MATH 600 Homework 8

Sam Hulse

December 5, 2017

Problem 1

Let $C_b(\mathbb{R})$ be the space of real-valued continuous and bounded function on $\mathbb{R}$ endowed with the sup-norm $\|\cdot\|_\infty$. Let $B \subset C_b(\mathbb{R})$ be

$$B = \{f \in C_b(\mathbb{R}) \mid 0 < f(x) < 2, \forall x \in \mathbb{R}\}$$

.

Let (f_n) be a convergent sequence of functions which converges to f_* such that $f_*(x) = 2 \forall x \in \mathbb{R}$. Then $f_* \notin B$, thus B does not contain all of it's limit points, therefore B is not closed.

Let $B^c := \{f \in C_b(\mathbb{R}) \mid f(x) \leq 0 \text{ or } f(x) \geq 2\}$. Let (f_n) be a convergent sequence in B^c which converges to f_* where $f_n \leq 0\ \forall x \in \mathbb{R}$ and every $n \in \mathbb{N}$. Since each f_n in (f_n) is continuous, and (f_n) converges uniformly, then $f_* \leq 0\ \forall x \in \mathbb{R}$.

Problem 2

Let $C([0,1])$ be the space of real-valued continuous functions on $[0,1]$ endowed with the sup-norm $\|\cdot\|_\infty$. Let $B \subset C([0,1])$ be

$$B = \{f \in C([0,1]) \mid 0 \leq f(x) \leq 2, \forall x \in [0,1]\}$$

.

Let (f_n) be a convergent sequence of functions in B which converges to f_*. Since (f_n) is uniformly convergent, then f_* is continuous. Then for every $n \in \mathbb{N}$ and every $x \in [0,1]$, $0 \leq f_n(x) \leq 2$. Therefore $0 \leq f_*(x) \leq 2$ for any fixed $x \in [0,1]$. Therefore $f_* \in B$, thus B is closed.

To show that B is not compact, I will demonstrate that it is not necesarily equi-continuous. Since compactness is equivalent to being closed, bounded, and equi-continuous, showing that B is not equicontinuous is sufficient to show that it is not compact. Let $f_n : [0,1] \to \mathbb{R}$ be $f_n(x) = x^n$. Since f_n is an exponential function, it is continuous, and since $0 \leq f_n(x) \leq 1 < 2$, $(f_n) \subseteq B$. However, we demonstrated in class that (f_n) is not equi-continuous. Therefore, there exits an $\epsilon > 0$, two sequences (x_n), $(y_n) \in [0,1]$, and a sequence $(f_n) \in B$ such that $|x_n - y_n| \to 0$ as $n \to \infty$ and $\|f_n(x) - f_n(y)\| \geq \epsilon$. Therefore B is not equi-continuous. This shows that B is not compact.

Problem 3

Let $A \subset \mathbb{R}$ be a bounded set, and the set $B \subset C(A, \mathbb{R})$ be

$$B = \{\frac{x^2}{\alpha^2 + x^2} : A \to \mathbb{R} | \alpha \geq 1\}.$$

To show that B is equi-continuous, we must show that for every $\epsilon > 0$, there exists a $\delta > 0$ such that $\|f_\alpha(x) - f_\alpha(y)\| < \epsilon$ for every $x, y \in A$ where $d(x,y) < \delta$. Then for any $f_\alpha \in B$ and $x, y \in A$, we have

$$\|f_\alpha(x) - f_\alpha(y)\| = \left\| \frac{x^2}{\alpha^2 + x^2} - \frac{y^2}{\alpha^2 + y^2} \right\|_\infty < f'(z)|x - y|$$

For some $x < z < y$ from the mean value theorem. $f'(z)|x-y| = \frac{2z\alpha^2}{(\alpha^2+z^2}|x-y|$. Since A is bounded, there exits a constant N such that $|z| \leq N$ for every $x, y \in A$. Therefore the expression $\frac{2z\alpha^2}{(\alpha^2+z^2}$ is bounded by some contant M. We can then say $f'(z)|x-y| < M|x-y|$ for some constant M. Combining these statements gives us $\|f_\alpha(x) - f_\alpha(y)\| < M|x-y|$. Let $\delta = \frac{\epsilon}{M}$, then when $|x-y| < \delta$, $\|f_\alpha(x) - f_\alpha(y)\| < M|x-y| < M\frac{\epsilon}{M} = \epsilon$. Therefore B is equi-continuous.

Problem 4

Consider the space $C([0,1])$ of real-valued continuous functitons on $[0,1]$ endowed with the sup-norm $\|\cdot\|_\infty$. Let $B \subset C([0,1])$ be

$$B = \{f \in C([0,1]) \mid f \text{ is differentiable on } [0,1], -1 \leq f'(x) \leq 2, \forall x \in [0,1], f(0) = 0\}$$

(1): From the fundamental theorem of calculus, we have

$$f(x) = \int_0^1 f'(x)dx \leq \int_0^1 2dx = 2 + a$$

For some constant a, and similarly

$$f(x) = \int_0^1 f'(x)dx \geq \int_0^1 -1dx = -1 + b$$

For some constant b. Since we know $f(0) = 0$, we can set a and b equal to 0. Thus $-1 \leq f(x) \leq 2$, so that $f(x)$ is bounded. Therefore $\|f(x)\|_\infty \leq 2$. This shows that for any $f \in B$, $f(x)$ is bounded under the sup-norm.

Let g be a limit point of B. Then for every $\epsilon > 0$, there exists an $f_0 \in B$ such that $\|f_0(x) - g(x)\|_\infty < \epsilon$. Since B is bounded, $\|f_0\|_\infty < M$ for some constant M. Therefore, using the triangle inequality, we have $\|f_0 - g\|_\infty \leq \|f_0\|_\infty + \|f - g\|_\infty < M + \epsilon$. Since ϵ can be arbirarily small, $M + \epsilon$ is a bound for $\operatorname{cl} B$, thus $\operatorname{cl} B$ is bounded.

(2): From the intermediate value theorem, we have $\|f(x) - f(y)\| \leq |f'(z)|x-y|$ for some $z \in [0,1]$ such that $x < z < y$. Since $-1 \leq f'(z) \leq 2$, then $\|f'(z)\|_\infty < M$ for some constant M. Let $\delta = \frac{\epsilon}{M}$, then when $|x-y| < \delta$, $\|f_\alpha(x) - f_\alpha(y)\| < M|x-y| < M\frac{\epsilon}{M} = \epsilon$. Therefore B is equi-continuous. Since $\operatorname{cl} B$ is bounded by necesity, $\operatorname{cl} B$ is bounded, closed, and equi-continuous, and thus compact.

Problem 5

Let $\mathbb{R}_+ := \{x \in \mathbb{R} | x \geq 0\}$, and consider the sequence of functions $f_n : \mathbb{R}_+ \to \mathbb{R}$ defined by

Let $\delta = \epsilon$. Then, if $|x-y| < \delta, |\frac{x-y}{n} < \delta = \epsilon$ since $n \geq 1$. Therefore for every $\epsilon > 0$ there exits a $\delta > 0$ such that $|x-y| < \delta$ implies that $|f_n(x) - f_n(y)| < \epsilon \ \forall n \in \mathbb{N}$. This shows that (f_n) is equi-continuous.

since, $\max f(x) > 1$, then f is bounded.

Let $x = 1$. Then $B_x = \{f(x) : f \in (f_n)\} = 1, \frac{1}{2}, \frac{1}{3}...$ Since $0 \notin B_x$, but is a limit point, then B_x is open, which means it is not compact. Therefore (f_n) is not pointwise compact, which by the Azela-Ascoli Theorem tells us that (f_n) is not compact.

Problem 6

Let (f_n) be an equi-continuous sequence of functions $f_n : (M,d) \to \mathbb{R}$, where (M,d) is compact. Suppose that (f_n) converges pointwise to f_* on M.

Since (f_n) is equi-continuous, then for every $\epsilon > 0$, there exists a $\delta > 0$ such that $|f_n(x), f_n(y)| < \epsilon$ for every $x, y \in M$ where $d(x,y) < \delta$. Since (f_n) is pointwise convergent, for every fixed $x \in M$, and any $\epsilon > 0$, there exists a $K(\epsilon, x) \in \mathbb{N}$ such that $d(f_n(x), f_*(x)) < \epsilon$ for every $n \geq K(\epsilon, x)$. Let $\epsilon > 0$ be arbitrary. Then there exists a $K \in \mathbb{N}$. Since M is compact, then $\min(f(x))$

Therefore, for every $\epsilon > 0$, there exists a $\delta > 0$ such that $|f_n(x) - f_n(y)| < \epsilon$ for every $x, y \in M$ where $d(x,y) < \delta$.