Converting nonlocality into contextuality

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Diagonalization of matrix pencils provide a uniform technique to transcribe operator-valued violations of Boole's 'conditions of possible experience' involving multipartite correlations into contextuality. They also provide structural analysis of the contexts involved, and thereby suggest compact forms of deviations of quantized systems from classical predictions.

Keywords: contextuality, two-valued states, quantum states, matrix pencil

MATRIX PENCILS

In heuristic terms, 'quantum contextuality' encompasses any aspect that contradicts classical predictions, with 'strong' types of contextuality entailing complete contradictions relative to classical expectations. In what follows we shall concentrate on 'strong quantum contextuality' rendered by operator-valued arguments exhibiting nonlocality. While the inverse problem—converting contextuality into nonlocality [1]—can be of empirical importance, the solution of the former task can identify the particular type of contextuality exhibited.

From a structural standpoint—that is, in terms of the quantum logical algebraic relations of the associated propositions—operator-valued arguments may be closely related, although they may formally appear to be very different. For instance, as observed by Cabello [2, 3], Hardy's theorem [4, 5] can, in quantum logical terms, be transcribed as a true-implies-false arrangement (in graph theoretical terms, a gadget) of observables [6, 7]. However, as we shall see in comparing Kochen-Specker (KS) and Greenberger-Horne-Zeilinger (GHZ) arguments, there need not be such a close relationship.

The algorithmic and thus constructive analysis of a transcription process for cases of operator-value arguments demonstrating nonclassical behavior is based on the proper spectral decomposition of the operators involved. Mutually commuting normal operators (such as Hermitian or unitary operators that commute with their respective adjoints) $A_1,\ldots,A_l$ share common projection operators. However, if their spectra are degenerate we need to find an orthonormal basis in which every single one of this collection of mutually commuting operators is diagonal. Although in principle well-known [8, Section 1.3], the standard procedure via block diagonalization can be rather involved [9]. Alternatively, we can diagonalize the matrix pencil:

$$P=\sum_{i=1}^{l} a_i A_i, \tag{1}$$

where a_i are scalars (for our purposes, real numbers). As P commutes with $A_1,\ldots,A_l$, they share a common set of projection operators. Moreover, since the scalar parameters a_i can be adjusted, and in particular, can be identified with Kronecker delta functions δ_{ij}, and as P commutes with each operator A_j for $1\le j\le l$, P and A_j share a common set of projection operators.

Equipped with these techniques, any collection of commeasurable multipartite observables corresponding to mutually commuting operators can be transcribed into projection operators in the spectrum of the operators of these observables. If these operators render a maximal resolution, the respective vectors correspond to an orthonormal basis called a context with respect to $A_1,\ldots,A_l$. The merging or pasting of possibly intertwining contexts then generates a quantum logic which can be analyzed to identify and characterize the contextual (nonclassical) predictions and features.

MATRIX PENCILS OF THE PERES-MERMIN SQUARE

Applying these techniques to the Peres-Mermin (PM) square [10–13] renders 24 propositions and 24 contexts, henceforth called the 24-24 configuration, that is the 'completion' of the (minimal in four dimensions [14]) 18-9 KS configuration comprising 18 vectors in 9 contexts [2]. In more detail, this configuration involves nine dichotomic observables with eigenvalues ±1 arranged in a 3×3 PM matrix (2). Its rows and columns are masking six four-element contexts, one per row and column ($\sigma_i\sigma_j$ stands for the tensor product of Pauli spin matrices $\sigma_i\otimes\sigma_j$, with similar notation for $\mathbb{1}_2$)

$$\begin{pmatrix} \sigma_z\mathbb{1}_2 & \mathbb{1}_2\sigma_z & \sigma_z\sigma_z \\ \mathbb{1}_2\sigma_x & \sigma_x\mathbb{1}_2 & \sigma_x\sigma_x \\ \sigma_z\sigma_x & \sigma_x\sigma_z & \sigma_y\sigma_y \end{pmatrix}. \tag{2}$$

To explicitly demonstrate the difficulties involved co-diagonalization of commuting degenerate matrices consider the last row of the PM square (2). Its operators $\sigma_z\sigma_x$, $\sigma_x\sigma_z$, and $\sigma_y\sigma_y$ mutually commute—for instance, $[\sigma_z\sigma_x,\sigma_y\sigma_y]=0$. However, a straightforward calculation of the eigenvectors of $\sigma_z\sigma_x$ yields: $\left(0,1,0,1\right)^\intercal$, $\left(-1,0,1,0\right)^\intercal$, $\left(0,-1,0,1\right)^\intercal$, and $\left(1,0,1,0\right)^\intercal$. None of these eigenvectors are eigenvectors of $\sigma_y\sigma_y$, and vice versa. This demonstrates the difficulties involved in co-diagonalizing commuting degenerate matrices.

Nonetheless, the 'joint' PM square contexts are revealed as the normalized eigenvectors of the respective matrix pencils (1). Table I enumerates those contexts, provided that the σ-matrices are encoded in the standard form.

TABLE I. Eigensystems of the matrix pencils of the rows and columns of the PM square (2) with normalization factors omitted. The eigenvectors corresponding to the last row and column are nonseparable and thus entangled, while all others are separable. This set of 24 vectors includes the 18 vectors of Cabello, Estebaranz and García-Alcaine [2]. As already noted by Peres [10], these six 'primary' contexts associated with orthogonal tetrads are disjoint (not intertwined). In the hypergraph representation depicted in Figure 1(a) they are represented as the 'small ovals' on the six edges of the hypergraph.

matrix pencils	eigenvalues			
	$a-b-c$	$-a+b-c$	$-a-b+c$	$a+b+c$
$a\sigma_z\mathbb{1}_2+b\mathbb{1}_2\sigma_z+c\sigma_z\sigma_z$	$\vert 7\rangle=\left(0,1,0,0\right)^\intercal$	$\vert 3\rangle=\left(0,0,1,0\right)^\intercal$	$\vert 1\rangle=\left(0,0,0,1\right)^\intercal$	$\vert 17\rangle=\left(1,0,0,0\right)^\intercal$
$a\mathbb{1}_2\sigma_x+b\sigma_x\mathbb{1}_2+c\sigma_x\sigma_x$	$\vert 20\rangle=\left(-1,-1,1,1\right)^\intercal$	$\vert 13\rangle=\left(-1,1,-1,1\right)^\intercal$	$\vert 11\rangle=\left(1,-1,-1,1\right)^\intercal$	$\vert 24\rangle=\left(1,1,1,1\right)^\intercal$
$a\sigma_z\sigma_x+b\sigma_x\sigma_z+c\sigma_y\sigma_y$	$\vert 21\rangle=\left(1,1,-1,1\right)^\intercal$	$\vert 14\rangle=\left(1,-1,1,1\right)^\intercal$	$\vert 23\rangle=\left(-1,1,1,1\right)^\intercal$	$\vert 10\rangle=\left(-1,-1,-1,1\right)^\intercal$
$a\sigma_z\mathbb{1}_2+b\mathbb{1}_2\sigma_x+c\sigma_z\sigma_x$	$\vert 12\rangle=\left(-1,1,0,0\right)^\intercal$	$\vert 4\rangle=\left(0,0,1,1\right)^\intercal$	$\vert 2\rangle=\left(0,0,-1,1\right)^\intercal$	$\vert 22\rangle=\left(1,1,0,0\right)^\intercal$
$a\mathbb{1}_2\sigma_z+b\sigma_x\mathbb{1}_2+c\sigma_x\sigma_z$	$\vert 15\rangle=\left(-1,0,1,0\right)^\intercal$	$\vert 8\rangle=\left(0,1,0,1\right)^\intercal$	$\vert 6\rangle=\left(0,-1,0,1\right)^\intercal$	$\vert 19\rangle=\left(1,0,1,0\right)^\intercal$
	$a-b-c$	$-a+b-c$	$-a-b+c$	$a+b+c$
$a\sigma_z\sigma_z+b\sigma_x\sigma_x+c\sigma_y\sigma_y$	$\vert 5\rangle=\vert\Psi_-\rangle=\left(0,1,-1,0\right)^\intercal$	$\vert 18\rangle=\vert\Phi_+\rangle=\left(1,0,0,1\right)^\intercal$	$\vert 16\rangle=\vert\Phi_-\rangle=\left(1,0,0,-1\right)^\intercal$	$\vert 9\rangle=\vert\Psi_+\rangle=\left(0,1,1,0\right)^\intercal$

Analysis of their orthogonality relations yields an adjacency matrix that, in turn, can be used to construct the respective (hyper)graph through the intertwining 24 cliques and thus contexts thereof. As can be expected, there are only four-cliques corresponding to orthonormal bases in four dimensional Hilbert space. Figure 1(a) depicts the hypergraph representing these intertwining contexts as unbroken smooth lines, and the vector labels as elements of these contexts, as enumerated in Table I.

The 24 rays were already discussed by Peres [10] as permutations of the vector components of $\left(1,0,0,0\right)^\intercal$, $\left(1,1,0,0\right)^\intercal$, $\left(1,-1,0,0\right)^\intercal$, $\left(1,1,1,1\right)^\intercal$, $\left(1,1,1,-1\right)^\intercal$, and $\left(1,1,-1,-1\right)^\intercal$. The 'full' 24-24 configuration was obtained by Pavičić [15] who reconstructed additional 18 contexts not provided in the original Peres paper [10] by hand [16]. Peres' 24-24 configuration is arranged in four-element contexts associated with four-dimensional Hilbert space, with vector components drawn from the set $\{-1,0,1\}$, that do not support any two-valued state. Pavičić, Megill and Merlet [17, Table 1] have demonstrated that Peres' 24-24 configuration contains 1,233 sets that do not support any two-valued states. Among these 1,233 sets are six 'irreducible' or 'critical' configurations which do not contain any proper subset that does not support two-valued states. Notably, among these configurations is the previously mentioned 18-9 configuration proposed by Cabello, Estebaranz and García-Alcaine [2]. Previously, Pavičić, Merlet, McKay, and Megill [18, 19, Section 5(viii)] had shown that, among all sets with 24 rays and vector components from the set $\{-1,0,1\}$, and 24 contexts, only one configuration does not allow any two valued state—and that one is isomorphic to Peres' 'full' (including 18 additional contexts) 24-24 configuration enumerated by Pavičić [15]. This computation had taken one year on a single CPU of a supercomputer [16]. More recently, Pavičić and Megill [20, Table 1] have demonstrated that the vector components from the set $\{-1,0,1\}$ vector-generate a 24-24 set, which contains all smaller KS sets and is simultaneously isomorphic to the 'completed' 24-24 configuration configuration.

We conjecture without providing a formal proof that if a 'larger' collection of quantum observables (such as 24-24) contains a 'smaller' collection of quantum observables (such as 18-9), then it inherits the scarcity or total absence of two-valued states of the latter: if the 'smaller' set cannot support features related to two-valued states, such as separability of propositions [21, Theorem 0], then intertwining or adding contexts can only impose further constraints, thereby exacerbating the situation by introducing new conditions.

BIPARTITE GREENBERGER-HORNE-ZEILINGER ARGUMENT

Based on the GHZ argument Mermin has suggested [12, 22] a "simple unified form for the major no-hidden-variables theorems" in which he identified four commuting three-partite operators: $\sigma_x\sigma_x\sigma_x$, $\sigma_x\sigma_y\sigma_y$, $\sigma_y\sigma_x\sigma_y$, and $\sigma_y\sigma_y\sigma_x$. A parity argument reveals a state-independent quantum contradiction to the classical existence of (local, noncontextual) elements of physical reality: The quantum mechanical expectation of the product of these four commuting three-partite operators for any quantum state is $-1=\langle -\mathbb{1}_8\rangle = \langle \mathbb{1}_2(-\mathbb{1}_2)\mathbb{1}_2\rangle = \langle(\sigma_x\cdot\sigma_x\cdot\sigma_y\cdot\sigma_y)(\sigma_x\cdot\sigma_y\cdot\sigma_x\cdot\sigma_y)(\sigma_x\cdot\sigma_y\cdot\sigma_y\cdot\sigma_x)\rangle = \langle(\sigma_x\sigma_x\sigma_x)\cdot(\sigma_x\sigma_y\sigma_y)\cdot(\sigma_y\sigma_x\sigma_y)\cdot(\sigma_y\sigma_y\sigma_x)\rangle = \langle\sigma_x\sigma_x\sigma_x\rangle\langle\sigma_x\sigma_y\sigma_y\rangle\langle\sigma_y\sigma_x\sigma_y\rangle\langle\sigma_y\sigma_y\sigma_x\rangle$. In this formulation, every operator σ_x and σ_y for each of the three particles occurs twice. Therefore, if classically all such single-particle observables would coexist as elements of physical reality and independent of what other measurements are made alongside, then their respective product must be 1, the exact negative of the quantum expectation.

Mermin's configuration can be analyzed in terms of its matrix pencil $a\sigma_x\sigma_x\sigma_x+b\sigma_x\sigma_y\sigma_y+c\sigma_y\sigma_x\sigma_y+d\sigma_y\sigma_y\sigma_x$, thereby

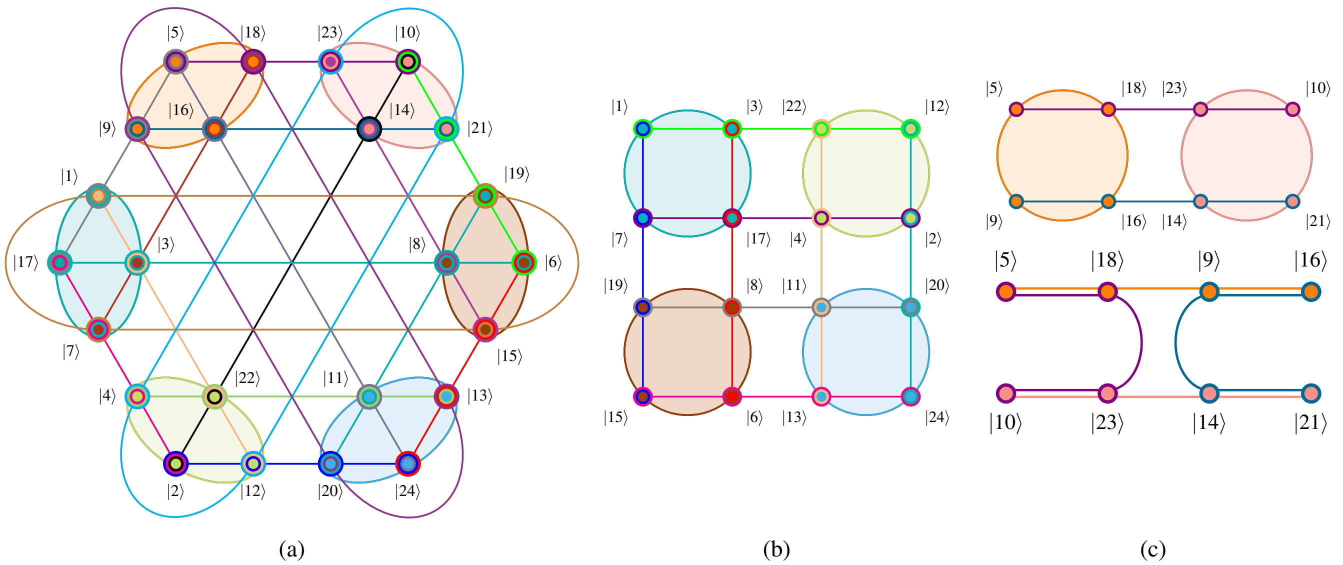

FIG. 1. (a) Hypergraph representing contexts (or cliques or orthonormal bases or maximal operators) as unbroken smooth lines. This is a 'orthogonal completion' [10, 15] of the KS set comprising 18 vectors in 9 contexts introduced by Cabello, Estebaranz and García-Alcaine [2]. The filled shaded small ovals on the edges correspond to the 'primary' isolated (nonintertwined) contexts from the matrix pencil calculations enumerated in Table I. (b) Hypergraph representing a 16-12 configuration: 16 elements in 12 contexts enumerated in the first, second, fourth, and fifth row of Table I. These vectors are separable and thus correspond to factorizable, nonentangled states. (c) Two equivalent hypergraph representations of a 8-4 configuration—8 elements in 4 contexts enumerated in the third and sixth row of Table I. These vectors are nonseparable and thus correspond to entangled states.

revealing the underlying, hidden context in terms of the simultaneous eigensystem of the four mutually commuting operators. These eight nonseparable vectors form an orthonormal basis of an eight-dimensional Hilbert space corresponding to an isolated single context [23, Table 1] of entangled states. Therefore, Mermin's configuration does not constitute a KS proof, as it still permits a separating set of eight two-valued states.

In view of this, how does one arrive at a complete GHZ contradiction with classical elements of physical reality, as outlined above? The criterion employed in an experimental corroboration [24] is to select any one of the eigenstates forming the orthonormal basis, such as $(1/\sqrt{2})\big(|z_+z_+z_+\rangle + |z_-z_-z_-\rangle\big)$. Since this is an eigenstate of all four terms of the matrix pencil, four separate measurements can be performed (possibly temporally separated) yielding the eigenvalues $+1$ for $\sigma_x\sigma_x\sigma_x$ as well as -1 for the three others. These three factors -1 and one factor $+1$ contribute to their product value -1, in total contradiction to the classical expectation $+1$. Note that similar contradictions arise if the seven other eigenstates of the matrix pencil are considered [23, Table 1].

Can an equally convincing argument be made for just two particles? Natural candidates would be the 'nonclassical' elements of the PM square (2). Note that its 'masked' or 'hidden' contexts, revealed by the matrix pencils, can be partitioned into four 'separable' type contexts depicted in Figure 1(b) containing only separable vectors—corresponding to the first and second rows and columns—and two 'nonclassical' contexts consisting of nonseparable vectors—corresponding to the last row and column, as depicted in Figure 1(c).

Concentrating on these two latter contexts consisting of nonseparable vectors, we make the following observations: Since the observables from the last row and last column (with the exception of $\sigma_y\sigma_y$) do not commute, they cannot be simultaneously measured. Nevertheless, by forming products within the last row and column, we may create two commuting operators $(\sigma_z\sigma_x)\cdot(\sigma_x\sigma_z) = -(\sigma_x\sigma_x)\cdot(\sigma_z\sigma_z) = (\sigma_z\cdot\sigma_x)(\sigma_x\cdot\sigma_z) = \sigma_y\sigma_y = \text{antidiag}\Big(-1,1,1,-1\Big)$. Their matrix pencil

$$a(\sigma_z\sigma_x)\cdot(\sigma_x\sigma_z) + b(\sigma_x\sigma_x)\cdot(\sigma_z\sigma_z) \tag{3}$$

has a degenerate spectrum with the Bell basis as eigenvectors—the same as the eigenvectors of the matrix pencil of the last column of the PM square. It is enumerated in Table II.

Hence, preparing a state in one Bell basis state and measuring (successively or separately) $(\sigma_z\sigma_x)\cdot(\sigma_x\sigma_z)$, and $(\sigma_x\sigma_x)\cdot(\sigma_z\sigma_z)$ or $\sigma_x\sigma_x$ as well as $\sigma_z\sigma_z$ separately, yields

$$\begin{aligned} -1 &= \langle -\mathbb{1}_4\rangle = \langle \mathbb{1}_2(-\mathbb{1}_2)\rangle \\ &= \langle (\sigma_z\cdot\sigma_x\cdot\sigma_x\cdot\sigma_z)(\sigma_x\cdot\sigma_z\cdot\sigma_x\cdot\sigma_z)\rangle \\ &= \langle (\sigma_z\sigma_x)\cdot(\sigma_x\sigma_z)\cdot(\sigma_x\sigma_x)\cdot(\sigma_z\sigma_z)\rangle \\ &= \langle (\sigma_z\sigma_x)\cdot(\sigma_x\sigma_z)\rangle\langle (\sigma_x\sigma_x)\cdot(\sigma_z\sigma_z)\rangle. \end{aligned} \tag{4}$$

In contrast, and in analogy to Mermin's version of the GHZ argument, the classical prediction is that the product of these terms always needs to be positive, as every alleged 'element

TABLE II. Eigensystem of the matrix pencil (3) associated with the commuting products of operators in the last (third) row and the last (third) column of the PM square, constituting the Bell basis. Inclusion of $(\sigma_y\sigma_y)\cdot(\sigma_y\sigma_y)=\mathbb{1}_4$ does not change the calculation and is therefore omitted. The values $+1$ and -1 represent the (co)measured values of the respective commuting operators.

value	vector	$(\sigma_z\sigma_x)\cdot(\sigma_x\sigma_z)$	$\sigma_x\sigma_x$	$\sigma_z\sigma_z$	$(\sigma_x\sigma_x)\cdot(\sigma_z\sigma_z)$
$a-b$	$\vert\Psi_+\rangle$	$+1$	$+1$	-1	-1
$a-b$	$\vert\Phi_-\rangle$	$+1$	-1	$+1$	-1
$-a+b$	$\vert\Psi_-\rangle$	-1	-1	-1	$+1$
$-a+b$	$\vert\Phi_+\rangle$	-1	$+1$	$+1$	$+1$

of reality', in particular corresponding to σ_x and σ_z, enters an even number of times (indeed, twice per particle).

I conclude with some comments and an outlook. A nonlocal measurement in quantum mechanics refers to the simultaneous measurement of properties of entangled particles that are—at least in principle—located in space-like separated regions (Einstein locality). We therefore suggest calling an operator, or a collection of mutually commuting operators, 'nonlocal' if they—or more generally, the eigensystem of their matrix pencil—allow entangled, that is, nonseparable, eigenstates after projective measurements. This is the case for the last row and column of the PM square, and also for the four three-partite operators suggested by Mermin in the context of the GHZ argument. I shall motivate and discuss these issues further in a later publication.

The matrix pencil method provides an elegant solution for simultaneously diagonalizing commuting operators with degenerate spectra. It offers a systematic approach for the application of 'contextual' nonclassical performance in quantized systems, particularly in delineating operator-valued arguments.

The PM square demonstrates a fundamental contradiction (quantum -1 versus classical $+1$) compared to classical existence in a dichotomic operator-valued formulation. By employing matrix pencils, this contradiction can be transcribed into a KS type argument with 24 vectors. This configuration, which does not support any binary (two-valued) state, consist of 6 'original' isolated contexts from the matrix pencils associated with every row and column of the PM square, as well as 18 'secondary' intertwining contexts obtained by studying orthogonalities.

Mermin's rendition of the GHZ operator-valued argument is indeed altogether different. When transcribed into quantum logic, it reveals a single isolated context that is perfectly set representable, for instance, by partition logic. Thereby, the quantum state becomes crucial for any experimental corroboration: if one takes any eigenstate of the matrix pencil it leads to a complete contradiction (again quantum -1 versus classical $+1$) when multiplying all the results and comparing the squares of operators in a parity argument.

In analyzing the 'entangled contexts' corresponding to the last row and column of the PM square and constructing mutually commuting products thereof, one arrives at a very similar argument as Mermin's rendition of the GHZ argument. It is also state-independent and operates within a single context. The operators are: $(\sigma_z\sigma_x)\cdot(\sigma_x\sigma_z)$ and alternatively, $(\sigma_x\sigma_x)\cdot(\sigma_z\sigma_z)$ or $\sigma_x\sigma_x$ and $\sigma_z\sigma_z$ and, although not needed for the constraction, $(\sigma_y\sigma_y)\cdot(\sigma_y\sigma_y)$. These operators commute, and for the Bell basis yield at a complete contradiction (quantum -1 versus classical $+1$) contingent on the assumption of noncontextual classical existence of those elements of physical reality. This reduces the eight-dimensional argument to a four-dimensional one. It might be interesting to probe the factors $\sigma_z\cdot\sigma_x$ and $\sigma_x\cdot\sigma_z$ of the tensor product $(\sigma_z\sigma_x)\cdot(\sigma_x\sigma_z)=(\sigma_z\cdot\sigma_x)(\sigma_x\cdot\sigma_z)$ by the Bell states $|\Psi_-\rangle$ and $|\Phi_+\rangle$ in an Einstein-Podolsky-Rosen configuration, because this alone could 'isolate' the 'rub', as the quantum prediction of the observed value would be -1. Likewise, application of the Bell states $|\Psi_+\rangle$ and $|\Phi_-\rangle$ on $(\sigma_z\sigma_z)\cdot(\sigma_x\sigma_x)=(\sigma_z\cdot\sigma_x)(\sigma_z\cdot\sigma_x)$ would result in an observed value -1.

Why or how can such operator-valued contradictions arise in the context of a single isolated context? Because measurements like $\sigma_x\sigma_y(\sigma_y)$ as part of a context from a matrix pencil should not be perceived as 'local' and cannot be performed as two (or three) single-qubit local measurements [13]. Such operator-valued arguments are based on a classically justified conviction that every two- (or three-) particle state can be composed of single-particle states in such a way that the former retains all properties of the latter. This is no longer true for entangled states, which encode relational information at the (unitary) cost of abandonment of local properties. From this perspective, both dichotomic operator-valued GHZ arguments as well as binary two-valued state KS arguments against noncontextuality share a nonoperational and thus (meta)physical presumption: the contingent use of counterfactuals.

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