Tau regularization of divergent series

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A toy model calculation in zero space-time dimensions compares the convergence of the asymptotic divergent perturbation series obtaind as “penalty’ for illegally exchanging limits with the convergent series obtained by using Tao-type test functions which legally allow an exchange of those limits.

Keywords: perturbation series, asymptodic divergence, Tao regularization

I. DIFFERENT ENCODINGS OF MATHEMATICAL ENTITIES

A formal entity such as the solution of an ordinary differential equation may have very different representations and encodings; some of them with problematic issues. This is often not a matter of choice but means relative, and therefore one of pragmatism or even desperation. In particular, theoretical physicists are often criticised for their “relaxed” stance on formal rigor. Dirac’s introduction of the needle-shaped delta function is often quoted as an example. Heaviside, in another instance, responded to critism for his use of the “highly nonsmooth” unit step function[1, p. 9, § 225]: *“But then the rigorous logic of the matter is not plain! Well, what of that? Shall I refuse my dinner because I do not fully understand the process of digestion? No, not if I am satisfied with the result.”*

This, in a nutshell, seems to be the attitude of field theorists with regards to use of perturbation series: It is well documented [2, 3] that the commonly used expansion in terms of the (square of the) coupling constant is divergent. Indeed the critical step in the derivation amounts to interchanging a sum with an integral in the case of nonuniform convergence of the former [4, Sect. II.A]. One may perceive asymptotic divergence as a “penalty” for such manipulations. Nevertheless, those calculations performed well with respect to empirical predictions.

Suppose it is unknown whether some mathematical entity has a representation in terms of common analytic functions. Nevertheless, in such cases often (power) series representations can be found. The partial sums of those series may converge – hopefully with a “good” rate of convergence – or diverge. In the latter case one can still hope for asymptotic [5–7] divergence which is heuristically characterised by reasonable, increasingly better estimates of the solution up to some “optimal” order of the (power) series, at which point the quality of the approximation deteriorates.

One way of coping with the apparent asymtotic divergence are resummation techniques, in particular, Borel (re)summations [8–13], which are in some instances capable to reconstruct an analytic function from its asymptotic expansion [14].

In what follows we shall adopt another strategy inspired by Ritt’s theorem [15, 16] stating that any (not necessarily asymptotic) divergent power series with arbitrary coefficients can be converted into nonunique analytic functions. Thereby, every summand is multiplied with a suitable nonunique *“convergence factor.”* (Conversely, every analytic function can be approximated by a unique asymptotic series.)

A general regularisation of divergent series utilizing such convergence factors, also called *cutoff functions*, has been recently introduced by Tao [17, Section 3.7]. The resulting smoothed sums may become uniformly convergent, thereby allowing interchanging a sum with an integral and avoiding the aforementioned issues while at the same time preserving inherent properties of the original divergent series. This is not dissimilar to the use of *test functions* in the theory of distributions.

A “canonical” example [18] is the Stieltjes function

$$S(x) = \int_0^\infty \frac{e^{-t}}{1+tx} dt \tag{1}$$

which is a solution of the ordinary differential equation

$$\left(\frac{d}{dx}x + \frac{1}{x}\right) S(x) = \frac{1}{x}. \tag{2}$$

It can be represented by power series in two different ways:

(i) by the asymptotic Stieltjes series

$$S(x) = \sum_{j=0}^{n} (-x)^j j! + (-x)^{n+1}(n+1)! \int_0^\infty \frac{e^{-t}}{(1+tx)^{n+2}} dt \tag{3}$$

as well as

(ii) by classical convergent Maclaurin series such as (Ramanujan found a series which converges even more rapidly)

$$S(x) = \frac{e^{\frac{1}{x}}}{x}\Gamma\left(0,\frac{1}{x}\right) = -\frac{e^{\frac{1}{x}}}{x}\left[\gamma - \log x + \sum_{j=1}^{\infty}\frac{(-1)^j}{j!\,jx^j}\right], \tag{4}$$

where

$$\gamma = \lim_{n\to\infty}\left(\sum_{j=1}^{n}\frac{1}{j} - \log n\right) \approx 0.5772 \tag{5}$$

is the Euler-Mascheroni constant [19]. $\Gamma(z,x)$ represents the upper incomplete gamma function).

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The Stieltjes function $S(x)$ for real positive $x > 0$ can be rewritten in terms of the exponential integral [20, formulæ 5.1.1, 5.1.2, 5.1.4, page 227]

$$\begin{aligned} E_1(y) &= -\mathrm{Ei}(-y) = \Gamma(0,y) \\ &= \int_1^\infty \frac{e^{-uy}}{u} du = \int_y^\infty \frac{e^{-u}}{u} du \end{aligned} \tag{6}$$

by first substituting $x = \frac{1}{y}$ in $S(x)$ as defined in (1), followed by the transformation of integration variable $t = y(u-1)$, so that, for $y > 0$,

$$\begin{aligned} S\left(\frac{1}{y}\right) &= \int_0^\infty \frac{e^{-t}}{1+\frac{t}{y}} dt \\ = ye^y \int_1^\infty \frac{e^{-yu}}{u} du &= ye^y E_1(y) = -ye^y \mathrm{Ei}(-y), \\ \text{or } S(x) &= \frac{e^{\frac{1}{x}}}{x} E_1\left(\frac{1}{x}\right) = \frac{e^{\frac{1}{x}}}{x} \Gamma\left(0, \frac{1}{x}\right). \end{aligned} \tag{7}$$

The asymptotic Stieltjes series (3) quoted in (i) as well as the convergent series (4) quoted in (ii) can, for positive (real) arguments, be obtained by substituting the respective series for the exponential integral [20, formulæ 5.1.51, page 231 and 5.1.10,5.1.11, page 229]:

$$\begin{aligned} E_1(y) &\sim \frac{e^{-y}}{y} \sum_{j=0}^\infty (-1)^j j! \frac{1}{y^j} \\ = \Gamma(0,y) &= -\gamma - \log y - \sum_{j=1}^\infty \frac{(-y)^j}{j(j!)}, \end{aligned} \tag{8}$$

where again γ stands for the Euler-Mascheroni constant and $\Gamma(z,x)$ represents the upper incomplete gamma function [20, 6.5.1, p 260].

There exists a vast literature [8, 11, 21, 22] on how to recover the exact entity from its asymptotic solution; one method employing Borel (re)summation. Thereby the (re)summation of the divergent perturbation theory in quantum electrodynamics [2, 23] may become justifiable and useful.

In what follows we shall pursue a different approach: as the divergence of the perturbation theory could be traced to an illegal exchange of two limits [4] we shall employ methods to formally justify and "legalize" this exchange of limits by suitable cutoff or test functions introduced by Terence Tao [17, § 3.7]. Like in the theorems of generalized functions the results do not depend on the explicit form of those test functions. Already Ritt's theorem – stating that a power series with *arbitrary* coefficients sequence a_i corresponds to a nonunique holomorphic function [11, 15, 16].

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