14.1 Introduction

"A wave is any disturbance from a normal or equilibrium condition that propagates without the transport of matter. In general, a wave transports both energy and momentum."

The disturbance created by a wave is represented by a wave function $y(x, t)$. For a string the wave function is a (vector) displacement; whereas for sound waves it is a (scalar) pressure or density fluctuation. In the case of light or radio waves, the wave function is either an electric or a magnetic field vector.

Wave motion appears in almost every branch of physics. We are all familiar with water waves, sound waves and light waves. Waves occur when a system is disturbed from its equilibrium position and this disturbance travels or propagates from one region of the system to other. Energy can be transmitted over considerable distances by wave motion. The waves requiring a medium are called **mechanical waves** and those which do not require a medium are called **non-mechanical waves**. Light waves and all other electromagnetic waves are nonmechanical. The energy in the mechanical waves is the kinetic and potential energy of the matter. In the propagation of mechanical waves elasticity and inertia of the medium play an important role. This is why mechanical waves sometimes are also referred to as **elastic waves.** Note that the medium itself does not move as a whole along with the wave motion. Apart from mechanical and nonmechanical waves there is also another kind of waves called **"matter waves"**. These represent wave like properties of particles.

14.2 Transverse and Longitudinal Waves

There are two distinct classes of wave motion :

(i) transverse and

(ii) longitudinal.

In a **transverse wave motion** the particles of the medium oscillate about their mean or equilibrium position at right angles to the direction of propagation of wave motion itself.

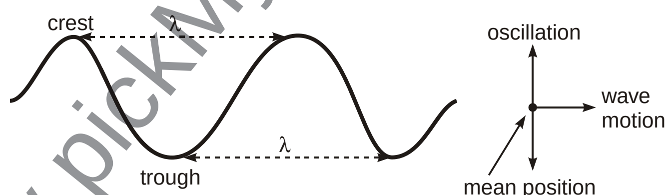


Fig. 14.1

This form of wave motion travels in the form of **crests and troughs**, as for example, waves travelling along a stretched string. This type of waves are possible in media which possess elasticity of shape or rigidity, *i.e.*, in solids. These are also possible on the surface of liquids also, even though they do not possess the property of rigidity. This is because they possess another equally effective property (surface tension) of resisting any vertical displacement of their particles (or keeping their level). Gases, however, possess neither rigidity nor do they resist any vertical displacement of particles (or keep their level). A transverse wave motion is therefore not possible in a gaseous medium. An electromagnetic wave is necessarily a transverse wave because of the electric and magnetic fields being perpendicular to its direction of propagation. The distance between two successive crests or troughs is known as the **wavelength (λ)** of the wave.

In a **longitudinal wave motion** the particles of the medium oscillate about their mean or equilibrium position along the direction of propagation of the wave motion itself. This type of wave motion travels in the form of **compressions** and **rarefactions** and is possible in media possessing elasticity of volume, *i.e.*, in solids, liquids and gases.

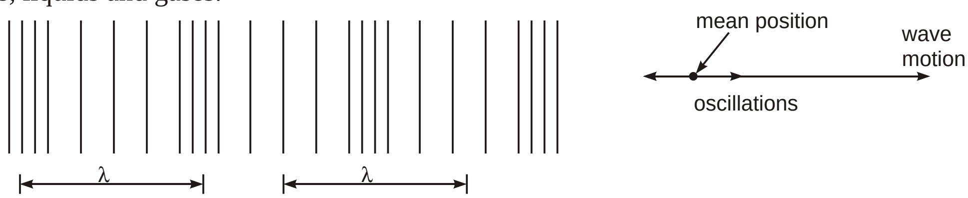


Fig. 14.2

The distance between two successive compressions or rarefactions constitute one wavelength. Sound waves in a gas are longitudinal in nature. In some cases, the waves are neither purely transverse nor purely longitudinal as, for example **ripples** on water surface (produced by dropping a stone in water), in which the particles of the medium (here water) oscillate across as well as along the direction of propagation of the wave motion describing elliptical paths. We need not however, bother ourselves with any such type of waves here.

Again waves may be **one dimensional, two dimensional or three dimensional** according as they propagate energy in just one, two or three dimensions.

Transverse waves along a string are one dimensional, ripples on water surface are two dimensional and sound waves proceeding radially from a point source are three dimensional.

14.3 The General Equation of Wave Motion

As we have already read, in a wave motion, some physical quantity (say y) is made to oscillate at one place and these oscillations of y propagate to other places. The y may be,

(i) displacement of particles from their mean position in case of transverse wave in a rope or longitudinal sound wave in a gas.

(ii) pressure difference (dP) or density difference ($d\rho$) in case of sound wave or

(iii) electric and magnetic fields in case of electromagnetic waves.

The oscillations of y may or may not be simple harmonic in nature. Now let us consider a one dimensional wave travelling along x-axis. In this case y is a function of position (x) and time (t). The reason is that one may be interested in knowing the value of y at a general point x at any time t. Thus, we can say that,

$$y = y(x, t)$$

But only those functions of x and t, represent a wave motion which satisfy the differential equation,

$$\frac{\partial^2 y}{\partial t^2} = k\,\frac{\partial^2 y}{\partial x^2}$$

Here k is a constant, which is equal to square of the wave speed, or

$$k = v^2$$

Thus, the above equation can be written as,

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2} \qquad \ldots(i)$$

The general solution of this equation is of the form

$$y(x, t) = f\,(ax \pm bt) \qquad \ldots(ii)$$

Thus, any function of x and t which satisfies Eq. (i) or which can be written as Eq. (ii) represents a wave. The only condition is that it should be finite everywhere and at all times. Further, if these conditions are satisfied, then speed of wave (v) is given by,

$$v = \frac{\text{coefficient of } t}{\text{coefficient of } x} = \frac{b}{a}$$

The plus (+) sign between ax and bt implies that the wave is travelling along negative x-direction and minus (–) sign shows that it is travelling along positive x-direction.

Sample Example 14.1 *Show that the equation,* $y = a\sin(\omega t - kx)$ *satisfies the wave equation* $\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$. *Find speed of wave and the direction in which it is travelling.*

Solution

$$\frac{\partial^2 y}{\partial t^2} = -\omega^2 a \sin(\omega t - kx)$$

and

$$\frac{\partial^2 y}{\partial x^2} = -k^2 a \sin(\omega t - kx)$$

We can write these two equations as,

$$\frac{\partial^2 y}{\partial t^2} = \frac{\omega^2}{k^2} \cdot \frac{\partial^2 y}{\partial x^2}$$

Comparing this with,

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$$

We get, wave speed $v = \frac{\omega}{k}$ **Ans.**

The negative sign between ωt and kx implies that wave is travelling along positive x-direction.

Sample Example 14.2 *Which of the following functions represent a wave*

(a) $(x - vt)^2$ *(b)* $\ln(x + vt)$ *(c)* $e^{-(x-vt)^2}$ *(d)* $\frac{1}{x + vt}$

Solution Although all the four functions are written in the form $f\,(ax \pm bt)$, only (c) among the four functions is finite everywhere at all times. Hence only (c) represents a wave.

Sample Example 14.3 $y(x, t) = \frac{0.8}{[(4x + 5t)^2 + 5]}$ *represents a moving pulse where x and y are in metre and t in second. Then choose the correct alternative(s):* **(JEE 1999)**

(a) pulse is moving in positive x-direction

(b) in 2 s it will travel a distance of 2.5 *m*
(c) its maximum displacement is 0.16 *m*
(d) it is a symmetric pulse

Solution (b), (c) and (d) are correct options.

The shape of pulse at $x = 0$ and $t = 0$ would be as shown in Fig. 14.3.

$$y(0, 0) = \frac{0.8}{5} = 0.16 \text{ m}$$

From the figure it is clear that $y_{\text{max}} = 0.16$ m

Pulse will be symmetric (symmetry is checked about y_{max}) if

At $t = 0$; $\quad y(x) = y(-x)$

From the given equation

$$\left.\begin{aligned} y(x) &= \frac{0.8}{16x^2 + 5} \\ \text{and} \quad y(-x) &= \frac{0.8}{16x^2 + 5} \end{aligned}\right\} \text{at } t = 0$$

or $\quad y(x) = y(-x)$

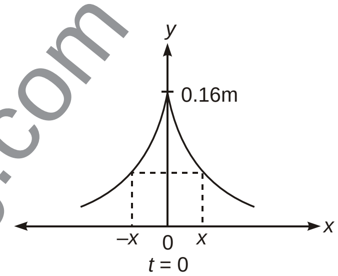


Fig. 14.3

Therefore, pulse is symmetric.

Speed of pulse : At $t = 1$s and $x = -1.25$ m

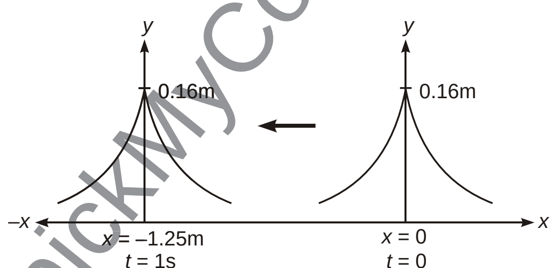


Fig. 14.4

value of y is again 0.16 m, *i.e.*, pulse has travelled a distance of 1.25 m in 1 second in negative x-direction or we can say that the speed of pulse is 1.25 m/s and it is travelling in negative x-direction. Therefore, it will travel a distance of 2.5 m in 2 seconds. The above statement can be better understood from Fig. 14.4.

Alternate method :

If equation of a wave pulse is

$$y = f(ax \pm bt)$$

the speed of wave is $\frac{b}{a}$ in negative x-direction for $y = f(ax + bt)$ and positive x-direction for $y = f(ax - bt)$. Comparing this from given equation we can find that speed of wave is $\frac{5}{4} = 1.25$ m/s and it is travelling in negative x-direction.

Sample Example 14.4 *In a wave motion $y = a \sin(kx - \omega t)$, y can represent:* **(JEE 1999)**

(a) electric field (b) magnetic field (c) displacement (d) pressure

Solution (a, b, c, d)

In case of sound wave, y can represent pressure and displacement, while in case of an electromagnetic wave it represents electric and magnetic fields.

- *In general, y is any general physical quantity which is made to oscillate at one place and these oscillations are propagated to other places.*

Introductory Exercise 14.1

1. Prove that the equation $y = a \sin \omega t$ does not satisfy the wave equation and hence it does not represent a wave.
2. A wave pulse is described by $y(x, t) = ae^{-(bx - ct)^2}$, where a, b and c are positive constants. What is the speed of this wave?
3. The displacement of a wave disturbance propagating in the positive x-direction is given by
$$y = \frac{1}{1 + x^2} \text{ at } t = 0 \quad \text{and} \quad y = \frac{1}{1 + (x - 1)^2} \text{ at } t = 2\text{ s}$$
where x and y are in metre. The shape of the wave disturbance does not change during the propagation. What is the velocity of the wave?
4. A travelling wave pulse is given by, $y = \dfrac{10}{5 + (x + 2t)^2}$
Here x and y are in metre and t in second. In which direction and with what velocity is the pulse propagating? What is the amplitude of pulse?
5. If at $t = 0$, a travelling wave pulse on a string is described by the function,
$$y = \frac{10}{(x^2 + 2)}$$
Here x and y are in metre and t in second. What will be the wave function representing the pulse at time t, if the pulse is propagating along positive x-axis with speed 2 m/s?

14.4 Plane Progressive Harmonic Wave

Consider a function $y = f(x)$, represented graphically by the solid curve shown in Fig. 14.5.

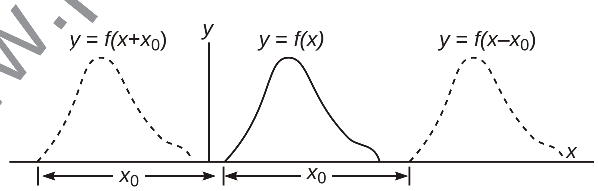


Fig. 14.5

If we replace x by $x - x_0$, we get the function,

$$y = f(x - x_0)$$

It is clear that shape of the curve has not changed, the same value of y occurs for values of x increased by the amount x_0. In other words, assuming that x_0 is positive, we see that the curve has been displaced to the right an amount x_0 without deformation. Similarly, $y = f(x + x_0)$ corresponds to a displacement of the curve to the left by an amount x_0.

For example if we have two functions:

$y_1 = x^2$ and $y_2 = (x - 5)^2$ and $y_1 = 16$ at $x = 4$ then y_2 has the same value, *i.e.*, 16 at $x = 4 + 5$ or $x = 9$.

Now, if $x_0 = vt$, where t is the time, we get a travelling curve. That is $y = f(x - vt)$ represents a curve moving to the right with a velocity v, called the **wave velocity** or **phase velocity**.

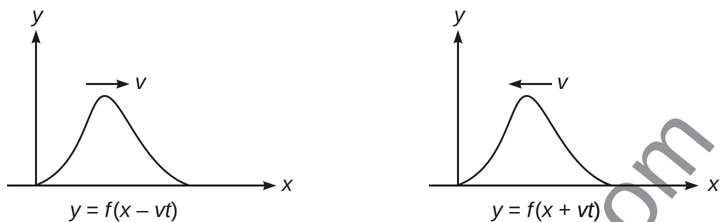


Fig. 14.6

Similarly $y = f(x + vt)$ represents a curve moving to the left with velocity v. Therefore, we conclude that a mathematical expression of the form,

$$y(x, t) = f(x \pm vt)$$

is adequate for describing a physical situation that 'travels' or 'propagates' without deformation along negative or positive x-axis. The quantity $y(x, t)$ may represent the deformation in a solid, the pressure in a gas, an electric or magnetic field, etc.

When $y(x, t)$ is a sine or cosine function such as,

$$y(x, t) = A \sin k(x - vt)$$

or

$$y(x, t) = A \cos k(x - vt)$$

it is called **plane progressive harmonic wave**. In plane progressive harmonic wave oscillations of y are simple harmonic in nature.

The quantity k has a special meaning. Replacing the value of x by $x + \frac{2\pi}{k}$, we get the same value of y, *i.e.*,

$$y\left(x + \frac{2\pi}{k}, t\right) = A \sin k\left(x + \frac{2\pi}{k} - vt\right) = A \sin[k(x - vt) + 2\pi]$$

$$= A \sin k(x - vt) = y(x, t)$$

The quantity,

$$\frac{2\pi}{k} = \lambda$$

designated as **wavelength,** is the **space period** of the curve, that is the curve repeats itself every length λ. The quantity $k = \frac{2\pi}{\lambda}$ represents the number of wavelengths in the distance 2π and is called the **wave number**.

Therefore,

$$y(x, t) = A \sin k(x - vt) = A \sin \frac{2\pi}{\lambda}(x - vt)$$

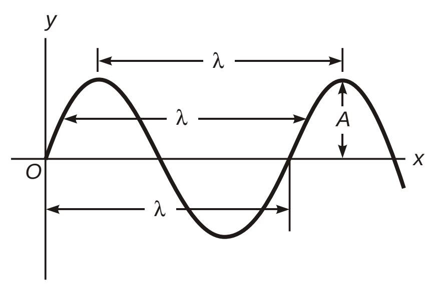


Fig. 14.7

represents a plane progressive harmonic wave of wavelength λ propagating towards the positive x-axis with speed v. The above equation can also be written as,

$$y(x, t) = A \sin (kx - \omega t)$$

where
$$\boxed{\omega = kv = \frac{2\pi v}{\lambda}}$$

gives the **angular frequency** of the wave. Further, $\omega = 2\pi f$, where f is the **frequency** with which y oscillates at every point x. We have the important relation,

$$\boxed{v = \lambda f}$$

Also if T is the **period** of oscillation then,

$$\boxed{T = \frac{2\pi}{\omega} = \frac{1}{f}}$$

We may also write,
$$y = A \sin 2\pi \left(\frac{x}{\lambda} - \frac{t}{T}\right)$$

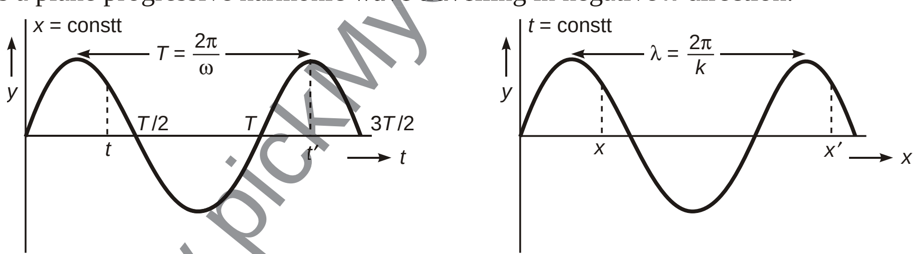
Thus, the equation of a plane progressive harmonic wave moving along positive x-direction can be written as,

$$\boxed{y = A \sin k\ (x - vt) = A \sin (kx - \omega t) = A \sin \frac{2\pi}{\lambda}(x - vt) = A \sin 2\pi \left(\frac{x}{\lambda} - \frac{t}{T}\right)}$$

Similarly, the expressions

$$y = A \sin k\ (x + vt) = A \sin (kx + \omega t) = A \sin 2\pi \left(\frac{x}{\lambda} + \frac{t}{T}\right)$$

represents a plane progressive harmonic wave travelling in negative x-direction.



Displacement of a particle at different instants

Position of different particles at same instant in a string.

Fig. 14.8

Note that as y propagates in the medium (or space), it repeats itself in space after one period, because

$$\lambda = vT$$

which shows that, **the wavelength is the distance advanced by the wave motion in one period.**

Therefore, in plane progressive harmonic wave we have two periodicities, one in time given by the period T, and one in space given by the wavelength λ, with the two related by

$$\boxed{\lambda = vT}$$

• Important points in Wave motion Read so far

1. As we have read in Art. 14.3, any function of x and t which satisfies equation number (i) of the same article or which can be written in the form of equation number (ii) represents a wave provided it is finite everywhere at all times. What we have read in Art. 14.4 is about plane progressive harmonic wave. If $f(ax \pm bt)$ is a sine or cosine function, it is called plane progressive harmonic wave. The only special characteristic of this wave is that oscillations of y are simple harmonic in nature.
2. The general expression of a plane progressive harmonic wave is,
$$y = A \sin(kx \pm \omega t \pm \phi)$$
or
$$y = A \cos(kx \pm \omega t \pm \phi)$$
Here ϕ represents the initial phase.
3. I have seen students often confused whether the equation of a plane progressive wave should be,
$$y = A \sin(kx - \omega t)$$
or
$$y = A \sin(\omega t - kx)$$
Because some books write the first while the others write the second. It hardly matters whether you write the first or the second. Both the equations represent a travelling wave travelling in positive x-direction with speed $v = \frac{\omega}{k} \cdot$ The difference between them is that they are out of phase, *i.e.*, phase difference between them is π. It means, if a particle in position $x = 0$ at time $t = 0$ is in its mean position and moving upwards (represented by first wave) then the same particle will be in its mean position but moving downwards (represented by the second wave). Similarly the waves, $y = A \sin(kx - \omega t)$ and $y = -A \sin(kx - \omega t)$ are also out of phase.
4. **Particle velocity (v_p) and acceleration (a_p) in a sinusoidal wave :** In plane progressive harmonic wave particles of the medium oscillate simple harmonically about their mean position. Therefore, all the formulae what we have read in SHM apply to the particles here also. For example, maximum particle velocity is $\pm A\omega$ at mean position and it is zero at extreme positions etc. Similarly maximum particle acceleration is $\pm \omega^2 A$ at extreme positions and zero at mean position. However the wave velocity is different from the particle velocity. This depends on certain characteristics of the medium. Unlike the particle velocity which oscillates simple harmonically (between $+A\omega$ and $-A\omega$) the wave velocity is constant for given characteristics of the medium.
Suppose the wave function is,
$$y(x, t) = A \sin(kx - \omega t) \quad \ldots(i)$$
Let us differentiate this function partially with respect to t and x.
$$\frac{\partial y(x,t)}{\partial t} = -A\omega \cos(kx - \omega t) \quad \ldots(ii)$$
$$\frac{\partial y(x,t)}{\partial x} = Ak \cos(kx - \omega t) \quad \ldots(iii)$$
Now, these can be written as,
$$\frac{\partial y(x,t)}{\partial t} = -\left(\frac{\omega}{k}\right)\frac{\partial y(x,t)}{\partial x}$$
Here, $\frac{\partial y(x,t)}{\partial t}$ = particle velocity v_P
$\frac{\omega}{k}$ = wave velocity v

and $$\frac{\partial y(x,t)}{\partial x} = \text{slope of the wave}$$

Thus, $$v_P = -v\,(\text{slope}) \qquad \ldots\text{(iv)}$$

i.e., particle velocity at a given position and time is equal to negative of the product of wave velocity with slope of the wave at that point at that instant.

The acceleration of the particle is the second partial derivative of $y(x,t)$ with respect to t,

$$\therefore \quad a_P = \frac{\partial^2 y(x,t)}{\partial t^2} = -\omega^2 A \sin(kx - \omega t) = -\omega^2 y(x,t)$$

i.e., the acceleration of the particle equals $-\omega^2$ times its displacement, which is the result we obtained for SHM. Thus,

$$a_P = -\omega^2\,(\text{displacement}) \qquad \ldots\text{(v)}$$

We can also show that,

$$\frac{\partial^2 y(x,t)}{\partial t^2} = \left(\frac{\omega^2}{k^2}\right) \cdot \frac{\partial^2 y(x,t)}{\partial x^2}$$

or $$\frac{\partial^2 y(x,t)}{\partial t^2} = v^2 \frac{\partial^2 y(x,t)}{\partial x^2} \qquad \ldots\text{(vi)}$$

which is also the wave equation.

Figure shows the velocity (v_P) and acceleration (a_P) given by Eqs. (iv) and (v) for two points 1 and 2 on a string as a sinusoidal wave is travelling in it along positive x-direction.

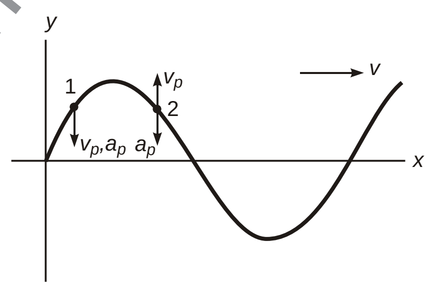


Fig. 14.9

At 1 : Slope of the curve is positive. Hence from Eq. (iv) particle velocity (v_P) is negative or downwards. Similarly displacement of the particle is positive, so from Eq. (v) acceleration will be negative or downwards.

At 2 : Slope is negative while displacement is positive. Hence v_P will be positive (upwards) and a_P is negative (downwards).

Note *The direction of v_P will change if the wave travels along negative x-direction.*

Sample Example 14.5 *The equation of a wave is,*

$$y(x,t) = 0.05 \sin\left[\frac{\pi}{2}(10x - 40t) - \frac{\pi}{4}\right] m$$

Find : (a) the wavelength, the frequency and the wave velocity

(b) the particle velocity and acceleration at $x = 0.5$ m and $t = 0.05$ s.

Solution (a) The equation may be rewritten as,

$$y(x,t) = 0.05 \sin\left(5\pi x - 20\pi t - \frac{\pi}{4}\right) \text{m}$$

Comparing this with equation of plane progressive harmonic wave,

$$y(x,t) = A \sin(kx - \omega t + \phi) \text{ we have,}$$

$$\text{wave number } k = \frac{2\pi}{\lambda} = 5\pi \text{ rad/m}$$

$$\therefore \qquad \lambda = 0.4 \text{ m} \qquad \textbf{Ans.}$$

The angular frequency is,

$$\omega = 2\pi f = 20\pi \text{ rad/s}$$

$$\therefore \qquad f = 10 \text{ Hz} \qquad \textbf{Ans.}$$

The wave velocity is,

$$v = f\lambda = \frac{\omega}{k} = 4 \text{ m/s in } +x \text{ direction} \qquad \textbf{Ans.}$$

(b) The particle velocity and acceleration are,

$$\frac{\partial y}{\partial t} = -(20\pi)(0.05)\cos\left(\frac{5\pi}{2} - \pi - \frac{\pi}{4}\right)$$

$$= 2.22 \text{ m/s} \qquad \textbf{Ans.}$$

$$\frac{\partial^2 y}{\partial t^2} = -(20\pi)^2(0.05)\sin\left(\frac{5\pi}{2} - \pi - \frac{\pi}{4}\right)$$

$$= 140 \text{ m/s}^2 \qquad \textbf{Ans.}$$

Sample Example 14.6 *Figure shows a snapshot of a sinusoidal travelling wave taken at $t = 0.3$ s. The wavelength is 7.5 cm and the amplitude is 2 cm. If the crest P was at $x = 0$ at $t = 0$, write the equation of travelling wave.*

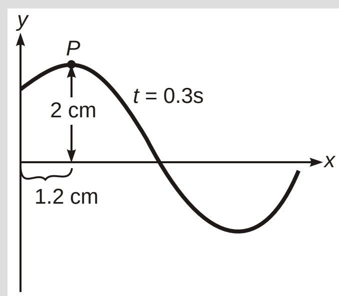


Fig. 14.10

Solution Given, $A = 2$ cm, $\lambda = 7.5$ cm

$$\therefore \qquad k = \frac{2\pi}{\lambda} = 0.84 \text{ cm}^{-1}$$

The wave has travelled a distance of 1.2 cm in 0.3 s. Hence, speed of the wave,

$$v = \frac{1.2}{0.3} = 4 \text{ cm/s}$$

$\therefore$ Angular frequency $\omega = (v)(k) = 3.36$ rad/s

Since the wave is travelling along positive x-direction and crest (maximum displacement) is at $x = 0$ at $t = 0$, we can write the wave equation as,

$$y(x, t) = A\cos(kx - \omega t)$$

or

$$y(x, t) = A\cos(\omega t - kx) \quad \text{as} \quad \cos(-\theta) = \cos\theta$$

Therefore, the desired equation is,

$$y(x, t) = (2 \text{ cm})\cos[(0.84 \text{ cm}^{-1})x - (3.36 \text{ rad/s})t] \text{ cm} \qquad \textbf{Ans.}$$

Introductory Exercise 14.2

1. The equation of a travelling wave is,

$$y(x, t) = 0.02 \sin\left(\frac{x}{0.05} + \frac{t}{0.01}\right) \text{ m}$$

Find :
(a) The wave velocity and
(b) the particle velocity at $x = 0.2$ m and $t = 0.3$ s.
Given $\cos\theta = -0.85$
where $\theta = 34$ rad

2. Is there any relationship between wave speed and the maximum particle speed for a wave travelling on a string? If so, what is it?

3. Consider a sinusoidal travelling wave shown in figure. The wave velocity is + 40 cm/ s.
Find :
(a) the frequency
(b) the phase difference between points 2.5 cm apart
(c) how long it takes for the phase at a given position to change by 60°
(d) the velocity of a particle at point P at the instant shown.

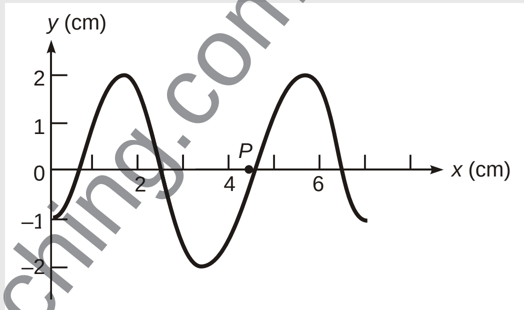


Fig. 14.11

4. Transverse waves on a string have wave speed 12.0 m/s, amplitude 0.05 m and wavelength 0.4 m. The waves travel in the + x direction and at $t = 0$ the $x = 0$ end of the string has zero displacement and is moving upwards.
(a) Write a wave function describing the wave.
(b) Find the transverse displacement of a point at $x = 0.25$ m at time $t = 0.15$ s.
(c) How much time must elapse from the instant in part (b) until the point at $x = 0.25$ m has zero displacement?

14.5 Speed of a Transverse Wave on a String

One of the key properties of any wave is the wave speed. In this section we'll see what determines the speed of propagation of transverse waves on a string. The physical quantities that determine the speed of transverse waves on a string are the **tension** in the string and its **mass per unit length** (also called linear mass density).

We might guess that increasing the tension showed increase the restoring forces that tend to straighten the string when it is disturbed, thus increasing the wave speed. We might also guess increasing the mass should make the motithat on more sluggish and decrease the speed. Both these guesses turn out to be right. We will develop the exact relationship between wave speed, tension and mass per unit length.

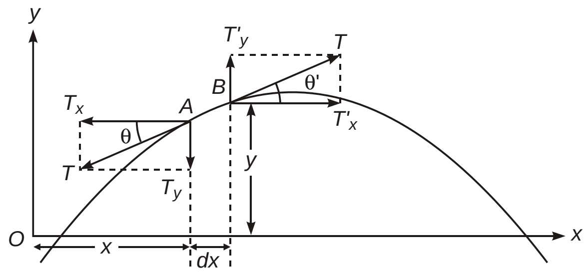


Fig. 14.12 Forces on a section of a transversely displaced string.

Under equilibrium conditions a string subject to a tension T is straight. Suppose that we now displace the string sidewise, or perpendicular to its length, by a small amount as shown in figure. Consider a small section AB of the string of length dx, that has been displaced a distance y from the equilibrium position. On each end a tangential force T is acting. Due to the curvature of the string, the two forces are not directly opposed but make angles θ and θ' with the x-axis. The resultant upward force on the section AB of the string is,

$$F_y = T_y' - T_y \qquad \ldots\text{(i)}$$

Under the acting of this force, the section AB of the string moves up and down.

Rewriting Eq. (i) we have,

$$F_y = T\,(\sin\theta' - \sin\theta)$$

Since θ and θ' are almost equal, we may write

$$F_y = Td\,(\sin\theta)$$

If the curvature of the string is not very large, the angles θ and θ' are small, and the sines can be replaced by their tangents. So the upward force is,

$$F_y = Td\,(\tan\theta) = T\,.\left\{\frac{d}{dx}(\tan\theta)\right\}.\,dx$$

But $\tan\theta$ is the slope of the curve adopted by the string, which is equal to $\frac{dy}{dx}$. Hence

$$F_y = T\left\{\frac{d}{dx}\left(\frac{dy}{dx}\right)\right\}dx = T\left(\frac{d^2y}{dx^2}\right)dx$$

This force must be equal to the mass of the section AB multiplied by its upward acceleration $\frac{d^2y}{dt^2}$. If μ is the linear density of the string, the mass of the section AB is μdx. We use the relation $F = ma$ and write the equation of motion of this section of the string as,

$$(\mu dx)\frac{d^2y}{dt^2} = T\left(\frac{d^2y}{dx^2}\right)dx$$

or

$$\frac{d^2y}{dt^2} = \frac{T}{\mu}\cdot\frac{d^2y}{dx^2}$$

Comparing this with wave equation,

$$\frac{\partial^2 y}{\partial t^2} = v^2\frac{\partial^2 y}{\partial x^2}$$

the wave speed,

$$\boxed{v = \sqrt{\frac{T}{\mu}}}$$

Alternate Method

Consider a pulse travelling along a string with a speed v to the right. If the amplitude of the pulse is small compared to the length of the string, the tension T will be approximately constant along the string. In the reference frame moving with speed v to the right, the pulse in stationary and the string moves with a speed v to the left. Figure shows a small segment of the string of length Δl. This segment forms part of a

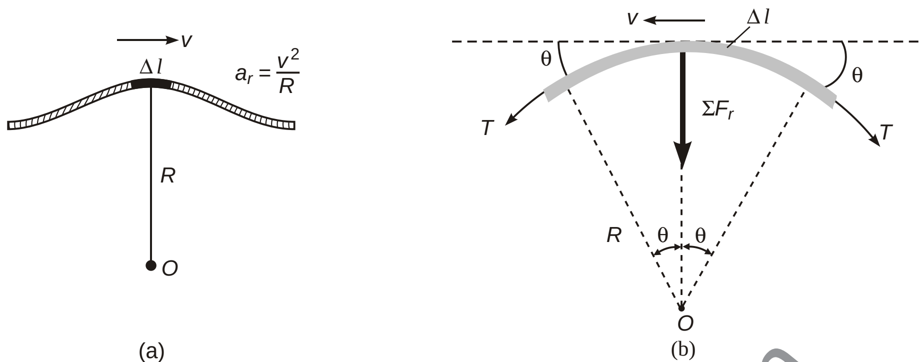


Fig. 14.13 (a) To obtain the speed v of a wave on a stretched string, it is convenient to describe the motion of a small segment of the string in a moving frame of reference.

(b) In the moving frame of reference, the small segment of length Δl moves to the left with speed v. The net force on the segment is in the radial direction because the horizontal components of the tension force cancel.

circular arc of radius R. Instantaneously the segment is moving with speed v in a circular path, so it has a centripetal acceleration $\frac{v^2}{R}$. The forces acting on the segment are the tension T at each end. The horizontal components of these forces are equal and opposite and thus cancel. The vertical components of these forces point radially inward toward the centre of the circular arc. These radial forces provide the centripetal acceleration. Let the angle subtended by the segment at centre be 2θ. The net radial force acting on the segment is

$$\Sigma F_r = 2T \sin\theta = 2T\theta$$

Where we have used the approximation $\sin\theta \approx \theta$ for small θ.

If μ is the mass per unit length of the string, the mass of the segment of length Δl is

$$m = \mu \Delta l = 2\mu R\theta \qquad (\text{as } \Delta l = 2R\theta)$$

From Newton's second law $\Sigma F_r = ma = \frac{mv^2}{R}$

or $$2T\theta = (2\mu R\theta)\left(\frac{v^2}{R}\right) \qquad \therefore \qquad v = \sqrt{\frac{T}{\mu}}$$

● Speed of Wave Motion

1. Speed of transverse wave on a string is given by,

$$v = \sqrt{\frac{T}{\mu}}$$

Here, μ = mass per unit length of the string

$$= \frac{m}{l}$$

$$= \frac{mA}{lA} \qquad (A = \text{area of cross-section of the string})$$

$$= \left(\frac{m}{V}\right) A \qquad (V = \text{volume of string})$$

$$= \rho A \qquad (\rho = \text{density of string})$$

Hence, the above expression can also be written as,

$$v = \sqrt{\frac{T}{\rho A}}$$

2. Speed of longitudinal wave through a gas (or a liquid) is given by,

$$v = \sqrt{\frac{B}{\rho}}$$

Here, B = Bulk modulus of the gas (or liquid)

and ρ = density of the gas (or liquid)

Now, Newton who first deduced this relation for v assumed that during the passage of a sound wave through a gas (or air), the temperature of the gas remains constant *i.e.*, sound wave travels under isothermal conditions and hence took B to be the isothermal elasticity of the gas and which is equal to its pressure P. So, **Newton's** formula for the velocity of a sound wave (or a longitudinal wave) in a gaseous medium becomes,

$$v = \sqrt{\frac{P}{\rho}}$$

If, however, we calculate the velocity of sound in air at NTP with the help of this formula by substituting.

$$P = 1.01 \times 10^5 \text{ N/m}^2 \quad \text{and} \quad \rho = 1.29 \times 10^{-3} \text{ kg/m}^3$$

then v comes out to be nearly 280 m/s. Actually the velocity of sound in air at NTP as measured by Newton himself, is found to be 332 m/s. Newton could not explain this large discrepancy between his theoretical and experimental results.

La'place after 140 years correctly argued that a sound wave passes through a gas (or air) very rapidly. So adiabatic conditions are developed. So, he took B to be the adiabatic elasticity of the gas, which is equal to γP where γ is the ratio of C_P (molar heat capacity at constant pressure) and C_V (molar heat capacity at constant volume). Thus, Newton's formula as corrected by La'place becomes,

$$v = \sqrt{\frac{\gamma P}{\rho}}$$

For air, $\gamma = 1.41$ So that in air,

$$v = \sqrt{\frac{1.41\, P}{\rho}}$$

which gives 331.6 m/s as the velocity of sound (in air) at NTP which is in agreement with Newton's experimental result.

Note *We will carry out the derivation of formula,*

$$v = \sqrt{\frac{B}{\rho}}$$

in the chapter of sound (Chapter-16).

3. Speed of longitudinal wave in a thin rod or wire is given by,

$$v = \sqrt{\frac{Y}{\rho}}$$

Here, Y is the Young's modulus of elasticity.

Sample Example 14.7 *One end of 12.0 m long rubber tube with a total mass of 0.9 kg is fastened to a fixed support. A cord attached to the other and passes over a pulley and supports an object with a mass of 5.0 kg. The tube is struck a transverse blow at one end. Find the time required for the pulse to reach the other end.* ($g = 9.8$ *m/s*^{2})

Solution Tension in the rubber tube AB, $T = mg$

or $$T = (5.0)(9.8) = 49 \text{ N}$$

Mass per unit length of rubber tube,

$$\mu = \frac{0.9}{12} = 0.075 \text{ kg/m}$$

∴ Speed of wave on the tube,

$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{49}{0.075}} = 25.56 \text{ m/s}$$

∴ The required time is,

$$t = \frac{AB}{v} = \frac{12}{25.56} = 0.47 \text{ s} \qquad \textbf{Ans.}$$

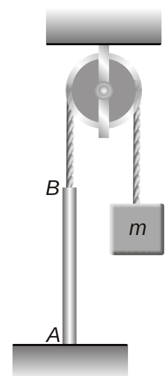


Fig. 14.14

Sample Example 14.8 *A uniform rope of mass 0.1 kg and length 2.45 m hangs from a ceiling.*

(a) Find the speed of transverse wave in the rope at a point 0.5 m distant from the lower end.
(b) Calculate the time taken by a transverse wave to travel the full length of the rope.

Solution (a) As the string has mass and it is suspended vertically, tension in it will be different at different points. For a point at a distance x from the free end, tension will be due to the weight of the string below it. So, if m is the mass of string of length l, the mass of length x of the string will be, $\left(\frac{m}{l}\right)x$.

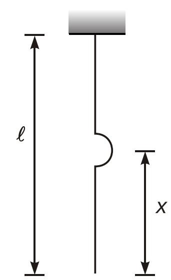


Fig. 14.15

∴ $$T = \left(\frac{m}{l}\right)xg = \mu xg \qquad \left(\frac{m}{l} = \mu\right)$$

∴ $$\frac{T}{\mu} = xg$$

or $$v = \sqrt{\frac{T}{\mu}} = \sqrt{xg} \qquad \ldots\text{(i)}$$

At $x = 0.5$ m, $$v = \sqrt{0.5 \times 9.8} = 2.21 \text{ m/s}$$

(b) From Eq. (i) we see that velocity of the wave is different at different points. So, if at point x the wave travels a distance dx in time dt, then

$$dt = \frac{dx}{v} = \frac{dx}{\sqrt{gx}}$$

$\therefore$
$$\int_0^t dt = \int_0^l \frac{dx}{\sqrt{gx}}$$

or
$$t = 2\sqrt{\frac{l}{g}} = 2\sqrt{\frac{2.45}{9.8}}$$
$$= 1.0 \text{ s}$$ **Ans.**

Introductory Exercise 14.3

1. Calculate the velocity of a transverse wave along a string of length 2 m and mass 0.06 kg under a tension of 500 N.
2. Calculate the speed of a transverse wave in a wire of 1.0 mm^2 cross-section under a tension of 0.98 N. Density of the material of wire is 9.8×10^3 kg/m^3.

14.6 Energy in Wave Motion

Every wave motion has energy associated with it. In wave motion, energy and momentum are transferred or propagated.

To produce any of the wave motions, we have to apply a force to a portion of the wave medium. The point where the force is applied moves, so we do work on the system. As the wave propagates, each portion of the medium exerts a force and does work on the adjoining portion. In this way a wave can transport energy from one region of space to other.

Regarding the energy in wave motion, we come across three terms namely, energy density (u), power (P) and intensity (I). Now let us take them one by one.

Energy density (u)

By the energy density of a plane progressive wave we mean **the total mechanical energy (kinetic + potential) per unit volume** of the medium through which the wave is passing. Let us proceed to obtain an expression for it.

Consider a string attached to a tuning fork. As the fork vibrates, it imparts energy to the segment of the string attached to it. For example, as the fork moves through its equilibrium position, it stretches the segment, increasing its potential energy, and the fork imparts transverse speed to the segment, increasing its kinetic energy. As a wave moves along the string, energy is imparted to the other segments of the string.

Kinetic energy per unit volume

We can calculate the kinetic energy of unit volume of the string from the wave function. Mass of unit volume is the density ρ. Its displacement from equilibrium is the wave function $y = A \sin (kx - \omega t)$. Its speed is $\frac{dy}{dt}$ where x is considered to be fixed. The kinetic energy of unit volume ΔK is then

$$\Delta K = \frac{1}{2} (\Delta m) v_y^2 = \frac{1}{2} \rho \left(\frac{dy}{dt}\right)^2$$

Using $y = A \sin (kx - \omega t)$, we obtain

$$\frac{dy}{dt} = -\omega A \cos (kx - \omega t)$$

So the kinetic energy per unit volume is

$$\Delta K = \frac{1}{2} \rho^2 \omega^2 A^2 \cos^2 (kx - \omega t) \quad \text{...(i)}$$

Potential energy per unit volume

The potential energy of a segment is the work done in stretching the string and depends on the slope $\frac{dy}{dx}$. The potential energy per unit volume of the string is related to the slope and tension T by (for small slopes)

$$\Delta U = \frac{1}{2} \rho v^2 \left(\frac{dy}{dx}\right)^2 \quad \text{...(ii)}$$

where v = wave speed $= \frac{\omega}{k}$

Using $\frac{dy}{dx} = kA \cos (kx - \omega t)$, we obtain for the potential energy

$$\Delta U = \frac{1}{2} \rho \omega^2 A^2 \cos^2 (kx - \omega t) \quad \text{...(iii)}$$

Which is the same as the kinetic energy. The total energy per unit volume is

$$\Delta E = \Delta K + \Delta U = \rho \omega^2 A^2 \cos^2 (kx - \omega t) \quad \text{...(iv)}$$

We see that ΔE varies with time. Since the average value of $\cos^2 (kx - \omega t)$ at any point is $\frac{1}{2}$, the average energy per unit volume (called the energy density u) is

$$u = \frac{1}{2} \rho \omega^2 A^2 \quad \text{...(v)}$$

Note *(i) Equation (v) is the same result as for a mass ρ attached to a spring and oscillating simple harmonic wave. However for a mass attached to a spring the potential energy is maximum when the displacement is maximum. For a string segment, the potential energy depends on the slope of the string and is maximum when the slope is maximum, which is at the equilibrium position of the segment, the same position for which the kinetic energy is maximum.*

At A : *Kinetic energy and potential energy both are zero.*

At B : *Kinetic energy and potential energy both are maximum.*

(ii) Equation (ii) has been derived in miscellaneous example number 13.

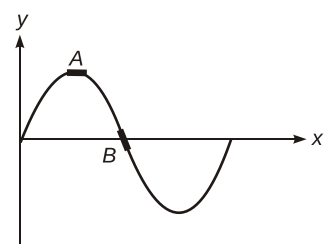


Fig. 14.16

Power (P)

Power is the instantaneous rate at which energy is transferred along the string (if we consider a transverse wave on a string).

Its value depends on the position x on the string at time t. Note that energy is being transferred only at points where the string has a nonzero slope $(\partial y/\partial x \neq 0)$, so that there is a transverse component of the tension force, and where the string has a nonzero velocity $(\partial y/\partial t \neq 0)$ so that the transverse force can do work. Let us obtain an expression for power transmitted through a string.

In unit time, the wave will travel a distance v. If s be the area of cross-section of the string, then volume of this length would be sv and energy transmitted per unit time (called power) would be,

$$P = \text{(energy density) (volume)}$$

$$\therefore \qquad \boxed{P = \frac{1}{2}\rho\omega^2 A^2 sv}$$

Note *This power is really the average power. The instantaneous power is given by,*

$$P(x, t) = F_y(x, t) \cdot v_y(x, t)$$

which comes out to be, $\rho\omega^2 A^2 sv \sin^2(kx - \omega t)$ *or* $\rho\omega^2 A^2 sv \cos^2(kx - \omega t)$ *depending on the displacement equation* $y(x, t)$. *The average value of* $\sin^2$ *or* $\cos^2$ *function, averaged over any whole number of cycles is* $\frac{1}{2}$. *Hence the average power will be* $\frac{1}{2}\rho\omega^2 A^2 sv$.

Intensity (I)

Flow of energy per unit area of cross-section of the string in unit time is known as the intensity of the wave. Thus,

$$I = \frac{\text{power}}{\text{area of cross-section}} = \frac{P}{s}$$

or

$$\boxed{I = \frac{1}{2}\rho\omega^2 A^2 v}$$

Again this is the average intensity transmitted through the string. The instantaneous intensity $\rho\omega^2 A^2 v \sin^2(kx - \omega t)$ or $\rho\omega^2 A^2 v \cos^2(kx - \omega t)$ depends on x and t.

Note *(i) Although the above relations for power and intensity have been discussed for a transverse wave on a string, they hold good for other waves also.*

(ii) ***Intensity due to a point source :*** *If a point source emits wave uniformly in all directions, the energy at a distance r from the source is distributed uniformly on a spherical surface of radius r and area* $S = 4\pi r^2$. *If P is the power emitted by the source, the power per unit area at a distance r from the source is* $\frac{P}{4\pi r^2}$. *The average power per unit area that is incident perpendicular to the direction of propagation is called the intensity. Therefore:*

$$\boxed{I = \frac{P}{4\pi r^2}}$$

or

$$\boxed{I \propto \frac{1}{r^2}}$$

Now, as amplitude $A \propto \sqrt{I}$, a spherical harmonic wave emanating from a point source can therefore, be written as

$$y(r,t) = \frac{A}{r}\sin(kr - \omega t)$$

Sample Example 14.9 *A stretched string is forced to transmit transverse waves by means of an oscillator coupled to one end. The string has a diameter of 4 mm. The amplitude of the oscillation is 10^{-4} m and the frequency is 10 Hz. Tension in the string is 100 N and mass density of wire is $4.2 \times 10^3\ kg/m^3$. Find :*

(a) the equation of the waves along the string
(b) the energy per unit volume of the wave
(c) the average energy flow per unit time across any section of the string and
(d) power required to drive the oscillator.

Solution (a) Speed of transverse wave on the string is,

$$v = \sqrt{\frac{T}{\rho S}} \qquad (\text{as } \mu = \rho S)$$

Substituting the values, we have

$$v = \sqrt{\frac{100}{(4.2\times 10^3)\left(\frac{\pi}{4}\right)(4.0\times 10^{-3})^2}}$$

$$= 43.53 \text{ m/s}$$

$$\omega = 2\pi f = 20\pi \frac{\text{rad}}{\text{s}} = 62.83\frac{\text{rad}}{\text{s}}$$

$$k = \frac{\omega}{v} = 1.44 \text{ m}^{-1}$$

$\therefore$ Equation of the waves along the string,

$$y(x, t) = A\sin(kx - \omega t)$$

$$= (10^{-4}\text{ m})\sin\left[(1.44\text{ m}^{-1})\,x - \left(62.83\frac{\text{rad}}{\text{s}}\right)t\right] \qquad \textbf{Ans.}$$

(b) Energy per unit volume of the string,

$$u = \text{energy density} = \frac{1}{2}\rho\omega^2 A^2$$

Substituting the values, we have

$$u = \left(\frac{1}{2}\right)(4.2\times 10^3)(62.83)^2(10^{-4})^2$$

$$= 8.29\times 10^{-2}\ \text{J/m}^3 \qquad \textbf{Ans.}$$

(c) Average energy flow per unit time,

$$P = \text{power}$$

$$= \left(\frac{1}{2}\rho\omega^2 A^2\right)(sv) = (u)\,(sv)$$

Substituting the values, we have

$$P = (8.29 \times 10^{-2})\left(\frac{\pi}{4}\right)(4.0 \times 10^{-3})^2\ (43.53)$$

$$= 4.53 \times 10^{-5}\ \text{J/s}$$ **Ans.**

(d) Power required to drive the oscillator is obviously 4.53×10^{-5} W. **Ans.**

Introductory Exercise 14.4

1. Spherical waves are emitted from a 1.0 W source in an isotropic non-absorbing medium. What is the wave intensity 1.0 m from the source?
2. A line source emits a cylindrical expanding wave. Assuming the medium absorbs no energy, find how the amplitude and intensity of the wave depend on the distance from the source?