I. THE EQUATIONS OF MOTION

$$m\ddot{v} = \sum_i \mathbf{F}_i$$

This gives us:

$$\Rightarrow m \begin{bmatrix} \ddot{\mathbf{x}} \\ \ddot{\mathbf{y}} \\ \ddot{z} \end{bmatrix} = \sum_i \begin{bmatrix} \mathbf{F}_x \\ \mathbf{F}_y \\ F_z \end{bmatrix} = \sum_i$$

$$m\ddot{x} = \sum_i F_{x,i}$$

Simple particle:

$$\Rightarrow \begin{bmatrix} \dot{x} \\ \dot{y} \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ B_0 \end{bmatrix} = \begin{bmatrix} B_0\dot{y} \\ B_0\dot{x} \\ 0 \end{bmatrix}$$

$$\begin{aligned} m\ddot{x} &= F_x \\ &= qE_x + (q\mathbf{V} \times B)_x \\ &= -q\frac{\partial v}{\partial x} + q \begin{bmatrix} \dot{x} \\ \dot{y} \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ B_0 \end{bmatrix} \quad = -q\left(-\frac{V_0}{d^2}x\right) + qB_0\ddot{y} \end{aligned}$$

$$\ddot{x} - \frac{qB_0}{m}\dot{y} - \frac{qV_0}{md^2}x = 0$$

$$\Rightarrow \ddot{x} - \omega_0\dot{y} - \frac{1}{2}\omega_z^2 x = 0$$

$$\begin{aligned} m\ddot{y} &= F_y \\ &= qE_y + (q\mathbf{V} \times B)_y \\ &= -q\frac{\partial v}{\partial y} - qB_o\dot{x} \end{aligned}$$

$$\Rightarrow \ddot{y} + \frac{qB_0}{m}\dot{x} - \frac{qV_0}{md^2} = 0$$

$$m\ddot{z} = qE_z = -q\frac{\partial v}{\partial z} = -\frac{4qv_0}{2d^2}z$$

$$\Rightarrow \ddot{x} + w_z^2 = 0$$

$$z = A\cos(\omega_z z) + B\sin(\omega_z z) \quad \square$$

II. SINGLE DIFFERENTIAL EQUATION

$$m\ddot{x} - \omega_0\dot{y} - \frac{1}{2}w_z^2 x + i(\ddot{y} + \omega_z^2) = 0$$

$$\Rightarrow \ddot{x} + i\ddot{y} + iw_0 dotx - \omega_0\dot{y} - \frac{1}{2}\omega_z^2(x + iy) = 0$$

$$\Rightarrow \ddot{f} + i\omega_0(\dot{x} + i\dot{y}) - \frac{1}{2}\omega_z^2 f = 0$$

$$\Rightarrow \ddot{f} + i\omega_0\dot{f} - \frac{1}{2}\omega_z^2 f = 0 \quad \square$$

III. NECESSARY CONSTRAINT ON ω_0 AND ω_z

Assuming $\omega\pm \in \mathbb{R}$:

$$\begin{aligned} &\Rightarrow \omega_0^2 - 2\omega_z^2 > 0 \\ &\Rightarrow \omega_0^2 > 2\omega z^2 \\ &\Rightarrow \left(\frac{qB_0}{m}\right)^2 > \left(\frac{2qv_0}{md^2}\right) \\ &\Rightarrow \frac{q}{m}B_0^2 > \frac{4v_0}{d^2} \quad \square \end{aligned}$$

IV. LOWER BOUNDS

$$A_+(\cos\omega_+ t + i\sin\omega_+ t) + A_-(\cos\omega_- t + i\sin\omega_- t)$$

$$\mathrm{Re}\,(f(t)) + A_+\cos\omega_+ t + A_-\cos\omega_- t$$

$$\mathrm{Im}\,(f(t)) + A_+\sin\omega_+ t + A_-\sin\omega_- t$$

We are getting upper bounds by when $x(t)$ and $y(t)$ are in upper phase:

$$R_+ = A_+ + A_-$$

and lower bounds when they are in opposite phase:

$$R_- = |A_+ + A_-|$$

V. SPECIFIC SOLUTION OF $z(t)$

$$x(0) = \mathrm{Re}\left(f(0)\right) = A_+ + A_- = x_0$$

$$y(0) = \mathrm{Im}\left(f(0)\right) = 0$$

$$\begin{aligned}\dot{f} &= \frac{d}{dt}\left[A_+ e^{-\omega_+ t} + A_- e^{-\omega_- t}\right]\\ &= -A_+ i\omega_+ e^{-i\omega t} - A_- i\omega e^{-i\omega t}\end{aligned}$$

$$\dot{y} = \mathrm{Im}(\dot{f}) = -A_+\omega_+ \cos\omega_+ t - A_-\omega_- \cos\omega_- t$$

$$\dot{y}(0) = v_0$$

$$\Rightarrow A_+\omega_+ - A_-\omega_- = v_0$$

$$\begin{aligned}\dot{x} &= \mathrm{Re}(\dot{f})\\ &= A_+\omega_+ \sin\omega_+ t + A_-\omega_- \sin\omega_- t\\ &= 0\end{aligned}$$

$$x(0) = x_0 \Rightarrow A_+ + A_- = x_0 \tag{1}$$

$$y(0) = v_0 \Rightarrow A_+\omega_+ - A_-\omega_- = v_0 \tag{2}$$

1 gives us

$$A_- = x_0 - A_+ \tag{3}$$

Sets 3 in to 2:

$$\begin{aligned}&\Rightarrow -A_+\omega_+ - x_0\omega_- + A_+\omega_- = v_0\\ &\Rightarrow A_+(omega_-\omega_+) = v_0 + x_0\omega_-\\ &\Rightarrow A_+ = \frac{v_0 + x_0\omega_-}{\omega_- - \omega_+}\end{aligned} \tag{4}$$

Sets 4 in to 3:

$$\begin{aligned}A_- &= x_0 - \frac{v_0 + x_0\omega_-}{\omega_- - \omega_+}\\ &= \frac{x_0(\omega_- - \omega_+) - v_0 - x_0\omega_-}{\omega_- - \omega_+}\\ &= \frac{-x_0\omega_+ - v_0}{\omega_- - \omega_+}\\ &= -\frac{v_0 + x_0\omega_+}{\omega_- - \omega_+}\end{aligned}$$

$$z(0) = z_0 \Rightarrow A\cos\omega_z\theta + B\sin\omega_z\theta = z_0 \Rightarrow A = z_0$$

$$\dot{z} = \omega_z(-z_0\sin\omega_z t + V\cos\omega_z t)$$

$$\dot{z}(0) = \omega_z(-z_0\sin\omega_z\dot{0} + B\cos\omega_z\theta) = w_z B = 0 \Rightarrow B = 0$$

$$z(t) = z_0\cos(\omega_z t) \quad \square$$