

Key researchable issues in physics that relate to randomness and miracles

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Physical randomness can be grouped into three categories [1]: (I) extreme sensitivity to initial values in classical continuum theories; (II) recursion theoretical undecidability applicable to physics (“by reduction” to some recursion theoretic incompleteness); (III) quantum randomness, which may be grouped into three subcategories: (i) the random occurrence of single events (e.g., particle decays, scattering of single quanta, beamsplitters); (ii) quantum complementarity; and (iii) quantum value indefiniteness (through Kochen-Specker- and Boole-Bell-type theorems). My subjective evaluations of the progressiveness of the research area will be given in terms of the common *Long Term Bond Ratings* from AAA to D.

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EXTREME SENSITIVITY TO INITIAL VALUES IN CLASSICAL CONTINUUM THEORIES (CCC)

The field of “deterministic chaos” or “symbolic dynamics” has been extensively searched for years now. I believe it is degenerative in the sense of Lakatos [2].

Already in 1873, Maxwell mentioned [3, 211-212]:

When an infinitely small variation in the present state may bring about a finite difference in the state of the system in a finite time, the condition of the system is said to be unstable. It is manifest that the existence of unstable conditions renders impossible the prediction of future events, if our knowledge of the present state is only approximate, and not accurate.

Maxwell also discussed unstable states of high potential energy whose spontaneous [4] decay or change [3, 212] “requires an expenditure of work, which in certain cases may be infinitesimally small, and in general bears no definite proportion to the energy developed in consequence thereof.”

In Poincaré’s words [5, chapt. 4, sect. 2, 56–57],

If we would know the laws of nature and the state of the Universe precisely for a certain time, we would be able to predict with certainty the state of the Universe for any later time. But ... it can be the case that small differences in the initial values produce great differences in the later phenomena; a small error in the former may result in a large error in the latter. The prediction becomes impossible and we have a “random phenomenon.”

RECURSION THEORETICAL UNDECIDABILITY APPLICABLE TO PHYSICS (BBB+)

This research area offers a plethora of formal results. It might be judged “progressive” but eventually might dry out and falter due to exhaustion. Maybe also some (pre-)Gödelian connections to quantum mechanics will be found.

Undecidability of the general prediction problem

For any deterministic system strong enough to support universal computation, the general forecast or prediction problem is provably unsolvable. This proposition will be argued by reduction to the halting problem, which is provably unsolvable. A straightforward embedding of a universal computer into a physical system results in the fact that, owing to the reduction to the recursive undecidability of the halting problem, certain future events cannot be predicted and are thus provably indeterministic.

A related question is the upper bound of running time – or, alternatively, recurrence time – of a program of length n bits before terminating or, alternatively, recurring? An answer to this question explains just how long we have to wait for the most time-consuming program of length n bits to halt. That, of course, is a worst-case scenario. We mention without proof [6, 7] that this bound can be represented by the busy beaver function, and therefore grows faster than any computable function of n .

Undecidability of the general induction problem

Induction, in physics, is the inference of general rules dominating and generating physical behaviors from these behaviors alone. For any deterministic system strong enough to support universal computation, the general induction problem is provable unsolvable. Induction is thereby reduced to the unsolvability of the rule inference problem [8–12] of identifying a rule or law reproducing the behavior of a deterministic system by observing its input-output performance by purely algorithmic means (not by intuition).

Impossibility

Physical tasks which would result in paradoxical behavior [13] are impossible to perform. One such task is the solution of the general halting problem, as discussed earlier. Thus omnipotence appears infeasible, at least as long as one sticks to the usual formal rules opposing inconsistencies [13, 163].

Another such paradoxical task (requiring substitution and self-reference) can be forced upon *La Bocca della Verità* (Mouth of Truth), located in the *portico* of the church of *Santa Maria in Cosmedin* in Rome. It is believed that if one tells a lie with one's hand in the mouth of the sculpture, the hand will be bitten off; another less violent legend has it that anyone sticking a hand in the mouth while uttering a false statement will never be able to pull the hand back out. Rucker [14, 178] once allegedly put in his hand in the sculpture's mouth uttering, "I will not be able to pull my hand back out." The author leaves it to the reader to imagine *La Bocca della Verità's* confusion when confronted with such a statement!

There is a *pandemonium* of conceivable physical tasks [15], some quite entertaining [16], which would result in paradoxical behavior and are thus impossible to perform. Some of these tasks are pre-Gödelian and merely require *substitution*.

Results in classical recursion theory with implications for theoretical physics

The following theorems of recursive analysis [17, 18] have some implications for theoretical physics [19]: (1) There exist recursive monotone bounded sequences of rational numbers whose limit is no computable number [20]. A concrete example of such a number is Chaitin's Omega number [21–23], the halting probability for a computer (using prefix-free code), which can be defined by a sequence of rational numbers with no computable rate of convergence. (2) There exist a recursive real function which has its maximum in the unit interval at no recursive real number [24]. This has implications for the

principle of least action. (3) There exists a real number r such that $G(r) = 0$ is recursively undecidable for $G(x)$ in a class of functions which involves polynomials and the sine function [25]. This, again, has some bearing on the principle of least action. (4) There exist incomputable solutions of the wave equations for computable initial values [26, 27]. (5) On the basis of theorems of recursive analysis [28, 29], many questions in dynamical systems theory are provable undecidable [30–33].

QUANTUM RANDOMNESS (A+)

I believe that our perception of the quantum phenomena will change dramatically, but I cannot predict when and where.

Random occurrence of single events

In 1926, Born [34, 866] [see an English translation in Wheeler and Zurek [35, 54]] postulated that

“from the standpoint of our quantum mechanics, there is no quantity which in any individual case causally fixes the consequence of the collision; but also experimentally we have so far no reason to believe that there are some inner properties of the atom which condition a definite outcome for the collision. Ought we to hope later to discover such properties . . . and determine them in individual cases? Or ought we to believe that the agreement of theory and experiment – as to the impossibility of prescribing conditions? I myself am inclined to give up determinism in the world of atoms.”

Furthermore, Born suggested that, though *individual particles behave irreducibly indeterministic*, the *quantum state evolves deterministically* in a strictly Laplacian causal way. Indeed, between (supposedly irreversible) measurements the (unitary) quantum state evolution is even reversible, that is, one-to-one, and amounts to a generalized (distance preserving) rotation in complex Hilbert space. In Born's [1926, 804] [see an English translation in Jammer [37, 302]] own words,

the motion of particles conforms to the laws of probability, but the probability itself is propagated in accordance with the law of causality. [This means that knowledge of a state in all points in a given time determines the distribution of the state at all later times.]

At present, indeterminism is clearly favored, the canonical position being expressed by Zeilinger [38]:

“The discovery that individual events are irreducibly random is probably one of the most significant findings of the twentieth century. . . . For the individual event in quantum physics, not only do we not know the cause, there is no cause.”

Quantum conundrum (AAA)

This distinction between a reversible, deterministic evolution of the quantum state, on one hand, and the irreversible measurement, on the other hand, has left some physicists with an uneasy feeling; in particular, because of the possibility to erase [39–47] measurements by reconstructing the quantum state, accompanied by a complete loss of the information obtained from the quantum state before the (undone) measurement – unlike in classical reversible computation [48–50], which still allows copying, that is, one-to-many operations, the quantum state evolution is strictly one-to-one. Indeed, the possibility to undo measurements on quantum states appears to be not bound by any fundamental principle, and limited merely by the experimenter’s technological capacities. Stated pointedly, it would in principle be possible to undo all measurements, yet this cannot be accomplished most of the time (for almost all measurements) for all practical purposes [51]. But then, one could speculate, Born’s statement seems to suggest that the deterministic state evolution uniformly prevails. Pointedly stated, if, at least in principle, there is no such thing as an irreversible measurement, and the quantum state evolves uniformly deterministically, why should there exist indeterministic individual events? In this view, the insistence in irreversible measurements as well as in an irreducible indeterminism associated with individual quantum events appears to be an idealistic, subjective illusion – in fact, this kind of indeterminism depends on measurement irreversibility and decays into thin air if the latter is denied.

Alas, the deterministic evolution of the quantum state could result in the *superposition* of classically contradictory states. One of the mind-boggling, perplexing and counterintuitive consequences associated with this coexistence of classical contradictions is Schrödinger’s [1935, 812] cat paradox implying the simultaneous coexistence of death and life of a macroscopic object such as a mammal. Another one is Everett’s [1957] aforementioned many-worlds interpretation suggesting that our universe perpetually branches off into zillions of consistent alternatives.

Thus one is faced with a *dilemma*: either to accept a somehow spurious nonuniformity in the evolution of the quantum state during (irreversible) measurement processes – an *ad hoc* assumption challenged by quantum erasure experiments – or being confronted with the counterintuitive decay of quantum states into superpositions

of classically mutually exclusive states – a sort of jelly – not backed by our everyday experience as conscious beings (although often ambivalent we usually don’t reside in mental ambiguity for too long). Schrödinger [54, 19–20] sharply addressed the difficulties of a quantum theorist coping with this aspect of the quantum formalism:

The idea that [the alternate measurement outcomes] be not alternatives but *all* really happening simultaneously seems lunatic to [the quantum theorist], just *impossible*. He thinks that if the laws of nature took *this* form for, let me say, a quarter of an hour, we should find our surroundings rapidly turning into a quagmire, a sort of a featureless jelly or plasma, all contours becoming blurred, we ourselves probably becoming jelly fish. It is strange that he should believe this. For I understand he grants that unobserved nature does behave this way – namely according to the wave equation. . . . according to the quantum theorist, nature is prevented from rapid jellification only by our perceiving or observing it.

Quantum complementarity

Complementarity is the impossibility of measuring two or more complementary observables with arbitrary precision simultaneously. In 1933, Pauli [55, 7] gave the first explicit definition of *complementarity* stating that [see the partial English translation in [37, 369]]

in the case of an indeterminacy of a property of a system at a certain configuration (at a certain state of a system), any attempt to measure the respective property (at least partially) annihilates the influence of the previous knowledge of the system on the (possibly statistical) propositions about possible later measurement results. . . . The impact on the system by the measurement apparatus for momentum (position) is such that within the limits of the uncertainty relations the value of the knowledge of the previous position (momentum) for the prediction of later measurements of position and momentum is lost.

Einstein, Podolsky, and Rosen [1935] challenged quantum complementarity (and doubted the completeness of quantum theory) by utilizing a configuration of two entangled [52, 57, 58] particles. They claimed to be able to empirically infer two different complementary contexts counterfactually simultaneously, thus circumventing quantum complementarity.

Quantum value indefiniteness

Still another quantum unknowable results from the fact that no global (in the sense of all or at least certain finite sets of complementary observables) classical truth assignment exists which is consistent with even a finite number of local (in the sense of comeasurable) ones, that is, no consistent classical truth table can be given by pasting together the possible outcomes of measurements of certain complementary observables. This phenomenon is also known as *value indefiniteness* or, by an option to interpret this result, *contextuality* (see later). Here the term *local* refers to a particular context [59] that, operationally, should be thought of as the collection of all comeasurable or copreparable [60] observables. The structure of quantum propositions [61–67] can be obtained by pasting contexts together.

As by definition, only *one* such context is directly measurable, arguments based on more than one context must necessarily involve counterfactuals [68, 69]. A *counterfactual* is a would-be-observable or *contrary-to-fact conditional* [70] which has not been measured but potentially could have been measured if an observer would have decided to do so; alas the observer decided to measure a different, presumably complementary, observable.

Already scholastic philosophy, for instance, Thomas Aquinas, considered similar questions such as whether God has knowledge of non-existing things [71, part one, question 14, article 9] or things that are not yet [71, part one, question 14, article 13]; see also Specker’s [1960, 243] reference to *infuturabilities*. Classical omniscience, at least its naive expression that, if a proposition is true, then an omniscient agent (such as God) knows that it is true, is plagued by controversies and paradoxes.

MIRACLES DUE TO GAPS IN CAUSAL DESCRIPTION (BB)

This topic is related to all others discussed so far. It is highly speculative and offers also potential scenarios for a dualistic mind-body link.

A different issue, discussed by Frank [4], is the possible occurrence of miracles in the presence of *gaps* of physical determinism. Already Maxwell has considered *singular points* [3, 212–213], “where prediction, except from absolutely perfect data, and guided by the omniscience of contingency, becomes impossible.” One might perceive individual events occurring outside the validity of classical and quantum physics without any apparent cause as miracles. For if there is no cause to an event, why should such an event occur altogether rather than not occur?

Although such thoughts remain highly speculative, miracles could be the basis for an operator-directed evolution in otherwise deterministic physical systems. Similar models have been applied to dualistic models of the

mind [73–75]. The objection that this scenario is unnecessarily complicating an otherwise monistic model should be carefully reevaluated in view of computer-generated *virtual realities* [76–78]. In such algorithmic universes, there are computable evolution laws as well as inputs from interfaces. From the intrinsic perspective [79], the inputs cannot be causally accounted for, and hence they remain irreducibly transcendental with respect to the otherwise algorithmic universe.

HARNESSING UNKNOWNABLES AND INDETERMINISM

Physical indeterminism need not necessarily be perceived negatively as the absence of causal laws but rather as a *valuable resource*. Indeed, ingenious quasi-programs to compute the *halting probability* [21, 23, 80] through summation of series without any computable rate of convergence could, at least in principle, and in the limit of unbounded computational resources, be interpreted as generating provable random sequences. However, as has already been expressed by von Neumann [81, 768], “anyone who considers arithmetical methods of producing random digits is, of course, in a state of sin.”

Besides recursion-theoretic undecidability, there appear to be at least two principal sources of indeterminism and randomness in physics: (1) one scenario is associated with instabilities of classical physical systems and with a strong dependence of future behaviors on the initial value, and (2) quantum indeterminism, which can be subdivided into three subcategories, including random outcomes of individual events, complementarity, and value indefiniteness.

The production of random numbers by physical generators has a long history [82]. The similarities and differences between classical and quantum randomness can be conceptualized in terms of two black boxes: the first of them, called the “*Poincaré box*,” containing a classical, deterministic, chaotic source of randomness and the second, called the “*Born box*,” containing a quantum source of randomness.

A Poincaré box could be realized by operating a classical dynamical system in the shift map region. Major principles for Born boxes utilizing beam splitters or parametric down conversion include the following: (1) there should be at least three mutually exclusive outcomes to ensure value indefiniteness [83–87]; (2) the states prepared and measured should be pure and in mutually [possibly interlinked [88]] unbiased bases or contexts; and (3) events should be independent to be able to apply proper normalization procedures [81, 89].

METAPHYSICAL STATUS OF (IN)DETERMINISM

Hilbert's [1902] sixth problem is about the axiomatization of physics. Regardless of whether this goal is achievable, omniscience cannot be gained via the formalized, syntactic route, which will remain blocked forever by the paradoxical self-reference to which intrinsic observers and operational methods are bound. Even if the universe were a computer [91–94], we would intrinsically experience unpredictability and complementarity.

With regard to conjectures about the (in)deterministic evolution of physical events, the situation is unsettled and can be expected to remain unsettled forever. The reason for this is the provable impossibility to formally prove (in)determinism: it is not possible to ensure that physical behaviors are causal and will remain so forever, nor is it possible to exclude all causal behaviors.

The opportunistic approach that (as historically, many ingenious scientists have failed to come up with a causal description) indeterminism will prevail appears to be anecdotal, at best, and misleading, at worst. Likewise, the advice of authoritative researchers to avoid asking questions related to completing a theory, or to avoid thinking about the meaning of quantum mechanics or any kind of rational interpretation, and to avoid searching for causal laws for phenomena which are, at the same time, postulated to occur indeterministically by the same authorities – even wisely and benevolently posted – hardly qualify as proof.

Any kind of lawlessness can thus be claimed only *with reference to*, and *relative to*, certain criteria, laws, or quantitative statistical or algorithmic tests. For instance, randomness could be established merely *with respect to* certain tests, such as some batteries of tests of randomness, for instance, *diehard* [95], *NIST* [96], *TestU01* [97], or algorithmic [98, 99] tests. Note, however, that even the decimal expansion of π , the ratio between the circumference and the diameter of an ideal circle [100, 101], behaves reasonably random [99]; π might even be a good source of randomness for many Monte Carlo calculations.

Thus, both from a formal as well as from an operational point of view, any rational investigation into, or claim of, absolute (in)determinism is metaphysical and can only be proved *relative to* a limited number of statistical or algorithmic tests which some specialists happen to choose; with very limited validity for the formal and the natural sciences.

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[1] Karl Svozil, “Physical unknowables,” in *Kurt Gödel and the Foundations of Mathematics*, edited by Matthias Baaz, Christos H. Papadimitriou, Hilary W. Putnam,

- and Dana S. Scott (Cambridge University Press, Cambridge, UK, 2011) pp. 213–251, [arXiv:physics/0701163](https://arxiv.org/abs/physics/0701163).
- [2] Imre Lakatos, *Philosophical Papers. 1. The Methodology of Scientific Research Programmes* (Cambridge University Press, Cambridge, 1978).
- [3] Lewis Campbell and William Garnett, *The life of James Clerk Maxwell. With a selection from his correspondence and occasional writings and a sketch of his contributions to science* (MacMillan, London, 1882).
- [4] Philipp Frank, *Das Kausalgesetz und seine Grenzen* (Springer, Vienna, 1932) English translation in Ref. [102].
- [5] Henri Poincaré, *Wissenschaft und Hypothese* (Teubner, Leipzig, 1914).
- [6] Gregory J. Chaitin, “Information-theoretic limitations of formal systems,” *Journal of the Association of Computing Machinery* **21**, 403–424 (1974), reprinted in Ref. [103].
- [7] Gregory J. Chaitin, “Computing the busy beaver function,” in *Open Problems in Communication and Computation*, edited by T. M. Cover and B. Gopinath (Springer, New York, 1987) p. 108, reprinted in Ref. [103].
- [8] Mark E. Gold, “Language identification in the limit,” *Information and Control* **10**, 447–474 (1967).
- [9] Lenore Blum and Manuel Blum, “Toward a mathematical theory of inductive inference,” *Information and Control* **28**, 125–155 (1975).
- [10] Dana Angluin and Carl H. Smith, “A survey of inductive inference: Theory and methods,” *Computing Surveys* **15**, 237–269 (1983).
- [11] Leonard M. Adleman and M. Blum, “Inductive inference and unsolvability,” *Journal of Symbolic Logic* **56**, 891–900 (1991).
- [12] M. Li and P. M. B. Vitányi, “Inductive reasoning and Kolmogorov complexity,” *Journal of Computer and System Science* **44**, 343–384 (1992).
- [13] David Hilbert, “Über das Unendliche,” *Mathematische Annalen* **95**, 161–190 (1926).
- [14] Rudy Rucker, *Infinity and the Mind* (Birkhäuser, Boston, 1982).
- [15] John D. Barrow, *Impossibility. The Limits of Science and the Science of Limits* (Oxford University Press, Oxford, 1998).
- [16] Raymond M. Smullyan, *What is the Name of This Book?* (Prentice-Hall, Inc., Englewood Cliffs, NJ, 1992).
- [17] Oliver Aberth, *Computable Analysis* (McGraw-Hill, New York, 1980).
- [18] Klaus Weihrauch, *Computable Analysis. An Introduction* (Springer, Berlin, Heidelberg, 2000).
- [19] Georg Kreisel, “A notion of mechanistic theory,” *Synthese* **29**, 11–26 (1974).
- [20] Ernst Specker, “Nicht konstruktiv beweisbare Sätze der Analysis,” *The Journal of Symbolic Logic* **14**, 145–158 (1949), reprinted in Ref. [104, pp. 35–48]; English translation: *Theorems of Analysis which cannot be proven constructively*.
- [21] Gregory J. Chaitin, *Algorithmic Information Theory* (Cambridge University Press, Cambridge, 1987).
- [22] Cristian Calude, *Information and Randomness—An Algorithmic Perspective*, 2nd ed. (Springer, Berlin, 2002).
- [23] Cristian S. Calude and Michael J. Dinneen, “Exact approximations of omega numbers,” *International Jour-*

- nal of Bifurcation and Chaos **17**, 1937–1954 (2007), CDMTCS report series 293.
- [24] Ernst Specker, “Der Satz vom Maximum in der rekursiven Analysis,” in *Constructivity in mathematics: proceedings of the colloquium held at Amsterdam, 1957*, edited by A. Heyting (North-Holland Publishing Company, Amsterdam, 1959) pp. 254–265, reprinted in Ref. [104, pp. 148–159]; English translation: *Theorems of Analysis which cannot be proven constructively*.
 - [25] Paul S. Wang, “The undecidability of the existence of zeros of real elementary functions,” *Journal of the Association for Computing Machinery (JACM)* **21**, 586–589 (1974).
 - [26] M. B. Pour-El and J. I. Richards, *Computability in Analysis and Physics* (Springer, Berlin, 1989).
 - [27] Douglas S. Bridges, “Can constructive mathematics be applied in physics?” *Journal of Philosophical Logic* **28**, 439–453 (1999).
 - [28] Bruno Scarpellini, “Zwei unentscheidbare Probleme der Analysis,” *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik* **9**, 265–289 (1963).
 - [29] Daniel Richardson, “Some undecidable problems involving elementary functions of a real variable,” *Journal of Symbolic Logic* **33**, 514–520 (1968).
 - [30] Morris W. Hirsch, “The chaos of dynamical systems,” in *Chaos, fractals, and dynamics (Guelph, Ont., 1981/1983)*, The chaos of dynamical systems. Lecture notes in pure and applied mathematics, Vol. 98 (Dekker, New York, 1985) pp. 189–196.
 - [31] N. C. A. da Costa, Francisco Antonio Doria, and A. F. Furtado do Amaral, “Dynamical system where proving chaos is equivalent to proving Fermat’s conjecture,” *International Journal of Theoretical Physics* **32**, 2187–2206 (1993).
 - [32] Ian Stewart, “Deciding the undecidable,” *Nature* **352**, 664–665 (1991).
 - [33] Cristian S. Calude, Elena Calude, and Karl Svozil, “The complexity of proving chaoticity and the Church–Turing thesis,” *Chaos: An Interdisciplinary Journal of Nonlinear Science* **20**, 037103 (2010).
 - [34] Max Born, “Zur Quantenmechanik der Stoßvorgänge,” *Zeitschrift für Physik* **37**, 863–867 (1926).
 - [35] John Archibald Wheeler and Wojciech Hubert Zurek, *Quantum Theory and Measurement* (Princeton University Press, Princeton, NJ, 1983).
 - [36] Max Born, “Quantenmechanik der Stoßvorgänge,” *Zeitschrift für Physik* **38**, 803–827 (1926).
 - [37] Max Jammer, *The Conceptual Development of Quantum Mechanics. The History of Modern Physics, 1800–1950; v. 12*, 2nd ed. (American Institute of Physics, New York, 1989).
 - [38] Anton Zeilinger, “The message of the quantum,” *Nature* **438**, 743 (2005).
 - [39] Asher Peres, “Can we undo quantum measurements?” *Physical Review D* **22**, 879–883 (1980).
 - [40] Marlan O. Scully and Kai Drühl, “Quantum eraser: A proposed photon correlation experiment concerning observation and “delayed choice” in quantum mechanics,” *Physical Review A* **25**, 2208–2213 (1982).
 - [41] Daniel M. Greenberger and Alain Aspect, ““Haunted” measurements in quantum theory,” *Foundation of Physics* **19**, 679–704 (1989).
 - [42] Marlan O. Scully, Berthold-Georg Englert, and Herbert Walther, “Quantum optical tests of complementarity,” *Nature* **351**, 111–116 (1991).
 - [43] A. G. Zajonc, L. J. Wang, X. Y. Zou, and L. Mandel, “Quantum eraser,” *Nature* **353**, 507–508 (1991).
 - [44] Paul G. Kwiat, Aephraim M. Steinberg, and Raymond Y. Chiao, “Observation of a “quantum eraser:” a revival of coherence in a two-photon interference experiment,” *Physical Review A* **45**, 7729–7739 (1992).
 - [45] T. Pfau, S. Spälter, Ch. Kurtsiefer, C. R. Ekstrom, and J. Mlynek, “Loss of spatial coherence by a single spontaneous emission,” *Physical Review Letters* **73**, 1223–1226 (1994).
 - [46] Michael S. Chapman, Troy D. Hammond, Alan Lenef, Jörg Schmiedmayer, Richard A. Rubenstein, Edward Smith, and David E. Pritchard, “Photon scattering from atoms in an atom interferometer: Coherence lost and regained,” *Physical Review Letters* **75**, 3783–3787 (1995).
 - [47] Thomas J. Herzog, Paul G. Kwiat, Harald Weinfurter, and Anton Zeilinger, “Complementarity and the quantum eraser,” *Physical Review Letters* **75**, 3034–3037 (1995).
 - [48] Charles H. Bennett, “Logical reversibility of computation,” *IBM Journal of Research and Development* **17**, 525–532 (1973), reprinted in Ref. [50, pp. 197–204].
 - [49] Charles H. Bennett, “The thermodynamics of computation—a review,” *International Journal of Theoretical Physics* **21**, 905–940 (1982), reprinted in Ref. [105, pp. 283–318].
 - [50] Harvey S. Leff and Andrew F. Rex, *Maxwell’s Demon* (Princeton University Press, Princeton, NJ, 1990).
 - [51] John S. Bell, “Against ‘measurement’,” *Physikalische Blätter* **48**, 267 (1992).
 - [52] Erwin Schrödinger, “Die gegenwärtige Situation in der Quantenmechanik,” *Naturwissenschaften* **23**, 807–812, 823–828, 844–849 (1935), English translation in Ref. [106] and in Ref. [35, pp. 152–167].
 - [53] Hugh Everett, ““relative state” formulation of quantum mechanics,” *Reviews of Modern Physics* **29**, 454–462 (1957), reprinted in Ref. [35, pp. 315–323].
 - [54] Erwin Schrödinger, *The Interpretation of Quantum Mechanics. Dublin Seminars (1949–1955) and Other Unpublished Essays* (Ox Bow Press, Woodbridge, Connecticut, 1995).
 - [55] Wolfgang Pauli, “Die allgemeinen Prinzipien der Wellenmechanik,” in *Handbuch der Physik. Band V, Teil 1. Prinzipien der Quantentheorie I*, edited by S. Flügge (Springer, Berlin, Göttingen and Heidelberg, 1958) pp. 1–168.
 - [56] Albert Einstein, Boris Podolsky, and Nathan Rosen, “Can quantum-mechanical description of physical reality be considered complete?” *Physical Review* **47**, 777–780 (1935).
 - [57] Erwin Schrödinger, “Discussion of probability relations between separated systems,” *Mathematical Proceedings of the Cambridge Philosophical Society* **31**, 555–563 (1935).
 - [58] Erwin Schrödinger, “Probability relations between separated systems,” *Mathematical Proceedings of the Cambridge Philosophical Society* **32**, 446–452 (1936).
 - [59] Karl Svozil, “Contexts in quantum, classical and partition logic,” in *Handbook of Quantum Logic and Quantum Structures*, edited by Kurt Engesser, Dov M. Gabbay, and Daniel Lehmann (Elsevier, Amsterdam, 2009) pp. 551–586, [arXiv:quant-ph/0609209](https://arxiv.org/abs/quant-ph/0609209).

- [60] Anton Zeilinger, “A foundational principle for quantum mechanics,” *Foundations of Physics* **29**, 631–643 (1999).
- [61] Garrett Birkhoff and John von Neumann, “The logic of quantum mechanics,” *Annals of Mathematics* **37**, 823–843 (1936).
- [62] Simon Kochen and Ernst P. Specker, “The calculus of partial propositional functions,” in *Proceedings of the 1964 International Congress for Logic, Methodology and Philosophy of Science, Jerusalem* (North Holland, Amsterdam, 1965) pp. 45–57, reprinted in Ref. [104, pp. 222–234].
- [63] Gudrun Kalmbach, *Orthomodular Lattices* (Academic Press, New York, 1983).
- [64] Gudrun Kalmbach, *Measures and Hilbert Lattices* (World Scientific, Singapore, 1986).
- [65] Pavel Pták and Sylvia Pulmannová, *Orthomodular Structures as Quantum Logics* (Kluwer Academic Publishers, Dordrecht, 1991).
- [66] Mirko Navara and Vladimír Rogalewicz, “The pasting constructions for orthomodular posets,” *Mathematische Nachrichten* **154**, 157–168 (1991).
- [67] Karl Svozil, *Quantum Logic* (Springer, Singapore, 1998).
- [68] Karl Svozil, “Quantum scholasticism: On quantum contexts, counterfactuals, and the absurdities of quantum omniscience,” *Information Sciences* **179**, 535–541 (2009).
- [69] Lev Vaidman, “Counterfactuals in quantum mechanics,” in *Compendium of Quantum Physics*, edited by Daniel Greenberger, Klaus Hentschel, and Friedel Weinert (Springer, Berlin, Heidelberg, 2007) pp. 132–136, [arXiv:0709.0340](https://arxiv.org/abs/0709.0340).
- [70] Roderick M. Chisholm, “The contrary-to-fact conditional,” *Mind* **55**, 289–307 (1946), reprinted in [107, pp. 482–497].
- [71] Thomas Aquinas, *Summa Theologica. Translated by Fathers of the English Dominican Province* (Christian Classics Ethereal Library, Grand Rapids, MI, 1981).
- [72] Ernst Specker, “Die Logik nicht gleichzeitig entscheidbarer Aussagen,” *Dialectica* **14**, 239–246 (1960), reprinted in Ref. [104, pp. 175–182]; English translation: *The logic of propositions which are not simultaneously decidable*, Reprinted in Ref. [108, pp. 135–140] and as an eprint [arXiv:1103.4537](https://arxiv.org/abs/1103.4537), <http://arxiv.org/abs/1103.4537>.
- [73] Karl Raimund Popper and John C. Eccles, *The Self and Its Brain* (Springer, Berlin, Heidelberg, London, New York, 1977).
- [74] John C. Eccles, “Do mental events cause neural events analogously to the probability fields of quantum mechanics?” *Proceedings of the Royal Society of London. Series B. Biological Sciences* **227**, 411–428 (1986).
- [75] John C. Eccles, “The mind-brain problem revisited: The microsite hypothesis,” in *The Principles of Design and Operation of the Brain*, edited by John C. Eccles and O. Creutzfeldt (Springer, Berlin, 1990) pp. 549–572.
- [76] Rene Descartes, *Meditation on First Philosophy* (1641).
- [77] H. Putnam, *Reason, Truth and History* (Cambridge University Press, Cambridge, 1981).
- [78] Karl Svozil, “A constructivist manifesto for the physical sciences — Constructive re-interpretation of physical undecidability,” in *The Foundational Debate, Complexity and Constructivity in Mathematics and Physics*, edited by Werner DePauli Schimanovich, Eckehart Köhler, and Friedrich Stadler (Kluwer, Dordrecht, Boston, London, 1995) pp. 65–88.
- [79] Karl Svozil, “Extrinsic-intrinsic concept and complementarity,” in *Inside versus Outside*, edited by H. Atmanspacher and G. J. Dalenoot (Springer, Heidelberg, 1994) pp. 273–288.
- [80] Cristian S. Calude and Gregory J. Chaitin, “What is . . . a halting probability?” *Notices of the AMS* **57**, 236–237 (2007).
- [81] John von Neumann, “Various techniques used in connection with random digits,” National Bureau of Standards Applied Math Series **12**, 36–38 (1951), reprinted in *John von Neumann, Collected Works*, (Vol. V), A. H. Traub, editor, MacMillan, New York, 1963, p. 768–770.
- [82] The RAND Corporation, *A Million Random Digits with 100,000 Normal Deviates Free Press Publishers* (Knolls Atomic Power Lab. Report KAPL-3147, Glencoe, Illinois, 1955).
- [83] Helle Bechmann-Pasquinucci and Asher Peres, “Quantum cryptography with 3-state systems,” *Physical Review Letters* **85**, 3313–3316 (2000).
- [84] Cristian S. Calude and Karl Svozil, “Quantum randomness and value indefiniteness,” *Advanced Science Letters* **1**, 165–168 (2008), [arXiv:quant-ph/0611029](https://arxiv.org/abs/quant-ph/0611029).
- [85] Karl Svozil, “Three criteria for quantum random-number generators based on beam splitters,” *Physical Review A* **79**, 054306 (2009), [arXiv:quant-ph/0903.2744](https://arxiv.org/abs/quant-ph/0903.2744).
- [86] T. Paterek, J. Kofler, R. Prevedel, P. Klimek, M. Aspelmeyer, A. Zeilinger, and Č Brukner, “Logical independence and quantum randomness,” *New Journal of Physics* **12**, 013019 (2010).
- [87] S. Pironio, A. Acín, S. Massar, A. Boyer de la Giroday, D. N. Matsukevich, P. Maunz, S. Olmschenk, D. Hayes, L. Luo, T. A. Manning, and C. Monroe, “Random numbers certified by Bell’s theorem,” *Nature* **464**, 1021–1024 (2010).
- [88] Karl Svozil, “Proposed direct test of a certain type of noncontextuality in quantum mechanics,” *Physical Review A* **80**, 040102 (2009).
- [89] Paul A. Samuelson, “Constructing an unbiased random sequence,” *Journal of the American Statistical Association* **63**, 1526–1527 (1968).
- [90] David Hilbert, “Mathematical problems,” *Bulletin of the American Mathematical Society* **8**, 437–479 (1902).
- [91] Konrad Zuse, *Calculating Space. MIT Technical Translation AZT-70-164-GEMIT* (MIT (Proj. MAC), Cambridge, MA, 1970).
- [92] Edward Fredkin, “Digital mechanics. an informational process based on reversible universal cellular automata,” *Physica* **D45**, 254–270 (1990).
- [93] Stephen Wolfram, *A New Kind of Science* (Wolfram Media, Inc., Champaign, IL, 2002).
- [94] Karl Svozil, “Computational universes,” *Chaos, Solitons & Fractals* **25**, 845–859 (2006), [physics/0305048](https://arxiv.org/abs/physics/0305048).
- [95] George Marsaglia, “The Marsaglia random number CDROM including the diehard battery of tests of randomness,” (1995), available from URL www.stat.fsu.edu/pub/diehard/.
- [96] Andrew Rukhin, Juan Soto, James Nechvatal, Miles Smid, Elaine Barker, Stefan Leigh, Mark Levenson, Mark Vangel, David Banks, Alan Hekert, James Dray, and San Vo, *A Statistical Test Suite for Random and Pseudorandom Number Generators for Cryptographic*

- Applications. NIST Special Publication 800-22* (National Institute of Standards and Technology (NIST), Gaithersburg, MD, 2001).
- [97] Pierre L'Ecuyer and Richard Simard, "TestU01: A C library for empirical testing of random number generators," *ACM Transactions on Mathematical Software (TOMS)* **33**, Article 22, 1–40 (2007).
 - [98] Cristian S. Calude and Michael J. Dinneen, "Is quantum randomness algorithmic random? a preliminary attack," in *Proceedings of the 1st International Conference on Algebraic Informatics*, edited by S. Bozapalidis, A. Kalampakas, and G. Rahonis (Aristotle University of Thessaloniki, Thessaloniki, Greece, 2005) pp. 195–196.
 - [99] Cristian S. Calude, Michael J. Dinneen, Monica Dumitrescu, and Karl Svozil, "Experimental evidence of quantum randomness incomputability," *Phys. Rev. A* **82**, 022102 (2010).
 - [100] David Bailey, Peter Borwein, and Simon Plouffe, "On the rapid computation of various polylogarithmic constants," *Mathematics of Computation* **66**, 903–913 (1997).
 - [101] David H. Bailey and Jonathan M. Borwein, "Experimental mathematics: Examples, methods and implications," *Notices of the American Mathematical Society* **52**, 502–514 (2005).
 - [102] Philipp Frank and R. S. Cohen (Editor), *The Law of Causality and its Limits (Vienna Circle Collection)* (Springer, Vienna, 1997).
 - [103] Gregory J. Chaitin, *Information, Randomness and Incompleteness*, 2nd ed. (World Scientific, Singapore, 1990) this is a collection of G. Chaitin's early publications.
 - [104] Ernst Specker, *Selecta* (Birkhäuser Verlag, Basel, 1990).
 - [105] Harvey S. Leff and Andrew F. Rex, *Maxwell's Demon 2. Entropy, Classical and Quantum Information, Computing* (Institute of Physics Publishing, Bristol and Philadelphia, 1990).
 - [106] J. D. Trimmer, "The present situation in quantum mechanics: a translation of Schrödinger's "cat paradox",", *Proceedings of the American Philosophical Society* **124**, 323–338 (1980), reprinted in Ref. [35, pp. 152–167].
 - [107] Roderick M. Chisholm, "The contrary-to-fact conditional," in *Readings in Philosophical Analysis*, edited by Herbert Feigl and Wilfrid Sellars (Appleton-Century-Crofts, New York, 1949) pp. 482–497.
 - [108] Clifford Alan Hooker, *The Logico-Algebraic Approach to Quantum Mechanics. Volume I: Historical Evolution* (Reidel, Dordrecht, 1975).