Advanced Statistical Methods.
P4-E1-K3.
Pointwise derivative estimation

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github.com/sverdoot/pointwise-derivative-estimation

Data generation

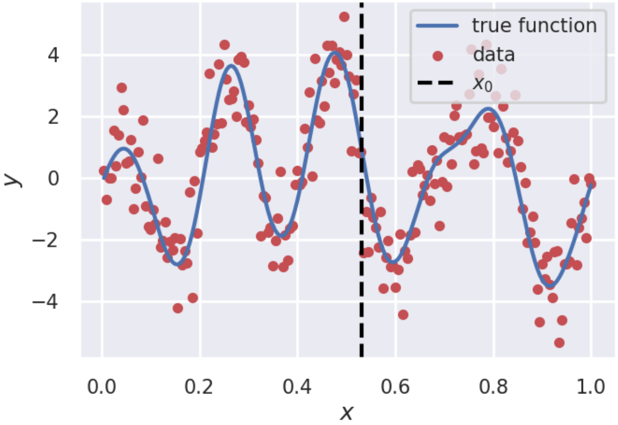

Figure 1: Function and sampled data

$$x_0 = 0.532$$

Local polynomial smoothing

$$K(x) = \exp(-x^2/2), \quad w_i(x) = K\left(\frac{1}{h}(X_i - x)\right)$$

$$\begin{aligned}
\tilde{\theta}(x) &= \mathcal{S}(x)\mathbf{Y} \\
S(x) &= \{\Psi(x)W(x)\Psi(x)^\top\}^{-1}\Psi(x)W(x) \\
W(x) &= \mathrm{diag}(\{w_i(x)\}_{i=1}^n) \\
\Psi(x) &= \begin{pmatrix} 1 & \cdots & 1 \\ (X_1 - x) & \cdots & (X_n - x) \end{pmatrix}
\end{aligned}$$

Unbiased risk estimation

$$\hat{\mathcal{R}}_m = \|\mathcal{K}_m \mathbf{Y} - \mathbf{Y}\|^2 + 2\sigma^2 \operatorname{tr}(\mathcal{K}_m)$$

$$\mathcal{K}_m = \begin{pmatrix} (\mathcal{S}(X_1))_0 \\ \cdots \\ (\mathcal{S}(X_n))_0 \end{pmatrix}$$

Unbiased risk estimation

Bandwidth	Function estimate	True value	Derivative estimate	True value	Unbiased risk
0.00028	0.196	0.898	-101	-86	225

Table 1: Result for chosen bandwidth

True risk

$$\mathcal{R}(\hat{f}) = \mathbb{E}((f^*)'(x_0) - \hat{f}'(x_0))^2 = \mathbb{E}((f^*)'(x_0) - S\left(\mathcal{S}(x_0)\right)_1^\top \mathbf{Y})^2 =$$
$$\mathbb{E}\left((f^*)'(x_0) - \left(\mathcal{S}(x_0)\right)_1^\top (\mathbf{f} + \boldsymbol{\varepsilon})\right)^2 =$$
$$\left((f^*)'(x_0) - \left(\mathcal{S}(x_0)\right)_1^\top \mathbf{f}\right)^2 + \left(\mathcal{S}(x_0)\right)_1^\top \mathbb{E}\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^\top \left(\mathcal{S}(x_0)\right)_1 -$$
$$2(f^*)'(x_0)\left(\mathcal{S}(x_0)\right)_1^\top (x_0)\mathbb{E}\boldsymbol{\varepsilon} =$$
$$\left((f^*)'(x_0) - \left(\mathcal{S}(x_0)\right)_1^\top \mathbf{f}\right)^2 + \sigma^2 \| \left(\mathcal{S}(x_0)\right)_1 \|^2$$

True risk

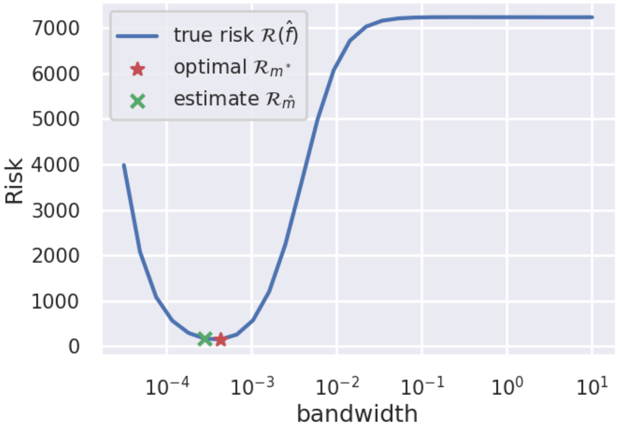

Figure 2: Dependence of true risk on bandwidth.

Comparison

Oracle bandwidth	Oracle risk	Chosen bandwidth	Estimate risk
0.00043	144	0.00028	163

Table 2: Comparison with oracle