion benutzen. Der Begriff *H-model theory* ist für diesen Zusammenschrieb frei erfunden.

The last 6 months, I was busy exploiting the mathematical properties of modified Schwarzschild-Blackholes in higher dimensions. The modification always entered by smearing the Heaviside Theta distribution $\Theta(r) \to H(r)$. Thereby,

$$\rho(r) = \frac{M}{\Omega_{2+n} r^{2+n}} \delta(r) \to \frac{M}{\Omega_{2+n} r^{2+n}} \frac{\mathrm{d}H(r)}{\mathrm{d}r} \tag{5}$$

with Ω_d the surface of an d-sphere (see e.g. Calc11 for a complete summary of my formalism). Note that $H(r)$ is a placehoder for a concrete function profile typically denoted by a lowercase letter like $h(r)$.

Lately I was supposed to bring my holographic and self-encoding approaches, which were modeled in 4d with models,

- Holographic: $H(r) = h(r) := r^2/(r^2 + L^2)$,
- Self-Encoding: $H(r) = h_\alpha(r) := r^3/(r^\alpha + L^\alpha/2)^{3/\alpha}$,
- Bardeen: $H(r) = h_e(r)$, a special choice of α in h_α,

to a more fundamental principle: By finding a bilocal smearing operator $\mathcal{A}^2(x-y) = \mathcal{A}^2(\Box)$ which acts on the Ricci scalar, one finds the modified Einstein Equations which contain delta-smearing according to the model choice of $H(r) \in \{h, h_\alpha, h_b, \dots\}$.

I found that one can connect that H-model framework with the determination of $\mathcal{A}^2$ by the expression

$$\mathcal{A}^{-2}(p^2) = \frac{1}{\Omega_{n+2}} \int \mathrm{d}^{3+n}z \underbrace{\frac{1}{z^{2+n}} \frac{\mathrm{d}H}{\mathrm{d}z}}_{:=V(z)} e^{-i\vec{p}\cdot\vec{z}} \tag{6}$$

I usually solved this $3+n$-dimensional Fourier Transformation by integrating out $n+1$ angels, which yields an effective one-dimensional Fourier integral with an assembled, artificially looking Fourier kernel. I worked on these expressions for one month. See Calc12 and Calc13 for the detailed derivation.

$$\mathcal{I} = \int \mathrm{d}^{3+n}\vec{z}\, V(|z|) e^{-i\vec{p}\vec{z}} = \frac{1}{2\Omega_{n+2}} \frac{2\pi i}{p} \int_{-\infty}^{\infty} \mathrm{d}z\, v(z) e^{-ipz} \tag{7a}$$

with the Fourier kernel

$$v(z) = z^{1+n} \left(V(z)\Theta(z) + (-1)^n V(-z)\Theta(-z) \right) \tag{7b}$$

In H-formalism, $V(z) := z^{-(2+n)} \frac{\mathrm{d}H}{\mathrm{d}z}$ and therefore

$$v(z) = z^{1+n} \left(\frac{1}{z^{n+2}} \frac{\mathrm{d}H}{\mathrm{d}z} \Theta(z) + \frac{(-1)^n}{(-z)^{n+2}} \frac{\mathrm{d}H}{\mathrm{d}z}(-z)\Theta(-z) \right) \tag{7c}$$

As told in Calc13, by construction $v(z)$ is always odd and therefore $\mathcal{I}$ always Real in any number of dimensions.

2 GUP: The Mass

The $\mathcal{A}^{-2}$-smearing in the Knipfer2014 paper was performed like

$$\mathcal{A}^{-2}(\Box)\delta(\vec{x}) \propto \int \frac{e^{i\vec{x}\vec{p}}}{1+\beta p^2} \mathrm{d}^3 p \tag{8}$$

Despite the fact that the integration direction is virtually inversed, I think this can be clearly identified to my calculations, defacto working with the integration kernel

$$p^{1+n} \frac{\mathrm{d}h(p)}{\mathrm{d}p} = \frac{1}{1+\beta p^2} = V(p) \tag{9}$$

2.1 GUP as H-model theory?

As it is described in the next sections, the integrals look really very similar to the Holographic Metric integrals. I wonder if there is any deeper connection.

2.2 Switching to $3+n$ dimensions in dimensionless coordinates

We now modify (8) in two ways: Tuning up the powers of p and absorbing $\sqrt{\beta} = L_p$ in the dimensionless variable q:

$$V(p) = \frac{1}{1+L_p^{n+2} p^{n+2}} = \frac{1}{1+q^{n+2}} \tag{10}$$

The coordinate shift $q = pL$ is also done in the integration measure: $\mathrm{d}p = \mathrm{d}q/L$. So I am about to solve the integral

$$\mathcal{I} = \underbrace{\frac{1}{(2\pi)^{3+n}}}_{\text{inv Four suppr.}} \frac{2\pi i}{r} \int_{-\infty}^{\infty} \mathrm{d}p \; p^{1+n} \left(\frac{1}{1+q^{n+2}} \Theta(q) + (-1)^n \frac{1}{1+(-q)^{n+2}} \Theta(-q) \right) e^{ipr} \tag{11a}$$

$$= \frac{2\pi i}{zL} \int_{-\infty}^{\infty} \mathrm{d}p \; \frac{q^{1+n}}{L^{1+n}} \left(\frac{1}{1+q^{n+2}} \Theta(q) + (-1)^n \frac{1}{1+(-q)^{n+2}} \Theta(-q) \right) e^{ipz} \tag{11b}$$

where the dimensionless $z = r/L$ is determined by requiring $e^{ipr} = e^{iqz}$. From (11a) to (11b) we note that the dimensionless calculation requires a global $\frac{1}{L^{2+n}}$ scaling.

2.3 The poles

Consider the poles of (10). They are given by

$$1 + q^{n+2} = 0 \quad \Leftrightarrow \quad q = (-1)^{\frac{1}{n+2}} = \exp\left\{ \frac{i\pi + 2\pi i k}{n+2} \right\} \quad \forall k \in \mathbb{N} \tag{12}$$

There are $n+1$ poles (different choices for k), given by $k = 0, 1, \ldots, n, n+1$.

As far as I see, these poles are absolutely equal to the Holographic poles. So perhaps see Calc11 for a nice plot of these poles in $n \in [0, 7]$ dimensions.

n	# poles	Wert für $A^{-2}(\square)\delta(z)$
0	2	$\frac{2 e^{z} \pi}{z}$
1	2	$\frac{2 e^{(-1)^{1/6} z} \pi}{3 z} - \frac{2 e^{-(-1)^{5/6} z} \pi}{3 z}$
2	2	$\frac{e^{(-1)^{1/4} z} \pi}{2 z} + \frac{e^{-(-1)^{3/4} z} \pi}{2 z}$
3	2	$\frac{2 e^{(-1)^{3/10} z} \pi}{5 z} - \frac{2 e^{-(-1)^{7/10} z} \pi}{5 z}$
4	4	$\frac{e^{-(-1)^{2/3} z} \pi}{3 z} + \frac{\left(2 e^{z} + e^{(-1)^{1/3} z}\right) \pi}{3 z}$
5	4	$-\frac{2 e^{-(-1)^{13/14} z} \pi}{7 z} + \frac{1}{7 z} 2 e^{-(-1)^{9/14} z} \left(-1 + e^{(-1)^{1/14} z + (-1)^{9/14} z} + e^{(-1)^{5/14} z + (-1)^{9/14} z}\right) \pi$
6	4	$\frac{e^{-(-1)^{7/8} z} \pi}{4 z} + \frac{1}{4 z} e^{-(-1)^{5/8} z} \left(1 + e^{(-1)^{1/8} z + (-1)^{5/8} z} + e^{(-1)^{3/8} z + (-1)^{5/8} z}\right) \pi$
7	4	$-\frac{2 e^{-(-1)^{5/6} z} \pi}{9 z} + \frac{1}{9 z} 2 e^{-(-1)^{11/18} z} \left(-1 + e^{(-1)^{1/6} z + (-1)^{11/18} z} + e^{(-1)^{7/18} z + (-1)^{11/18} z}\right) \pi$

Table 1: Values for $\mathcal{A}^{-2}$ as derived in section 2.4 for different number of extra dimensions n.

2.4 Performing Cauchy theorem

In Calc13, the Cauchy theorem steps are explained in detail. I won't retrace them here. Actually, for $n = 0$ the integral can be easily made by hand because there is only one pole to consider. One archieves

$$\mathcal{I} = \frac{2\pi e^{z}}{L^2 z} = \frac{2\pi e^{r/L}}{Lr} = \frac{2\pi e^{r/\sqrt{\beta}}}{\sqrt{\beta} r} \tag{13}$$

This is the same as in equation (7) in Knipfer2014.

The expressions for higher n get really long. I have not yet found short summation or recursion relations. I think the most outstanding problem is that the expressions are purely $\in \mathbb{C} \setminus \mathbb{R}$, c.f. table 1.

The further calculation of $\mathcal{M}(R)$ can be in principle made (I actually think one even can express that in terms of $\gamma(s, x)$ functions), but they inherit the complex value.

Calc15

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Abstract

This document effective Quantum Gravity approaches investigated in the Calc series so far in respect to their divergence curing behaviour.

This is done for the Journal Club presentation at 18. June 2014.

Contents

1 Divergence of General Relativity

GR exhibits an ultra-violet divergence, that is, expressions typically diverge when $r \to 0 \Leftrightarrow p \to \infty$ in spherical symmetry. Consider the spatial flat integration measure in four Spacetime dimensions:

$$\int \mathrm{d}^3 p \propto \int_{-\infty}^{\infty} p^2 \,\mathrm{d}p = \lim_{\Lambda\to\infty} \int_{-\Lambda}^{\Lambda} p^2 \,\mathrm{d}p \propto \Lambda^3 \tag{1}$$

When examining self-regular black hole solutions, we examine expressions which cure this divergencies. They typically look like

$$\int \frac{\mathrm{d}^3 p}{f(p)} \tag{2}$$

with polynomial functions $f(p)$ that manages a »soft cutoff«. For example, in the GUP principle [Kempf2005], it is $f(p) = 1 + \beta p^2$. The series expansion of $1/f(p)$ at $p \to \infty$ (which corresponds with $\beta \to 0$) is

$$\frac{1}{1+\beta p^2} \approx \frac{1}{\beta p^2} - \mathcal{O}\left(\frac{1}{\beta^2 p^4}\right) \tag{3}$$

Therefore, we can understand the integration modification as

$$\int \frac{\mathrm{d}^3 p}{1+\beta p^2} \propto \int_{-\infty}^{\infty} \frac{p^2\,\mathrm{d}p}{1+\beta p^2} \stackrel{(3)}{\approx} \int_{-\infty}^{\infty} \frac{\mathrm{d}p}{\beta} \tag{4}$$

This is good. We like that.

1.1 What $f(p)$ has to archive in higher dimensions

It is obvious that $f(p)$ must scale with the number of extra dimensions, because (1) gets

$$\int \mathrm{d}^{3+n} p \approx \int_{-\infty}^{\infty} p^{2+n}\,\mathrm{d}p \tag{5}$$

Thus the most simple extension of Kempf would be $f(p) = 1 + L^{2+n} p^{2+n}$, with $\beta = L^2$ and L the reduced higher dimensional Planck length. It is easy to show that, using this approach, (4) again gets $\propto \int \mathrm{d}p$.

1.2 How $f(p)$ is archieved with my H-models

This section ties on the formalism I introduced in Calc14 – which is merely the name »H-model« for the approach of talking about the holographic metric ($h(r)$ profile), self-encoding metric ($h_\alpha(r)$ profile) and eventually the Bardeen metric ($h_e(r)$ profile).

In my work, a fourier transformation is typically introduced like

$$\mathcal{A}^{-2}(p^2) = \int \mathrm{d}^{3+n}r \, \left(\frac{1}{r^{n+2}} \frac{\mathrm{d}H(r)}{\mathrm{d}r} \right) \, e^{-ipr} \tag{6}$$

The factor r^{n+2} in the denominator is placed there »by design«, as all H-models have a matter density

$$\rho(r) = \frac{M}{\Omega_{n+2} r^{n+2}} H'(r) \tag{7}$$

with the $(n+2)$-surface (spatial surface) in the denominator. When inserting $H(r) = \Theta(r)$, $H'(r) = \delta(r)$, one ends up in the Schwarzschild(-Tangherlini) case. That is, everything is fine in ordinary Schwarzschild:

$$\int \mathrm{d}^{3+n}r \left(\frac{1}{r^{n+2}} \delta(r) \right) \propto \int_{-\infty}^{\infty} \mathrm{d}r \, r^{2+n} \left(\frac{1}{r^{n+2}} \delta(r) \right) = \int \mathrm{d}r \, \delta(r) \tag{8}$$

Caution must be made when performing the $(3+n) \to 1$ dimensional integral rewrite, since an alternating $(-1)^n$ inserts the integrand. This technical detail was first found in Calc13 and discussed in Calc14.

So it looks like in my calculations, $H(r)$ does not need to scale with the number of extradimensions n. This is really weird, I always thought it has to scale. Hm.

Modified GUP in Extra Dimensions

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Abstract

This document subsumes calculations done based on the paper *Self-Completeness and the Generalized Uncertainty Principle in Extra Dimensions* calculated by Maximiliano Isi and Marco Knipfer [1]. Their use of the GUP in Large Extra Dimensions (LXDs) as proposed by Achim Kempf 1995 allows non-convergent (aka Schwarzschild-Tangherlini like) metrics at the origin.

Achim Kempf provided the mathematical framework for generic GUP modifications [2]. It is easy to choose one which features regular solutions for any number of dimensions, but Marcos integral solving approach (Schwinger Operator representation and identification as higher dimensional Gaussian integral) fails for that Ansatz. In this document I will present another approach.

The issue was presented by Marco in Journal Club at 14-06-2014 and by Sven at 18-06-2014.

Internal working title: CALC17

Contents

1 Framework

This document follows the reasoning of the 2013 JHEP paper [3]. Skip section 1.1 if you are familiar with the notation of the work in progress [1].

1.1 No extra dimensions

I copy the notational introduction by Maximiliano from [1] here. In $N+1 = 3+1$ space-time dimensions, we consider the GUP as modification of the canonical commutation relations ($p = |\mathbf{p}|$):

$$[x^i, p_j] = i\delta^i_j(1+\beta p^2) \tag{1}$$

Kempf [2] showed how this results in a modified momentum integration measure,

$$\int \frac{\mathrm{d}^3 p}{1+\beta p^2} |p\rangle\langle p| = 1, \tag{2}$$

which is used to determine the modified Einstein equations for with the momentum representation of the Dirac delta distribution:

$$R_{\mu\nu} = \frac{1}{2} g_{\mu\nu} R = 8\pi G \mathcal{T}_{\mu\nu} \tag{3}$$

$$\mathcal{T}_{\mu\nu} = \mathcal{A}^{-2}(\Box) T_{\mu\nu} \tag{4}$$

$$\mathcal{A}^{-2}(\Box)\delta(x) = (2\pi)^{-3} \int \frac{\mathrm{d}^3 p}{1+\beta p^2} e^{i\mathbf{x}\cdot\mathbf{p}} \tag{5}$$

Our investigations work on static spherically symmetric sources $T^0_0 = M\delta(x)$ that lead to the well-known Einstein(-Tangherlini) modified metric. For convenience, we choose to sum up all non-local effects in the mass, that is,

$$\mathcal{M}\delta(x) = M\mathcal{A}^{-2}(\Box)\delta(x), \tag{6}$$

and except the new curly mass symbol $\mathcal{M}$, the metric looks exactly like the $N+1$ dimensional Einstein(-Tangherlini) metric.

1.2 Extradimensions

With the presence of n large extra dimensions, $N+1 = 4+n$ total space-time dimensions, Kempf proposes [2]

$$\int \frac{\mathrm{d}^N p}{1+\beta p^2} |p\rangle\langle p| = 1. \tag{7}$$

It is easy to see that the powers of p in the denominator must be increased for extra dimensions, for example by taking the series expansion of $1/f(p)$ at $p \to \infty$ (which corresponds with $\beta \to 0$):

$$\frac{1}{1+\beta p^2} \approx \frac{1}{\beta p^2} - \mathcal{O}\left(\frac{1}{\beta^2 p^4}\right) \tag{8}$$

Only in the ordinary $N=4$ case, after radial composition of the integral (2), the result looks like

$$\int \frac{\mathrm{d}^3 p}{1+\beta p^2} \propto \int_{-\infty}^{\infty} \frac{p^2\,\mathrm{d}p}{1+\beta p^2} \overset{(8)}{\approx} \int_{-\infty}^{\infty} \frac{\mathrm{d}p}{\beta}. \tag{9}$$

The simplest possible extension of this approach is a modification of the GUP in $N+1$ dimensions

$$[x^i, p_j] = i\delta^i_j(1+L_n^{N-2}|\mathbf{p}|^{N-2}) \tag{10}$$

with the reduced planck scale in n extradimensions L_n and $\sqrt{\beta} = L_0$ the conventional Planck scale (in further equations, I will suppress the index of L for readibility).

Defacto, the choice of arbitary GUPs was also handled in [2], Section 6.1.

2 Solving the energy density integral

We now proceed to compute the line element

$$\mathcal{T}_0^0 = M\frac{1}{(2\pi)^N}\int \frac{\mathrm{d}^N p\ e^{i\mathbf{x}\cdot\mathbf{p}}}{1+L^{N-2}|\mathbf{p}|^{N-2}} \tag{11}$$

My method does not make use of the Schwinger operator representation used in earlier papers, but residue theorem and Jordan's lemma to solve the integral (11). To apply them, I reduce the N-dimensional Fourier transformation to an effective one dimensional one.

Let's identify the integral to be solved with $\mathcal{I}$ and the kernel with $V(\mathbf{p})$:

$$\mathcal{T}_0^0 = M\frac{1}{(2\pi)^N}\mathcal{I}, \quad \mathcal{I} = \int V(\mathbf{p})\ e^{i\mathbf{x}\cdot\mathbf{p}}\mathrm{d}^N p, \tag{12}$$

Using dimensionless coordinates $q = pL$ we write

$$V(p) = \frac{1}{1+L_p^{n+2}p^{n+2}} = \frac{1}{1+q^{n+2}} \tag{13}$$

Using (24) from the appendix, we can integrate out all angles, resulting in

$$\mathcal{I} = \underbrace{\frac{1}{(2\pi)^{3+n}}}_{\text{inv Four suppr.}} \frac{2\pi i}{r}\int_{-\infty}^{\infty}\mathrm{d}p\ p^{1+n}\left(\frac{1}{1+q^{n+2}}\Theta(q) + (-1)^n\frac{1}{1+(-q)^{n+2}}\Theta(-q)\right)\ e^{ipr} \tag{14a}$$

$$= \frac{2\pi i}{zL}\int_{-\infty}^{\infty}\mathrm{d}q\ \frac{q^{1+n}}{L^{1+n}}\left(\frac{1}{1+q^{n+2}}\Theta(q) + (-1)^n\frac{1}{1+(-q)^{n+2}}\Theta(-q)\right)\ e^{iqz} \tag{14b}$$

where the dimensionless $z = r/L$ is determined by requiring $e^{ipr} = e^{iqz}$. From (14a) to (14b) we note that the dimensionless calculation requires a global $\frac{1}{L^{2+n}}$ scaling.

We now plan to solve (14b) by cauchy theorem.

2.1 The poles

Consider the poles of (13). They are given by

$$1+q^{n+2} = 0 \quad \Leftrightarrow \quad q = (-1)^{\frac{1}{n+2}} = \exp\left\{\frac{i\pi + 2\pi i k}{n+2}\right\} \quad \forall k \in \mathbb{N}_0 \tag{15}$$

There are $n+2$ unique poles, given e.g. by $k \in \{0, 1, \ldots, n, n+1\}$.

The integration contour is determined by requiring

$$\lim_{p\to\pm i\infty} e^{ipz} = 0 \ \Rightarrow\ e^{i(+i\infty)(+\#)} = e^{-i\infty\#} = 0 \ \Rightarrow\ p \to i\infty \tag{16}$$

as $z = |\mathbf{z}|$ is positive semi-definite. Therefore, only $\frac{n+2}{2}$ poles are taken into account.

One can check that no poles are on the real axis. For such poles,

$$\mathrm{Im}(q) = 0 \Leftrightarrow \sin\left(\frac{\pi + 2\pi k}{n+2}\right) = 0 \Leftrightarrow \frac{1+2k}{n+2} = 1 + j \tag{17a}$$

must hold in the domain of integral $k, j \in \mathbb{Z}_0$ for a given (integral) dimension n. The only solution is $q = -1$ which is not visible due to the $\Theta(q)$ in (14b). On the other hand, for some n, there are poles on the imaginary axis:

$$\mathrm{Re}(q) = 0 \Leftrightarrow \cos\left(\frac{\pi + 2\pi k}{n+2}\right) = 0 \Leftrightarrow \frac{1+2k}{n+2} = \frac{1}{2} + j. \tag{17b}$$

is solved e.g. for $n = 0, 4, 8$ and some k, j (see Appendix A.1 why that could be problematic).

2.2 Performing Residue summation

Factorizing the poles (15) in the integral (14b), suppressing prefactors, gives

$$\mathcal{I} \propto \int \mathrm{d}q \; q^{1+n} \; e^{iqz} \left(\frac{\Theta(q)}{\prod\limits_{k=0}^{n+1} e^{\frac{i\pi+2\pi ik}{2+n}} - q} + \frac{(-1)^n \Theta(-q)}{\prod\limits_{k=0}^{n+1} e^{\frac{i\pi+2\pi ik}{2+n}} + q} \right) \tag{18}$$

Actually, for all n,

$$\underset{q_0}{Res} \frac{q^{1+n}}{1+q^{n+2}} := r_0 \in \mathbb{Q} \quad \forall q_0 \in \{q : q^{n+2}+1=0\}, \tag{19}$$

that is, the complex phases in the fourier coefficient e^{iqz} survive in the residues except the very special situation $n=0$ where $q_0 = \pm i$. Due to the symmetry of the poles (by construction), the complex phases of each two residues for poles $q_0 = (-1)^\alpha$ and $\bar{q}_0 = (-1)^{1-\alpha}$ kill each other: With the rational number r_0 as defined in (19),

$$r_0 e^{iq_0 z} + r_0 e^{i\bar{q}_0 z} = 2 r_0 e^{-z\sin(\pi\alpha)} \cos\left(z\cos(\pi\alpha)\right). \tag{20}$$

For the full results of the energy density, see table 1. Note that in the $n=0$ case, the result reduces to the well-known result from [3]. See 1 for a plot of the energy densities.

2.3 Mass and Metric behaviour

Table 1 motivates for a lookout about the black hole properties induced by the energy densities. For convenience I have sketched the metric coefficient in figure 2, given by integrating the mass $\mathcal{M}(R) \propto \int_0^R \mathcal{T}_0^0 r^{2+n} \mathrm{d}r$ (c.f. [1] for notations).

n	$\mathcal{I} \propto \mathcal{T}_0^0$
0	$-4/z\pi^2 e^{-z}$
1	$-\frac{8}{3z}\pi^2 e^{-\frac{\sqrt{3}z}{2}} \cos\left(\frac{z}{2}\right)$
2	$-2/z\pi^2 e^{-\frac{z}{\sqrt{2}}} \cos\left(\frac{z}{\sqrt{2}}\right)$
3	$-\frac{8}{5z}\pi^2 e^{-\frac{1}{2}\sqrt{\frac{1}{2}(5-\sqrt{5})}z} \cos\left(\frac{1}{4}\left(1+\sqrt{5}\right)z\right)$
4	$-\frac{4}{3z}\pi^2 e^{-z}\left(e^{z/2}\cos\left(\frac{\sqrt{3}z}{2}\right)+1\right)$
5	$\frac{8}{7z}\pi^2\left(-e^{-z\sin\left(\frac{\pi}{7}\right)}\cos\left(z\cos\left(\frac{\pi}{7}\right)\right) - e^{-z\cos\left(\frac{\pi}{14}\right)}\cos\left(z\sin\left(\frac{\pi}{14}\right)\right)\right)$
6	$\frac{\pi^2}{z}\left(-e^{-z\sin\left(\frac{\pi}{8}\right)}\cos\left(z\cos\left(\frac{\pi}{8}\right)\right) - e^{-z\cos\left(\frac{\pi}{8}\right)}\cos\left(z\sin\left(\frac{\pi}{8}\right)\right)\right)$
7	$\frac{8}{9z}\pi^2\left(-e^{-\frac{\sqrt{3}z}{2}}\cos\left(\frac{z}{2}\right) - e^{-z\sin\left(\frac{\pi}{9}\right)}\cos\left(z\cos\left(\frac{\pi}{9}\right)\right)\right)$

Table 1: Exact results of (14b) in n large extra dimensions (only missing the dimensionful $1/L^{2+n}$ scaling)

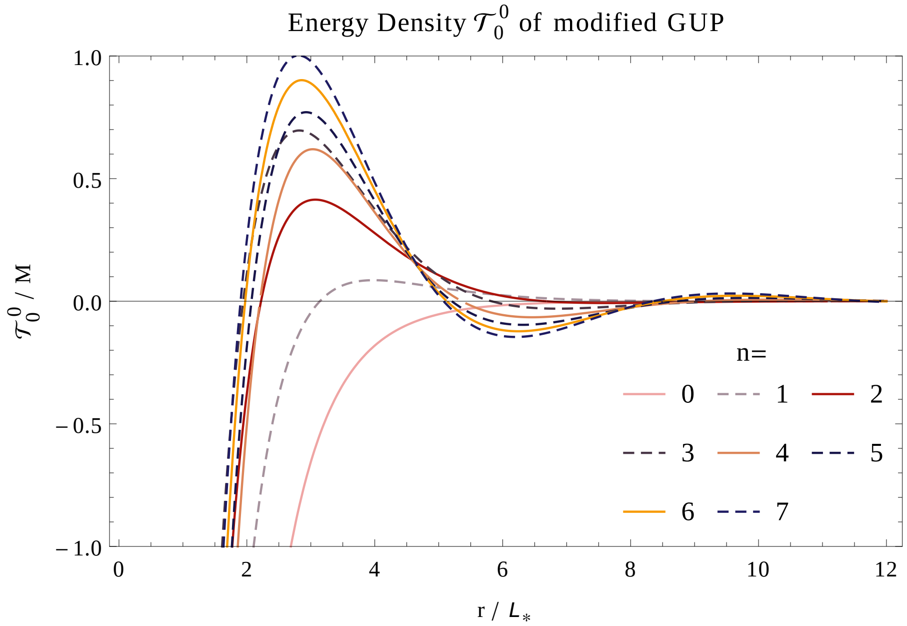

Figure 1: Plot of the functions in table 1.

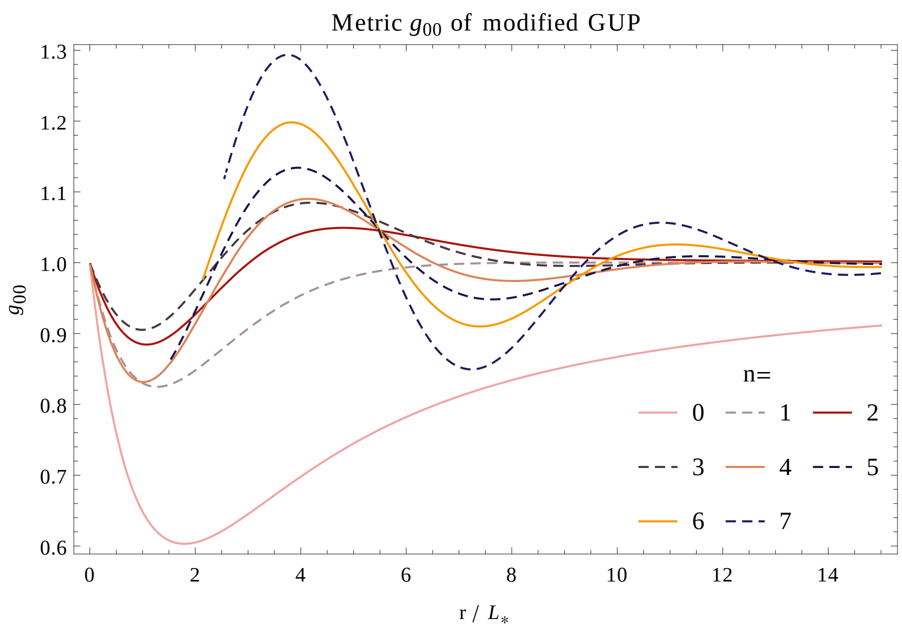

Figure 2: Plot of the line element of the modified GUP metrics in higher dimensions n.

Appendices

The following calculations retrace reducing higher dimensional Fourier transformations to one dimensional ones. While the 3d case (c.f. section A.1) is really well known, in literature I didn't found the derivation of the higher dimensional case (c.f. section A.2). Anyway, qualitatively the result is the same as in 3 dimensions and therefore the calculations are not that exciting but given for completeness.

A Review of the radial symmetric d-dimensional FT

Having the Fourier transformation $\mathcal{F}_d$ in d dimensions ($\mathbf{x} \in \mathbb{R}^d$) defined as

$$\mathcal{F}_d \{f\} (\mathbf{p}) = \tilde{f}(\mathbf{p}) = \frac{1}{(2\pi)^d} \int \mathrm{d}^d x \; e^{-i\mathbf{p}\cdot\mathbf{x}} f(\mathbf{x}) \tag{21a}$$

$$\mathcal{F}_d^{-1} \left\{\tilde{f}\right\} (\mathbf{x}) = f(\mathbf{x}) = \int d^d p \; e^{+i\mathbf{p}\cdot\mathbf{x}} \tilde{f}(\mathbf{p}) \tag{21b}$$

For shortness of notation, I will supress the leading $(2\pi)^d$ in the following equations.

The following calculations compute $\hat{V} = \mathcal{F}\{V\}$, that is, going from position to momentum space. That is because I used them to extend [4]. Of course they also hold for the inverse, as used in section 2.

A.1 3d Fourier transformation

We start the derivation in $d = 3$ total spatial dimensions ($\vec{r} \in \mathbb{R}^3$). Let $V = V(r)$ with $r = |\vec{r}|$ be a radially symmetric potential. Then it's fourier transformation is given by:

$$\hat{V}(p) = \int \mathrm{d}^3 r \; e^{-i\vec{r}\cdot\vec{p}} \; V(r) \tag{22a}$$

$$= \int_0^\infty \mathrm{d}r \int_0^\pi r^2 \sin\theta \; \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\varphi \; V(r) \; e^{-ipr\cos\theta_2} \tag{22b}$$

In line (22b) we already wrote the scalar product with an inner angle θ_2. We now substitute the radial angle θ (the θ which is part of $\vec{r} = (r, \theta, \varphi)$) integration with a $\cos\theta$ integration. This can be done because $\frac{\mathrm{d}\cos\theta}{\mathrm{d}\theta} = -\sin\theta$ and so $\int_0^\pi \sin\theta \mathrm{d}\theta = -\int_{-1}^1 \mathrm{d}\cos\theta = \int_{-1}^1 \mathrm{d}\cos\theta := \int_{-1}^1 \mathrm{d}x$. We now identify $\cos\theta := x$ with $\cos\theta_1$ because they share the same domain, actually $\theta, \theta_1 \in \{0, \pi\}$ (this is a standard procedure, one can also argue with rotating the coordinate systems. I think nobody doubts this substitution). We continue (naturally, $\int_0^{2\pi} \mathrm{d}\varphi = 2\pi$ was already integrated out in the next line):

$$= 2\pi \int_{-1}^{+1} \mathrm{d}x \int_0^\infty \mathrm{d}r \; r^2 \; e^{-irpx} V(r) \tag{22c}$$

$$= 2\pi \int_0^\infty r^2 \; \mathrm{d}r \; V(r) \left[\frac{1}{-ipr} e^{-iprx} \right]_{-1}^{+1} \tag{22d}$$

$$= \frac{2\pi i}{p} \int_0^\infty r \; \mathrm{d}r \; V(r) \left\{ e^{-ipr} - e^{+ipr} \right\} \tag{22e}$$

$$= \frac{2\pi i}{p} \left\{ \int_0^\infty r \; \mathrm{d}r \; V(r) e^{-ipr} - \int_0^\infty r \; \mathrm{d}r \; V(r) e^{+ipr} \right\} \tag{22f}$$

In line (22f), we splitted the integral, and we now make two recastings: At first, switching the integral borders, which inserts one minus: $\int_a^b = -\int_b^a$ in (22g). Second, another substitution of the integration parameter $r := -r'$ and therefore $\mathrm{d}r = -\mathrm{d}r'$. The two minus signs kill each other in (22h), so $r\mathrm{d}r = r'\mathrm{d}r'$. After substitution, we will call r' again r, which is totally valid.

$$= \frac{2\pi i}{p}\left\{\int_0^\infty r\ \mathrm{d}r\ V(r)e^{-ipr} + \int_\infty^0 r\ \mathrm{d}r\ V(r)e^{+ipr}\right\} \tag{22g}$$

$$= \frac{2\pi i}{p}\left\{\int_0^\infty r\ \mathrm{d}r\ V(r)e^{-ipr} + \int_{-\infty}^0 r'\ \mathrm{d}r'\ V(-r')e^{-ipr'}\right\} \tag{22h}$$

$$= \frac{2\pi i}{p}\int_{-\infty}^\infty r\ \mathrm{d}r\ e^{-ipr}\ \{V(r)\Theta(r) + V(-r)\Theta(-r)\} \tag{22i}$$

$$= \frac{2\pi i}{p}\int_{-\infty}^\infty \mathrm{d}r\ \{rV(|r|)\}\ e^{-ipr} \tag{22j}$$

Basically (22j) is our final result. The more extensive eq. (22i) may be better to argue why I think the new *effective* one dimensional function ("kernel") $v(r) \sim r\ V(|r|)$ can still be treated as a holomorphic function when Θ is implemented as a smeared distribution. Discontinuities of $v(r)$ at $r = 0$ may be discussed. If there are no poles at $r = 0$, it should be always possible to let $V(-r)$ and $V(r)$ blend into each other in a continous way.

Whats about the real and complex parts of this fourier transformation? By construction, $V(|r|)$ is an even function (definition: $f(x) = f(-x)$ is even, $-f(x) = f(-x)$ is odd). Therefore $r\ V(|r|)$ is an odd function. By eulers formula $e^{i\varphi} = \cos\varphi + i\sin\varphi$, one quickly finds that the Fourier Transform of an even function includes only (also even) cos terms and the complex part vanishes, while the FT of an odd function only contains sin terms and the real part vanishes. The integral in (22j) is therefore ony complex, $\int \mathrm{d}r\ rV(|r|)e^{-ipr} \in \mathbb{C}\setminus\mathbb{R}$. But the prefactor makes the final result in (22j) completely real again. This can help as a quick check wether the computed result of the integral is correct.

The analytic continuation of the Heaviside function may be another issue. I think it can be again motivated by continuing the integral of a Dirac delta approximation like the Cauchy distribution. Anyway I always use the complex Heaviside function like $\Theta(z) = \Theta(Re\, z)$ which looks intrinsically non-holomorphic. Since there is no ordering relation $\leq_{\mathbb{C}}$ in the complex numbers, the theta is likely to behave differently on the complex plane. Anyway in three dimensions this approach is well known and works, so it should work in any number of dimensions.

A.2 From 3 to (3+n) dimensions

Consider the Fourier transformation in $d = 3 + n$ dimensions. Following the same steps as in section A.1, we integrate out the angles of the measure (omitting the fourier kernel for shortness)

$$\int \mathrm{d}^d r = \int_0^\infty \mathrm{d}r\ r^{d-1}\int_0^{2\pi}\mathrm{d}\phi\prod_{i=1}^{d-2}\int_0^\pi \mathrm{d}\theta_i \sin^i(\theta_i) := \int_0^\infty \mathrm{d}r\ \Omega_{d-1}r^{d-1} \tag{23a}$$

$$= \frac{\Omega_{d-1}}{2}\underbrace{\int_0^\pi \mathrm{d}\theta_1 \sin(\theta_1)}_{=2}\int_0^\infty \mathrm{d}r\ r^{d-1} \qquad \text{with} \quad \Omega_{n+2} = 2\frac{\pi^{\frac{n+3}{2}}}{\Gamma\left(\frac{n+3}{2}\right)} \tag{23b}$$

The θ_1 integral in (23b) can only be evaluated to $\int \mathrm{d}\theta_1 \cdots = 2$ if the integrand (which is ommitted in these equations) is not dependend of θ_1. In our calculation, of course, this is not the case.

When doing the $\int_0^\infty \to \int_{-\infty}^\infty$ trick, an *alternating minus* enters the effective fourier transformation kernel $v(r)$, taking account for the even/odd behaviour of the integrand. We end up with

$$\hat{V}(p) = \frac{\Omega_{2+n}}{2}\frac{2\pi i}{p}\int_{-\infty}^\infty \mathrm{d}r\ r^{1+n}\left(V(r)\Theta(r) + (-1)^n V(-r)\Theta(-r)\right) \tag{24}$$

Pay attention the toggling minus $(-1)^n = (-1)^{2+n}$, this does **not** allow writing the effective integrand function $v(r)$ in a short way like $v(r) \neq r^{1+n} V(|r|)$ as the naive extension of (22j) could suggest. Why is it not $(-1)^{1+n}$? Because when substituting $r \to -r'$ and $\mathrm{d}r \to -\mathrm{d}r'$,

$$r^{1+n}\mathrm{d}r = (-1)^{1+n}(r')^{1+n}(-1)\mathrm{d}r' = (-1)^n r' \mathrm{d}r' \tag{25}$$

We end up with the nice and expected result that $v(r)$ is **always odd** for all n, therefore $\forall n$:

$$\int \mathrm{d}r \; v(r) \in \mathbb{C} \setminus \mathbb{R} \tag{26a}$$

$$\mathcal{F}_{n+3}\left\{V(|\vec{r}|)\right\} \in \mathbb{R} \tag{26b}$$

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Modified GUP in Extra Dimensions, update^c

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Abstract

This document is an addition for my proposal from July 6, 2014. It brings corrected prefactors, a more compact result for $\mathcal{T}^{00}$, numerical results for r_0 and M_*, corrected plots of the metric and the black hole temperatures. It ends with an outlook about the Heat capacity and stability.

This document is written in the context of the currently prepared paper *Self-Completeness and the Generalized Uncertainty Principle in Extra Dimensions* calculated by Maximiliano Isi and Marco Knipfer [1]. My proposal from July introduced a way to solve higher dimensional fourier transformations that occur in the computation, where Marcos approach (Schwinger Operator representation and identification as higher dimensional Gaussian integral) fails.

Internal working title: Calc18-update c

Contents

Mini symbols key

- $\lfloor x \rfloor := \max\{k \in \mathbb{N}_0 : k \leq x\}$: Gaussian step function
- $z = x + iy$, $\bar{z} = x - iy$ the complex conjugate
- $\Omega_d = \frac{\pi^{(d-1)/2}}{\Gamma\left(\frac{d-1}{2}+1\right)}$: Surface factor of d-sphere, the surface is given by $A_d = \Omega_d r^{d-1}$
- $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \mathrm{d}t$ the Gamma function
- $\Gamma(a,z) = \int_z^\infty t^{a-1} e^{-t} \mathrm{d}t$ the upper Gamma function
- $\gamma(a,z) = \int_0^z t^{a-1} e^{-t} \mathrm{d}t$ the lower Gamma function.
- $M^2_{\text{4d Planck}} = V_n M_*^{n+2}$ as defined in [5]: M_* is the reduced Planck length and V_n the volume of the compactified dimensions, e.g. in a torus $V_n = (2\pi R_c)^n$ with R_c the compactification radius.
- $L_* = 1/M_*$ the reduced Planck length and $M_{Pl}^2 = 1/(8\pi G)$ the link to newtonian Physics (the 8π must be preserved somewhere)

1 Framework

This document follows the reasoning of the 2013 JHEP paper [3], the work in progress [1] and my last proposal from July 2014. I won't repeat the details but write down the improved results.

1.1 The modified GUP

We want to modify the GUP relation discussed in [3] to higher dimensions in a way that without extra dimensions, it reduces to the ordinary case. Consider the "ordinary" GUP as modification of the canonical commutation relations ($p = |\mathbf{p}|$):

$$[x^i, p_j] = i\delta^i_j (1 + \beta p^2) \tag{1}$$

Now our improved versions in total $N + 1 = 4 + n$ space-time dimensions[1] looks like

$$[x^i, p_j] = i\delta^i_j (1 + L^{2+n} p^{2+n}) \tag{2}$$

with $L^2 = \beta$. The modified energy-momentum tensor is given by smearing the classical energy-momentum tensor (Schwarzschild static isotropic point-like matter source $T^0_0 = M\delta^N(\mathbf{x})$) with a bilocal function $\mathcal{A}^{-2}$.

$$\mathcal{T}^\mu_\nu = \mathcal{A}^{-2}(\Box) T^\mu_\nu = M\mathcal{A}^{-2}(\Box)\delta(\mathbf{x}) \tag{3}$$

Representing the Dirac in momentum space, the usual approach is

$$\mathcal{A}^{-2}(\Box)\delta(\mathbf{x}) = \frac{1}{(2\pi)^N} \int \mathrm{d}^N p\, \mathcal{A}^{-2}(\Box)\, e^{i\mathbf{x}\cdot\mathbf{p}} = \frac{1}{(2\pi)^N} \int \mathrm{d}^N p\, \mathcal{A}^{-2}(p^2)\, e^{i\mathbf{x}\cdot\mathbf{p}} := \mathcal{F}^{-1}_N\{\mathcal{A}^{-2}(p^2)\} \tag{4}$$

with $\mathcal{F}^{-1}_N$ the N-dimensional inverse fourier transformation. Using the modified momentum integration measure given by [2],

$$\int \frac{\mathrm{d}^3 p}{1 + L^{2+n} p^{2+n}} |p\rangle\langle p| = 1, \tag{5}$$

we end up determining the smeared matter density by

$$\mathcal{T}^0_0 = \frac{M}{(2\pi)^N} \int \frac{\mathrm{d}^3 p}{1 + L^{2+n} p^{2+n}} e^{i\mathbf{x}\cdot\mathbf{p}}. \tag{6}$$

In this text, we will solve $\mathcal{T}^0_0$, afterwards derive $g_{\mu\nu}$ and be happy.

1.2 The effective 1-dimensional FT

In order to solve (6), I already proposed an approach using Heaviside-step functions $\Theta(z) = \Theta(\mathrm{Re}\ z)$ on the complex plane.

Computing the Fourier transformation

$$\hat{V}(\mathbf{r}) = \frac{1}{(2\pi)^N} \int \mathrm{d}^N p\ e^{+i\mathbf{r}\cdot\mathbf{p}}\ V(p) \tag{7}$$

we integrate out all angles up to one, used to make the identification with the $\mathbf{x}\cdot\mathbf{p}$ scalar product:

$$\hat{V}(\mathbf{r}) = \frac{1}{(2\pi)^N} \frac{\Omega_{N-1}}{2} \int_0^\infty \mathrm{d}p\ p^{N-1}\ V(p)\ \frac{e^{+irp} - e^{-irp}}{irp} \tag{8}$$

Following the reasoning in the Jul 2014 proposal, we end up with an one dimensional integration

$$\hat{V}(\mathbf{r}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}p\ v(p)\ e^{+irp} \tag{9}$$

with the effective 1d Fourier kernel $v(r)$, given by

$$v(p) := \frac{1}{(2\pi)^{N-1}} \frac{\Omega_{N-1}}{2} \frac{1}{ir} p^{N-2} \left[V(p)\Theta(p) + (-1)^{N-1} V(-p)\Theta(-p)\right] \tag{10}$$

[1]Remember: N = Number of spatial dimensions, n = number of large extra dimensions

2 Properties of the modified GUP

2.1 Gaining the matter density

With $V(p) = 1/(1 + L^{2+n}p^{2+n})$, following (10) gives us the integral

$$\mathcal{T}_0^0 = \frac{M}{(2\pi)^N}\frac{\Omega_{N-1}}{2}\frac{1}{ir}\int_{-\infty}^{\infty} \mathrm{d}p\; p^{N-2}\left[\frac{\Theta(p)}{1+L^{N-1}p^{N-1}} + \frac{(-1)^{N-1}\Theta(-p)}{1+L^{N-1}(-p)^{N-1}}\right] e^{ipr}. \tag{11}$$

For easier evaluation of the poles, I prefer using dimensionless units $q = pL$ and $z = r/L$:

$$\mathcal{T}_0^0 = \underbrace{\frac{M}{(2\pi)^N}\frac{\Omega_{N-1}}{2L^N}\frac{1}{iz}}_{f_0}\int_{-\infty}^{\infty} \mathrm{d}q\;\left[\underbrace{\frac{q^{N-2}}{1+q^{N-1}}}_{f_+(q)}\Theta(q) + \underbrace{\frac{q^{N-2}(-1)^{N-1}}{1+(-q)^{N-1}}}_{f_-(q)}\Theta(-q)\right]e^{iqz} \tag{12}$$

The explicit determination of the poles of $f_\pm(q)$ was already given in the July proposal. Note that the solution set of the equation $1/f_+(q) = 0$ is just the negative of the solution set of $1/f_-(q) = 0$. The poles of $f_+(q)$ were already given in the July proposal by

$$1 + q^{n+2} = 0 \quad\Leftrightarrow\quad q = (-1)^{\frac{1}{n+2}} = \exp\left\{\frac{i\pi + 2\pi ik}{n+2}\right\} \quad \forall k \in \mathbb{N}_0, \tag{13}$$

and we also already summed up the residues $f_\pm(q_0)e^{iq_0z}$ of all eligible poles q_0. In the July proposal, the result was lacking all prefactors, called f_0 in (12), or to be more specific, I worked with the wrong $f_0 = 2\pi i/z$.

All our poles q_0, as given in (13), have the same value for $\mathrm{Res}_{q_0} f_\pm(q_0) = \frac{1}{2+n}$. Furthermore, for a given root $q_0 = (-1)^\alpha$ there is always a partner $-\bar{q}_0 = -(-1)^{-\alpha}$ and the two exponential factors from the corresponding residues combine to an exponentially suppressed cosine,

$$e^{iqz} + e^{-i\bar{q}z} = 2e^{-z\sin(\alpha)}\cos(z\cos(\alpha)), \quad q = e^{i\alpha}, \tag{14}$$

as introduced in the July proposal. We get the overall real result

$$\boxed{\mathcal{T}_0^0 = \frac{M}{(2+n)r}\frac{\Omega_{N-1}}{(2\pi L)^{N-1}}\sum_{\varphi\in\Phi_n} e^{-r/L\sin(\varphi)}\cos\left(r/L\cos(\varphi)\right),} \tag{15}$$

with Φ_n the phases of the poles taken into account for n extra dimensions, given as

$$\Phi_n = \left\{\varphi = \arg(q) \;:\; 1 + q^{n+2} = 0 \wedge \mathrm{Im}(q) \geq 0 \wedge \mathrm{Re}(q) \geq 0\right\} \tag{16}$$

$$= \left\{\varphi = \pi\frac{1+2k}{n+2} \;:\; k \in \mathbb{N}_0 \wedge k \leq \frac{n}{4}\right\}. \tag{17}$$

Since the number of angles $|\Phi_n| = \lfloor \frac{n}{4} \rfloor$ is a step-function, in this notation there is no way to write (15) in a more compact way. Note that the metric has a $e^{-r}\cos(r)/r \approx 1/r - 1$ behaviour around $r \to 0$ and no regular core, as expected.

For $n = 0$, our result reduces to [3], as can bee seen when computing $\Phi_0 = \{\pi/2\}$, $\Omega_2 = 2\pi$:

$$\mathcal{T}_0^0 = \frac{M}{2r}\frac{\Omega_2}{(2\pi L)^2}e^{-z} = \frac{M}{\beta r 4\pi}e^{-r/\sqrt{\beta}}. \tag{18}$$

2.2 The metric

Following the $N+1$-dimensional solution of the Einstein Equations made by Rizzo 2005 [5], the correct line element is given by the solution of the first order differential equation

$$V'(r) + \frac{n+1}{r}V(r) = \frac{1}{M_*^{n+2}}\frac{2r\rho(r)}{n+2} \tag{19}$$

with $\rho(r) = \mathcal{T}_0^0$, in a way that the line element is then given by

$$ds^2 = (1 - V(r))\mathrm{d}t^2 - (1 - V(r))^{-1}\mathrm{d}r^2 - \mathrm{d}\Omega^2. \tag{20}$$

The general solution of (19) is given by

$$V(r) = \frac{1}{r^{n+1}} \left(\frac{2}{(n+2)M_*^{n+2}} \int_{c_1}^{r} x^{n+2}\rho(x)\mathrm{d}x + c_2 \right) \quad \text{with} \quad c_1, c_2 = \text{const}, \tag{21}$$

and after inserting our density $\rho = \mathcal{T}_0^0$ given by (15), with $p_0 = e^{i\varphi}$,

$$V(r) = \frac{1}{r^{n+1}} \frac{2M}{(n+2)^2 M_*^{n+2}} \frac{\Omega_{n+2}}{(2\pi L)^{n+2}} \sum_{\varphi \in \Phi_n} \int_0^r \mathrm{d}x \; x^{n+1} \left(e^{ip_0 x} + e^{i\bar{p}_0 x} \right). \tag{22}$$

By substitution $x' = x/(-ip_0)$ or $x' = x/(-i\bar{p}_0)$, respectively, the occurring integrals can be rewritten to lower gamma functions. Note that $q_0^{2+n} \equiv \bar{q}_0^{2+n} \equiv -1$. Our final result for the metric is

$$\boxed{V(r) = \frac{1}{r^{n+1}} \frac{2M}{(n+2)^2 M_*^{n+2}} \frac{\Omega_{n+2}}{(2\pi L)^{n+2}} \frac{(-1)^{1+n}}{i^{2+n}} \sum_{\varphi \in \Phi_n} \gamma(2+n, -ip_0 r) + \gamma(2+n, -i\bar{p}_0 r).} \tag{23}$$

Let's check the result for $n \to 0$: The sum only contains two times the same $p_0 = \bar{p}_0 = i/L$ and $1/M_* = 8\pi G$, so we derived the 4d GUP metric of [3],

$$V(r) = \frac{2GM}{r\beta} \gamma(2, {}^{r}/\!\sqrt{\beta}). \tag{24}$$

Figure 1 shows the metric with the special properties which are basically the same as in $n = 0$, that is, three possible situations (no black hole, extremal horizon r_0 or two horizons $r_\pm$), the presence of a remnant for a special mass, etc.

2.3 Self-completeness

Casting $L = L_*$ the reduced Planck mass, the special mass $M = M_*$ exhibits the extremal configuration at $r = r_0$ (actually no self-encoding), see e.g. figure 1 and table 1 for numerical values. Actually, the extremal black hole masses get enormous values, compared to other models I know (e.g. holographic ones in LXDs, [4]).

2.4 Black Hole Temperature

Computing the temperatures $T_H = \frac{1}{4\pi} \partial_r g_{00}|_{r=r_H}$ is a straightforward process and generates a long expression, c.f. the length of the temperature expressions of the non-modified GUPs in [1]. In figure 2, the temperature is plotted for the GUP-modified Black Holes. Further computation may be done, but not in this document.

n	0	1	2	3	4	5	6	7
r_0	1.79328	1.27534	1.07714	0.993701	1.01592	0.981144	0.953649	0.932502
M_*	3.35092	53.0073	621.491	6536.7	35182.9	359680.	$3.69058 \cdot 10^6$	$3.8323 \cdot 10^7$

Table 1: Self-encoding horizon radius $r_0 = L_*$ and Remnant masses $M_* = 1/L_*$, in 4d Planck Units for the modified GUP

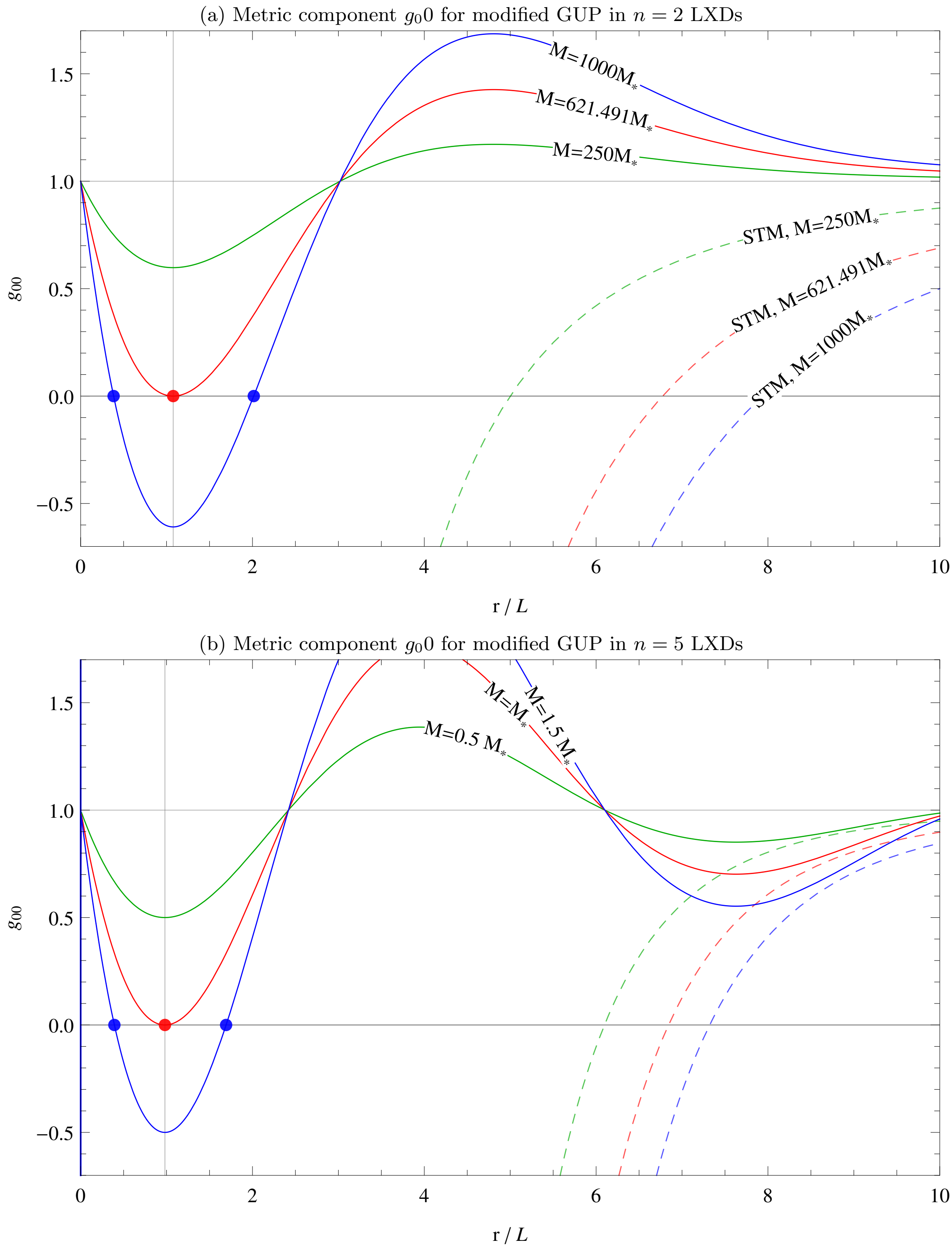

Figure 1: The metric behaviour for the modified GUP $[x^i, p_j] = i\delta^i_j(1+L^{2+n}p^{2+n})$ in n large extra dimensions, compared to the Schwarzschild-Tangherlini metric in n large extradimensions (dashed lines). The red dot indicates the extremal radius r_0, for small r, this metric behaves always like the $n = 0$ GUP.

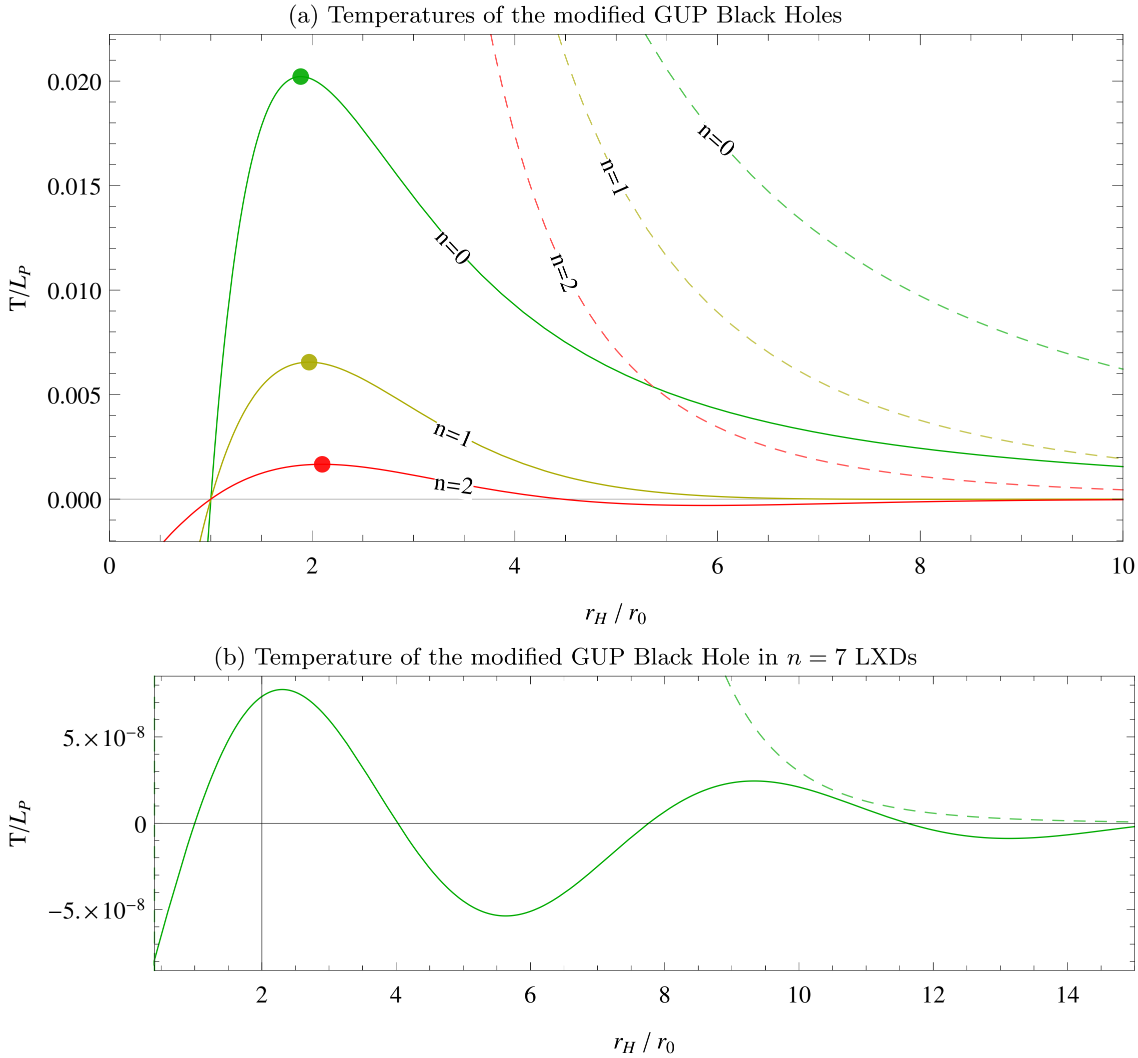

Figure 2: Temperatures of the modified GUP Black holes in n large extra dimensions, compared to Schwarzschild-Black holes (dashed lines). The x-axis is scaled by the extremal radius r_0. In figure (a), the circles indicate the (lowest) critical radii r_C where the Heat Capacity diverges. With increasing $n > 0$, the temperature fluctuates more and more around $T = 0$. The model therefore suffers negative temperatures which have actually the same order of magnitude as the remnant temperature for large n, as shown in figure (b). With each inflexion point, a diverging heat capacity is associated. Therefore the Black holes should exhibit a rich phase structure with alternating stable and unstable phases.

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Another modified GUP in Extra Dimensions

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Abstract

We present here another modified GUP that exposes cold remnants in any number of dimensions, we will call that approach "backward-modified" since we start with the metric and derive the nonlocal operator which is tested if it is a good GUP modification or not (it turns out it is not).

We also discuss the last kind of "naively" modified GUP from Svens July 6, 2014 proposal in terms of a long-distance (i.e. Schwarzschild) expansion.

This document contains ideas and thoughts developed by Marco and Sven while the DPG physics school *GR@99* in Bonn/Bad Honnef, Sept 2014.

This document is written in the context of the currently prepared paper *Self-Completeness and the Generalized Uncertainty Principle in Extra Dimensions* calculated by Maximiliano Isi and Marco Knipfer [1]. *Internal working title:* CALC19

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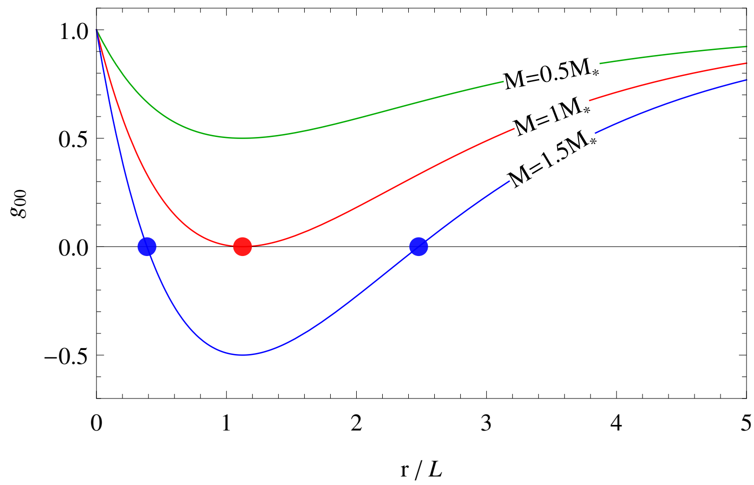

Figure 1: The remnant picture for $n = 7$ ($N = 10$) for our modified GUP as given in equation (3), here $M_* = 10.95 M_P$ like estimated in table 1. The plot basically looks the same in any number of dimensions.

1 Another modified GUP ("backward-modified")

We present a way to compute the GUP function $f(\mathbf{p})$ in the Kempf notation

$$\Delta x \Delta p \geq \frac{\hbar}{2} (1 + f(\mathbf{p})) \tag{1}$$

in N spatial dimensions. For $N = 3$ and $f(\mathbf{p}) = \beta \mathbf{p}^2$, in 2013 Max et al [3] derived the metric

$$g_{00} = 1 - \frac{2GM}{r} \gamma(2; {}^r/\sqrt{\beta}) \tag{2}$$

with the lower gamma function $\gamma(n, z)$. Marco and Max did not found remnants for some big N any more, so I proposed a modified GUP $f(p) = L^{N-1} p^{N-1}$ which produced a repulsive gravitational potential which discussion is proceeded in section 2.

In this section, we search for another modified GUP, in a "reverse" fashion by starting from the metric: With an "educated guess" while looking at metric (2), we propose an N-dimensional extension by means of

$$\boxed{g_{00} = 1 - \frac{2GM}{r^{N-2}} \gamma(N-1; {}^r/\sqrt{\beta}).} \tag{3}$$

We checked that this metric exposes remnants in any dimension N with appropriate masses M_*, as given in table 1. See figure 1 for the plot of g_{00} for an arbitrary dimension N.

n	0	1	2	3	4	5	6	7
r_0	1.79328	1.45123	1.31433	1.24065	1.19472	1.1634	1.1407	1.1235
M_*	1.67546	2.9412	4.24808	5.57428	6.9109	8.25373	9.60055	10.9501

Table 1: Minimal length $r_0 = L$ and memnant masses in 4d Planck units for the "backward-modified" GUP

1.1 Derivation of the stress tensor

We now derive the nonlocal operator $\mathcal{A}^{-2}$ from our modified metric (3). To do so, we identify the modified mass $\mathcal{M}$ of the Black hole with event horizon r_H by

$$g_{00} = 1 - \frac{2G\mathcal{M}(r_H)}{r_H^{N-2}}, \quad \mathcal{M}(r_H) = M\gamma(N-1; {}^{r_H}/_{\beta}). \tag{4}$$

On the other hand, it is linked to the mass density by the full space integration

$$\mathcal{M}(R) = \int_{\mathcal{B}(R)} \mathrm{d}^N r \mathcal{T}_0^0(r) = \int_0^R \mathrm{d}r\ r^{N-1}\mathcal{T}_0^0(r) A_{N-2} \tag{5}$$

with $\mathcal{B}(R)$ being the ball with volume R, A_{N-2} the surface area of that ball and $\mathcal{T}_0^0$ the mass density (first entry of modified energy momentum tensor) we are looking for. We compute it by deriving the gamma function:

$$\mathcal{T}_0^0(r) = \frac{\partial_r \mathcal{M}(r)}{r^{N-1}A_{N-2}} = \frac{M}{r^{N-1}A_{N-2}}\partial_r\gamma(N-1; {}^{r}/_{\sqrt{\beta}}). \tag{6}$$

Inserting the definition of the gamma function, the derivate is given by

$$\gamma(N-1; {}^{r}/_{\sqrt{\beta}}) = \int_0^{r/\sqrt{\beta}} t^{N-2}e^{-t}\mathrm{d}t \quad \Rightarrow \quad \partial_r\gamma(N-1; {}^{r}/_{\sqrt{\beta}}) = ({}^{r}/_{\sqrt{\beta}})^{N-2}e^{-r/\sqrt{\beta}}, \tag{7}$$

so we determined the mass density for the modified metric to

$$\boxed{\mathcal{T}_0^0 = \frac{M}{A_{N-2}(\sqrt{\beta})^{N-2}}\frac{e^{-r/\sqrt{\beta}}}{r}.} \tag{8}$$

We note that, except for propertionality factors, this energy density is *exactly the same* in any number of dimensions.

1.2 Derivation of the nonlocal operator

The N-dependence of $\mathcal{A}^{-2}$ arises from the following steps: The higher dimensional fourier transformation that is used to transform the $\mathcal{A}^{-2}$ defining equation

$$\mathcal{T}_0^0 = \mathcal{A}^{-2}(\Box)M\delta^N(\mathbf{x}) = \mathcal{A}^{-2}(\Box)M\mathcal{F}_N\{1\} = M\mathcal{F}_N\{\mathcal{A}^{-2}(p)\}, \tag{9}$$

with $\mathcal{F}_N$ being the forward fourier transformation in N dimensions. This equation immediately yields

$$\mathcal{A}^{-2}(p) = \frac{1}{M}\mathcal{F}_N^{-1}\{\mathcal{T}_0^0\} = \frac{1}{M}\int \mathcal{T}_0^0(r)e^{-i\mathbf{x}\cdot\mathbf{p}}\mathrm{d}^N x \tag{10}$$

Note that the inverse fourier transformation $\mathcal{F}^{-1}$ prefactor convention here follows Max [1] from April 2014. Now we use the effective dimensional reduction of the fourier transformation that I derived multiple times:

$$\int \mathrm{d}^N r\ V(\|\mathbf{r}\|)e^{-i\mathbf{r}\cdot\mathbf{p}} = \int_{-\infty}^{\infty}\mathrm{d}r\ v(r)e^{-ipr} \tag{11}$$

with the effective 1d Fourier kernel $v(r)$. For details, see e.g. *Calc18* I sent around. For $V(r) = \mathcal{T}_0^0(r)$ the effective function is given by

$$v(r) = \mathrm{pre}\frac{A_{N-2}}{2}\frac{i}{p}r^{N-2}\left[\mathcal{T}_0^0(r)\Theta(r) + (-1)^{N+1}\mathcal{T}_0^0(-r)\Theta(-r)\right], \tag{12}$$

with "pre" being typically the fourier prefactors which are 1 here in the Max convention.

Inserting $v(r)$ in the operator defining integral (10) let's us compute

$$\boxed{\mathcal{A}^{-2}(p)} = \int_{-\infty}^{\infty} \frac{1}{(\sqrt{\beta})^{N-2}} \frac{1}{2} \frac{i}{p} r^{N-2} \left[\frac{1}{r} e^{-r/\sqrt{\beta}} \Theta(r) + (-1)^{N+1} \frac{1}{-r} e^{+r/\sqrt{\beta}} \Theta(-r) \right] e^{-irp} \mathrm{d}r \tag{13}$$

$$= \frac{1}{(\sqrt{\beta})^{N-2}} \frac{1}{2} \frac{i}{p} \left(\int_0^{\infty} r^{N-3} e^{-r/\sqrt{\beta}} e^{-irp} \mathrm{d}r + (-1)^{N+2} \int_{-\infty}^{0} r^{N-3} e^{+r/\sqrt{\beta}} e^{-irp} \mathrm{d}r \right) \tag{14}$$

$$= \frac{1}{(\sqrt{\beta})^{N-2}} \frac{1}{2} \frac{i}{p} \left(\int_0^{\infty} r^{N-3} e^{-r\kappa_+} \mathrm{d}r + (-1)^{N+2} \int_{-\infty}^{0} r^{N-3} e^{+r\kappa_-} \mathrm{d}r \right) \tag{15}$$

with $\kappa_\pm := \frac{1}{\sqrt{\beta}} \pm ip$. Now we do a "derivation trick" to solve the integrals with $\kappa_\pm$, which gives

$$= \frac{1}{(\sqrt{\beta})^{N-2}} \frac{1}{2} \frac{i}{p} \left(\partial_{\kappa_+}^{N-3} (-1)^{N-3} \int_0^{\infty} e^{-r\kappa_+} \mathrm{d}r + (-1)^{N+2} \partial_{\kappa_-}^{N-3} (-1) \int_{-\infty}^{0} e^{+r\kappa_-} \mathrm{d}r \right) \tag{16}$$

$$= \frac{1}{(\sqrt{\beta})^{N-2}} \frac{(-1)^{N-3}}{2} \frac{i}{p} \left(\partial_{\kappa_+}^{N-3} \left[\frac{1}{-\kappa_+} e^{-r\kappa_+} \right]_0^{\infty} - \partial_{\kappa_-}^{N-3} \left[\frac{1}{+\kappa_-} e^{+r\kappa_-} \right]_{-\infty}^{0} \right) \tag{17}$$

$$= \frac{1}{(\sqrt{\beta})^{N-2}} \frac{(-1)^{N-3}}{2} \frac{i}{p} \left(\partial_{\kappa_+}^{N-3} \frac{1}{\kappa_+} - \partial_{\kappa_-}^{N-3} \frac{1}{\kappa_-} \right). \tag{18}$$

Using the derivation $\partial_x^m x^{-1} = (-1)^m m! x^{-(1+m)}$, we can resolve the derivatives to

$$= \frac{(N-3)!}{(\sqrt{\beta})^{N-2}} \frac{(-1)^{N-3}}{2} \frac{i}{p} \left(\frac{1}{\left(\frac{1}{\sqrt{\beta}} + ip\right)^{N-2}} - \frac{1}{\left(\frac{1}{\sqrt{\beta}} - ip\right)^{N-2}} \right) \tag{19}$$

$$= \boxed{(-1)^{N-3} \frac{(N-3)!}{2} \frac{i}{p\sqrt{\beta}} \left(\frac{1}{(1+i\sqrt{\beta}p)^{N-2}} - \frac{1}{(1-i\sqrt{\beta}p)^{N-2}} \right).} \tag{20}$$

This is our final result. You might the inserted blue $\sqrt{\beta}$ which is correct there for a dimensionless operator $\mathcal{A}^{-2}$ and was somewhere missed on the way down to the result.

Note that the operator is *real* in any number of dimensions. Until now, we found a compact way to write the result (20) only for fixed N. Table 2 lists such expressions.

n	$\mathcal{A}^{-2}$
0	$\frac{1}{\beta p^2+1}$
1	$-\frac{2}{(\beta p^2+1)^2}$
2	$\frac{6-2\beta p^2}{(\beta p^2+1)^3}$
3	$-\frac{24(1-\beta p^2)}{(\beta p^2+1)^4}$
4	$\frac{24(\beta p^2(\beta p^2-10)+5)}{(\beta p^2+1)^5}$
5	$-\frac{240(\beta p^2-3)(3\beta p^2-1)}{(\beta p^2+1)^6}$
6	$\frac{720(7-\beta p^2(\beta p^2(\beta p^2-21)+35))}{(\beta p^2+1)^7}$
7	$-\frac{40320(1-\beta p^2)(\beta p^2(\beta p^2-6)+1)}{(\beta p^2+1)^8}$

Table 2: The value for $\mathcal{A}^{-2}(p)$ in n large extra dimensions, as can be found when inserting $N = n + 3$ into equation (20).

(a) Backwards modified GUP (sec. 1) (b) Ordinary modified GUP (sec. 2)

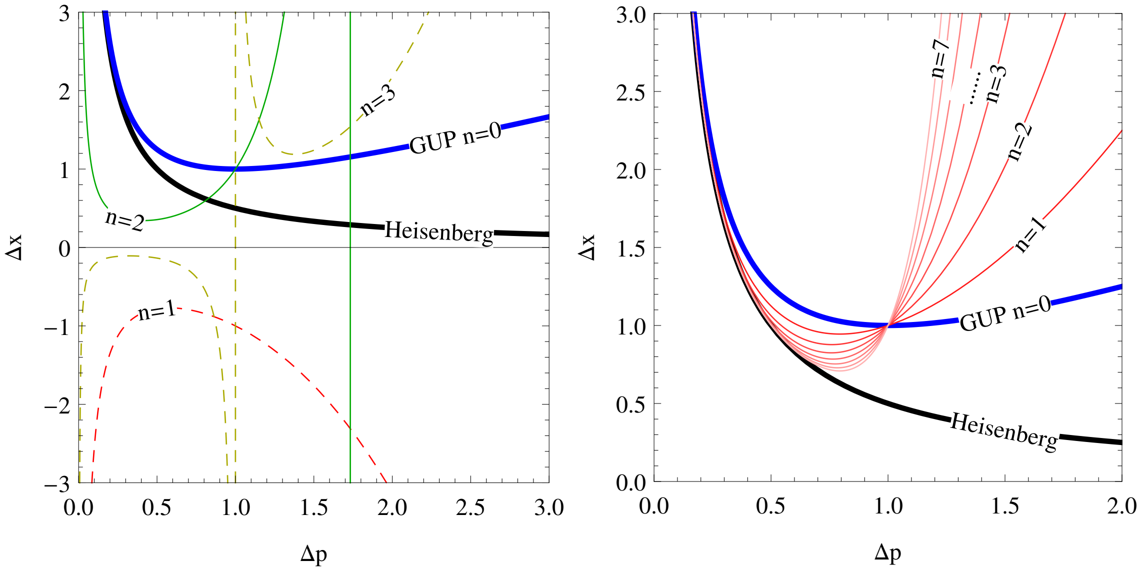

Figure 2: GUP modifications in comparison with ordinary uncertainity principle ($f(p) = 0$), ordinary GUP in 4d ($f(p) = p^2$). Here, $\beta = L = 1$. The pictures actually shows the function $x = \frac{1+f(p)}{2p}$.

1.3 The modified GUP

From $\mathcal{A}^{-2}$ we get the modified GUP expression by backwards following Kempf's integral measure

$$\mathcal{A}^{-2}\delta(x) = \int \frac{\mathrm{d}^N p}{1 + f(\mathbf{p})} \tag{21}$$

This function $f(p)$ is the one that enters the GUP as

$$\Delta x \Delta p \geq \frac{\hbar}{2}(1 + f(p)) \tag{22}$$

So we just insert

$$\boxed{f(p) = \frac{1}{\mathcal{A}^{-2}(p)} - 1 = \frac{(-1)^N}{(N-3)!} \frac{2ip\beta}{\left(1 + i\sqrt{\beta}p\right)^{-(N-2)} - \left(1 - i\sqrt{\beta}p\right)^{-(N-2)}}.} \tag{23}$$

For $N = 3$, this gives the well-known $f(p) = \beta p^2$, while for example for $N = 4$, the GUP is given by

$$\Delta x \Delta p \geq \frac{\hbar}{2}\left(p^2\beta^2 - \frac{1}{2}\right), \tag{24}$$

while for $N = 5$, things get even worse

$$\Delta x \Delta p \geq \frac{\hbar}{2}\left(\frac{(1 + p^2\beta)^3}{6 - 2p^2\beta}\right). \tag{25}$$

It seems that those expressions does not reduce to the ordinary Heisenberg principle for small p (or rather β). The minimal length picture is also lost, as the relationship is for non-even n negative and thus physically meaningless. See figure 2a for a plot for the first extra dimensions $n = N - 3$ in the $\Delta x \Delta p$ plane.

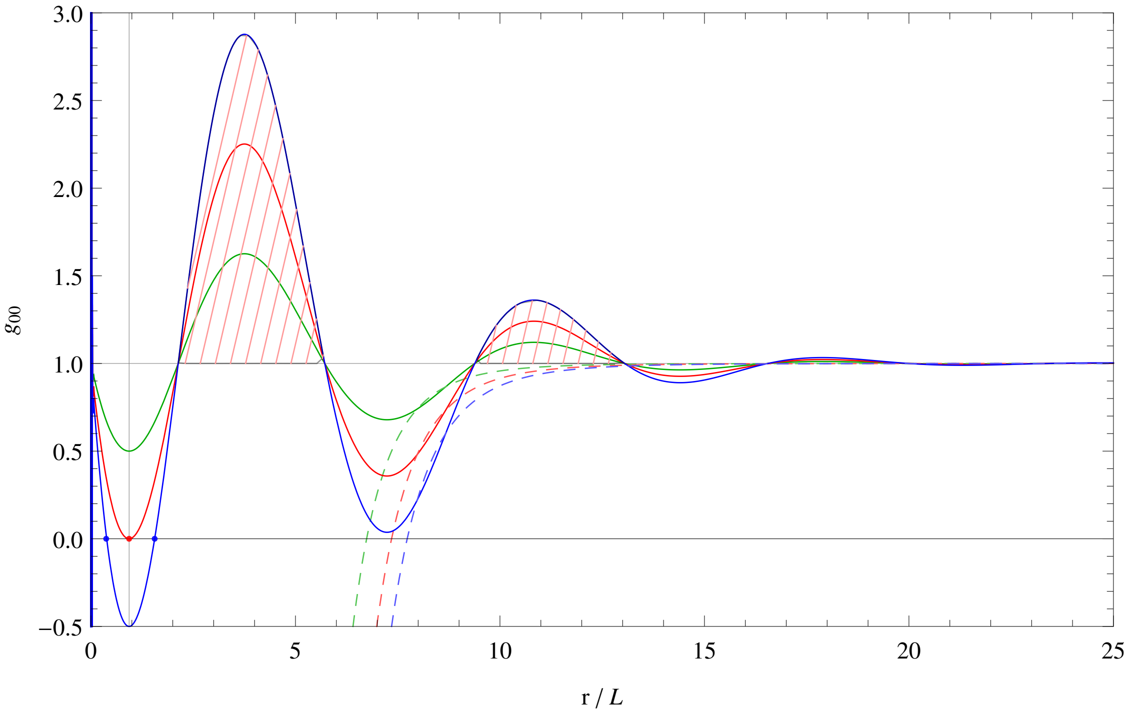

Figure 3: The modified GUP metric in $n = 5$ large extra dimensions exhibits three notable repulsive spheres, defined by $g_{00} > 1$ and shaded red in this plot. Since the wiggling is exponentially surpressed with increasing radius, c.f. (27), repulsive regions do not appear at bigger distances, that is, the Schwarzschild limit holds. Anyway, the shown metric differs heavily from the Schwarzschild limit already $r = 17L$.

2 Weak-field approximation of the ordinary modified GUP

In this section, we again consider the *original* modified GUP proposed by Sven in April 2014. It reads in $n = N - 3$ large extra dimensions ($p = |\mathbf{p}|$):

$$[x^i, p_j] = i\delta^i_j(1 + L^{2+n}p^{2+n}), \tag{26}$$

see figure 2b for a plot of the corresponding GUP $\Delta x \Delta p \geq \frac{\hbar}{2}(1 + f(p))$. Sven found that this approach produces smeared matter densities that induce metrics exposing repulsive zones (thick spheres around the Black Hole, see figure 3) at short scales. This is based on the fluctuating matter density that predicts ranges of negative matter density due to the cosine terms, as derived in *Calc18* and approved by Maximiliano:

$$\mathcal{T}^0_0 \approx \sum_{\varphi \in \Phi_n} e^{-r/L \sin(\varphi)} \cos\left(r/L \cos(\varphi)\right). \tag{27}$$

It is worth investigating the limits of the theories. This is done by taylor expansions. We could for example investigate the long-distance behaviour of the lower Gamma function that appears in the gravitational potential $V(r) = 1 - g_{00}$,

$$\lim_{z\to\infty} \gamma(n, z) = e^{-z} z^n \left(\frac{1}{z} + \frac{n}{z^2} + \mathcal{O}\left(\frac{1}{z^3}\right)\right). \tag{28}$$

I didn't yet compute these limits because they were clearly visible at the figures in *Calc18* (or e.g. figure 3), as the proposed Black Holes always have a clear Schwarzschild/flat space limit.

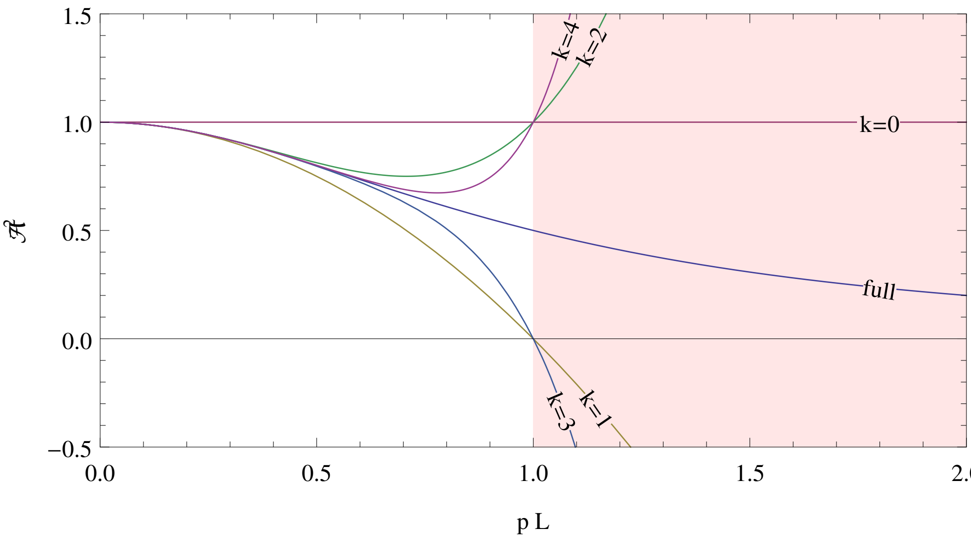

Figure 4: Plot of the k. series approximation of $\mathcal{A}^{-2} = 1/(1 + (Lp)^{2+n})$ for an arbitrary extra dimension n (here $n = 0$), as given in equation (32). The convergence radius for the series is always L, that is, the approximation breaks down for $r > L$ (red area). As these functions are the amplitude for the fourier coefficients, this motivates truncating the fourier transformation (33) to the interval $p \in [-L, L]$.

In September, P.N. proposed investigating the limits of the nonlocal operator $\mathcal{A}$, which means in the derivation of the GUP metric [1] at the *start* of the calculation, not at the *end*. In practical terms this means approximating the deformed matter density

$$\mathcal{T}_0^0 = \frac{M}{(2\pi)^N} \int \frac{\mathrm{d}^N p}{1 + L^{2+n} p^{2+n}} e^{i\mathbf{x}\cdot\mathbf{p}}, \tag{29}$$

with a series expansion of the nonlocal action, using the well-known identities

$$\lim_{x\to 0} \frac{1}{1+x} = 1 - x + x^2 + \mathcal{O}(x^3) \tag{30}$$

$$\text{and} \quad \lim_{x\to\infty} \frac{1}{1+x} = \frac{1}{x} - \frac{1}{x^2} + \mathcal{O}\left(\frac{1}{x^3}\right). \tag{31}$$

With the rule of thumb "*big momenta correspond to small distances*" in mind, we investigate the limit

$$\boxed{\lim_{p\to 0} \frac{1}{1 + L^{N-1} p^{N-1}} = 1 - L^{N-1} p^{N-1} + \mathcal{O}(p^2)} \tag{32}$$

which corresponds to far distances from the black hole, i.e. the Schwarzschild limit, with a first order correction term. We could also interpret this limit in terms of $L \to 0$, that is, $\beta \to 0$, or say that we expand the GUP effects *around* the Schwarzschild solution.

It turns out that this expansion is not only an exercise like tayloring the gamma (28) in $V(r)$, but requires a special treatment of the divergences. Consider the approach just inserting the first two orders of the approximation (32) into the matter density (29):

$$\mathcal{T}_0^0 \approx \frac{M}{(2\pi)^N} \int \mathrm{d}^N p \; e^{i\mathbf{x}\cdot\mathbf{p}} + \frac{ML^{N-1}}{(2\pi)^N} \int \mathrm{d}^N p \; p^{N-1} e^{i\mathbf{x}\cdot\mathbf{p}} \tag{33}$$

$$= M\delta(x) + \frac{ML^{N-1}}{(2\pi)^N} \int \mathrm{d}^N p \; p^{N-1} e^{i\mathbf{x}\cdot\mathbf{p}}. \tag{34}$$

We see that this integral diverges for $N > 1$. This is not surprising as integrating a truncated taylor series with finite convergence radius p_c over the full space is always diverging. Defining the series as (with $m = N - 1$)

$$\mathcal{A}^{-2}(p) := \frac{1}{1 + (Lp)^m} = \lim_{k\to\infty} \sum_{n=0}^{k} (-1)^n (Lp)^{nm} \tag{35}$$

allows directly reading the series coefficients $c_n = (-1)^n L^{nm}$ and determining the convergence radius, as shown in figure 4 as the red shaded region:

$$p_c = \lim \left| \frac{c_{n+1}}{c_n} \right| = \lim \left| \frac{\pm L^{m(n+1)}}{\mp L^{mn}} \right| = L^m. \tag{36}$$

We present here two methods how to deal with the divergence: Regularisation (section 2.1) and another approach of deriving the matter density, inspired by Kempfs Delta representations (section 2.2). In both approaches, we work with the effective one dimensional integral, which is basically the same formalism as in section 1.2. Since the following section, the fourier transformation direction reversed in respect to section 1.2, the effective kernel for an approximation of order k is given by

$$v(p) = -\frac{\Omega_N}{2L^N} \frac{i}{z} q^{N-2} \left[\sum_{n=0}^{k} (-1)^n q^{nm} \Theta(p) + (-1)^{N+1} \sum_{n=0}^{k} (-1)^n (-q)^{nm} \Theta(-p) \right]. \tag{37}$$

with dimensionless units $q = LP$, $z = r/L$ and the shorthand $m = N - 1$. Ω_N collects propertionality factors like 2π which will be neglected here.

2.1 Regularized Fourier transformation

Accepting the fact of a finite convergence radius of the $\mathcal{A}^{-2}$ series expansion motivates to handle the infinities by limiting the fourier transformation (29) in a way that, in 1 dimension, with (37),

$$\int_{-\infty}^{\infty} \left(1 - q^m + q^{2m} + \mathcal{O}(q^3)\right) \mathrm{d}q\ q^{m-1}\ e^{izq} \to \int_{-a}^{a} \left(1 - q^m + q^{2m} + \mathcal{O}(q^3)\right) \mathrm{d}q\ q^{m-1}\ e^{izq}. \tag{38}$$

Identifying $a = L$ would be appealing, but allows no more checking the effects of the truncation, because $a \to \infty$ would then also mean $L \to \infty$. In dimensionless coordinates, as used in eq (38), $a \leq 1$ is the meaningful domain where divergences due to the series truncation are suppressed.

The boxing causes serious difficulties in the interpretation of the results. Consider the most simple case $k = 1$ and $N = 3$, that is, the Dirac Delta

$$\int_{-\infty}^{\infty} e^{i\mathbf{x}\cdot\mathbf{p}} \mathrm{d}^3 p = \delta^3(\mathbf{x}) \tag{39}$$

in the "boxed volume a", more precisely the N-sphere with radius a, called $\mathcal{B}(a)$, gets

$$\int_{\mathcal{B}(a)} e^{i\mathbf{x}\cdot\mathbf{p}} \mathrm{d}^3 p = \frac{\Omega_N}{2L^3} \frac{i}{z} \int_{-a}^{a} q\ e^{ipz} \mathrm{d}q = \frac{\Omega_N}{2L^3} \frac{i}{z^3} \left(\Gamma(2, iaz) - \Gamma(2, -iaz)\right), \tag{40}$$

with the upper gamma function $\Gamma(a, x) = \int_x^\infty t^{a-1} e^{-t} \mathrm{d}t$, which shall be treated in a box with (dimensionless) extend $a \leq 1$, c.f. figure 5. Evventually for $a \to \infty$, equation (40) is a delta approximation.

As we expect for more and more orders k in the series (35) the non-local effects to appear, we expect the first order ($k = 0$) term, as shown in (40), to be the *sharpest* and most-Dirac-like object. But regarding figure 5, the graph of this function already shows wiggling around $T_{00} = 0$. The failure of this Ansatz is already encoded here.

It is easy to compute the $\mathcal{T}_{00}$ terms for the k. order by partial integration. Figure 6 shows some results. Qualitatively, the 2. to 4. order change really few (not to say nothing) on the Delta disaster. We cannot identify the source of nonlocal effects in this figure. This approach tells us *nothing*.

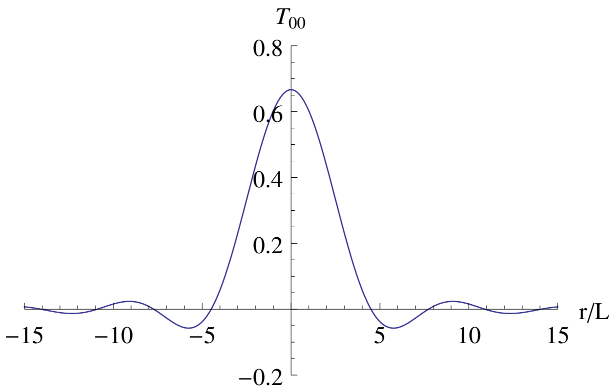

Figure 5: The "sharpest object" in an $a = L$ box: Plot of equation (40).

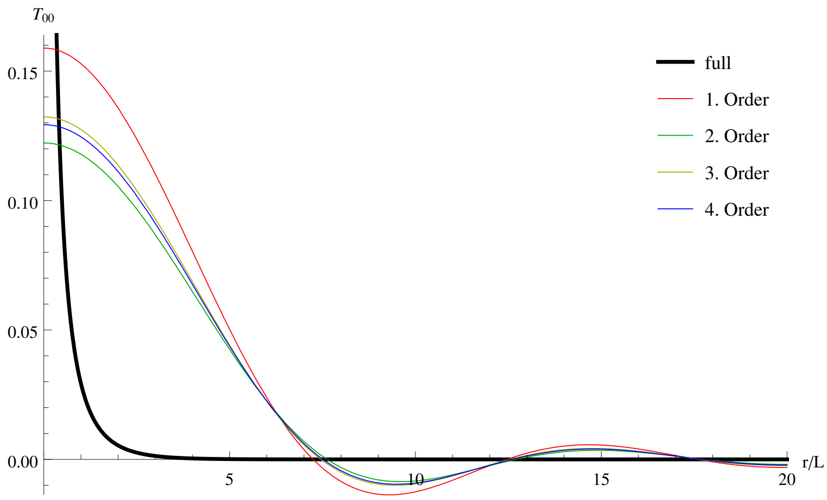

Figure 6: $\mathcal{T}_{00}/(M/L^3)$ in approximations of k. order. The red line corresponds to the dirac delta (fig 5), while the next orders give nonlocal corrections. For $k \to \infty$, the full form as derived in [3] shall be arrived. Actually I didn't compute the full result in the limited volume $a < \infty$, so $k \to \infty$ will clearly be not be identical to the black curve.

2.2 The Kempf representation

In this final section we choose another approach to solve an integral like (33), which follows Kempfs Delta distribution representations [6] from 2014. He proposes writing

$$\delta(x) = \frac{1}{\sqrt{2\pi}} \frac{1}{g(-i\partial_x)} \tilde{g}(x), \tag{41}$$

with a "suitably well-behaved" function $g(x)$ and it's fourier transformed $\tilde{g}(x)$. Note that in the Kempf paper, function arguments are taken literally as *placeholders* (slots), so one must not be confused that $\tilde{g}(x)$ and $\tilde{g}(y)$ describe the same function with different arguments putted in.

The authors of [6] quickly extend their formalism on functionals. In this spirit, we extend it to extra dimensions, where the delta representation reads

$$\delta^N(\mathbf{x}) = \frac{1}{(2\pi)^{N/2}} \frac{1}{g(-i\nabla_{\mathbf{x}})} \tilde{g}(\mathbf{x}) \tag{42}$$

By following Kempfs brave recasting of the dirac delta (41), we get a "new" representation of

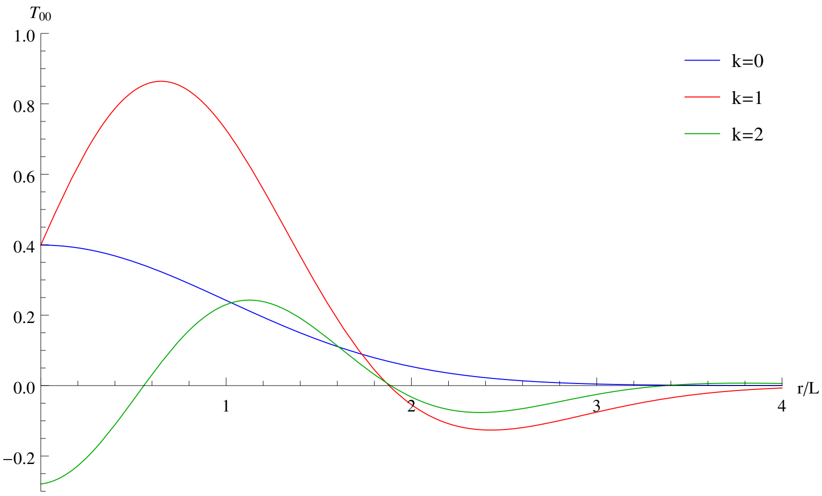

Figure 7: The lower order approximations of $\mathcal{T}_{00}/(M/L^3)$ with the Kempf approach, for $N = 3$, $\sigma = 1$. Smaller σ make a better Delta approximation.

the N-dimensional Fourier transformation

$$\boxed{\tilde{g}(\mathbf{x}) = (2\pi)^{\frac{N}{2}} g(-i\nabla_{\mathbf{x}})\delta^N(\mathbf{x}).} \tag{43}$$

When choosing $g(|\mathbf{p}|) = \mathcal{A}^{-2}(p) = 1/(1 + (pL)^{N-1})$, this looks very much as the well-known starting point for the nonlocal action on the Dirac delta,

$$\mathcal{T}_0^0 = \frac{M}{A_{N-1} r^{N-1}} \mathcal{A}^{-2}(\Box)\delta(\mathbf{x}), \tag{44}$$

where there is a mismatch with $\sqrt{\Box_{\mathbf{x}}} \leftrightarrow -i\nabla_{\mathbf{x}}$. This shall be ignored at this point.

Basically, the authors of [6] propose new methods how to solve an equation like (44): Instead of going to momentum space to get the action of the operator, they propose to solve such equations in pure position space, by letting the differential operator act on a delta representation. This might be the gaussian approximation

$$\delta_\sigma(r) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{r^2}{2\sigma^2}\right), \quad \delta(r) = \lim_{\sigma\to 0} \delta_\sigma(r). \tag{45}$$

Now we can compute the k. matter density approximation by derivation:

$$\mathcal{T}_0^0 = \frac{M}{A_{N-1} r^{N-1}} \frac{1}{\sqrt{2\pi}\sigma} \sum_{n=0}^{k} (-1)^n (-iL)^{mn} \nabla_{\mathbf{x}}^{mn} \exp\left(-\frac{|\mathbf{x}|^2}{2\sigma^2}\right) \tag{46}$$

The $k = 0$ and $k = 1$ results are shown in figure 7. In contrast to the regularisation approach 2.1, this looks probabily more appealing.

Integrating up these results to a unit mass distribution $\mathcal{M} = \int_0^r \mathcal{T}_0^0(r') r'^{N-1} \mathrm{d}r$ and composing the Schwarzschild metric gives figure 8. This is our final result in this document.

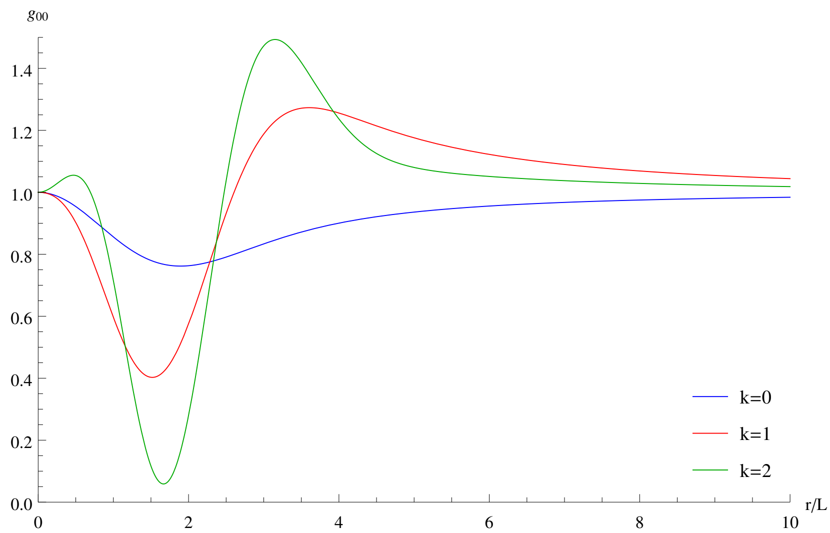

Figure 8: The metric in k. order approximation of the nonlocal operator, using the Kempf approach. This plot is in $N = 4$ dimensions, with "unit mass" $M = 1$ and $\sigma = 1$.

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