Obligatorisk oppgave 3
Ma-103

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Matrise 1

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$$

Egenverdiene: Vi finner først det karakteristiske polynomet.

$$C_x(A) = \det(xI - A) = \det \begin{bmatrix} x-2 & -1 \\ 1 & x-2 \end{bmatrix} = (x-2)^2 + 1 = x^2 - 4x + 5$$

Vi faktoriserer dette utrykket ved å løse $x^2 - 4x + 5 = 0$

$$x = \frac{4 \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot 5}}{2} = \frac{4 \pm \sqrt{-4}}{2} = \frac{4 \pm \sqrt{4(i)^2}}{2} = \frac{4 \pm 2i}{2} = 2 \pm i$$

Vi får da at

$$C_x(A) = \big(x - (2+i)\big)\big(x - (2-i)\big)$$

Egenverdiene til A er da de komplekse tallene $\lambda_1 = 2 + i$ og $\lambda_2 = 2 - i$. Vi finner så matrisens egenvektorer.

Egenrommet E_{2-i}: Vi er her ute etter X som løser ligningen $\big((2-i)I - A\big)X = 0$. Dette gjøres vha. gaussisk eliminasjon:

$$\left[\begin{array}{cc|c} (2-i)-2 & -1 & 0 \\ 1 & (2-i)-2 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} -i & -1 & 0 \\ 1 & -i & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -i & 0 \\ 1 & -i & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Som gir oss ligningene

$$\begin{aligned} x_2 &= s \\ x_1 - ix_2 = 0 \Leftrightarrow x_1 &= is \end{aligned}$$

og vi får løsningen $X = sX_1$, hvor $X_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$, og følgelig $E_{2-i} = \operatorname{span}\{X_1\}$.

Egenrommet E_{2+i}: Vi løser ligningen $\big((2+i)I - A\big)X = 0$ vha gaussisk eliminasjon:

$$\left[\begin{array}{cc|c} (2+i)-2 & -1 & 0 \\ 1 & (2+i)-2 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} i & -1 & 0 \\ 1 & i & 0 \end{array}\right] \to \left[\begin{array}{cc|c} -1 & -i & 0 \\ 1 & -i & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Som gir oss ligningene

$$\begin{aligned} x_2 &= s \\ x_1 + ix_2 = 0 \Leftrightarrow x_1 &= -is \end{aligned}$$

og vi får løsningen $X = sX_2$, hvor $X_2 = \overline{X_1} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$, og følgelig $E_{2+i} = \operatorname{span}\{X_2\}$.

Diagonaliseringsmatrisen P: Egenvektorene er i følge theorem 2§6.2 linært uavhengie, og P er derfor gitt ved $P = \begin{bmatrix} X_1 & X_2 \end{bmatrix} = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$.

Den inverse P^{-1}:

$$\left[\begin{array}{c|c} P & I \end{array}\right] = \left[\begin{array}{cc|cc} i & -i & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} -1 & 1 & i & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 0 & 2 & i & 1 \end{array}\right] \to$$
$$\left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 0 & 1 & \frac{i}{2} & \frac{1}{2} \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & -\frac{i}{2} & \frac{1}{2} \\ 0 & 1 & \frac{i}{2} & \frac{1}{2} \end{array}\right] = \left[\begin{array}{c|c} I & P^{-1} \end{array}\right]$$

Altså har vi at $P^{-1} = \begin{bmatrix} -\frac{i}{2} & \frac{1}{2} \\ \frac{i}{2} & \frac{1}{2} \end{bmatrix}$.

Diagonal matrisen D:

$$D = P^{-1}AP = \begin{bmatrix} -\frac{i}{2} & \frac{1}{2} \\ \frac{i}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -(\frac{1}{2}+i) & 1-\frac{i}{2} \\ -\frac{1}{2}+i & 1+\frac{i}{2} \end{bmatrix} \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} -(\frac{1}{2}+i)i+(1-\frac{i}{2}) & (\frac{1}{2}+i)i+(1-\frac{i}{2}) \\ (-\frac{1}{2}+i)i+(1+\frac{i}{2}) & (\frac{1}{2}+i)i+(1+\frac{i}{2}) \end{bmatrix} = \underline{\underline{\begin{bmatrix} 2-i & 0 \\ 0 & 2+1 \end{bmatrix}}}$$

Matrise 2

$$A = \begin{bmatrix} 10 & -4 & -1 \\ 24 & -10 & -2 \\ 3 & -1 & 2 \end{bmatrix}$$

Egenverdiene: Vi finner først det karakteristiske polynomet.

$$C_x(A) = \det(xI-A) = \det \begin{bmatrix} x-10 & 4 & 1 \\ -24 & x+10 & 2 \\ -3 & 1 & x-2 \end{bmatrix}$$

$$= \det \begin{bmatrix} x-10 & 4 & 1 \\ -24-2(x-10) & x+10-2\cdot 4 & 2-2\cdot 1 \\ -3-(x-2)(x-10) & 1-(x-2)\cdot 4 & (x-2)-(x-2)\cdot 1 \end{bmatrix}$$

$$= \det \begin{bmatrix} x-10 & 4 & 1 \\ -2(x+2) & x+2 & 0 \\ -x^2+12x-23 & -4x+9 & 0 \end{bmatrix} = \det \begin{bmatrix} -2(x+2) & x+2 \\ -x^2+12x-23 & -4x+9 \end{bmatrix}$$

$$= \det \begin{bmatrix} -2(x+2)+2(x+2) & x+2 \\ -x^2+12x-23+2(-4x+9) & -4x+9 \end{bmatrix} = \det \begin{bmatrix} 0 & x+2 \\ -x^2+4x-5 & -4x+9 \end{bmatrix}$$

$$= (x+2)(x^2-4x+5)$$

Vi faktoriserer dette utrykket ved å løse $x^2-4x+5=0$

$$x = \frac{4 \pm \sqrt{(-4)^2-4\cdot 1\cdot 5}}{2} = \frac{4\pm\sqrt{-4}}{2} = \frac{4\pm\sqrt{4(i)^2}}{2} = \frac{4\pm 2i}{2} = 2\pm i$$

Vi får dermed at:

$$C_x(A) = (x+2)\big(x-(2+i)\big)\big(x-(2-i)\big)$$

Egenverdiene til A er da $\lambda_1=-2$, $\lambda_2 = 2+i$ og $\lambda_3 = 2-i$. Vi kan nå finne matrisens egenvektorer.

Egenrommet E_{-2}**:** Vi er her ute etter X som løser ligningen $\big((-2)I-A\big)X=0$:

$$\left[\begin{array}{ccc|c} -2-10 & 4 & 1 & 0 \\ -24 & -2+10 & 2 & 0 \\ -3 & 1 & -2-2 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} -12 & 4 & 1 & 0 \\ -24 & 8 & 2 & 0 \\ -3 & 1 & -4 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} -12 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -3 & 1 & -4 & 0 \end{array}\right]$$
$$\to \left[\begin{array}{ccc|c} 0 & 0 & 17 & 0 \\ 0 & 0 & 0 & 0 \\ -3 & 1 & -4 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 3 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Som gir oss ligningene

$$\begin{aligned} x_2 &= s \\ x_3 &= 0 \\ 3x_1 - x_2 = 0 \Leftrightarrow x_1 &= x_2/3 = s/3 \end{aligned}$$

og vi får løsningen $X = sX_1$, hvor $X_1 = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} s$, og følgelig $E_{-2} = \operatorname{span}\{X_1\}$.

Egenrommet E_{2+i}**:** Vi er her ute etter X som løser ligningen $\big((2+i)I-A\big)X=0$:

$$\left[\begin{array}{ccc|c} (2+i)-10 & 4 & 1 & 0 \\ -24 & (2+i)+10 & 2 & 0 \\ -3 & 1 & (2+i)-2 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} -8+i & 4 & 1 & 0 \\ -24 & 12+i & 2 & 0 \\ -3 & 1 & i & 0 \end{array}\right]$$
$$\to \left[\begin{array}{ccc|c} -8+i & 4 & 1 & 0 \\ -24 & 12+i & 2 & 0 \\ 1 & -1/3 & -i/3 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} -8+i & 4 & 1 & 0 \\ 0 & 12+i+24\cdot\frac{-1}{3} & 2+24\cdot\frac{-i}{3} & 0 \\ 1 & -1/3 & -4i/3 & 0 \end{array}\right]$$
$$= \left[\begin{array}{ccc|c} -8+i & 4 & 1 & 0 \\ 0 & 4+i & 2-8i & 0 \\ 1 & -1/3 & -i/3 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 0 & 4+\frac{-8+i}{3} & 1+\frac{i(-8+i)}{3} & 0 \\ 0 & 4+i & 2-8i & 0 \\ 1 & -1/3 & -i/3 & 0 \end{array}\right]$$
$$\to \left[\begin{array}{ccc|c} 0 & 12-8+i & 3-8i-1 & 0 \\ 0 & 4+i & 2-8i & 0 \\ 1 & -1/3 & -i/3 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 0 & 4+i & 2-8i & 0 \\ 0 & 4+i & 2-8i & 0 \\ 1 & -1/3 & -i/3 & 0 \end{array}\right]$$
$$\to \left[\begin{array}{ccc|c} 0 & 4+i & 2-8i & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -1/3 & -i/3 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 3 & 0 & -i+\frac{2-8i}{4+i} & 0 \\ 0 & 1 & \frac{2-8i}{4+i} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -i & 0 \\ 0 & 1 & -2i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Som gir oss ligningene

$$\begin{aligned} x_3 &= s \\ x_1 - ix_3 = 0 &\Leftrightarrow x_1 = is \\ x_2 - 2ix_3 = 0 &\Leftrightarrow x_2 = 2is \end{aligned}$$

Som gir oss løsningen $X = sX_2$, hvor $X_2 = \begin{bmatrix} i \\ 2i \\ 1 \end{bmatrix} s$, og følgelig $E_{2+i} = \text{span}\{X_2\}$.

Egenrommet E_{2-i}: Vi er her ute etter X som løser ligningen $\big((2-i)I - A\big)X = 0$:

$$\left[\begin{array}{ccc|c} (2-i)-10 & 4 & 1 & 0 \\ -24 & (2-i)+10 & 2 & 0 \\ -3 & 1 & (2-i)-2 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} -8-i & 4 & 1 & 0 \\ -24 & 12-i & 2 & 0 \\ -3 & 1 & -i & 0 \end{array}\right]$$
$$\to \left[\begin{array}{ccc|c} -8-i & 4 & 1 & 0 \\ -24 & 12-i & 2 & 0 \\ 1 & -1/3 & i/3 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 0 & 4+\frac{-8-i}{3} & 1-\frac{i(-8-i)}{3} & 0 \\ -24 & 12-i & 2 & 0 \\ 1 & -1/3 & i/3 & 0 \end{array}\right]$$
$$\to \left[\begin{array}{ccc|c} 0 & 12-8-i & 3+8i-1 & 0 \\ -24 & 12-i & 2 & 0 \\ 1 & -1/3 & i/3 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 0 & 4-i & 2+8i & 0 \\ 0 & 4-i & 2+8i & 0 \\ 1 & -1/3 & i/3 & 0 \end{array}\right]$$
$$\to \left[\begin{array}{ccc|c} 1 & -1/3 & i/3 & 0 \\ 0 & 1 & 2i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & i & 0 \\ 0 & 1 & 2i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Som gir oss ligningene

$$\begin{aligned} x_3 &= s \\ x_1 + ix_3 = 0 &\Leftrightarrow x_1 = -is \\ x_2 + 2ix_3 = 0 &\Leftrightarrow x_2 = -2is \end{aligned}$$

som gir oss løsningen $X = sX_3$, hvor $X_3 = \overline{X_2} = \begin{bmatrix} -i \\ -2i \\ 1 \end{bmatrix} s$, og følgelig $E_{2-i} = \text{span}\{X_3\}$.

Diagonaliseringsmatrisen P: Egenvektorene er i følge theorem 2§6.2 linært uavhengie, og P er derfor gitt ved $P = \begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix} = \begin{bmatrix} 1 & i & -i \\ 3 & 2i & -2i \\ 0 & 1 & 1 \end{bmatrix}$.

Den inverse P^{-1}:

$$\left[\begin{array}{c|c} P & I \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & i & -i & 1 & 0 & 0 \\ 3 & 2i & -2i & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & i & -i & 1 & 0 & 0 \\ 3 & 2i & -2i & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & i & -i & 1 & 0 & 0 \\ 0 & 2i-3i & -2i+3i & -3 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & i & -i & 1 & 0 & 0 \\ 0 & -i & i & -3 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 1 & 0 \\ 0 & -i & i & -3 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 1 & 0 \\ 0 & 1 & -1 & -3i & i & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3i/2 & -i/2 & 1/2 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 1 & 0 \\ 0 & 1 & 0 & -3i/2 & i/2 & 1/2 \\ 0 & 0 & 1 & 3i/2 & -i/2 & 1/2 \end{array}\right] = \left[\begin{array}{c|c} I & P^{-1} \end{array}\right]$$

Altså har vi $P^{-1} = \begin{bmatrix} -2 & 1 & 0 \\ -3i/2 & i/2 & 1/2 \\ 3i/2 & -i/2 & 1/2 \end{bmatrix}$.

Diagonal matrisen D:

$$D = P^{-1}AP = \begin{bmatrix} -2 & 1 & 0 \\ -3i/2 & i/2 & 1/2 \\ 3i/2 & -i/2 & 1/2 \end{bmatrix} \begin{bmatrix} 10 & -4 & -1 \\ 24 & -10 & -2 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & i & -i \\ 3 & 2i & -2i \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -2 & 0 \\ \frac{3}{2}-3i & -\frac{1}{2}+i & 1+\frac{1}{2}i \\ \frac{3}{2}+3i & -\frac{1}{2}-i & 1-\frac{1}{2}i \end{bmatrix} \begin{bmatrix} 1 & i & -i \\ 3 & 2i & -2i \\ 0 & 1 & 1 \end{bmatrix} = \underline{\underline{\begin{bmatrix} -2 & 0 & 0 \\ 0 & 2+i & 0 \\ 0 & 0 & 2-i \end{bmatrix}}}$$

Matrise 3

$$A = \begin{bmatrix} 1-5i & 6 \\ 3 & 1+4i \end{bmatrix}$$

Egenverdiene: Vi finner først det karakteristiske polynomet.

$$C_x(A) = \det(xI - A) = \det \begin{bmatrix} x-1+5i & -6 \\ -3 & x-1-4i \end{bmatrix} = (x-1+5i)(x-1-4i) - 18$$
$$= x^2 - x - 4xi - x + 1 + 4i + 5xi - 5i + 20 - 18 = x^2 - (2-i)x + 3 - i$$

Vi faktoriserer dette utrykket ved å løse $x^2 - (2-i)x + 3 - i = 0$

$$x = \frac{(2-i) \pm \sqrt{\left(-(2-i)\right)^2 - 4 \cdot 1 \cdot (3-i)}}{2} = \frac{(2-i) \pm \sqrt{4 - 4i - 1 - 12 + 4i}}{2}$$
$$= \frac{(2-i) \pm \sqrt{-9}}{2} = \frac{(2-i) \pm 3i}{2} = \begin{cases} 1-2i \\ 1+i \end{cases}$$

Vi får dermed at

$$C_x(A) = \big(x - (1+i)\big)\big(x - (1-2i)\big)$$

Egenverdiene til A er da de komplekse tallene $\lambda_1 = 1+i$ og $\lambda_2 = 1-2i$. Vi finner så matrisens egenvektorer.

Egenrommet E_{1+i}: Vi er her ute etter X som løser ligningen $\big((1+i)I - A\big)X = 0$.

$$\left[\begin{array}{cc|c} 1+i-1+5i & -6 & 0 \\ -3 & 1+i-1-4i & 0 \end{array}\right] = \left[\begin{array}{cc|c} 6i & -6 & 0 \\ -3 & -3i & 0 \end{array}\right] \to \left[\begin{array}{cc|c} i & -1 & 0 \\ i & -1 & 0 \end{array}\right]$$
$$\to \left[\begin{array}{cc|c} i & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Som gir oss ligningene

$$x_2 = s$$
$$ix_1 - x_2 = 0 \Leftrightarrow x_1 = \frac{s}{i} = -s \cdot i$$

og vi får løsningen $X = sX_1$, hvor $X_1 = i \begin{bmatrix} -i \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ i \end{bmatrix}$, og $E_{1+i} = \operatorname{span}\{X_1\}$.

Egenrommet E_{1-2i}: Vi er her ute etter X som løser ligningen $\big((1-2i)I - A\big)X = 0$.

$$\left[\begin{array}{cc|c} 1-2i-1+5i & -6 & 0 \\ -3 & 1-2i-1-4i & 0 \end{array}\right] \to \left[\begin{array}{cc|c} i & -2 & 0 \\ 1 & 2i & 0 \end{array}\right] \to \left[\begin{array}{cc|c} i & -2 & 0 \\ i & -2 & 0 \end{array}\right]$$
$$\to \left[\begin{array}{cc|c} i & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Som gir oss ligningene

$$x_2 = s$$
$$ix_1 - 2x_2 = 0 \Leftrightarrow x_1 = \frac{2}{i}s = -2i \cdot s$$

og vi får løsningen $X = sX_2$, hvor $X_2 = i\begin{bmatrix} -2i \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ i \end{bmatrix}$, og $E_{1-2i} = \text{span}\{X_2\}$.

Diagonaliseringsmatrisen P: Egenvektorene er i følge theorem 2§6.2 linært uavhengie, og P er derfor gitt ved $P = \begin{bmatrix} X_1 & X_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ i & i \end{bmatrix}$.

Den inverse P^{-1}:

$$\left[\begin{array}{c|c} P & I \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ i & i & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & -i \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & i \end{array}\right]$$
$$\to \left[\begin{array}{cc|cc} 1 & 0 & -1 & -2i \\ 0 & 1 & 1 & i \end{array}\right] = \left[\begin{array}{c|c} I & P^{-1} \end{array}\right]$$

Altså har vi at $P^{-1} = \begin{bmatrix} -1 & -2i \\ 1 & i \end{bmatrix}$.

Diagonal matrisen D:

$$D = P^{-1}AP = \begin{bmatrix} -1 & -2i \\ 1 & i \end{bmatrix}\begin{bmatrix} 1-5i & 6 \\ 3 & 1+4i \end{bmatrix}\begin{bmatrix} 1 & 2 \\ i & i \end{bmatrix} = \begin{bmatrix} -1-i & 2-2i \\ 1-2i & 2+i \end{bmatrix}\begin{bmatrix} 1 & 2 \\ i & i \end{bmatrix}$$
$$= \underline{\underline{\begin{bmatrix} 1+i & 0 \\ 0 & 1-2i \end{bmatrix}}}$$