An Exact Jumper Insertion Algorithm for Antenna Effect Avoidance/Fixing

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ABSTRACT

As the process technology enters the nanometer era, reliability has become a major concern in the design and manufacturing of VLSI circuits. In this paper we focus on one reliability issue—jumper insertion in routing trees for avoiding/fixing antenna effect violations at the routing/post-layout stages. We formulate the jumper insertion for antenna avoidance/fixing as a tree-cutting problem. We show that the tree-cutting problem exhibits the properties of optimal substructures and greedy choices. With these properties, we present an $O(V \lg V)$ -time exact jumper insertion algorithm that uses the optimum number of jumpers to avoid/fix the antenna violations in a routing tree with V vertices. Experimental results show the superior effectiveness and efficiency of our algorithm.

Categories and Subject Descriptors: J.6 [Computer-Aided Engineering]: Computer-Aided Design

General Terms: Algorithms, Performance Keywords: Antenna Effect, Jumper Insertion

1. INTRODUCTION

As the process technology enters the nanometer era, product reliability and manufacturing yield have become major concerns in the design and manufacturing of VLSI circuits. The fine feature size of modern IC technologies is typically achieved by using plasma-based processes. However, these plasma-based processes will charge conducting components of a fabricated structure. As a result, the accumulated charges may affect the quality of IC's. This is called the *antenna effect*.

During metalization, long floating interconnects act as temporary capacitors and accumulate charges gained from the energy provided by fabrication steps such as plasma etching. A random discharge of the floating node due to subsequent process steps could permanently damage transistors in the IC [4, 5]. For instance, the exposed polysilicon and metal structures connected to a thin-oxide transistor will collect charge from the processing environment and damage the transistor when the discharging current flows through the thin oxide. Charging occurs when conductor layers not covered by a shielding layer of oxide are directly exposed to plasma. The amount of such charging is proportional

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to this plasma-exposed area. If conductor layers are connected to a diffusion layer pattern, such charges are discharged to the substrate through the diffusion; see Figures 1(b), (c), and (d) for illustrations. On the other hand, if the charged conductor layers are connected only to the gate oxide, Fowler-Nordheim (F-N) tunneling current through thin oxide discharges such charges and causes damage to the thin oxide [4]; see Figures 1(b) and (c). As shown in Figure 1, interconnects are manufactured layer by layer. Before a conducting path to the diffusion is formed in metal 2 layer pattern etching (see Figure 1(d)), the interconnects in the poly and metal 1 layers might have accumulated so many charges that they cause damage on the gate in the left of Figure 1(c). (Note that there will not be any antenna violation after a conducting path to the diffusion is formed.)

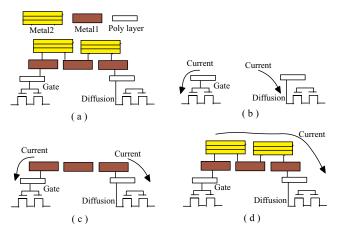


Figure 1: Antenna effect. (a) An example routing. (b) Late stage of poly layer pattern etching of (a). Charge on the left poly pattern is discharged through the gate while charge on the right is discharged through the diffusion. (c) Late stage of metal 1 layer pattern etching of (a). Charge on the left metal 1 pattern is discharged through the gate while charge on the right is discharged through the diffusion. (d) Late stage of metal 2 layer pattern etching. Charges on all the metal 2 patterns are discharged through the diffusion.

There are three kinds of solutions to reduce the antenna effect [1]:

- Jumper insertion: Break the wires with antenna violations and route them to the highest layers through vias.
- 2. Embedded protection diode: Add protection diodes on every input port of a standard cell.
- 3. Diode insertion during layout design: Fix those violated wires by "under-the-wire" diode insertion.

Comparing the three methods, for method 2 of embedded protection diode, since these diodes are embedded and fixed, they

consume unnecessary areas when there is no violation at the connecting wire. In the third method, we need extra space in the chip to place the diodes. Because the number of diodes needed for fixing antenna violations grows dramatically as the feature shrinks, it is hard to preserve enough space for diodes in nanometer IC designs. As a result, jumper insertion becomes the most popular approach for avoiding/fixing antenna violations. The function of jumper insertion can be explained using Figure 2. In Figure 2(a), when the metal 1 layer is manufactured, the gate on the right might be damaged because the large area of the metal 1 interconnection can accumulate sufficient charges to damage the gate. However, if we insert a jumper to route the interconnect on the metal 2 layer as shown in Figure 2(b), the effective conductor layer becomes smaller. Therefore, the stored charge is not enough to damage the gate on the right, and thus we can avoid the antenna violation.

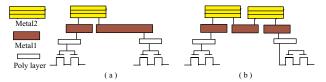


Figure 2: Jumper insertion. (a) Stage before inserting a jumper. (b) Stage after inserting a jumper from the metal 1 layer to the metal 2 layer.

Although jumper insertion is currently the most popular approach for antenna avoidance/fixing, jumpers induce vias that will consume silicon areas and reduce circuit performance. Therefore, it is desired to fix antenna violations by using the least jumpers. Recently, Ho, Chang, and Chen in [3] proposed a bottom up approach to insert jumpers in a routing tree for antenna avoidance. The work inserts jumpers only beside tree nodes, and its optimality holds only for this special condition of inserting jumpers right beside tree nodes. As an example shown in Figure 3, the wire segment is of $1.3L_{max}$ long, where L_{max} denotes the upper bound for antenna (i.e., any wire longer than L_{max} will violate the antenna rule). For this wire segment, the work in [3] needs two jumpers to fix the antenna violation (see Figure 3(a)) while a single jumper suffices to fix the violation (see Figure 3(b)).



Figure 3: Jumper insertion for a wire of $1.3L_{max}$ long. (a) Two jumpers are needed for fixing the antenna violation if jumpers can be inserted only beside tree nodes. b) One jumper suffices to fix the problem if it can be inserted at an arbitrary position of the wire.

In this paper, we consider the general case of inserting jumpers at arbitrary positions (e.g., in any position of a tree edge). We formulate the general jumper insertion for antenna avoidance (applicable at the routing stage) and/or fixing (applicable at the post-layout stage) as a tree-cutting problem. We show that the tree-cutting problem exhibits the properties of optimal substructures and greedy choices. With these properties, a greedy algorithm suffices to find an optimal solution [2]. Based on the theory, we present an $O(V \lg V)$ -time exact jumper insertion algorithm that uses the minimum number of jumpers to fix the antenna violations in a routing tree with V vertices. Compared with the previous work in [3], our algorithm outperforms the method by large margins.

2. PROBLEM DEFINITION

To avoid/fix the antenna violation, we require that the total effective conductor connecting to a gate is less than or equal to

a threshold, L_{max} . The threshold can be wire length limit, wire area limit, or any model of the strength of antenna effect caused by conductors. Typically, a net is modeled as a routing tree, where a node in the tree denotes a circuit terminal (a gate or a diffusion) and an edge denotes the interconnection between two circuit terminals. Since the interconnection connecting to a diffusion terminal will not cause any antenna violation, we shall focus on those connecting to gate terminals.

Let T = (V, E) be a routing tree, which can be a Steiner tree or a spanning tree of any form. The set V of nodes represents all gate terminals, the set E of edges denotes the wires connecting the circuit terminals, and an edge weight gives the measure of the wires with the same unit as L_{max} . A gate will violate the antenna rule if the effective conductor incident on the gate (i.e., the effective weight—the sum of the weights of the edges incident on the corresponding node) is larger than L_{max} . To reduce the antenna effects on a gate, we can apply the technique illustrated in Figure 2 by adding a jumper on a wire connecting to the gate to reduce the effective conductor. This operation is modeled as adding a cutting node on the tree edge corresponding to the wire to reduce the effective edge weight associated with the gate node. As aforementioned, jumpers are implemented by vias which will consume silicon areas and reduce circuit performance. Therefore, it is desired to fix antenna violations by using the least jumpers. In other words, given a routing tree T = (V, E) and an upper bound on the antenna L_{max} , we intend to add the minimum number of cutting nodes so that the effective edge weight associated with each node is smaller than L_{max} . Let L(u) denote the sum of edge weights (lengths, wire areas, etc) between the node u and all its neighbors. We formulate the problem of jumper insertion on a routing tree for antenna avoidance/fixing as a tree cutting problem as follows:

• Problem JITA (Jumper Insertion on a Routing Tree for Antenna Avoidance): Given a routing tree T=(V,E) and an upper bound L_{max} , find the minimum set C of cutting nodes, $c \neq u$ for any $c \in C$ and $u \in V$, so that $L(u) \leq L_{max}$, $\forall u \in V$.

3. JUMPER INSERTION ALGORITHM

For the JITA problem, we present in this section an $O(V \lg V)$ -time exact algorithm, named BUJI (Bottom Up Jumper Insertion), for finding the minimum cutting set C for a given routing tree T=(V,E) with V nodes. (Note that we use V to denote the set or the number of nodes in a routing tree.) Algorithm BUJI is summarized in Figure 4. To simplify the presentation, we assume that the antenna bound L_{max} is measured by wire length. Let l(e) (or l(u,v)) be the length (i.e., weight) of the edge e=(u,v) in T. In the BUJI algorithm, we add the cutting nodes into the original tree in a bottom-up manner. We first define a subleaf node as follows:

Definition 1. A subleaf is a node for which all its children are leaf nodes, and all the edges between it and its children have lengths $\leq L_{max}$.

We derive the algorithm based on the following two steps:

- $\bullet\,$ Step 1 (lines 2–12 of Algorithm BUJI): Deal with every leaf node.
 - In this case, our main goal is to prevent every leaf node from antenna violation. Obviously, if we have dealt with a leaf node, we need not consider it any more. Therefore, line 3 of the BUJI algorithm marks these nodes to ensure that every leaf node is processed only once. If $l(u,p(u)) \leq L_{max}$, the leaf node u satisfies the antenna rule, and thus we do nothing. However, if $l(u,p(u)) > L_{max}$, we must insert at least one cutting node to satisfy that $L(u) \leq L_{max}$. For this case, we can further divide it into two subcases as follows:
 - 1. $u \in C$: In this subcase, we need at least one cutting node to satisfy the rule that $L(u) \leq L_{max}$. We claim that l(u,c) = l(u,p(u)) (and thus l(c,p(u)) = 0) gives the best position for inserting the cutting node; see Figure 5(a) for an illustration. Therefore, we add c into C and cut the edge e(u,p(u)) from the original tree T (lines 5–8).

- 2. $u \notin C$: In this subcase, we need at least one cutting node to prevent u from antenna violation. We claim that $l(u,c) = L_{max}$ (and thus $l(c,p(u)) = l(u,p(u)) L_{max}$) gives the best position for inserting the cutting node (the proof is given in the next section); see Figure 5(b) for an illustration. Therefore, we add c into C, add c into V, and cut the node u and edge e(u,c) from the original tree T (lines 9–12).
- Step 2 (lines 13–19 of BUJI): Deal with every subleaf node. In this case, our main goal is to prevent every subleaf node from antenna violation. Moreover, we delete some nodes and edges to make each subleaf node as a leaf node. We classify the subleaf nodes into two categories by the sum of lengths between the node and its children. Let u_p be a subleaf node and $u_i, \forall 1 \leq i \leq k$, be its children. Let $totallen = \sum_{i=1}^k l(u_i, u_p)$.
 - 1. Case 1: $totallen \leq L_{max}$ (Subroutine LessEqual) If u_p and its children form an isolated component, they must satisfy the antenna rule, and thus we are done with the subroutine. If $totallen + l(u_p, p(u_p)) \leq L_{max}, u_p$ will not violate the antenna rule. Therefore, we simply cut u_p 's children from the original tree to make u_p as a leaf node (lines 3–5 in LessEqual). Otherwise, we must add at least one cutting node c to prevent u_p from antenna violation. We claim that $l(c, u_p) + totallen = L_{max}$ gives the best position for inserting the cutting node; see Figure 6(a). Therefore, we add c into C, and cut u_p and all its children from the original tree T (lines 6–9 in LessEqual).
 - 2. Case 2: $totallen > L_{max}$ (Subroutine More) We first sort edges from u_p and its children by their lengths. Let $A[i] = l(u_i, u_p), \forall 1 \leq i \leq k$. Then we find the maximum number s such that $\sum_{j=1}^{s} A[j] \leq L_{max}$. We claim that c_{s+1}, \ldots, c_k on edge $e(u_i, u_p)$ with $l(c_i, u_p) = 0$ (and thus $l(c_i, u_i) = l(u_p, u_i)$) give the best positions for inserting the cutting nodes; see Figure 6(b). Therefore, we add c_{s+1}, \ldots, c_k into C, and cut u_{s+1}, \ldots, u_k from the original tree T (lines 1–9 in More). Moreover, we call subroutine LessEqual to further reduce u_p into a leaf node (line 10 in More)

When the total length of the tree T is $\leq L_{max}$, Algorithm BUJI terminates and C is a cutting set of the minimum size.

4. PROOF OF THE OPTIMALITY

Algorithm BUJI is greedy in nature. To prove that Algorithm BUJI finds the optimal cutting set (of the minimum size), therefore, we can show that the JITA problem exhibits optimal substructure and has the greedy-choice property [2]. A problem exhibits optimal substructure if an optimal solution to the problem contains within it optimal solutions to the subproblems; a problem has the greedy-choice property if a globally optimal solution can be arrived at by making a locally optimal (greedy) choice [2]. Due to the limitation of space, we shall omit the proofs.

Theorem 1. The JITA problem exhibits optimal substructure.

Now we show that the JITA problem has the greedy-choice property, and Algorithm BUJI finds the best solution in each step. First, we show that Algorithm BUJI has greedy choice property among all leaf nodes. Then, we show that BUJI has greedy choice property among all subleaf nodes.

LEMMA 1. Lines 2–12 of Algorithm BUJI finds the best cutting set so that every leaf node u satisfies the antenna rule (i.e., $L(u) \leq L_{max}, \forall$ leaf nodes u).

We proceed to show that lines 13–19 in BUJI finds the best cutting set for each subleaf node u_p . In this step, we classify the subleaf nodes into two categories based on the sum of lengths between u_p and its children u_i : $\sum_{i=1}^k l(u_p, u_i) \leq L_{max}$, and $\sum_{i=1}^k l(u_p, u_i) > L_{max}$. Therefore, we show that each case is with the greedy-choice property, and we find the best cutting set in each case.

```
Algorithm: BUJI(T, L_{max}, t_{total}, C)
Input: T = (V, E) /* The given tree. */
            L_{max} /* Upper Bound on antenna */
            C /* Cutting set */
            t_{total} /* Total Edge Length in T */
      while (t_{total} > L_{max})
\frac{2}{3}
            for each leaf node u \in T not having been processed
                {\it Mark}\ u\ {\it as}\ processed;
                if l(u, p(u)) > L_{max}
5
                     \mathbf{if}\;u\in C
                          Let c be the node between u and p(u) with
                          l(u, c) = l(u, p(u)) and l(c, p(u)) = 0;
                          C \leftarrow C \cup \{c\};
6
                          t_{total} \leftarrow t_{total} - l(u, p(u));

T(V, E) \leftarrow T(V - \{u\}, E - \{e(u, p(u))\});
7
8
9
                           Let c be the node between u and p(u) with
                     l(c, u) = L_{max} and l(c, p(u)) = l(u, p(u)) - L_{max};
10
                          C \leftarrow C \cup \{c\};
            t_{total} \leftarrow t_{total} - L_{max};
T(V, E) \leftarrow T(V + \{c\} - \{u\}, E - \{e(u, c)\});
for each subleaf node u_p \in T
11
12
13
                Let u_1, u_2, \ldots, u_k denote all children nodes of u_p. totallen \leftarrow \sum_{i=1}^k l(u_p, u_i); if totallen \leq L_{max}
14
15
16
17
                     LessEqual(T, t_{total}, C, u_p, totallen, L_{max}) ;
18
19
                     More(T, t_{total}, C, u_p, totallen, L_{max});
```

Figure 4: Algorithm BUJI.

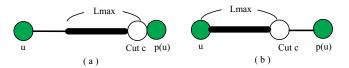


Figure 5: Illustration of lines 2-12 in the BUJI algorithm. (a) Case $u \in C$. (b) Case $u \notin C$.

LEMMA 2. Subroutine LessEqual finds the best cutting set so that every subleaf node u_p satisfies the antenna rule (i.e., $L(u_p) \leq L_{max}$, \forall subleaf nodes u_p satisfying $\sum_{i=1}^k l(u_p, u_i) \leq L_{max}$, where $u_p = p(u_i)$).

Lemma 3. Subroutine More finds the best cutting set so that every subleaf node u_p satisfies the antenna rule (i.e., $L(u_p) \leq L_{max}$, \forall subleaf nodes u_p satisfying $\sum_{i=1}^k l(u_p, u_i) > L_{max}$, where $u_p = p(u_i)$).

Based on the above theorem and lemmas, we have the following theorem:

THEOREM 2. Algorithm BUJI optimally solves the JITA problem in $O(V \lg V)$ time using O(V) space, where V is the number of vertices in the routing tree.

5. EXPERIMENTAL RESULTS

We implemented the BUJI algorithms in the C++ language on a 2.4 GHz Intel Pentium PC with 256 MB memory under the Windows XP operating system. For the JITA problem, we compared our algorithm with the ISPD-04 work [3].

For comparative study, we first randomly generated tree nodes on grid planes of the dimension $10^5 \mu m \times 10^5 \mu m$, assuming that each node is a gate terminal. Then, we constructed a routing tree (a minimum spanning tree) for the nodes. We performed the following two experiments for our BUJI algorithm and the ISPD-04 work [3]:

(1) First, for a given routing tree, we find the minimum number of jumpers required for fixing all antenna violations for various L_{max} values.

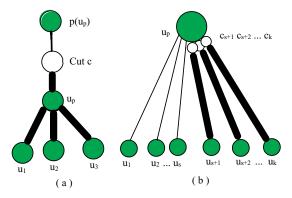


Figure 6: (a) Illustration of the LessEqual Subroutine. Here, $l(u_p,u_1)+l(u_p,u_2)+l(u_p,u_3)+l(u_p,c)=L_{max}$. (b) Illustration of the More Subroutine.

```
Subroutine: LessEqual(T, t_{total}, C, u_p, totallen, L_{max})
1
         if p(u_p) does not exist
2
               return
         \textbf{if } totallen + l(u_p, p(u_p)) \leq L_{max}
3
                \begin{aligned} t_{total} \leftarrow t_{total} - totallen; \\ T(V, E) \leftarrow T(V - \cup_{i=1}^{k} \{u_i\}, E - \cup_{i=1}^{k} \{e(u_i, u_p)\}); \end{aligned}
4
5
6
               Let c be the node on e(u_p, p(u_p))
                with l(c, u_p) + totallen = L_{max};
                C \leftarrow \hat{C} \cup \{\hat{c}\}\;;
               \begin{split} C \leftarrow C \cup \{c\} \ , \\ t_{total} \leftarrow t_{total} - L_{max}; \\ T(V, E) \leftarrow T(V - \cup_{i=1}^k \{u_i\} + \{c\} - \{u_p\}, \\ E - \cup_{i=1}^k \{e(u_i, u_p)\} - \{e(u, c)\}); \end{split}
8
9
```

Figure 7: Case when $totallen \leq L_{max}$.

(2) Second, giving L_{max} as a constant, we find the running times for the algorithms to fix all antenna violations for routing trees with various numbers of nodes.

Table 1 shows the number of jumpers required for fixing all antenna violations for a routing tree with 500000 nodes by changing L_{max} from 100 μm to 800 μm . Note that the L_{max} range is typical for 90 nm to 250 nm CMOS technologies. Column 1 gives the L_{max} value, and Columns 2 and 3 list the numbers of jumpers required (#J) for fixing the antenna violations for each L_{max} for the BUJI algorithm and the ISPD-04 work [3], respectively. Columns 4 gives the percentage of additional jumpers required (%More) for the ISPD-04 method over the BUJI algorithm to fix all antenna violations, i.e., $\%More=(\#\mathrm{Jumpers}\ of\ \mathrm{the}\ \mathrm{heuristic}-\#\mathrm{Jumpers}\ of\ \mathrm{BUJI})/\ \#\mathrm{Jumpers}\ of\ \mathrm{BUJI}.$

It is not surprised that BUJI performs much better than the ISPD-04 method [3]. When we deal with nodes in a bottom-up manner, BUJI always pushes the jumper upward until at a position that just satisfies the antenna rule, adding more freedom and thus reducing the chance of antenna violations for the upper nodes. Therefore, BUJI can save a significant number of jumpers for antenna avoidance/fixing. In contrast, the ISPD-04 method [3] does not have such an optimization scheme.

Table 2 shows the CPU times required for antenna fixing on routing trees of 100000 to 900000 nodes with $L_{max}=50\mu m$. Column 1 gives the numbers of nodes in the trees. The 2nd column (File) gives the CPU times for reading the input files. The (Main) columns in each algorithm give the respective CPU times for executing the main body of the algorithm. As shown in the table, the empirical running times for the two methods are close to linear. (Note that the time complexities of the two methods are all $O(V \lg V)$.) In particular, BUJI requires only 3.8 sec to find an optimal solution for a routing tree of 0.9 million nodes. Therefore, the BUJI algorithm can handle a test case of a very huge number of nodes in very short time.

```
Subroutine: More(T, t_{total}, C, u_p, totallen, L_{max})

1 Let A[i] \leftarrow l(u_i, u_p), \forall 1 \leq i \leq k.

2 Sort A in non-decreasing order;

3 s \leftarrow 1;

4 while (\sum_{j=1}^{s+1} A[j] \leq L_{max})

5 s \leftarrow s+1;

Let c_{s+1}, \ldots, c_k be the nodes between u_p and u_{s+1}, \ldots, u_k with l(c_i, u_p) = 0 and l(c_i, u_i) = l(u_p, u_i), \forall s+1 \leq i \leq k;

6 C \leftarrow C \cup \{c_i\} \ \forall s+1 \leq i \leq k;

7 T(V, E) \leftarrow T(V - \bigcup_{i=s+1}^k \{u_i\}, E - \bigcup_{i=s+1}^k \{e(u_i, u_p)\});

8 minuslen \leftarrow \sum_{i=s+1}^k A[i];

9 t_{total} \leftarrow t_{total} - minuslen;

10 LessEqual(T, t_{total}, C, u_p, totallen - minuslen);
```

Figure 8: Case when totallen > L_{max} .

L_{max}	BUJI	ISPD04	
(um)	#J	#J	%More
100	517851	729048	+40.8%
200	232063	336084	+44.8%
300	98465	129891	+31.9%
400	35178	41136	+16.9%
500	9873	10658	+7.95%
600	1925	1978	+2.75%
700	319	321	+0.63%
800	40	40	0.00%

Table 1: Comparisons of the numbers of jumpers for fixing antenna violations based on a tree of 500000 nodes.

6. CONCLUSION

We have presented a loglinear-time exact jumper insertion algorithms for avoiding/fixing antenna violations on routing trees. Empirical results have shown that our algorithm approaches linear and obtains solutions of very high quality. Our work can apply to any routing trees (could be a net to be globally routed or a net after detailed routing) and thus readily be incorporated into a router for antenna effect avoidance or a post-layout optimizer for antenna violation fixing.

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node		BUJI	ISPD-04
number	File (s)	Main (s)	Main (s)
100000	1.062	0.295	0.284
300000	3.208	1.041	1.031
500000	5.363	1.893	1.872
700000	7.498	2.808	2.798
900000	9.654	3.828	3.754

Table 2: Comparisons of the CPU times ($L_{max} = 50 \mu m$).