Some remarks on generalized probabilities

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Abstract

Three distinct modifications of nonclassical probability theory are discussed. First, we propose to generalize the probability axiom of quantum mechanics to self-adjoint positive operators of trace one and to normal operators, in particular ones which maximally represent the context. In the second part, we discuss Pitowsky polytopes for automaton logic as well as for generalized urn models. Finally, so-called "parameter cheats" are introduced, whereby parameters are transformed bijectively and nonlinearly in such a way that classical systems mimic quantum correlations and *vice versa*.

1 Extending the axioms of quantum mechanics

Consider finite dimensional Hilbert spaces. The quantum probability $P(\psi, A)$ of a proposition A given a state ψ is usually introduced as the trace of the product of the state operators ρ_ψ and the projection operator E_A; i.e., $P(\psi, A) = \mathrm{Tr}(\rho_\psi E_A)$. The state operator ρ_ψ must be (i) self-adjoint; i.e., $\rho_\psi = \rho_\psi^\dagger$, (ii) positive; i.e., $(\rho_\psi x, x) = \langle x \mid \rho_\psi \mid x\rangle \geq 0$ for all x, and (iii) of trace one; i.e., $\mathrm{Tr}(\rho_\psi) = 1$. Since one criterion for a pure state is its idempotence; i.e., $\rho_\varphi \rho_\varphi = \rho_\varphi$, one way to interpret E_A is a measurement apparatus in a pure state $\rho_\varphi = E_A$. But while pure states can be interpreted as a system being in a given property, not every state is pure and thus corresponds to a projection.

A rather trivial generalization of quantum probabilities is

$$P(\psi, \varphi) = \mathrm{Tr}(\rho_\psi \rho_\varphi), \tag{1}$$

where again we require that ρ_ψ as well as ρ_φ is self-adjoint, positive and of trace one. That is, we relax the requirement to consider only properties which are known perfectly: the "property" ρ_φ is "unsharp", as compared to a "sharp" property corresponding to a projection operator. One property of the extended probability measure is its positive definiteness and boundedness; i.e., $0 \leq P(\psi, \varphi) \leq 1$. The former bound follows from positivity. The latter bound by 1 can be easily proved for finite dimensions by making a unitary basis transformation such that ρ_φ (or ρ_ψ) is diagonal.

A very simple example of an extended probability is the case of the total ignorance of the state of the measurement apparatus as well as of the measured system. Take n nondegenerate possible outcomes for the apparatus and the state, then $\rho_\psi = \rho_\varphi = \mathbf{1}/n = (1/n)\,\mathrm{diag}(1, 1, \ldots, 1)$, and the probability to find any combination thereof is $P(\psi, \varphi) = 1/n^2$. The extended probability reduces to the standard form if one assumes total knowledge of the state of the measurement apparatus, since then ρ_φ is pure and thus a projection.

A well known fact is the Cartesian and polar decomposition of an arbitrary operator A into operators B, C and D, E such that

$$\begin{aligned} A &= B + iC = DE, \\ B &= \frac{A + A^\dagger}{2}, \quad C = \frac{A - A^\dagger}{2i}, \\ E &= \sqrt{A^\dagger A}, \quad D = AE^{-1}, \end{aligned}$$

where B, C are self-adjoint, E is positive and D is unitary (i.e., an isometry). The last two equations are for invertible operators A. These are just the matrix equivalents of the decompositions of complex numbers.

If A is a *normal* operator; i.e., $AA^\dagger = A^\dagger A$, then B and C commute (i.e., $[B, C] = 0$) and are thus co-measurable. (All unitary and self-adjoint operators are normal.) In this case, also the operators of the polar decomposition D and E are unique and commute; i.e., $[D, E] = 0$, and are thus co-measurable. We have thus reduced the issue of operationalizability of normal operators to the self-adjoint case.

Hence, normal operators are operationalizable either by a simultaneous measurement of the summands in the Cartesian decomposition or of the factors in a polar decomposition. Indeed, all operators are "measurable" if one assumes EPR's elements of counterfactual physical reality [1, p. 108f]. In this case, one makes use of the Cartesian decomposition, where B and C not necessarily can be diagonalized simultaneously and thus need not commute. Nevertheless, one may devise a singlet state of two particles with respect to the observables B and C, and measure B on one particle and C on the other one.

As an example for the case of a normal operator which is neither self-adjoint nor unitary, consider

$$\begin{aligned} \text{diag}(2, i) &= \mathbf{1} + \sigma_3 + \frac{i}{2}(\mathbf{1} - \sigma_3) \\ &= \left[\frac{1+i}{2}\mathbf{1} + \frac{1-i}{2}\sigma_3\right]\left[\frac{3}{2}\mathbf{1} + \frac{1}{2}\sigma_3\right], \end{aligned} \tag{2}$$

where $\sigma_3 = \text{diag}(1, -1)$ and both summands and factors commute and thus are co-measurable.

Let us define the *context* as the set of all co-measurable properties of a physical system. By a well-known theorem, any context has associated with it a single (though not unique) observable represented by a self-adjoint operator C such that all other observables represented by self-adjoint A_i within a given context are merely functions (in finite dimensions polynomials) $A_i = f_i(C)$ thereof. We shall call C the *context operator*. Context operators are maximal in the sense that they exhaust their context but they are not unique, since any one-to-one transformation of C such as an isometry yields a context operator as well.

Different operators A_i may belong to different contexts. Actually, the proof of Kochen and Specker [2] (of the nonexistence of consistent global truth values by associating such valuations locally) is based on a finite chain of contexts linked together at one operator per junction which belongs to the two contexts forming that junction. This fact suggests that—rather than considering single operators which may belong to different contexts—it is more appropriate to consider context operators instead. By definition, they carry the entire context and thus cannot belong to different ones. A graphical representation of context operators has been given by Tkadlec [3], who suggested to consider dual Greechie diagrams which represent context operators as vertices and links between different contexts by edges. A typical application would be the measurement of all the N contexts necessary for a Kochen-Specker contradiction in an entangle N particle singlet state. In such a case, there should exist at least one observable belonging to two different contexts whose outcomes are different (cf. also [4] for a similar reasoning).

2 Pitowsky polytopes for automaton logics and generalized urn models

In the middle of the 19th century the English mathematician George Boole formulated a theory of "conditions of possible experience" (CPE) [5, 6, 7]. These conditions are related to the joint probabilities of events and are expressed by certain equations or inequalities. More recently, similar equations for a particular setup which are relevant in the quantum mechanical context have been discussed by Bell, Clauser and Horne and others. Itamar Pitowsky has given a geometrical interpretation of classical Boole-Bell CPE's in terms of correlation polytopes [8, 9, 10, 11]: Take the probabilities $P_1, \ldots, P_n$ of some events $1, 2, \ldots n$ and some (or all) of the joint probabilities $P_1 \wedge P_2, \ldots, P_{n-1} \wedge P_n, P_1 \wedge P_2 \wedge P_3, \ldots$ and write them in vector form $\mathbf{x} = (P_1, \ldots, P_n, P_1 \wedge P_2, \ldots, P_{n-1} \wedge P_n, P_1 \wedge P_2 \wedge P_3, \ldots)$. All possible combinations of valuations[1] of the n Boolean algebras corresponds to one of the 2^n vertices (i.e., extreme points) of a classical correlation polytope

$$\left\{ \lambda_1 \mathbf{x}_1 + \cdots + \lambda_l \mathbf{x}_l \;\middle|\; l = 2^n, \lambda_j \geq 0, \sum_{j=1}^{2^n} \lambda_j = 1 \right\}, \tag{3}$$

where $\mathbf{x}_i$ stands for the truth function corresponding to the ith valuation. Thus, the vector components of $\mathbf{x}_i$ are either 0 or 1, and the first n components contain all 2^n possible distinct combinations thereof.

Every convex polytope in an Euclidean space has a dual description: either as the convex hull of its vertices as in Eq. (3) (V-representation), or as the intersection of a finite number of half-spaces, each one given by a linear inequality (H-representation) This equivalence is known as the *Weyl-Minkowski* theorem. The problem to obtain all inequalities from the vertices of a convex polytope is known as the *hull problem.* One solution strategy is the Double Description Method which we shall use but not review here.

What Boole did not foresee, however, is that certain events in one and the same inequality may be operationally incompatible, and that the event structure may not be a Boolean algebra. This applies to quantum mechanics as well as to Wright's generalized urn models [12, 13], as well as to automaton partition logics [14, 15, 1].

The importance of correlation polytopes lies in the fact that they fully exploit all consistently conceivable probabilities. Their border faces correspond to Boole-Bell type inequalities. If dispersion-free states exist, every vertex corresponds to a dispersion-free state. In this sense, correlation polytopes define the probabilities of a given formal structure completely.

Whereas for Hilbert logics of Hilbert spaces with dimension higher than or equal to three, no dispersion-free state exists [2], for generalized urn logics and automaton partition logics, two-valued states exist and can be used for an explicit construction of the respective models. Just as for Boolean algebras, every probability can be composed by the convex combination of two-valued states, any probability on orthologics (a bounded, orthocomplemented poset in which orthocomplemented joins exist) (admitting a two-valued state) is a convex combination of two-valued states [12, Theorems 1.6, 1.7]. We shall restrict our attention to orthologics L with a

[1] In what follows, the terms "two-valued (probability) measure", "two-valued state", "valuation", and "dispersion-free measure (state)" will be used synonymously for a lattice homomorphism $P : L \to 0, 1$ such that $P(L) = 1$.

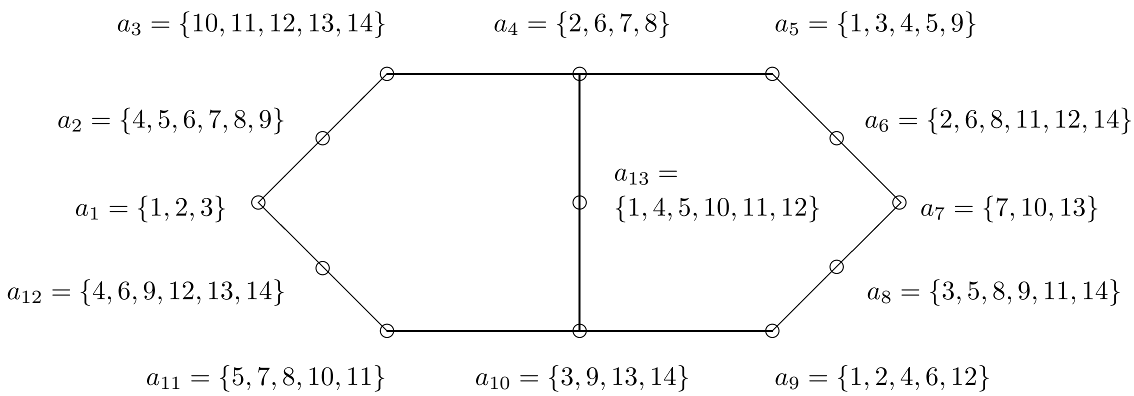

Figure 1: Greechie diagram of automaton partition logic with a nonfull set of dispersion-free measures.

separating set of states; i.e., for every $s, t \in L$ with $s \neq t$, there is a two-valued state P such that $P(s) \neq P(t)$ (an even weaker criterion would be unitality).

One immediate question is the following one: how do such nonclassical correlation polytopes of orthologics admitting two-valued states relate to classical correlation polytopes? The nonclassical correlation polytope $\mathcal{C}(L)$ corresponding to some nonboolean lattice L can be defined as the convex hull of all two-valued states thereon. That is, Eq. (3) also applies for the nonclassical case; with the generalization that the set of vectors $\mathbf{x}_i$ corresponds to the set of all two-valued states thereon. (In the quantum mechanical case, no valuations exist for Hilbert spaces of dimension bigger than two; thus the definition cannot be applied to quantum correlation polytopes.)

Separability (unitality) implies embedability of a the orthologic L into a Boolean algebra $B = 2^n$ with n atoms. We may consider the corresponding correlation polytope $\mathcal{C}_n$ generated by the subset of its 2^n vertices, which are its extreme points if the truth assignments are identified by vector components. The dimension of the vector space depends on the number of propositions involved. Since not all valuations of the Boolean algebra 2^n need to be valuations of L, $\mathcal{C}(L)$ is a subset of $\mathcal{C}_n$.

Consider, as an example, a logic already mentioned by Kochen and Specker [2] (this is a subgraph of their Γ_1) whose automaton partition logic is depicted in Fig. 1. The correlation polytope of this lattice consists of 14 vertices listed in Table 1, where the 14 rows indicate the vertices corresponding to the 14 dispersion-free states. The columns represent the partitioning of the automaton states. The solution of the hull problem by the `LPoly` package due to Maximian Kreuzer and Harald Skarke [16] yields the equalities

$$
\begin{array}{l}
1 = P_1 + P_2 + P_3 = P_4 + P_{10} + P_{13}, \\
1 = P_1 + P_2 - P_4 + P_6 + P_7 = -P_2 + P_4 - P_6 + P_8 - P_{10} + P_{12}, \\
1 = P_1 + P_2 - P_4 + P_6 - P_8 + P_{10} + P_{11}, \\
0 = P_1 + P_2 - P_4 - P_5 = -P_1 - P_2 + P_4 - P_6 + P_8 + P_9.
\end{array}
\tag{4}
$$

The operational meaning of $P_i = P_{a_i}$ is "the probability to find the automaton in state a_i." Eqs. (4) are equivalent to all probabilistic conditions on the contexts (subalgebras) $1 = P_1 + P_2 + P_3 = P_3 + P_4 + P_5 = P_5 + P_6 + P_7 = P_7 + P_8 + P_9 = P_9 + P_{10} + P_{11} = P_4 + P_{10} + P_{13}$.

Let us now turn to the joint probability case. Notice that formally it is possible to form a statement such as $a_1 \wedge a_{13}$ (which would be true for measure number

#	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}	$a_1 \wedge a_2$	$\cdots$
1	1	0	0	0	1	0	0	0	1	0	0	0	1	0	
2	1	0	0	1	0	1	0	0	1	0	0	0	0	0	
3	1	0	0	0	1	0	0	1	0	1	0	0	0	0	
4	0	1	0	0	1	0	0	0	1	0	0	1	1	0	
5	0	1	0	0	1	0	0	1	0	0	1	0	1	0	
6	0	1	0	1	0	1	0	0	1	0	0	1	0	0	
7	0	1	0	1	0	0	1	0	0	0	1	0	0	0	
8	0	1	0	1	0	1	0	1	0	0	1	0	0	0	
9	0	1	0	0	1	0	0	1	0	1	0	1	0	0	
10	0	0	1	0	0	0	1	0	0	0	1	0	1	0	
11	0	0	1	0	0	1	0	1	0	0	1	0	1	0	
12	0	0	1	0	0	1	0	0	1	0	0	1	1	0	
13	0	0	1	0	0	0	1	0	0	1	0	1	0	0	
14	0	0	1	0	0	1	0	1	0	1	0	1	0	0	

Table 1: Truth table of a logic with 14 dispersion-free states. The rows, interpreted as vectors, are just the vertices of the corresponding correlation polytope.

1 and false otherwise), but this is not operational on a single automaton, since no experiment can decide such a proposition on a single automaton. Nevertheless, if one considers a a "singlet state" of two automata which are in an unknown yet identical initial state, then an expression such as $a_1 \wedge a_{13}$ makes operational sense if property a_1 is measured on the first automaton and property a_{13} on the second automaton. Indeed, all joint probabilities $a_i \wedge a_j \wedge \ldots a_n$ make sense in an n-automaton singlet context.

3 Parameter cheats

In this section, certain bijective (one-to-one) parameter transformations will be performed which artificially give classical systems a quantum flavor; and conversely, seemingly make quantum systems behave classically, at least with respect to joint probabilities. As such transformations have other, undesirable features, we shall call them "parameter cheats."

Consider a singlet state for which the sum of all angular momenta and spins is zero. In the quantum mechanical case, let us assume two particles of spin 1/2 in an EPR-Bohm configuration. Then the probability $P^{=}(\theta)$ to find the angular momentum or spin of both particles measured along two axis which are an angle θ apart in the same direction is given by [17]

$$P^{=}_{qm}(\theta) \quad = \quad \sin^2(\theta/2) \tag{5}$$

$$P^{=}_{cl}(\theta) \quad = \quad \theta/\pi \tag{6}$$

$$\begin{aligned} P^{=}_{s}(\theta) \quad &= \quad \frac{1}{2} + \frac{2}{\pi}\sum_{k=0}^{n>1} \frac{\sin\left[(2k+1)\left(2\Delta/\pi - 1\right)\right]}{2k+1} \\ &\stackrel{n\rightarrow\infty}{\longrightarrow} \quad H(2\theta/\pi - 1) = (1/2)(1 + \mathrm{sgn}(2\theta/\pi - 1)) \end{aligned} \tag{7}$$

for $0 \leq \theta \leq \pi$. $P^{=}_{qm}(\theta)$, $P^{=}_{cl}(\theta)$, $P^{=}_{s}(\theta)$ stand for the joint classical, quantum and stronger-than-quantum probabilities, respectively. Figure 2 represents different joint probability functions of the parameter θ.

Figure 2: Different joint probability functions of the parameter θ. The solid, dashed and dot-dashed lines indicate classical, quantum and stronger-than-quantum behavior ($n = 11$), respectively.

3.1 Quantum cheat for classical system

Then, in order to be able to fake a quantum form of the classical expression, we introduce a "cheat parameter" δ, which is obtained from the angular parameter θ by a nonlinear transformation $T : \theta \mapsto \delta$ from the Ansatz

$$P_{cl}^{=}(\theta(\delta)) = P_{cl}^{=}(\delta) = \frac{\theta(\delta)}{\pi} = \sin^2\left(\frac{\delta}{2}\right). \tag{8}$$

The right hand side of Eq. (8) yields

$$\begin{aligned} \theta &= \pi \sin^2\left(\frac{\delta}{2}\right) && (9) \\ \delta &= 2 \arcsin \sqrt{\frac{\theta}{\pi}} && (10) \end{aligned}$$

where $0 \le \delta \le \pi$. Figure 3 represents a numerical evaluation of the deformed parameter scale δ in terms θ.

3.2 Classical cheat for quantum system

In order to be able to fake a classical form of the quantum expression, we introduce a "cheat parameter" ϕ, which is obtained from the angular parameter θ by a nonlinear transformation $T : \theta \mapsto \phi$ from the Ansatz

$$P_{qm}^{=}(\theta(\phi)) = P_{qm}^{=}(\phi) = \frac{\phi}{\pi} = \sin^2\left(\frac{\theta(\phi)}{2}\right). \tag{11}$$

The right hand side of Eq. (11) yields

$$\begin{aligned} \theta &= 2 \arcsin \sqrt{\frac{\phi}{\pi}} && (12) \\ \phi &= \pi \sin^2\left(\frac{\theta}{2}\right) && (13) \end{aligned}$$

where $0 \le \phi \le \pi$. Figure 4 represents a numerical evaluation of the deformed parameter scale ϕ in terms θ.

3.3 Stronger-than-quantum (STQ) cheat for classical system

In order to be able to fake a STR form of the classical expression, we introduce a "cheat parameter" Δ, which is obtained from the angular parameter θ by a nonlinear transformation $T : \theta \mapsto \Delta$ from the Ansatz

$$\begin{aligned} P_{cl}^{=}(\theta(\Delta)) &= \\ P_{cl}^{=}(\Delta) &= \frac{1}{2} + \frac{2}{\pi} \sum_{k=0}^{n} \frac{\sin\left[(2k+1)\left(2\Delta/\pi - 1\right)\right]}{2k+1} = \frac{\delta(\Delta)}{\pi}, && (14) \end{aligned}$$

where $n \ge 1$. In the limit,

$$\lim_{n\to\infty} \frac{4}{\pi} \sum_{k=0}^{n} \frac{\sin\left[(2k+1)\left(\frac{2\Delta}{\pi} - 1\right)\right]}{2k+1} = \operatorname{sgn}\left(\frac{2\Delta}{\pi} - 1\right).$$

The right hand side of Eq. (14) yields

$$\theta = \frac{\pi}{2} + 2 \sum_{k=0}^{n} \frac{\sin\left[(2k+1)\left(2\Delta/\pi - 1\right)\right]}{2k+1} \tag{15}$$

a)

b)

Figure 3: a) Evaluation of the deformed parameter scale δ versus θ. b) Evaluation of the linear reference parameter δ.

Figure 4: Evaluation of the deformed parameter scale ϕ versus θ.

3.4 How do the cheats perform?

Cheats perform in a very simple way, which can be best understood if one considers the "proper" physical parameter and compares it to the "cheat" parameter. The cheat parameter effectively deforms the proper parameter range in that measures therein pretend to be in a different parameter range than the one in which the proper parameter is. It is quite clear then, that cheats can mimic almost any behavior as long as the parameter transformation remains one-to-one.

Let us consider the Clauser-Horne (CH) inequality

$$-1 \leq P(A_1B_1) + P(A_1B_2) + P(A_2B_2) - P(A_2B_1) - P(A_1) - P(B_2) \leq 0 \qquad (16)$$

and a classical system on which a quantum cheat has been applied. Let the angles be $A_1 : \delta_1 = 0$, $B_1 : \delta_2 = \pi/4$, $A_2 : \delta_3 = \pi/2$, $B_2 : \delta_4 = 3\pi/4$. Identify $P(A_i) = P(B_i) = 1/2$ and

$$\begin{aligned}
P(A_1B_1) &= P^{=}_{cl}((\delta_2 - \delta_1)/2 = \pi/8),\\
P(A_2B_2) &= P^{=}_{cl}((\delta_4 - \delta_3)/2 = \pi/8),\\
P(A_1B_2) &= P^{=}_{cl}((\delta_4 - \delta_1)/2 = 3\pi/8),\\
P(A_2B_1) &= P^{=}_{cl}((\delta_3 - \delta_2)/2 = \pi/8).
\end{aligned}$$

With a choice of these angles, the right hand side of Eq. (16) is violated.

Of course, we cannot expect from the cheat parameter to inherit the linear behavior of the old parameter; in particular $\delta_3(\theta_3) = \delta_1(\theta_1) + \delta_2(\theta_2)$ does not imply $\theta_3 = \theta_1 + \theta_2$, and $\delta(\theta_1) + \delta(\theta_2) = \delta(\theta_3)$; i.e.,

$$\delta_3(\theta_1 + \theta_2) \neq \delta_1(\theta_1) + \delta_2(\theta_2) \qquad (17)$$

Therefore, one might call a parameter description to be "proper" if it is isotropic and linear with respect to a reference scheme. Of course, this leaves open the question whether or not it makes sense to refer to particular parameter descriptions as absolute ones; yet at least in the typical experimental physical context this seems evident and appropriate for most physical purposes. In such a scheme, the above mentioned cheat parameters are improper.

4 Summary

There exist extensions of quantum mechanics guided by Hilbert space theory which may be considered as generalizations of the standard formalism. All these extensions are operationalizable and may thus contribute to a better understanding of the quantum phenomena.

In the second section we discussed Boole-Bell type inequalities and Pitowsky correlation polytopes as a criterion for probabilities of automaton partition logics and generalized urn models. We find that, although the event structure is non-boolean, the corresponding probabilities can be represented as linear combinations of dispersion-free states and thus by the hull of the vertices defined by them. The corresponding correlation polytope is a subset of the classical correlation polytope of the Boolean algebra in which these logics can be embedded. The issue of Pitowsky correlation polytopes which exceed their classical counterpart remains open.

Finally, parameter transformation have been discussed which translate classical correlations into nonclassical ones and *vice versa*. While at the first glance the possibility for such a representation seems counterintuitive, a more detailed analysis reveals that the corresponding parameters can be used consistently but have undesirable features such as nonuniformity.

References

[1] Karl Svozil. *Quantum Logic*. Springer, Singapore, 1998.

[2] Simon Kochen and Ernst P. Specker. The problem of hidden variables in quantum mechanics. *Journal of Mathematics and Mechanics*, 17(1):59–87, 1967. Reprinted in [18, pp. 235–263].

[3] Josef Tkadlec. Greechie diagrams of small quantum logics with small state spaces. Jan 1998.

[4] Peter Heywood and Michael L. G. Redhead. Nonlocality and the kochen-specker paradox. *Foundations of Physics*, 13(5):481–499, 1983.

[5] George Boole. *An investigation of the laws of thought*. Dover edition, New York, 1958.

[6] George Boole. On the theory of probabilities. *Philosophical Transactions of the Royal Society of London*, 152:225–252, 1862.

[7] Theodore Hailperin. *Boole's logic and probability (Studies in logic and the foundations of mathematics ; 85)*. North-Holland, Amsterdam, 1976.

[8] Itamar Pitowsky. From George Boole to John Bell: The origin of Bell's inequality. In M. Kafatos, editor, *Bell's Theorem, Quantum Theory and the Conceptions of the Universe*. Kluwer, Dordrecht, 1989.

[9] Itamar Pitowsky. *Quantum Probability—Quantum Logic*. Springer, Berlin, 1989.

[10] Itamar Pitowsky. Correlation polytopes their geometry and complexity. *Mathematical Programming*, 50:395–414, 1991.

[11] Itamar Pitowsky. George Boole's 'conditions od possible experience' and the quantum puzzle. *Brit. J. Phil. Sci.*, 45:95–125, 1994.

[12] Ron Wright. The state of the pentagon. A nonclassical example. In A. R. Marlow, editor, *Mathematical Foundations of Quantum Theory*, pages 255–274. Academic Press, New York, 1978.

[13] Ron Wright. Generalized urn models. *Foundations of Physics*, 20:881–903, 1990.

[14] Karl Svozil. *Randomness & Undecidability in Physics*. World Scientific, Singapore, 1993.

[15] Martin Schaller and Karl Svozil. Automaton logic. *International Journal of Theoretical Physics*, 35(5):911–940, May 1996.

[16] Maximilian Kreuzer and Harald Skarke. LPoly: A package for lattice polyhedra and applications to toric geometry. TUW preprint, 2000.

[17] Günther Krenn and Karl Svozil. Stronger-than-quantum correlations. *Foundations of Physics*, 28(6):971–984, 1998.

[18] Ernst Specker. *Selecta*. Birkhäuser Verlag, Basel, 1990.