Mathematical Modeling and Simulation Project - II

Synchronization and anti-synchronization between Lorenz and Rossler systems using Function Projective Synchronization method

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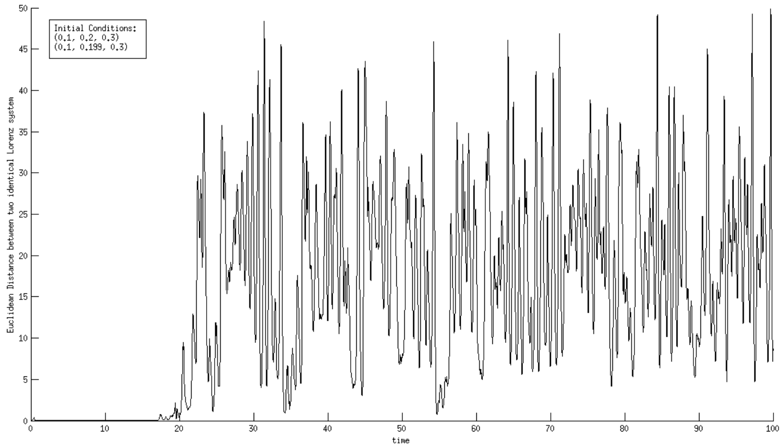

Figure 1: Evolution of two chaotic Lorenz systems with time

1 Introduction

A chaotic system is a dynamical system which is highly sensitive to the initial conditions. As a result the future states of such a system can't be calculated with any considerable degree of accuracy, unless the initial conditions of the system are measured with infinite precision. Since such a measurement is not possible, such a system becomes unpredictable, hence chaotic.

Figure 1. shows the euclidean error between two identical Lorenz Systems with a very minor change in initial conditions. This shows the fundamental charecteristic of chaotic systems: high dependence on initial conditions.

This project is concerned with two popular chaotic systems. The Lorenz system and the Rossler system. The Lorenz system is defined by the following equations:

$$\dot{x_1} = 10(x_2 - x_1) \tag{1}$$
$$\dot{x_2} = 28x_1 - x_1x_3 - x_2 \tag{2}$$
$$\dot{x_3} = x_1x_2 - \frac{8}{3}x_3 \tag{3}$$

The Rossler system is defined by the following equations:

$$\dot{y_1} = -y_2 - y_3 \tag{4}$$
$$\dot{y_2} = y_1 + 0.2y_2 \tag{5}$$
$$\dot{y_3} = 0.2 + y_3(y_1 - 5.7) \tag{6}$$

Figure 2. shows the 3-D graph of these two systems.

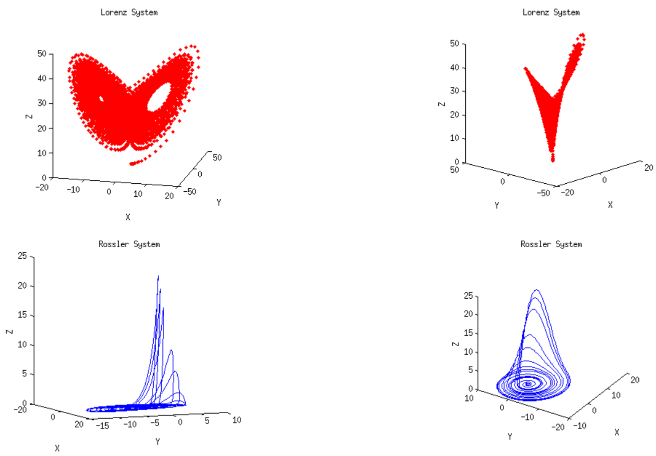

Figure 2: Lorenz and Rossler systems

The aim of this project is two study the synchronization between these two chaotic systems. Synchronization of two chaotic systems occurs when the two systems become coupled. This means that given a long enough time interval, the trajectories of both the systems completely match each other. Or in a more general sense of the term, a functional relation is developed between the trajectories of the two chaotic systems.

2 Synchronization and anti-synchronization of chaotic systems

This section explains the synchronization and anti-synchronization of two chaotic systems. Consider two chaotic systems, represented by $\dot{\mathbf{x}} = \mathrm{F}(\mathbf{x}, t)$ and $\dot{\mathbf{y}} = \mathrm{F}(\mathbf{y}, t)$. The state of each system is specified by vectors $\mathbf{x} = (x_1(t) \dots x_m(t))^T$ and $\mathbf{y} = (y_1(t) \dots y_m(t))^T$.

Let the first system act as the drive system, and the second one as the response or slave system. This means that the first system X, follows its own natural trajectory. The trajectory of the response system on the other hand Y, is modified by a controller function $\mathbf{U} = (u_1(\mathbf{x}, \mathbf{y}), u_2(\mathbf{x}, \mathbf{y}), \dots, u_m(\mathbf{x}, \mathbf{y}))$ which strongly depends on the drive system's state. Thus the following equations for represent the chaotic systems in consideration:

$$X \equiv \dot{\mathbf{x}} = \mathrm{F}(\mathbf{x}, t) \tag{7}$$

$$Y \equiv \dot{\mathbf{y}} = \mathrm{F}(\mathbf{y}, t) + U \tag{8}$$

The definition of synchronization and anti-synchronization between these two systems can now be given.

Let the error $e_i = x_i - f_i(x)y_i \quad (i = 1 \ldots m)$ where $f_i(x)$ are functions of x. Further let $\mathbf{e} = (e_1 \ldots e_m)$. Then if $\lim_{t\to\infty}|\mathbf{e}| = 0$, the two chaotic systems X and Y are function projectively synchronized. If each of $f_i(x) = 1$ then the systems are in simple synchronization.

Similarly, if the error is defined by $e_i = x_i + y_i \quad (i = 1 \ldots m)$ and $\mathbf{e} = (e_1 \ldots e_m)$, $\lim_{t\to\infty}|\mathbf{e}| = 0$ implies that the two chaotic systems X and Y are anti-synchronized.

If the state equations for the two chaotic systems are written by breaking them explicitly into a linear and non-linear component:

$$X \equiv \dot{\mathbf{x}} = A_1\mathbf{x} + h_1(x,t) \tag{9}$$

$$Y \equiv \dot{\mathbf{y}} = A_2\mathbf{y} + h_2(y,t) + U \tag{10}$$

where A_1, A_2 are $m\times m$ constant matrices and $h_1, h_2 : \mathbb{R}^m \to \mathbb{R}^m$ are non-linear functions. Then the controller function [1] U such that $\lim_{t\to\infty}|\mathbf{e}| = 0$ can be found as:

$$U = f^{-1}h_1 + (f^{-1}A_1f - A_2)y + f^{-1}B(x - fy) - h_2 - f^{-1}gy \tag{11}$$

The matrix $B \in \mathbb{R}^{m\times m}$ is chosen so that the real parts of all the eigenvalues of the matrix $A_1 - B$ are negative and $g = \mathrm{diag}(\dot{f_1} \ldots \dot{f_m})$.

The next section applies this method to synchronization and anti-synchronization of a Lorenz and Rossler chaotic system.

3 Synchronization and anti-synchronization of Lorenz and Rossler systems

To demonstrate the synchronization of Lorenz and Rossler system, let the Lorenz system be the driving system and Rossler system be the response system. The equations of both the systems are presented here again for quick reference. The drive system (Lorenz System) is:

$$\begin{aligned}
\dot{x_1} &= 10(x_2 - x_1)\\
\dot{x_2} &= 28x_1 - x_1x_3 - x_2\\
\dot{x_3} &= x_1x_2 - \frac{8}{3}x_3
\end{aligned}$$

And the response system (Rossler system) is as follows:

$$\begin{aligned}
\dot{y_1} &= -y_2 - y_3 + u_1\\
\dot{y_2} &= y_1 + 0.2y_2 + u_2\\
\dot{y_3} &= 0.2 + y_3(y_1 - 5.7) + u_3
\end{aligned}$$

where $u_i s$ are control functions. The error functions are chosen, for function projective synchronization, as:

$$e_1 = x_1 - y_1 \tanh(x_3) \tag{12}$$
$$e_2 = x_2 - y_2 \tanh(x_3) \tag{13}$$
$$e_3 = x_3 - y_3 \tag{14}$$

For this system of equations we determine

$$A_1 = \begin{bmatrix} -10 & 10 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -\frac{8}{3} \end{bmatrix}; \quad A_2 = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0.2 & 0 \\ 0 & 0 & -5.7 \end{bmatrix};$$
$$h_1 = \begin{bmatrix} 0 \\ -x_1x_3 \\ x_1x_2 \end{bmatrix}; \quad h_2 = \begin{bmatrix} 0 \\ 0 \\ y_1y_3 + 0.2 \end{bmatrix};$$
$$f = \begin{bmatrix} \tanh(x_3) & 0 & 0 \\ 0 & \tanh(x_3) & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad B = \begin{bmatrix} 0 & 10 & 0 \\ 28 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix};$$

and $g = \mathrm{diag}[(\frac{8}{3}x_3 - x_1x_2)(\tanh^2(x_3) - 1), (\frac{8}{3}x_3 - x_1x_2)(\tanh^2(x_3) - 1), 0]$. Note that all the eigenvalues of $A_1 - B$ are negative. Next step is to determine the control function U, using eqn. (11). It is:

$$u_1 = 11y_2 - 10y_1 + y_3 + \frac{10(x_2 - y_2\tanh(x_3))}{\tanh(x_3)} - \frac{y_1(\frac{8}{3}x_3 - x_1x_2)(\tanh^2(x_3) - 1)}{\tanh(x_3)} \tag{15}$$

$$u_2 = 27y_1 - 1.2y_2 + \frac{28x_1 - y_1\tanh(x_3)}{\tanh(x_3)} - \frac{x_1x_3}{\tanh(x_3)} - \frac{y_2(\frac{8}{3}x_3 - x_1x_2)(\tanh^2(x_3) - 1)}{\tanh(x_3)} \tag{16}$$

$$u_3 = 3.033y_3 - y_1y_3 + x_1x_2 - 0.2 \tag{17}$$

With these control functions, the two systems are synchronized; see Figure 3 for the synchronized systems with initial conditions: $(0.1, 0.2, 0.3)$ for Lorenz system and $(0.2, 0.3, 0.5)$ for Rossler system.

Proceeding similarly, keep the Lorenz system as the driving system and Rossler system as the response system. New error function is defined for anti-synchronization between the two chaotic systems as:

$$e_1 = x_1 + y_1 \tag{18}$$
$$e_2 = x_2 + y_2 \tag{19}$$
$$e_3 = x_3 + y_3 \tag{20}$$

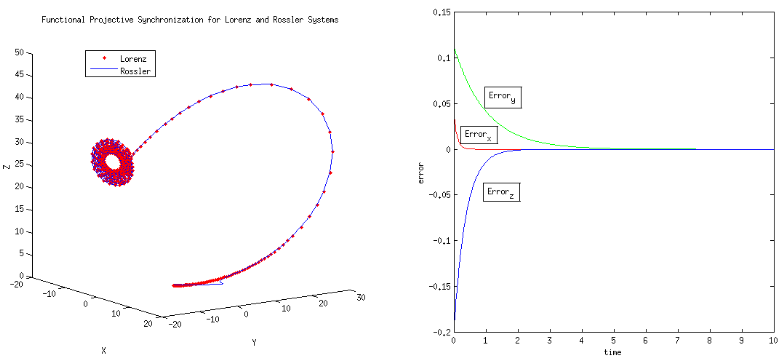

Figure 3: Function projective synchronization between Lorenz and Rossler system

For this system of equations A_1, A_2, h_1, h_2 and B remain the same as in previous case. New f and g are:

$$f = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}; \quad g = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix};$$

Again using eqn. (11), the control function U, given by its components, is:

$$u_1 = y_2 - 10y_1 - 10x_2 + y_3 \tag{21}$$

$$u_2 = x_1x_3 - y_1 - 1.2y_2 - 28x_1 \tag{22}$$

$$u_3 = 3.033y_3 = x_1x_2 - y_1y_3 - 0.2 \tag{23}$$

These control functions make the two systems anti-synchronized; see Figure 4 for the anti-synchronized systems with initial conditions: $(0.1, 0.2, 0.3)$ for Lorenz system and $(0.2, 0.3, 0.5)$ for Rossler system.

4 Results and Conclusions

Figure 3. and Figure 4. show the synchronization and anti-synchronization between the Lorenz and Rossler system. These figures were generated using MATLAB. The differential equations were solved using 'ode45' function of MATLAB which uses a variable step Runge-Kutta Method. Both the graphs were drawn for time 0 to 10 units.

Figure 3. shows the function projective synchronization for error functions given in eqn. (12) to (14). The figure clearly shows how the two

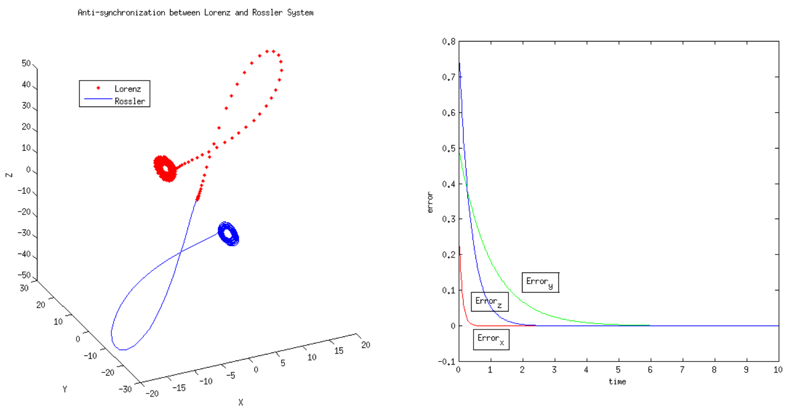

Figure 4: Anti-synchronization between Lorenz and Rossler system

systems both follow the same trajectory. Since the $\tanh(x)$ curve gives a $+1$ for any value of x that is not small, the eqn. (12) to (14) reduce to $u_i = x_i - y_i$, which is same as simple synchronization. Also the error functions drop to zero for all the three components as time progresses and the systems are synchronized.

Figure 4. shows anti-synchronization, given by error functions in eqn. (18) to (20). The figure illustrates that anti-synchronization drives the reponse system in an exact same shape as the drive system, albeit in a completely opposite direction. The errors in this case are sums of x_is and y_is, which also drop to zero for all the three components as time progesses.

Synchronization and anti-synchronization of chaotic systems have wide applications, as in physiology, nonlinear optics and fluid dynamics. It is a very popular topic in physics. Almost every complex phenomena can be modeled as being chaotic. Some people even go the extent of saying that fluttering of a butterfly over one region may lead up to the development of a hurricane in some far off region, the so-called ‘butterfly effect’.

Future works in the project could involve extending this approach to more complex systems, and predicting the system behaviour for a real world system, such as those mentioned above.

5 References

1 Xin Li, Yong Chen: *Function Projective Synchronization and Its Applications*, International Journal of Modern Physics C (2007)

2 Kim et al.: *Anti-synchronization of chaotic oscillators*, Physics Letters A (2003)