MATH 600 Homework 6

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Problem 1

(1): Let $f : (M, d_x) \to (N, d_y)$ be a continuous function on a metric space (M, d_x) and A be a nonempty set in M.

Let $p, q \in A$. Since $A \subseteq \operatorname{cl} A$, $p, q \in A \Rightarrow p, q \in \operatorname{cl} A$. Since f is uniformly continuous on $\operatorname{cl} A$, then for every $\epsilon > 0$, there exists a $\delta > 0$ such that $d_y(f(p), f(q)) < \epsilon$ for every $p, q \in \operatorname{cl} A$ such that $d(f(p), f(q)) < \delta$. Therefore, since $A \subseteq \operatorname{cl} A$, for every $\epsilon > 0$, we can use the same δ as we did for $\operatorname{cl} A$ to say that $d(p, q) < \delta$ when $d_y(f(p), f(q)) < \epsilon$ for every $p, q \in A$. This also implies that if f is uniformly continuous on A, then f is uniformly continuous on any nonempty subset of A.

(2): Let $g : \mathbb{R}^2 \to \mathbb{R}$ be continuous on $\mathbb{R}^2$. Let $(a, b]$ and (c, d) be two intervals in $\mathbb{R}$.

Since g is a continuous function on R^2, g is uniformly continuous on a compact subset of $\mathbb{R}^2$. Since $\operatorname{cl}((a, b] \times (c, d))$ is a closed and bounded subset of R^n, then it is compact from Heine-Borel. Therefore g is uniformly continuous on $\operatorname{cl}((a, b] \times (c, d))$. From part 1, we know this implies g is uniformly continuous on $(a, b] \times (c, d)$.

Problem 2

Let $f : (M, d) \to \mathbb{R}^n$ be Lipschitz continuous on M, and $f(A)$ be a closed and bounded set in R^n for some set $A \subset M$. Let $g : \mathbb{R}^n \to \mathbb{R}$ be continuous on $\mathbb{R}^n$. Show that the commposition $g \circ f$ is uniformly continuous on A.

Since g is continuous on $\mathbb{R}^n$, and $f(A)$ is a compact subset of $\mathbb{R}^n$, then g is uniformly continuous on $f(A)$. Therefore, for every $\epsilon > 0$, there exists a $\gamma > 0$ such that $|g(p) - g(q)| < \epsilon$ for every $p, q \in f(A)$ where $d_{\mathbb{R}^n}(p, q) < \gamma$. Let $\delta = \gamma/K$, where K is the Lipschitz constant of f. Then for any $p, q \in M$ such that $d(p, q) < \delta$, $d(f(p), f(q)) \leq K \cdot d(p, q) < K \cdot \delta = \gamma$. Therefore, if $d(p, q) < \delta$, $d_{\mathbb{R}^n}(f(p), f(q)) < \gamma \Rightarrow |f(g(p)) - f(g(q))| < \epsilon$, thus $g \circ f$ is uniformly continuous.

Problem 3

Let $f_n(x) = \sin(nx)/(1 + nx)$, and $A = [0, \infty)$.

(1): Let $f_*(x) = 0, x \in A$. Let $x \in A$ be fixed, and $\epsilon < 0$. Then $d(f_n(x), f_*(x)) = |f_n(x) - 0| = |\sin(nx)/(1 + nx)| \leq |1/(1 + nx)| = 1/(1 + nx)$. Let $K = (1 - \epsilon)/\epsilon x$ Then in $n > K$, $1/(1 + nx) < \epsilon \Rightarrow d(f_n(x), f_*(x)) < \epsilon$. Therefore (f_n) is pointwise convergent.

(2): If $x \in [a, \infty]$, we have $|f_n(x) - f_*(x)| = |f_n(x) - 0| = |f_n(x)| < 1/(1 + na)$.

(3): To demonstrate that (f_n) does not converge uniformly on $[0, \infty]$, we must show there exits an $\epsilon > 0$ such that for any $K \in \mathbb{N}$, there exits an $x_* \in [0, \infty]$

Problem 4

Let $f_n(x) = x^n/(1 + x^n)$, and $A = [0, \infty)$.

(1): Let $f_*(x) =$
$0 : 0 \leq x < 1$,
$1/2 : x = 1$,

$1 : x > 1$

(2):

(3):

Problem 5

Suppose a sequence of continuous function (f_n) converges pointwise to f_* on a compact set A. If f_* is continuous on A, does this imply that (f_n) always converge uniformly to f_* on A?

This statement does not hold in general. Take for example the the following function $f : \mathbb{R} \to \mathbb{R}$ on the compact set $[0, 1]$.

$$f_n(x) = \begin{cases} n \cdot x & x \in [0, \frac{1}{n}] \\ 2 - n \cdot x & x \in (\frac{1}{n}, \frac{2}{n}] \\ 0 & x \in (\frac{2}{n}, 1] \end{cases}$$

Let $x \in [0, 1]$ be arbitary. If $x \in (0, 1]$ we can then find some point in the sequence after which $f_n(x)$ is always 0 by letting $n > 2/x$. If $x = 0$, then $x = n \cdot x = 0$, therefore $f_n \to f_*(x) = 0$, and (f_n) is pointwise convergent. In each case of $f_n(x)$, it is either a linear function, and thus continuous, or a constant function, thus continuous. At $x = \frac{1}{n}$, $f_n(x) = 1 = n \cdot x = 2 - n \cdot x$. At $x = \frac{2}{n}$, $f_n(x) = 0 = 2 - n \cdot x = 0$. This shows that at the intersections of the cases, f_n has the same value in each case. Therefore $f_n(x)$ is pointwise continuous on $[0, 1]$.

However, (f_n) is not uniformly convergent since $\sup_{x\in[0,1]} |f_n(x) - f_*(x)| = \sup_{x\in[0,1]} |f_n(x)| = 1 \ \forall n \in \mathbb{N}$. Therefore there does not exist any $N \in \mathbb{N}$ such that $n \geq N$ implies $|f_n(x) - f_*(x)| \leq \frac{1}{2}$.

Problem 6

Suppose that each f_n is continuous on the set A, and (f_n) converges to f_* uniformly on A. Suppose the sequence (x_n) in A converges to $x_* \in A$. Show that $(f_n(x_n))$ converges to $f_*(x_*)$.

Let $\epsilon > 0$ be arbitraty. Since each f_n is continuous, we can choose $\delta > 0$ such that $d(f_n(x_n), f_n(x_*)) < \epsilon/2$ when $d(x_n, x_*) < \delta$. Since (x_n) is convergent, we can choose $N \in \mathbb{N}$ such that $d(x_n, x_*) < \delta$ when $n \geq N$. Therefore, when $n \geq N$, $d(f_n(x_n), f(x_*)) < \epsilon/2$.

Since (f_n) conveges uniformly, we can find an $M \in \mathbb{N}$ such that $m \geq M$ implies that $d(f_n(x_*), f_*(x_*)) \leq \epsilon/2$ for every x, and every $m \geq M$. Let $L = \max(N, M)$. From the triangle inquality, we have $d(f_l(x_l), f_*(x_*)) \leq d(f_l(x_l), f_l(x_*)) + d(f_l(x_*), f_*(x_*)) > \epsilon/2 + \epsilon/2 = \epsilon$ for any $l \geq L$. Therefore, $(f_n(x_n))$ converges to $f_*(x_*)$.

Problem 7

Let $f_n : \mathbb{R} \to \mathbb{R}$ and $g_n : \mathbb{R} \to \mathbb{R}$ be two sequences of bounded functions that converge uniformly on the set A to f_* and g_*, respectively. Show that $(f_n \cdot g_n)$ converges uniformly to $f_* \cdot g_*$ on A.

Since f_n and g_n are bounded functions on $\mathbb{R}$, there exist a B_1, B_2 such that $|f_n(x)| < B_1$ and $|g_n(x)| < B_2$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Since (f_n) and (g_n) are uniformly convergent, for any $\epsilon > 0$, there exist an $N_1, N_2 \in \mathbb{N}$ such that $|f_n(x) - f_*(x)| \leq \epsilon/2B_1$ for all $x \in \mathbb{R}$ and $n > N_1$ and $|g_n(x) - g_*(x)| \leq \epsilon/2B_2$ for all $x \in \mathbb{R}$ and all $n > N_2$. Take $M = \max(N_1, N_2)$. We can then proceed analagously to the sequential case.

$|f_n(x)g_n(x) - f_*(x)g_*(x)| \leq |f_n(x)g_n(x) - f_n(x)g_*(x)| + |f_n(x)g_*(x) - f_*(x)g_*(x)| \leq M|g_n(x) - g_*(x)| + M|f_n(x) - f_*(x)| < M\frac{\epsilon}{2M} + M\frac{\epsilon}{2M} = \epsilon$. Therefore $(f_n \cdot g_n)$ converges uniformly to $f_* \cdot g_*$ on A.