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Extensions of Hardy-type true-implies-false gadgets to classically obtain indistinguishability

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In quantum logical terms, Hardy-type arguments can be uniformly presented and extended as collections of intertwined contexts and their observables. If interpreted classically, those structures serve as graph-theoretic “gadgets” that enforce correlations on the respective preselected and postselected observable terminal points. The method allows the generalization and extension to other types of relational properties, in particular, to novel joint properties predicting classical equality of quantum mechanically distinct observables. It also facilitates finding faithful orthogonal representations of quantum observables.

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I. CERTIFICATION OF NONCLASSICALITY

When it comes to certifying nonclassical observance of quantized systems, at least three types of approaches have been suggested: (i) Bell-type theorems, related to Boole’s “conditions of possible experience” for observables in disjoint [1–3] or intertwined [4–7] contexts (also known as the maximal collection of compatible observables organized in a Boolean subalgebra), present empirical evidence involving statistical terms which (due to complementarity) cannot be obtained simultaneously, but are obtained from sequential “one term at a time” measurements: whatever the sampling, the events contributing to each term need to be temporally (mutually) apart from the other terms. (ii) Kochen-Specker-type theorems [8–12] are theoretical proofs by contradiction employing finite sets of intertwined quantum observables [13] which have no classical interpretation in terms of two-valued (truth) assignments. Indirect empirical corroborations of Kochen-Specker-type theorems (and, thereby, quantum contextuality) amount to violations of local Bell-type classical predictions [6]. (iii) A third, statistical method [7,14] is based on preselected (prepared) quantum states which are sequentially measured or postselected in terms of suitably chosen quantum observables: prepare one state, measure another. Thereby, pre- and postselection (and their respective observables) are imagined to be logically connected by suitable finite collections of hypothetical counterfactual [15] intertwined contexts, with their choice being motivated by their predictive capacities and yet remaining arbitrary [16]. In particular, particles are prepared in such states and observable properties and are logically connected to other observables in certain ways, such that their (non)occurrence is classically mandatory, but quantum mechanically unrestricted, and occasionally violates the respective classical predictions. By choosing different patterns of connection, this method could be strengthened to the point that any singular outcome contradicts the respective

classical predictions. With this (counterfactual) adaptive modification, any such stochastic argument turns definite [16].

Common to all these cases is their reliance on complementary counterfactuals [15] because they suppose the simultaneous existence of more than one context: Except for common observables of intertwining [13] contexts, all observables in any such context are mutually complementary to all observables in a different context, and there is no physically feasible way of simultaneously measuring them.

For the sake of obtaining discrepancies between classical and quantum predictions, these conglomerates of contexts, and thereby the quantum observables they consist of, are interpreted “as if” they could have a classical interpretation. That is, a classical interpretation is forced upon such collections of (intertwining) observables. In quantum logic, a classical interpretation amounts to a two-valued (also known as 0 – 1, false-true) state (frame function [13]), which is context independent; that is, its value on observables does not depend on the particular context (also known as maximal observable [17], orthonormal basis) in which it occurs.

Kochen-Specker-type theorems employ configurations of intertwining contexts for which these classical interpretations (in terms of two-valued measures interpreted as dichotomic observables) fail: There does not exist any consistent classical interpretation for these conglomerates of contexts and the observables they hold.

Type-(iii) configurations exhibiting finite collections of observables (in intertwining contexts) may still allow classical interpretations, but the predictions based upon them statistically directly contradict the quantum predictions and without the need of type-(i) inequalities [18]. Indeed, Kochen and Specker used the latter [19] and constructed the former, stronger result [8] without explicitly mentioning the empirical opportunities of such configurations. Appendix B in Ref. [20], and Ref. [21], provide early discussions [7] of type-(iii) nonclassicality.

In what follows, we shall extend a particular type-(iii) instance involving two two-state particles, proposed by Hardy [18,22], to configurations that contain distinct quantum observables that cannot be “resolved” or distinguished

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classically. En route, we shall study configurations [14] of contexts enforcing classical predictions which are the “inverse” of the original relational properties resulting from Hardy-type arguments. The latter are often synonymously referred to as “Hardy’s theorem” [10,23], “Hardy’s proof” [14,24–26], or “Hardy’s paradox” [27,28]. “Hardy’s wonderful trick” [29], also called “Hardy’s beautiful example” [30], and attempts to make it accessible to a wider audience abound [31,32]. Nevertheless, a detailed account in terms of the quantum logical structure of the counterfactual argument results in a better comprehension of the resources and assumptions involved; in particular, when it comes to related proposals. Such an account suggests extensions to other type-(iii) configurations with different relational properties, and also allows empirical predictions by yielding a systematic way of finding quantum realizations, in particular, in regard to desiderata such as (in)distinguishability of the associated quantized entities.

II. HYPERGRAPH NOMENCLATURE

The counterfactual arguments (ii) and (iii) mentioned earlier can be depicted structurally transparent by the use of hypergraphs [7,14,33,34] introduced by Greechie, drawing contexts as smooth lines. [In what follows, we shall use the following terms synonymously, thereby having in mind the different areas in which they occur: context, maximal observable, orthonormal basis, block, (Boolean) subalgebra, (maximal) clique, and complete graph.] In particular, Greechie has suggested to (amendments are indicated by square brackets) [35, p. 120] “[...] present [...] lattices as unions of [contexts] intertwined or pasted together in some fashion [...] by replacing, for example, the 2^n elements in the Hasse diagram of the power set of an n -element set with the context complete graph $[K_n]$ on n elements. The reduction in numbers of elements is considerable but the number of remaining ‘links’ or ‘lines’ is still too cumbersome for our purposes. We replace the context complete graph on n elements by a single smooth curve (usually a straight line) containing n distinguished points. Thus we replace $n(n+1)/2$ ‘links’ with a single smooth curve. This representation is propitious and uncomplicated provided that the intersection of any pair of blocks contains at most one atom.”]

In what follows, we shall refer to such a general representation of observables as the (orthogonality) hypergraph [33]. The term should be understood in the broadest possible consistent sense. Most of our arguments will be in four-dimensional state space. An exception will be our mentioning the gadget¹ [36–38], called “Käfer” (German for “bug”) by Specker, which was introduced in 1965 [19] and used in the Kochen-Specker proof [8], serving as a true-implies-false configuration. The Specker bug is the three-dimensional analog of the four-dimensional Hardy configuration [14]. Its hypergraph is depicted in Fig. 1(b).

¹A clarification with regards to the use of the technical term “gadget” seems in order: this denomination is frequently used in graph theory to indicate “useful subgraphs.” It is not meant to be polemic.

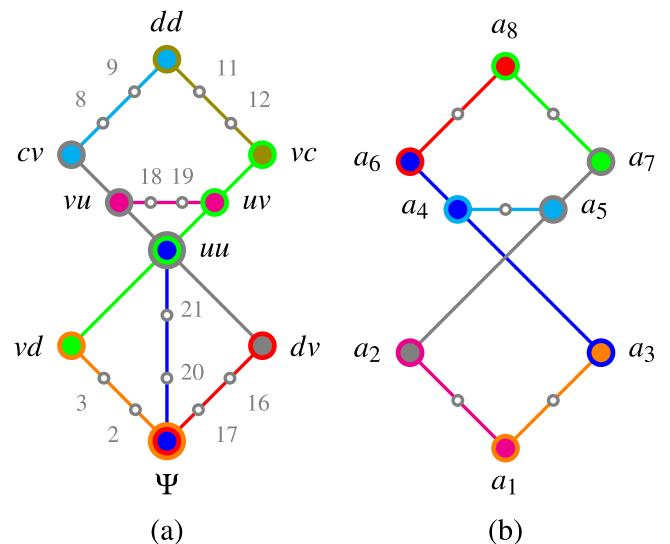


FIG. 1. Orthogonality hypergraphs of (a) the Hardy gadget with 8 contexts and 21 atoms $\{\{dd, 8, 9, cv\}, \{dd, 11, 12, vc\}, \{cv, vu, uu, dv\}, \{vu, 18, 19, uv\}, \{vd, 2, 3, \Psi\}, \{uu, 20, 21, \Psi\}, \{dv, 16, 17, \Psi\}\}$; (b) rendition of the true-implies-false Specker bug gadget with 7 contexts and 13 atoms $\{\{a_8, \dots, a_6\}, \{a_8, \dots, a_7\}, \{a_6, a_4, a_3\}, \{a_7, a_5, a_2\}, \{a_4, \dots, a_5\}, \{a_2, \dots, a_1\}, \{a_3, \dots, a_1\}\}$. Small circles indicate “auxiliary” observables which can be chosen freely, subject to orthogonality constraints: all smooth lines indicate respective contexts representing orthonormal bases. Larger circles indicate observables common to two or more intertwining contexts.

We shall concentrate on orthogonality hypergraphs which are pasting [39] constructions [40] of a homogeneous single type of contexts K_n , where the (maximal) clique number n is fixed. (Note that other authors use related definitions for Greechie diagrams [41] and McKay-Megill-Pavicic (MMP) diagrams [42].) If interpreted as representing some configuration of mutually orthogonal vectors, the (maximal) clique number n equals the dimension of the Hilbert space.

III. FAITHFUL ORTHOGONAL REPRESENTATIONS OF (HYPER)GRAPHS

In what follows, ket vectors, which are usually represented by column vectors, will be represented by the respective transposed row vectors. In all our examples, hypergraphs have a faithful orthogonal representation [43–46] in terms of vectors which are mutually orthogonal within (maximal) cliques or contexts. The phrases faithful orthogonal representation, coordinatization [47], or vector encoding will be used synonymously.

When drawing hypergraphs, some of the atomic propositions will be omitted (or only drawn lightly) if they are not essential to the argument. In particular, in three and four dimensions, given two orthogonal (in general, noncollinear) vectors, it is always possible to “complete” this partially defined context by a Gram-Schmidt process [34,48]. Indeed, given two (orthogonal) noncollinear vectors, in three dimensions the span of the “missing” vector is uniquely determined by the span of the cross product of those two vectors. (A

generalized cross product of $n - 1$ vectors in n -dimensional space can be written as a determinant; that is, in the form of a Levi-Civita symbol.) This “lack of freedom” is in one dimension; in particular, whenever the missing vector is collinear to some vector occurring in the faithful orthogonal representation of the incomplete hypergraph that one is attempting to complete. The most elementary such counterexample is a triangular hypergraph with three cyclically connected contexts $\{\{1, 2, 3\}, \{3, 4, 5\}, \{5, 6, 1\}\}$. Consider any incomplete faithful orthogonal of its intertwining atoms such as $1 = (0, 0, 1)$, $3 = (0, 1, 0)$, $5 = (1, 0, 0)$: any conceivable completion fails because the missing vectors would result in duplicates in the faithful orthogonal representation, that is, in $2 = 5$, $4 = 1$, and $6 = 3$.

Nevertheless, in four dimensions, given at least two (orthogonal) noncollinear vectors, the two-dimensional orthogonal subspace is spanned by a continuity of (e.g., rotated) bases. Therefore, in such a case, there is always “enough room for breathing”; that is, for accommodating the basis vectors and thereby transforming them if necessary such that any hypergraph can be properly completed without duplicates. I encourage the reader to try to find a faithful orthogonal representation of the cyclic triangular shaped hypergraph $\{\{1, \dots, 4\}, \{4, \dots, 7\}, \{7, \dots, 1\}\}$ in four dimensions.

Whether or not such faithful orthogonal representations can be given in terms of decomposable or indecomposable vectors associated with factorizable or entangled states is an entirely different issue. In four dimensions, a careful analysis [49] yields a no-go theorem for four-dimensional coordinatizations of the triangle hypergraph by allowing a maximal number seven of nine decomposable vectors.

In general and for arbitrary dimensions, as long as there are two or more “free” (without any strings and intertwining contexts attached) vectors per context missing from a faithful orthogonal representation of a hypergraph, its completion is always possible. Stated differently, any faithful orthogonal representation of an incomplete hypergraph can be straightforwardly extended (without reshuffling of vector components) to a faithful orthogonal representation in a completed hypergraph (e.g., by a Gram-Schmidt process) if at least two or more nonintertwining vectors per context in that hypergraph are missing. Indeed, one may even drop an already existing vector “blocking” a faithful orthogonal representation of an entire (hyper)graph if the associated atom is not intertwining in two or more contexts, and if the new freedom facilitates continuous bases instead of a single vector whose addition may result in duplicates through collinear vectors (we shall mention such an instance later).

In the case of two or more “missing” vectors, any completion involves a two- or higher-dimensional subspace. Any such subspace $\mathbb{R}^{k \geq n-2}$ or $\mathbb{C}^{k \geq n-2}$ of the n -dimensional continua \mathbb{R}^n or \mathbb{C}^n is spanned by a continuity of (orthogonal) bases. A typical example is an incomplete faithful orthogonal representation of a basis of \mathbb{R}^4 rotated into a form $\{(1, 0, 0, 0), (0, 1, 0, 0)\}$. Its completion is then given by the continuity of bases $\{(0, 0, \cos \theta, \sin \theta), (0, 0, -\sin \theta, \cos \theta)\}$, with $0 \leq \theta < \pi$.

A completion should even be possible if one merely allows sets of bases which are denumerable—or even finitely but “sufficiently” many bases with respect to the hyper-

graph encoded. From this viewpoint, four dimensions offer a much wider variety of completions as compared to the three-dimensional case—indeed, the difference results from the abundance offered by a continuum of bases versus a single vector spanning the respective subspaces, a fact which is very convenient for all kinds of constructions. However, as has been mentioned earlier, the completion of coordinatizations of hypergraphs by (in)decomposable vectors—in particular, if one desires to maintain (non)decomposability—is an altogether different issue [49].

This possibility to complete incomplete contexts is also the reason why practically all papers introducing and reviewing Hardy’s configuration operate not with the complete eight contexts including 21 atomic vertices, but merely with the nine vectors or vertices in which those eight contexts intertwine. Nevertheless, for tasks such as determining whether a particular configuration of observables supports or does not allow a classical two-valued state, as well as for determining the set of two-valued states and their properties (e.g., separable, unital), the nonintertwining atomic propositions matter.

IV. QUANTUM LOGICAL FORMULATION OF HARDY’S ARGUMENT

For the sake of being able to delineate Hardy’s rather involved original derivation [22], let us stick to his nomenclature as much as possible. We shall, however, drop the particle index as it is redundant. So, for instance, Hardy’s $|+\rangle_1|+\rangle_2$ will be written as $|+\rangle|+\rangle = |++\rangle$. Later, we shall be very explicit and identify the respective entities in terms of Hardy’s Ansatz, but let us study Hardy’s schematics in some generality first:

(i) It begins with a specific entangled state of two two-state particles $|\Psi\rangle$.

(ii) Then the argument suggests measuring two dichotomic (i.e., two-valued) observables \hat{U} (exclusive) or \hat{D} on each one of the two particles. This results in four measurement configurations $\hat{U} \otimes \hat{U}$, $\hat{U} \otimes \hat{D}\hat{D} \otimes \hat{U}\hat{D} \otimes \hat{D}$ —that is, effectively, the two-particle observable $\hat{U} \otimes \hat{D}$ is measured “in Einstein-Podolsky-Rosen (EPR) terms of” $\hat{U} \otimes \mathbb{I}_2$ and $\mathbb{I}_2 \otimes \hat{D}$.

(iii) As both of these dichotomic observables \hat{U} and \hat{D} have two possible outcomes called u and v for \hat{U} and c and d for \hat{D} , respectively, there are $2^2 \times 2^2 = 2^4 = 16$ different outcomes that are denoted by the ordered pairs uu , uv , uc , ud , vu , vv , vc , vd , cu , cv , cc , cd , du , dv , dc , and dd .

(iv) From these 16 outcomes, 5 groups of (incomplete if not all atoms or vertices are specified—yet, as discussed earlier, a completion is straightforward if desired) contexts, which consist of simultaneously measurable and mutually exclusive observables, can be formed, namely, $\{dd, \dots, cv\}$, $\{dd, \dots, vc\}$, $\{cv, vu, uu, dv\}$, $\{vc, uv, uu, vd\}$, and $\{vu, \dots, uv\}$.

(v) Finally, this collection of five contexts are “bundled with” or “tied to” the (projection) observable corresponding to the original entangled state $|\Psi\rangle$ introduced in (i) by the three (incomplete) contexts $\{vd, \dots, \Psi\}$, $\{uu, \dots, \Psi\}$, and $\{dv, \dots, \Psi\}$.

As a result, these (incomplete) contexts, if pasted [39] together at their respective intertwining observables, result in

TABLE I. Partition logic representing classical probabilities of the Hardy configuration [22], whose intertwined contexts are enumerated in Eq. (1), obtained from the separating or distinguishing set of all 186 two-valued states it supports. Note that the intersection of $\Psi \cap dd = \{1, 2, 3, 4, 5, 6\} \cap \{11, 16, 21, 26, 55, 60, 73, 78, 83, 88, 117, 122, 135, 140, 145, 150, 155, 164, 173, 182\} = \emptyset$ is empty, yielding true-implies-false relations among Ψ and dd , and vice versa, respectively.

$\Psi = \{1, 2, 3, 4, 5, 6\},$
$2 = \{7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39,$
$40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68\},$
$3 = \{69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106,$
$107, 108, 109, 110, 111, 112, 113, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129, 130\},$
$vd = \{131, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143, 144, 145, 146, 147, 148, 149, 150, 151, 152, 153, 154, 155, 156, 157, 158, 159, 160,$
$161, 162, 163, 164, 165, 166, 167, 168, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178, 179, 180, 181, 182, 183, 184, 185, 186\},$
$uu = \{7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88\},$
$vu = \{1, 2, 35, 36, 39, 40, 43, 44, 47, 48, 97, 98, 101, 102, 105, 106, 109, 110, 151, 152,$
$153, 154, 155, 160, 161, 162, 163, 164, 169, 170, 171, 172, 173, 178, 179, 180, 181, 182\},$
$cv = \{3, 4, 5, 6, 37, 38, 41, 42, 45, 46, 49, 50, 61, 62, 63, 64, 65, 66, 67, 68, 99, 100, 103, 104, 107, 108, 111, 112, 123, 124, 125, 126,$
$127, 128, 129, 130, 156, 157, 158, 159, 165, 166, 167, 168, 174, 175, 176, 177, 183, 184, 185, 186\},$
$8 = \{1, 7, 8, 12, 13, 17, 18, 22, 23, 27, 29, 31, 33, 35, 39, 43, 47, 51, 52, 56, 57, 69, 70, 74, 75, 79, 80, 84, 85, 89, 91, 93, 95, 97, 101, 105, 109, 113,$
$114, 118, 119, 131, 132, 136, 137, 141, 142, 146, 147, 151, 152, 160, 161, 169, 170, 178, 179\},$
$9 = \{2, 9, 10, 14, 15, 19, 20, 24, 25, 28, 30, 32, 34, 36, 40, 44, 48, 53, 54, 58, 59, 71, 72, 76, 77, 81, 82, 86, 87, 90, 92, 94, 96, 98, 102, 106, 110, 115,$
$116, 120, 121, 133, 134, 138, 139, 143, 144, 148, 149, 153, 154, 162, 163, 171, 172, 180, 181\},$
$dd = \{11, 16, 21, 26, 55, 60, 73, 78, 83, 88, 117, 122, 135, 140, 145, 150, 155, 164, 173, 182\},$
$11 = \{5, 7, 9, 12, 14, 17, 19, 22, 24, 51, 53, 56, 58, 61, 63, 65, 67, 69, 71, 74, 76, 79, 81, 84, 86, 113, 115, 118, 120, 123, 125, 127, 129, 131, 133, 136, 138,$
$141, 143, 146, 148, 151, 153, 156, 158, 160, 162, 165, 167, 169, 171, 174, 176, 178, 180, 183, 185\},$
$12 = \{6, 8, 10, 13, 15, 18, 20, 23, 25, 52, 54, 57, 59, 62, 64, 66, 68, 70, 72, 75, 77, 80, 82, 85, 87, 114, 116, 119, 121, 124, 126, 128, 130, 132, 134, 137, 139,$
$142, 144, 147, 149, 152, 154, 157, 159, 161, 163, 166, 168, 170, 172, 175, 177, 179, 181, 184, 186\},$
$vc = \{1, 2, 3, 4, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 89, 90, 91, 92, 93, 94, 95, 96, 97,$
$98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112\},$
$uv = \{5, 6, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 113, 114, 115, 116,$
$117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129, 130\},$
$dv = \{27, 28, 29, 30, 31, 32, 33, 34, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 89, 90, 91, 92, 93, 94, 95, 96, 113, 114, 115, 116, 117, 118, 119, 120, 121, 122,$
$131, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143, 144, 145, 146, 147, 148, 149, 150\},$
$16 = \{7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 35, 36, 37, 38, 39, 40, 41, 42, 61, 62, 63, 64, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 97, 98, 99, 100, 101,$
$102, 103, 104, 123, 124, 125, 126, 151, 152, 153, 154, 155, 156, 157, 158, 159, 160, 161, 162, 163, 164, 165, 166, 167, 168\},$
$17 = \{17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 43, 44, 45, 46, 47, 48, 49, 50, 65, 66, 67, 68, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 105, 106, 107, 108, 109,$
$110, 111, 112, 127, 128, 129, 130, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178, 179, 180, 181, 182, 183, 184, 185, 186\},$
$18 = \{3, 7, 8, 9, 10, 11, 17, 18, 19, 20, 21, 27, 28, 31, 32, 37, 41, 45, 49, 69, 70, 71, 72, 73, 79, 80, 81, 82, 83, 89, 90, 93, 94, 99, 103, 107, 111,$
$131, 132, 133, 134, 135, 141, 142, 143, 144, 145, 156, 157, 165, 166, 174, 175, 183, 184\}$

285 a collection of eight (incomplete) contexts,

$$\begin{aligned} \{\{dd, \dots, cv\} &= \{dd, 8, 9, cv\}, \\ \{dd, \dots, vc\} &= \{dd, 11, 12, vc\}, \\ \{cv, vu, uu, dv\} &= \{cv, vu, uu, dv\}, \\ \{vc, uv, uu, vd\} &= \{vc, uv, uu, vd\}, \\ \{vu, \dots, uv\} &= \{vu, 18, 19, uv\}, \\ \{vd, \dots, \Psi\} &= \{vd, 2, 3, \Psi\}, \\ \{uu, \dots, \Psi\} &= \{uu, 20, 21, \Psi\}, \\ \{dv, \dots, \Psi\} &= \{dv, 16, 17, \Psi\}, \end{aligned} \quad (1)$$

286 whose orthogonality hypergraph is depicted in Fig. 1(a).

A. Classical realization and predictions

288 In what follows, we shall prove the following:

289 (i) Hardy's configuration (1) allows a classical interpretation as it supports a distinguishing (often termed "separable")

291 set of two-valued states. A "canonical" classical representation will be explicitly enumerated.

292 (ii) All classical interpretations of Hardy's configuration
293 (1) enumerated in (i) predict that if the system is prepared in
294 state Ψ , then the observable dd never occurs. That is, Hardy's
295 setup is a gadget graph [36–38] with a "true-implies-false
296 (classical) set of two-valued states" (TIFS). Indeed, it is one
297 out of three minimal nonisomorphic true-implies-false config-
298 urations in four dimensions [14, Fig. 4(a)].

299 Hardy's configuration (1) allows a classical interpretation
300 because it supports a set of 186 two-valued states that distin-
301 guishes different observables from one another. This means
302 that the elements of every pair of distinct observables can be
303 "separated" or "distinguished" by (at least) one two-valued
304 state such that the respective state values of these elements
305 are different. Therefore, by Kochen and Specker's Theorem 0
306 [8], the structure of its observables can be embedded in some
307 Boolean algebra, which indicates classical representability.

308 An explicit construction of a classical model of a proposi-
309 tional structure corresponding to Hardy's 1993 configuration
310 [22] is enumerated in Table I. Its realization is in terms of

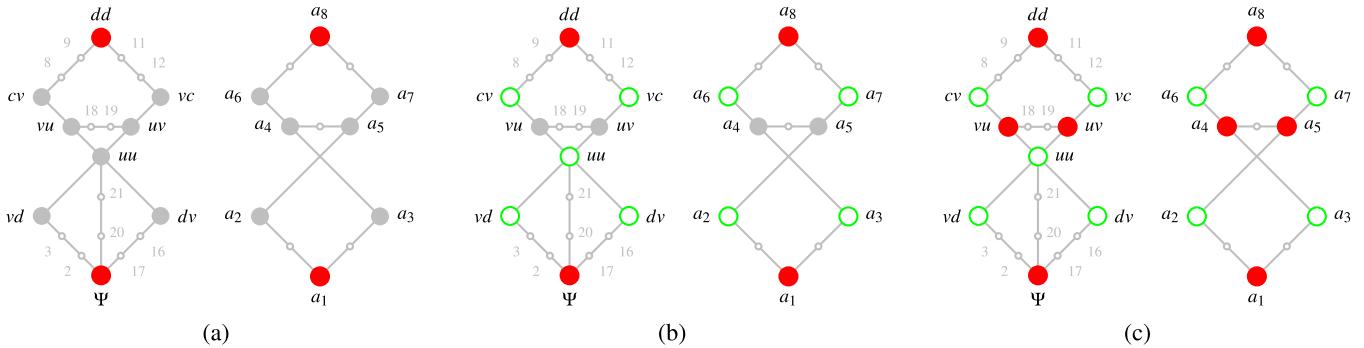


FIG. 2. Graphical presentation of a three-step proof by contradiction that from the pairs of observables $\{\Psi, dd\}$ and $\{a_1, a_8\}$, only one element can have assigned the classical value 1: (a) suppose otherwise; that is, $\Psi = dd = 1$ and $a_1 = a_8 = 1$; (b) then, by exclusivity, $vd = dv = uu = vc = cv = 0$ and $a_2 = a_3 = a_6 = a_7 = 0$; (c) then, by completeness, $vu = uv = 1$ and $a_4 = a_5 = 1$, contradicting exclusivity. Small circles indicate “auxiliary” observables which can be chosen freely, subject to orthogonality constraints: all smooth lines indicate respective contexts representing orthonormal bases.

312 eight partitions (corresponding to the eight contexts) of the
 313 index set $\{1, 2, \dots, 185, 186\}$ of 186 two-valued states. The
 314 elements of the partitions corresponding to the 21 atomic
 315 propositions [which are obtained by “completing” the context
 316 as enumerated in Eq. (1)] are the index sets of all two-valued
 317 states which obtain the value “1” on the respective atoms.
 318 A detailed description of this construction can be found in
 319 Refs. [7,50,51].

320 Next, we shall elaborate on a classical prediction which
 321 is violated by quantum predictions: If Ψ is assumed to be
 322 true—that is, if a classical system is prepared (also known as
 323 preselected) in the state corresponding to observable Ψ —then
 324 the outcome corresponding to the observable dd cannot occur.

325 For a proof by contradiction depicted in Fig. 2, (wrongly)
 326 suppose that both Ψ as well as dd were both true sim-
 327 taneously. Then, by the standard admissibility criteria for
 328 two-valued states [12,52] (also denoted as completeness and
 329 exclusivity [26,53,54]), $cv = vc = vd = dv = uu = 0$, en-
 330 forcing $vu = uv = 1$, which contradicts admissibility (com-
 331 pleteness and exclusivity).

332 The only remaining possibility is that ψ and dd have op-
 333 posite values if one of them is true (they still may both be 0).
 334 Therefore, any two-valued state for which Ψ is 1—that is, in
 335 which the observable corresponding to Ψ occurs—must clas-
 336 sically result in nonoccurrence of the outcome corresponding
 337 to the observable dd , and vice versa. This particular relation
 338 between the input and output ports of gadget graphs [16] has
 339 been called the 1-0 property [55], or one dominated by a
 340 true-implies-false set of two-valued states (TIFS) [14].

341 As mentioned earlier, the first true-implies-false gadget
 342 seems to have been introduced by Kochen and Specker
 343 ([19, Fig. 1] and used by them as a subgraph of Γ_1 [8] in
 344 three dimensions. Its orthogonality hypergraph is depicted in
 345 Fig. 1(b). As mentioned earlier, this gadget seems to have
 346 been independently discussed by, among others, Pitowsky,
 347 who called it the “cat’s cradle” [56,57]. See, also, Fig. 1 in
 348 [52] (reprinted in Ref. [58]), a subgraph in Fig. 21 in Ref. [59],
 349 Fig. B.1 in [20], [21], Fig. 2 in [60], and Fig. 2.4.6 in [61] for
 350 early discussions of the true-implies-false prediction.

351 The full nuances of the predictions are revealed when the
 352 classical probabilities are computed. As the classical prob-
 353 ability distributions are just the convex combinations of all

354 two-valued states [62], it is easy to read them off from the
 355 canonical partition logic enumerated in Table I. In particular,
 356 the true-implies-false gadget behavior at the terminals Ψ and
 357 dd can be directly read off from

$$\begin{aligned} P_\Psi &= \sum_{i \in \Psi} \lambda_i = \lambda_1 + \lambda_2 + \dots + \lambda_6, \\ P_{dd} &= \sum_{i \in dd} \lambda_i = \lambda_{11} + \lambda_{16} + \dots + \lambda_{182}, \\ \text{with } \lambda_i &\geq 0, \text{ and } \sum_{i=1}^{186} \lambda_i = 1. \end{aligned} \quad (2)$$

358 Since the intersection of the index sets Ψ and dd is empty,
 359 $P_{dd} = 0$ whenever $P_\Psi = 1$, and vice versa. For the sake of the
 360 example, all six two-valued measures assigning 1 to Ψ are
 361 depicted in Fig. 3.

362 One equivalent alternative way to characterize the classical
 363 probabilities completely would be to exploit the Minkowski-
 364 Weyl “main” representation theorem [63–69] and consider the
 365 classical convex polytope spanned by the 186 21-dimensional
 366 vectors whose components are the values in $\{0, 1\}$ of the
 367 two-valued states on the atomic propositions of the Hardy
 368 gadget. From these vertices (V-representation), the 35 half
 369 spaces that are the bounds of the polytope (H-representation)
 370 can be computed [1,62]. But due to space restrictions, we omit
 371 this discussion, although it might reveal quantum violations of
 372 Boole’s (classical) “conditions of experience” [3].

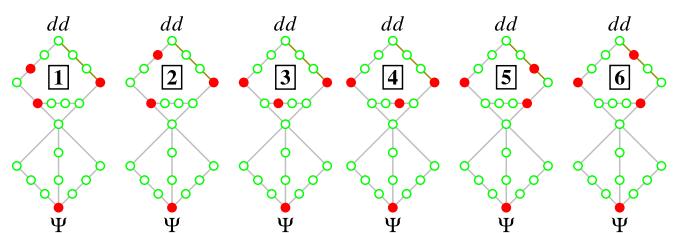


FIG. 3. Orthogonality hypergraphs of the Hardy gadget with overlaid six two-valued states which are 1 at Ψ .

373

B. Original quantum realization

Hardy's original quantum realization in terms of a particular type of faithful orthogonal representation is quite involved, but for the sake of delineating it, we shall mostly stick to the nomenclature of the 1993 paper [22]. Suppose two two-state particles and, for each one of the two particles, consider three orthonormal bases of its two-dimensional Hilbert space, namely,

$$\begin{aligned} B_1 &= \{|+\rangle, |-\rangle\} \equiv \{\mathbf{e}_1, \mathbf{e}_2\}, \\ B_2 &= \{|u\rangle, |v\rangle\} \equiv \{\mathbf{f}_1, \mathbf{f}_2\}, \\ B_3 &= \{|c\rangle, |d\rangle\} \equiv \{\mathbf{g}_1, \mathbf{g}_2\}. \end{aligned} \quad (3)$$

The components of the respective unitary transformations "rotating" these orthonormal bases into each other are defined by [70]

$$\begin{aligned} B_1 \leftrightarrow B_2 : \quad &\mathbf{f}_j = \mathbf{U}_{ji}^{12} \mathbf{e}_i, \quad \text{and } \mathbf{e}_j = (\mathbf{U}^{12})_{ji}^\dagger \mathbf{f}_i, \\ B_2 \leftrightarrow B_3 : \quad &\mathbf{g}_k = \sum_{i=1}^2 \mathbf{U}_{kj}^{23} \mathbf{f}_i, \quad \text{and } \mathbf{f}_k = \sum_{i=1}^2 (\mathbf{U}^{23})_{ki}^\dagger \mathbf{g}_i, \\ B_1 \leftrightarrow B_3 : \quad &\mathbf{g}_k = \sum_{i=1}^2 \mathbf{U}_{ki}^{13} \mathbf{e}_i = \sum_{i,j=1}^2 \mathbf{U}_{kj}^{23} \mathbf{U}_{ji}^{12} \mathbf{e}_i, \\ &\text{and } \mathbf{e}_k = \sum_{i,j=1}^2 (\mathbf{U}^{12})_{kj}^\dagger (\mathbf{U}^{23})_{ji}^\dagger \mathbf{g}_i. \end{aligned} \quad (4)$$

One further ingredient of Hardy's configuration is a pure entangled state of two two-state particles which can be parameterized by [71] (the relative order of states matter, therefore, as pointed out earlier, we shall omit a subscript referring to the first and second particle, respectively)

$$\begin{aligned} |\Psi\rangle &= \alpha|++\rangle - \beta|--\rangle \quad \text{with } \alpha, \beta \in \mathbb{R}, \\ \alpha^2 + \beta^2 &= \cos^2 \phi + \sin^2 \phi = 1 \quad \text{with } 0 \leq \phi \leq \pi/4. \end{aligned} \quad (5)$$

The minus sign (indicating a phase $\varphi = \pi$ for which $e^{i\varphi} = -1$) has been chosen for the sake of conforming to Hardy's conventions. Note that $0 \leq \alpha, \beta \leq 1$.

So far, the transformation matrices in (4) have not yet been specified, but in order for the argument to work, they should yield a faithful orthogonal representation of the orthogonality hypergraph depicted in Fig. 1(a). In particular, one needs to assure that

$$\langle \Psi | uu \rangle = \langle \Psi | vd \rangle = \langle \Psi | dv \rangle = 0. \quad (6)$$

At the same time, and in order to obtain a contradiction with the classical prediction "if Ψ is true then dd must be false" or, in physical terms, "if a system is prepared or (pre)selected in state Ψ then an event or outcome associated with dd cannot occur" (and vice versa), one needs to define those transformations such that, in addition to (6),

$$\langle \Psi | dd \rangle \neq 0 \quad \text{and "as great as possible".} \quad (7)$$

In order to facilitate these desiderata (6) and (7), suppose *ad hoc* that

$$\begin{aligned} (\mathbf{U}^{12})^\dagger &= -\frac{i}{\sqrt{\alpha + \beta}} \begin{pmatrix} \sqrt{\beta} & \sqrt{\alpha} \\ \sqrt{\alpha} & -\sqrt{\beta} \end{pmatrix}, \\ (\mathbf{U}^{23})^\dagger &= \frac{1}{\sqrt{1 - \alpha\beta}} \begin{pmatrix} \sqrt{\alpha\beta} & -\alpha + \beta \\ \alpha - \beta & \sqrt{\alpha\beta} \end{pmatrix}, \\ (\mathbf{U}^{13})^\dagger &= (\mathbf{U}^{12})^\dagger \cdot (\mathbf{U}^{23})^\dagger. \end{aligned} \quad (8)$$

Assume further, without loss of generality, that the first basis B_1 in (3) is identified with the Cartesian basis; that is, $|+\rangle = (1, 0)$ and $|-\rangle = (0, 1)$. Consequently, the vectors of the other bases B_2 and B_3 are obtained by applying the respective transformations (4) and (8),

$$\begin{aligned} |u\rangle &= \frac{i(\sqrt{1 - \alpha^2}, \sqrt{\alpha})}{\sqrt{\sqrt{1 - \alpha^2} + \alpha}}, \\ |v\rangle &= \frac{i(\sqrt{\alpha}, -\sqrt{1 - \alpha^2})}{\sqrt{\sqrt{1 - \alpha^2} + \alpha}}, \\ |c\rangle &= \frac{i(\alpha^{3/2}, (1 - \alpha^2)^{3/4})}{\sqrt{\alpha^3 - \sqrt{1 - \alpha^2}\alpha^2 + \sqrt{1 - \alpha^2}}}, \\ |d\rangle &= \frac{i((1 - \alpha^2)^{3/4}, -\alpha^{3/2})}{\sqrt{\alpha^3 - \sqrt{1 - \alpha^2}\alpha^2 + \sqrt{1 - \alpha^2}}}. \end{aligned} \quad (9)$$

We are only dealing with pure states represented as normalized vectors which are (the sum of) the Kronecker (that is, "delinedated" outer or tensor) products [72], e.g., $|\Psi\rangle = \alpha(1, 0) \otimes (1, 0) - \sqrt{1 - \alpha^2}(0, 1) \otimes (0, 1) = (\alpha, 0, 0, -\sqrt{1 - \alpha^2})$. The associated propositional observables can then be written in terms of the orthogonal projections formed as dyadic products $|x\rangle\langle x|$ of the unit (state) vectors $|x\rangle$.

By applying the transformations (4), $|\Psi\rangle$ can be rewritten in terms of either (i) the second basis B_2 for the first particle and the second basis B_2 for the second particle, (ii) the second basis B_2 for the first particle and the third basis B_3 for the second particle, (iii) the third basis B_3 for the first particle and the second basis B_2 for the second particle, or (iv) the third basis B_3 for the first particle and the third basis B_3 for the second particle. That is,

$$|\Psi\rangle = -|uv\rangle\sqrt{\alpha\beta} - |v\rangle[|u\rangle\sqrt{\alpha\beta} + |v\rangle(\alpha - \beta)] \quad (10)$$

$$= \frac{1}{\sqrt{1 - \alpha\beta}} [|uc\rangle(\sqrt{\alpha\beta^3} - \sqrt{\alpha^3\beta}) - |vc\rangle(\alpha^2 - \alpha\beta + \beta^2) - |ud\rangle\alpha\beta] \quad (11)$$

$$= \frac{1}{\sqrt{1 - \alpha\beta}} [|cu\rangle(\sqrt{\alpha\beta^3} - \sqrt{\alpha^3\beta}) - |cv\rangle(\alpha^2 - \alpha\beta + \beta^2) - |du\rangle\alpha\beta] \quad (12)$$

$$= -\frac{1}{1 - \alpha\beta} [|cc\rangle(\alpha - \beta)(\alpha^2 + \beta^2) + (|cd\rangle + |dc\rangle)(\alpha\beta)^{3/2} + |dd\rangle\alpha\beta(\beta - \alpha)]. \quad (13)$$

425 As can be readily read off from these representations of
 426 $|\Psi\rangle$, the conditions (6) and desideratum (7) are satisfied: (10)
 427 has no term proportional to $|uu\rangle$, (11) has no term proportional
 428 to $|vd\rangle$, (12) has no term proportional to $|dv\rangle$, and (13) has a
 429 term proportional to $|dd\rangle$.

430 To complete Hardy's original argument, we compare the
 431 classical prediction of "zero outcome" (nonoccurrence) for
 432 observable dd to the quantum prediction probability,

$$|\langle dd|\Psi\rangle|^2 = \left\{ \frac{\alpha[\alpha(\sqrt{1-\alpha^2} + \alpha) - 1]}{\alpha\sqrt{1-\alpha^2} - 1} \right\}^2, \quad (14)$$

433 obtained from preparing (also known as preselecting) two
 434 entangled particles in state $|\Psi\rangle$ and measuring (e.g., by post-
 435 selection) the nonvanishing probability to find them in state
 436 $|dd\rangle$ (thus contradicting the aforementioned classical predic-
 437 tions). $|\langle dd|\Psi\rangle|^2$ acquires its maximal value $\frac{1}{2}(5\sqrt{5} - 11) \approx$
 438 0.09 at $\alpha_{\pm} = \sqrt{1 \pm \sqrt{6\sqrt{5} - 13}}/\sqrt{2}$. This is slightly below
 439 the maximal violation of the three-dimensional "minimal"
 440 true-implies-false case [the Specker bug [7,14,19] depicted in
 441 Fig. 1(b)] with probability $1/9 \approx 0.1$ [20,34,59,73,74].

V. VARIETIES OF COORDINATIZATION

443 In what follows, we shall enumerate a few faithful or-
 444 thogonal representations of the Hardy gadget. Presently, no
 445 general analytic construction for finding even a single faithful
 446 orthogonal representation of a (hyper)graph (if any) exists, let
 447 alone a method for finding all such coordinatizations. Nev-
 448 ertheless, *ad hoc* faithful orthogonal representations can be
 449 generated in extenso by heuristic algorithms. With regards to
 450 (in)decomposability, the Hardy gadget allows almost all types
 451 of faithful orthogonal representations: "mixed" ones which
 452 have entangled as well as factorizable states, and ones which
 453 use entangled states. Entirely decomposable configurations
 454 are prohibited for geometric reasons.

455 From now on, the observables need not be formed by some
 456 sort of composition, and therefore two symbols such as " uv "
 457 should only be understood as a label. Note that indecom-
 458 posable vectors can be interpreted as pure entangled states.
 459 Likewise, decomposable vectors represent pure factorizable
 460 states. The coordinatizations will not be enumerated com-
 461 pletely, as only the intertwining vertices will be explicitly
 462 mentioned. Nevertheless, completions are straightforward and
 463 have been discussed earlier. Reference [49] contains a careful
 464 categorization with respect to (in)decomposability.

A. Mixed (in)decomposability

466 Previous parametrizations [10,18,22,24,27] of Hardy's
 467 (minimal with respect to the number of vertices in four dimen-
 468 sions [14]) true-implies-false gadget, depicted in Fig. 1(a), in
 469 terms of four-dimensional vectors appear to be motivated by
 470 high yield—that is, by maximizing the quantum predictions
 471 of occurrence of the output (postselection) port dd , as well as
 472 by (in)decomposability of the associated vectors. This is moti-
 473 vated by what is sometimes referred to as "demonstrations
 474 of nonlocal contextuality"; that is, the "spread" of the rela-
 475 tional information [75] among pairs of (spacelike) separated
 476 particles.

477 The first, mixed with respect to (in)decomposability of the
 478 vectors, type of coordinatization can almost directly be read
 479 off from the orthogonality hypergraph of the Hardy gadget
 480 depicted in Fig. 1(a). Note that the two "central full contexts"
 481 $\{|cv\rangle, |vu\rangle, |uu\rangle, |dv\rangle\}$ and $\{|vc\rangle, |uv\rangle, |uu\rangle, |vd\rangle\}$ intertwine
 482 at one common element $|uu\rangle$ and are actually "generated"
 483 by the flattened tensor products of two nonidentical two-
 484 dimensional contexts representable by the two orthonormal
 485 bases $\{|c\rangle, |d\rangle\}$ and $\{|u\rangle, |v\rangle\}$, respectively. Hence all that is
 486 necessary is to make sure that $|\Psi\rangle$ is orthogonal to three
 487 vectors $|vd\rangle$, $|uu\rangle$, and $|dv\rangle$ of four-dimensional space (and
 488 no multiplicities occur), as already encoded in Eqs. (6):

$$\begin{aligned} |\Psi\rangle \propto & (d_2 u_2^2 v_1 - 2d_2 u_1 u_2 v_2 + d_1 u_2^2 v_2, \\ & d_2 u_1^2 v_2 - d_1 u_2^2 v_1, \\ & d_2 u_1^2 v_2 - d_1 u_2^2 v_1, \\ & 2d_1 u_1 u_2 v_1 - d_2 u_1^2 v_1 - d_1 u_1^2 v_2), \end{aligned} \quad (15)$$

489 where x_i stands for the i th component of the vector x with
 490 respect to some basis common to all vectors.

491 In order to be able to claim nonlocality, additional
 492 constraints can be required from the components of $|\Psi\rangle$.
 493 Suppose one desires $|\Psi\rangle$ to be entangled. Then the prod-
 494 uct of its outer components should not be equal to the
 495 product of its inner components; that is, $\Psi_1 \Psi_4 \neq \Psi_2 \Psi_3$
 496 because every decomposable product state of two vec-
 497 tors with components (a, b) and (c, d) is of the (delin-
 498 eated) form $(x_1 = ac, x_2 = ad, x_3 = bc, x_4 = bd)$, so that
 499 because of commutativity of scalars, $x_1 x_4 = (ac)(bd) =$
 500 $abcd = (ad)(bc) = x_2 x_3$. If one prefers the tensor product in
 501 matrix notation, then $\begin{pmatrix} x_1 = ac & x_2 = ad \\ x_3 = bc & x_4 = bd \end{pmatrix}$ and the criterion
 502 for factorizability and decomposability is a vanishing deter-
 503 minant, $x_1 x_4 - x_2 x_3 = 0$. Applying this constraint to Eq. (15)
 504 results in

$$(d_2 u_1 - d_1 u_2)(u_1 v_2 - u_2 v_1) \neq 0. \quad (16)$$

505 The third and the fourth rows of Table II containing vectors
 506 present two *ad hoc* configurations satisfying this "indecom-
 507 posability" constraint.

B. Indecomposable configurations

508 The last two rows of Table II contain faithful orthogonal
 509 representations of the Hardy gadget in which all intertwining
 510 vectors are entangled (e.g., in the complex realization because
 511 the number of components of any vector with imaginary units
 512 i and $-i$ is odd; that is, either one or three). These coordi-
 513 natisations have been obtained with a heuristic algorithm
 514 developed by McKay *et al.* [47].

C. Impossibility of complete decomposability

516 Conversely, it might be desirable to keep $|\Psi\rangle$ decompos-
 517 able and factorizable; that is, all entities should be in a product
 518 state. In this case, the product of the outer components of
 519 $|\Psi\rangle$ should be equal to the product of its inner components;
 520 that is, $\Psi_1 \Psi_4 = \Psi_2 \Psi_3$. This results in the constraint $d_1 =$
 521 $d_2 u_1 / u_2$, with $u_2 \neq 0$ from Eq. (15), and consequently in a
 522 multiplicity of vectors; more explicitly, $d \propto u$. The resulting

TABLE II. Tabulation of some faithful orthogonal representations of the Hardy gadget. Two missing vectors per context as well as normalizations can be completed with a little effort. Labels of the form “ $a\bar{b}$ ” should not be understood as product states, but have been used merely to conform to Hardy’s original nomenclature.

	ψ	dv	vd	uu	vu	cv	vc	uc	dd
CEG-A 1996 [10]	(1, -1, -1, 0)	(1, 0, 1, 0)	(1, 1, 0, 0)	(0, 0, 0, 1)	(0, 0, 1, 0)	(0, 1, 0, 0)	(1, 0, -1, 0)	(1, -1, 0, 0)	(1, 1, 1, 1)
Cabello 1997 [24]	AB	β_+	β_-	(0, $\frac{1}{2}$, $\frac{-\sqrt{3}}{2}$, 0)	(0, $\frac{1}{2}$, $\frac{\sqrt{3}}{2}$, 0)	(0, 0, 0, 1)	δ_+	γ_+	aB
$A = B = (1, 0), a = (\frac{1}{\sqrt{3}}, \frac{-2\sqrt{2}}{\sqrt{3}})$	(1, 0, 0, 0)	(0, $\frac{1}{2}$, $\frac{-\sqrt{3}}{2}$, 0)	(0, $\frac{1}{2}$, $\frac{\sqrt{3}}{2}$, 0)	(0, 0, 0, 1)	($\frac{1}{\sqrt{3}}$, $\frac{1}{\sqrt{2}}$, $\frac{-1}{\sqrt{6}}$, 0)	($\frac{1}{\sqrt{3}}$, $\frac{1}{\sqrt{2}}$, $\frac{1}{\sqrt{6}}$, 0)	($\sqrt{\frac{2}{3}}$, $\frac{1}{2}$, $\frac{-1}{2\sqrt{3}}$, 0)	($\sqrt{\frac{2}{3}}$, $\frac{1}{2}$, $\frac{1}{2\sqrt{3}}$, 0)	($\frac{1}{3}$, 0, $\frac{-2\sqrt{2}}{3}$, 0)
BBCGL 2011 [27]	Ψ	$\bar{a}_2\bar{b}_1$	$\bar{a}_1\bar{b}_2$	$\bar{a}_2\bar{b}_2$	$\bar{a}_2\bar{b}_2$	$\bar{a}_2\bar{b}_1$	$\bar{a}_1\bar{b}_2$	$\bar{a}_1\bar{b}_1$	a_1b_1
$u = (1, 0), c = (1, 1)$	(0, -1, -1, -1)	(0, 1, 0, -1)	(0, 0, 1, -1)	(1, 0, 0, 0)	(0, 0, 1, 0)	(0, 1, 0, 0)	(0, 1, 0, 1)	(0, 0, 1, 1)	(1, -1, -1, 1)
$u = (1, 0), c = (2, 3)$	(0, -2, -2, -3)	(0, 3, 0, -2)	(0, 0, 3, -2)	(1, 0, 0, 0)	(0, 0, 1, 0)	(0, 0, 1, 0)	(0, 2, 0, 3)	(0, 0, 2, 3)	(9, -6, -6, 4)
VECFIND [47] {0, 1, -2, $\sqrt{2}$ }	(1, -2, $\sqrt{2}$, 0)	(-2, 0, $\sqrt{2}$, 0)	(0, 1, $\sqrt{2}$, 0)	(0, 0, 0, 1)	(1, 0, 0, 0)	(0, 1, 0, 0)	(1, 0, $\sqrt{2}$, 0)	(0, -2, $\sqrt{2}$, 0)	(-2, 1, $\sqrt{2}$, 0)
VECFIND {1, 2, $\frac{-1}{2}$, 3, 5, $\pm i$ }	(i , 3, 3, 5)	(i , $\frac{-1}{2}$, 1, $\frac{-1}{2}$)	(i , 1, $\frac{-1}{2}$, $\frac{1}{2}$)	(1, i , i , $-i$)	(1, i , i , i)	(1, $-i$, $-i$, i)	(i , 2, 1, 2)	(i , 1, 2, 2)	(5, i , i , i)
VECFIND {1, -1, 2, $\frac{1}{2}$, 3, 5}	(-1, 3, 3, 5)	(1, -1, $\frac{1}{2}$, $\frac{1}{2}$)	(1, 1, $\frac{1}{2}$, $\frac{1}{2}$)	(1, -1, 1, 1)	(1, -1, 1, 1)	(1, -1, 1, 2)	(-1, 1, 2, 2)	(1, 1, -1, 1)	(1, $\frac{1}{2}$, -1, $\frac{1}{2}$)

unattainability of a coordinatization with purely decomposable vectors should come as no surprise as the hypergraph of the Hardy gadget depicted in Fig. 1(a) contains three triangular subgraphs, namely, the cyclically intertwining sets of contexts $\{\{\Psi, 2, 3, vd\}, \{vd, uv, vc, uu\}, \{uu, 20, 21, \Psi\}\}$, $\{\{\Psi, 16, 17, dv\}, \{dv, vu, cv, uu\}, \{uu, 20, 21, \Psi\}\}$, as well as $\{\{vu, dv, cv, uu\}, \{uu, vd, vc, uv\}, \{vu, 18, 19, uv\}\}$. Triangular hypergraphs have no faithful orthogonal representation by purely decomposable vectors [49].

VI. EXTENSIONS TO TRUE-IMPLIES-TRUE GADGETS

We now turn to important extensions of the Hardy gadget which have a classical true-implies-true structure, as already employed in Kochen and Specker’s Γ_1 [8] and discussed in Ref. [14]. A further escalation is a combination of these true-implies-true gadgets, similar to Kochen and Specker’s Γ_3 , which delivers a truly nonclassical performance on the

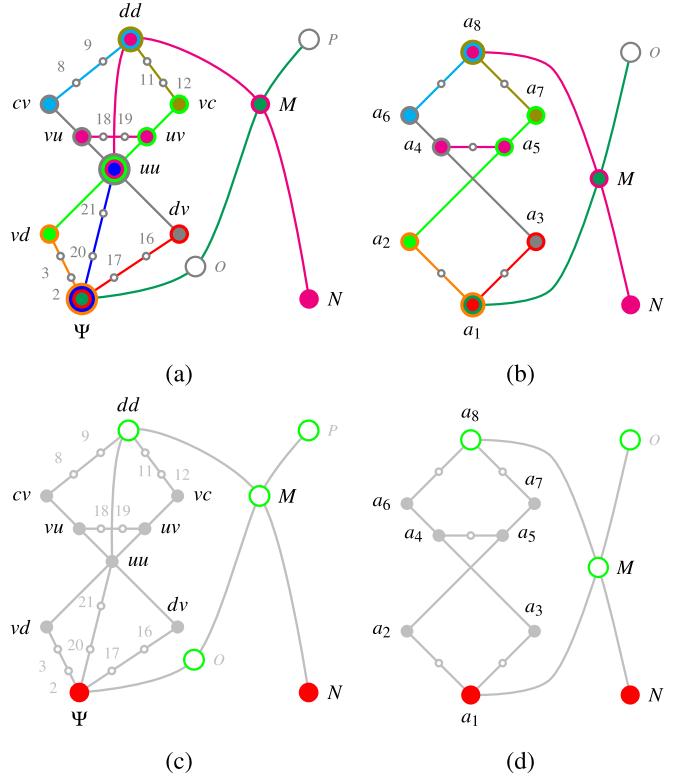


FIG. 4. Orthogonality hypergraphs of (a) the Hardy gadget extended to a true-implies-true gadget [14], as enumerated in (c), with $8 + 2 = 10$ contexts and $21 + 4 = 25$ atoms $\{\{dd, 8, 9, cv\}, \{dd, 11, 12, vc\}, \{cv, vu, uu, dv\}, \{vc, uv, uu, vd\}, \{vu, 18, 19, uv\}, \{vd, 2, 3, \Psi\}, \{uu, 20, 21, \Psi\}, \{dv, 16, 17, \Psi\}, \{5, dd, M, N\}, \{1, O, P, M\}\}$; (b) rendition based on the true-implies-true extended Specker bug or cat’s cradle gadget [7,8], as enumerated in (c), with $7 + 2 = 9$ contexts and $13 + 3 = 17$ atoms $\{a_8, \dots, a_6\}, \{a_8, \dots, a_7\}, \{a_6, a_4, a_3\}, \{a_7, a_5, a_2\}, \{a_4, \dots, a_5\}, \{a_2, \dots, a_1\}, \{a_3, \dots, a_1\}, \{a_8, M, N\}, \{a_1, \dots, M\}\}$. Due to the way these true-implies-false gadgets are constructed, N always turns out to be 1 if Ψ or a_1 are supposed to be 1. Small circles indicate “auxiliary” observables, which can be chosen freely, subject to orthogonality constraints: all smooth lines indicate respective contexts representing orthonormal bases.

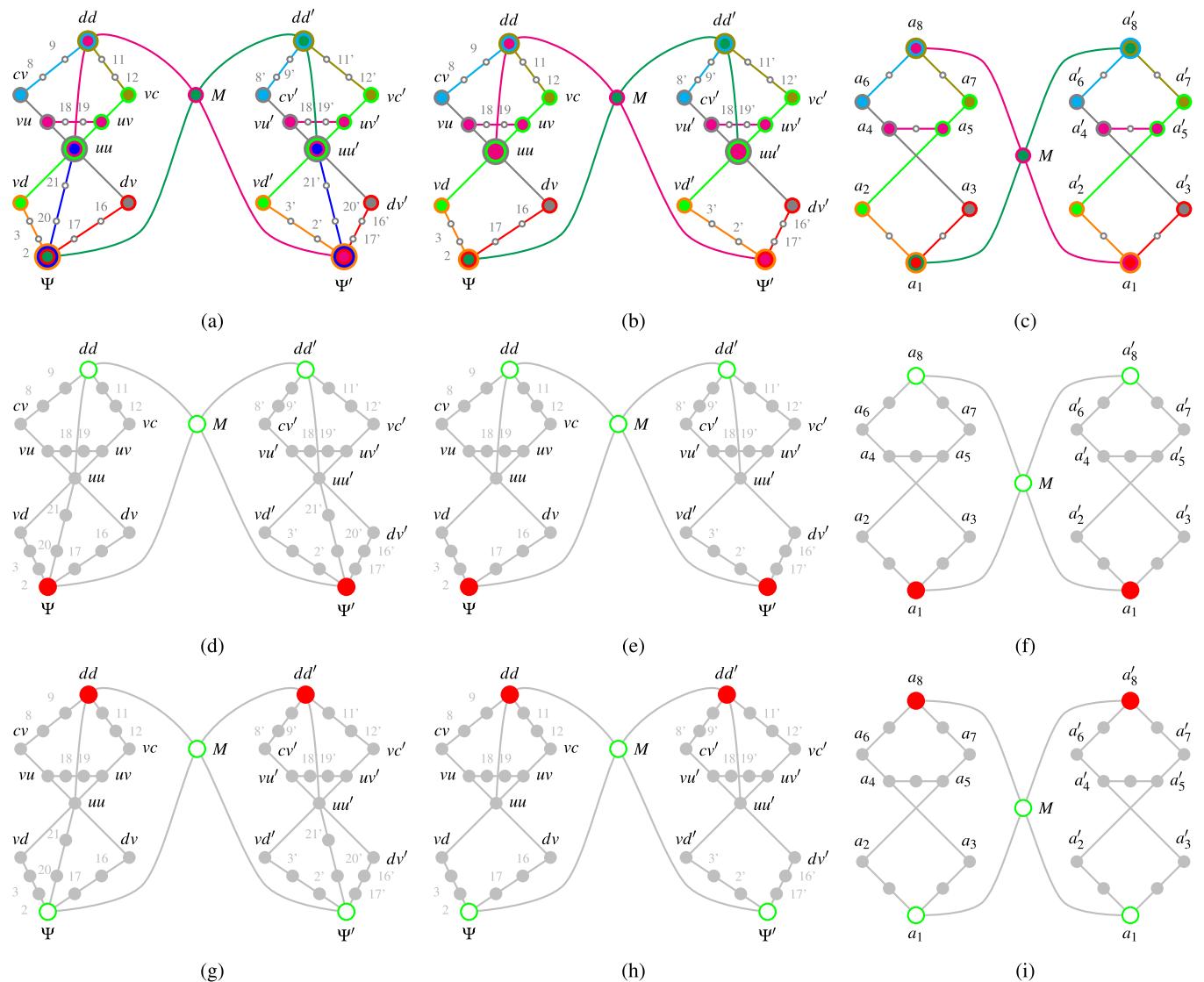


FIG. 5. Orthogonality hypergraphs of (a) a combination of two extended Hardy gadgets form a structure of quantum observables which cannot be classically embedded with $2 \times 8 + 2 = 18$ contexts and $2 \times 21 + 1 = 43$ atoms that cannot be classically embedded because of the indistinguishability with classical means (two-valued states) of the two pairs of atoms $\{\psi_1, \psi_2\}$ as well as $\{dd_1, dd_2\}$, respectively; (b) a combination of two extended Hardy-like gadgets first introduced in Fig. 4(b) of Ref. [14] form a structure of quantum observables with $2 \times 7 = 14$ contexts and $2 \times 19 + 1 = 39$ atoms that cannot be classically embedded because of the indistinguishability with classical means (two-valued states) of the two pairs of atoms $\{\psi_1, \psi_2\}$ as well as $\{dd_1, dd_2\}$, respectively; (c) a combination of two extended Specker bug or cat's cradle gadgets [7,8] with $2 \times 7 + 2 = 16$ contexts and $2 \times 13 + 1 = 27$ atoms that cannot be classically embedded because of the indistinguishability with classical means (two-valued states) of the two pairs of atoms $\{a_1, a'_1\}$ as well as $\{a_8, a'_8\}$, respectively. (d)–(i) depict the associated two-valued states which are not 0 on all four observables $\{\psi_1, \psi_2, dd_1, dd_2\}$ as well as $\{a_1, a'_1, a_8, a'_8\}$, respectively. Only valuations that are relevant for the proof are drawn. Small circles indicate “auxiliary” observables which can be chosen freely, subject to orthogonality constraints: all smooth lines indicate respective contexts representing orthonormal bases.

algebraic level of the propositional observables (and not just probabilistic predictions based upon classical probabilities): Unlike the Hardy and its extended true-implies-true gadgets, those observables can no longer be faithfully embedded into any Boolean algebra ([8], Theorem 0).

Figure 4 depicts the extension of the Hardy gadget which delivers a classical true-implies-true prediction at its terminal points Ψ and N . A faithful orthogonal representation of the extended Hardy gadget can be obtained *ad hoc* by the heuristic algorithm VECFIND [47] in the coordinate basis $\{0, \pm 1, \pm 2, 3\}$ and the -nk

option, which is capable of finding “almost all” vectors, including the true-implies-true terminal points Ψ and N *ex machina*, and (for this coordinate basis) needs a little helping hand (or the additional component basis elements $\{-3, 5, 7, \sin \theta, \cos \theta\}$ with $\theta \neq n\pi/4$, $n \in \mathbb{Z}$) to find the complete set, given by $\Psi = (0, 1, 1, -1)$, $2 = (2, 2, -1, 1)$, $3 = (3, -2, 1, -1)$, $vd = (0, 0, 1, 1)$, $uu = (1, 0, 0, 0)$, $vu = (0, 0, 0, 1)$, $cv = (0, 1, 1, 0)$, $8 = (3, 1, -1, 2)$, $9 = (-2, 1, -1, 2)$, $dd = (0, -1, 1, 1)$, $11 = (3, -2, -1, -1)$, $12 = (2, 2, 1, 1)$, $vc = (0, 0, 1, -1)$, $uv = (0, 1, 0, 0)$, $dv = (0, 1, -1, 0)$, $16 = (-2, 1, 1, 2)$, $17 = (3, 1, 1, 2)$,

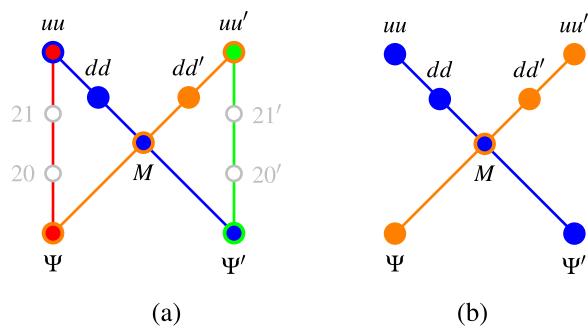


FIG. 6. Hypergraphs of the “orthogonality backbones” of (a) Fig. 5(a) and (b) Fig. 5(b) supporting the two-valued states depicted in Figs. 5(c) and 5(d), respectively.

18 = $(\cos \theta, 0, \sin \theta, 0)$, 19 = $(-\sin \theta, 0, \cos \theta, 0)$,
 20 = $(0, 4, -3, 1)$, 21 = $(0, 2, 5, 7)$, $M = (0, 1, 0, 1)$,
 $N = (0, 1, 2, -1)$, $O = (2, -1, 2, 1)$, $P = (3, 1, -2, -1)$, where $\theta \neq n\pi/4$, $n \in \mathbb{Z}$. [The original coordinatization suggested for atom 20 was $(0, 1, -1, 0)$, but a completion would have resulted in duplicities, namely, $21 = (0, 1, 1, 2) = dv$; and therefore the original suggestion had to be dropped.] Although we do not concentrate on maximal violations of classical predictions by quantum probabilities, for reasons mentioned later, it is worth noting that as $|\langle \psi | N \rangle|^2 = 8/9$, the quantum violation of the classical predictions will, in this particular configuration, occur in one out of nine times; that is, with probability 0.1̄ (proper normalization is always assumed). Fortunately, if one concentrates on the quantum signal for observable $|dd\rangle\langle dd|$, then one obtains the same quantum prediction $|\langle \psi | dd \rangle|^2 = 1/9$ for this outcome—although classically it should never occur.

VII. EXTENSIONS TO GADGETS WITH INDISTINGUISHABLE CLASSICAL TRUTH ASSIGNMENTS

One way to proceed would be what Kochen and Specker did with their true-implies-true gadget Γ_1 , and serially compose them at their respective (properly parametrized) terminal points often enough to obtain Γ_2 , which renders a complete contradiction with exclusivity [8]. Instead of this head-on strategy for obtaining complete contradictions with classical noncontextual hidden-variable models, we shall use a more subtle approach and consider a hypergraph which, again in analogy with Kochen and Specker’s Γ_3 in three dimensions, cannot be classically embedded in a Boolean algebra. The construction uses two true-implies-true extended Hardy gadgets to construct two pairs of observable propositions which cannot be differentiated by classical two-valued measures—and thus by any classical probability distributions—although “plenty” such two-valued states still exist (but their set is “too meager” to allow mutual distinguishability of all pairs of distinct propositions).

Searches for a faithful orthogonal representation of an extensions using two (a combination) of the “original” version of the Hardy gadget, as depicted in Fig. 6(a), have been inconclusive so far. Nevertheless, as it turns out, this task can be completed by using (a combination of) a slight

modification of Hardy’s gadget introduced in Fig. 4(b) of Ref. [14], in which the original context $\{\Psi, \dots, uu\}$ is “relocated” or “reshuffled” into the context $\{uu, \dots, dd\}$. The resulting gadget, depicted in Fig. 5(b), not only has less atoms but, most importantly, has a less tight “orthogonality backbone” structure, depicted in Fig. 6(b), of just two contexts intertwined in a single atom M , namely, $\{uu_1, dd_1, M, \Psi_2\}$, $\{uu_2, dd_2, M, \Psi_1\}$, as compared to the tight configuration resulting from a composition of two of Hardy’s original gadgets $\{P\psi_1, \dots, uu_1\}$, $\{uu_1, dd_1, M, \Psi_2\}$, $\{uu_2, dd_2, M, \Psi_1\}$, $\{P\psi_2, \dots, uu_2\}$ depicted in Fig. 6(a).

More explicitly, VECFIND [47], with the component basis $\{0, \pm 1, 2, -3, 4, 5\}$, yields an *ad hoc* coordinatization of the intertwine atoms $\psi = (1, 0, 0, 0)$, $vd = (0, 2, -1, 1)$, $uu = (0, 0, 1, 1)$, $vu = (1, -1, 1, -1)$, $cv = (-3, -1, 1, -1)$, $dd = (1, -3, 0, 0)$, $vc = (-3, -1, -1, 1)$, $uv = (1, -1, -1, 1)$, $dv = (0, 2, 1, -1)$, $\psi' = (-3, -1, 0, 0)$, $vd' = (1, -3, 4, -1)$, $uu' = (0, 1, 1, 1)$, $vu' = (-3, -1, 2, -1)$, $cv' = (1, 0, 1, -1)$, $dd' = (0, 2, -1, -1)$, $vc' = (5, 0, -1, 1)$, $uv' = (-1, -3, -1, 4)$, $dv' = (1, -3, 1, 2)$, and $M = (0, 0, 1, -1)$, which can be readily completed into a faithful orthogonal representation of the hypergraph depicted in Fig. 5(b). Note that in this particular configuration, because of indistinguishability, the classical prediction to find a particle prepared in a state Ψ in the state Ψ' is one (certainty), whereas quantum mechanics predicts nonoccurrence of the elementary propositional observable $|\Psi'\rangle\langle\Psi'|$ given a preselected, prepared state $|\Psi\rangle$ with probability $|\langle\Psi|\Psi'\rangle|^2 = 9/10$; that is, the violation of the classical prediction by quantum mechanics occurs in this case in one out of ten experimental runs.

VIII. SUMMARY AND CAUTIONARY REMARKS

Hardy-type configurations have been extended to configurations of contexts which show a different nonclassical performance: they contain distinct quantum observables that cannot be distinguished from one another by any classical (noncontextual) means. To appreciate the difference of this aspect beyond the realization of just another relational property among some prepared state and its measurement, it is important to keep in mind that according to a finding by Kochen and Specker [8, Theorem 0], indistinguishability serves a demarcation criterion for strong forms of nonclassicality: The absence of classical distinguishability indicates a stronger constraint on hidden-variable theories (relative to the assumptions) than, say, exploitation of true-implies-{true, false} properties [14] which still allow faithful classical embeddability of the quantum observables (albeit with different statistical predictions), and merely requires complementarity.

Indistinguishability by classical means indicates a rather strong form of nonclassicality—that is, the impossibility to faithfully embed the quantum mechanical observables in classical Boolean structures—while still allowing the direct experimental falsification of the respective quantum and classical predictions. Therefore, it is not affected by questions related to the empirical pertinence of Kochen-Specker proofs (by contradiction) of the absence of any classical interpretation, stated pointedly by Clifton, “how can you measure a contradiction?” [60].

Further efforts could advance by “improving the nonclassical performance” of gadgets not in terms of the number of (counterfactual, complementary) observables, but in terms of quantum-to-classical discords in four dimensions. The hypergraph method developed earlier might suggest such advancements by their emphasis on the logico-algebraic structure, thereby making possible a more systematic exploitation of feasible configurations of observables. There already exist true-implies-{true, false} gadgets which yield high performance in three dimensions [38,76,77].

Nevertheless, once the vectors corresponding to pre- and postselected states are fixed, it is always possible to find any kind of conforming or disagreeing classical-versus-quantum behavior. As I have pointed out elsewhere [16], these kind of statements are contingent on the chosen gadget consisting of mostly counterfactual observables “in the mind” of the observer [78,79]. Nevertheless, any such considerations raise fascinating, challenging issues in a variety of fields which might have been perceived unrelated so far: graph theory, (linear) algebra, functional analysis, geometry, automated the-

orem proving, and—last but not least—quantum physics and quantum information (processing) technology.

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The logic of propositions which are not simultaneously decidable

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