Chapter 4: Trigonometric Interpolation

In polynomial interpolation, given a set of data points $[t_i, f_i]$, $i = 1, 2, \ldots, n$, we determine a polynomial of degree at most $n-1$, $p_{n-1}(t)$, such that $p_{n-1}(t_i) = f_i$, $i = 1, 2, \ldots, n$.

In some applications, the given data may be function values of a periodic function. In such cases, it is preferable to use periodic functions as well in the interpolation, e.g., the trigonometric functions.

Let us now consider the case when

1 Some basics of complex numbers

Let $z = x + iy$ be a complex value, where x is the real part of z and y is the imaginary part of z. The complex conjugate of z is defined by $\bar{z} = x - iy$, and $|z|^2 = x^2 + y^2 = \bar{z} \cdot z$. Here i represents the imaginary unit and $i^2 = -1$.

The complex exponential e^{ix}, where x is a real number, is defined by

$$e^{ix} = \cos x + i \sin x \quad .$$

e^{ix} is a periodic function with period 2π, since $e^{i(2\pi+x)} = e^{ix}$. e^{ix} also represents the points on the unit circle with center at the origin in the complex number plane, where x represents the angle. For example,

$$e^{i\pi} = -1, \quad e^{2i\pi} = 1, \quad e^{i\pi/2} = i.$$

We also have

$$\overline{e^{ix}} = \overline{\cos x + i \sin x} = \cos x - i \sin x = e^{-ix}.$$

The formulas $e^{a+b} = e^a \cdot e^b$ and $(e^a)^b = e^{a \cdot b}$ apply here as well. For example

$$e^{i(z+\pi)} = e^{iz} \cdot e^{i\pi} = -e^{iz}, \quad e^{2i\pi} = \left(e^{i\pi}\right)^2 = (-1)^2 = 1.$$

The rules of integration also apply, e.g.,

$$\int_a^b e^{inz} dz = \frac{1}{in}\left(e^{inb} - e^{ina}\right) \qquad \text{if} \ \ n \neq 0 \ .$$

The rules for summing geometric series also apply to complex exponentials. Consider the geometric series

$$S = z^k + z^{k+1} + \cdots + z^n \quad .$$

Multiplying this by z gives

$$zS = z^{k+1} + z^{k+2} + \cdots + z^{n+1} \quad .$$

Subtracting these expressions gives

$$S - zS = z^k - z^{n+1} \quad .$$

Solving for S gives the eventual formula for the geometric sum

$$S = z^k + z^{k+1} + \cdots + z^n = \frac{z^k - z^{n+1}}{1 - z} \quad \text{if } z \neq 1.$$

In particular,

$$1 + z + \cdots + z^{N-1} = \frac{1 - z^N}{1 - z} \quad \text{if } z \neq 1.$$

2 Phase polynomials

When we discuss the polynomial interpolation problem, for any given set of points $[t_k, y_k]$, $k = 0, 1, 2, \ldots, n-1$, with $t_k \neq t_j$ if $k \neq j$, there always exists a unique polynomial $p(t)$ of degree at most $n-1$, such that $p(t_k) = y_k$, $k = 0, 1, 2, ..., n-1$. There we consider the case that all t_k, y_k, and the coefficients of $p(t)$ are real values.

In fact there is no limitation that the polynomial interpolation problem has to be in the real number field. In general, given any set of data $[w_k, f_k]$, $k = 0, 1, 2, \ldots, N-1$, where $w_k \neq w_j$ if $k \neq j$ and both w_k and f_k can be complex, there always exists a unique polynomial of degree at most $N-1$,

$$p(w) = \beta_0 + \beta_1 w + \beta_2 w^2 + \ldots + \beta_{N-1} w^{N-1},$$

satisfying $p(w_k) = f_k$, $k = 0, 1, 2, ..., N-1$, where the coefficients β_j of $p(w)$ are complex in general.

Let us consider an equally spaced partition of the interval $[0, 2\pi]$, by $x_k = k\dfrac{2\pi}{N}$, for $k = 0, 1, 2, \ldots, N-1, N$, where we can see that $x_0 = 0$ and $x_N = 2\pi$. Those x_k separate the angle 2π equally into N angles.

Let us introduce another variable $w = e^{ix}$. Corresponding to the partition $x_k = k\dfrac{2\pi}{N}$, $k = 0, 1, 2, \ldots, N-1$, of the interval $[0, 2\pi]$, we have

$$w_k = e^{ix_k} = e^{i\frac{2\pi k}{N}}, \quad k = 0, 1, 2, \ldots, N-1,$$

which represents an equally spaced partition of the unit circle in the complex number plane. We also see that $w_k \neq w_j$, if $k \neq j$ and $k, j = 0, 1, \ldots, N-1$.

Given a set of function values f_k of a periodic function with period 2π, corresponding to the partition points x_k, $k = 0, 1, 2, \ldots, N-1, N$, of the interval $[0, 2\pi]$, we can consider an interpolation of those function values by using trigonometric functions of different frequency. Here the function values are periodic, i.e., $f_0 = f_N$. Therefore only (x_k, f_k), for $k = 0, 1, 2, \ldots, N-1$, are needed.

Let us first consider the polynomial interpolation of (w_k, f_k), $k = 0, 1, 2, \ldots, N-1$, where $w_k = e^{ix_k}$. Using the interpolation theorem, we know that there exists a unique polynomial in w of degree at most $N-1$,

$$P(w) = \beta_0 + \beta_1 w + \beta_2 w^2 + \ldots + \beta_{N-1} w^{N-1},$$

satisfying $P(w_k) = f_k$, $k = 0, 1, 2, \ldots, N-1$, where the polynomial coeffcients β_j, $j = 0, 1, 2, \ldots, N-1$ are uniquely determined and are in general complex. Written in the variable x, we have a unique

$$p(x) = P(w) = \beta_0 + \beta_1 w + \beta_2 w^2 + \ldots + \beta_{N-1} w^{N-1} = \beta_0 + \beta_1 e^{ix} + \beta_2 e^{2ix} + \ldots + \beta_{N-1} e^{(N-1)ix}$$

satisfying $p(x_k) = f_k$, $k = 0, 1, 2, ..., N-1$. $p(x)$ is called the phase polynomial. Here $p(x)$ is a certain combination of the periodic functions $1, e^{ix}, e^{2ix}, \cdots, e^{(N-1)ix}$ with different frequencies. In another word, here we consider to interpolate the function $f(x)$ by periodic functions (essentially trigonometric functions here) of different frequencies on the interval $[0, 2\pi]$.

In the following, we discuss how to determine the phase polynomial $p(x)$ for any given set of (x_k, f_k), $k = 0, 1, 2, \ldots, N-1$, where $x_k = k\dfrac{2\pi}{N}$. Let us define a vector

$$\mathbf{w} = [w_0, w_1, \ldots, w_{N-1}] = \left[1, e^{ix_1}, \ldots, e^{ix_{N-1}}\right]$$

and the h-th power of $\mathbf{w}$ by

$$\mathbf{w}^{(h)} = \left[w_0^h, w_1^h, \ldots, w_{N-1}^h\right] = \left[1, e^{ihx_1}, \ldots, e^{ihx_{N-1}}\right].$$

Given any two vectors $\mathbf{u} = [u_0, u_1, \ldots, u_{N-1}]$ and $\mathbf{v} = [v_0, v_1, \ldots, v_{N-1}]$, define the inner product of $\mathbf{u}$ and $\mathbf{v}$ by

$$(\mathbf{u}, \mathbf{v}) = \sum_{k=0}^{N-1} u_k \overline{v_k}.$$

Then we have the following orthogonality properties on the powers of $\mathbf{w}$:

$$\left(\mathbf{w}^{(j)}, \mathbf{w}^{(h)}\right) = \begin{cases} N, & j = h, \\ 0, & j \neq h, \end{cases}$$

where $j, h = 0, 1, 2, \ldots, N-1$. To prove this orthogonality, we observe that

$$\left(\mathbf{w}^{(j)}, \mathbf{w}^{(h)}\right) = \sum_{k=0}^{N-1} w_k^j w_k^{-h} = \sum_{k=0}^{N-1} w_k^{j-h} = \sum_{k=0}^{N-1} e^{i(j-h)x_k},$$

which equals N, if $j = h$. If $j \neq h$, where $j, h = 0, 1, 2, \ldots, N-1$, denote $z = e^{\frac{i(j-h)2\pi}{N}} \neq 1$,

$$\sum_{k=0}^{N-1} e^{i(j-h)x_k} = \sum_{k=0}^{N-1} \left(e^{\frac{i(j-h)2\pi}{N}}\right)^k = \sum_{k=0}^{N-1} z^k = \frac{1 - z^N}{1 - z} = 0,$$

since $z^N = 1$.

Using this orthogonality property, we are able to determine the coefficients in the phase polynomial

$$p(x) = \beta_0 + \beta_1 e^{ix} + \beta_2 e^{2ix} + \ldots + \beta_{N-1} e^{(N-1)ix},$$

such that $p(x_k) = f_k$, $k = 0, 1, 2, ..., N-1$. Denote

$$\begin{aligned} \mathbf{f} &= [f_0, f_1, \ldots, f_{N-1}] = [p(x_0), p(x_1), \ldots, p(x_{N-1})] \\ &= \left[\sum_{k=0}^{N-1} \beta_k w_0^k,\ \sum_{k=0}^{N-1} \beta_k w_1^k,\ \ldots,\ \sum_{k=0}^{N-1} \beta_k w_{N-1}^k\right] = \sum_{k=0}^{N-1} \beta_k \mathbf{w}^{(k)}. \end{aligned}$$

Then, for $j = 0, 1, 2, ..., N-1$,

$$\left(\mathbf{f}, \mathbf{w}^{(j)}\right) = \left(\sum_{k=0}^{N-1} \beta_k \mathbf{w}^{(k)}, \mathbf{w}^{(j)}\right) = \sum_{k=0}^{N-1} \beta_k \left(\mathbf{w}^{(k)}, \mathbf{w}^{(j)}\right) = N\beta_j,$$

i.e.,

$$\beta_j = \frac{1}{N}\left(\mathbf{f}, \mathbf{w}^{(j)}\right) = \frac{1}{N}\sum_{k=0}^{N-1} f_k w_k^{-j} = \frac{1}{N}\sum_{k=0}^{N-1} f_k e^{-ijx_k} = \frac{1}{N}\sum_{k=0}^{N-1} f_k e^{-i\frac{2\pi jk}{N}}.$$

There is a minimum property of the phase polynomial. Let us define the s-segments of the phase polynomial $p(x)$ by

$$p_s(x) = \beta_0 + \beta_1 e^{ix} + \beta_2 e^{2ix} + \ldots + \beta_s e^{six},$$

for $0 \le s \le N-1$, which is at most of order s. Typically, on the N interpolation knots,

$$p_s(x_k) \neq f_k, \quad k = 0, 1, 2, ..., N-1.$$

But we can prove that the difference between the function values f_k and $p_s(x_k)$ is the smallest among all phase polynomials $q(x) = \gamma_0 + \gamma_1 e^{ix} + \gamma_2 e^{2ix} + \ldots + \gamma_s e^{six}$ of order at most s, in the sense that

$$\sum_{k=0}^{N-1} \left(f_k - p_s(x_k)\right)^2 = \min_{q(x)} \sum_{k=0}^{N-1} \left(f_k - q(x_k)\right)^2,$$

To prove it, let us denote vectors

$$\begin{aligned} \mathbf{p}_s &= [p_s(x_0), p_s(x_1), \ldots, p_s(x_{N-1})] = \left[\sum_{k=0}^{s} \beta_k w_0^k,\ \sum_{k=0}^{s} \beta_k w_1^k,\ \ldots,\ \sum_{k=0}^{s} \beta_k w_{N-1}^k\right] = \sum_{k=0}^{s} \beta_k \mathbf{w}^{(k)} \\ \mathbf{q} &= [q(x_0), q(x_1), \ldots, q(x_{N-1})] = \left[\sum_{k=0}^{s} \gamma_k w_0^k,\ \sum_{k=0}^{s} \gamma_k w_1^k,\ \ldots,\ \sum_{k=0}^{s} \gamma_k w_{N-1}^k\right] = \sum_{k=0}^{s} \gamma_k \mathbf{w}^{(k)}. \end{aligned}$$

We first note that, for $k = 0, 1, 2, ..., N-1$,

$$\left(\mathbf{f} - \mathbf{p}_s,\ \mathbf{w}^{(k)}\right) = \left(\mathbf{f} - \sum_{j=0}^{s} \beta_j \mathbf{w}^{(j)},\ \mathbf{w}^{(k)}\right) = \left(\mathbf{f},\ \mathbf{w}^{(k)}\right) - \left(\beta_k \mathbf{w}^{(k)},\ \mathbf{w}^{(k)}\right) = N\beta_k - N\beta_k = 0,$$

from which we have

$$(\mathbf{f} - \mathbf{p}_s,\ \mathbf{p}_s - \mathbf{q}) = \left(\mathbf{f} - \mathbf{p}_s,\ \sum_{k=0}^{s} (\beta_k - \gamma_k)\mathbf{w}^{(k)}\right) = \sum_{k=0}^{s} (\beta_k - \gamma_k) \left(\mathbf{f} - \mathbf{p}_s,\ \mathbf{w}^{(k)}\right) = 0.$$

Then

$$\begin{aligned} (\mathbf{f} - \mathbf{q},\ f - \mathbf{q}) &= (\mathbf{f} - \mathbf{p}_s + \mathbf{p}_s - \mathbf{q},\ \mathbf{f} - \mathbf{p}_s + \mathbf{p}_s - \mathbf{q}) \\ &= (\mathbf{f} - \mathbf{p}_s,\ \mathbf{f} - \mathbf{p}_s) + (\mathbf{p}_s - \mathbf{q},\ \mathbf{p}_s - \mathbf{q}) \geq (\mathbf{f} - \mathbf{p}_s,\ \mathbf{f} - \mathbf{p}_s)\,. \end{aligned}$$

The minimum is achieved when $\mathbf{q} = \mathbf{p}_s$. The minimum is unique, since if $\mathbf{q} = \mathbf{p}_s$, then $p_s(x_k) = q(x_k)$, for $k = 0, 1, 2, ..., N-1$, and the uniqueness of the interpolation implies that the phase polynomials $q(x)$ and $p_s(x)$ are the same.

3 Trigonometric interpolation

The phase polynomial $p(x)$ is in general a complex function, even though the functions values f_k, for $k = 0, 1, 2, ..., N-1$, may be all real values. For example, consider the following given partition of the interval $[0,\ 2\pi]$ by two points $x_0 = 0$ and $x_1 = \pi$, with function values $f_0 = 0$ and $f_1 = 1$. Applying the above formulas for β_0 and β_1, we obtain the phase polynomial $p(x) = \dfrac{1}{2} - \dfrac{e^{ix}}{2}$, which is not real even though $p(0) = 0$ and $p(\pi) = 1$. This lead to consideration of generating interpolations by using real periodic functions, e.g., the trigonometric functions to interpolate real periodic functions values.

Given a partition of the interval $[0,\ 2\pi]$ by a set of N equally spaced knots $x_k = k\dfrac{2\pi}{N}$, $k = 0, 1, 2, ..., N-1$, and a set of real function values f_k, we can determine a function of the form

$$\psi(x) = \frac{A_0}{2} + \sum_{h=1}^{M} \left(A_h \cos hx + B_h \sin hx\right), \quad \text{if} \quad N = 2M+1;$$

or

$$\psi(x) = \frac{A_0}{2} + \sum_{h=1}^{M-1} \left(A_h \cos hx + B_h \sin hx\right) + \frac{A_M}{2} \cos Mx, \quad \text{if} \quad N = 2M;$$

with real coefficients, such that $\psi(x_k) = f_k$, $k = 0, 1, 2, ..., N-1$. We can this the trigonometric interpolation.

To determine the coefficients in the trigonometric interpolation, we need to look at its connection with the phase polynomial

$$p(x) = \beta_0 + \beta_1 e^{ix} + \beta_2 e^{2ix} + \ldots + \beta_{N-1} e^{(N-1)ix},$$

where it also holds that $p(x_k) = f_k$, $k = 0, 1, 2, ..., N-1$.

For example for the case $N = 2M$, at each x_k, $k = 0, 1, 2, ..., N-1$,

$$\begin{aligned}
p(x_k) &= \beta_0 + \beta_1 e^{ix_k} + \beta_2 e^{i2x_k} + \ldots + \beta_M e^{iMx_k} + \ldots + \beta_{N-1} e^{i(N-1)x_k} \\
&= \beta_0 + \sum_{h=1}^{M-1} \beta_h e^{ihx_k} + \sum_{h=1}^{M-1} \beta_{N-h} e^{i(N-h)x_k} + \beta_M e^{iMx_k} \\
&= \beta_0 + \sum_{h=1}^{M-1} \left(\beta_h e^{ihx_k} + \beta_{N-h} e^{-ihx_k}\right) + \beta_M e^{iMx_k} \\
&= \beta_0 + \sum_{h=1}^{M-1} \left[(\beta_h + \beta_{N-h}) \cos hx_k + i\,(\beta_h - \beta_{N-h}) \sin hx_k\right] + \beta_M e^{iMx_k},
\end{aligned}$$

where we used that $e^{iNx_k} = 1$, $e^{ihx_k} = \cos hx_k + i \sin hx_k$, and $e^{ihx_k} = \cos hx_k - i \sin hx_k$.

Comparing it with

$$\psi(x_k) = \frac{A_0}{2} + \sum_{h=1}^{M-1} (A_h \cos hx_k + B_h \sin hx_k) + \frac{A_M}{2} \cos Mx_k,$$

we have

$$A_0 = 2\beta_0, \quad A_M = 2\beta_M, \quad A_h = \beta_h + \beta_{N-h}, \quad B_h = i\,(\beta_h - \beta_{N-h}), \quad \text{for} \quad h = 1, 2, \ldots, M-1.$$

Similarly, for the case $N = 2M+1$, we have

$$A_0 = 2\beta_0, \quad A_h = \beta_h + \beta_{N-h}, \quad B_h = i\,(\beta_h - \beta_{N-h}), \quad \text{for} \quad h = 1, 2, \ldots, M.$$

The uniqueness of β_j in the phase polynomial $p(x)$ implies the uniqueness of A_h and B_h in the trigonometric interpolation. Substituting

$$\beta_j = \frac{1}{N} \sum_{k=0}^{N-1} f_k e^{-i\frac{2\pi jk}{N}},$$

into the above formulas, we have

$$\begin{aligned}
A_h &= \frac{1}{N} \sum_{k=0}^{N-1} f_k \left(e^{-i\frac{2\pi hk}{N}} + e^{-i\frac{2\pi(N-h)k}{N}}\right) = \frac{1}{N} \sum_{k=0}^{N-1} f_k \left(e^{-i\frac{2\pi hk}{N}} + e^{i\frac{2\pi hk}{N}}\right) \\
&= \frac{2}{N} \sum_{k=0}^{N-1} f_k \cos \frac{2\pi hk}{N} = \frac{2}{N} \sum_{k=0}^{N-1} f_k \cos hx_k \\
B_h &= \frac{i}{N} \sum_{k=0}^{N-1} f_k \left(e^{-i\frac{2\pi hk}{N}} - e^{-i\frac{2\pi(N-h)k}{N}}\right) = \frac{i}{N} \sum_{k=0}^{N-1} f_k \left(e^{-i\frac{2\pi hk}{N}} - e^{i\frac{2\pi hk}{N}}\right) \\
&= \frac{2}{N} \sum_{k=0}^{N-1} f_k \sin \frac{2\pi hk}{N} = \frac{2}{N} \sum_{k=0}^{N-1} f_k \sin hx_k
\end{aligned}$$

We can see that if all the function values f_k are real, then the coefficients in the trigonometric interpolation $\psi(x)$ are also real.

In the previous example, for given $x_0 = 0$ and $x_1 = \pi$, with function values $f_0 = 0$ and $f_1 = 1$, we found the phase polynomial

$$p(x) = \beta_0 + \beta_1 e^{ix} = \frac{1}{2} - \frac{1}{2} e^{ix},$$

from which we have $N = 2$, and the trigonometric interpolation is

$$\psi(x) = \frac{A_0}{2} + \frac{A_1}{2} \cos x = \frac{1}{2} - \frac{1}{2} \cos x.$$

4 Connection with Fourier series, discrete Fourier transform

Let us consider that $f(x)$ is periodic defined on the interval $[0, 2\pi]$ with period of 2π. Any periodic function $f(x)$ with period 2π can be represented by its *Fourier series* as:

$$f(x) = \sum_{\alpha=-\infty}^{\infty} \hat{f}_\alpha e^{i\alpha x} \quad .$$

The sum is over all integer values of α, both positive and negative. The numbers $\hat{f}_\alpha$ are called the *Fourier coefficients* of the function $f(x)$. The equation expresses the function $f(x)$ as a sum of the fundamental functions $e^{i\alpha x}$, corresponding to different frequencies.

To determine the Fourier coefficients $\hat{f}_\alpha$, let us first define an inner product of functions by

$$(f(x), g(x)) = \int_0^{2\pi} f(x)\overline{g(x)} dx.$$

We say two functions $f(x)$ are $g(x)$ are orthogonal if $(f(x), g(x)) = 0$. We observe that for any two integers α and β,

$$\left(e^{i\alpha x}, e^{i\beta x}\right) = \int_0^{2\pi} e^{i(\alpha-\beta)x} dx = \begin{cases} 2\pi & \text{if } \alpha = \beta, \\ 0 & \text{if } \alpha \neq \beta, \end{cases}$$

i.e., those fundamental functions $e^{i\alpha x}$ of difference frequency are orthogonal to each other. Using such orthogonality, for any given integer α,

$$\left(f(x), e^{i\alpha x}\right) = \int_0^{2\pi} \left(\sum_{\beta=-\infty}^{\infty} \hat{f}_\beta e^{i\beta x} \right) e^{-i\alpha x} dx = \sum_{\beta=-\infty}^{\infty} \hat{f}_\beta \left(e^{i\beta x}, e^{i\alpha x}\right) = 2\pi \hat{f}_\alpha,$$

i.e.,

$$\hat{f}_\alpha = \frac{1}{2\pi} \left(f(x), e^{i\alpha x}\right) = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-i\alpha x} dx \quad .$$

The discrete Fourier transform may be dereived as a discrete approximation of the above Fourier coefficients. Cut the interval $[0, 2\pi]$ into N subinterval of the same size by equally spaced nodes

$$x_k = k\frac{2\pi}{N} \quad \text{for} \quad k = 0,\ 1,\ \ldots,\ N-1.$$

Applying the rectangle rule to approximate the above integral, we have

$$\tilde{f}_\alpha = \frac{1}{2\pi}\,\frac{2\pi}{N}\,\sum_{k=0}^{N-1} f_k\ e^{-i\alpha x_k} = \frac{1}{N}\sum_{k=0}^{N-1} f_k\ e^{-i\frac{2\pi\alpha k}{N}} \quad .$$

where $f_k = f(x_k)$. We call such $\tilde{f}_\alpha$, for any integer α, the *discrete Fourier coefficients.*

We can see that such discrete Fourier coefficients $\tilde{f}_\alpha$, for $\alpha = 0,\ 1,\ \ldots,\ N-1$, are just the same as the coefficients

$$\beta_j = \frac{1}{N}\sum_{k=0}^{N-1} f_k e^{-i\frac{2\pi jk}{N}},$$

for $j = 0,\ 1,\ \ldots,\ N-1$, of the phase polynomial

$$p(x) = \beta_0 + \beta_1 e^{ix} + \beta_2 e^{2ix} + \ldots + \beta_{N-1} e^{(N-1)ix},$$

which satisfies

$$f_k = p(x_k) = \beta_0 + \beta_1 e^{ix} + \beta_2 e^{2ix} + \ldots + \beta_{N-1} e^{(N-1)ix} = \sum_{j=0}^{N-1} \beta_j e^{ijx_k},$$

for $k = 0, 1, 2, \ldots, N-1$. Here we derive exactly the same relationship, but based on the above discrete Fourier coefficients, i.e., we need to establish that for $k = 0, 1, 2, \ldots, N-1$,

$$f_k = \sum_{\alpha=0}^{N-1} \tilde{f}_\alpha e^{i\alpha x_k}.$$

Then we can see that the discrete Fourier coefficients are exactly the coefficients of the phase polynomial discussed earlier.

Let us treat the sample values, f_k, $k = 0,\ 1,\ \ldots,\ N-1$, as representing a discretely defined but periodic function on the interval $[0, 2\pi]$. That is, for any integer multiple of N, $f_{k\pm mN} = f_k$, i.e., f_k repeats its values after every N positions. The periodicity of the *discrete Fourier coefficients* $\tilde{f}_\alpha$ is also obvious from its formula. For any integer multiple of N,

$$\tilde{f}_{\alpha\pm mN} = \tilde{f}_\alpha \quad .$$

Due to such a periodicity, the *discrete Fourier coefficients* $\tilde{f}_\alpha$ can be represented by just N values. We call the vector of N discrete Fourier coefficients $\tilde{f} = (\tilde{f}_0, \tilde{f}_1, \ldots, \tilde{f}_{N-1})$ the Discrete Fourier Transform (DFT) of the N samples $f = (f_0, f_1, \ldots, f_{N-1})$.

This Discrete Fourier Transform (DFT) can be represented in the following matrix form

$$\tilde{f} := \begin{bmatrix} \tilde{f}_0 \\ \tilde{f}_1 \\ \vdots \\ \tilde{f}_{N-1} \end{bmatrix} = \begin{bmatrix} w_{11} & w_{12} & \dots & w_{1N} \\ w_{21} & w_{22} & \dots & w_{2N} \\ & & & \\ w_{N1} & w_{N2} & \dots & w_{NN} \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_{N-1} \end{bmatrix} =: W\ f \ \ ,$$

where W is an $N \times N$ matrix whose (α, k) entry is the complex number $\frac{1}{N}e^{-i\frac{2\pi\alpha k}{N}}$.

The above equation implies that $f = W^{-1}\tilde{f}$. The inverse of W can be found as in the following. Denote the conjugate transpose of W by W^*; the (j, β) entry of W^* is the conjugate of the (β, j) entry of W, i.e., $w^*_{j\beta} = \frac{1}{N}e^{i\frac{2\pi j\beta}{N}}$. Therefore the (j, k) element of W^*W is

$$(W^*W)_{jk} = \sum_{\beta=0}^{N-1} w^*_{j\beta}w_{\beta k} = \sum_{\beta=0}^{N-1} \frac{1}{N}e^{i\frac{2\pi j\beta}{N}} \frac{1}{N}e^{-i\frac{2\pi\beta k}{N}} = \frac{1}{N^2}\sum_{\beta=0}^{N-1} e^{i\frac{2\pi\beta(j-k)}{N}} \ \ .$$

To compute the sum, let us denote $z = i\frac{2\pi(j-k)}{N}$. If $j = k$, the sum equals N. If $j \neq k$,

$$\sum_{\beta=0}^{N-1} e^{i\frac{2\pi\beta(j-k)}{N}} = 1 + z + z^2 + \ldots + z^{N-1} = \frac{1 - z^N}{1 - z} = 0.$$

Therefore

$$(W^*W)_{jk} = \begin{cases} \frac{1}{N} & \text{if } j = k, \\ 0 & \text{if } j \neq k, \end{cases}$$

i.e.,

$$W^*W = \frac{1}{N}I \ \ ,$$

which means that

$$W^{-1} = NW^* \ \ .$$

Writing out the relation $f = W^{-1}\tilde{f} = NW^*\tilde{f}$ in component form, we have

$$f_k = \sum_{\alpha=0}^{N-1} e^{i\frac{2\pi\alpha k}{N}}\tilde{f}_\alpha = \sum_{\alpha=0}^{N-1} \tilde{f}_\alpha e^{i\alpha x_k} \ \ , \quad k = 0, 1, ..., N - 1.$$

In summary, given a set of data, f_k, for $k = 0, 1, \ldots, N-1$, corresponding to $x_k = k\dfrac{2\pi}{N}$, we have

$$f_k = \sum_{\alpha=0}^{N-1} \tilde{f}_\alpha e^{i\alpha x_k}, \quad \text{where} \quad \tilde{f}_\alpha = \frac{1}{N}\sum_{k=0}^{N-1} f_k\ e^{-i\alpha x_k}.$$

In matrix form, $\tilde{f} = Wf$, and $f = W^{-1}\tilde{f} = NW^*\tilde{f}$, which are called the Discrete Fourier Transform (DFT) and the inverse DFT.

5 Discrete Fourier Sine and Cosine Transform

Here we assume that the given sequence of values f_k, $k = 0, 1, ..., N-1$, are all real values, and $N = 2M + 1$ is an odd number (it is equally valid with minor modification for an even value N).

In this case, the discrete Fourier transform have the following properties, for $\beta = 0, 1, ..., N-1$,

$$\tilde{f}_\beta = \frac{1}{N}\sum_{k=0}^{N-1} f_k\ e^{-i\frac{2\pi\beta k}{N}} = \overline{\frac{1}{N}\sum_{k=0}^{N-1} f_k\ e^{-i\frac{2\pi(-\beta)k}{N}}} = \overline{\frac{1}{N}\sum_{k=0}^{N-1} f_k\ e^{-i\frac{2\pi(N-\beta)k}{N}}} = \bar{\tilde{f}}_{N-\beta}.$$

Then the discrete Fourier transform of f_k, for $k = 0, 1, ..., N-1$, can be written as

$$\begin{aligned}
f_k &= \sum_{\beta=0}^{N-1} e^{i\beta x_k}\tilde{f}_\beta = \tilde{f}_0 + \sum_{\beta=1}^{M} e^{i\beta x_k}\tilde{f}_\beta + \sum_{\beta=M+1}^{N-1} e^{i\beta x_k}\tilde{f}_\beta = \tilde{f}_0 + \sum_{\beta=1}^{M} e^{i\beta x_k}\tilde{f}_\beta + \sum_{\beta=1}^{M} e^{i(N-\beta)x_k}\tilde{f}_{N-\beta} \\
&= \tilde{f}_0 + \sum_{\beta=1}^{M}\left(e^{i\beta x_k}\tilde{f}_\beta + e^{-i\beta x_k}\bar{\tilde{f}}_\beta\right) = \tilde{f}_0 + 2\ Re\left\{\sum_{\beta=1}^{M} e^{i\beta x_k}\tilde{f}_\beta\right\} \\
&= \tilde{f}_0 + \frac{2}{N}\ Re\left\{\sum_{\beta=1}^{M} e^{i\beta x_k}\left(\sum_{j=0}^{N-1} f_j\ e^{-i\beta x_j}\right)\right\} \\
&= \tilde{f}_0 + \frac{2}{N} Re\left\{\sum_{\beta=1}^{M}\sum_{j=0}^{N-1} f_j e^{i\beta(x_k - x_j)}\right\} = \tilde{f}_0 + \frac{2}{N}\sum_{\beta=1}^{M}\sum_{j=0}^{N-1} f_j\cos\beta(x_k - x_j) \\
&= \tilde{f}_0 + \frac{2}{N}\sum_{\beta=1}^{M}\left(\sum_{j=0}^{N-1} f_j(\cos\beta x_j\ \cos\beta x_k + \sin\beta x_j\ \sin\beta x_k)\right) \\
&= \frac{A_0}{2} + \sum_{\beta=1}^{M}\left\{A_\beta\cos\beta x_k + B_\beta\sin\beta x_k\right\},
\end{aligned}$$

where

$$A_\beta = \frac{2}{N}\sum_{j=0}^{N-1} f_j\cos\beta x_j\ , \quad \text{and} \quad B_\beta = \frac{2}{N}\sum_{j=0}^{N-1} f_j\sin\beta x_j\ , \quad \beta = 0, 1, ..., M.$$

We call this the Discrete Fourier Sine and Cosine Transforms.

If we define a trigonometric function by

$$\phi(x) = \frac{A_0}{2} + \sum_{\beta=1}^{M}\left\{A_\beta\cos\beta x + B_\beta\sin\beta x\right\},$$

then $\phi(x_k) = f(x_k) = f_k$, for $k = 0, 1, \ldots, N-1$, and $\phi(x)$ can be regarded as an approximation of the original function $f(x)$.

6 The Fast Fourier Transformation (FFT) algorithm

Computation of the DFT requires multiply the $N \times N$ matrix W with the N-dimensional vector f. The straightforward matrix-vector multiplication would requires $O(N^2)$ flops. However, due to the special properties of the DFT, a faster algorithm, FFT, can compute the DFT in $O(N \log(N))$ work.

The FFT is based on the "divide and conquer" strategy. The computation of a size-N DFT is reduced to computation of two size-$N/2$ DFTs, plus some postprocessing.

Assume that $N = 2^l$, i.e., $l = \log_2 N$. More general N can also be accommodated with little modification of the algorithm. Denote $P(N)$ be flops required to compute a size-N DFT. The reduction leads to

$$P(N) = 2P(N/2) + C \cdot N \ , \tag{1}$$

where C is a constant independent of N. Applying such reduction repeatedly, we have

$$\begin{aligned} P(N) &= 2\,(2P(N/4) + C \cdot N/2) + C \cdot N \\ &= 4P(N/4) + C \cdot N + C \cdot N \\ &= 8P(N/8) + C \cdot N + C \cdot N + C \cdot N \\ &\ \ . \\ &\ \ . \\ &\ \ . \\ &= NP(1) + \overbrace{C \cdot N + \cdots + C \cdot N}^{\log_2(N) \text{ terms}} \end{aligned}$$

i.e.,

$$P(N) = O(N \log(N)).$$

What underlies (1) is some interesting manipulations with exponentials. Suppose that $N = 2M$. From the vector f we form two half size vectors g and h containing the even and odd numbered components of f, respectively:

$$g_k = f_{2k} \quad k = 0, \ldots, M-1 \ ,$$

$$h_k = f_{2k+1} \quad k = 0, \ldots, M-1 \ .$$

The two size $M = N/2$ DFT's of g and h are given by

$$\tilde{g}_\alpha = \frac{1}{M} \sum_{k=0}^{M-1} g_k e^{-i\alpha x_{2k}} \quad , \quad \tilde{h}_\alpha = \frac{1}{M} \sum_{k=0}^{M-1} h_k e^{-i\alpha x_{2k}} \quad ,$$

for $\alpha = 0, 1, \ldots, M-1$. We know that both $\tilde{g}_\alpha$ and $\tilde{h}_\alpha$ are periodic, i.e., $\tilde{g}_{\alpha+M} = \tilde{g}_\alpha$ and $\tilde{h}_{\alpha+M} = \tilde{h}_\alpha$. Also note that when computing $\tilde{h}_\alpha$, we still use the nodes, x_{2k}, $k = 0, \ldots, M-1$, i.e., starting at $x_0 = 0$, as usual.

Once we have $\tilde{g}$ and $\tilde{h}$, we can construct $\tilde{f}$ in $O(N)$ operations as following. For each $\alpha = 0, 1, \ldots, N-1$,

$$\begin{aligned}
\tilde{f}_\alpha &= \frac{1}{N}\sum_{j=0}^{N-1} f_j e^{-i\alpha x_j} = \frac{1}{N}\left[\sum_{k=0}^{M-1} f_{2k} e^{-i\alpha x_{2k}} + \sum_{k=0}^{M-1} f_{2k+1} e^{-i\alpha x_{2k+1}}\right] \\
&= \frac{1}{2}\left[\frac{1}{M}\sum_{k=0}^{M-1} g_k e^{-i\alpha x_{2k}} \;+\; \frac{1}{M}\sum_{k=0}^{M-1} h_k e^{-i\alpha x_{2k}} \cdot \exp\left(\frac{-i\alpha 2\pi}{N}\right)\right] \\
&= \frac{1}{2}\left(\tilde{g}_\alpha + \exp\left(\frac{-i2\pi\alpha}{N}\right)\tilde{h}_\alpha\right) \;,
\end{aligned}$$

which just takes three additional flops for each $\tilde{f}_\alpha$.