MATH 600 Homework 6

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Problem 1

Let $f_n : [1,2] \to \mathbb{R}$ be $f_n(x) = \frac{x}{(x+1)^n}$.

(1): Since f is defined on a compact set which maps to $\mathbb{R}$, the min-max theorem tells us that f_n for every $n \in \mathbb{N} \geq 1$. For any n, x, f_n achieves its maximum at 1. Thus $\max_{x\in[1,2]} f_n(x) = \frac{1}{2^n}$. Then, since $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a convergent geometric series, the Weierstrass M-Test tells us that $\sum_{n=1}^{\infty} f_n$ converges uniformly on $[1,2]$.

(2): Since f_n is continuous on $[1,2]$, and the series $\sum_{n=1}^{\infty} f_n$ converges uniformly on $[1,2]$, then $\int_1^2 \left(\sum_{n-1}^{\infty} f_n(x)\right) dx = \sum_{n-1}^{\infty} \left(\int_1^2 f_n(x)\right) dx$. This is a direct implication of the corollary to the proof that the integral of a uniformly convergent function is equal to the integral of its limiting function.

Problem 2

Let $A = [-a, a] \subset \mathbb{R}$ with $a > 0$, and let

$$f_n(x) = \frac{-1^{n-1} x^{2n-1}}{(2n-1)!}, x \in \mathbb{R}$$

(1):

(2):

Problem 3

Let A be a bounded set in $\mathbb{R}$, and $f_n : \mathbb{R} \to \mathbb{R}$ be

$$f_n(x) = \frac{(-1)^{n+1} x}{\sqrt{n}}.$$

(1):
To show that the series $\sum f_n$ is uniformly convergent, we must show that it is Cauchy.

$S_n = \sum_{i=1}^{n} \frac{(-1)^{i+1} x}{\sqrt{i}} = x \sum \frac{(-1)^{i+1}}{\sqrt{i}}$

(2):

Problem 4

Let $f_n : \mathbb{R} \to \mathbb{R}$ be $f_n(x) = \frac{x}{n^2+x^2}$.

(1):

(2):

Problem 5

Let $(V, ||\cdot||)$ be a complete normed vector space and its induced metric $d(x,y) = ||x-y||$ for $x, y \in V$. Let $f : V \to V$ be a linear mapping for all $x \in V$.

(1):

$(\Rightarrow)$: If f is a contraction, then for any $x, y \in V$, $d(f(x), f(y)) \leq d(x,y)$. Let $x \in V$ be arbitrary and let $y = 0$. Then $d(f(x), f(y)) \leq C \cdot d(x,y) \Rightarrow ||f(x) - f(y)|| \leq C \cdot ||x-y|| \Rightarrow ||f(x) - f(0)|| \leq C||x-0|| \Rightarrow ||f(x)|| \leq C||x||$. Thus there exists a constant C such that $||f(x)|| \leq C||x||$ for all $x \in V$.

$(\Leftarrow)$: Let $x, y \in V$, since V is closed under scalar multiplication and addition $x + (-1)y = x - y \in V$. Therefore $||f(x-y)|| \leq C||x-y|| \Rightarrow ||f(x) - f(y)|| \leq C||x-y|| \Rightarrow d(f(x), f(y)) \leq C \cdot d(x,y)$. Since $0 < C < 1 \subset [0,1)$, then f is a contraction.

(2): Since $(V, ||\cdot||)$ is complete, if f is a contraction, then the recursive sequence $x_n = f(x_{n-2})$ is Cauchy, and converges to a fixed point $x_* \in V$, as demonstrated in class. Assume $x_* \neq 0$. Since f is linear and $x_* \neq 0$, then $f(\alpha x_*) = \alpha f(x_*) = \alpha x_*$. Therefore αx_* is also a fixed point of f. However, this means that f has multiple fixed points, which contradicts Banach's contraction mapping theorem. Therefore $x_* = 0$.

Problem 6

Let the constant K satisfy $0 < K < 1$. Consider the linear function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$f(x) = \tfrac{K}{\sqrt{2}}(x_1 + x_2, x_2 - x_1), \forall x = (x_1, x_2) \in \mathbb{R}^2.$$

(1): With the 2-norm, f is a contraction if there exists a constant $C \in [0,1)$ such that

$$||f(x) - f(y)||_2 \leq C||x-y||_2 \ \forall x, y \in \mathbb{R}^2.$$

Let $z = x - y$. Since f is a linear function, $||f(x) - f(y)||_2 = ||f(x-y)||_2 = ||f(z)||_2$. Therefore $||f(x) - f(y)||_2 = \sqrt{\left(\tfrac{K}{\sqrt{2}}(z_1 + z_2)\right)^2 + \left(\tfrac{K}{\sqrt{2}}(z_2 - z_1)\right)^2} = \sqrt{\tfrac{K^2}{2}(z_1^2 + 2z_1z_2 + z_2^2) + \tfrac{K^2}{2}(z_2^2 - 2z_1z_2 + z_1^2)} = \sqrt{\tfrac{K^2}{2}(z_1^2 + 2z_1z_2 + z_2^2 + z_2^2 - 2z_1z_2 + z_1^2)} = \sqrt{\tfrac{K^2}{2}(2z_1^2 + 2z_2^2)} = \sqrt{K^2(z_1^2 + z_2^2)} = \sqrt{K^2}\sqrt{z_1^2 + z_2^2} = K\sqrt{z_1^2 + z_2^2} = K||z||_2 = K||x-y||_2$. Therefore, $||f(x) - f(y)||_2 = K||x-y||_2$.

Since $K < 1$, we can find a C such that $K \leq C < 1$, then $||f(x) - f(y)||_2 \leq C||x-y||_2$. Therefore, f is a contraction with the 2-norm.

(2): With the 1-norm, f is a contraction if there exists a constant $C \in [0,1)$ such that

$$|f(x) - f(y)| \leq C|x-y| \ \forall x, y \in \mathbb{R}^2.$$

Let $z = x - y$. Since f is a linear function, $||f(x) - f(y)||_1 = ||f(x-y)||_1 = ||f(z)||_1$. Therefore $||f(x) - f(y)||_1 = |\tfrac{K}{\sqrt{2}}(z_1 + z_2)| + |\tfrac{K}{\sqrt{2}}(z_2 - z_1)| = \tfrac{K}{\sqrt{2}}(|z_1 + z_2| + |z_2 - z_1|)$ since $K > 0$. Since $K > \tfrac{1}{\sqrt{2}}$, then $||f(z)||_1 < |z_1 + z_2| + |z_2 - z_1|$

(3):

(4): Since norms are equivalent, there exists constants C_1 and C_2 such that $C_1||x||_2 / leq ||x||_1 \leq C_2||x||_2$. Since we know (x^k) is convergent with the 2-norm, then for every $\epsilon > 0$, there exits an $N \in \mathbb{N}$ such that $n \geq N$ implies $||x^n - x_*|| < \epsilon$. Let $\delta = \epsilon / C_2$. Then there exists an $M \in N$ such that $m \geq M$ implies $||x^m - x_*|| < \epsilon / C_2$. Therefore $||x^m - x_*||_1 \leq C_2||x^m - x_*||_2 \leq C_2 \cdot \tfrac{\epsilon}{C_2} = \epsilon$. Thus (x^k) is convergent under the 1-norm.