MATH 600 Homework 4

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0.0.1 Problem 1

Let (M, d) be a metric space, and $E \subseteq X \subseteq M$, where X is a totally bounded set. Since X is totally bounded, for every $\epsilon > 0$, there is a finite union of open balls of radius ϵ which covers X. Since $E \subset X$, this collection is also a cover of E. Therefore E is totally bounded.

0.0.2 Problem 2

In $\mathbb{R}$, Heine-Borel tells us a compact set is a closed and bounded set. A connected set on $\mathbb{R}$ if and only if for any set $E \subseteq R$, for any $x, y \in E$, $z \in E$ if $x < z < y$. Therefore a connected set in $\mathbb{R}$ is either a closed or open interval (x, y) or $[x, y]$. Therefore a compact and connected set in $\mathbb{R}$ is a closed interval $[x, y]$.

0.0.3 Problem 3

A metric space (M, d) is connected if and only if the only open and closed sets in M are M and the empty set.

$(\Rightarrow)$: Let (M, d) be a connected, and suppose there exists a nonempty clopen set $E \subset M$. Then E^c is also nonempty, closed and open and $E \cup E^c = M$. Since both E and E^c are closed, $E = cl(E)$ and $E^c = cl(E^c)$. Therefore $E \cap cl(E^c) = \emptyset$, $cl(E) \cap E^c = \emptyset$, which shows E and E^c are seperated. Since M is the union of two serpated sets, (M, d) is not connected. The contradiction implies that the only clopen sets in a connected space are M and $\emptyset$.

$(\Leftarrow)$: Suppose (M, d) is not connected, but the only two clopen sets are $\emptyset$ and M. Then there exist two nonempty sets A and B such that $M \subseteq A \cup B$ and A and B are disjoint. Then $M/A = B$

0.0.4 Problem 4

Let (M, d) be a metric space. Fix $x \in M$ and define $f : M \to \mathbb{R}$ by $f := d(z, x)$. Show that f is continuous on M.

Let $p \in M$, and $\epsilon > 0$. Then if $d_y(f(p), f(y)) < \epsilon \Rightarrow d_y(d(p, x), d(y, x)) < \epsilon \Rightarrow \sqrt{d(y, x)^2 + d(x, p)^2} < \epsilon \Rightarrow \sqrt{d(y, p)^2} < \epsilon \Rightarrow d(y, p) < \epsilon$. Let $\delta = \epsilon$. Then if $d(y, p) < \delta$, $d_y(f(p), f(y)) < \epsilon$, therefore f is continuous.

0.0.5 Problem 5

(1): Let $f : \mathbb{R}^2 \to \mathbb{R}$ be $f(x_1, x_2) = x_1$. Let $\epsilon > 0$ be arbitrary. For any $x \in f($

0.0.6 Problem 6

Let $f : (M, d) \to (N, \rho)$ be continuous on M, and $B \subseteq M$.