General Facts

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Contents

1 Algebra

Definition 1 (Novikov Ring)**.** The Novikov ring over a base field $\mathbb{K}$ is defined by

$$\Lambda_0 := \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} \middle| a_i \in \mathbb{K}, \lambda_i \in \mathbb{R}_{\geq 0}, \lim_{i \to \infty} \lambda_i = +\infty \right\}.$$

The Novikov field is the field of fractions of Λ_o, i.e.

$$\Lambda := \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} \middle| a_i \in \mathbb{K}, \lambda_i \in \mathbb{R}, \lim_{i \to \infty} \lambda_i = +\infty \right\}.$$

2 Analysis

2.1 Basics

Definition 2 (Equicontinuos Family)**.** Let X be a compact Hausdorff space and $C(X)$ denote the space of real-valued continous funtions on X. A subset $F \subset C(X)$ is called *equicontinous* if for every $x \in X$ and every $\epsilon > 0$ there exists a neighborhood U of x such that for all $y \in U$ and all $f \in F$ we have $|f(y) - f(x)| < \epsilon$.

Definition 3 (Pointwise Bounded Family)**.** A set $F \subset C(X)$ of continous real-valued functions on some compact Hausdorff space X is *pointwise bounded* if for every $x \in X$ we have $\sup\{|f(x)| : f \in F\} < \infty$.

Theorem 1 (Arzéla–Ascoli)**.** Let X be a compact Hausdorff space. Then $F \subset C(X)$ is relatively compact in the topology induced by uniform norm if and only if it is equicontinous and pointwise bounded.

Corollary 1. Consider a sequence of real-valued continous functions $\{f_n\}_{n \in \mathbb{N}}$ defined on a closed and bounded interval $[a, b]$ of the real line. There exists a subsequence of $\{f_n\}$ which converges uniformly if and only if this sequence is uniformly bounded and equicontinous.

2.2 Functional Analysis

Definition 4 (Weak Topologies)**.** Let X be a topological vector space. Then the *weak* topology on X is defined by $x_n \longrightarrow x$ if and only if $\phi(x_n) \longrightarrow \phi(x)\ \forall \phi \in X^*$, where X^* is the topological dual of X, i.e. the space of all continuous linear functionals on X. The *weak-$\star$ topology* on X^* is defined by $\phi_n \longrightarrow \phi$ if and only if $\phi_n(x) \longrightarrow \phi(x)\ \forall x \in X$.

Remark 1. The weak-$\star$ topology on X^* is weaker than the weak topology on X^*, because in general $X \longrightarrow X^{**}$ is an injective linear map and the weak-$\star$ topology is defined as the coarsest topology such that the image of X in X^{**} still consists of continuous maps $X^* \longrightarrow \mathbb{R}$.

Theorem 2 (Open Mapping Theorem)**.** Let X and Y be Banach (or Fréchet) spaces and $A : X \longrightarrow Y$ a surjective continous linear operator. Then A is an open map.

Theorem 3 (Bounded Inverse Theorem)**.** If $A : X \longrightarrow Y$ is a bijective continous linear operator between the Banach spaces X and Y, then the inverse operator $A^{-1} : Y \longrightarrow X$ is continous as well.

Theorem 4 (Closed Graph Theorem)**.** If $A : X \longrightarrow Y$ is a linear operator between the Banach spaces X and Y, and if for every sequence (x_n) in X with $x_n \longrightarrow 0$ and $Ax_n \longrightarrow 0$ it follows that $y = 0$, then A is continous.

Definition 5 (Fréchet Space)**.** A topological vector space X is called a *Fréchet* space if and only if it satisfies one of the following equivalent triples of conditions:

1. X is locally convex, its topology can be induced by a translation invariant metric and it is a complete metric space.
2. X is a Hausdorff space, its topology may be induced by a countable family of semi-norms and it is complete with respect to the family of semi-norms.

Definition 6 (Baire Space)**.** A *Baire space* is a topological space with the property that for each countable collection of open dense sets their intersection is also dense.

Theorem 5 (Baire Category Theorem)**.** 1. Every complete metric space is Baire.

2. Every locally compact Hausdorff space is Baire.
3. A non-empty complete metric space is not the countable union of nowhere-dense closed sets.

Definition 7 (Comeagre or Residual Set)**.**

Definition 8 (Compact Operator)**.**

2.3 Fixed Point Theorems

Theorem 6 (Brouwer Fixed Point Theorem)**.** Every continous function from a convex compact subset K of a Euclidean space to K itself has a fixed point.

Theorem 7 (Schauder Fixed Point Theorem)**.** Every continous function from a convex compact subest K of a Banach space to K itself has a fixed point.

Definition 9 (Contraction Mapping)**.** Let (X, d) be a metric space. Then a map $T : X \longrightarrow X$ is called a contraction mapping on X if there exists a $q \in [0, 1]$ such that $d(T(x), T(y)) \leq qd(x, y)$ for all $x, y \in X$.

Theorem 8 (Banach Fixed Point Theorem)**.** Let (X, d) be an non-empty complete metric space with a contraction mapping $T : X \longrightarrow X$. Then T admits a unique fixed-point $x^* \in X$. Furthermore, x^* can be found as follows: start with an arbitrary element $x_0 \in X$ and define a sequence $\{x_n\}$ by $x_{n+1} = T(x_{n-1})$, then $x_n \longrightarrow x^*$.

Definition 10 (Lefschetz Number)**.** Let $f : X \longrightarrow X$ be a continous map from a compact triangulizable space X to itself. Define the *Lefschetz number* Λ_f by

$$\Lambda_f := \sum_{k \geq k} (-1)^k \operatorname{Tr}(f_*|_{H_k(X,\mathbb{Q})})$$

Theorem 9 (Lefschetz Fixed Point Theorem)**.** If $\Lambda_f \neq 0$ then f has at least one fixed point. Furthermore, if you denote by $i(f, x)$ the index of the fixed point x and if f has only finitely many fixed points, then

$$\sum_{x \in \operatorname{Fix}(f)} i(f, x) = \Lambda_f.$$

2.4 Fredholm Theory

Definition 11 (Fredholm Operator)**.**

Proposition 1 (Properties of Fredholm Operators)**.**

Theorem 10 (Elliptic Regularity)**.**

2.5 Sobolev Spaces

Definition 12 (Sobolev Spaces)**.**

Theorem 11 (Sobolev Embedding Theorems)**.**

3 Topology

Remark 2. A topology τ_1 is called weaker or coarser than τ_2 if τ_1 contains less open sets than τ_2. If $\tau_2 \subset \tau_1$ then τ_1 is called stronger or finer. This means that if a sequence converges in one topology then it also converges in every weaker topology as there are less open sets to test the condition on.

4 Algebraic Topology

Theorem 12 (Alexander Duality)**.** Let $K \subset S^n$ be a compact, locally contractible, nonempty, proper subspace. Then $\widetilde{H}_i(S^n \setminus K; \mathbb{Z}) \simeq \widetilde{H}^{n-i-1}(K; \mathbb{Z})$ for all i.

5 Differential Topology

Definition 13 (Ruled 4-Manifold)**.** A manifold M of dimension 4 is called *ruled* if it is a S^2-bundle over a closed Riemann surface.

Definition 14 (Fibered Knot)**.** A knot $K \subset S^3$ is called *fibered* if there exists a S^1-family F_t with $t \in S^1$ of Seifert surfaces for K such that $F_s \cap F_t = K$ for all $s \neq t$.

Proposition 2. A knot is fibered if and only if it is the binding of some open book decomposition of S^3.

Definition 15 (Heegard Splittings and Diagrams)**.**

6 Riemannian Geometry

6.1 Hypersurfaces

Definition 16 (Shape operator or Weingarten map)**.** Let $S \subset \mathbb{R}^n$ be a smooth hypersurface in Euclidean n-space. Then the *shape operator* or *Weingarten map* S_p is defined by

$$\langle S_p(v), w \rangle = \langle \mathrm{d}\nu(v), w \rangle$$

for all $v, w \in \mathrm{T}_p S$, where $\nu : S \longrightarrow S^{n-1}$ is the Gauss map, i.e. it is given by $\nu(p) = N_p$, where N_p is a normal vector to S at p.

6.2 Hyperbolic Geometry

Definition 17 (Geodesic lamination)**.** A *geodesic lamination* on a complete hyperbolic surface S is a closed subset of S foliated by complete simple geodesics.

Definition 18 (Transversal measures on laminations)**.** A *transversal measure* on a lamination λ is a measure on the collection of arcs on S transversal to λ which is invariant under isotopies of S preserving λ. A *measured lamination* is a pair of a lamination and a transversal measure.

Remark 3. Let $\widetilde{S}_\infty$ be the boundary at infinity of $\mathbb{H}^2$. Then $\mathcal{GL}(S)$ denotes the subset of closed subsets of $\widetilde{S}_\infty \times \widetilde{S}_\infty /\sim$ parametrizing geodesics (and thus laminations) with the Hausdorff topology. It is compact. A geodesic lamination is called *minimal* if every leaf is dense. See [?].

7 Riemann Surfaces

8 Fibre Bundles

8.1 Definitions

8.2 Existence

Lemma 1 (Ehresmann's lemma)**.** Let M and N be smooth manifolds and $f : M \longrightarrow N$ a smooth map. If f is a proper surjective submersion then f is a locally trivial fibration.

9 Symplectic Geometry

9.1 Basics

Definition 19 (Hamiltonian Diffeomorphism)**.** A Hamiltonian diffeomorphism $\Psi \in \mathrm{Ham}(M, \omega)$ of a symplectic manifold is a time-one map of a time-dependent Hamiltonian flow.

Definition 20 (Various Equivalences of Symplectic Structures)**.**

9.2 Examples

9.3 Lagrangian Submanifolds

Definition 21 (Properties of Lagrangians)**.** Let $L \subset (M, \omega)$ be a Lagrangian submanifold. We call L

- monotone, if there exists a $\tau > 0$ such that $\omega = \tau\mu$, where $\omega : \pi_2(M, L) \longrightarrow \mathbb{R}$ is the symplectic form as a map on $\pi_2(M, L)$ and $\mu : \pi_2(M, L) \longrightarrow \mathbb{Z}$ is the Maslov index,
- exact, if $\omega = \mathrm{d}\lambda$ is exact and if $[\lambda|_L] = 0 \in H^1(L)$,
- displacable, if there exists $\Psi \in \mathrm{Ham}(M, \omega)$ such that $\Psi(L) \cap L = \emptyset$,
- semi-monotone, if .

Definition 22 (Lagrangian Cobordism)**.** Let (M, ω) be a symplectic manifold, denote by $\pi : \mathbb{R}^2 \times M \longrightarrow \mathbb{R}^2$ the projection and equip $\mathbb{R}^2$ with the standard symplectic structure. A *Lagrangian cobordism* $V : (L'_j) \rightsquigarrow (L_i)$ between two families of closed Lagrangian submanifolds $(L_i)_{1 \leq i \leq k_-}$ and $(L'_j)_{1 \leq j \leq k_+}$ is a Lagrangian embedding $V \subset [0, 1] \times \mathbb{R} \times M$ such that for some $\epsilon > 0$ we have

$$\begin{aligned} V \cap \pi^{-1}([0, \epsilon) \times \mathbb{R}) &= \coprod_i ([0, \epsilon) \times \{i\}) \times L_i \\ V \cap \pi^{-1}([1-\epsilon, 1) \times \mathbb{R}) &= \coprod_j ([1-\epsilon, 1) \times \{j\}) \times L'_j \end{aligned}$$

Definition 23 (Fukaya Category)**.** Let (M, ω) be a symplectic manifold with $2c_1(\mathrm{T}M) = 0$. The objects of the compact Fukaya category $\mathcal{F}(M, \omega)$ are compact, closed, oriented, spin Lagrangian submanifolds $L \subset M$ such that $[\omega]|_{\pi_2(M,L)} = 0$ and vanishing Maslov class $\mu_L = 0 \in H^1(L, \mathbb{Z})$ together with the choice of a spin structure and a graded lift of L.
For every pair of objects (L, L') we choose perturbation data $H_{L,L'} \in C^\infty([0, 1] \times M, \mathbb{R})$ and $J_{L,L'} \in C^\infty([0, 1], \mathcal{J}(M, \omega))$ and for all tuples of objects $(L_0, \dots, L_k)$ and all moduli spaces of discs we choose consistent perturbation data (H, J) compatible with the choices made for the pairs of objects (L_i, L_j) such that we have transversality for alle moduli spaces of perturbed J-holomorphic discs.
We set $\hom(L, L') := CF(L, L'; H_{L,L'}, J_{L,L'})$ and the differential μ^1 and composition μ^2 and higher operations μ^k are given by counts of perturbed J-holomorphic discs with boundary on the k arguments. This makes $\mathcal{F}(M, \omega)$ a Λ-linear, $\mathbb{Z}$-graded, non-unital (but cohomologically unital) A_∞-category.

Definition 24 (Lagrangian Suspension Construction)**.**

Definition 25 (Lagrangian Surgery)**.**

Definition 26 (Symplectic Folding)**.**

Definition 27 (Lagrangian Correspondence)**.**

Example 1 (Lagrangian Correspondences).

9.4 Various Topics

Definition 28 (Symplectic and Stein Cobordism, [**?**])**.** A contact 3-manifold (M_1, ξ_1) is *symplectically* (resp. *Stein*) cobordant to (M_2, ξ_2) if there exists a symplectic (resp. Stein) 4-manifold (X, ω) with $\partial X = M_2 - M_1$ and a vector field V defined on a neighborhood of $M_1 \cup M_2 \subset X$ for which $\mathcal{L}_V \omega = \omega, V \pitchfork M_1 \cup M_2$ and the normal orientation of $M_1 \cup M_2$ agrees with V.

Remark 4. Symplectic and Stein cobordisms are not a symmetric relation, see [**?**].

Definition 29 (Weinstein manifold)**.** A *Weinstein manifold* is a tuple (V, ω, X, ϕ), where (V, ω) is a symplectic manifold , $\phi : V \longrightarrow \mathbb{R}$ is an exhausting Morse funtion and X is a complete Liouville vector field which is gradient-like for ϕ.

Remark 5. A function $\phi : V \longrightarrow \mathbb{R}$ is called *exhausting* if it is proper and bounded from below.

Definition 30 (Stein manifold)**.** The following statements are equivalent for a non-compact complex manifold (V, J):

1. (V, J) admits a proper holomorphic embedding into some $\mathbb{C}^N$
2. V admits an exhausting J-convex function $\phi : V \longrightarrow \mathbb{R}$
3. V is holomorphically convex, $\forall x \in V \exists f_1, \ldots, f_n : V \longrightarrow \mathbb{C}$ holomorphic such that they form a holomorphic coordinate system at x and $\forall x \neq y \in V \exists f : V \longrightarrow \mathbb{C}$ holomorphic s.t. $f(x) \neq f(y)$.

Any complex manifold satisfying one (and thus all) of the above is called a *Stein manifold*.

Definition 31 (Liouville Domain)**.** Let (M, θ) be a compact manifold with boundarz and $\theta \in \Omega^1(M)$ with $\mathrm{d}\theta$ symplectic and $\theta = r\alpha$ close to the boundary, where we identify a neighborhood of the boundary with $[1 - \epsilon, 1] \times \partial M$, r is the coordinate in the first factor and α is a contact form on the boundary. Such a $(M\theta)$ is called a *Liouville domain*.

Definition 32 (Open Book Decomposition)**.** An open book (K, θ) of a manifold V^{2n+1} consists of a submanifold $K \subset V$ of codimension 2 with trivial normal bundle and a fibration $\theta : V \setminus K \longrightarrow S^1$ which on a neighborhood $K \times D^2$ of $K \times \{0\}$ is given by the angle coordinate on D^2.

Definition 33. A contact structure ξ on a manifold V^{2n+1} is *supported* by an open book (K, θ) if there exists a contact form such that $\xi = \ker \alpha$ and if

1. α induces a contact form on K,
2. $\mathrm{d}\alpha$ induces a symplectic form on each fibre F of θ and
3. orientation of K induced by α equals the orientation of K as the boundary of $(F, \mathrm{d}\alpha)$.

Remark 6. K is called the *binding*, $\overline{F}^V$ is called the *page*.

Theorem 13 (Giroux)**.** Every contact manifold is supported by by an open book whose fibres are Weinstein.

Theorem 14 (Giroux Correspondence)**.** If M is a closed oriented 3-manifold there is a one-to-one correspondence between

$$\{\text{oriented contact structures on } M \text{ up to isotopy}\}$$
$$\text{and}$$
$$\{\text{open book decompositions of } M \text{ up to positive stabilization}\}.$$

Definition 34 (Positive Stabilization)**.**

Definition 35 (Lefschetz Fibration)**.** Let (V, ω) be a symplectic manifold. A topological Lefschetz fibration is a tuple $(A, \{x_\alpha\}, f)$, where $A \subset V$ is a codimension-2 symplectic submanifold, $x_\alpha \in V \setminus A$ are finitely many points in V and $f : V \setminus A \longrightarrow S^2$ is a submersion on $V \setminus (A \cup \{x_\alpha\}$ and $f(x_\alpha) \neq f(x_\beta)$ for all $\alpha \neq \beta$ which satisfy the following:

1. at each $a \in A$ there exist local compatible complex coordinates z_i such that A is locally defined by $z_1 = z_2 = 0$ and f is given locally by $(z_1, \dots, z_n) \longmapsto \dfrac{z_1}{z_2} \in \mathbb{C}P^1 \cong S^2$ and

2. at a point x_α there exist local compatible complex coordinates z_i such that f is given locally by $(z_1, \dots, z_n) \longmapsto f(x_\alpha) + z_1^2 + \dots + z_n^2$.

Remark 7. A system of local complex coordinates $(z_1, \dots, z_n)$ on a compact symplectic manifold of dimension $2n$ is called compatible if ω is in those coordinates a positive form of type $(1,1)$ at the origin.

Definition 36 (Thimbles or Vanishing Spheres).

Theorem 15 (Lefschetz). Suppose (V, ω) is a compact symplectic manifold such that $[\omega] \in H^2(V; \mathbb{Z})$. For a sufficiently large integer $k \in \mathbb{N}$ there is a topological Lefschetz pencil on V whose fibres are symplectic (outside the singularities) and homologous to k times the Poincaré dual of $[\omega]$.

Theorem 16 (Lefschetz Hyperplane Theorem).

1. If $M \subset \mathbb{C}^N$ is a non-singular affine algebraic variety with real dimension $2k$ then $H_i(M; \mathbb{Z}) \cong 0$ for $i > k$.

2.

Definition 37 (Symplectically Aspherical). A symplectic manifold (M, ω) is called *symplectically aspherical* if for any smooth map $f : S^2 \longrightarrow M$ one has $\displaystyle\int_{S^2} f^* M = 0$ or equivalently $\omega|_{\pi_2(M)} = 0$ or $[\omega]|_{\mathrm{im}(\mathrm{hur}_2)} = 0$, where hur_2 denotes the Hurewicz homomorphism $\pi_2(M) \longrightarrow H_2(M)$.

Definition 38 (Stable Hamiltonian Structure).

Remark 8. Relations of stable Hamiltonian structures to other things

Definition 39 (Dehn Twist).

Definition 40 (Asymptotic Operators).

9.5 J-holomorphic Curves

Definition 41 (Properties of J-holomorphic Curves). A J-holomorphic curve $u : \Sigma \longrightarrow X$ is called

(i) *simple*

(ii) *somewhere injective*

(iii) *multiply covered*

Remark 9. Relation between properties.

Theorem 17 (Micaleff–White).

Example 2 (Lantern Example).

Theorem 18 (Automatic Transversality).

9.6 Compactness Results

Theorem 19 (Compactness of Morse Gradient Flow Lines).

Theorem 20 (Gromov Compactness).

Theorem 21 (SFT Compactness).

9.7 Conjectures

Conjecture 1 (Conley's Conjecture)**.** A Hamiltonian diffeomorphism of a suitable (e.g. surface, torus, closed symplectically aspherical or cotangent bundle) symplectic manifold has infinitely many simple periodic points.

Conjecture 2 (Weinstein Conjecture)**.** If M is a closed oriented odd-dimensional manifold with a contact form λ then the associated Reeb vector field has a closed orbit.

Conjecture 3 (Arnold Conjecture)**.** A Hamiltonian diffeomorphism on a symplectic manifold M has at least as many fixed points as the minimal number of critical points of a Morse function on M.

Conjecture 4 (Arnold–Givental Conjecture)**.**

9.8 Dynamics

Definition 42 (Types of Orbits)**.** A periodic orbit $\gamma : S^1 \longrightarrow M$ of a flow ϕ_t is called

- nondegenerate, if the linearized flow after one period on a transversal space $\Psi : V \longrightarrow V$ (with $V \subset \mathrm{T}_p M$ such that $V \pitchfork \mathbb{R}\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \phi_t(p)$) has no eigenvalue equal to 1, or equivalently $\det(\Psi - \mathrm{id}) \neq 0$,
- elliptic, if every eigenvalue of Ψ is in the unit circle,
- hyperbolic, if every eigenvalue of Ψ has norm diferrent from 1,
- (un-)stable, if every eigenvalue of Ψ has norm (bigger) smaller than 1.

Definition 43 (Hofer's Metric)**.** Let (m, ω) be a connected symplectic manifold without boundary. Denote by $\mathrm{Ham}^c(M, \omega)$ all Hamiltonian diffeomorphisms with compact support. Given a path $\{\phi_t\}_{0 \leq t \leq 1} \subset \mathrm{Ham}^c(M, \omega)$ and a family of Hamiltonian functions $\{H_t\}$ generating this flow we define

$$\mathcal{L}(\{\phi_t\}) := \int_0^1 \left(\sup_{z \in M} H_t(z) - \inf_{z \in M} H_t(z) \right) \mathrm{d}t.$$

Define the *Hofer metric* on $\mathrm{Ham}^c(M, \omega)$ by

$$\rho(\phi, \psi) := \inf_{\substack{\{\phi_t\} \subset \mathrm{Ham}^c(M,\omega) \\ \phi_0 = \phi, \phi_1 = \psi}} \mathcal{L}(\{\phi_t\}).$$

Definition 44 (Objects for Conley–Zehnder Index)**.**

- Let $(V\omega)$ be a symplectic vector space and $\mathrm{Sp}(V)$ the symplectic group. Then for $A \in \mathrm{Sp}(V)$ the graph $\Gamma_A \subset (V \oplus V, \omega \oplus -\omega)$ is Lagrangian, in particular the diagonal $\Delta = \Gamma_{\mathrm{id}}$ is Lagrangian.
- Let $\lambda : [0,1] \longrightarrow \Lambda$ be a smooth path of Lagrangian subspaces in $(\mathbb{R}^{2n}, \omega_{\mathrm{std}})$ and $L \in \Lambda$. Define the crossing form $\Gamma(\lambda, L, t) := Q^{\dot{\lambda}(t)}|_{\lambda(t) \cap L}$ which is a quadratic form on the vector space $\lambda(t) \cap L$.
- $t \in \lambda^{-1}(\bigcup_{k=1}^{n} \Lambda_L^k)$ is called a *regular crossing* if and only if $\Gamma(\lambda, L, t)$ is nonsingular.
- If $\lambda : [0,1] \longrightarrow \Lambda$ has only regular crossings define
$$\mu_L(\lambda) := \frac{1}{2} \operatorname{sign} \Gamma(\lambda, L, 0) + \sum_{0 < t < 1} \operatorname{sign} \Gamma(\lambda, L, t) + \frac{1}{2} \operatorname{sign} \Gamma(\lambda, L, 1) \in \frac{1}{2}\mathbb{Z}.$$
- $\Psi : [0,1] \longrightarrow \mathrm{Sp}(V)$ is called *nondegenerate* if and only if $\det(\Psi(1) - \mathrm{id}) \neq 0$.
- Let $L_0 \in \Lambda$ be a Lagrangian. Then $\Lambda_{L_0}^k := \{L \in \Lambda : \dim(L \cap L_0) = k\}$ is a codimension $\frac{k(k+1)}{2}$ submanifold of Λ.

- Let $L_1 \in \Lambda^0_{L_2}$. Then there exists a map $\Lambda^0_{L_2} \longrightarrow S^2(L_1)$ given by $L \longmapsto Q^{L_1,L_2}_L$ defined by $Q^{L_1,L_2}_L(v) := \omega(v, w_v)$ where for $L \in \Lambda^0_{L_2}$ and $v \in L_1, w_v \in L_2$ is the unique vector such that $v + w_v \in L$. This gives a vector space isomorphism $\mathrm{T}_{L_0}\Lambda \longrightarrow S^2(L_0)$ by mapping $L \longmapsto Q^{L_0,L_1}_L$ which is in fact independent of L_1. Also the coorientation of $\Lambda^1 \in \Lambda$ is well-defined. Therefore for $L \in \Lambda$ and $\widehat{L} \in \mathrm{T}_L\Lambda$ we obtain $Q^{\widehat{L}} \in S^2(L)$ which is well-defined.

Definition 45 (Conley–Zehnder Index)**.** Let $\Psi : [0,1] \longrightarrow \mathrm{Sp}(V)$ be a smooth nondegenerate path of symplectic transformations such that $\Psi(0) = \mathrm{id}$. Then

$$\mu_{\mathrm{CZ}}(\Psi) := \mu_\Delta(\Gamma_\Psi).$$

Proposition 3 (Properties of Conley–Zehnder Index)**.** The Conley–Zehnder Index for paths of Lagrangians $\lambda_i : [0,1] \longrightarrow \Lambda$ satisfies

Invariance: λ_0 and λ_1 homotopic with fixed endpoints $\Longrightarrow \mu_L(\lambda_0) = \mu_L(\lambda_1)$

Concatenation: $\lambda_0(1) = \lambda_1(0) \Longrightarrow \mu_L(\lambda_0 \# \lambda_1) = \mu_L(\lambda_0) + \mu_L(\lambda_1)$

Loop: $\lambda(0) = \lambda(1) \Longrightarrow \mu_L(\lambda) = \mu(\lambda)$ is the Maslov index.

Definition 46 (Bad Orbits)**.** Let (N, ξ) be a contact manifold and $\gamma : S^1 \longrightarrow N$ a simple periodic orbit for α such that $\xi = \ker \alpha$. γ^{2k} for $k > 0$ is called *bad* if for any (and thus all) $i > 0$ $\mathrm{CZ}(\gamma^{2i})$ and $\mathrm{CZ}(\gamma^{2i-1})$ have different parities.

Definition 47 (Asymptotic Operators)**.**

9.9 Contact Geometry

Definition 48 (Contactomorphism)**.** Two contact manifolds (N_1, ξ_1) and (N_2, ξ_2) are called *contactomorphic* if there exists a diffeomorphism $f : N_1 \longrightarrow N_2$ such that $f_*(\xi_1) = \xi_2$ or for contact forms α_i such that $\xi_i = \ker \alpha_i$ there exists a nowhere zero function $\lambda : N_1 \longrightarrow \mathbb{R}$ such that $f^*\alpha_2 = \lambda\alpha_1$.

Definition 49 (Contact Hamiltonian)**.** Let (N, α) be a contact manifold and $H : N \longrightarrow \mathbb{R}^+$ a smooth function. Then we define the *contact Hamiltonian* X_H of H by

$$\alpha(X_H) = H \text{ and } i_{X_H}\mathrm{d}\alpha = \mathrm{d}H(R_\alpha)\alpha - \mathrm{d}H.$$

Definition 50 (Contact Embedding)**.** An embedding $j : (A, \eta) \longrightarrow (N, \xi)$ is called a *contact embedding* if the image of j is a contact submanifold, i.e. $\mathrm{T}j(A) \cap \xi|_{j(A)} = \mathrm{d}j(\eta)$, or for α such that $\xi = \ker \alpha$ we have $\ker j^*\alpha = \eta$.

Definition 51 (Symplectic and Stein Fillings)**.**

Definition 52 (Contact Isotopy)**.** Denote by $\mathrm{Cont}_0(N, \xi)$ the space of all contactomorphisms of (N, ξ) which are contact isotopic to the identity, i.e. $\psi \in \mathrm{Cont}_0(N, \xi)$ if and only if $\exists \psi_t : N \longrightarrow N : \psi_t^*\alpha = \lambda_t\alpha$ for some smooth family of functions $\lambda_t : N \longrightarrow \mathbb{R}^+$ and a contact form α such that $\xi = \ker \alpha$ and such that $\psi_0 = \mathrm{id}$ and $\psi_1 = \psi$.

Definition 53 (Translated Points)**.** Let $\phi \in \mathrm{Cont}_0(N, \xi)$ be a contactomorphism contact isotopic to the identity and α a contact form such that $\xi = \ker \alpha$. A point $x \in N$ is called a *translated point of ϕ with respect to α* if there exists $t \in \mathbb{R}$ such that $\phi(x) = \phi^t_{R_\alpha}(x)$ and $(\phi^*\alpha)_x = \alpha_x$.

Definition 54 (Hofer Energy of J-holomorphic Cylinders in the Symplectization)**.** Consider the symplectization $(\mathbb{R} \times N, \mathrm{d}(e^t\lambda))$ of a contact manifold $(N, \xi = \ker \lambda)$. Then define $\Sigma := \{\phi \in C^\infty(\mathbb{R}, [\frac{1}{2}, 1]) \mid \phi' \geq 0\}$. Now define the *Hofer energy* of a map $u : \mathbb{C} \longrightarrow \mathbb{R} \times N$ by

$$E_\Sigma(u) := \sup_{\phi \in \Sigma} \int_{\mathbb{C}} u^*\mathrm{d}(\phi\lambda).$$

9.10 Theorems

Theorem 22 (Floer–McDuff–Eliashberg)**.** Let (Z, ω) be an asymptotically flat symplectic manifold, i.e. outside of a compact subset it is symplectomorphic to a neighborhood of infinity in $(\mathbb{R}^{2n}, \omega_{\text{std}})$ and such that $H_2(Z, \mathbb{R}) = 0$. Then Z is diffeomorphic to $\mathbb{R}^{2n}$.

Remark 10. The diffeomorphism from Theorem 22 does not need to extend the given symplectomorphism.

9.11 Homology Theories

9.11.1 Quantum Homology

9.11.2 Contact Homology

9.11.3 Cylindrical Contact Homology

9.11.4 Linearized Contact Homology

9.11.5 Embedded Contact Homology

9.11.6 Symplectic Homology

9.11.7 S^1-equivariant Symplectic Homology

9.11.8 Hamiltonian Floer Homology

9.11.9 Lagrangian Floer Homology

9.11.10 Rabinowitz–Floer Homology

9.11.11 Knot-Contact Homology

9.11.12 Instanton–Floer Homology

9.11.13 Khovanov Homology

9.11.14 Heegard–Floer Homology