Joint Operators and Eigensystems for CHSH Inequality

This document provides the joint measurement operators and their eigensystems for the optimal angles achieving the maximal CHSH violation of $2\sqrt{2}$. The operators are defined for Alice's measurements at $\theta_A = 0°$ (σ_z) and $90°$ (σ_x), and Bob's measurements at $\theta_B = 45°$ and $-45°$.

Single-Qubit Operators

- $A_1 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.
- $A_2 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
- $B_1 = \frac{1}{\sqrt{2}}(\sigma_z + \sigma_x) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$.
- $B_2 = \frac{1}{\sqrt{2}}(\sigma_z - \sigma_x) = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$.

Joint Operators

The joint operators are tensor products acting on the two-qubit Hilbert space (basis $|00\rangle, |01\rangle, |10\rangle, |11\rangle$).

- $A_1 \otimes B_1$:

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

- $A_1 \otimes B_2$:

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

- $A_2 \otimes B_1$:

$$\begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \end{pmatrix}$$

- $A_2 \otimes B_2$:

$$\begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \end{pmatrix}$$

Eigensystems

Each joint operator has eigenvalues $+1$ and -1, each with multiplicity 2. The eigenvectors are constructed from the single-qubit eigenvectors:

- σ_z: $\lambda = +1$: $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$; $\lambda = -1$: $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.
- σ_x: $\lambda = +1$: $|+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$; $\lambda = -1$: $|-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

- B_1: $\lambda = +1$: $|v_1\rangle = \frac{1}{\sqrt{4-2\sqrt{2}}}\begin{pmatrix} 1 \\ \sqrt{2}-1 \end{pmatrix}$; $\lambda = -1$: $|v_2\rangle = \frac{1}{\sqrt{4+2\sqrt{2}}}\begin{pmatrix} 1 \\ -1-\sqrt{2} \end{pmatrix}$.
- B_2: $\lambda = +1$: $|w_1\rangle = \frac{1}{\sqrt{4-2\sqrt{2}}}\begin{pmatrix} 1 \\ 1-\sqrt{2} \end{pmatrix}$; $\lambda = -1$: $|w_2\rangle = \frac{1}{\sqrt{4+2\sqrt{2}}}\begin{pmatrix} 1 \\ 1+\sqrt{2} \end{pmatrix}$.
- $A_1 \otimes B_1$:
  - Eigenvalue $+1$:
$$\frac{1}{\sqrt{4-2\sqrt{2}}}\begin{pmatrix} 1 \\ \sqrt{2}-1 \\ 0 \\ 0 \end{pmatrix}, \quad \frac{1}{\sqrt{4+2\sqrt{2}}}\begin{pmatrix} 0 \\ 0 \\ 1 \\ -1-\sqrt{2} \end{pmatrix}.$$
  - Eigenvalue -1:
$$\frac{1}{\sqrt{???}\sqrt{4+2\sqrt{2}}}\begin{pmatrix} 1 \\ -1-\sqrt{2} \\ 0 \\ 0 \end{pmatrix}, \quad \frac{1}{\sqrt{4-2\sqrt{2}}}\begin{pmatrix} 0 \\ 0 \\ 1 \\ \sqrt{2}-1 \end{pmatrix}.$$
- $A_1 \otimes B_2$:
  - Eigenvalue $+1$:
$$\frac{1}{\sqrt{4-2\sqrt{2}}}\begin{pmatrix} 1 \\ 1-\sqrt{2} \\ 0 \\ 0 \end{pmatrix}, \quad \frac{1}{\sqrt{4+2\sqrt{2}}}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1+\sqrt{2} \end{pmatrix}.$$
  - Eigenvalue -1:
$$\frac{1}{\sqrt{4+2\sqrt{2}}}\begin{pmatrix} 1 \\ 1+\sqrt{2} \\ 0 \\ 0 \end{pmatrix}, \quad \frac{1}{\sqrt{4-2\sqrt{2}}}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1-\sqrt{2} \end{pmatrix}.$$
- $A_2 \otimes B_1$:
  - Eigenvalue $+1$:
$$\frac{1}{\sqrt{8-4\sqrt{2}}}\begin{pmatrix} 1 \\ \sqrt{2}-1 \\ 1 \\ \sqrt{2}-1 \end{pmatrix}, \quad \frac{1}{\sqrt{8+4\sqrt{2}}}\begin{pmatrix} 1 \\ -1-\sqrt{2} \\ -1 \\ 1+\sqrt{2} \end{pmatrix}.$$
  - Eigenvalue -1:
$$\frac{1}{\sqrt{8+4\sqrt{2}}}\begin{pmatrix} 1 \\ -1-\sqrt{2} \\ 1 \\ -1-\sqrt{2} \end{pmatrix}, \quad \frac{1}{\sqrt{8-4\sqrt{2}}}\begin{pmatrix} 1 \\ \sqrt{2}-1 \\ -1 \\ 1-\sqrt{2} \end{pmatrix}.$$
- $A_2 \otimes B_2$:
  - Eigenvalue $+1$:
$$\frac{1}{\sqrt{8-4\sqrt{2}}}\begin{pmatrix} 1 \\ 1-\sqrt{2} \\ 1 \\ 1-\sqrt{2} \end{pmatrix}, \quad \frac{1}{\sqrt{8+4\sqrt{2}}}\begin{pmatrix} 1 \\ 1+\sqrt{2} \\ -1 \\ -1-\sqrt{2} \end{pmatrix}.$$
  - Eigenvalue -1:
$$\frac{1}{\sqrt{8+4\sqrt{2}}}\begin{pmatrix} 1 \\ 1+\sqrt{2} \\ 1 \\ 1+\sqrt{2} \end{pmatrix}, \quad \frac{1}{\sqrt{8-4\sqrt{2}}}\begin{pmatrix} 1 \\ 1-\sqrt{2} \\ -1 \\ \sqrt{2}-1 \end{pmatrix}.$$

Comparison of Eigenvectors Across CHSH Joint Operator Eigensystems

This analysis examines whether eigenvectors from the eigensystems of the joint operators $A_1 \otimes B_1$, $A_1 \otimes B_2$, $A_2 \otimes B_1$, and $A_2 \otimes B_2$ are identical or orthogonal across different eigensystems, for the optimal CHSH measurement angles achieving the maximal violation $2\sqrt{2}$.

Eigenvectors

The eigenvectors for each operator are as follows (in the basis $|00\rangle, |01\rangle, |10\rangle, |11\rangle$):

- $A_1 \otimes B_1$:
  - Eigenvalue $+1$:

$$|v_{11}^+\rangle = \frac{1}{\sqrt{4-2\sqrt{2}}} \begin{pmatrix} 1 \\ \sqrt{2}-1 \\ 0 \\ 0 \end{pmatrix}, \quad |v_{12}^+\rangle = \frac{1}{\sqrt{4+2\sqrt{2}}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1-\sqrt{2} \end{pmatrix}.$$

  - Eigenvalue -1:

$$|v_{11}^-\rangle = \frac{1}{\sqrt{4+2\sqrt{2}}} \begin{pmatrix} 1 \\ -1-\sqrt{2} \\ 0 \\ 0 \end{pmatrix}, \quad |v_{12}^-\rangle = \frac{1}{\sqrt{4-2\sqrt{2}}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ \sqrt{2}-1 \end{pmatrix}.$$

- $A_1 \otimes B_2$:
  - Eigenvalue $+1$:

$$|v_{21}^+\rangle = \frac{1}{\sqrt{4-2\sqrt{2}}} \begin{pmatrix} 1 \\ 1-\sqrt{2} \\ 0 \\ 0 \end{pmatrix}, \quad |v_{22}^+\rangle = \frac{1}{\sqrt{4+2\sqrt{2}}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1+\sqrt{2} \end{pmatrix}.$$

  - Eigenvalue -1:

$$|v_{21}^-\rangle = \frac{1}{\sqrt{4+2\sqrt{2}}} \begin{pmatrix} 1 \\ 1+\sqrt{2} \\ 0 \\ 0 \end{pmatrix}, \quad |v_{22}^-\rangle = \frac{1}{\sqrt{4-2\sqrt{2}}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1-\sqrt{2} \end{pmatrix}.$$

- $A_2 \otimes B_1$:
  - Eigenvalue $+1$:

$$|v_{31}^+\rangle = \frac{1}{\sqrt{8-4\sqrt{2}}} \begin{pmatrix} 1 \\ \sqrt{2}-1 \\ 1 \\ \sqrt{2}-1 \end{pmatrix}, \quad |v_{32}^+\rangle = \frac{1}{\sqrt{8+4\sqrt{2}}} \begin{pmatrix} 1 \\ -1-\sqrt{2} \\ -1 \\ 1+\sqrt{2} \end{pmatrix}.$$

  - Eigenvalue -1:

$$|v_{31}^-\rangle = \frac{1}{\sqrt{8+4\sqrt{2}}} \begin{pmatrix} 1 \\ -1-\sqrt{2} \\ 1 \\ -1-\sqrt{2} \end{pmatrix}, \quad |v_{32}^-\rangle = \frac{1}{\sqrt{8-4\sqrt{2}}} \begin{pmatrix} 1 \\ \sqrt{2}-1 \\ -1 \\ 1-\sqrt{2} \end{pmatrix}.$$

- $A_2 \otimes B_2$:

– Eigenvalue $+1$:

$$|v_{41}^{+}\rangle = \frac{1}{\sqrt{8-4\sqrt{2}}}\begin{pmatrix} 1 \\ 1-\sqrt{2} \\ 1 \\ 1-\sqrt{2} \end{pmatrix}, \quad |v_{42}^{+}\rangle = \frac{1}{\sqrt{8+4\sqrt{2}}}\begin{pmatrix} 1 \\ 1+\sqrt{2} \\ -1 \\ -1-\sqrt{2} \end{pmatrix}.$$

– Eigenvalue -1:

$$|v_{41}^{-}\rangle = \frac{1}{\sqrt{8+4\sqrt{2}}}\begin{pmatrix} 1 \\ 1+\sqrt{2} \\ 1 \\ 1+\sqrt{2} \end{pmatrix}, \quad |v_{42}^{-}\rangle = \frac{1}{\sqrt{8-4\sqrt{2}}}\begin{pmatrix} 1 \\ 1-\sqrt{2} \\ -1 \\ \sqrt{2}-1 \end{pmatrix}.$$

Identical Vectors

No eigenvectors from different eigensystems are identical (up to a phase). The vectors have distinct component structures due to the different measurement bases (σ_z, σ_x, B_1, B_2), with differing normalization constants and non-zero component patterns (e.g., A_1-based vectors have two zeros, while A_2-based vectors have four non-zero components).

Orthogonal Vectors

Representative inner product calculations show that eigenvectors from different eigensystems are not orthogonal. For example:

- $\langle v_{11}^{+}|v_{21}^{+}\rangle \approx 1.366 \neq 0$.
- $\langle v_{11}^{+}|v_{31}^{+}\rangle \approx 0.686 \neq 0$.
- $\langle v_{31}^{+}|v_{41}^{+}\rangle \approx 1.354 \neq 0$.
- $\langle v_{31}^{+}|v_{42}^{-}\rangle \approx 0.323 \neq 0$.

The non-zero inner products arise because the eigenvectors involve components like $\sqrt{2}-1$ and $1-\sqrt{2}$, which do not cancel out across different operators. The operators do not commute in general, confirming that their eigenspaces are not orthogonal.

Conclusion

There are no identical or orthogonal eigenvectors across the eigensystems of different joint operators, due to the distinct measurement angles and resulting tensor product structures.

Eigensystem of the CHSH Operator

The CHSH operator is defined as $S = A_1 \otimes B_1 + A_1 \otimes B_2 + A_2 \otimes B_1 - A_2 \otimes B_2$, where:

- $A_1 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$,
- $A_2 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$,
- $B_1 = \frac{1}{\sqrt{2}}(\sigma_z + \sigma_x) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$,
- $B_2 = \frac{1}{\sqrt{2}}(\sigma_z - \sigma_x) = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$.

CHSH Operator

The operator S in the two-qubit basis $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ is:

$$S = \begin{pmatrix} \sqrt{2} & 0 & 0 & \sqrt{2} \\ 0 & -\sqrt{2} & \sqrt{2} & 0 \\ 0 & \sqrt{2} & -\sqrt{2} & 0 \\ \sqrt{2} & 0 & 0 & \sqrt{2} \end{pmatrix}.$$

Eigensystem

The eigenvalues and corresponding eigenvectors are:

- Eigenvalue $\lambda = 2\sqrt{2}$:

$$|v_1\rangle = \frac{1}{2\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 3 \\ 1 \end{pmatrix}.$$

- Eigenvalue $\lambda = -2\sqrt{2}$:

$$|v_2\rangle = \frac{1}{2\sqrt{3}} \begin{pmatrix} -3 \\ 1 \\ -1 \\ 1 \end{pmatrix}.$$

- Eigenvalue $\lambda = 0$ (multiplicity 2):

$$|v_3\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad |v_4\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$