

I. THE EQUATIONS OF MOTION

$$m\ddot{\mathbf{v}} = \sum_i \mathbf{F}_i$$

This gives us:

$$\Rightarrow m \begin{bmatrix} \ddot{\mathbf{x}} \\ \ddot{\mathbf{y}} \\ \ddot{\mathbf{z}} \end{bmatrix} = \sum_i \begin{bmatrix} \mathbf{F}_x \\ \mathbf{F}_y \\ \mathbf{F}_z \end{bmatrix} = \sum_i$$

$$m\ddot{x} = \sum_i F_{x,i}$$

Simple particle:

$$\Rightarrow \begin{bmatrix} \dot{x} \\ \dot{y} \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ B_0 \end{bmatrix} = \begin{bmatrix} B_0 \dot{y} \\ B_0 \dot{x} \\ 0 \end{bmatrix}$$

$$\begin{aligned} m\ddot{x} &= F_x \\ &= qE_x + (q\mathbf{V} \times \mathbf{B})_x \\ &= -q \frac{\partial v}{\partial x} + q \begin{bmatrix} \dot{x} \\ \dot{y} \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ B_0 \end{bmatrix} = -q \left(-\frac{V_0}{d^2} x \right) + qB_0 \dot{y} \end{aligned}$$

$$\ddot{x} - \frac{qB_0}{m} \dot{y} - \frac{qV_0}{md^2} x = 0$$

$$\Rightarrow \ddot{x} - \omega_0 \dot{y} - \frac{1}{2} \omega_z^2 x = 0$$

$$\begin{aligned} m\ddot{y} &= F_y \\ &= qE_y + (q\mathbf{V} \times \mathbf{B})_y \\ &= -q \frac{\partial v}{\partial y} - qB_0 \dot{x} \end{aligned}$$

$$\Rightarrow \ddot{y} + \frac{qB_0}{m} \dot{x} - \frac{qV_0}{md^2} = 0$$

$$m\ddot{z} = qE_z = -q \frac{\partial v}{\partial z} = -\frac{4qv_0}{2d^2} z$$

$$\Rightarrow \ddot{z} + \omega_z^2 z = 0$$

$$z = A \cos(\omega_z z) + B \sin(\omega_z z) \quad \square$$

II. SINGLE DIFFERENTIAL EQUATION

$$m\ddot{x} - \omega_0 \dot{y} - \frac{1}{2} \omega_z^2 x + i(\ddot{y} + \omega_z^2 y) = 0$$

$$\Rightarrow \ddot{x} + i\ddot{y} + i\omega_0 \dot{x} - \omega_0 \dot{y} - \frac{1}{2} \omega_z^2 (x + iy) = 0$$

$$\Rightarrow \ddot{f} + i\omega_0 \dot{f} - \frac{1}{2} \omega_z^2 f = 0$$

$$\Rightarrow \ddot{f} + i\omega_0 \dot{f} - \frac{1}{2} \omega_z^2 f = 0 \quad \square$$

III. NECESSARY CONSTRAINT ON ω_0 AND ω_z

Assuming $\omega \pm \in \mathbb{R}$:

$$\begin{aligned} \Rightarrow \omega_0^2 - 2\omega_z^2 &> 0 \\ \Rightarrow \omega_0^2 &> 2\omega_z^2 \\ \Rightarrow \left(\frac{qB_0}{m} \right)^2 &> \left(\frac{2qv_0}{md^2} \right) \\ \Rightarrow \frac{q}{m} B_0^2 &> \frac{4v_0}{d^2} \quad \square \end{aligned}$$

IV. LOWER BOUNDS

$$A_+(\cos \omega_+ t + i \sin \omega_+ t) + A_-(\cos \omega_- t + i \sin \omega_- t)$$

$$\text{Re}(f(t)) + A_+ \cos \omega_+ t + A_- \cos \omega_- t$$

$$\text{Im}(f(t)) + A_+ \sin \omega_+ t + A_- \sin \omega_- t$$

We are getting upper bounds by when $x(t)$ and $y(t)$ are in upper phase:

$$R_+ = A_+ + A_-$$

and lower bounds when they are in opposite phase:

$$R_- = |A_+ - A_-|$$

V. SPECIFIC SOLUTION OF $z(t)$

$$y(0) = v_0 \Rightarrow A_+\omega_+ - A_-\omega_- = v_0 \quad (2)$$

1 gives us

$$x(0) = \text{Re}(f(0)) = A_+ + A_- = x_0$$

$$y(0) = \text{Im}(f(0)) = 0$$

$$\begin{aligned} \dot{f} &= \frac{d}{dt} [A_+e^{-\omega_+t} + A_-e^{-\omega_-t}] \\ &= -A_+i\omega_+e^{-i\omega_+t} - A_-i\omega_-e^{-i\omega_-t} \end{aligned}$$

$$\dot{y} = \text{Im}(\dot{f}) = -A_+\omega_+ \cos \omega_+t - A_-\omega_- \cos \omega_-t$$

$$\dot{y}(0) = v_0$$

$$\Rightarrow A_+\omega_+ - A_-\omega_- = v_0$$

$$\begin{aligned} \dot{x} &= \text{Re}(\dot{f}) \\ &= A_+\omega_+ \sin \omega_+t + A_-\omega_- \sin \omega_-t \\ &= 0 \end{aligned}$$

$$x(0) = x_0 \Rightarrow A_+ + A_- = x_0 \quad (1)$$

Sets **3** in to **2**:

$$\begin{aligned} &\Rightarrow -A_+\omega_+ - x_0\omega_- + A_+\omega_- = v_0 \\ &\Rightarrow A_+(\omega_- - \omega_+) = v_0 + x_0\omega_- \\ &\Rightarrow A_+ = \frac{v_0 + x_0\omega_-}{\omega_- - \omega_+} \end{aligned} \quad (4)$$

Sets **4** in to **3**:

$$\begin{aligned} A_- &= x_0 - \frac{v_0 + x_0\omega_-}{\omega_- - \omega_+} \\ &= \frac{x_0(\omega_- - \omega_+) - v_0 - x_0\omega_-}{\omega_- - \omega_+} \\ &= \frac{-x_0\omega_+ - v_0}{\omega_- - \omega_+} \\ &= -\frac{v_0 + x_0\omega_+}{\omega_- - \omega_+} \end{aligned}$$

$$z(0) = z_0 \Rightarrow A \cos \omega_z \theta + B \sin \omega_z \theta = z_0 \Rightarrow A = z_0$$

$$\dot{z} = \omega_z(-z_0 \sin \omega_z t + B \cos \omega_z t)$$

$$\dot{z}(0) = \omega_z(-z_0 \sin \omega_z \dot{0} + B \cos \omega_z \theta) = \omega_z B = 0 \Rightarrow B = 0$$

$$z(t) = z_0 \cos(\omega_z t) \quad \square$$