Homework 5, due Apr. 5

Consider the solution to the following integral equation

$$\int_0^\pi e^{s\cos(t)} x(t)dt = \frac{e^s - e^{-s}}{s}, \quad \forall s \in [0,\ \pi/2].$$

We know that the exact solution is $x(t) = \sin(t)$, for $t \in [0,\ \pi]$. In this homework, we find the solution numerically.

Partition the interval $[0,\ \pi/2]$ by a set of n equally spaced points, $s_i = (i-1)\frac{\pi}{2(n-1)}$, $i = 1, 2, \ldots, n$. Consider the following n equations at those n discrete points,

$$\int_0^\pi e^{s_i\cos(t)} x(t)dt = \frac{e^{s_i} - e^{-s_i}}{s_i}, \quad i = 1, 2, \ldots, n.$$

Approximate the integrals in the above n equations by applying the composite trapezoid rule on the $n-1$ subintervals of $[0,\ \pi]$ partitioned by n equally spaced points $t_i = (i-1)\frac{\pi}{(n-1)}$, $i = 1, 2, \ldots, n$. Denote the approximate values of $x(t)$ at those discrete points t_i by x_i, and denote by $\mathbf{x}$ the n–dimensional column vector containing x_i, $i = 1, 2, \ldots, n$. This leads to a system of linear equations for $\mathbf{x}$

$$A\mathbf{x} = \mathbf{b}.$$

1. Solve the above system of linear equations directly by using the MATLAB '\' operator. Take $n = 10, 20, 50$, respectively. For each n, draw the numerical solution on the interval $[0,\ \pi]$ and compare it with the true solution $x(t) = \sin(t)$. Explain what you observed.

2. Solve the same system by using the truncated SVD, as discussed in the lecture. You can take the tolerance value as 10^{-6} to filter out the smaller singular values in the summation. For each of $n = 10, 20, 50$, draw the numerical solution on the interval $[0,\ \pi]$ and compare it with the true solution $x(t) = \sin(t)$.

3. Solve the system by using the Tikhonov regularization approach, as discussed in the lecture. You can take the parameter $\mu = 10^{-3}$ in the algorithm. For $n = 10, 20, 50$, draw the numerical solution on the interval $[0,\ \pi]$ and compare it with the true solution $x(t) = \sin(t)$.