FMA-PG Notes

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1 Softmax PPO Closed Form Update

We will consider direct functional representation with tabular parameterization, i.e. $\pi \equiv p^\pi$ is essentially an S $\times$ A table satisfying the constraints

$$\sum_a p^\pi(a|s) = 1, \qquad \forall s \in \mathcal{S}$$
$$p^\pi(a|s) \geq 0, \qquad \forall s \in \mathcal{S},\ \forall a \in \mathcal{A}.$$

Our goal is to find the closed form solution to the following optimization problem (from Eq. 6, Sharan et al., 2021):

$$\pi_{t+1} = \arg\max_{\pi \in \Pi} \left[\sum_s d^{\pi_t}(s) \sum_a p^{\pi_t}(a|s) \left(A^{\pi_t}(s,a) + \frac{1}{\eta} \right) \log \frac{p^\pi(s,a)}{p^{\pi_t}(s,a)} \right], \tag{1}$$

subject to the above constraints on p^π. The above equation is obtained by setting the mirror map to the weighted exponential function.

We begin by formulating this problem using Lagrange multipliers λ_s, $\lambda_{s,a}$ for all states s and actions a (and with a slight abuse of the notation, i.e. we write λ_s for all the λ_ss, etc.):

$$\begin{aligned} \mathcal{L}(p^\pi, \lambda_s, \lambda_{s,a}) = & \sum_s d^{\pi_t}(s) \sum_a p^{\pi_t}(a|s) \left(A^{\pi_t}(s,a) + \frac{1}{\eta} \right) \log \frac{p^\pi(a|s)}{p^{\pi_t}(a|s)} \\ & - \sum_{s,a} \lambda_{s,a} p^\pi(a|s) - \sum_s \lambda_s \left(\sum_a p^\pi(a|s) - 1 \right). \end{aligned} \tag{2}$$

KKT conditions for this problem are:

$$\nabla_{p^\pi(x,b)} \mathcal{L}(p^\pi, \lambda_s, \lambda_{s,a}) = 0, \qquad \forall x \in \mathcal{S},\ \forall b \in \mathcal{A} \tag{C1}$$
$$\sum_a p^\pi(a|s) = 1, \qquad \forall s \in \mathcal{S} \tag{C2}$$
$$p^\pi(a|s) \geq 0, \qquad \forall s \in \mathcal{S},\ \forall a \in \mathcal{A} \tag{C3}$$
$$\lambda_s \geq 0, \qquad \forall s \in \mathcal{S} \tag{C4}$$
$$\lambda_s \left(\sum_a p^\pi(a|s) - 1 \right) = 0 \qquad \forall s \in \mathcal{S} \tag{C5}$$
$$\lambda_{s,a} p^\pi(a|s) = 0, \qquad \forall s \in \mathcal{S},\ \forall a \in \mathcal{A}. \tag{C6}$$

Let us now try to solve this system. Solving the first equation for arbitrary state-action pair (x,b), gives us:

$$\nabla_{p^\pi(b|x)} \mathcal{L}(p^\pi, \lambda_s, \lambda_{s,a}) = d^{\pi_t}(x) p^{\pi_t}(b|x) \left(A^{\pi_t}(x,b) + \frac{1}{\eta} \right) \frac{1}{p^\pi(b|x)} - \lambda_{x,b} - \lambda_x = 0$$

$$\Rightarrow \qquad p^\pi(b|x) = \frac{d^{\pi_t}(x) p^{\pi_t}(b|x)(1 + \eta A^{\pi_t}(x,b))}{\eta(\lambda_x + \lambda_{x,b})}. \tag{3}$$

Let us set

$$\lambda_{s,a} = 0, \qquad \forall s \in \mathcal{S},\ \forall a \in \mathcal{A}. \tag{4}$$

Combining Eq. 3 with the second KKT condition gives us

$$\lambda_s = \frac{1}{\eta} \sum_a d^{\pi_t}(s) p^{\pi_t}(a|s)(1 + \eta A^{\pi_t}(s, a)). \tag{5}$$

Therefore, with the additional assumption $d^{\pi_t}(s) > 0$, $p^{\pi}(a|s)$ becomes

$$p^{\pi}(a|s) = \frac{p^{\pi_t}(a|s)(1 + \eta A^{\pi_t}(s, a))}{\sum_b p^{\pi_t}(b|s)(1 + \eta A^{\pi_t}(s, b))}. \tag{6}$$

Note that $d^{\pi_t}(s), p^{\pi_t}(a|s) \geq 0$ for any state-action pair, since they are proper measures. All that remains is to ensure that

$$1 + \eta A^{\pi_t}(s, a) \geq 0$$

to satisfy the third and fourth KKT conditions. But how to do that? One straightforward way is to define $p^{\pi}(a|s) = 0$ whenever $1 + \eta A^{\pi_t}(s, a) \leq 0$, and accordingly re-define λ_s. This gives us the final solution to our original optimization problem (Eq. 1):

$$\pi_{t+1} = p^{\pi}(s, a) = \frac{p^{\pi_t}(a|s) \max(1 + \eta A^{\pi_t}(s, a), 0)}{\sum_b p^{\pi_t}(b|s) \max(1 + \eta A^{\pi_t}(s, b), 0)}. \tag{7}$$

This leaves one last problem: Is it always true that given any state s, there exists atleast one action a, such that $1 + \eta A^{\pi_t}(s, a) \geq 0$? Because otherwise, we would fail to satisfy the second KKT condition.

2 MDPO Closed Form Update

The paper (Sharan et al., 2021) considers the direct representation along with tabular parameterization of the policy, albeit with a small change in notation as compared to the previous section: $\pi(a|s) \equiv p^{\pi}(a|s, \theta)$. However, since this notation is more cumbersome, we will stick to our old one: $\pi(a|s) \equiv p^{\pi}(a|s)$. The constraints on these parameters are the same as before: $\sum_a p^{\pi}(a|s) = 1,\ \forall s \in \mathcal{S}$; and $p^{\pi}(a|s) \geq 0,\ \forall s \in \mathcal{S},\ \forall a \in \mathcal{A}$. Our goal, this time, is to solve the following optimization problem (see Eq. 9, Sharan et al., 2021)

$$\pi_{t+1} = \arg\max_{\pi \in \Pi} \left[\sum_s d^{\pi_t}(s) \sum_a p^{\pi_t}(a|s) \left(Q^{\pi_t}(s, a) \frac{p^{\pi}(a|s)}{p^{\pi_t}(a|s)} - \frac{1}{\eta} D_{\phi}(p^{\pi}(\cdot|s), p^{\pi_t}(\cdot|s)) \right) \right], \tag{8}$$

with the mirror map as the negative entropy (Eq. 5.27, Beck and Teboulle, 2002). This particular choice results in

$$D_{\phi}(p^{\pi}(\cdot|s), p^{\pi_t}(\cdot|s)) = \mathrm{KL}(p^{\pi}(\cdot|s) \| p^{\pi_t}(\cdot|s)) := \sum_a p^{\pi}(a|s) \log \frac{p^{\pi}(a|s)}{p^{\pi_t}(a|s)}. \tag{9}$$

The optimization problem then simplifies to

$$\pi_{t+1} = \arg\max_{\pi \in \Pi} \left[\sum_s d^{\pi_t}(s) \sum_a p^{\pi_t}(a|s) \left(Q^{\pi_t}(s, a) \frac{p^{\pi}(a|s)}{p^{\pi_t}(a|s)} - \frac{1}{\eta} \sum_{a'} p^{\pi}(a'|s) \log \frac{p^{\pi}(a'|s)}{p^{\pi_t}(a'|s)} \right) \right]. \tag{10}$$

Proceeding analogously to the previous section, we use Lagrange multipliers λ_s, $\lambda_{s,a}$ for all states s and actions a to obtain the function

$$\begin{aligned}\mathcal{L}(p^\pi, \lambda_s, \lambda_{s,a}) = \sum_s d^{\pi_t}(s) \sum_a p^{\pi_t}(a|s) Q^{\pi_t}(s,a) \frac{p^\pi(a|s)}{p^{\pi_t}(a|s)} - \frac{1}{\eta} \sum_s d^{\pi_t}(s) \sum_{a'} p^\pi(a'|s) \log \frac{p^\pi(a'|s)}{p^{\pi_t}(a'|s)} \\ - \sum_{s,a} \lambda_{s,a} p^\pi(a|s) - \sum_s \lambda_s \Big(\sum_a p^\pi(a|s) - 1 \Big). \qquad (11)\end{aligned}$$

The KKT conditions are exactly the same as before (Eq. C1 to Eq. C6).

Again, we begin by solving the first KKT condition:

$$\begin{aligned}\nabla_{p^\pi(b|x)} \mathcal{L}(p^\pi, \lambda_s, \lambda_{s,a}) &= d^{\pi_t}(x) p^{\pi_t}(b|x) \frac{Q^{\pi_t}(x,b)}{p^{\pi_t}(b|x)} - \frac{d^{\pi_t}(x)}{\eta} \left[\log \frac{p^\pi(b|x)}{p^{\pi_t}(b|x)} + 1 \right] - \lambda_{x,b} - \lambda_x \\ &= \frac{d^{\pi_t}(x)}{\eta} \left[\eta Q^{\pi_t}(x,b) - \log \frac{p^\pi(b|x)}{p^{\pi_t}(b|x)} - 1 - \frac{\eta(\lambda_{x,b} + \lambda_x)}{d^{\pi_t}(x)} \right] \\ &= 0 \\ \Rightarrow \qquad \log \frac{p^\pi(b|x)}{p^{\pi_t}(b|x)} &= \eta Q^{\pi_t}(x,b) - \frac{\eta(\lambda_{x,b} + \lambda_x)}{d^{\pi_t}(x)} - 1 \\ \Rightarrow \qquad p^\pi(b|x) &= p^{\pi_t}(b|x) \cdot \exp\Big(\eta Q^{\pi_t}(x,b) \Big) \cdot \exp\left(-\frac{\eta(\lambda_{x,b} + \lambda_x)}{d^{\pi_t}(x)} - 1 \right), \qquad (12)\end{aligned}$$

where in the fourth line, we made the assumption that $d^{\pi_t}(x) > 0$ for all states x. We again set

$$\lambda_{s,a} = 0, \qquad \forall s \in \mathcal{S},\ \forall a \in \mathcal{A}. \qquad (13)$$

Yet again, we put Eq. 12 in the second KKT condition to get

$$\exp\left(-\frac{\eta \lambda_x}{d^{\pi_t}(x)} - 1 \right) = \left(\sum_b p^{\pi_t}(b|x) \cdot \exp\Big(\eta Q^{\pi_t}(x,b) \Big) \right)^{-1}. \qquad (14)$$

Therefore, we obtain

$$p^\pi(a|s) = \frac{p^{\pi_t}(a|s) \cdot \exp\Big(\eta Q^{\pi_t}(s,a) \Big)}{\sum_b p^{\pi_t}(b|x) \cdot \exp\Big(\eta Q^{\pi_t}(x,b) \Big)}. \qquad (15)$$

Again, this leaves one last problem: Can we ensure that $\lambda_s \geq 0$ for all states s, so that the fourth KKT condition is also satisfied?