Converting nonlocality into contextuality

Karl Svozil*
Institute for Theoretical Physics, TU Wien, Wiedner Hauptstrasse 8-10/136, 1040 Vienna, Austria
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Diagonalization of matrix pencils provide a uniform technique to transcribe operator based violations of Boole's 'conditions of possible experience' involving multipartite correllations into contextuality. They also provide structural analysis of the contexts involved, and thereby suggest compact forms of deviations of quantized systems from classical predictions.

Keywords: contextuality, two-valued states, quantum states, matrix pencil

In what follows we shall concentrate on the problem of analyzing the particular type of 'contextuality' rendered by Einstein-Podolsky-Rosen type configurations exhibiting nonlocality. While the inverse problem—converting contextuality into nonlocality [1]—may be of empirical importance, only the solution of the former task can identify the particular type of contextuality exhibited.

From a purely structural point of view—that is, in terms of the algebraic relations of quantum observables—both views are strictly equivalent, although they may appear to be very different. For instance, as observed by Cabello [2–4], Hardy's theorem [5, 6] can, in quantum logical terms, be transcribed as a true-implies-false arrangement (in graph theoretical terms, a gadget) of observables [7, 8].

In what follows, we shall present a systematic, algorithmic, and thus constructive, analysis of similar transcription processes for general cases of operator-based arguments demonstrating nonclassical behavior. It is based on the proper spectral decomposition of the operators involved. For multipartite observables, this amounts to a translation from quantum nonlocality into quantum contextuality.

Whether one 'image' (in the sense of Hertz [9]) is to be preferred over the other is a matter of preference. One could argue that contextuality is 'stronger' if it resides in a single particle or if it is spatiotemporally 'spread' among multiple particles. In any case, as pointed out in a recent review [10], measurements of a respective nonlocal observable cannot be performed as single-qubit local measurements instead of a global measurement.

The algebraic techniques used in the proposed method are fairly standard for mathematicians (although often rederived in the physical context). Let us mention them without providing proofs for an n-dimensional Hilbert space.

Normal operators A (such as Hermitian or unitary operators that commute with their respective adjoints) can be represented as a unique spectral sum $A=\sum_{i=1}^{k\leq n}\lambda_i E_i$ of products of distinct eigenvalues λ_i and orthogonal (that is, self-adjoint) projection operators E_i. Furthermore, the projection operators are mutually orthogonal, such that $E_iE_j=\delta_{ij}E_i$, and they form a resolution of the identity operator, $\sum_{i=1}^{k\leq n} E_i = \mathbb{1}_n$.

If $k=n$, the spectrum is called nondegenerate, and the projection operators are one-dimensional, written as the dyadic products $E_i=|e_i\rangle\langle e_i|$ of an orthonormal basis, with basis vectors $|e_i\rangle$, $1\leq i\leq n$. Otherwise, for $k<n$, the spectrum is called degenerate. If all eigenvalues λ_i of A are known, then the orthogonal projection $E_i=p_i(A)$ operators can be directly calculated from the polynomials $p_i(t)=\prod_{j\neq i}(t-\lambda_j)/(\lambda_i-\lambda_j)$ [11, Sec. 79].

Mutually commuting normal operators $A_1,\ldots,A_l$ share common projection operators $E_1,\ldots,E_k$ and can be written as functions $A_m=f_m(M)=\sum_{i=1}^{k\leq n} f_m(\lambda_i)E_i$ of a single 'maximal' (with respect to $A_1,\ldots,A_l$) normal operator $M=\sum_{i=1}^{k\leq n}\lambda_i E_i$; moreover, $M=g(A_1,\ldots,A_l)$ is a function of $A_1,\ldots,A_l$ [12, Satz 8]. If $k=n$, then M is nondegenerate and renders the finest and thus maximal resolution. The calculation of the common projection operators for nondegenerate $A_1,\ldots,A_l$ is straightforward: all that is needed is the formation of $p_{i,m}(A_m)$ for any one of these operators A_m, with $1\leq m\leq l$.

For degenerate $A_1,\ldots,A_l$, we need to find an orthonormal basis in which every single one of this collection of mutually commuting operators is diagonal. Although in principle well-known [13, Section 1.3], the standard procedure via block diagonalization can be rather involved [14]. Alternatively, and for our purposes, we shall diagonalize the matrix pencil:

$$P=\sum_{i=1}^{l} a_i A_i, \tag{1}$$

where a_i are scalars. As P commutes with $A_1,\ldots,A_l$, they share a common set of projection operators. Moreover, since the scalar parameters a_i can be adjusted, and in particular, can be identified with Kronecker delta functions δ_{ij}, and as P commutes with every single operator A_j for $1\leq j\leq l$, P and A_j share a common set of projection operators.

Equipped with these techniques, any collection of commeasurable multipartite observables corresponding to mutually commuting operators can be transcribed into vectors $|e_i\rangle$ corresponding to the projection operators $E_i=|e_i\rangle\langle e_i|$ in the spectrum of the operators of these observables. If these operators render a maximal resolution, the respective vectors correspond to an orthonormal basis called a context. The merging or pasting of possibly intertwining contexts then generates a quantum logic which can be analyzed to identify and characterize the contextual (nonclassical) predictions and features.

The matrix pencil of $A_1,\ldots,A_l$ yields a maximal resolution (which not necessarily is unique if $A_1,\ldots,A_l$ is not sufficiently resolving) in terms of common eigenvectors. The orthogonal projection operators corresponding to these eigenvectors can be bundled into a single maximal nondegenerate operator $M=\sum_{i=1}^{n}\mu_i E_i$ with n mutually different eigenvalues μ_i (of multiplicity one), which can be conceived as containing the entirety of possible information available through $A_1,\ldots,A_l$, which is all that can be known quantum mechanically [15]. In this formal sense, the collection of observables $A_1,\ldots,A_l$ may be considered to 'correspond to a context'. This bridges the gap between general operator-valued arguments on the one hand, and quantum logics on the other hand, as the latter is based on the identification of propositions with projection operators [16].

Applying these techniques to the Peres-Mermin square [10, 17–19] renders an (un)expected result: the resulting 24 propositions and 24 contexts are a 'completion' of the (minimal in four dimensions [20]) 18-9 Kochen-Specker set comprising 18 vectors in 9 contexts [2]. In more detail, this configuration involves nine dichotomic observables with eigenvalues ± 1 arranged in a 3×3 matrix. The rows and columns of the matrix form (2), which are masking six four-element contexts, one per row and column ($\sigma_i\sigma_j$ stands for $\sigma_i\otimes\sigma_j$, same for products involving factors $\mathbb{1}_2$):

$$\begin{pmatrix} \sigma_z\mathbb{1}_2 & \mathbb{1}_2\sigma_z & \sigma_z\sigma_z \\ \mathbb{1}_2\sigma_x & \sigma_x\mathbb{1}_2 & \sigma_x\sigma_x \\ \sigma_z\sigma_x & \sigma_x\sigma_z & \sigma_y\sigma_y \end{pmatrix}. \tag{2}$$

To explicitly demonstrate the difficulties involved co-diagonalization of commuting degenerate matrices consider the last row of the Peres-Mermin square (2). Its operators $\sigma_z\sigma_x$, $\sigma_x\sigma_z$, and $\sigma_y\sigma_y$ mutually commute—for instance, $[\sigma_z\sigma_x,\sigma_y\sigma_y]=0$. However, a straightforward calculation of the eigenvectors of $\sigma_z\sigma_x$ yields: $\left(0,1,0,1\right)^\intercal$, $\left(-1,0,1,0\right)^\intercal$, $\left(0,-1,0,1\right)^\intercal$, and $\left(1,0,1,0\right)^\intercal$. None of these eigenvectors are eigenvectors of $\sigma_y\sigma_y$, and vice versa. This demonstrates the difficulties involved in co-diagonalizing these commuting degenerate matrices.

Nonetheless, the 'joint' Peres-Mermin square contexts are revealed as the normalized eigenvectors of the respective matrix pencils (1). Table I enumerates those contexts, provided that the σ-matrices are encoded in the standard form $\sigma(\theta,\varphi)=\sigma_x\sin\theta\cos\varphi+\sigma_y\sin\theta\sin\varphi+\sigma_z\cos\theta$, where $0\le\theta\le\pi$ is the polar angle in the x–z-plane from the z-axis, and $0\le\varphi<2\pi$ is the azimuthal angle in the x–y-plane from the x-axis, with $\sigma_x=\sigma\left(\frac{\pi}{2},0\right)=\text{antidiag}\left(1,1\right)$, $\sigma_y=\sigma\left(\frac{\pi}{2},\frac{\pi}{2}\right)=\text{antidiag}\left(-i,i\right)$, and $\sigma_z=\sigma\left(0,0\right)=\text{diag}\left(1,-1\right)$. Then the resulting set of 24 vectors contains all 18 vectors enumerated by Cabello, Estebaranz and García-Alcaine [2], but the contexts are enumerated differently and need to be reconstructed through orthogonality relations. Both the Peres-Mermin square–through the hypothetical existence of uniform dichotomic, operator based values, as well as the Kochen-Specker set of 18 elements in 9 contexts—through the hypothetical existence of uniform binary two-valued states, support parity proofs of nonclassicality.

'Directing' or 'orienting' the 24 different eigenvectors of Table I such that their first non-negative entry is positive, and subsequently lexicographically ordering them, results in a list enumerated in Table II.

Analysis of their orthogonality relations yields an adjacency matrix that, in turn, can be used to construct the respective (hyper)graph through the intertwining cliques and thus contexts thereof. As can be expected, there are only four-cliques corresponding to orthonormal bases in four dimensional Hilbert space. They are enumerated in Table III. Indeed, they represent the completed and extended elements and orthogonality relationships of the Cabello, Estebaranz and García-Alcaine [2] set of 18 vector labels in 9 contexts (also depicted in [21, Figure 4(c)] and mentioned in [22], as well as [23, Figure 2]). Figure 1 depicts the hypergraph representing these intertwining contexts as unbroken smooth lines, and the vector labels as elements of these contexts, as enumerated in Tables II and III.

The 24 rays were already discussed by Peres [19] as permutations of the vector components of $\left(1,0,0,0\right)^\intercal$, $\left(1,1,0,0\right)^\intercal$, $\left(1,-1,0,0\right)^\intercal$, $\left(1,1,1,1\right)^\intercal$, $\left(1,1,1,-1\right)^\intercal$, and $\left(1,1,-1,-1\right)^\intercal$. The 'full' 24-24 configuration was obtained by Pavičić [21] who reconstructed additional 18 contexts not provided in the original Peres paper [19] by hand [24]. Peres' 24-24 configuration is arranged in four-element contexts associated with four-dimensional Hilbert space, with vector components drawn from the set $\{-1,0,1\}$, that do not support any two-valued state. An early incomplete representation of the Peres configuration [19] was presented by Tkadlec [25, Figure 1] who mentions 24 elements in 22 contexts. Pavičić, Megill and Merlet [22, Table 1] have demonstrated that Peres' 24-24 configuration contains 1,233 sets that do not support any two-valued states. Among these 1,233 sets are six 'irreducible' or 'critical' configurations which do not contain any proper subset that does not support two-valued states. Notably, among these configurations is the previously mentioned 18-9 configuration proposed by Cabello, Estebaranz and García-Alcaine [2]. Previously, Pavičić, Merlet, McKay, and Megill [26, 27, Section 5(viii)] had shown that, among all sets with 24 rays and vector components from the set $\{-1,0,1\}$, and 24 contexts, only one configuration does not allow any two valued state—and that one is isomorphic to Peres' 'full' (incluiding 18 additional contexts) 24-24 configuration enumerated by Pavičić [21]. This computation had taken one year on a single CPU of a supercomputer [24]. More recently, Pavičić and Megill [28, Table 1] have demonstrated that the vector components from the set $\{-1,0,1\}$ vector-generate a 24-24 set, which contains all smaller Kochen-Specker sets and is simultaneously isomorphic to the 'completed' 24-24 configuration configuration.

We state here, without providing a formal proof, that if a 'larger' collection of quantum observables encompasses another 'smaller' collection of quantum observables, and all contexts are complete such that no element of the contexts is lacking, then the former inherits the classical state-based properties of the latter. In particular, it inherits the scarcity

TABLE I. Eigensystems of the matrix pencils of the rows and columns of Equation (2). The set of 24 vectors includes the 18 vectors of Cabello, Estebaranz and García-Alcaine [2], also enumerated in [10, Figure 3]. As already noted by Peres [19], these six 'primary' contexts associated with orthogonal tetrads are disjoint (not intertwined). In the hypergraph representation depicted in Figure 1 they are represented as the 'small ovals' on the six edges of the hypergraph. This arrangement of contexts is thus not isomorphic to the (biconnected) 18-9 configuration of Cabello, Estebaranz and García-Alcaine [2]. The latter matrix pencil yields the Bell basis $\{|\Psi_+\rangle, |\Psi_-\rangle, |\Phi_+\rangle, |\Phi_-\rangle\}$.

matrix pencils	eigenvalues			
	$a-b-c$	$-a+b-c$	$-a-b+c$	$a+b+c$
$a\sigma_z\mathbb{1}_2 + b\mathbb{1}_2\sigma_z + c\sigma_z\sigma_z$	$\left(0,1,0,0\right)^\intercal$	$\left(0,0,1,0\right)^\intercal$	$\left(0,0,0,1\right)^\intercal$	$\left(1,0,0,0\right)^\intercal$
$a\mathbb{1}_2\sigma_x + b\sigma_x\mathbb{1}_2 + c\sigma_x\sigma_x$	$\left(-1,-1,1,1\right)^\intercal$	$\left(-1,1,-1,1\right)^\intercal$	$\left(1,-1,-1,1\right)^\intercal$	$\left(1,1,1,1\right)^\intercal$
$a\sigma_z\sigma_x + b\sigma_x\sigma_z + c\sigma_y\sigma_y$	$\left(1,1,-1,1\right)^\intercal$	$\left(1,-1,1,1\right)^\intercal$	$\left(-1,1,1,1\right)^\intercal$	$\left(-1,-1,-1,1\right)^\intercal$
$a\sigma_z\mathbb{1}_2 + b\mathbb{1}_2\sigma_x + c\sigma_z\sigma_x$	$\left(-1,1,0,0\right)^\intercal$	$\left(0,0,1,1\right)^\intercal$	$\left(0,0,-1,1\right)^\intercal$	$\left(1,1,0,0\right)^\intercal$
$a\mathbb{1}_2\sigma_z + b\sigma_x\mathbb{1}_2 + c\sigma_x\sigma_z$	$\left(-1,0,1,0\right)^\intercal$	$\left(0,1,0,1\right)^\intercal$	$\left(0,-1,0,1\right)^\intercal$	$\left(1,0,1,0\right)^\intercal$
	$a-b-c$	$-a+b-c$	$-a-b+c$	$a+b+c$
$a\sigma_z\sigma_z + b\sigma_x\sigma_x + c\sigma_y\sigma_y$	$\vert\Psi_-\rangle = \left(0,1,-1,0\right)^\intercal$	$\vert\Phi_+\rangle = \left(1,0,0,1\right)^\intercal$	$\vert\Phi_-\rangle = \left(1,0,0,-1\right)^\intercal$	$\vert\Psi_-\rangle = \left(0,1,1,0\right)^\intercal$

TABLE II. The 24 vectors specifying the binary observables (propositions) of the Peres-Mermin square, with normalization factors omitted.

$\vert 1\rangle = \left(0,0,0,1\right)^\intercal$,	$\vert 2\rangle = \left(0,0,1,-1\right)^\intercal$,	$\vert 3\rangle = \left(0,0,1,0\right)^\intercal$,	$\vert 4\rangle = \left(0,0,1,1\right)^\intercal$,	$\vert 5\rangle = \vert\Psi_-\rangle = \left(0,1,-1,0\right)^\intercal$,
$\vert 6\rangle = \left(0,1,0,-1\right)^\intercal$,	$\vert 7\rangle = \left(0,1,0,0\right)^\intercal$,	$\vert 8\rangle = \left(0,1,0,1\right)^\intercal$,	$\vert 9\rangle = \vert\Psi_+\rangle = \left(0,1,1,0\right)^\intercal$,	$\vert 10\rangle = \left(1,-1,-1,-1\right)^\intercal$,
$\vert 11\rangle = \left(1,-1,-1,1\right)^\intercal$,	$\vert 12\rangle = \left(1,-1,0,0\right)^\intercal$,	$\vert 13\rangle = \left(1,-1,1,-1\right)^\intercal$,	$\vert 14\rangle = \left(1,-1,1,1\right)^\intercal$,	$\vert 15\rangle = \left(1,0,-1,0\right)^\intercal$,
$\vert 16\rangle = \vert\Phi_-\rangle = \left(1,0,0,-1\right)^\intercal$,	$\vert 17\rangle = \left(1,0,0,0\right)^\intercal$,	$\vert 18\rangle = \vert\Phi_+\rangle = \left(1,0,0,1\right)^\intercal$,	$\vert 19\rangle = \left(1,0,1,0\right)^\intercal$,	$\vert 20\rangle = \left(1,1,-1,-1\right)^\intercal$,
$\vert 21\rangle = \left(1,1,-1,1\right)^\intercal$,	$\vert 22\rangle = \left(1,1,0,0\right)^\intercal$,	$\vert 23\rangle = \left(1,1,1,-1\right)^\intercal$,	$\vert 24\rangle = \left(1,1,1,1\right)^\intercal$	

(such as nonseparability [29, Theorem 0]) or a total absence of two-valued states.

The underlying rationale is that, informally speaking, if the 'smaller' set cannot support features related to two-valued states, such as separability of propositions, then intertwining or adding contexts can only impose further constraints, thereby exacerbating the situation or 'making things worse', by introducing new conditions.

Mermin has suggested [18, 30] a "simple unified form for the major no-hidden-variables theorems" in which he identified four commuting three-partite operators: $\sigma_x\sigma_x\sigma_x$, $\sigma_x\sigma_y\sigma_y$, $\sigma_y\sigma_x\sigma_y$, and $\sigma_y\sigma_y\sigma_x$. A parity argument reveals a state-independent quantum contradiction to the classical existence of (local, noncontextual) elements of physical reality: whereas the product $-\mathbb{1}_8 = \sigma_x\sigma_x\sigma_x = -(\sigma_x\sigma_y\sigma_y)\cdot(\sigma_y\sigma_x\sigma_y)\cdot(\sigma_y\sigma_y\sigma_x)$, and thus the expectation for any quantum state is $\langle -\mathbb{1}_8\rangle = \langle \mathbb{1}_2\otimes(-\mathbb{1}_2)\otimes\mathbb{1}_2\rangle = \langle(\sigma_x\cdot\sigma_x\cdot\sigma_y\cdot\sigma_y)(\sigma_x\cdot\sigma_y\cdot\sigma_x\cdot\sigma_y)(\sigma_x\cdot\sigma_y\cdot\sigma_y\cdot\sigma_x)\rangle = \langle(\sigma_x\sigma_x\sigma_x)\cdot(\sigma_x\sigma_y\sigma_y)\cdot(\sigma_y\sigma_x\sigma_y)\cdot(\sigma_y\sigma_y\sigma_x)\rangle = \langle\sigma_x\sigma_x\sigma_x\rangle\langle\sigma_x\sigma_y\sigma_y\rangle\langle\sigma_y\sigma_x\sigma_y\rangle\langle\sigma_y\sigma_y\sigma_x\rangle = -1$, every observable σ_x and σ_y for each of the three particles occurs twice; thus, if classically all such single-particle observables would coexist, their respective product would be expected to be 1.

With the help of the matrix pencil:

$$a\sigma_x\sigma_x\sigma_x + b\sigma_x\sigma_y\sigma_y + c\sigma_y\sigma_x\sigma_y + d\sigma_y\sigma_y\sigma_x \tag{3}$$

the simultaneous eigensystem is enumerated in Table IV.

The eight vectors resulting from the matrix pencil calculation form an orthogonal basis of an eight-dimensional Hilbert space, which, in graph theoretical language, represents an isolated clique. However, this configuration does not constitute a Kochen-Specker proof, as it still permits a separating set of eight two-valued states. The question arises: How does one arrive at a complete Greenberger-Horne-Zeilinger contradiction with classical elements of physical reality, as outlined above?

The criterion employed in an experimental corroboration [31] is to select any one of the eigenstates constructed from entangled three-partite states along the z-axis, such as $|\Gamma_8\rangle = (1/\sqrt{2})\left(|z_+z_+z_+\rangle + |z_-z_-z_-\rangle\right) \equiv \left(1,0,0,0,0,0,0,1\right)^\intercal$, where $|z_+\rangle \equiv \left(1,0\right)^\intercal$ and $|z_-\rangle \equiv \left(0,1\right)^\intercal$. Since $|\Gamma_8\rangle$ is an eigenstate of all four terms of the matrix pencil, four separate measurements can be performed (possibly temporally separated) yielding the eigenvalues $+1$ for $\sigma_x\sigma_x\sigma_x$ as well as -1 for the three others, namely $\sigma_x\sigma_y\sigma_y$, $\sigma_y\sigma_x\sigma_y$, and $\sigma_y\sigma_y\sigma_x$, respectively. Note that similar contradictions arise if the seven other eigenstates of the matrix pencil are considered [32, Table 1]. These quantum-versus-classical discrepancies are independent of the specific state. We chose the particular state $|\Gamma_8\rangle$ because it is—among the eight eigenstates enzumerated in Table IV—an eigenstate of the individ-

TABLE III. The 24 cliques or contexts or orthonormal bases or maximal operators of the Peres-Mermin square. Numbers $1 \le i \le 24$ refer to the vectors $|i\rangle$ defined in Table II, or, equivalently, to the onedimensional orthogonal projection operator $E_i = (|i\rangle\langle i|)/\langle i|i\rangle$.

$C_1 = \{5,9,16,18\}$,	$C_2 = \{3,7,16,18\}$,	$C_3 = \{9,14,16,21\}$,	$C_4 = \{9,13,18,20\}$,	$C_5 = \{11,13,20,24\}$,	$C_6 = \{5,11,16,24\}$,
$C_7 = \{10,14,21,23\}$,	$C_8 = \{5,10,18,23\}$,	$C_9 = \{8,14,15,23\}$,	$C_{10} = \{8,11,19,20\}$,	$C_{11} = \{6,13,15,24\}$,	$C_{12} = \{6,10,19,21\}$,
$C_{13} = \{6,8,15,19\}$,	$C_{14} = \{3,6,8,17\}$,	$C_{15} = \{4,12,21,23\}$,	$C_{16} = \{4,11,13,22\}$,	$C_{17} = \{2,12,20,24\}$,	$C_{18} = \{2,10,14,22\}$,
$C_{19} = \{2,4,12,22\}$,	$C_{20} = \{2,4,7,17\}$,	$C_{21} = \{1,7,15,19\}$,	$C_{22} = \{1,5,9,17\}$,	$C_{23} = \{1,3,12,22\}$,	$C_{24} = \{1,3,7,17\}$

TABLE IV. Eigensystem of the matrix pencil (3) associated with the Mermin configuration [18, 30], constituting an orthogonal basis in an eight-dimensional Hilbert space. The values '+' and '−' represent the measured vales +1 and −1 of the respective operators.

eigenvalue	eigenvector	$\sigma_x\sigma_x\sigma_x$	$\sigma_x\sigma_y\sigma_y$	$\sigma_x\sigma_y\sigma_y$	$\sigma_y\sigma_y\sigma_x$
$-a+b-c-d$	$\vert\Gamma_1\rangle = \frac{1}{\sqrt{2}}(\vert z_+z_-z_-\rangle - \vert z_-z_+z_+\rangle) \equiv (0,0,0,1,-1,0,0,0)^\intercal$	−	+	−	−
$a-b+c+d$	$\vert\Gamma_2\rangle = \frac{1}{\sqrt{2}}(\vert z_+z_-z_-\rangle + \vert z_-z_+z_+\rangle) \equiv (0,0,0,1,1,0,0,0)^\intercal$	+	−	+	+
$-a-b+c-d$	$\vert\Gamma_3\rangle = \frac{1}{\sqrt{2}}(\vert z_+z_-z_+\rangle - \vert z_-z_+z_-\rangle) \equiv (0,0,1,0,0,-1,0,0)^\intercal$	−	−	+	−
$a+b-c+d$	$\vert\Gamma_4\rangle = \frac{1}{\sqrt{2}}(\vert z_+z_-z_+\rangle + \vert z_-z_+z_-\rangle) \equiv (0,0,1,0,0,1,0,0)^\intercal$	+	+	−	+
$-a-b-c+d$	$\vert\Gamma_5\rangle = \frac{1}{\sqrt{2}}(\vert z_+z_+z_-\rangle - \vert z_-z_-z_+\rangle) \equiv (0,1,0,0,0,0,-1,0)^\intercal$	−	−	−	+
$a+b+c-d$	$\vert\Gamma_6\rangle = \frac{1}{\sqrt{2}}(\vert z_+z_+z_-\rangle + \vert z_-z_-z_+\rangle) \equiv (0,1,0,0,0,0,1,0)^\intercal$	+	+	+	−
$-a+b+c+d$	$\vert\Gamma_7\rangle = \frac{1}{\sqrt{2}}(\vert z_+z_+z_+\rangle - \vert z_-z_-z_-\rangle) \equiv (1,0,0,0,0,0,0,-1)^\intercal$	−	+	+	+
$a-b-c-d$	$\vert\Gamma_8\rangle = \frac{1}{\sqrt{2}}(\vert z_+z_+z_+\rangle + \vert z_-z_-z_-\rangle) \equiv (1,0,0,0,0,0,0,1)^\intercal$	+	−	−	−

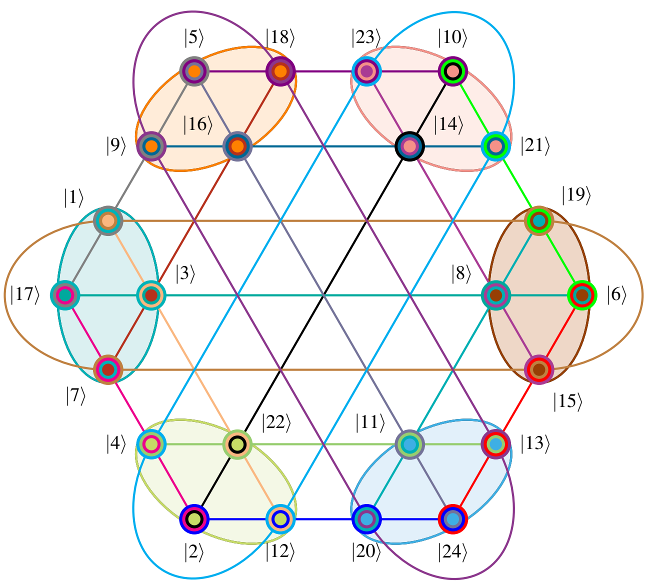

FIG. 1. Hypergraph representing contexts (or cliques or orthonormal bases or maximal operators) enumerated in Table III as unbroken smooth lines. This is a 'orthogonal completion' [19, 21] of the Kochen-Specker set comprising 18 vectors in 9 contexts introduced by Cabello, Estebaranz and García-Alcaine [2]. The filled shaded small ovals on the edges correspond to the 'primary' isolated (nonintertwined) contexts from the matrix pencil calculations enumerated in Table I.

ual terms involved in the matrix pencil.

The quantum logic and hypergraph representation of the Greenberger-Horne-Zeilinger-Mermin argument is elementary—indeed it amounts to a single context, clique, block, or subalgebra, hypergraph representable by a line [32, Figure 2(a)]. The 'strength' of its conviction lies in a particular form of 'operational realizability' as a collection of four, preferably nonlocal (under strict Einstein separability) measurements of $\sigma_x\sigma_x\sigma_x$, $\sigma_x\sigma_y\sigma_y$, $\sigma_y\sigma_x\sigma_y$, and $\sigma_y\sigma_y\sigma_x$ applied to one of the eigenstates $|\Gamma_i\rangle$, $1 \le i \le 8$; historically, $|\Gamma_8\rangle$ [31]. As all the operators involved mutually commute, they could even be serially composed—one measurement after the other, in arbitrary order—yielding the quantum prediction −1, but this would spoil the nonlocality as the respective input and output ports would have to be aligned.

One wonders whether a similarly convincing argument exists for just two particles. A natural candidate would be the Peres-Mermin square (2) quoted earlier. Note that its 'masked' or 'hidden' contexts, revealed by the matrix pencils, can be partitioned into four 'separable' type contexts depicted in Figure 2(a) containing only separable vectors—corresponding to the first and second rows and columns—and two contexts consisting of nonseparable vectors—corresponding to the last row and column of (2). More importantly, Figure 2(b) is the four-dimensional equivalent of the (v-shaped) 'firefly logic' [33, pp. 21-22] in three dimensions with dichotomic degenerate operators taking on the role of quantum logical propositions: the 'intertwining' observable $\sigma_y\sigma_y$ 'connects' two contexts, namely the one formed by $\{\sigma_x\sigma_z, \sigma_y\sigma_y, \sigma_z\sigma_x\}$, and $\{\sigma_x\sigma_x, \sigma_y\sigma_y, \sigma_z\sigma_z\}$.

Concentrating on these two latter contexts consisting of nonseparable vectors, we make the following observations: (i) Since the respective observables (for instance, $\sigma_x\sigma_x$ and $\sigma_x\sigma_z$) do not all commute, they cannot be as nicely and conveniently measured separately as would be the case if they would be located within the same context. (ii) However, we may convince ourselves that the product of the row and column elements contain, in classical terms, the same 'elements of physical reality', namely $(\sigma_z\sigma_x)\cdot(\sigma_x\sigma_z)\cdot(\sigma_y\sigma_y)$ and $(\sigma_z\sigma_z)\cdot(\sigma_x\sigma_x)\cdot(\sigma_y\sigma_y)$. Within the contexts, that is, if they belong to the same product, any of these factors can be simultaneously measured. And yet, quantum mechanically, the first product yields $\mathbb{1}_4$ and the second $-\mathbb{1}_4$, and thus their respective expectations (on any state) are $+1$ and -1 This latter prediction is in total contradiction with classical expectations.

The product of these two products contains an even number of classical supposedly 'elements of physical reality' for both particles, similarly to the Greenberger-Horne-Zeilinger-Mermin argument for three particles. Dissimilar to the latter it involves two contexts rather than just one. One may object that the two-partite 'analog' is trivial in the sense that one effectively 'merely measures identities' and not the individual expectations. This is due to complementarity, as, say, $\sigma_x\sigma_x$ is not in the same context as $\sigma_x\sigma_z$. But it should be kept in mind that the basic presumptions remain the same in both arguments.

The argument could be made shorter and possibly more convincing by realizing that the following operators commute because of equality: $(\sigma_z\sigma_x)\cdot(\sigma_x\sigma_z) = -(\sigma_x\sigma_x)\cdot(\sigma_z\sigma_z) = (\sigma_z\cdot\sigma_x)(\sigma_x\cdot\sigma_z) = \sigma_y\sigma_y = \text{antidiag}\left(-1,1,1,-1\right)$. In more detail the matrix pencil

$$a(\sigma_z\sigma_x)\cdot(\sigma_x\sigma_z)+b(\sigma_x\sigma_x)\cdot(\sigma_z\sigma_z) \tag{4}$$

has a degenerate spectrum with the Bell basis as eigenvectors. It is enumerated in Table V.

TABLE V. Eigensystem of the matrix pencil (4) associated with the products of operators in the last (third) row and column of the Peres-Mermin square, constituting the Bell basis. Inclusion of $\sigma_y\sigma_y$ or $(\sigma_y\sigma_y)\cdot(\sigma_y\sigma_y)=\mathbb{1}_4$ does not change the calculation and is therefore omitted. The values $+$ and $-$ represent the measured values $+1$ and -1 of the respective operators. $\sigma_y\sigma_y$ always yields $+$.

eigenvalue	eigenvector	$(\sigma_z\sigma_x)\cdot(\sigma_x\sigma_z)$	$(\sigma_x\sigma_x)\cdot(\sigma_z\sigma_z)$
$a-b$	$\vert\Psi_+\rangle$, $\vert\Phi_-\rangle$	$+$	$-$
$-a+b$	$\vert\Psi_-\rangle$, $\vert\Phi_+\rangle$	$-$	$+$

Hence, preparing a state in one Bell basis state and measuring (successively or separately) $(\sigma_z\sigma_x)\cdot(\sigma_x\sigma_z)$ and $(\sigma_x\sigma_x)\cdot(\sigma_z\sigma_z)$ yields a quantum expectation

$$\begin{aligned}
&\langle(\sigma_z\sigma_x)\cdot(\sigma_x\sigma_z)\rangle\langle(\sigma_x\sigma_x)\cdot(\sigma_z\sigma_z)\rangle\\
&=\langle(\sigma_z\sigma_x)\cdot(\sigma_x\sigma_z)\cdot(\sigma_x\sigma_x)\cdot(\sigma_z\sigma_z)\rangle\\
&=\langle(\sigma_z\cdot\sigma_x\cdot\sigma_x\cdot\sigma_z)(\sigma_x\cdot\sigma_z\cdot\sigma_x\cdot\sigma_z)\rangle\\
&=\langle\mathbb{1}_2\otimes(-\mathbb{1}_2)\rangle=\langle-\mathbb{1}_4\rangle=-1.
\end{aligned} \tag{5}$$

And yet, the classical prediction is that the product of these terms always needs to be positive, as every alleged 'element of reality', in particular corresponding to σ_x and σ_z, enters an even number of times (indeed twice per particle). Again, the product observables $(\sigma_z\sigma_x)\cdot(\sigma_x\sigma_z)$ and $(\sigma_x\sigma_x)\cdot(\sigma_z\sigma_z)$ commute, but their factors from different products belong to different contexts and therefore do not commute. Nevertheless, these commuting dichotomic operators, together with $(\sigma_y\sigma_y)\cdot(\sigma_y\sigma_y)$, form a simultanuously measurable set of operators $\{(\sigma_z\sigma_x)\cdot(\sigma_x\sigma_z),(\sigma_x\sigma_x)\cdot(\sigma_z\sigma_z),(\sigma_y\sigma_y)\cdot(\sigma_y\sigma_y)\}$ that 'masks' a single context, namely the Bell basis.

I conclude with some comments and an outlook. A nonlocal measurement in quantum mechanics refers to the simultaneous measurement of properties of entangled particles that are—at least in principle—located in space-like separated regions (Einstein locality). We therefore suggest calling an operator, or a collection of mutually commuting operators, 'nonlocal' if they—or more generally, the eigensystem of their matrix pencil—allow entangled, that is, nonseparable, eigenvalues. This is the case for the last row and column of the Peres-Mermin square, and also for the four three-partite operators suggested by Mermin in the context of the Greenberger-Horne-Zeilinger argument. I shall motivate and discuss these issues further in a later publication.

We have concentrated on transcriptions of operator-based arguments against the classical performance of quantized systems with dichotomic arguments. These methods allow the generalization to non-dichotomic outcomes. A converse transcription in terms of a hypergraph representing (possibly intertwining) contexts into nonlocal measurements can be achieved if this hypergraph allows a faithful orthogonal representation [34, 35] (or, equivalently, a coordinatization [28]) in terms of vector labels that are entangled (or, equivalently, nonfactorizable).

The sum of noncommuting operators such as the Clauser-Horne-Shimony-Holt operator $\sigma_1\sigma_3+\sigma_1\sigma_4+\sigma_2\sigma_3-\sigma_2\sigma_4$ with $\sigma_i=\sigma(\theta_i,\varphi_i)$ can be analized by acknowledging that the sum of products of scalars with normal operators is also a normal operator. Therefore, this sum has a (not necessarily unique if the sum is degenerate) spectral decomposition in terms of projection operators corresponding to some orthonormal basis. This orthonormal basis forms an isolated context whose elements may be simultaneously measured and optimized [36]. Alternatively, the contexts formed by the spectral decompositions of the individual summands can be considered as the basis of the analysis.

Let me summarize the means presented here by pointing out that the matrix pencil method introduced in the context of nonlocal multipartite measurements is a very elegant way of simultaneously diagonalizing commuting operators with degenerate spectra. It provides general grounds for the systematic application of nonclassical performance of quantized systems; in particular, for operator based arguments.

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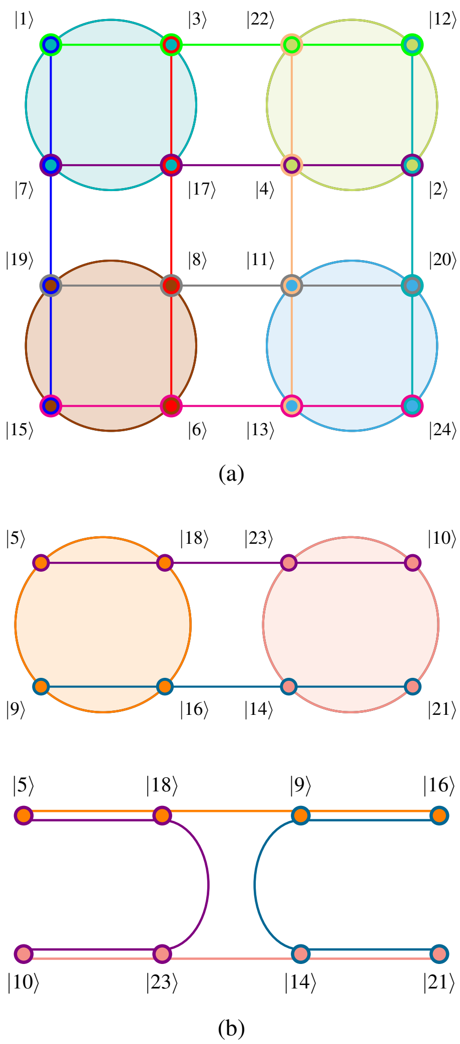

FIG. 2. (a) Hypergraph representing a 16-12 configuration—16 elements in 12 contexts (or cliques or orthonormal bases or maximal operators) enumerated in the first, second, fourth and fifth row of Table III as unbroken smooth lines, with corresponding vector labels. All these vectors are separable and thus correspond to factorizable, noentangled states. (b) Two equivalent hypergraph representations of a 8-4 configuration—8 elements in 4 contexts (or cliques or orthonormal bases or maximal operators) enumerated in the third and sixth row of Table III. The filled shaded small ovals on the edges correspond to the 'primary' isolated (nonintertwined) contexts from the matrix pencil calculations enumerated in Table I. The respective vectors are nonseparable and thus correspond to entangled states.

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* karl.svozil@tuwien.ac.at; http://tph.tuwien.ac.at/~svozil

[1] A. Cabello, Physical Review Letters **127**, 070401 (2021), arXiv:2011.13790, URL `https://link.aps.org/doi/10.1103/PhysRevLett.127.070401`.

[2] A. Cabello, J. M. Estebaranz, and G. García-Alcaine, Physics Letters A **212**, 183 (1996), arXiv:quant-ph/9706009, URL `https://doi.org/10.1016/0375-9601(96)00134-X`.

[3] A. Cabello, Physical Review A **55**, 4109 (1997), URL `https://doi.org/doi/10.1103/PhysRevA.55.4109`.

[4] A. Cabello, P. Badziag, M. Terra Cunha, and M. Bourennane, Physical Review Letters **111**, 180404 (2013), URL `https://doi.org/10.1103/PhysRevLett.111.180404`.

[5] L. Hardy, Physical Review Letters **68**, 2981 (1992), URL `http://dx.doi.org/10.1103/PhysRevLett.68.2981`.

[6] L. Hardy, Physical Review Letters **71**, 1665 (1993), URL `http://dx.doi.org/10.1103/PhysRevLett.71.1665`.

[7] A. Cabello, J. R. Portillo, A. Solís, and K. Svozil, Physical Review A **98**, 012106 (2018), arXiv:1805.00796, URL `https://doi.org/10.1103/PhysRevA.98.012106`.

[8] K. Svozil, Physical Review A **103**, 022204 (2021), arXiv:2006.11396, URL `https://doi.org/10.1103/PhysRevA.103.022204`.

[9] H. Hertz, *The principles of mechanics presented in a new form* (MacMillan and Co., Ltd., London and New York, 1899), with a foreword by H. von Helmholtz, translated by D. E. Jones and J. T. Walley, URL `https://archive.org/details/principlesofmech00hertuoft`.

[10] C. Budroni, A. Cabello, O. Gühne, M. Kleinmann, and J.-A. Larsson, Reviews of Modern Physics **94**, 045007 (2022), arXiv:2102.13036, URL `https://https://doi.org/10.1103/RevModPhys.94.045007`.

[11] P. R. Halmos, *Finite-Dimensional Vector Spaces*, Undergraduate Texts in Mathematics (Springer, New York, 1958), ISBN 978-1-4612-6387-6,978-0-387-90093-3, URL `https://doi.org/10.1007/978-1-4612-6387-6`.

[12] J. von Neumann, Annalen der Mathematik (Annals of Mathematics) **32**, 191 (1931), URL `https://doi.org/10.2307/1968185`.

[13] R. A. Horn and C. R. Johnson, *Matrix Analysis* (Cambridge University Press, New York, NY, 2013), 2nd ed., ISBN 9780521839402,9780521548236,9781139785884, URL `https://www.cambridge.org/9780521548236`.

[14] R. P. Nordgren, *Simultaneous diagonalization and SVD of commuting matrices* (2020), arXiv:2006.16364, URL `https://doi.org/10.48550/arXiv.2006.16364`.

[15] A. Zeilinger, Foundations of Physics **29**, 631 (1999), URL `https://doi.org/10.1023/A:1018820410908`.

[16] G. Birkhoff and J. von Neumann, Annals of Mathematics **37**, 823 (1936), URL `https://doi.org/10.2307/1968621`.

[17] A. Peres, Physics Letters A **151**, 107 (1990), URL `https://doi.org/10.1016/0375-9601(90)90172-k`.

[18] D. N. Mermin, Physical Review Letters **65**, 3373 (1990), URL `https://doi.org/10.1103/PhysRevLett.65.3373`.

[19] A. Peres, Journal of Physics A: Mathematical and General **24**, L175 (1991), URL `https://doi.org/10.1088/0305-4470/24/4/003`.

[20] M. Pavičić, J.-P. Merlet, B. McKay, and N. D. Megill, Journal of Physics A: Mathematical and General **38**, 1577 (2005),

arXiv:quant-ph/0409014, URL https://doi.org/10.1088/0305-4470/38/7/013.
[21] M. Pavičić, AIP Conference Proceedings **734**, 195 (2004), URL https://doi.org/10.1063/1.1834414.
[22] M. Pavičić, N. D. Megill, and J.-P. Merlet, Physics Letters A **374**, 2122 (2010), URL https://doi.org/10.1016/j.physleta.2010.03.019.
[23] K. Svozil, Theoretical Computer Science **560**, 82 (2014), ISSN 0304-3975, arXiv:0903.0231, URL https://doi.org/10.1016/j.tcs.2014.09.019.
[24] M. Pavičić, *Exhaustive search of 24-24 configurations* (2024), private communication in an email, April 3rd, 2024.
[25] J. Tkadlec, International Journal of Theoretical Physics **39**, 921 (2000), URL https://doi.org/10.1023/A:1003695317353.
[26] M. Pavičić, J.-P. Merlet, B. D. McKay, and N. D. Megill, Journal of Physics A: Mathematical and General **38**, 1577 (2005), URL https://doi.org/10.1088/0305-4470/38/7/013.
[27] M. Pavičić, J.-P. Merlet, B. D. McKay, and N. D. Megill, Journal of Physics A: Mathematical and General **38**, 3709 (2005), URL https://doi.org/10.1088/0305-4470/38/16/c01.
[28] M. Pavičić and N. D. Megill, Entropy **20** (2018), ISSN 1099-4300, program code at https://puh.srce.hr/s/Qegixzz2BdjYwFL, accessed on June 12th, 2020, arXiv:1905.01567, URL https://doi.org/10.3390/e20120928.
[29] S. Kochen and E. P. Specker, Journal of Mathematics and Mechanics (now Indiana University Mathematics Journal) **17**, 59 (1967), ISSN 0022-2518, URL https://doi.org/10.1512/iumj.1968.17.17004.
[30] D. N. Mermin, Physics Today **43**, 9 (1990), URL https://doi.org/10.1063/1.2810588.
[31] J.-W. Pan, D. Bouwmeester, M. Daniell, H. Weinfurter, and A. Zeilinger, Nature **403**, 515 (2000), URL https://doi.org/10.1038/35000514.
[32] K. Svozil, Foundations of Physics **52**, 1 (2022), arXiv:2006.14623, URL https://doi.org/10.1007/s10701-021-00515-z.
[33] D. W. Cohen, *An Introduction to Hilbert Space and Quantum Logic*, Problem Books in Mathematics (Springer, New York, 1989), ISBN 978-1-4613-8841-8,978-1-4613-8843-2, URL https://doi.org/10.1007/978-1-4613-8841-8.
[34] L. Lovász, IEEE Transactions on Information Theory **25**, 1 (1979), ISSN 0018-9448, URL https://doi.org/10.1109/TIT.1979.1055985.
[35] A. Solís-Encina and J. R. Portillo, *Orthogonal representation of graphs* (2015), arXiv:1504.03662, URL https://doi.org/10.48550/arXiv.1504.03662.
[36] S. Filipp and K. Svozil, Physical Review Letters **93**, 130407 (2004), arXiv:quant-ph/0403175, URL https://doi.org/10.1103/PhysRevLett.93.130407.