MLPM Tutorial 1

October 8, 2019

1. Prove

   (a)
   $$p(x, y|z) = p(x|z)p(y|x, z)$$

   (b)
   $$p(x|y, z) = \frac{p(y|x, z)p(x|z)}{p(y|z)}$$

   (c)
   $$p(x|z) = \sum_y p(x|y, z)p(y|z) = \sum_{y,w} p(x|w, y, z)p(w|y, z)p(y|z)$$

   (d)
   $$\sum_x p(x|y)f(y) = f(y)$$
   for any function $f(y)$. Why
   $$\sum_x p(x|y)f(x, y) = \sum_x f(x, y)$$
   is **not** true in general?

2. A secret government agency has developed a scanner which determines whether a person is a terrorist. The scanner is fairly reliable; 95% of all scanned terrorists are identified as terrorists, and 95% of all non-terrorists are identified as such. An informant tells the agency that exactly one passenger of 100 aboard an aeroplane which you are seated in is a terrorist. The police arrests the one person which the scanner tested positive. What is the probability that the arrested person is in fact a terrorist?

3. The weather in London can be summarised as: if it rains one day there's a 70% chance it will rain the following day; if it's sunny one day there's a 40% chance it will be sunny the following day.

   (a) Assuming that the prior probability it rained yesterday is 0.5, what is the probability that it was raining yesterday given that it's sunny today?

   (b) If the weather follows the same pattern as above, day after day, what is the probability that it will rain on any day (based on an effectively infinite number of days of observing the weather)?

   (c) Use the result from part 2 above as a new prior probability of rain yesterday and recompute the probability that it was raining yesterday given that it's sunny today.

4. Consider a sphere of radius a in D-dimensions together with the concentric hypercube of side $2a$, so that the sphere touches the hypercube at the centres of each of its sides. By using the fact that the volume of a D-dimensional sphere with radius R is given by

$$V_S = \frac{\pi^{D/2}}{\Gamma(\frac{D}{2}+1)} R^D$$

   where Γ is the gamma function, and the approximation (Stirling's formula)

$$\Gamma(x+1) \approx (2\pi)^{1/2} e^{-x} x^{x+1/2}$$

   which is valid for $x >> 1$, find

$$\lim_{D\to\infty} \frac{V_S}{V_C} = ?$$

   What is the ratio of the distance from the centre of the hypercube to one of the corners, divided by the distance to one of the sides of the cube?

   How are these results related to the curse of dimensionality?