Farewell to quantum contextuality?

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A *principle of limited quantum noncontuality* is formulated. Experimental tests based on entanglement of the orbital angular momentum states of photons [1–3] as well as on multiport beam splitters [4–6] are proposed. Problems associated with a direct operationalization of the configurations employed for proofs of the Kochen-Specker theorem will be discussed; in particular the impossibility of (entangled) states allowing an "explosion view" of the proof.

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Contextuality [7–9] contends that, in Bell's words [7, Sect. 5], the "*... result of an observation may reasonably depend not only on the state of the system ... but also on the complete disposition of the apparatus*." This concept has been developed in reaction to the theorems by Gleason [10–12], Bell [7, 13–15] and Kochen and Specker [7, 16–26], and rests upon the notion that counterfactual observables are not physically irrelevant. (Cf. also the scholastic "infuturabilities" discussed by Specker [16].)

Classical Boole-Kolmogorovian probability theory is based on the assumption that the probability of the occurrence of pair-wise "disjoint" independent events e_i, $i = 1, 2, \ldots$ is the sum of those probabilities, and that the probability is bounded by zero and one; i.e., $0 \leq P(e_1, e_2, \ldots) = P(e_1) + P(e_2) + \cdots \leq 1$. For Hilbert spaces of dimension greater than two, Gleason's theorem ensures that, as long as these properties persist for *comeasurable* events associatiated with commuting observables, then the Born-Von Neumann rule of computing quantum probabilities and expectation values [27] hold; i.e., the expectation value of a self-adjoint operator A is $\langle A\rangle = \mathrm{Trace}(\rho A)$ with ρ being a (self-adjoint, positive semidefinite, and of trace class) quantum state operator. Stimulated by this result, Specker [16] and subsequently others developed finite, constructive proofs of the inconsistency and thus nonexistence of classical truth values for certain propositions representable by systems of tightly interconnected tripodes (or higher dimensional orthogonal bases). Formally, the Kochen-Specker theorem amounts to the "scarcity" and even nonexistence of dispersionless two-valued states, whose "abundance" would be required for a structurally faithful (homomorphic) embedding of the associated quantum propositions into a classical Boolean algebra. Pointedly stated, quantum systems do not seem to be able to accommodate "most" (potentially counterfactual) observables, save but a tiny fraction of maximal comeasurable ones. It appears natural and even evident (albeit not necessary) to hold that the lack of value definiteness of certain quantum observables implied by the Kochen-Specker theorem and the violation of Bell inequalities could be explained by their dependence on the global measurement context.

In what follows, the operational content of the concept of contextuality will be studied in some detail. We shall first analyze specific configurations which can be operationalized, and then turn to the general problem to empirically test contextuality for nonclassical configurations of (counterfactual) observables which do not allow a classical interpretation.

Proofs of the Kochen-Specker theorem consider systems of orthogonal tripods (bases) in real Hilbert spaces of dimension three interlinked at one common leg. The simplest such system consists of just two orthogonal tripods, whose common leg is located along the x_3-axis; the other orthogonal legs both lie in the x_1-x_2–plane, such as the tripods spanned by the two bases $\{(1,0,0)^T,(0,1,0)^T,(0,0,1)^T\}$ and $\{(\cos\varphi,0,0)^T,(0,\sin\varphi,0)^T,(0,0,1)^T\}$, there the superscript "T" indicates transposition. This configuration is depicted in Fig. 1a), together with its representation in a Greechie (orthogonality) diagram [28] in Fig. 1b), which represents orthogonal tripods by points symbolizing individual legs that are connected by smooth curves. Thereby, every context can formally be identified with a single *maximal* nondegenerate hermitean operator, of which the single projecors are functions [20, 29].

For the above configuration, noncontextuality should hold; i.e., intuitively the outcome of a measurement of an observable associated with the ray and thus the projector along $x_3 = x_3'$ should be indifferent to the choice of the context $C \equiv \{x_1, x_2, x_3\}$ or $C' \equiv \{x_1', x_2', x_3'\}$. More precisely, consider a rotation of $\varphi = \pi/4$ along the x_3-axis in $x_1 - x_2$-plane, such that $x_1' = (1/\sqrt{2})(1,1,0)^T$, $x_2' = (1/\sqrt{2})(-1,1,0)^T$, and let the maximal context operators be $C = \alpha[{x_1}^T, x_1] + \beta[{x_2}^T, x_2] + \gamma[{x_3}^T, x_3]$, $C' = \alpha[{x_1'}^T, x_1'] + \beta[{x_2'}^T, x_2'] + \gamma[{x_3}^T, x_3]$, with $\alpha \neq \beta \neq \gamma \neq \alpha$; $[x^T, x] \equiv |x\rangle\langle x|$ represents the dyadic product of the vector x with itself.

Consider the observables corresponding to projectors along the common direction represented by the projector $[{x_3}^T, x_3] \propto E_z C = E_z C' \propto [{x_3'}^T, x_3']$. Then, for all states ρ, the quantum expectation values $\mathrm{Trace}(\rho E_z C) = \mathrm{Trace}(E_z C \rho) = \mathrm{Trace}(\rho E_z C') = \mathrm{Trace}(E_z C' \rho)$ are the same and independent of the choice of the maximal context operators C and C'.

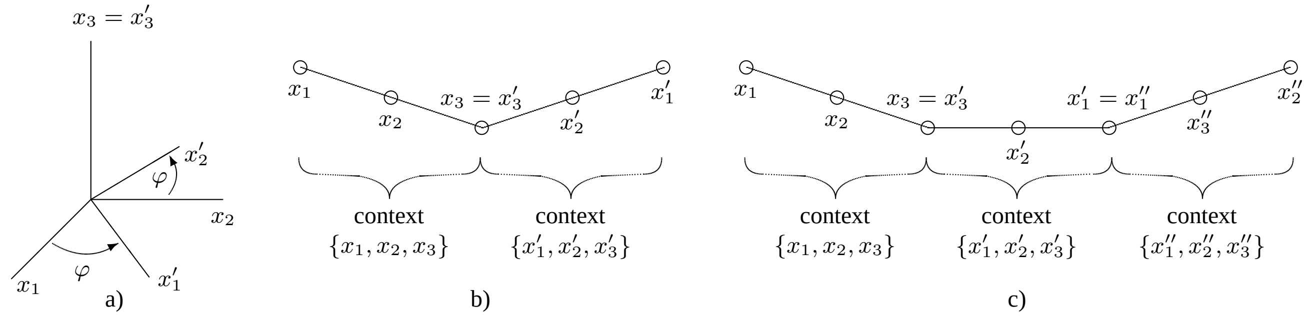

FIG. 1: a) Two tripods with a common leg spanning two measurement contexts; b) Greechie (orthogonality) diagram: points stand for individual basis vectors, and orthogonal tripods are drawn as smooth curves; c) Greechie diagram of three tripods interconnected at two legs.

More explicitly, for arbitrary ρ,

$$
\begin{aligned}
E_z C\rho &= \operatorname{diag}(0,0,1)\cdot\operatorname{diag}(\alpha,\beta,\gamma)\cdot\rho = \gamma\rho_{33},\\
E_z C'\rho &= \tfrac{1}{2}\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix} \alpha+\beta & \alpha-\beta & 0\\ \alpha-\beta & \alpha+\beta & 0\\ 0 & 0 & \gamma\end{pmatrix}\rho = \gamma\rho_{33}.
\end{aligned}
\tag{1}
$$

This holds for single particles as well as for correlated particles; in particular for the singlet state of two spin-1 particles [30, 31] $|\Psi_2\rangle = (1/\sqrt{3})(|+-\rangle + |-+\rangle - |00\rangle)$. When measuring the two contexts C and C' and thus also the projections corresponding to $x_3 = x_3'$ at two locations, the outcomes of the latter observables should be independent of the contexts. Due to the lack of experimental realizations of singlet states of spin-1 particles, a realization in terms of multiport interferometers [4, 5] is proposed; in [6], the explicit setup for preparation and measurement of C as well as of C' is enumerated. Tha corresponding interferometric setup is depicted in Fig. 2

A generalization of the above argument for arbitrary dimensions and an arbitrary number of observables yields the *principle of limited quantum noncontextuality:* Given a set of contexts in which one or more observables $A, \ldots$ coincide. Then, within that set of contexts, the outcomes of $A, \ldots$ do not depend on the context; i.e., which other observables are measured alongside.

Note that the Kochen-Specker set of contexts has an empty set of coinciding observables. Indeed, already a three-tripod configuration $\{x_1, x_2, x_3 = x_3'\} - \{x_3 = x_3', x_2', x_1' = x_1''\} - \{x_1' = x_1'', x_2'', x_3''\}$ as depicted in Fig. 1c), with two interconnections has an empty set of contexts and thus cannot be directly operationalized.

Early on, Kochen [8] has suggested to consider entangled multiparticle systems measured at different locations. In its extreme form, this approach may yield an "explosion view" of the Kochen-Specker proof by requiring (at least) one particle per context; all these particles should be "suitable" entangled (see below) to allow a counterfactual inference similar to the Einstein-Podolsky-Rosen (EPR) argument. For instance, a physical realization, if it existed, of the observables in Peres' form of the Kochen-Specker proof [23, 25] would require $N = 40$ different, interconnected contexts and thus an entangled state of just as many particles.

In order to be able to reach conclusions at one location about the elements of physical reality [32] at all the other $N-1$ locations, the following *uniqueness property* must hold: the N particles must be in a state Ψ such that (i) Ψ is invariant under the unitary transformations u^N (identical transformations u for every particle), while at the same time meeting the requirement that (ii) a partial measurement at only one location must fix a *unique* term in the expansion of Ψ. The uniqueness property asserts that measurement of the state of one particle in an N-particle (entangled) state defines the state of all the other $N-1$ particles instantaneously, although neither particle needs to possess its own well-defined state before the measurement.

As a group theoretic argument [33] shows, this is impossible for the spin-1 multipartite ($N > 2$) case. Indeed, already for $N = 3$, the only singlet state is (see also [34])

$$
|\Psi_3\rangle = \frac{1}{\sqrt{6}}(|-+0\rangle - |-0+\rangle + |+0-\rangle - |+-0\rangle + |0-+\rangle - |0+-\rangle). \tag{2}
$$

From (2) it can be inferred that any partial measurement at only one location cannot fix a unique term in the expansion of Ψ_3. Suppose, for instance, that the first particle is measured to be in state "$-$;" then the second and third particle may either be in the state "0" or in state "$+$" in that particular direction. Thus, due to the nonuniqueness property, the counterfactual inference of a context $\{x_1, x_2, x_3\}$ in the three-particle spin-1 case (2) is impossible, since if the property corresponding to x_3 is fixed, the properties corresponding to x_2 and x_3 need not be—indeed, they are only fixed if one more particle spin is measured in the *same*

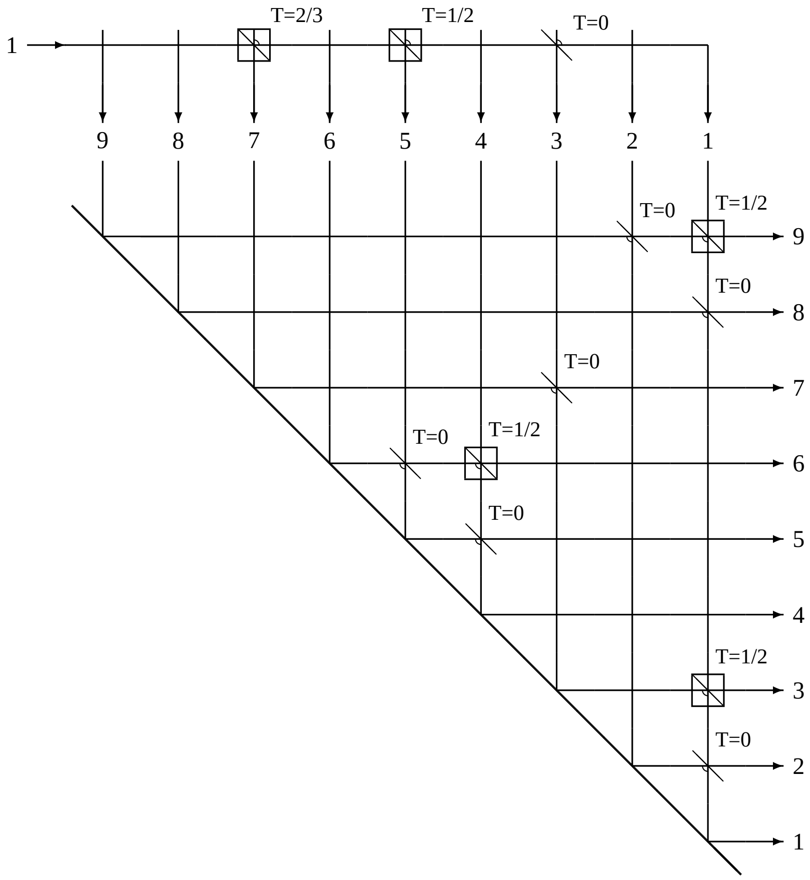

FIG. 2: Experimental setup for a multiport interferometric analogue of the measurement of two three-state particles in the singlet state, measured by C, C' in Eq. (1) and depicted in Fig. 1a-b). T denotes the transmittance of the beam splitters. The upper part represents the preparation stage, the lower part the analyzation stage. In the analizing stage, not all beam splitters are required for the singlet state.

direction as the spin measurement of x_1. Again, Ref. [6] contains a complete construction of a realization in terms of multiport interferometers.

Note that the uniqueness property holds for $N = 2$; i.e., for the two spin-1 particle singlet state Ψ_2. For $N = 4$, there are three different singlet states. One of these states is given by

$$|\Psi_4^1\rangle = \frac{1}{\sqrt{9}} \left(|+-+-\rangle + |-++-\rangle - |00+-\rangle + |+--+\rangle + |-+-+\rangle - |00-+\rangle - |+-00\rangle - |-+00\rangle + |0000\rangle \right), \tag{3}$$

for which again the uniqueness property does not hold. For $N = 5, 6, 7, 8, 9, 10, 11, 12$, the number of singlet states is 6, 15, 36, 91, 232, 603, 1585, 4213, 11298, 30537, 83097, 227475, respectively, In general, despite the abundance of singlet states, the number of coherent terms in the sum at least doubles with any additional particle, thus contributing to the multitude of possibilities which spoil the uniqueness required for value definiteness. This nonuniqueness seems to be the way quantum mechanics “avoids” inconsistency forced upon it by the assumption of value definiteness in the Kochen-Specker argument (which is a proof by contradiction).

Suppose one insisted on measuring all observables in the explosion view of the Kochen-Specker argument. This will result in the measurement values $v_1, \ldots, v_N$, $v_i \in \{+, -, 0\}$ for $i = 1, \ldots, N$ different measurement directions. From these findings it cannot be inferred that every single one of the N particles has the properties $v_1, \ldots, v_N$; only one of the properties being measured, the others inferred counterfactually. This is due to the fact that, because of failure of the uniqueness property, such a counterfactual inference is impossible. Nonetheless, this does not exclude that certain (non)singlet states of multipartite systems, for which the uniqueness property holds, can be utilized for the sake of similar arguments than the Kochen-Specker proof. In variants of the Greenberger-Horne-Zeilinger theorem [35–37], such three- and four-particle states with the uniqueness property have been used to derive complete contradictions by a counterfactual argument. In none of these arguments, contextuality is an essential feature.

In summary and stated pointedly, despite its appeal as a metaphor, quantum contextuality defies operationalization. Unlike complementarity, randomness and interference, it seems to be no valuable computational resource in quantum information and computation theory. We are thus led to the conclusion that, as far as experimental tests are possible, quantum mechanics is noncontextual, and as far as contextuality is suggested as a possible quantum mechanical feature, it does not refer to any operationalizable scheme.

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