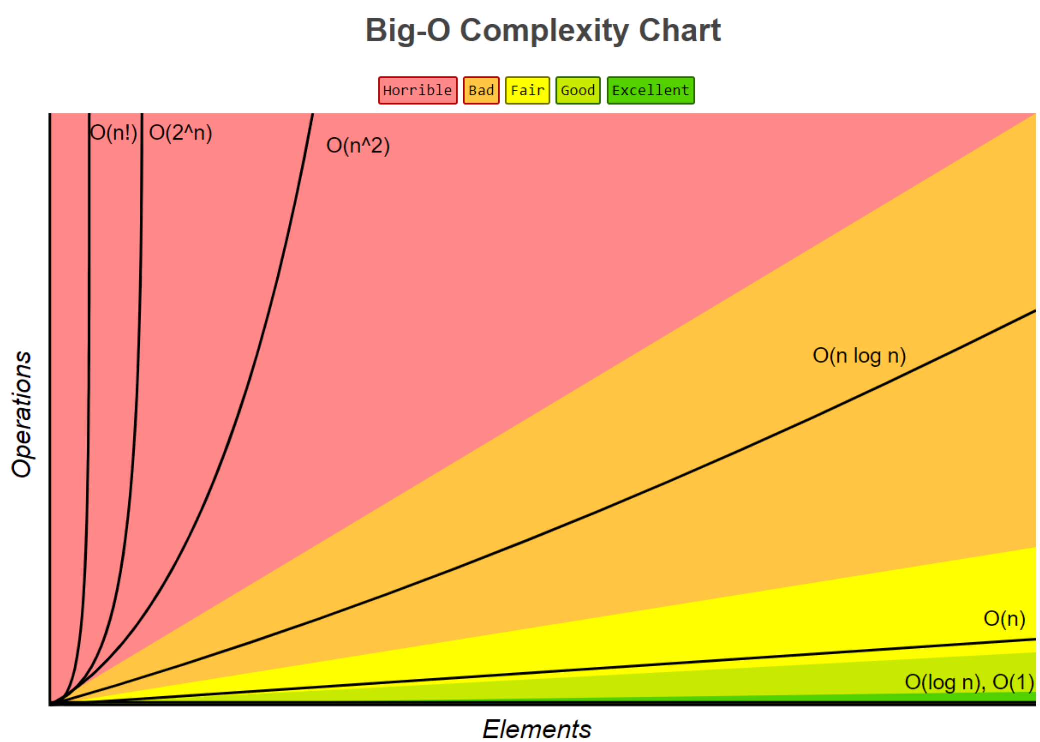
1. Algorithm Complexities

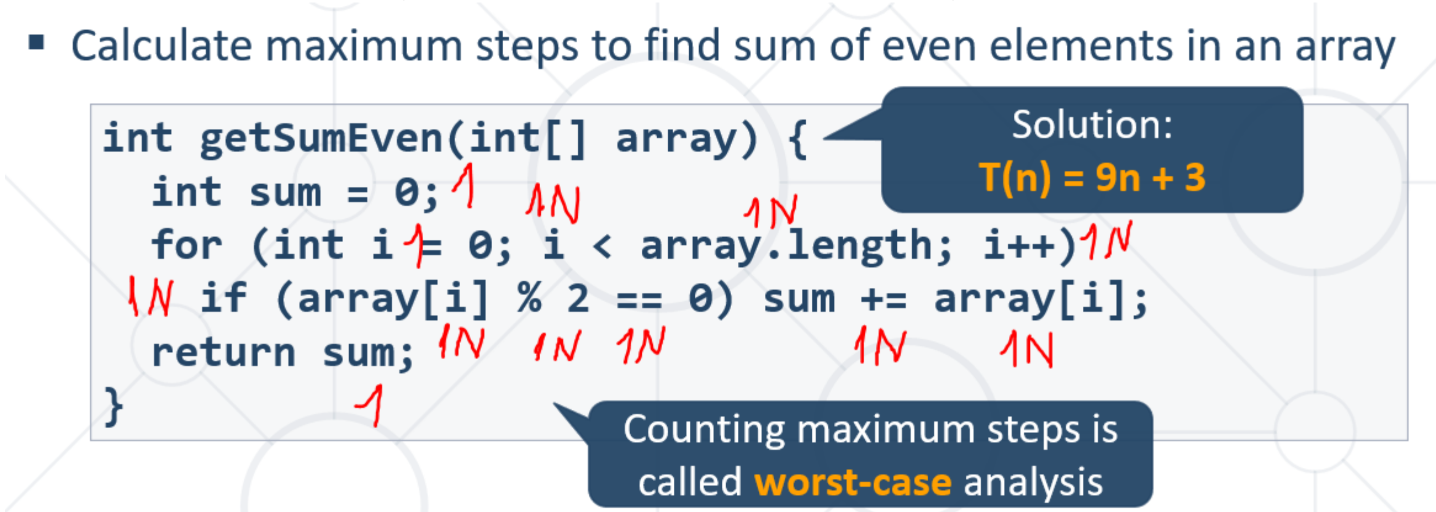
Asymptotic notations:

- Big **O – O(f(n)) – worst case - in our course**
- Big **Theta – Θ(f(n)) – амортизиран amortized constant time**
- Big **Omega – Ω(f(n))**



- O(1) – Constant time – time does not depend on **N**
- O(log(N)) – Logarithmic time – grows with rate as **log(N)**   *log2 64 = 6 (2^6 = 64)*
- O(N) – Linear time grows at the same rate as **N**
- O(N^2),O(N^3) – Quadratic, Cubic grows as square or cube of **N**
- O(2^N) – Exponential grows as **N** becomes the exponent worst algorithmic complexity

Assume that a **single step** is a single CPU instruction.
Гледаме колко стъпки има алгортъма ни, а не колко памет е заета (което е различно)



Инструкция на процесора можем да кажем, че е код завършващ с точка и запетая накрая.
Реално повече инструкции на процесора има на 1 ред код -аритметични/логически и т.н.

Typical Complexities

Complexity	Notation	Description
constant	O(1)	n = 1 000 → 1-2 operations
logarithmic	O(log n)	n = 1 000 → 10 operations
linear	O(n)	n = 1 000 → 1 000 operations
linearithmic	O(n*log n)	n = 1 000 → 10 000 operations
quadratic	O(n2)	n = 1 000 → 1 000 000 operations
cubic	O(n3)	n = 1 000 → 1 000 000 000 operations
exponential	O(n^n)	n = 10 → 10 000 000 000 operations

Brute-Force Algorithms – преминава през всички случаи/варианти – не е ефективен, но се ползва

2. Recursion

За да разберете какво е рекурсия, първо трябва да разберете какво е рекурсия 😊

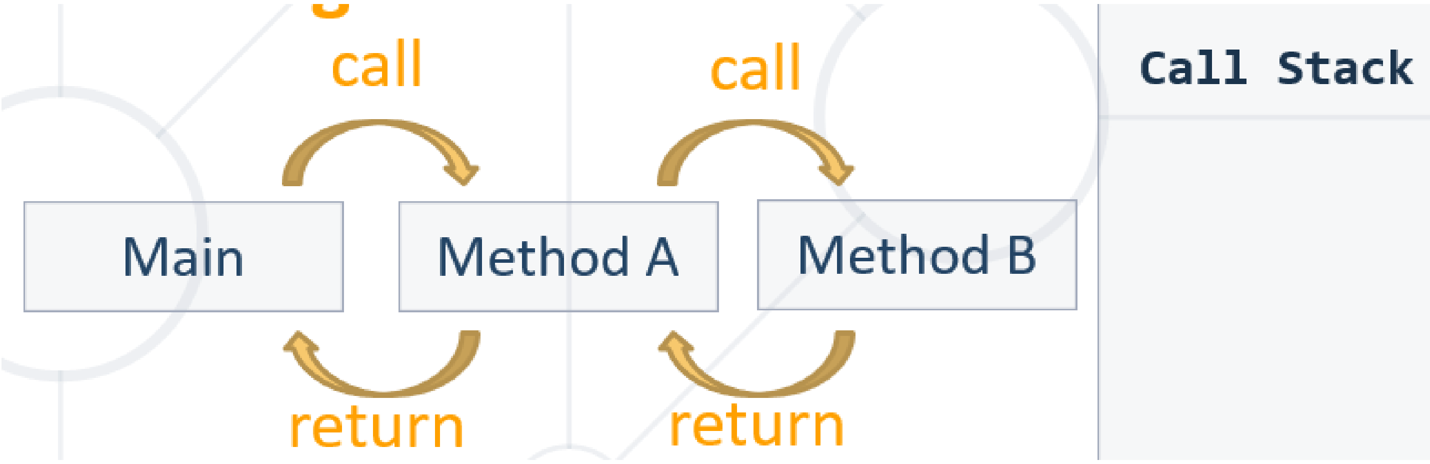
- **Method** of solving a problem where the solution depends on solutions to smaller instances of the same problem
- A common **computer programing tactic** is to **divide** a problem into **sub-problems** of the same type as the original, **solve** those sub-problems, and **combine** the **results**
- A function or a method that calls itself one or more times until a specified condition is met
- After the recursive call the rest code is processed from the last one called to the first
- Дефиницията за рекурсия, е че може да няма дъно

Recursive methods have 3 parts:

- **Pre-actions** (before calling the recursion)
- **Recursive calls** (step-in)
- **Post-actions** (after returning from recursion)

```
static void recursion() {
  // Pre-actions
  recursion();
  // Post-actions
}
```

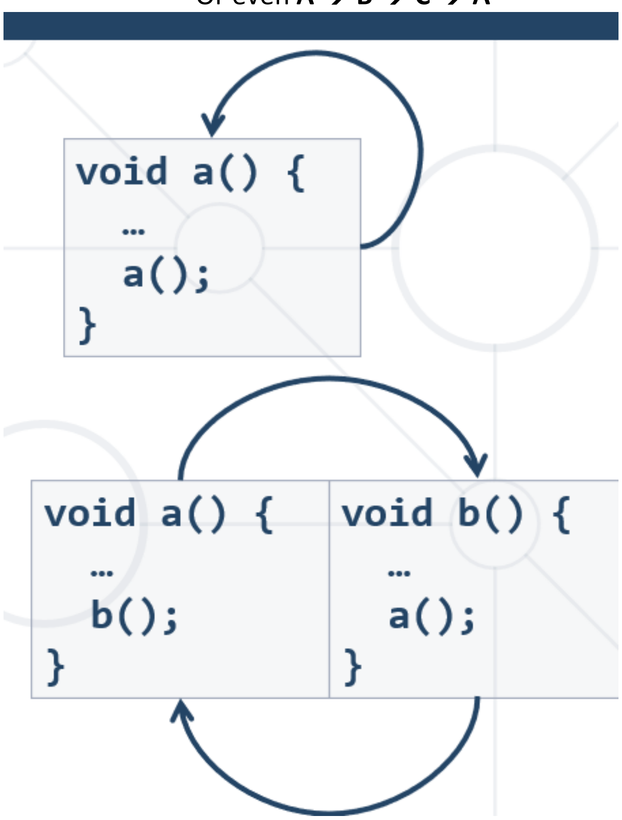
Работа на Call Stack



Other Definition of Recursion:

- Involves a **function calling itself**
- The function should have a **base case /дъно/**
- **Each step** of the recursion should **move towards** the **base case /дъно/**
- **Обаждаме инстанциите до дъното, и след това връщаме от дъното към върха резултата**

- Direct recursion
  - A method directly calls itself
- Indirect recursion
  - Method **A** calls **B**, method **B** calls **A**
  - Or even **A → B → C → A**




```
private static void drawFigure(int n) {
        if (n == 0) {
            return;
```

```java
        }

        for (int i = 0; i < n; i++) {
            System.out.print("*");
        }
        System.out.println();
        drawFigure(n-1); // до тук правим Call до дъното

        for (int i = 0; i < n; i++) {//оттук нататък правим Return  от дъното нагоре - post action
            System.out.print("#");
        }
        System.out.println();
    }
}
```

3. Backtracking

find all paths from Source to Destination

Рекурсивно извикваме алгоритъма от всяка една точка

We use recursion and Backtracking for branches problems.
Otherwise, we should use linear iterative algorithm

4. Memoization – not to be confused with memorization

Когато искаме да спестим вече изчислената част от дървото на рекурсията -да не я изчислява за всяко клонче същата част отново и отново.

Записваме в масив или друг тип колекция.

1, 1, 2, 3, 5, 8, 13, 21, 34 – нулевият елемент е 1, първият елемент е също 1, вторият елемент е 2, третият елемент е 3, четвъртият елемент е 5, петият елемент е 8

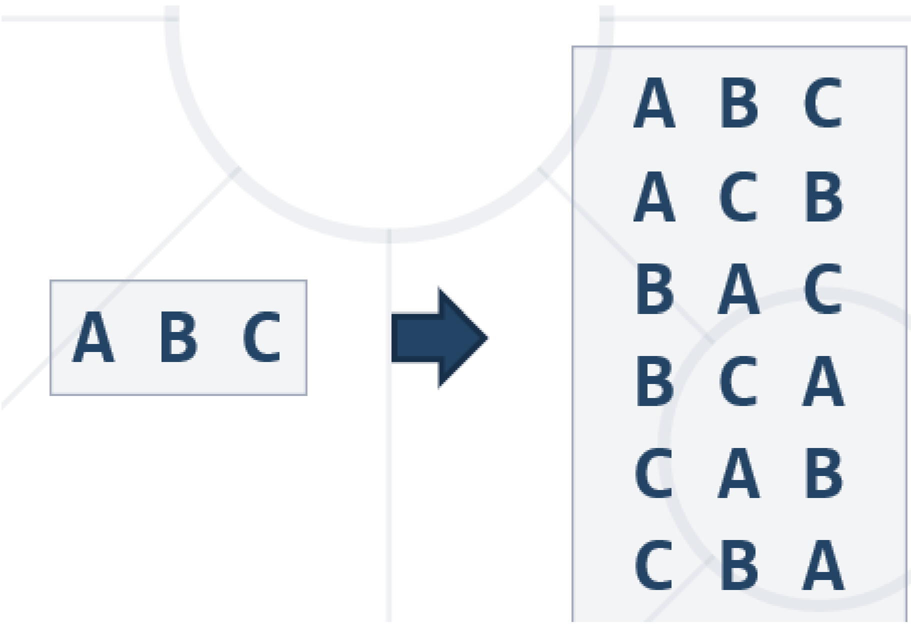
```java
public class RecursiveFibonacci {
    public static Long[] storedFibbonachiNumbers;

    public static void main(String[] args) {
        Scanner sc = new Scanner(System.in);
        int n = Integer.parseInt(sc.nextLine());
        storedFibbonachiNumbers = new Long[n + 1];

        for (int i = 0; i < n+1; i++) { //запълваме си масива с нули
            storedFibbonachiNumbers[i] = 0L;
        }


        if (n >= 1) { //нулвеият и първият елемент от масива са 1
            storedFibbonachiNumbers[0] = 1L;
            storedFibbonachiNumbers[1] = 1L;
        } else { //когато n е нула, то дължината на масива е 1. Нулевият елемент на масива е 1
            storedFibbonachiNumbers[0] = 1L;
        }

        Long fib = fibonacci(n);
        System.out.println(fib);
//        System.out.println(fibonacci(50)); // This will hang!
    }

    static long fibonacci(int n) {
        if (n == 0) {
            return storedFibbonachiNumbers[0];
        } else if (n == 1) {
```

```
            return storedFibbonachiNumbers[1];
        } else {
            if (storedFibbonachiNumbers[n] > 0L) {
                return storedFibbonachiNumbers[n];
            }

            return storedFibbonachiNumbers[n] = fibonacci(n - 1) + fibonacci(n - 2);
        }
    }
}
```

5. Combinatorial Problems

5.1. Permutations

Permutation of a set of items is arrangement all the items in the set, linear, in all possible ways



Permutations Count - при неповтарящи се елементи

= n! = 3! factorial = 6 possible ways –

Нормална пермутация – без повторения

```
import java.util.Scanner;

public class PermutationsWithoutRepetitions {
    public static String[] elements;
    public static String[] permutes;
    public static boolean[] used;

    public static void main(String[] args) {
        Scanner sc = new Scanner(System.in);
        elements = sc.nextLine().split("\\s+");
        permutes = new String[elements.length];
        used = new boolean[elements.length];

        permute(0);
    }

    private static void permute(int index) {
        if (index == elements.length) {
```

```java
            print();
            return;
        }

        for (int i = 0; i < elements.length; i++) {
            if (!used[i]) { //ако не е използван дадения елемент
                used[i] = true;
                permutes[index] = elements[i];
                permute(index + 1);
                used[i] = false;  // the Backtracking
            }
        }
    }

    private static void print() {
        System.out.println(String.join(" ", permutes));
    }
}
```

Optimize permutations – Swap algorithm: - ползва по-малко памет

```java
private static void permute(int index) {
    if (index == elements.length) {
        print(elements);
        return;
    }

    permute(index + 1);
    for (int i = index + 1; i < elements.length; i++) {
        swap(elements, index, i);
        permute(index + 1);
        swap(elements, index, i); //backtracking (unswapping)
    }
}

private static void swap(String[] arr, int first, int second) {
    String temp = arr[first];
    arr[first] = arr[second];
    arr[second] = temp;

}
```

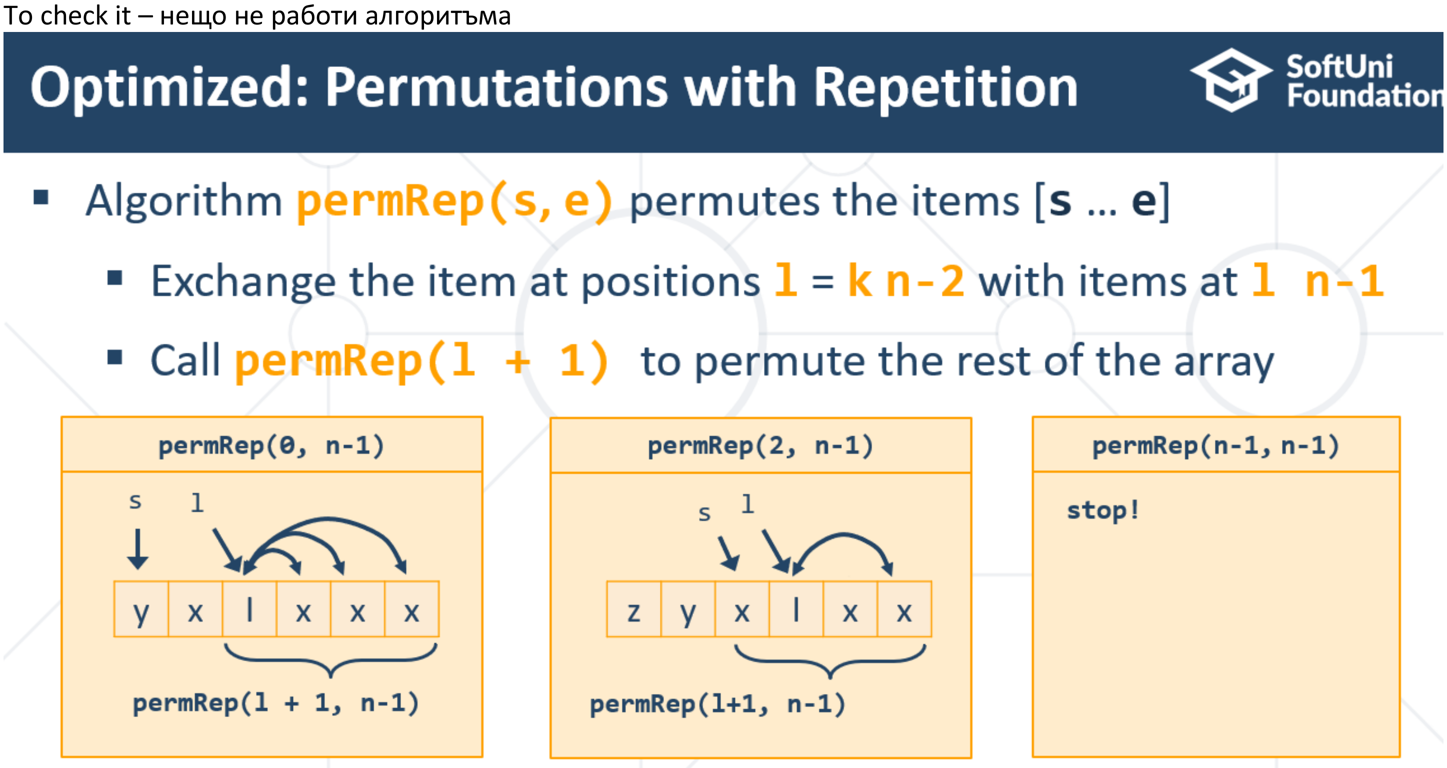
Permutations with Repetitions and Swap algorithm

```java
public class PermutationsWithRepetitionsSwap {
    public static String[] elements;

    public static void main(String[] args) {
        Scanner sc = new Scanner(System.in);
        elements = sc.nextLine().split("\\s+");

        permute(0);
    }

    private static void permute(int index) {
        if (index == elements.length) {
            print(elements);
            return;
        }
        permute(index + 1);
        HashSet<String> swapped = new HashSet<>();
        swapped.add(elements[index]);
```

```java
        for (int i = index + 1; i < elements.length; i++) {
            if (!swapped.contains(elements[i])) {
                swap(elements, index, i);
                permute(index + 1);
                swap(elements, index, i); //backtracking (unswapping)
                swapped.add(elements[i]);
            }

        }
    }

    private static void swap(String[] arr, int first, int second) {
        String temp = arr[first];
        arr[first] = arr[second];
        arr[second] = temp;

    }

    private static void print(String[] arr) {
        System.out.println(String.join(" ", arr));
    }
}
```

Permutations with Repetitions and Swap algorithm – по-комплексен алгоритъм (когато имаме много повтарящи се елементи)

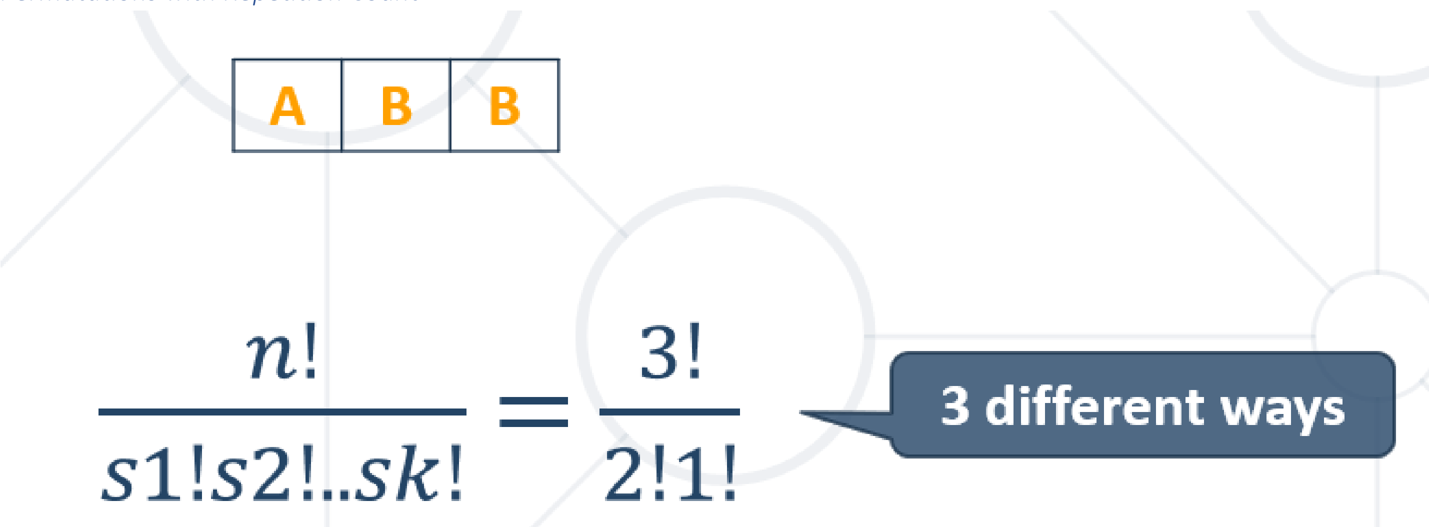
To check it – нещо не работи алгоритъма




```java
public static void main(String[] args) {
    Scanner sc = new Scanner(System.in);

    String[] arr = {"3", "5", "1", "5", "5"};
    Arrays.sort(arr); // 1 3 5 5 5
    permuteRep(arr, 0, arr.length - 1);
}

static void permuteRep(String[] arr, int start, int end) {
```

```java
    print(arr);
    for (int left = end - 1; left >= start; left--)
        for (int right = left + 1; right <= end; right++) {
            if (!arr[left].equals(arr[right])) {
                swap(arr, left, right);
                permuteRep(arr, left + 1, end);
            }
            String firstElement = arr[left];
            for (int i = left; i <= end - 1; i++) {
                arr[i] = arr[i + 1];
            }
            arr[end] = firstElement;
        }
}
```

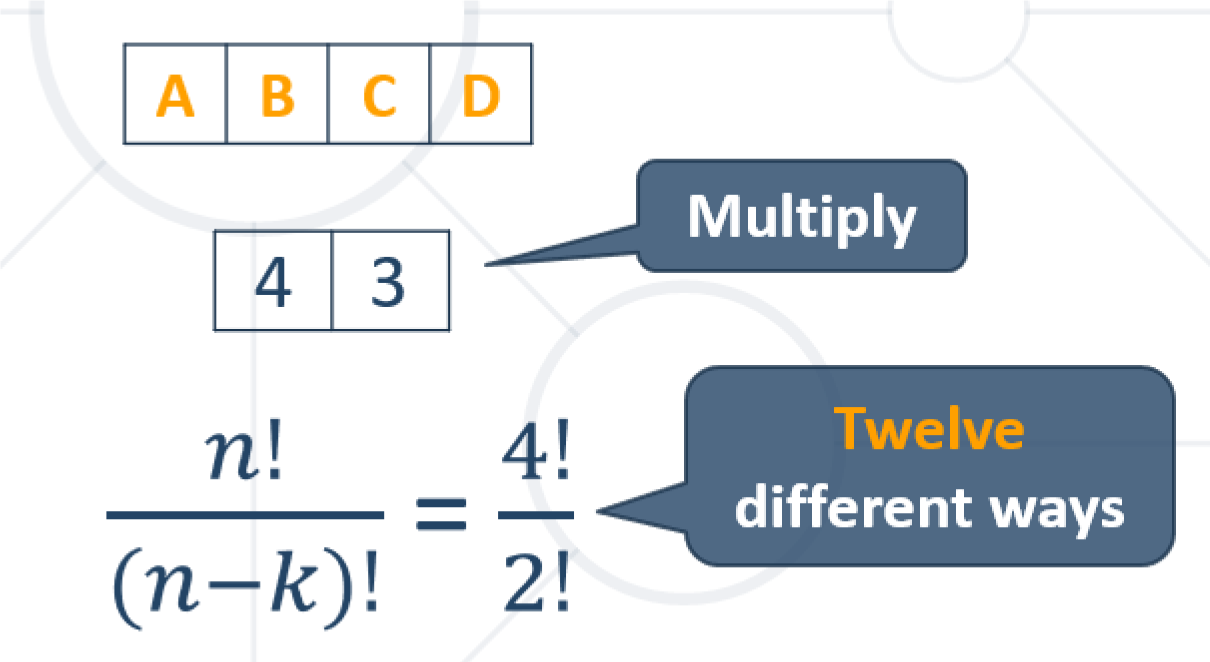
Permutations with Repetition Count



5.2. Variations

Given set of elements N and K slots:
order the N elements in all the possible ways inside the K slots

Variations Count - при неповтарящи се елементи



Нормална вариация – без повторение

```
public class VariationsWithoutRepetition {
    public static String[] elements;
    public static String[] variations;
    public static boolean[] used;

    public static void main(String[] args) {
        Scanner sc = new Scanner(System.in);
        elements = sc.nextLine().split("\\s+");

        int k = Integer.parseInt(sc.nextLine()); // k Slots
        variations = new String[k];
        used = new boolean[elements.length];

        variationsMethod(0);
    }

    private static void variationsMethod(int index) {
        if (index == variations.length) {
            print(variations);
            return;
        }

        for (int i = 0; i < elements.length; i++) {
            if (!used[i]) {
                used[i] = true;
                variations[index] = elements[i];
                variationsMethod(index + 1);
                used[i] = false; // backtracking
            }
        }

    }

    private static void print(String[] arr) {
        System.out.println(String.join(" ", arr));
```

```
    }
}
```

Вариация с повторение – повтаряме всеки един елемент в слотовете К

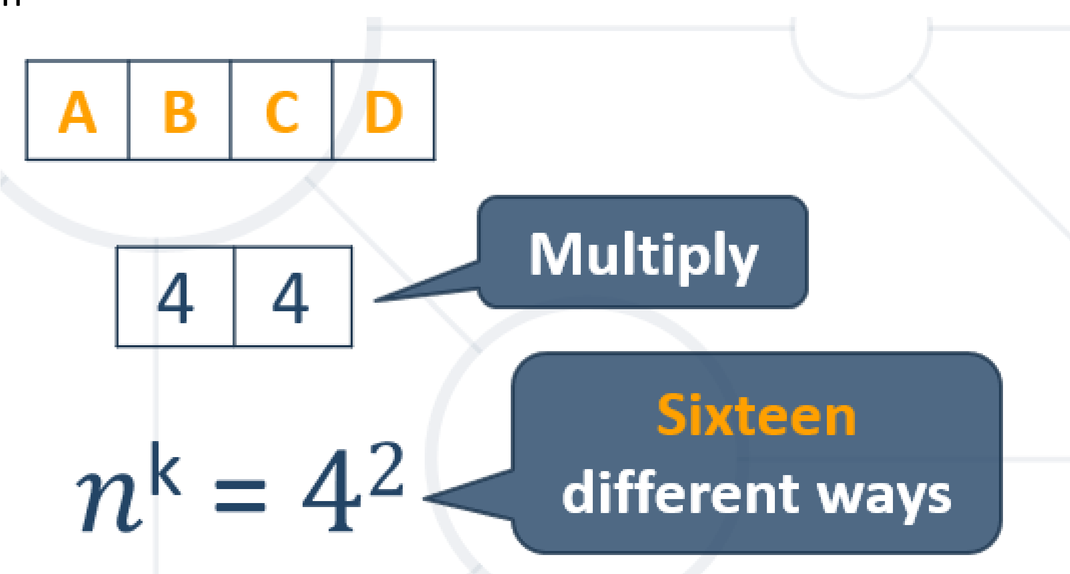
```
private static void variationsMethod(int index) {
    if (index == variations.length) {
        print(variations);
        return;
    }

    for (int i = 0; i < elements.length; i++) {
        variations[index] = elements[i];
        variationsMethod(index + 1);

    }
}
```

Вариация с повторение – с използваме на итерация, т.е. без рекурсия

```
int n = 5;
int k = 3;
int[] arr = new int[k];

while (true) {
    print(arr);
    int index = k - 1;
    while (index >= 0 && arr[index] == n-1)
        index--;
    if (index < 0)
        break;
    arr[index]++;
    for (int i = index + 1; i < k; i++)
        arr[i] = 0;
}
```

Variations Count - при повтаряне на елементи

n^k



5.3. Combinations

Изпринтирай всички k сетове образувани от общо n елемента.

Един сет от K елемента е примерно:
При k ==2, то A B и B A се брои за една комбинация/сет, а не за две.

Combinations Count - без повтаряне на елементи

$$\frac{n!}{k!\,(n-k)!}$$

Нормална комбинация – без повторение

```
public class CombinationsWithoutRepetition {
    public static String[] elements;
    public static String[] variations;

    public static void main(String[] args) {
        Scanner sc = new Scanner(System.in);
        elements = sc.nextLine().split("\\s+");

        int k = Integer.parseInt(sc.nextLine());
        variations = new String[k];

        combinations(0, 0);
    }

    private static void combinations(int index, int start) {
        if (index == variations.length) { //дъно
            print(variations);
        } else {
            for (int i = start; i < elements.length; i++) {
                variations[index] = elements[i];
                combinations(index +1, i+1);
            }
        }
    }

    private static void print(String[] arr) {
        System.out.println(String.join(" ", arr));
    }
}
```

Комбинация с повторение – един елемент от n-те може да се съдържа в сета/комбинацията с K-слота K пъти

```
public class CombinationsWithRepetition {
    public static String[] elements;
    public static String[] variations;

    public static void main(String[] args) {
        Scanner sc = new Scanner(System.in);
        elements = sc.nextLine().split("\\s+");

        int k = Integer.parseInt(sc.nextLine());
        variations = new String[k];

        combinations(0, 0);
    }

    private static void combinations(int index, int start) {
        if (index == variations.length) { //дъно
            print(variations);
        } else {
            for (int i = start; i < elements.length; i++) {
                variations[index] = elements[i];
                combinations(index + 1, i);
            }
        }
    }

    private static void print(String[] arr) {
        System.out.println(String.join(" ", arr));
    }
}
```

Combinations Count - с повтаряне на елементи:

to select r things from n possibilities

$$C(n+r-1,r) = \frac{(n+r-1)!}{r!(n-1)!}$$

5.4. N Choose K Count

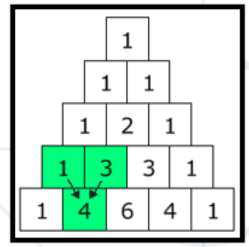
Начин на решаване на формулата за комбинации – от гледна точка на спестяване на компютърна памет/операции

$$C_n^k = \binom{n}{k} = \frac{n!}{(n-k)!\,k!}$$

Използваме пирамидата на Паскал

Binomial Coefficients: Calculation

$$C_n^k = \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$



Base cases (дъна):

if k > n → 0

if k == 0 → 1

if k == n → 1

```java
private static int binom(int n, int k) {
    if (k > n) {
        return 0;
    }
    if (k == 0 || k == n) {
        return 1;
    }

    return binom(n - 1, k - 1) + binom(n - 1, k);
}
```

```java
public static binom(int n, int k) {
  if (k > n)
    return 0;
  if (k == 0 || k == n)
    return 1;
  return binom(n - 1, k - 1) + binom(n - 1, k);
}
```

This is exponential, we can do way better with dynamic programming

6. Searching, Sorting and Greedy Algorithms

6.1. Searching

6.1.1 Linear search

Worst & average performance: O(n)

```java
public class LinearSearch {
    public static void main(String[] args) {
        Scanner sc = new Scanner(System.in);
        int[] arr = {13, 2, 34, 73, 24, 86};
        System.out.println(indexOf(arr, 37));
    }

    private static int indexOf(int[] arr, int key) {
        for (int i = 0; i < arr.length; i++) {
```

```
            if (arr[i] == key) {
                return i;
            }
        }
        return -1;
    }

}
```

6.1.1 Binary search

finds an item within a ordered data structure – търси половината от половината от половината от ……

Average performance: **O(log(n))**

```
public class BinarySearchIterative {
    public static void main(String[] args) {
        Scanner sc = new Scanner(System.in);
        int[] arr = Arrays.stream(sc.nextLine().split("\\s+"))
                .mapToInt(x -> Integer.parseInt(x))
                .toArray();
        Arrays.sort(arr);

        int key = Integer.parseInt(sc.nextLine());
        System.out.println(indexOf(arr, key));
    }

    private static int indexOf(int[] arr, int key) {
        int start = 0;
        int end = arr.length - 1;

        while (start <= end) {
            int mid = (start + end) / 2;
            int curr = arr[mid];
            if (key < curr) {
                end = mid - 1;
            } else if (key > curr) {
                start = mid + 1;
            } else {
                return mid;
            }
        }
        return -1;
    }

}
```

6.2. Sorting

https://visualgo.net/ - сайт за визуализация на сортировки и други структури от данни с техните алгоритми

Efficient sorting algorithms

Sorting algorithms are often classified by:

- Computational **complexity** and memory usage
- **Recursive** / non-recursive
- **Stability** – stable / unstable – стабилен например при срещане на еднакви елементи, то няма да ги swap-не
- **Comparison-based sort** / non-comparison based (като броим например)
- Sorting **method**: insertion, exchange (bubble sort and quicksort), selection (heapsort), merging, serial / parallel, etc.

6.2.1. Simple algorithms

I. Selection sort – не е много ефективен, unstable

Swap the first with the min element on the right, then the second, etc

Като първи елемент отива най-малкия, след това гледаме за всички елементи след 1ият, след това за всички елементи след 2рият

```
public class SelectionSort {
    public static void main(String[] args) {
        Scanner sc = new Scanner(System.in);
        int[] arr = {5, 4, 3, 2, 1};
        sort(arr);

        for (int i : arr) {
            System.out.print(i + " ");
        }
    }

    private static void sort(int[] arr) {
        for (int index = 0; index < arr.length; index++) {
            int min = index;
            for (int curr = index + 1; curr < arr.length; curr++) {
                if (arr[curr] < arr[min]) {
                    min = curr;
                }
            }
            swap(arr, index, min);
        }
    }

    private static void swap(int[] arr, int first, int second) {
        int temp = arr[first];
        arr[first] = arr[second];
        arr[second] = temp;
    }
}
```

II. Bubble sort – simple, but inefficient algorithm, but **stable**

Сравнява две съседни и влачи най-тежкия елемент накрая, и не прави swap

```
private static void sort(int[] arr) {
    for (int i = 0; i < arr.length; i++) { //брой мехурчета изплували / най-тежкия накрая
        for (int j = 1; j < arr.length - i; j++) { //самото сравнение на всички съседни
            if (arr[j - 1] > arr[j]) { //възходящо сравнение
                swap(arr, j - 1, j);
            }
        }
    }
}
```

Или

```
public static void sort(int[] arr) {
    int n = arr.length;

    for (int k = 0; k < n - 1; k++) { //брой операции

        for (int i = 0; i < n - k - 1; i++) { //размени два съседни, като всеки път следващи два 
съседни взема
            if (arr[i] > arr[i + 1]) {
                int temp = arr[i];
                arr[i] = arr[i + 1];
                arr[i + 1] = temp;
```

```
            }
        }
    }
}
```

III. Insertion sort - simple, but inefficient algorithm, but **stable**

Move the first unsorted element left to its place

Name	Best	Average	Worst	Memory	Stable	Method
Selection	n^2	n^2	n^2	1	No	Selection
Bubble	n	n^2	n^2	1	Yes	Exchanging
Insertion	n	n^2	n^2	1	Yes	Insertion

6.2.2. Shuffling

Randomizing the order of items in a collection - Generate a random permutation

```
public class Shuffling {
    public static void main(String[] args) {
        Scanner sc = new Scanner(System.in);

        int[] arr = {13, 15, 12, 24, 59};
        Arrays.sort(arr);

        getAsRand(arr);

        for (int i : arr) {
            System.out.print(i + " ");
        }
    }

    private static void getAsRand(int[] arr) {
        Random rand = new Random();
        for (int i = 0; i < arr.length; i++) {
            swap(arr, i, rand.nextInt(arr.length - 1)); // може и без минус 1
        }
   }

    private static void swap(int[] arr, int first, int second) {
        int temp = arr[first];
        arr[first] = arr[second];
        arr[second] = temp;
    }
}
```

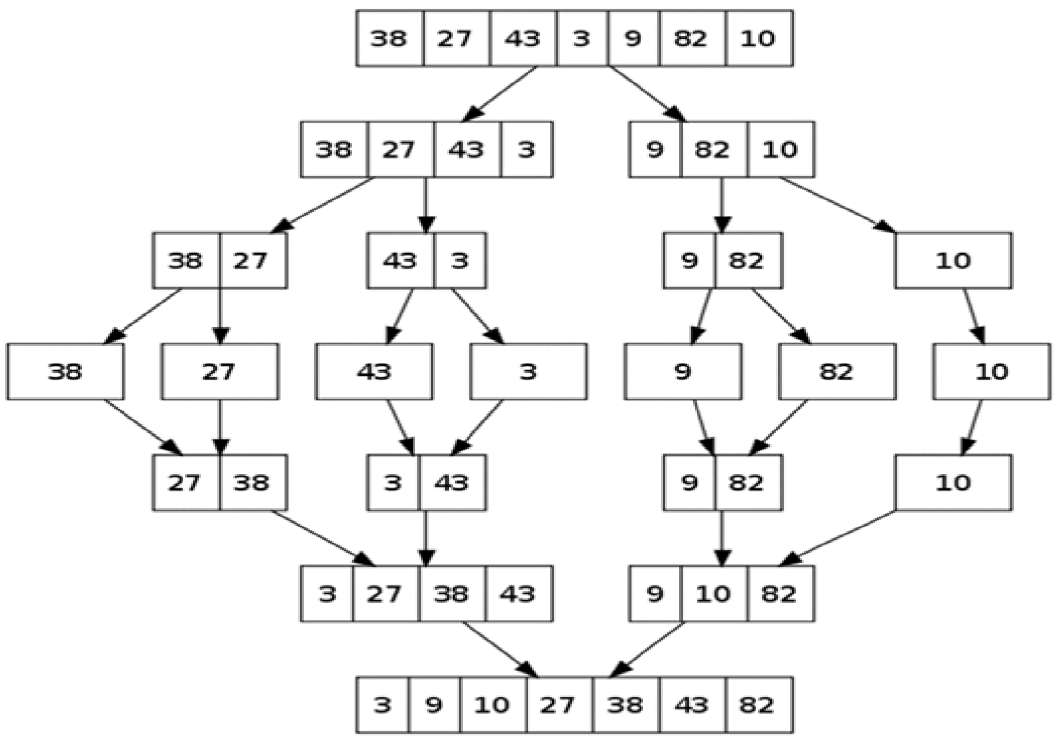
6.2.3. Advanced Sorting Algorithms – recursive, saving iterations and memory/time

I. Merge Sort

Efficient- from O(n * log(n)) up to O(log(n))

Divide the list into sub-lists (typically 2 sub-lists):

- Sort each sub-list (recursively call merge-sort)
- Merge the sorted sub-lists into a single list



```
public class MergeSort {
    public static void main(String[] args) {
        Scanner sc = new Scanner(System.in);
        int[] arr = Arrays.stream(sc.nextLine().split("\\s+")).mapToInt(x ->
Integer.parseInt(x)).toArray();
        mergeSort(arr, 0, arr.length - 1);

        StringBuilder builder = new StringBuilder();
        for (int num : arr) {
            builder.append(num).append(" ");
        }
        System.out.println(builder.toString());
    }

    private static void mergeSort(int[] arr, int begin, int end) {
        if (begin >= end) {
            return;
        }
        int mid = (begin + end) / 2;

        mergeSort(arr, begin, mid);
        mergeSort(arr, mid + 1, end);

        merge(arr, begin, mid, end); //backtracking
    }

    private static void merge(int[] arr, int begin, int mid, int end) {
        if (mid < 0 || mid >= arr.length || arr[mid] < arr[mid + 1]) { // ако последния елемент на
сортирания вече събмасив е по-малък от първият елемент на следващия сортиран събмасив, то
пропускаме сортировката
            return;
        }
        int left = begin;
        int right = mid + 1;

        int[] helper = new int[arr.length];
```

```java
        for (int i = begin; i <= end; i++) {
            helper[i] = arr[i];
        }

        for (int i = begin; i <= end; i++) {
            if (left > mid) { //when 1st substring is over
                arr[i] = helper[right++]; //we take next element from the sorted 2nd substring
            } else if (right > end) { //when 2nd substring is over
                arr[i] = helper[left++]; //we take next element from the sorted 1st substring
            } else if (helper[left] < helper[right]) { //when the element of 1st substring is
lower than the element of 2nd substing
                arr[i] = helper[left++]; //arr[i] is with new value helper[left]
            } else {
                arr[i] = helper[right++]; //arr[i] is with new value helper[right]
            }
        }
    }
```

II. Quick Sort

Best & average case: **O(n*log(n))**; Worst: **O(n2)**

The algorithm in short:

- Quicksort takes unsorted partitions of an array and sorts them
- We choose the **pivot**
  - We pick the first element from the unsorted partition and move it in such a way, that all smaller elements are on its left and all greater, to its right
- With pivot moved to its correct place, we now have two unsorted partitions – one to the left of it and one to the right
- **Call the procedure recursively** for each partition

In Lomuto partition scheme, the starting **pivot** is the last element of an array / the **last (high)** element

The bottom of the recursion is when a partition has a size of 1, which is by definition sorted

```java
public class Qucksort {
    public static void main(String[] args) {
        Scanner sc = new Scanner(System.in);
        int[] arr = Arrays.stream(sc.nextLine().split("\\s+")).mapToInt(x ->
Integer.parseInt(x)).toArray();
        quickSort(arr, 0, arr.length - 1);

        StringBuilder builder = new StringBuilder();
        for (int num : arr) {
            builder.append(num).append(" ");
        }
        System.out.println(builder.toString());
    }

    // low is the start index
    // high is the end index
    private static void quickSort(int[] arr, int low, int high) {
        if (low < high) {
            int pi = partition(arr, low, high);

            // Recursively sort elements before partition and after partition
            quickSort(arr, low, pi - 1);
            quickSort(arr, pi + 1, high);
        }
    }
```

```java
    /*This method takes last element as pivot, places the pivot at its correct position in sorted array,
    and places all smaller (smaller than pivot) to left of pivot, and all greater elements to right of pivot
    */
    private static int partition(int[] arr, int low, int high) {
        int pivot = arr[high];
        int i = (low - 1); // index of the smaller element
        for (int j = low; j < high; j++) {
            //If current element is smaller or equal to pivot
            if (arr[j] <= pivot) {
                i++;
                swap(arr, i, j);
            }
        }

        swap(arr, i + 1, high);

        return i + 1;
    }

    private static void swap(int[] arr, int first, int second) {
        int temp = arr[first];
        arr[first] = arr[second];
        arr[second] = temp;
    }

}
```

III. Counting Sort - sorting without comparison, using counts only - very efficient and stable – iterative algorithm

Sorts small integers by counting their occurrences

Създаваме масив counts с дължина най-големият елемент от arr. И всеки елемент на масива counts е 0 в началото. Реално масива counts е сортиран по подразбиране възходящо(индекси 0, 1, 2….). На индекс 2 има 2 броя, на индекс 5 има 2 броя, и на идекс 13 има 1 брой.

```java
public class SortWithoutComparison {
    public static int[] counts;

    public static void main(String[] args) {
        int[] arr = {13, 5, 2, 2, 5};
        int max = Arrays.stream(arr).max().getAsInt();

        counts = new int[max + 1];
        sort(arr);

        for (int index = 0; index < counts.length; index++) {
            if (counts[index] != 0) {
                for (int i = 0; i < counts[index]; i++) {
                    System.out.print(index + " ");
                }
            }
        }
    }

    private static void sort(int[] arr) {
        for (int i = 0; i < arr.length; i++) {
            counts[arr[i]]++;
        }
    }
```

}

IV. Bucket Sort

Bucket sort partitions an array into a number of buckets:
- Each bucket is then sorted individually with a different algorithm
- Not a comparison-based sort

V. Heap Sort

VI. Bogo Sort – прави безброй много пермутации, докато видите ли елементите не се окаже, че са се подредили сами

6.2.4. Summary

Effect

Name	Best	Average	Worst	Memory	Stable	Method
Selection	n^2	n^2	n^2	1	No	Selection
Bubble	n	n^2	n^2	1	Yes	Exchanging
Insertion	n	n^2	n^2	1	Yes	Insertion
Quick	n * log(n)	n * log(n)	n^2	1	Depends	Partitioning
Merge	n * log(n)	n * log(n)	n * log(n)	1 (or n)	Yes	Merging
Heap	n * log(n)	n * log(n)	n * log(n)	1	No	Selection
Bogo	n	n * n!	n * n!	1	No	Luck

Counting Sort is also high effective/efficient

6.3. Greedy Algorithms

Usually more efficient than the other algorithms
Pick the best local solution
Greedy algorithms assume that always choosing a local optimum leads to the global optimum **– което не винаги е верно**

Examples:
- Find the **shortest** path from Sofia to Varna
- Find the **maximum increasing subsequence**
- Find the shortest route that visits each city and returns to the origin city

Greedy Failure Cases - Greedy Algorithms Often Fail, when all local optimal results do not give the optimal global maximum!

When we can use Greedy?
- **Greedy choice property -** A global optimal solution can be obtained by greedily selecting a locally optimal choice

Случаят когато имаме монета със стойност 4, то 18 постигаме с 3 стъпки: 10 +4 +4.
А иначе постигаме 18 с 5 стъпки: 10+5+1+1+1
- **Optimal substructure -** An optimal global solution contains the optimal solutions of all its sub-problems

Any problem having the above properties (**Greedy choice property & Optimal substructure**) is guaranteed to have an optimal greedy solution

Интересен сайт за това кой алгоритъм къде се използва най-добре:
https://stackoverflow.com/questions/1933759/when-is-each-sorting-algorithm-used/1934004#1934004

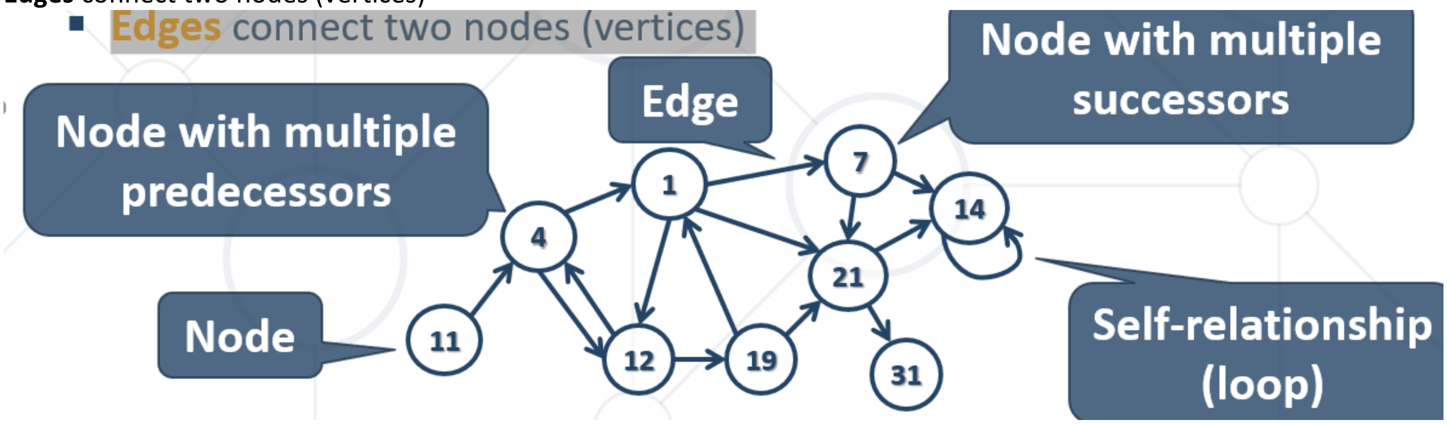
7. Graph Theory, Traversal and Shortest Paths

7.1. Graphs

Graph, denoted as **G(Vertice, Edge):**

- Node / Vertice – върхове / връх
- Edges – ребра/стрелки свързващи върховете

Each **node** (**vertex**) has **multiple** predecessors(предшественици) and **multiple** successors(наследници)
Edges connect two nodes (vertices)

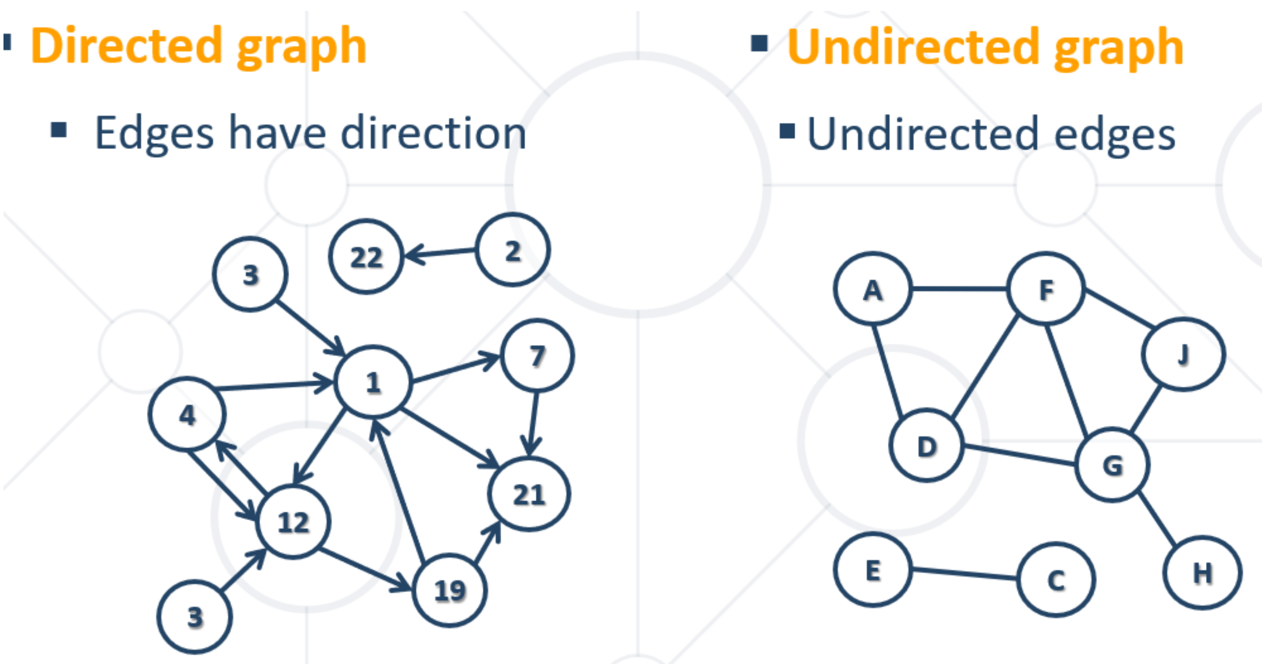


Node (vertex):

- Element of a graph
- Can have name / value
- Keeps a list of adjacent nodes

Edge:

- Connection between two nodes
- Can be directed / undirected
- Can be weighted / unweighted
- Can have name / value

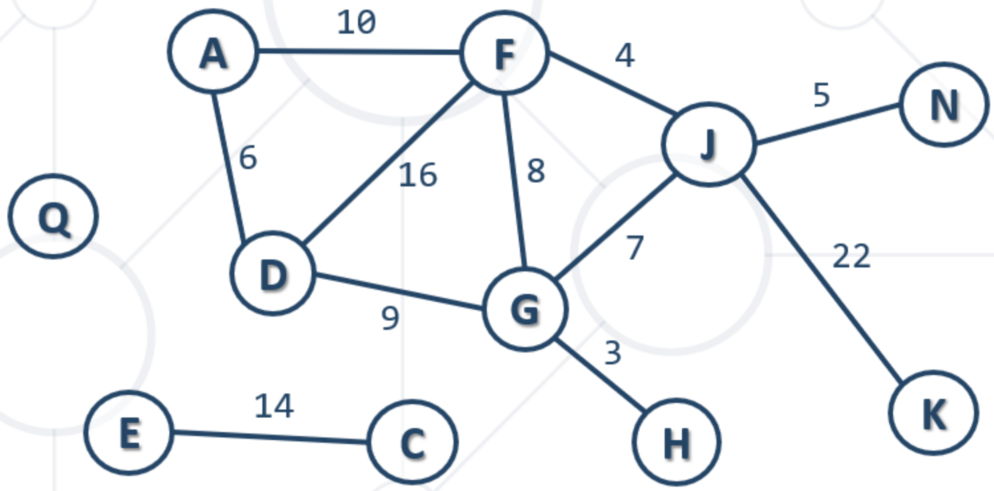


Идеята на графите е да не се връщаме на вече посетено място/Node.

В този курс, ще работим само с непретеглени графи.

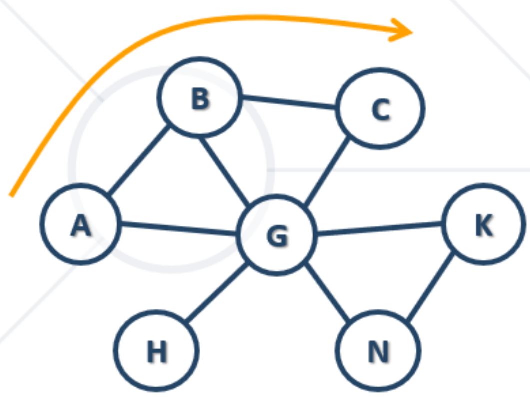
Weighted graph

- Weight (cost) is associated with each edge



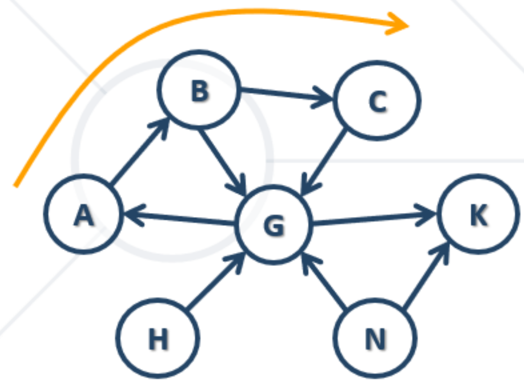
Path (in undirected graph)

- Sequence of nodes $n_1, n_2, \dots n_k$
- Edge exists between each pair of nodes n_i, n_{i+1}
- Examples:
  - A, B, C is a path
  - A, B, G, N, K is a path
  - H, K, C is not a path
  - H, G, G, B, N is not a path



Path (in directed graph)

- Sequence of nodes n_1, n_2, … n_k
- Directed edge exists between each pair of nodes n_i, n_{i+1}
- Examples:
  - A, B, C is a path
  - N, G, A, B, C is a path
  - A, G, K is not a path
  - H, G, K, N is not a path



Cycle

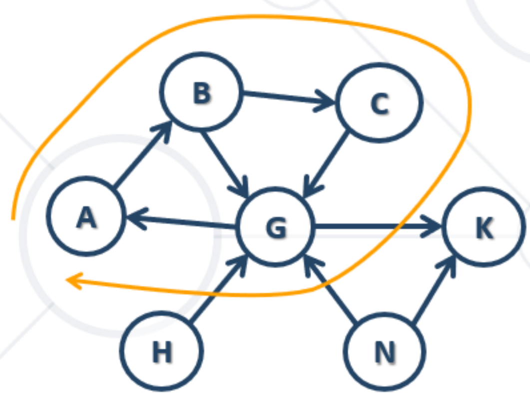
- Path that ends back at the starting node
- Example of cycle: A, B, C, G, A

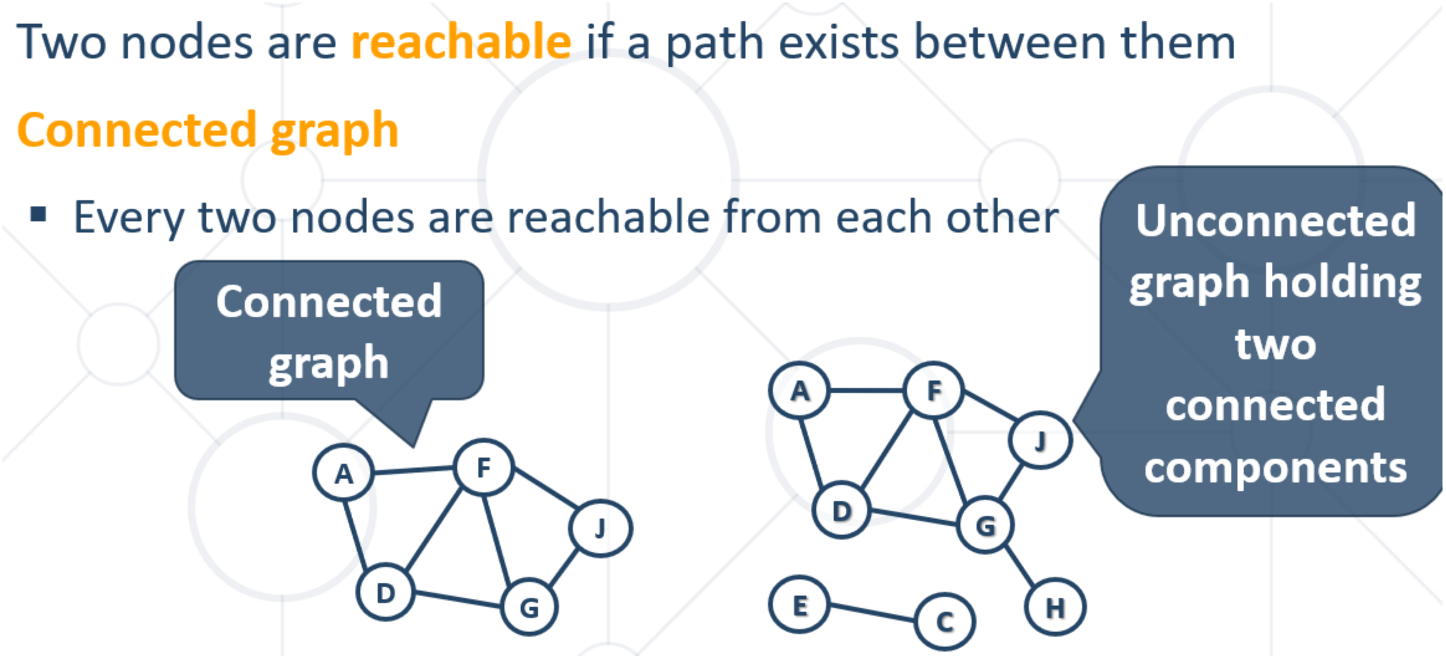
Simple path

- No cycles in path

Acyclic graph

- Graph with no cycles
- Acyclic undirected graphs are trees





7.2. Representing graphs

При графи, много често пропускаме нулевият елемент, затова добавяме +1 за броя на елементите

Adjacency list - *Each node holds a list of its neighbors*

```
List<List<Integer>> graph = new ArrayList<>();
for (int i = 0; i < 10 + 1; i++) {
    graph.add(new ArrayList<>());
}

graph.get(1).addAll(Arrays.asList(9, 8, 5));
graph.get(9).add(1);  // от 1 ходим към 9 и от 9 ходим към 1 – това е неподреден граф, има ненасочено ребро
```

Adjacency Matrix – матрица на съседство

```
int nodes = 10;
int[][] graph = new int[nodes + 1][nodes + 1];

graph[3][6] = 1;
int[][] graph = new int[][] {
   // 0  1  2  3  4  5  6
    { 0, 0, 0, 1, 0, 0, 1 }, // node 0
    { 0, 0, 1, 1, 1, 1, 1 }, // node 1
    { 0, 1, 0, 0, 1, 1, 0 }, // node 2
    { 1, 1, 0, 0, 0, 1, 0 }, // node 3
    { 0, 1, 1, 0, 0, 0, 1 }, // node 4
    { 0, 1, 1, 1, 0, 0, 0 }, // node 5
    { 1, 1, 0, 0, 1, 0, 0 }, // node 6
};
// Add an edge { 3 -> 6 }
   graph[3][6] = 1;
// List the children of node #1
int[] childNodes = graph[1];
```

List of edges

```
{1,2}, {1,4}, {2,3}, {3,1}, {4,2}
```

Representing with a Class

```java
public static class Edge {
    public int source;
    public int destination;

    public Edge(int source, int destination) {
        this.source = source;
        this.destination = destination;
    }
}

public static void main(String[] args) {
    List<Edge> graph = new ArrayList<>();
    graph.add(new Edge(1, 2));
    graph.add(new Edge(1, 3));
    graph.add(new Edge(1, 4));
    graph.add(new Edge(1, 5));
    graph.add(new Edge(1, 6));
}
```

ИЛИ

```java
public static class Graph {
    int source;
    List<Edge> edges;

    public Graph(int source) {
        this.source = source;
        this.edges = new ArrayList<>();
    }
}

public static class Edge {
    public int source;
    public int destination;

    public Edge(int source, int destination) {
        this.source = source;
        this.destination = destination;
    }
}

public static void main(String[] args) {
    Graph graph = new Graph(1);
    graph.edges.add(new Edge(1, 2));
    graph.edges.add(new Edge(1, 3));
    graph.edges.add(new Edge(1, 4));
}

// List the children of node #1
List<Edge> childNodes = graph.stream().filter(e -> e.source == 1).collect(Collectors.toList());
```

Матрица на теглото – като има ребро, то не е 1-ца, а има стойност/тежест

```java
int[][] graph = new int[2 + 1][2 + 1];
graph[1][2] = 12; //тежест 12
```

7.3. Graphs Traversals

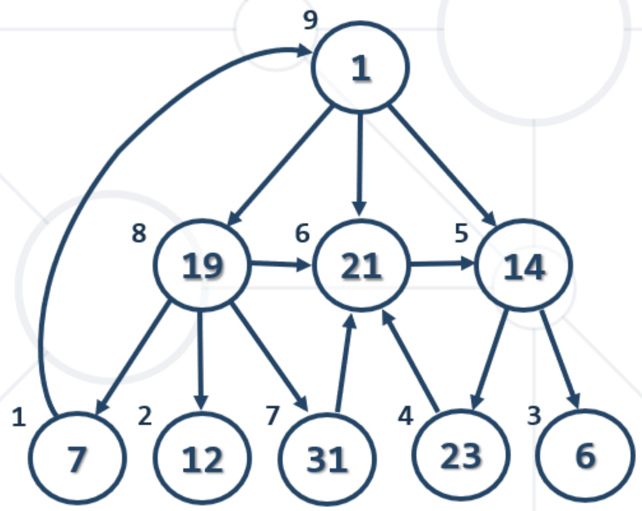
Traversing a graph means to visit each of its nodes exactly once.

The order of visiting nodes may vary on the traversal algorithm

Depth-First Search (DFS) – на нива дълбочина

first visits all descendants of given node recursively, finally visits the node itself

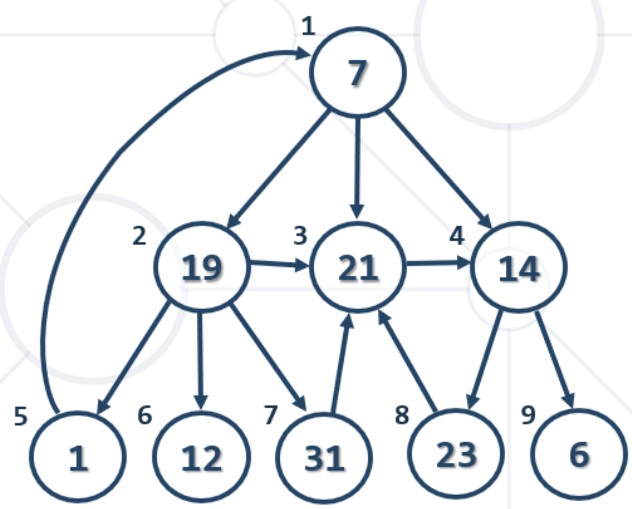
```
visited[0 … n-1] = false;
for (v = 0 … n-1) dfs(v)
dfs (node) {
  if not visited[node] {
    visited[node] = true;
    for each child c of node
      dfs(c);
    print node;
  }
}
```



Breadth-First Search (BFS) – на вълни

first visits the neighbor nodes, then the neighbors of neighbors, then their neighbors, etc.

```
bfs(node) {
  queue ⟸ node
  visited[node] = true
  while queue not empty
    v ⟸ queue
    print v
    for each child c of v
      if not visited[c]
        queue ⟸ c
        visited[c] = true
}
```



Iterative DFS and BFS

What will happen if in the **Breadth-First Search** (**BFS**) algorithm we change the **queue** with a **stack**?

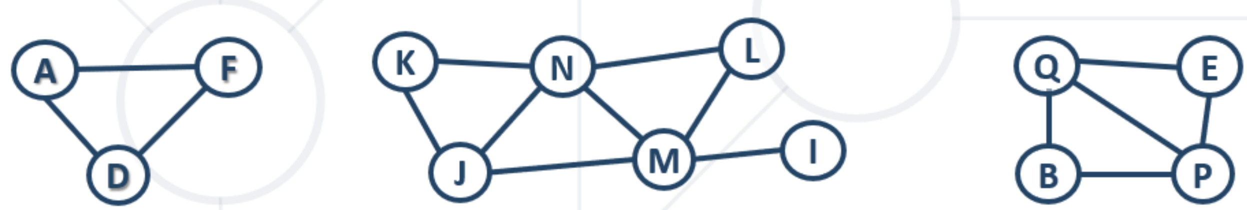
- An iterative stack-based **Depth-First Search** (**DFS**)

```
bfs(node) {
  queue ⇐ node
  visited[node] = true
  while queue not empty
    v ⇐ queue
    print v
    for each child c of v
      if not visited[c]
        queue ⇐ c
        visited[c] = true
}
```

```
dfs(node) {
  stack ⇐ node
  visited[node] = true
  while stack not empty
    v ⇐ stack
    print v
    for each child c of v
      if not visited[c]
        stack ⇐ c
        visited[c] = true
}
```

7.4. Graph Connectivity

- E.g. the graph below consists of 3 connected components



Finding the connected components - Loop through all nodes and start a **DFS** / **BFS** traversing from any **unvisited** node

Всички свързани компоненти на даден граф – **DFS:**

```java
public class Main {

    public static void main(String[] args) {
        Scanner sc = new Scanner(System.in);

        int n = Integer.parseInt(sc.nextLine());
        List<List<Integer>> graph = new ArrayList<>();

        for (int i = 0; i < n; i++) {
            String nextLine = sc.nextLine();
            if (nextLine.trim().equals("")) {
                graph.add(new ArrayList<>());
            } else {
                List<Integer> nextNodes = Arrays.stream(nextLine.split("\\s+"))
                        .map(Integer::parseInt)
                        .collect(Collectors.toList());
```

```java
                graph.add(nextNodes);
            }
        }

        List<Deque<Integer>> connectedComponents = getConnectedComponents(graph);
        System.out.println();
    }

    public static List<Deque<Integer>> getConnectedComponents(List<List<Integer>> graph) {
        boolean[] visited = new boolean[graph.size()];
        List<Deque<Integer>> components = new ArrayList<>();

        for (int start = 0; start < graph.size(); start++) {
            if (!visited[start]) {
                dfs(start, components, graph, visited);
                System.out.println();
            }
        }
        dfs(0, components, graph, visited);

        return components;
    }

    private static void dfs(int node, List<Deque<Integer>> components, List<List<Integer>> graph,
boolean[] visited) {
        if (!visited[node]) {
            visited[node] = true;
            for (int child : graph.get(node)) {
                dfs(child, components, graph, visited);
            }
            System.out.print(node + " ");
        }
    }
}
```

Всички свързани компоненти на даден граф – **BFS:**

```java
public class ConnectedComponentsBFS {
    public static void main(String[] args) {
        Scanner sc = new Scanner(System.in);

        int n = Integer.parseInt(sc.nextLine());
        List<List<Integer>> graph = new ArrayList<>();

        for (int i = 0; i < n; i++) {
            String nextLine = sc.nextLine();
            if (nextLine.trim().equals("")) {
                graph.add(new ArrayList<>());
            } else {
                List<Integer> nextNodes = Arrays.stream(nextLine.split("\\s+"))
                        .map(Integer::parseInt)
                        .collect(Collectors.toList());

                graph.add(nextNodes);
            }
        }

        List<Deque<Integer>> connectedComponents = getConnectedComponents(graph);
        for (Deque<Integer> connectedComponent : connectedComponents) {
            System.out.print("Connected component: ");
            for (int intNum : connectedComponent) {
```

```
                System.out.print(intNum + " ");
            }
            System.out.println();
        }
    }

    public static List<Deque<Integer>> getConnectedComponents(List<List<Integer>> graph) {
        boolean[] visited = new boolean[graph.size()];
        List<Deque<Integer>> components = new ArrayList<>();

        for (int start = 0; start < graph.size(); start++) {
            if (!visited[start]) {
                components.add(new ArrayDeque<>());

                bfs(start, components, graph, visited);
            }
        }

        return components;
    }

    private static void bfs(int start, List<Deque<Integer>> components, List<List<Integer>> graph,
boolean[] visited) {
        Deque<Integer> queue = new ArrayDeque<>();
        visited[start] = true;
        queue.offer(start);

        while (!queue.isEmpty()) {
            int node = queue.poll();

            components.get(components.size() - 1).offer(node);

            for (int child : graph.get(node)) {
                if (!visited[child]) {
                    visited[child] = true;
                    queue.offer(child);
                }
            }
        }
    }
}
```

7.5. Topological Sorting

Ordering a Graph by Set of Dependencies

Rules:

- Undirected graphs cannot be sorted
- Graphs with cycles cannot be sorted
- Sorting is not unique
- Various sorting algorithms exists and they give different results

7.5.1. Source removal top-sort algorithm – без рекурсия

Source removal top-sort algorithm + cycle detect

Create an empty list

Repeat until the graph is empty:

- Find a node without incoming edges
- Print this node
- Remove the edge from the graph

L ← empty list that will hold the sorted elements (output)
S ← set of all nodes with no incoming edges
while S is non-empty do
   remove some node n from S
   append n to L
   for each node m with an edge e: { n through m }
      remove edge e from the graph
      if m has no other incoming edges then
         insert m into S
if graph is empty
  return L (a topologically sorted order)
else
  return "Error: graph has at least one cycle"

```java
public static Collection<String> topSort(Map<String, List<String>> graph) {
    Map<String, Integer> dependenciesCount = getDependenciesCount(graph);

    List<String> sorted = new ArrayList<>();

    while (!graph.isEmpty()) {
        String key = graph.keySet()
                .stream()
                .filter(k -> dependenciesCount.get(k) == 0)
                .findFirst()
                .orElse(null);

        if (key == null) {
            break;
        }

        for (String child : graph.get(key)) {
            dependenciesCount.put(child, dependenciesCount.get(child) - 1);
        }
        graph.remove(key);
        sorted.add(key);
    }

    if (!graph.isEmpty()) { // детектване за цикличност
        throw new IllegalArgumentException();
    }

    return sorted;
}

private static Map<String, Integer> getDependenciesCount(Map<String, List<String>> graph) {
    Map<String, Integer> dependenciesCount = new LinkedHashMap<>();

    for (Map.Entry<String, List<String>> node : graph.entrySet()) {
        dependenciesCount.putIfAbsent(node.getKey(), 0);
        for (String child : node.getValue()) {
            dependenciesCount.putIfAbsent(child, 0);
            dependenciesCount.put(child, dependenciesCount.get(child) + 1);
        }
    }

    return dependenciesCount;
}
```

7.5.2. DFS Algorithm – с рекурсия

sortedNodes = { } // linked list to hold the result
visitedNodes = { } // set of already visited nodes
foreach node in graphNodes
topSortDFS(node)
topSortDFS(node)
if node ∉ visitedNodes
visitedNodes ← node
for each child c of node
TopSortDFS(c)
insert node upfront in the sortedNodes

```
public static Collection<String> topSort(Map<String, List<String>> graph) {
    List<String> sorted = new ArrayList<>();

    Set<String> visited = new HashSet<>();

    for (Map.Entry<String, List<String>> node : graph.entrySet()) {
        dfs(node.getKey(), visited,graph, sorted);
    }

    Collections.reverse(sorted);

    return sorted;
}

public static void dfs(String key, Set<String> visited, Map<String, List<String>> graph,
List<String> sorted){
    if (!visited.contains(key)) {
        visited.add(key);
        for (String child : graph.get(key)) {
            if (!visited.contains(child)) {
                dfs(child, visited, graph, sorted);
            }
        }
        sorted.add(key);
    }
}
```

7.5.3. DFS Algorithm + Cycle Detection

sortedNodes = { } // linked list to hold the result
visitedNodes = { } // set of already visited nodes
cycleNodes = { } // set of nodes in the current DFS cycle
foreach node in graphNodes
topSortDFS(node)
topSortDFS(node)
if node ϵ cycleNodes
return "Error: cycle detected"
if node ∉ visitedNodes
visitedNodes ← node
cycleNodes ← node
for each child c of node
topSortDFS(c)

remove node from cycleNodes
insert node upfront in the sortedNodes

```
public static Collection<String> topSort(Map<String, List<String>> graph) {
    List<String> sorted = new ArrayList<>();

    Set<String> visited = new HashSet<>();
    Set<String> detectCycles = new HashSet<>(); // детектване за цикличност

    for (Map.Entry<String, List<String>> node : graph.entrySet()) {
        dfs(node.getKey(), visited, graph, sorted, detectCycles);
    }

    Collections.reverse(sorted);

    return sorted;
}

public static void dfs(String key, Set<String> visited, Map<String, List<String>> graph,
List<String> sorted, Set<String> detectCycles) {
    if (detectCycles.contains(key)) { // детектване за цикличност
        throw new IllegalArgumentException();
    }
    if (!visited.contains(key)) {
        visited.add(key);
        detectCycles.add(key); // детектване за цикличност
        for (String child : graph.get(key)) {
            dfs(child, visited, graph, sorted, detectCycles);
        }
        detectCycles.remove(key); // детектване за цикличност
        sorted.add(key);
    }
}
```

7.6. Shortest Path

In **unweighted** graphs finding the **shortest path** can be done with **BFS** (all edges have the same weight)

```
bfs(G, start, end)
  visited[start] = true
  queue.offer(start)
  while (!queue.isEmpty())
     v = queue.poll()
     if v is end
     return v
     for all edges from v to w in G.adjacentEdges(v) do
        if w is not labeled as discovered then
          label w as discovered
          w.parent = v
          queue.offer(w)
```

8. Dynamic Programming == Динамично оптимиране

- "**Controlled**" brute force / exhaustive search – от всички обхождания, по някакъв начин избираме добрите brute force
- **Subproblems**: like original problem, but smaller

- Write solution to one **subproblem** in terms of solutions to smaller **acyclic** subproblems – избягваме откличност в подпроблема!
- **Memoization**: remember the **solution** to subproblems we've already solved, and **re-use - Avoid exponentials**
- **Guessing**: if you don't know something, **guess it!** (try all possibilities)

8.1. Fibonacci Sequence

Recursive mathematical formula:

F_0 = 0, **F1 = 1**

Fn = **Fn-1** + **Fn-2**

```
public class Fibonacci {
    public static void main(String[] args) {
        Scanner sc = new Scanner(System.in);

        int n = Integer.parseInt(sc.nextLine());

        long fib = calcFib(n);

        System.out.println(fib);
    }

    private static long calcFib(int n) {
        if (n <= 2) {
            return 1;
        }

        return calcFib(n-1) + calcFib(n-2);
    }
}
```

Memoization

- DP → sub-problems **overlap**
- In order to **avoid solving** problems **multiple times**, memorize
  - **Memoization** → **save/cache** sub-problem solutions **for later use**
- Typically using an **array**, **matrix** or a **hash table**

- Рекурсивно решение - **Top down** approach - Solve **recursively** by **breaking down** the problem further and further

```
public class Fibonacci {
    public static long[] dp;
    public static void main(String[] args) {
        Scanner sc = new Scanner(System.in);

        int n = Integer.parseInt(sc.nextLine());
        dp = new long[n+1];

        long fib = calcFib(n);

        System.out.println(fib);
    }

    private static long calcFib(int n) {
        if (n <= 2) {
            return 1;
        }

        if (dp[n] != 0) {
```

```java
            return dp[n];
        }

        return dp[n] = calcFib(n-1) + calcFib(n-2);
    }
}
```

- Итеративно решение - **Bottom up** approach

```java
public class Fibonacci {
    public static long[] dp;

    public static void main(String[] args) {
        Scanner sc = new Scanner(System.in);

        int n = Integer.parseInt(sc.nextLine());
        dp = new long[n + 1];

        dp[1] = 1;
        dp[2] = 1;

        for (int i = 3; i <= n; i++) {
            dp[i] = dp[i - 1] + dp[i - 2];
        }

        System.out.println(dp[n]);
    }
}
```

8.2. Five "Easy" Steps to DP

Define **subproblems**
Guess part of the solution
Relate subprolems and solutions
Recurse and **memoization** or build **DP table** bottom-up
Check subproblems **acyclic/topological** order

Solve original problem:
- A **subproblem**
- Or **combination** of subproblems

Useful Subproblems for Sequences – използваме или suffixes или prefixes, но не и двете едновременно
- **Suffixes** x[i… n-1] – от първия до последния - наставка
- **Prefixes** x[n-1… i] – от последния до първия – представка
- Both approaches usually run in **Θ(x)**

Substrings (subsequences) x[i…j] - два вложени цикъла
- Usually runs in **Θ(x^2)**

8.3. Longest Increasing Subsequence

LIS Optimal Substructure - Recursive Top-Down Approach
…….. – как става с рекурсия

LIS - Iterative Bottom-Up Approach – **left-most or right-most**

index	0	1	2	3	4	5	6	7	8	9	10
S[]	**3**	**14**	**5**	**12**	**15**	**7**	**8**	**9**	**11**	**10**	**1**

len[]	1	2	2	3	4	3	4	5	6	6	1
prevIndex[]	-1	0	0	2	3	2	5	6	7	7	-1
LIS	{3}	{3,14}	{3,5}	{3,5,12}	{3,5,12,15}	{3,5,7}	{3,5,7,8}	{3,5,7,8,9}	{3,5,7,8,9,11}	{3,5,7,8,9,10}	{1}

```
public class LIS_iterative {
    public static void main(String[] args) {
        Scanner sc = new Scanner(System.in);

        int[] sequence =
Arrays.stream(sc.nextLine().split("\\s+")).mapToInt(Integer::parseInt).toArray();
        int[] length = new int[sequence.length];
        int[] prevIndex = new int[sequence.length];
        Arrays.fill(prevIndex, -1);

        int maxLength = 0, maxIndex = -1;

        for (int i = 0; i < sequence.length; i++) {
            int current = sequence[i];
            int bestLength = 1;
            int bestIndex = -1;

            for (int j = i - 1; j >= 0; j--) {
                if (sequence[j] < current && length[j] + 1 >= bestLength) {  //= дали е leftmost
или rightmost
                    bestLength = length[j] + 1;
                    bestIndex = j;
                }
            }

            prevIndex[i] = bestIndex;
            length[i] = bestLength;
            if (maxLength < bestLength) {
                maxLength = bestLength;
                maxIndex = i;
            }
        }

        List<Integer> LIS = new ArrayList<>();

        int index = maxIndex;
        while (index != -1) {
            LIS.add(sequence[index]);
            index = prevIndex[index];
        }

        for (int i = LIS.size() - 1; i >= 0 ; i--) {
            System.out.print(LIS.get(i) + " ");
        }
    }
}
```

8.4. Move Down/Right Sum – iterative approach, има опция за разписване и с рекурсия

```
public class MoveDown_Right {
    public static void main(String[] args) {
        Scanner sc = new Scanner(System.in);
```

```java
        int rows = Integer.parseInt(sc.nextLine());
        int cols = Integer.parseInt(sc.nextLine());

        int[][] elements = new int[rows][cols];

        for (int row = 0; row < rows; row++) {
            elements[row] = Arrays.stream(sc.nextLine().split("\\s+"))
                    .mapToInt(Integer::parseInt)
                    .toArray();
        }

        int[][] dpTable = new int[rows][cols];

        dpTable[0][0] = elements[0][0];

        for (int col = 1; col < cols; col++) {
            dpTable[0][col] = dpTable[0][col - 1] + elements[0][col];
        }

        for (int row = 1; row < rows; row++) {
            dpTable[row][0] = dpTable[row - 1][0] + elements[row][0];
        }

        for (int row = 1; row < rows; row++) {
            for (int col = 1; col < cols; col++) {
                dpTable[row][col] = Math.max(dpTable[row - 1][col] + elements[row][col],
                        dpTable[row][col - 1] + elements[row][col]);
            }
        }

        int row = rows - 1;
        int col = cols - 1;

        List<String> path = new ArrayList<>();

        path.add(formatOutput(row, col));
        while (row > 0 || col > 0) {
            int top = -1;
            if (row > 0) {
                top = dpTable[row - 1][col];
            }

            int left = -1;
            if (col > 0) {
                left = dpTable[row][col - 1];
            }

            if (top > left) {
                row--;
            } else {
                col--;
            }
            path.add(formatOutput(row, col));
        }

        Collections.reverse(path);
        System.out.println(String.join(" ", path));
    }

    private static String formatOutput(int row, int col) {
        return "[" + row + ", " + col + "]";
```

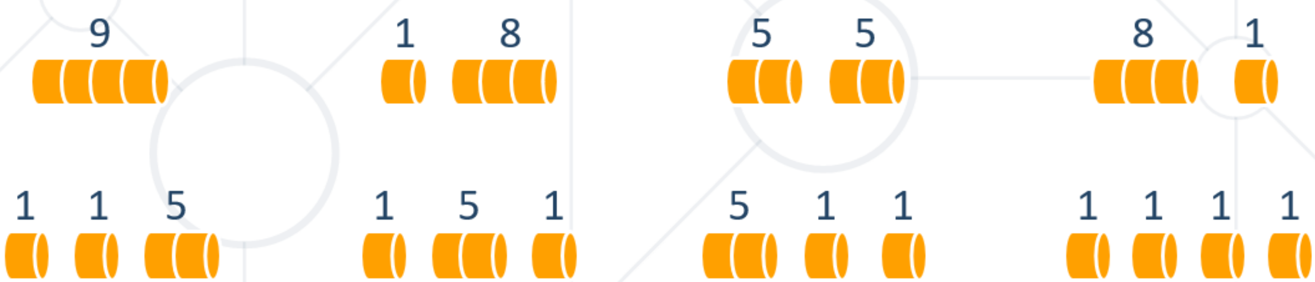
```
    }
}
```

8.5. Rod Cutting Problem

- Find the **best way** to cut up a rod, given the prices below

Length	0	1	2	3	4	5	6	7	8	9	10
Price	0	1	5	8	9	10	17	17	20	24	30

- Example: All possible ways to **cut rod** with **length = 4**



Length	0	1	2	3	4	5	6	7	8	9	10
Price	0	1	5	8	9	10	17	17	20	24	30

Best Price	0	1	5	8	10	10	17	18	20	24	30
Best Prev	0	1	2	3	2	2	6	6	8	9	10

Recursive solution

```
public class RodCutting {
    public static int[] bestPrices;
    public static int[] prevIndex;
    public static int[] prices;

    public static void main(String[] args) {
        Scanner sc = new Scanner(System.in);

        prices = Arrays.stream(sc.nextLine().split("\\s+"))
                .mapToInt(Integer::parseInt)
                .toArray();
        int length = Integer.parseInt(sc.nextLine());
        bestPrices = new int[length + 1];
        prevIndex = new int[length + 1];

        int maxProfit = cutRope(length);
        System.out.println(maxProfit);
```

```java
        reconstructSolution(length);
    }

    private static int cutRope(int length) {
        if (length == 0) {
            return 0;
        }
        if (bestPrices[length] != 0) {
            return bestPrices[length];
        }
        int currentBest = bestPrices[length];

        for (int i = 1; i <= length; i++) {
            currentBest = Math.max(currentBest, prices[i] + cutRope(length - i));
            if (currentBest > bestPrices[length]) {
                bestPrices[length] = currentBest;
                prevIndex[length] = i;
            }
        }

        return bestPrices[length];
    }

    private static void reconstructSolution(int n) {
        while (n - prevIndex[n] != 0) {
            System.out.print(prevIndex[n] + " ");
            n = n - prevIndex[n];
        }
        System.out.println(prevIndex[n]);
    }
}
```

Iterative solution

```java
private static int cutRod(int n) {
  for (int i = 1; i <= n; i++) {
    int currentBest;
    for (int j = 1; j <= i; j++) {
      currentBest =
      Math.max(bestPrice[i], price[j] + bestPrice[i - j]);
      if (currentBest > bestPrice[i]) {
        bestPrice[i] = currentBest;
        bestCombo[i] = j;
      }
    }
  }
  return bestPrice[n];
}
```