Chromatic "Operator-Valued" Contextuality

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Shannon Entropy: Quantifying Information

- Shannon Entropy (H) measures the average uncertainty or information content of a random variable.
- The more uncertain an outcome, the higher its entropy, and the more information we gain upon observation.
- It's typically measured in bits (when using $\log_2$).

Formula for Shannon Entropy

For a discrete random variable X with outcomes x_i and probabilities $P(x_i)$:

$$H(X) = -\sum_{i=1}^{n} P(x_i) \log_2 P(x_i)$$

Core Assumption

For the following examples, we assume the initial underlying states are **equiprobable**.

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3-State System: Full Resolution

- **System:** 3 distinct, equiprobable states.
- **Observed Outcomes:** $\{0, 1, 2\}$
- **Probabilities:** $P(0) = P(1) = P(2) = 1/3$

Calculating Information (H_{full})

$$H_{\text{full}} = -\sum_{i=0}^{2} \frac{1}{3} \log_2 \left(\frac{1}{3}\right)$$

$$= -3 \times \frac{1}{3} \log_2 \left(\frac{1}{3}\right) = \log_2(3)$$

$$\approx 1.585 \text{ bits}$$

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3-State System: Aggregated Resolution

- **Mapping (Aggregation):** $0 \to 0_{\text{obs}}, \quad 1 \to 1_{\text{obs}}, \quad 2 \to 1_{\text{obs}}$
- **New Observed Outcomes:** $\{0_{\text{obs}}, 1_{\text{obs}}\}$
- **New Probabilities:**

$$P(0_{\text{obs}}) = P(0) = 1/3$$
$$P(1_{\text{obs}}) = P(1) + P(2) = 2/3$$

Calculating Information (H_{coll})

$$H_{\text{coll}} = -\left[\frac{1}{3}\log_2\left(\frac{1}{3}\right) + \frac{2}{3}\log_2\left(\frac{2}{3}\right)\right]$$
$$\approx 0.918 \text{ bits}$$

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- **Full Resolution:**
  $H_{\text{full}} \approx 1.585\,\text{bits}$
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Information Loss

The aggregation reduces the information obtained.

$$\begin{aligned} \text{Loss} &= H_{\text{full}} - H_{\text{coll}} \\ &\approx 1.585 - 0.918 \\ &= 0.667\,\text{bits} \end{aligned}$$

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4-State System: Full Resolution

- **System:** 4 distinct, equiprobable states.
- **Observed Outcomes:** $\{0, 1, 2, 3\}$
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Calculating Information (H_{full})

$$H_{\text{full}} = -\sum_{i=0}^{3} \frac{1}{4} \log_2\left(\frac{1}{4}\right) = \log_2(4)$$
$$= 2\,\text{bits}$$

A 4-state equiprobable system perfectly encodes 2 bits.

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4-State System: Aggregated Cases

Case A: Symmetric Aggregation

$\{0, 1\} \to 0_{\text{obs}}$
$\{2, 3\} \to 1_{\text{obs}}$

$$P(0_{\text{obs}}) = 1/2$$
$$P(1_{\text{obs}}) = 1/2$$
$$H_{\text{case A}} = 1 \text{ bit}$$

Like a fair coin flip.

Case B: Asymmetric Aggregation

$\{0, 1, 2\} \to 0_{\text{obs}}$
$\{3\} \to 1_{\text{obs}}$

$$P(0_{\text{obs}}) = 3/4$$
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$$H_{\text{case B}} \approx 0.811 \text{ bits}$$

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Observation

Each state aggregation leads to a reduction in measurable information. The more states are merged and the more skewed the resulting probabilities, the lower the entropy.

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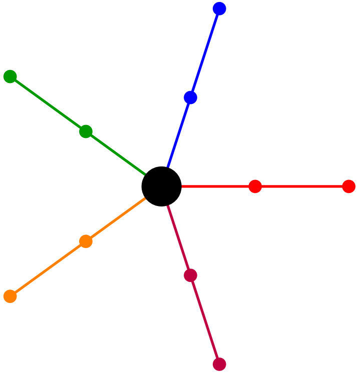
2-Valued States in 3 Dimensions



Interpretation

This represents a aggregated system:

- The black state maps to value 1.
- All non-black states map to value 0.

In Hilbert space, this corresponds to projecting onto a 1D subspace vs. its orthogonal (d-1)D complement.

Spectral Decomposition: Maximal vs. Degenerate, "full operator-valued versus two-valued"

Let $\{|\mathbf{e}_i\rangle \mid 1 \leq i \leq d\}$ be an orthonormal basis.

Maximal Operator (von Neumann, 1931)

Outcomes λ_i are mutually distinct (unique "colors"):

$$A = \sum_{i=1}^{d} \lambda_i \left|\mathbf{e}_i\right\rangle \left\langle\mathbf{e}_i\right|$$

Degenerate Operator (Projector)

Only two outcomes (e.g., 1 for state j, 0 otherwise):

$$P_j = \left|\mathbf{e}_j\right\rangle \left\langle\mathbf{e}_j\right| = \sum_{i=1}^{d} \delta_{ij} \left|\mathbf{e}_i\right\rangle \left\langle\mathbf{e}_i\right|$$

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Postulate/Presumption of Classicality

- **Chromatic Noncontextuality:** The color (value) of intertwining observables is independent of the (hyper)edge.
- **Chromatic Reality:** Existence of classical d-ary elements of physical reality for certain d-uniform "chromatic Kochen-Specker" hypergraphs.

Results on Chromatic Contextuality

- If a (d-uniform hyper)graph has chromatic number d, it has at least d two-valued states (by aggregation).
(M. H. Shekarriz and KS, JMP 63, 032104, 2022)
- The Yu-Oh 3-uniform (hyper)graph has clique number 3 but chromatic number 4, yet is set representable. This is a "Chromatic Kochen-Specker theorem".
(KS, Entropy 27, 387, 2025)
- The house/pentagon/pentagram d-uniform hypergraph has one "exotic" 2-valued state that cannot be obtained by aggregating one of its 5 non-equivalent colorings.
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Summary

Colorings are a formidable tool to investigate quantum contextuality.