General Facts

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Contents

1 Algebra

Definition 1 (Novikov Ring)**.** The Novikov ring over a base field $\mathbb{K}$ is defined by

$$\Lambda_0 := \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} \middle| a_i \in \mathbb{K}, \lambda_i \in \mathbb{R}_{\geq 0}, \lim_{i\to\infty} \lambda_i = +\infty \right\}.$$

The Novikov field is the field of fractions of Λ_o, i.e.

$$\Lambda := \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} \middle| a_i \in \mathbb{K}, \lambda_i \in \mathbb{R}, \lim_{i\to\infty} \lambda_i = +\infty \right\}.$$

2 Analysis

2.1 Basics

Definition 2 (Equicontinuos Family)**.** Let X be a compact Hausdorff space and $C(X)$ denote the space of real-valued continous funtions on X. A subset $F \subset C(X)$ is called *equicontinous* if for every $x \in X$ and every $\epsilon > 0$ there exists a neighborhood U of x such that for all $y \in U$ and all $f \in F$ we have $|f(y) - f(x)| < \epsilon$.

Definition 3 (Pointwise Bounded Family)**.** A set $F \subset C(X)$ of continous real-valued functions on some compact Hausdorff space X is *pointwise bounded* if for every $x \in X$ we have $\sup\{|f(x)| : f \in F\} < \infty$.

Theorem 1 (Arzéla–Ascoli)**.** Let X be a compact Hausdorff space. Then $F \subset C(X)$ is relatively compact in the topology induced by uniform norm if and only if it is equicontinous and pointwise bounded.

Corollary 1. Consider a sequence of real-valued continous functions $\{f_n\}_{n\in\mathbb{N}}$ defined on a closed and bounded interval $[a, b]$ of the real line. There exists a subsequence of $\{f_n\}$ which converges uniformly if and only if this sequence is uniformly bounded and equicontinous.

2.2 Functional Analysis

Definition 4 (Weak Topologies)**.** Let X be a topological vector space. Then the *weak* topology on X is defined by $x_n \longrightarrow x$ if and only if $\phi(x_n) \longrightarrow \phi(x)\ \forall \phi \in X^*$, where X^* is the topological dual of X, i.e. the space of all continuous linear functionals on X. The *weak-⋆ topology* on X^* is defined by $\phi_n \longrightarrow \phi$ if and only if $\phi_n(x) \longrightarrow \phi(x)\ \forall x \in X$.

Remark 1. The weak-⋆ topology on X^* is weaker than the weak topology on X^*, because in general $X \longrightarrow X^{**}$ is an injective linear map and the weak-⋆ topology is defined as the coarsest topology such that the image of X in X^{**} still consists of continuous maps $X^* \longrightarrow \mathbb{R}$.

Theorem 2 (Open Mapping Theorem)**.** Let X and Y be Banach (or Fréchet) spaces and $A : X \longrightarrow Y$ a surjective continous linear operator. Then A is an open map.

Theorem 3 (Bounded Inverse Theorem)**.** If $A : X \longrightarrow Y$ is a bijective continous linear operator between the Banach spaces X and Y, then the inverse operator $A^{-1} : Y \longrightarrow X$ is continous as well.

Theorem 4 (Closed Graph Theorem)**.** If $A : X \longrightarrow Y$ is a linear operator between the Banach spaces X and Y, and if for every sequence (x_n) in X with $x_n \longrightarrow 0$ and $Ax_n \longrightarrow 0$ it follows that $y = 0$, then A is continous.

Definition 5 (Fréchet Space)**.** A topological vector space X is called a *Fréchet* space if and only if it satisfies one of the following equivalent triples of conditions:

1. X is locally convex, its topology can be induced by a translation invariant metric and it is a complete metric space.

2. X is a Hausdorff space, its topology may be induced by a countable family of semi-norms and it is complete with respect to the family of semi-norms.

Definition 6 (Baire Space)**.** A *Baire space* is a topological space with the property that for each countable collection of open dense sets their intersection is also dense.

Theorem 5 (Baire Category Theorem)**.** 1. Every complete metric space is Baire.

2. Every locally compact Hausdorff space is Baire.

3. A non-empty complete metric space is not the countable union of nowhere-dense closed sets.

Definition 7 (Comeagre or Residual Set)**.**

Definition 8 (Compact Operator)**.**

2.3 Fixed Point Theorems

Theorem 6 (Brouwer Fixed Point Theorem)**.** Every continous function from a convex compact subset K of a Euclidean space to K itself has a fixed point.

Theorem 7 (Schauder Fixed Point Theorem)**.** Every continous function from a convex compact subest K of a Banach space to K itself has a fixed point.

Definition 9 (Contraction Mapping)**.** Let (X, d) be a metric space. Then a map $T : X \longrightarrow X$ is called a contraction mapping on X if there exists a $q \in [0, 1]$ such that $d(T(x), T(y)) \leq qd(x, y)$ for all $x, y \in X$.

Theorem 8 (Banach Fixed Point Theorem)**.** Let (X, d) be an non-empty complete metric space with a contraction mapping $T : X \longrightarrow X$. Then T admits a unique fixed-point $x^* \in X$. Furthermore, x^* can be found as follows: start with an arbitrary element $x_0 \in X$ and define a sequence $\{x_n\}$ by $x_{n+1} = T(x_{n-1})$, then $x_n \longrightarrow x^*$.

Definition 10 (Lefschetz Number)**.** Let $f : X \longrightarrow X$ be a continous map from a compact triangulizable space X to itself. Define the *Lefschetz number* Λ_f by

$$\Lambda_f := \sum_{k \geq k} (-1)^k \operatorname{Tr}(f_*|_{H_k(X,\mathbb{Q})})$$

Theorem 9 (Lefschetz Fixed Point Theorem)**.** If $\Lambda_f \neq 0$ then f has at least one fixed point. Furthermore, if you denote by $i(f, x)$ the index of the fixed point x and if f has only finitely many fixed points, then

$$\sum_{x \in \operatorname{Fix}(f)} i(f, x) = \Lambda_f.$$

2.4 Fredholm Theory

Definition 11 (Fredholm Operator)**.**

Proposition 1 (Properties of Fredholm Operators)**.**

Theorem 10 (Elliptic Regularity)**.**

2.5 Sobolev Spaces

Definition 12 (Sobolev Spaces)**.**

Theorem 11 (Sobolev Embedding Theorems)**.**

3 Topology

Remark 2. A topology τ_1 is called weaker or coarser than τ_2 if τ_1 contains less open sets than τ_2. If $\tau_2 \subset \tau_1$ then τ_1 is called stronger or finer. This means that if a sequence converges in one topology then it also converges in every weaker topology as there are less open sets to test the condition on.

4 Algebraic Topology

Theorem 12 (Alexander Duality)**.**

5 Differential Topology

Definition 13 (Ruled 4-Manifold)**.** A manifold M of dimension 4 is called *ruled* if it is a S^2-bundle over a closed Riemann surface.

Definition 14 (Fibered Knot)**.** A knot $K \subset S^3$ is called *fibered* if there exists a S^1-family F_t with $t \in S^1$ of Seifert surfaces for K such that $F_s \cap F_t = K$ for all $s \neq t$.

Proposition 2. A knot is fibered if and only if it is the binding of some open book decomposition of S^3.

Definition 15 (Heegard Splittings and Diagrams)**.**

6 Riemannian Geometry

6.1 Hypersurfaces

Definition 16 (Shape operator or Weingarten map)**.** Let $S \subset \mathbb{R}^n$ be a smooth hypersurface in Euclidean n-space. Then the *shape operator* or *Weingarten map* S_p is defined by

$$\langle S_p(v), w \rangle = \langle \mathrm{d}\nu(v), w \rangle$$

for all $v, w \in \mathrm{T}_pS$, where $\nu : S \longrightarrow S^{n-1}$ is the Gauss map, i.e. it is given by $\nu(p) = N_p$, where N_p is a normal vector to S at p.

6.2 Hyperbolic Geometry

Definition 17 (Geodesic lamination)**.** A *geodesic lamination* on a complete hyperbolic surface S is a closed subset of S foliated by complete simple geodesics.

Definition 18 (Transversal measures on laminations)**.** A *transversal measure* on a lamination λ is a measure on the collection of arcs on S transversal to λ which is invariant under isotopies of S preserving λ. A *measured lamination* is a pair of a lamination and a transversal measure.

Remark 3. Let $\widetilde{S}_\infty$ be the boundary at infinity of $\mathbb{H}^2$. Then $\mathcal{GL}(S)$ denotes the subset of closed subsets of $\widetilde{S}_\infty \times \widetilde{S}_\infty /\sim$ parametrizing geodesics (and thus laminations) with the Hausdorff topology. It is compact. A geodesic lamination is called *minimal* if every leaf is dense. See [**?**].

7 Riemann Surfaces

8 Fibre Bundles

8.1 Definitions

8.2 Existence

Lemma 1 (Ehresmann's lemma)**.** Let M and N be smooth manifolds and $f : M \longrightarrow N$ a smooth map. If f is a proper surjective submersion then f is a locally trivial fibration.

9 Symplectic Geometry

9.1 Basics

Definition 19 (Hamiltonian Diffeomorphism)**.** A Hamiltonian diffeomorphism $\Psi \in \mathrm{Ham}(M, \omega)$ of a symplectic manifold is a time-one map of a time-dependent Hamiltonian flow.

9.2 Examples

9.3 Lagrangian Submanifolds

Definition 20 (Properties of Lagrangians)**.** Let $L \subset (M, \omega)$ be a Lagrangian submanifold. We call L

- monotone, if there exists a $\tau > 0$ such that $\omega = \tau\mu$, where $\omega : \pi_2(M, L) \longrightarrow \mathbb{R}$ is the symplectic form as a map on $\pi_2(M, L)$ and $\mu : \pi_2(M, L) \longrightarrow \mathbb{Z}$ is the Maslov index,
- exact, if $\omega = \mathrm{d}\lambda$ is exact and if $[\lambda|_L] = 0 \in H^1(L)$,
- displacable, if there exists $\Psi \in \mathrm{Ham}(M, \omega)$ such that $\Psi(L) \cap L = \emptyset$,
- semi-monotone, if .

Definition 21 (Lagrangian Cobordism)**.** Let (M, ω) be a symplectic manifold, denote by $\pi : \mathbb{R}^2 \times M \longrightarrow \mathbb{R}^2$ the projection and equip $\mathbb{R}^2$ with the standard symplectic structure. A *Lagrangian cobordism* $V : (L'_j) \rightsquigarrow (L_i)$ between two families of closed Lagrangian submanifolds $(L_i)_{1 \le i \le k_-}$ and $(L'_j)_{1 \le j \le k_+}$ is a Lagrangian embedding $V \subset [0, 1] \times \mathbb{R} \times M$ such that for some $\epsilon > 0$ we have

$$V \cap \pi^{-1}([0, \epsilon) \times \mathbb{R}) = \coprod_i ([0, \epsilon) \times \{i\}) \times L_i$$
$$V \cap \pi^{-1}([1 - \epsilon, 1) \times \mathbb{R}) = \coprod_j ([1 - \epsilon, 1) \times \{j\}) \times L'_j$$

Definition 22 (Fukaya Category)**.** Let (M, ω) be a symplectic manifold with $2c_1(\mathrm{T}M) = 0$. The objects of the compact Fukaya category $\mathcal{F}(M, \omega)$ are compact, closed, oriented, spin Lagrangian submanifolds $L \subset M$ such that $[\omega]|_{\pi_2(M,L)} = 0$ and vanishing Maslov class $\mu_L = 0 \in H^1(L, \mathbb{Z})$ together with the choice of a spin structure and a graded lift of L.
For every pair of objects (L, L') we choose perturbation data $H_{L,L'} \in C^\infty([0, 1] \times M, \mathbb{R})$ and $J_{L,L'} \in C^\infty([0, 1], \mathcal{J}(M, \omega))$ and for all tuples of objects $(L_0, \dots, L_k)$ and all moduli spaces of discs we choose consistent perturbation data (H, J) compatible with the choices made for the pairs of objects (L_i, L_j) such that we have transversality for alle moduli spaces of perturbed J-holomorphic discs.
We set $\hom(L, L') := CF(L, L'; H_{L,L'}, J_{L,L'})$ and the differential μ^1 and composition μ^2 and higher operations μ^k are given by counts of perturbed J-holomorphic discs with boundary on the k arguments. This makes $\mathcal{F}(M, \omega)$ a Λ-linear, $\mathbb{Z}$-graded, non-unital (but cohomologically unital) A_∞-category.

Definition 23 (Lagrangian Suspension Construction)**.**

Definition 24 (Lagrangian Surgery)**.**

Definition 25 (Symplectic Folding)**.**

Definition 26 (Lagrangian Correspondence)**.**

Example 1 (Lagrangian Correspondences).

9.4 Various Topics

Definition 27 (Symplectic and Stein Cobordism, [**?**])**.** A contact 3-manifold (M_1, ξ_1) is *symplectically* (resp. *Stein*) cobordant to (M_2, ξ_2) if there exists a symplectic (resp. Stein) 4-manifold (X, ω) with $\partial X = M_2 - M_1$ and a vector field V defined on a neighborhood of $M_1 \cup M_2 \subset X$ for which $\mathcal{L}_V \omega = \omega, V \pitchfork M_1 \cup M_2$ and the normal orientation of $M_1 \cup M_2$ agrees with V.

Remark 4. Symplectic and Stein cobordisms are not a symmetric relation, see [**?**].

Definition 28 (Weinstein manifold)**.** A *Weinstein manifold* is a tuple (V, ω, X, ϕ), where (V, ω) is a symplectic manifold , $\phi : V \longrightarrow \mathbb{R}$ is an exhausting Morse funtion and X is a complete Liouville vector field which is gradient-like for ϕ.

Remark 5. A function $\phi : V \longrightarrow \mathbb{R}$ is called *exhausting* if it is proper and bounded from below.

Definition 29 (Stein manifold)**.** The following statements are equivalent for a non-compact complex manifold (V, J):

1. (V, J) admits a proper holomorphic embedding into some $\mathbb{C}^N$
2. V admits an exhausting J-convex function $\phi : V \longrightarrow \mathbb{R}$
3. V is holomorphically convex, $\forall x \in V \exists f_1, \ldots, f_n : V \longrightarrow \mathbb{C}$ holomorphic such that they form a holomorphic coordinate system at x and $\forall x \neq y \in V \exists f : V \longrightarrow \mathbb{C}$ holomorphic s.t. $f(x) \neq f(y)$.

Any complex manifold satisfying one (and thus all) of the above is called a *Stein manifold.*

Definition 30 (Liouville Domain)**.** Let (M, θ) be a compact manifold with boundarz and $\theta \in \Omega^1(M)$ with $\mathrm{d}\theta$ symplectic and $\theta = r\alpha$ close to the boundary, where we identify a neighborhood of the boundary with $[1-\epsilon, 1] \times \partial M$, r is the coordinate in the first factor and α is a contact form on the boundary. Such a $(M\theta)$ is called a *Liouville domain.*

Definition 31 (Open Book Decomposition)**.** An open book (K, θ) of a manifold V^{2n+1} consists of a submanifold $K \subset V$ of codimension 2 with trivial normal bundle and a fibration $\theta : V \setminus K \longrightarrow S^1$ which on a neighborhood $K \times D^2$ of $K \times \{0\}$ is given by the angle coordinate on D^2.

Definition 32. A contact structure ξ on a manifold V^{2n+1} is *supported* by an open book (K, θ) if there exists a contact form such that $\xi = \ker \alpha$ and if

1. α induces a contact form on K,
2. $\mathrm{d}\alpha$ induces a symplectic form on each fibre F of θ and
3. orientation of K induced by α equals the orientation of K as the boundary of $(F, \mathrm{d}\alpha)$.

Remark 6. K is called the *binding*, $\overline{F}^V$ is called the *page.*

Theorem 13 (Giroux)**.** Every contact manifold is supported by by an open book whose fibres are Weinstein.

Theorem 14 (Giroux Correspondence)**.** If M is a closed oriented 3-manifold there is a one-to-one correspondence between

$$\{\text{oriented contact structures on } M \text{ up to isotopy}\}$$
$$\text{and}$$
$$\{\text{open book decompositions of } M \text{ up to positive stabilization}\}.$$

Definition 33 (Positive Stabilization)**.**

Definition 34 (Lefschetz Fibration)**.** Let (V, ω) be a symplectic manifold. A topological Lefschetz fibration is a tuple $(A, \{x_\alpha\}, f)$, where $A \subset V$ is a codimension-2 symplectic submanifold, $x_\alpha \in V \setminus A$ are finitely many points in V and $f : V \setminus A \longrightarrow S^2$ is a submersion on $V \setminus (A \cup \{x_\alpha\}$ and $f(x_\alpha) \neq f(x_\beta)$ for all $\alpha \neq \beta$ which satisfy the following:

1. at each $a \in A$ there exist local compatible complex coordinates z_i such that A is locally defined by $z_1 = z_2 = 0$ and f is given locally by $(z_1, \ldots, z_n) \longmapsto \dfrac{z_1}{z_2} \in \mathbb{C}P^1 \cong S^2$ and
2. at a point x_α there exist local compatible complex coordinates z_i such that f is given locally by $(z_1, \ldots, z_n) \longmapsto f(x_\alpha) + z_1^2 + \cdots + z_n^2$.

Remark 7. A system of local complex coordinates $(z_1, \dots, z_n)$ on a compact symplectic manifold of dimension $2n$ is called compatible if ω is in those coordinates a positive form of type $(1,1)$ at the origin.

Definition 35 (Thimbles or Vanishing Spheres)**.**

Theorem 15 (Lefschetz)**.** Suppose (V, ω) is a compact symplectic manifold such that $[\omega] \in H^2(V; \mathbb{Z})$. For a sufficiently large integer $k \in \mathbb{N}$ there is a topological Lefschetz pencil on V whose fibres are symplectic (outside the singularities) and homologous to k times the Poincaré dual of $[\omega]$.

Theorem 16 (Lefschetz Hyperplane Theorem)**.**

1. If $M \subset \mathbb{C}^N$ is a non-singular affine algebraic variety with real dimension $2k$ then $H_i(M; \mathbb{Z}) \cong 0$ for $i > k$.
2.

Definition 36 (Symplectically Aspherical)**.** A symplectic manifold (M, ω) is called *symplectically aspherical* if for any smooth map $f : S^2 \longrightarrow M$ one has $\int_{S^2} f^* M = 0$ or equivalently $\omega|_{\pi_2(M)} = 0$ or $[\omega]|_{\mathrm{im}(\mathrm{hur}_2)} = 0$, where hur_2 denotes the Hurewicz homomorphism $\pi_2(M) \longrightarrow H_2(M)$.

Definition 37 (Stable Hamiltonian Structure)**.**

Remark 8. Relations of stable Hamiltonian structures to other things

Definition 38 (Dehn Twist)**.**

Definition 39 (Asymptotic Operators)**.**

9.5 J-holomoprhic Curves

Definition 40 (Properties of J-holomorphic Curves)**.** A J-holomorphic curve $u : \Sigma \longrightarrow X$ is called

(i) *simple*

(ii) *somewhere injective*

(iii) *multiply covered*

Remark 9. Relation between properties.

Theorem 17 (Micaleff–White)**.**

Example 2 (Lantern Example)*.*

Theorem 18 (Automatic Transversality)**.**

9.6 Compactness Results

Theorem 19 (Compactness of Morse Gradient Flow Lines)**.**

Theorem 20 (Gromov Compactness)**.**

Theorem 21 (SFT Compactness)**.**

9.7 Conjectures

Conjecture 1 (Conley's Conjecture)**.** A Hamiltonian diffeomorphism of a suitable (e.g. surface, torus, closed symplectically aspherical or cotangent bundle) symplectic manifold has infinitely many simple periodic points.

Conjecture 2 (Weinstein Conjecture)**.** If M is a closed oriented odd-dimensional manifold with a contact form λ then the associated Reeb vector field has a closed orbit.

Conjecture 3 (Arnold Conjecture)**.** A Hamiltonian diffeomorphism on a symplectic manifold M has at least as many fixed points as the minimal number of critical points of a Morse function on M.

Conjecture 4 (Arnold–Givental Conjecture)**.**

9.8 Dynamics

Definition 41 (Types of Orbits)**.** A periodic orbit $\gamma : S^1 \longrightarrow M$ of a flow ϕ_t is called

- nondegenerate, if the linearized flow after one period on a transversal space $\Psi : V \longrightarrow V$ (with $V \subset \mathrm{T}_p M$ such that $V \pitchfork \mathbb{R}\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \phi_t(p)$) has no eigenvalue equal to 1, or equivalently $\det(\Psi - \mathrm{id}) \neq 0$,
- elliptic, if every eigenvalue of Ψ is in the unit circle,
- hyperbolic, if every eigenvalue of Ψ has norm diferrent from 1,
- (un-)stable, if every eigenvalue of Ψ has norm (bigger) smaller than 1.

Definition 42 (Hofer's Metric)**.** Let (m, ω) be a connected symplectic manifold without boundary. Denote by $\mathrm{Ham}^c(M, \omega)$ all Hamiltonian diffeomorphisms with compact support. Given a path $\{\phi_t\}_{0 \leq t \leq 1} \subset \mathrm{Ham}^c(M, \omega)$ and a family of Hamiltonian functions $\{H_t\}$ generating this flow we define

$$\mathcal{L}(\{\phi_t\}) := \int_0^1 \left(\sup_{z \in M} H_t(z) - \inf_{z \in M} H_t(z) \right) \mathrm{d}t.$$

Define the *Hofer metric* on $\mathrm{Ham}^c(M, \omega)$ by

$$\rho(\phi, \psi) := \inf_{\substack{\{\phi_t\} \subset \mathrm{Ham}^c(M,\omega) \\ \phi_0 = \phi, \phi_1 = \psi}} \mathcal{L}(\{\phi_t\}).$$

Definition 43 (Bad Orbits)**.**

Definition 44 (Conley–Zehnder Index)**.**

Proposition 3 (Properties of Conley–Zehnder Index)**.**

9.9 Contact Geometry

Definition 45 (Contact Embedding)**.**

Definition 46 (Symplectic and Stein Fillings)**.**

9.10 Theorems

Theorem 22 (Floer–McDuff–Eliashberg)**.**

9.11 Homology Theories

9.11.1 Quantum Homology

9.11.2 Contact Homology

9.11.3 Cylindrical Contact Homology

9.11.4 Linearized Contact Homology

9.11.5 Embedded Contact Homology

9.11.6 Symplectic Homology

9.11.7 S^1-equivariant Symplectic Homology

9.11.8 Hamiltonian Floer Homology

9.11.9 Lagrangian Floer Homology

9.11.10 Rabinowitz–Floer Homology

9.11.11 Knot-Contact Homology

9.11.12 Instanton–Floer Homology

9.11.13 Khovanov Homology

9.11.14 Heegard–Floer Homology