Problem 1b

From the table in a)

$$Z = \sum_s e^{-\beta E(s)} = e^{8\beta J} + 4 + 2e^{-8\beta J} + 4 + 4 + e^{8\beta J}$$
$$= 2e^{8\beta J} + 2e^{-8\beta J} + 12$$
$$= 4\cosh(8\beta J) + 12$$

Notice that $\langle E \rangle = \sum_s E(s)\, p(s:1)$

$$= \sum_s \frac{E(s)}{Z} e^{-\beta E(s)}$$
$$= -\frac{1}{Z} \sum_s \frac{\partial}{\partial \beta} e^{-\beta E(s)}$$
$$= -\frac{1}{Z} \frac{\partial}{\partial \beta} \sum_s e^{-\beta E(s)}$$
$$= -\frac{1}{Z} \frac{\partial}{\partial \beta} Z = -8 \frac{J \sinh(8\beta J)}{\cosh(8\beta J) + 3}$$

Hence

$$\langle \varepsilon \rangle = \langle \frac{E}{N} \rangle = \frac{\langle E \rangle}{N} = -2 \frac{J \sinh(8\beta J)}{\cosh(8\beta J) + 3}$$

$$\langle \varepsilon^2 \rangle = \frac{1}{N^2 Z} \frac{\partial^2}{\partial \beta^2} Z = -4J^2 \frac{\cosh(8\beta J)}{\cosh(8\beta J) + 3}$$

$$\langle |m| \rangle = \frac{\langle |M| \rangle}{N} = \frac{1}{NZ} \sum_s |M(s)| e^{-\beta E(s)}$$

$$= \frac{1}{4Z}\left(4e^{8\beta J} + 8 + 8 + 4e^{8\beta J}\right)$$

$$= \frac{2e^{8\beta J} + 4}{4\cosh(8\beta J) + 12} = \frac{e^{8\beta J} + 2}{2\cosh(8\beta J) + 6}$$

$$\langle m^2 \rangle = \frac{\langle M^2 \rangle}{N^2} = \frac{1}{N^2 Z} \sum_s M(s)^2 e^{-\beta E(s)}$$

$$= \frac{1}{16Z}\left(16e^{8\beta J} + 16 + 16 + 16e^{8\beta J}\right)$$

$$= \frac{e^{8\beta J} + 1}{2\cosh(8\beta J) + 6}$$

$$E_{\vec{s}} = E_{ext} + \left(-J\sum_{j=1}^{4} s_i s_j\right)$$

$$\Delta E = E_{after} - E_{before}$$

$$= E_{ext} - J\sum_{j=1}^{4} s_{after} s_j - E_{ext} + J\sum_{j=1}^{4} s_{before} s_j$$

$$= J\left((-s_{after} + s_{before})\, 2\sum_{j=1}^{4} s_j\right), \quad s_{after} = -s_{before}$$

$$= 2 s_{before} \sum_{j=1}^{4} s_j$$

Mapping: $\Delta E/4 + 2$