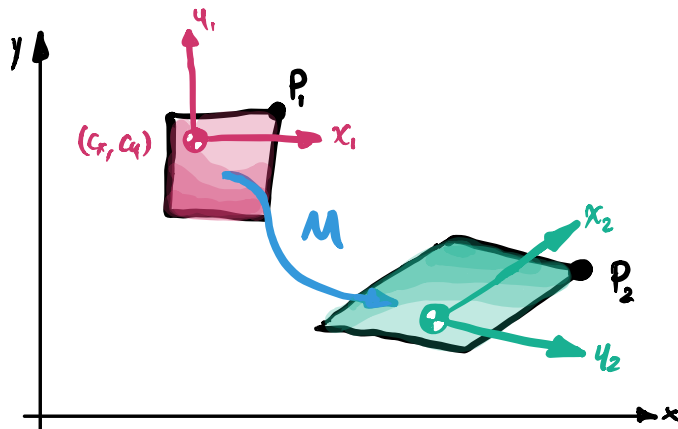


If we have an affine transform, we can decompose it into a number of transforms, to keep things general, we select a **transformation origin** (c_x, c_y) to rotate the object around. So we have:



Now we would like to express the matrix M in terms of a set of affine transforms, so we should be able to write

$$M = T R S \Gamma$$

$$T(T_x, T_y) = \begin{bmatrix} 1 & 0 & T_x \\ 0 & 1 & T_y \\ 0 & 0 & 1 \end{bmatrix}$$

$$R(\theta, c_x, c_y) = C^{-1} \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} C$$

$$S(S_x, S_y, c_x, c_y) = C^{-1} \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix} C$$

$$\Gamma(\lambda_x, \lambda_y, c_x, c_y) = C^{-1} \begin{bmatrix} 1 & \lambda_x & 0 \\ \lambda_y & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} C$$

Define a matrix to move our center to the origin so that:

$$C = \begin{bmatrix} 1 & 0 & -c_x \\ 0 & 1 & -c_y \\ 0 & 0 & 1 \end{bmatrix}$$

Now if we are careful, we notice two direct representations for the matrix M . Note that I've used the fact that $CC^{-1} = I$ between R and S and also between S and Γ :

$$M = \begin{matrix} & \begin{matrix} 1 & 0 & T_x \\ 0 & 1 & T_y \\ 0 & 0 & 1 \end{matrix} & \begin{matrix} \begin{matrix} 1 & 0 & C_x \\ 0 & 1 & C_y \\ 0 & 0 & 1 \end{matrix} \\ C^{-1} & \begin{matrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{matrix} \\ R & \begin{matrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{matrix} \\ S & \begin{matrix} 1 & \lambda_x & 0 \\ \lambda_y & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \\ \Gamma & \begin{matrix} 1 & 0 & -C_x \\ 0 & 1 & -C_y \\ 0 & 0 & 1 \end{matrix} \\ C \end{matrix}$$

Lets do some multiplying: (define $C_\theta = \cos\theta$ $S_\theta = \sin\theta$)

$$M = \begin{matrix} & \begin{matrix} 1 & 0 & T_x+C_x \\ 0 & 1 & T_y+C_y \\ 0 & 0 & 1 \end{matrix} \\ TC^{-1} & \begin{matrix} S_x C_\theta & -S_y S_\theta & 0 \\ S_x S_\theta & S_y C_\theta & 0 \\ 0 & 0 & 1 \end{matrix} \\ RS & \begin{matrix} 1 & \lambda_x & -C_x - C_y \lambda_x \\ \lambda_y & 1 & -C_x \lambda_y - C_y \\ 0 & 0 & 1 \end{matrix} \\ TC \end{matrix}$$

Now is the time to make a critical observation:

$$M = \begin{bmatrix} a & c & e \\ b & d & f \\ 0 & 0 & 1 \end{bmatrix} \leftarrow \text{there are six degrees of freedom here.}$$

But:

$$M = T(T_x, T_y) C^{-1}(C_x, C_y) R(\theta) S(S_x, S_y) \Gamma(\lambda_x, \lambda_y) C(C_x, C_y)$$

we are proposing to have seven degrees of freedom here.

So we have ● - free ● - fixed ● - extra.

Since it seems that we are proposing seven degrees of freedom, we must fix at least one of them based on our taste. I think few people will be showing svgs constantly so we should fix $\lambda_y = 0$. whilst this choice is arbitrary, it should work well in practice.

Now we can apply this simplification to our matrix, and continue multiplying it out:

$$M = \begin{matrix} \begin{bmatrix} s_x c_\theta & -s_y s_\theta & T_x + C_x \\ s_x s_\theta & s_y c_\theta & T_y + C_y \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & \lambda_x & -C_x - C_y \lambda_x \\ 0 & 1 & -C_y \\ 0 & 0 & 1 \end{bmatrix} \\ TC^{-1}RS & TC \end{matrix}$$

Now:

$$M = \begin{bmatrix} s_x c_\theta & \lambda s_x c_\theta - s_y s_\theta & C_x - C_x \lambda c_\theta s_x - C_y (\lambda c_\theta s_x - s_\theta s_y) + T_x \\ s_x s_\theta & \lambda s_x s_\theta + s_y c_\theta & C_y - C_x s_\theta s_x - C_y (\lambda s_\theta s_x + c_\theta s_y) + T_y \\ 0 & 0 & 1 \end{bmatrix}$$

So we can directly write that

$$a = s_x c_\theta$$

$$b = s_x s_\theta$$

$$c = \lambda \overset{a}{s_x c_\theta} - s_y s_\theta$$

$$d = \lambda \overset{b}{s_x s_\theta} + s_y c_\theta$$

$$e = C_x - C_x \lambda c_\theta s_x - C_y (\lambda c_\theta s_x - s_\theta s_y) + T_x$$

$$f = C_y - C_x s_\theta s_x - C_y (\lambda s_\theta s_x + c_\theta s_y) + T_y$$

But we want

$$(T_x, T_y, \theta, s_x, s_y, \lambda_x)$$

so lets solve them

Solving for s_x and θ :

$$a^2 + b^2 = s_x^2 (c_\theta^2 + s_\theta^2) = s_x^2 \rightarrow s_x = \sqrt{a^2 + b^2}$$

$$\frac{b}{a} = \tan(\theta) \rightarrow \theta = \text{atan2}(b, a)$$

Now we solve for λ

$$c c_\theta = \lambda a c_\theta - s_y c_\theta s_\theta$$

$$d s_\theta = \lambda b s_\theta + s_y c_\theta s_\theta \oplus$$

$$c c_\theta + d s_\theta = \lambda (a c_\theta + b s_\theta)$$

$$\lambda = \frac{c c_\theta + d s_\theta}{a c_\theta + b s_\theta}$$

Now for S_y :

$$S_y^2 s_\theta^2 + S_y^2 c_\theta^2 = S_y^2 = (a\lambda - c)^2 + (d - b\lambda)^2$$

↙

$$\therefore S_y = \sqrt{(a\lambda - c)^2 + (d - b\lambda)^2}$$

Now for T_x and T_y :

$$e = c_x - c_x c_\theta s_x - c_y (\lambda c_\theta s_x - s_\theta s_y) + T_x$$

$$f = c_y - c_x s_\theta s_x - c_y (\lambda s_\theta s_x + c_\theta s_y) + T_y$$

$$T_x = e - c_x + c_x c_\theta s_x + c_y (\lambda c_\theta s_x - s_\theta s_y)$$

$$T_y = f - c_y + c_x s_\theta s_x + c_y (\lambda s_\theta s_x + c_\theta s_y)$$

Proof Complete.