Problem 3

$$MSE_{mod}(i,j) = \frac{\sum_{l=-M}^{M}\sum_{k=-M}^{M} w(k,l)\left(x(i+k,j+l) - \mu_x(i,j) - y(i+k,j+l) + \mu_y(i,j)\right)^2}{\sigma_x^2(i,j) + \sigma_y^2(i,j) + C_2}$$

Numerator: $\sum_{l=-M}^{M}\sum_{k=-M}^{M} w(k,l)\left[\left(x(i+k,j+l) - \mu_x(i,j)\right) - \left(y(i+k,j+l) - \mu_y(i,j)\right)\right]^2$

$$= \sum_{l=-M}^{M}\sum_{k=-M}^{M}\left[w(k,l)\left[x(i+k,j+l) - \mu_x(i,j)\right]^2 + w(k,l)\left[y(i+k,j+l) - \mu_y(i,j)\right]^2\right]$$

$$-2\sum_{l=-M}^{M}\sum_{k=-M}^{M} w(k,l)\left(x(i+k,j+l) - \mu_x(i,j)\right)\left(y(i+k,j+l) - \mu_y(i,j)\right)$$

$$= \sigma_x^2(i,j) + \sigma_y^2(i,j) - 2\sigma_{xy}(i,j)$$

$$\boxed{MSE_{mod}(i,j) = \frac{\sigma_x^2(i,j) + \sigma_y^2(i,j) - 2\sigma_{xy}(i,j)}{\sigma_x^2(i,j) + \sigma_y^2(i,j) + C_2}}$$

$$CS(i,j) = \left(\frac{\cancel{2\sigma_x(i,j)\sigma_y(i,j) + C_2}}{\sigma_x^2(i,j) + \sigma_y^2(i,j) + C_2}\right)\left(\frac{2\sigma_{xy(i,j)} + C_2}{\cancel{2\sigma_{x(i,j)}\sigma_{y(i,j)} + C_2}}\right) \quad \left[\text{Replaced } C_3 = C_2/2\right]$$

$$\boxed{CS(i,j) = \frac{2\sigma_{xy}(i,j) + C_2}{\sigma_x^2(i,j) + \sigma_y^2(i,j) + C_2}}$$

$$MSE_{mod}(i,j) = \frac{\sigma_x^2(i,j) + \sigma_y^2(i,j) + C_2 - \cancel{(2\sigma_{xy}(i,j) + C_2)}}{\sigma_x^2(i,j) + \sigma_y^2(i,j) + C_2}$$

$$= 1 - \frac{2\sigma_{xy}(i,j) + C_2}{\sigma_x^2(i,j) + \sigma_y^2(i,j) + C_2}$$

$$\boxed{MSE_{mod}(i,j) = 1 - CS(i,j)}$$

$(P_3)$ SROCC $= \dfrac{\sum_{i=1}^{N} (p_i-\bar{p})(q_i-\bar{q})}{\sqrt{\sum_{i=1}^{N}(p_i-\bar{p})^2} \cdot \sqrt{\sum_{i=1}^{N}(q_i-\bar{q})^2}}$

We know from last derivation that

$MSE_{mod} = 1 - CS$

So the rank given by $MSE_{mod}$ will be the reverse of what will be given by CS.

Let $\cdot\, p_i \Rightarrow$ rank of $i^{th}$ image acc. to $MSE_{mod}$

$r_i \Rightarrow$ rank of $i^{th}$ image acc. to CS

$r_i = (N+1) - p_i$ . where N is the total number of images

$$\boxed{\bar{r} = (N+1) - \bar{p}}$$

SROCC for $MSE_{mod}$ $= \dfrac{\sum_{i=1}^{N} (p_i-\bar{p})(q_i-\bar{q})}{\sqrt{\sum_{i=1}^{N}(p_i-\bar{p})^2} \cdot \sqrt{\sum_{i=1}^{N}(q_i-\bar{q})^2}}$

SROCC for CS $= \dfrac{\sum_{i=1}^{N} (r_i-\bar{r})(q_i-\bar{q})}{\sqrt{\sum_{i=1}^{N}(r_i-\bar{r})^2} \cdot \sqrt{\sum_{i=1}^{N}(q_i-\bar{q})^2}}$

$$= \frac{\sum_{i=1}^{N} \left( (N+1-p_i) - (N+1-\bar{p}) \right)(q_i-\bar{q})}{\sqrt{\sum_{i=1}^{N}(N+1-p_i-(N+1-\bar{p}))^2} \cdot \sqrt{\sum_{i=1}^{N}(q_i-\bar{q})^2}}$$

$$= \frac{\sum_{i=1}^{N} (-p_i+\bar{p})(q_i-\bar{q})}{\sqrt{\sum_{i=1}^{N}(-p_i+\bar{p})^2} \cdot \sqrt{\sum_{i=1}^{N}(q_i-\bar{q})^2}}$$

$$= - \frac{\sum_{i=1}^{N} (p_i-\bar{p})(q_i-\bar{q})}{\sqrt{\sum_{i=1}^{N}(p_i-\bar{p})^2} \cdot \sqrt{\sum_{i=1}^{N}(q_i-\bar{q})^2}} = -\text{SROCC for } MSE_{mod}$$