(Non)contextual coloring of orthogonality hypergraphs

Mohammad Hadi Shekarriz and Karl Svozil*
Institute for Theoretical Physics, Vienna University of Technology, Wiedner Hauptstrasse 8-10/136, 1040 Vienna, Austria
(Dated: November 20, 2020)

Chromatic constructions on orthogonality hypergraphs which are classical set representable or have a faithful orthogonal representation are discussed. The latter ones have a quantum mechanical realization as a context or maximal observable.

PACS numbers: 03.65.Ca, 02.50.-r, 02.10.-v, 03.65.Aa, 03.67.Ac, 03.65.Ud
Keywords: Quantum mechanics, Gleason theorem, Kochen-Specker theorem, Born rule, gadget graphs

I. NOMENCLATURE

In what follows w eshall use the following terms will be used synonymuously: context, block, (Boolean) subalgebra, (maximal) clique, complete graph.

Greechie has suggested [1] to (amendmends are indicated by square brackets "[...]")

> [...] present [...] lattices as unions of [contexts] intertwined or pasted together in some fashion [...] by replacing, for example, the 2^n elements in the Hasse diagram of the power set of an n-element set with the [context aka] complete graph [K_n] on n elements. The reduction in numbers of elements is considerable but the number of remaining "links" or "lines" is still too cumbersome for our purposes. We replace the [context aka] complete graph on n elements by a single smooth curve (usually a straight line) containing n distinguished points. Thus we replace $n(n+1)/2$ "links" with a single smooth curve. This representation is propitious and uncomplicated provided that the intersection of any pair of blocks contains at most one atom.

In what follows we shall refer to such a hypergraph representation as Greechie diagram [2].

We shall concentrate on Greechie diagrams which are pasting [3] constructions [4, Chapter 2] of a homogenuous single type of contexts K_n where the clique number n is fixed. In what follows we shall refer to the Greechie diagrammatical representation of such, possibly intertwined, collection of blocks, as *hypergraph*.

A hypergraph coloring is a uniform coloring which associates n mutually different colors to every atomic element of each context. That is, the n distinguished points of any single smooth curve in the hypergraph have n different colors. The coloring is noncontextual; that is, the coloring of atomic elements common to two or more contexts (intertwining there) is independent of the context. In requirng uniformity we shall implicitly also exclude partial colorings [5–7] whers partiality is understood as allowing for undefinedness in the sense of Kleene [8].

* svozil@tuwien.ac.at; http://tph.tuwien.ac.at/~svozil

The chromatic number m of a hypergraph is the minimal number of mutually different colors in any coloring of this hypergraph. It is bound from below by the clique number n. If these numbers are the same, that is, if $n = m$, then one could obtain two-valued measures from colorings by "projecting" one of the colors into the value 1, and all the other $m-1$ colors into the value 0 [9–11].

Finite examples for which the chromatic number exceeds the clique number, that is, $m > n$, are the logical structures involved in proofs of the Kochen-Specker theorem. Explicit constructions are, for instance, Γ_2 of Ref. [12], as well as the configurations enumerated in Fig. 9 of [13], Fig. 1–3 of [14], Ref. [15], as well as Table I, Fig. 2 of Ref. [7], among numerous others which have a faithful orthogonal representation [16–18] in "small dimensions" greater than two.

II. CHROMATIC CONSTRUCTION FROM TWO-VALUED STATES

Conjecture: the following statements are equivalent:

(i) The chromatic number of the hypergraph equals the clique number n; that is, the associated graph is colorable by n distinct colors.

(ii) The set of two-valued states contains n states which correspond to a partitioning of all elements of the partition logic; the equivalence relation defined by each one of these n states evaluating to 1 on some element of every context. That is, those n states are 1 on different atoms of every context.

Conjecture: the following statements are equivalent:

(i) Whenever the chromatic number of the hypergraph equals the clique number n, then this collection of (intertwined) contexts possesses a separating set of two-valued states. This means that it is homomorphically embeddable into a Boolean algebra [12, Theorem 0], which in turn means that it is set representable as a partition logic [19].

(ii) There exists a (nonunique) "canonical construction" of a partition logic from its set of two-valued states [20] facilitating such a coloring with n colors.

III. BABYLONIAN EVIDENCE [21] BY ANECDOTAL (COUNTER-)EXAMPLES

A (nonunique) coloring can (at least for some hypergraphs) be effectively constructed as follows:

(1) Choose two arbitrary contexts $C_1 = \{a_1, \ldots, a_n\}$ and $C_2 = \{b_1, \ldots, b_n\}$ of the logic (represented by the respective hypergraph).

(2) In C_1 choose an arbitrary atom, say a_i; and identify the first color χ_1 with any single one non-vanishing (on a_i) two-valued state $s_j(a_i) = 1$ of choice. Moreover, assign a_i this first color χ_1.

(3) Associate within C_2 the unique single atom b_j for which s_j does not vanish – that is, $s_j(b_j) = 1$ – and assign b_j the first color χ_1.

(4) Discard all two-valued states s which:

 (4.1) either do not vanish on a_i; that is, only states $s \in S$ remain with $s(a_i) = 0$. [By the assumption of separability the set of remaining states is not empty (this is crucial).]

 (4.2) or do not vanish on any atom e_{kl} on each context C_k (there will be one atom e_{kl} per context C_k) for which $s_j(e_{kl}) = 1$; that is, only states $s \in S$ remain with $s(e_{kl}) = 0$ for all contexts labelled by k. [Note: (4.1) is a subcase of (4.2), so (4.1) is redundant and only (4.2) needs to be mentioned.]

(5) Then repeat (2) and choose a second atom $a_{i'}$ from C_1; and associate the second color χ_2 with any single one non-vanishing (on $a_{i'}$) remaining two-valued state $s_{j'}(a_{i'}) = 1$ of choice.

(6) Then repeat (3) and associate within C_2 those singe atom $b_{j'}$ for which $s_{j'}$ does not vanish – that is, $s_{j'}(b_{j'}) = 1$ – and color it with the second color χ_2.

 By construction (elimination of all two-valued states which do not vanish on all previously colored atoms) this atom $b_{j'}$ must be different from b_j (both b_j as well as $b_{j'}$ are in C_2).

(7) Then repeat (4).

(8) Repeat (2)-(4) until all atoms of the first context C_1 are covered.

This completes the coloring of allatoms of the hypergraph, and thus the direct proof.

A. Examples

1. Triangle logic

The coloring procedure of the triangle hypergraph is depicted in Fig. 1. Consider the set of all four two-valued states

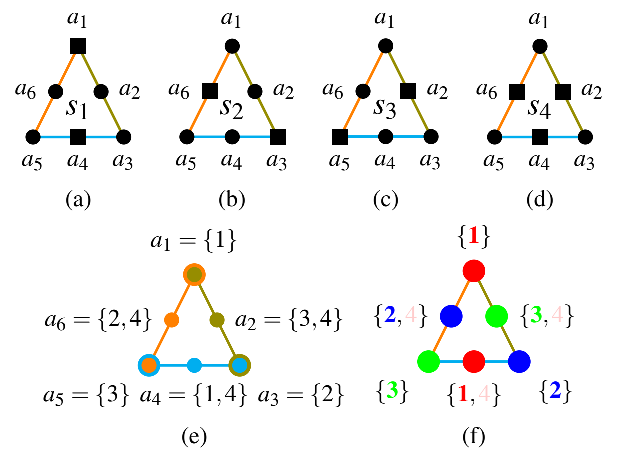

FIG. 1. One (nonunique) coloring (f) construction of the triangle hypergraph of the logic: first compose a (nonunique) canonical partition logic (e) from enumerating the set of all 4 two-valued states depicted in (a)–(d). Then chose the context $\{a_1, a_2, a_3\}$, and from this context choose the atom $a_1 = \{1\}$. Now identify the first color (red) with the index 1, thereby identifying $a_1 = \{1\}$ as well as $a_4 = \{1,4\}$ with red. Then delete the index number 4 from every atom; that is, $a_2 = \{3,4\} \to \{3\}$ and $a_6 = \{2,4\} \to \{2\}$. Finally identify 3 with the second color (green) and 2 with the third color (blue), thereby identifying a_2 and a_5 with green, and a_3 and a_6 with blue, respectively. Note that s_1, s_2, and s_3 "generate" a 3-partitioning of the set of atoms $\{a_1, \ldots, a_6\}$ of this logic.

on the six atoms which can be tabulated by a (compactified) Travis [22] matrix [4] T_{ij} whose rows indicate the ith state s_i and whose columns indicate the atoms a_j, respectively; that is, $T_{ij} = s_i(a_j)$:

$$T_{ij} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}. \tag{1}$$

It is not too dificult to see that the first three measures, represented by the first three row vectors of the Travis matrix, add up to $\left(1,1,1,1,1,1\right)$. They can thus be taken as the basis of a coloring.

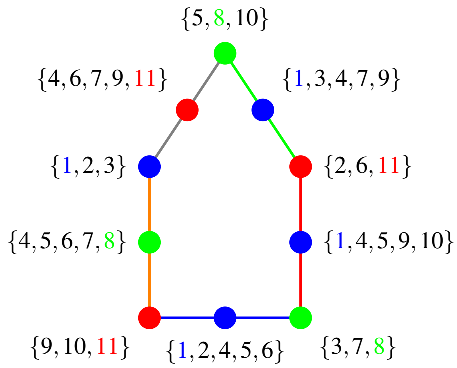

FIG. 2. Coloring scheme of the house/pentagon/pentagram logic from the set of two-valued states.

2. *House/pentagon/pentagram logic*

The Travis matrix of the house/pentagon/pentagram logic is a matrix representation of its 11 dispersion free states [23]

$$T_{ij} = \begin{pmatrix} 1&0&0&1&0&1&0&1&0&0\\ 1&0&0&1&0&0&1&0&0&0\\ 1&0&0&0&1&0&0&1&0&0\\ 0&1&0&1&0&1&0&1&0&1\\ 0&1&0&1&0&1&0&0&1&0\\ 0&1&0&1&0&0&1&0&0&1\\ 0&1&0&0&1&0&0&1&0&1\\ 0&1&0&0&1&0&0&0&1&0\\ 0&0&1&0&0&1&0&1&0&1\\ 0&0&1&0&0&1&0&0&1&0\\ 0&0&1&0&0&0&1&0&0&1 \end{pmatrix}. \qquad (2)$$

A coloring can be constructed with the earlier mentioned construction which results in three states partitioning all 10 atoms. The associated 1st, the8th and the 11th row vectors of T_{ij} are partitioning the 10 atoms.

3. *"Specker bug" gadget*

The hypergraph depicted in Fig. 3 is a minimal [24] true-implies false gadget introduced by Kochen and Specker [25, Fig. 1, p. 182] (see also [26, Fig. 1, p. 123], among others). It is a subgraph of G_{32} introduced later in Fig. 4. Its Travice

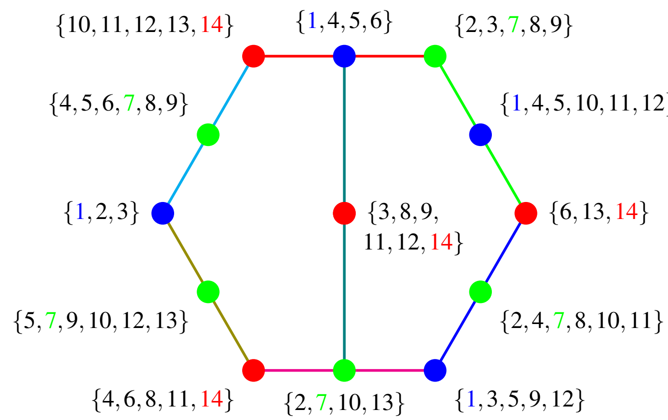

FIG. 3. Coloring scheme of the "Specker bug" gadget [25, 26] from two-valued states. The set theoretic representation is in terms of the canonical partition logic as an equipartitioning of the set $\{1,2,\ldots,14\}$ obtained from all 14 two-valued states on this gadget.

matrix is

$$T_{ij} = \begin{pmatrix} 1&0&0&1&0&1&0&0&1&0&0&0&0\\ 1&0&0&0&1&0&0&1&0&1&0&0&0\\ 1&0&0&0&1&0&0&0&1&0&0&0&1\\ 0&1&0&1&0&1&0&1&0&0&1&0&0\\ 0&1&0&1&0&1&0&0&1&0&0&1&0\\ 0&1&0&1&0&0&1&0&0&0&1&0&0\\ 0&1&0&0&1&0&0&1&0&1&0&1&0\\ 0&1&0&0&1&0&0&1&0&0&1&0&1\\ 0&1&0&0&1&0&0&0&1&0&0&1&1\\ 0&0&1&0&0&1&0&1&0&1&0&1&0\\ 0&0&1&0&0&1&0&1&0&0&1&0&1\\ 0&0&1&0&0&1&0&0&1&0&0&1&1\\ 0&0&1&0&0&0&1&0&0&1&0&1&0\\ 0&0&1&0&0&0&1&0&0&0&1&0&1 \end{pmatrix}. \qquad (3)$$

IV. INHERITED CHROMATIC PROPERTIES

A. Color-(in)separability of nonadjacent atoms

An intertwined combo of Specker bugs – Kochen & Specker's Γ_3 [12] – is still colorable because the two bugs therein "inherit" the colorings of the single bugs. Yet, any set of such colorings is no longer *separable* in that two nonadjacent ("complementary") atoms allow different colors.

B. Colorings from non-unital set of two-valued states

In a similar way, non-unital sets of two-valued states "fix" the variety of colorings by allowing only a single color at certain atoms.

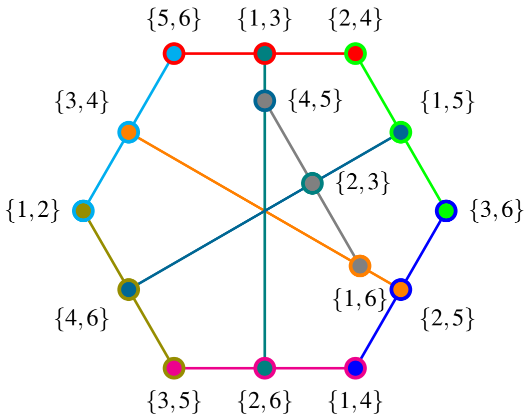

FIG. 4. Greechie diagram of G_{32} introduced by Greechie [1, Fig. 6, p. 121]. The overlaid set theoretic representation is in terms of the canonical partition logic as an equipartitioning of the set $\{1,2,3,4,5,6\}$ obtained from all 6 two-valued states on G_{32}.

C. Counterexamples

1. *Greechie's* G_{32}

It is quite straightforward to demonstrate that the logic G_{32} introduced by Greechie [1, Fig. 6, p. 121] (see also Refs. [26–29]) whose hypergraph is depicted in Fig. 4 has a chromatic number larger than three; and, in particular, cannot be colored by two-valued states. Consider the set of all six two-valued states which can be tabulated by the Travis matrix

$$T_{ij} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}. \tag{4}$$

There is no way how three of these six row vectors add up to a vector whose components are all one; that is, $(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1)$. "Completing" the partition logic and "extending" G_{32} by adding five more contexts $\{\{1,2\},\{3,6\},\{4,5\}\}$, $\{\{1,4\},\{2,3\},\{5,6\}\}$, $\{\{1,3\},\{2,5\},\{4,6\}\}$, $\{\{1,5\},\{2,6\},\{3,4\}\}$, and $\{\{1,6\},\{2,4\},\{3,5\}\}$ does not change the set of two-valued states and thus the Travis matrix.

Another way of seing this is to associate a color to, say, the first state. As a consequence, all other states, namely states number 2, 3, 4, 5, and 6, need to be eliminated, leaving no state which can be associated with another color.

ACKNOWLEDGMENTS

The author acknowledges the support by the Austrian Science Fund (FWF): project I 4579-N and the Czech Science Foundation (GAČR): project 20-09869L.

The author declares no conflict of interest.

I kindly acknowledge enlightening discussions with Adan Cabello, José R. Portillo, and Mohammad Hadi Shekarriz. I am grateful to Josef Tkadlec for providing a *Pascal* program which computes and analyses the set of two-valued states of collections of contexts. All misconceptions and errors are mine.

[1] R. J. Greechie, Journal of Combinatorial Theory. Series A **10**, 119 (1971), URL https://doi.org/10.1016/0097-3165(71)90015-X.

[2] G. Kalmbach, *Orthomodular Lattices*, vol. 18 of *London Mathematical Society Monographs* (Academic Press, London and New York, 1983), ISBN 0123945801,9780123945808.

[3] R. J. Greechie, Journal of Combinatorial Theory **4**, 210 (1968), URL https://doi.org/10.1016/s0021-9800(68)80002-x.

[4] R. J. Greechie, Ph.D. thesis, University of Florida, Florida, USA (1996), URL https://ufdc.ufl.edu/UF00097858/00001/pdf.

[5] A. A. Abbott, C. S. Calude, and K. Svozil, Mathematical Structures in Computer Science **24**, e240303 (2014), ISSN 1469-8072, arXiv:1012.1960, URL https://doi.org/10.1017/S0960129512000692.

[6] A. A. Abbott, C. S. Calude, and K. Svozil, Physical Review A **89**, 032109 (2014), arXiv:1309.7188, URL https://doi.org/10.1103/PhysRevA.89.032109.

[7] A. A. Abbott, C. S. Calude, and K. Svozil, Journal of Mathematical Physics **56**, 102201 (2015), arXiv:1503.01985, URL https://doi.org/10.1063/1.4931658.

[8] S. C. Kleene, Mathematische Annalen **112**, 727 (1936), ISSN 1432-1807, URL https://doi.org/10.1007/BF01565439.

[9] C. D. Godsil and J. Zaks, *Coloring the sphere* (1988, 2012), University of Waterloo research report CORR 88-12, arXiv:1201.0486, URL https://arxiv.org/abs/1201.0486.

[10] D. A. Meyer, Physical Review Letters **83**, 3751 (1999), arXiv:quant-ph/9905080, URL https://doi.org/10.1103/PhysRevLett.83.3751.

[11] H. Havlicek, G. Krenn, J. Summhammer, and K. Svozil, Journal of Physics A: Mathematical and General **34**, 3071 (2001), arXiv:quant-ph/9911040, URL https://doi.org/10.1088/0305-4470/34/14/312.

[12] S. Kochen and E. P. Specker, Journal of Mathematics and Mechanics (now Indiana University Mathematics Journal) **17**, 59 (1967), ISSN 0022-2518, URL https://doi.org/10.1512/iumj.1968.17.17004.

[13] K. Svozil and J. Tkadlec, Journal of Mathematical Physics **37**,

5380 (1996), URL https://doi.org/10.1063/1.531710.
[14] J. Tkadlec, International Journal of Theoretical Physics **39**, 921 (2000), URL https://doi.org/10.1023/A:1003695317353.
[15] A. Cabello, J. M. Estebaranz, and G. García-Alcaine, Physics Letters A **212**, 183 (1996), arXiv:quant-ph/9706009, URL https://doi.org/10.1016/0375-9601(96)00134-X.
[16] L. Lovász, IEEE Transactions on Information Theory **25**, 1 (1979), ISSN 0018-9448.
[17] L. Lovász, M. Saks, and A. Schrijver, Linear Algebra and its Applications **114-115**, 439 (1989), ISSN 0024-3795, special Issue Dedicated to Alan J. Hoffman, URL https://doi.org/10.1016/0024-3795(89)90475-8.
[18] A. Solís-Encina and J. R. Portillo, *Orthogonal representation of graphs* (2015), arXiv:1504.03662, URL https://arxiv.org/abs/1504.03662.
[19] M. Schaller and K. Svozil, Il Nuovo Cimento B **109**, 167 (1994), URL https://doi.org/10.1007/BF02727427.
[20] K. Svozil, International Journal of Theoretical Physics **44**, 745 (2005), arXiv:quant-ph/0209136, URL https://doi.org/10.1007/s10773-005-7052-0.
[21] O. Neugebauer, *Vorlesungen über die Geschichte der antiken mathematischen Wissenschaften. 1. Band: Vorgriechische Mathematik* (Springer, Berlin, Heidelberg, 1934), ISBN 978-3-642-95096-4,978-3-642-95095-7, URL https://doi.org/10.1007/978-3-642-95095-7.
[22] R. D. Travis, Master's thesis, Wayne State University, Detroit, Michigan, USA (1962), Master's Thesis under the supervision of David J. Foulis.
[23] R. Wright, in *Mathematical Foundations of Quantum Theory*, edited by A. R. Marlow (Academic Press, New York, 1978), pp. 255–274, ISBN 9780323141185, URL https://www.elsevier.com/books/mathematical-foundations-of-quantum-theory/marlow/978-0-12-473250-6.
[24] A. Cabello, J. R. Portillo, A. Solís, and K. Svozil, Physical Review A **98**, 012106 (2018), arXiv:1805.00796, URL https://doi.org/10.1103/PhysRevA.98.012106.
[25] S. Kochen and E. P. Specker, in *The Theory of Models, Proceedings of the 1963 International Symposium at Berkeley* (North Holland, Amsterdam, New York, Oxford, 1965), pp. 177–189, ISBN 0720422337,9781483275345, reprinted in Ref. [30, pp. 209-221], URL https://www.elsevier.com/books/the-theory-of-models/addison/978-0-7204-2233-7.
[26] R. J. Greechie, Synthese **29**, 113 (1974), URL https://doi.org/10.1007/bf00484954.
[27] S. S. Holland, in *The Logico-Algebraic Approach to Quantum Mechanics: Volume I: Historical Evolution*, edited by C. A. Hooker (Springer Netherlands, Dordrecht, 1975), pp. 437–496, ISBN 978-94-010-1795-4, URL https://doi.org/10.1007/978-94-010-1795-4_25.
[28] M. K. Bennett, SIAM Review **12**, 267 (1970), URL https://doi.org/10.1137/1012047.
[29] R. J. Greechie, in *Logic and Probability in Quantum Mechanics*, edited by P. Suppes (Springer Netherlands, Dordrecht, 1976), pp. 105–119, URL https://doi.org/10.1007/978-94-010-9466-5.
[30] E. Specker, *Selecta* (Birkhäuser Verlag, Basel, 1990), ISBN 978-3-0348-9966-6,978-3-0348-9259-9, URL https://doi.org/10.1007/978-3-0348-9259-9.