Conventions in relativity theory and quantum mechanics

Karl Svozil

Institut für Theoretische Physik, University of Technology Vienna

Wiedner Hauptstraße 8-10/136, A-1040 Vienna, Austria

e-mail: svozil@tuwien.ac.at

Abstract

The conventionalistic aspects of physical world perception are reviewed with an emphasis on the constancy of the speed of light in relativity theory and the irreversibility of measurements in quantum mechanics. An appendix contains a complete proof of Alexandrov's theorem using mainly methods of affine geometry.

1

1 Know thyself

This inscription on the Oracle of Apollo at Delphi, Greece, dates from 6th century B.C., and it is still of tremendous importance today. For we do not and never will see the world "as is," but rather as we perceive it. And how we perceive the world is mediated by our senses which serve as interfaces to the world "out there" (if any); but not to a small extend also by what we project onto it. Conventions are projections which we have to adopt in order to be able to cope with the phenomena "streaming in" from the senses. Conventions are a necessary and indispensable part of operationalizable[1] phenomenology and tool-building. There is no perception and intervening without conventions. They lie at the very foundations of our world conceptions. Conventions serve as a sort of "scaffolding" from which we construct our scientific worldview. Yet, they are so simple and almost self-evident that they are hardly mentioned and go unreflected.

To the author, this unreflectedness and unawareness of conventionality appears to be the biggest problem related to conventions, especially if they

---

[1]In what follows we shall adopt Bridgman's concept of "operational" as one of quite simple-minded experimental testability, even in view of its difficulties which this author approached later on [1, 2, 3, 4, 5].

are mistakenly considered as physical "facts" which are empirically testable. This confusion between assumption and observational, operational fact seems to be one of the biggest impediments for progressive research programs, in particular if they suggest postulates which are based on conventions different from the existing ones.

In what follows we shall mainly review and discuss conventions in the two dominating theories of the 20th century: quantum mechanics and relativity theory.

2 Conventionality of the constancy of the characteristic speed

Suppose two observers called Alice and Bob measure space and time in two coordinate frames. Operationally their activities amount to the following. They have constructed "identical" clocks and scales of "equal" length which they have compared in the distant past; when Bob lived together with Alice. Then they have separated. Alice has decided to depart from Bob and, since then, is moving with constant speed away from him. How do Bob's and

Alice's scales and clocks compare now? Will they be identical, or will they dephase?

These are some of the questions which "relativity" theory deals with. It derives its name from Poicaré's 1904 "principle of relativity" stating that [6, p. 74] *"the laws of physical phenomena must be the same for a stationary observer as for an observer carried along in a uniform translation; so that we have not and can not have any means of discerning whether or not we are carried along in such a motion."* Formally, this amounts to the requirement of form invariance or covariance of the physical equations of motion.

One of the seemingly mindboggling features of the theory of relativity is the fact that simultaneity and even the time order of two events needs no longer be conserved. It may indeed happen that Alice perceives the first event before the second, while Bob perceives both events as happening at the same time; or even the second event ahead of the first. Simultaneity can only be defined "relative" to a particular reference frame. If true there, it is false in any different frame.

The first part of Einstein's seminal paper [7] is dedicated to a detailed study of the intrinsically operational procedures and methods which are necessary to conceptualize the comparison of Alice's and Bob's reference

frames. This part contains certain "reasonable" conventions for defining clocks, scales, velocities and in particular simultaneity, without which no such comparison could ever be achieved. These conventions appear to be rather evident and natural, almost trivial, and yield a convenient formalization of space-time transformations, but they are nevertheless arbitrary. The simultaneity issue has been much debated in the contemporary discussions on conventionality [8, 9, 10].

There is another element of conventionality present in relativity theory which has gotten less attention [11]. It is the assumption of the constancy of the speed of light. Indeed, the International System of units assumes light to be constant. It was decided in 1983 by the General Conference on Weights and Measures that the accepted value for the speed of light would be exactly 299,792,458 meters per second. The meter is now thus defined as the distance traveled by light in a vacuum in 1/299,792,458 second. With the speed of light no longer being an empirical fact but a convention, the proper empirical statement is that the length of a rod neither depends on the orientation nor on the inertial frame where that body is at rest [12].

Despite the obvious conventionality of the constancy of the speed of light, many introductions to relativity theory present this proposition not as a con-

vention but rather as an important empirical finding. Indeed, it is historically correct to claim that experiments like the ones of Fizeau, Hoek and Michelson-Morley, which produced a null result by attempting to measuring the earth's motion against some kind of "ether," preceded Einstein's special theory of relativity.

But this may be misleading. First of all, Einstein's major reason for introducing the Lorentz transformation seems to have been the elimination of asymmetries which appeared in the electromagnetic formalism of the time but are not inherent in the phenomena, thereby unifying electromagnetism. Secondly, not too much consideration has been given to the possibility that experiments like the one of Michelson and Morley may be a kind of "self-fulfilling prophecy," a circular, closed tautologic exercise. If the very instruments which should indicate a change in the velocity of light are themselves dilated, then any dilation effect will be effectively nullified. This possibility has already been imagined in the 18th century by Boskovich [13] and was later put forward by FitzGerald [14] (see also John Bell [15, 16]), Lorentz, Poincaré and others in the context of the ether theory [6].[2]

---

[2]In stressing the conventionality aspect of these effects, the author would like to state that he does not want to promote any ether-type theory, nor is he against any such

But what is the point in arguing that the constancy of the speed of light is a convention rather than an empirical finding? Is this not a vain question; devoid of any operational testability?

The answer to this concern is twofold. First, a misinterpretation might give rise to a doctrinaire and improper preconception of relativity theory by limiting the scope of its applicability. Indeed, as it turns out, for reasons mentioned below [11], the special theory of relativity is much more generally applicable as is nowadays appreciated. It applies also to situations in which the velocity of light needs not necessarily be the highest possible limit velocity for signaling and travel. Secondly, it is not totally unreasonable to ask the following question. What if one adopts a different convention by assuming a different velocity than that of light to be the basis for frame generation? Such a velocity may be anything, sub- but also superluminal. What will be the changes to Alice's and Bob's frames, and how do these new coordinates relate to the usual "luminal" frames?

These issues have been discussed by the author [11] on the basis of a geo-metrical theorem by Alexandrov [17, 18, 19, 20] and Borchers and Hegerfeldt

attempts.

[21] reviewed in [22, 23] (see also previous articles [24, 25]). Alexandrov's theorem requires the convention that some speed is the same in Bob's and Alices's frames. Furthermore, if two space-time points are different in Alice's frame, then these points must also be mapped into different points in Bob's frame and vice versa; i.e., the mapping must be one-to-one, a bijection. It can be proven that under these conditions, the mapping relating Alice's and Bob's frames must be an affine Lorentz transformation, with some fundamental speed playing the role of light in the usual Lorentz transformations of relativity theory. The nontrivial geometric part of a proof uses the fundamental theorem of affine geometry, which results in the linearity of the transformation equations. No Poicaré's 1904 "principle of relativity," no relativistic form invariance or covariance is needed despite the postulate or convention of equality of a single speed in all reference frames. The derivation uses geometry, not physics. The Appendix contains a detailed derivation of Alexandrov's theorem which should be conprehensive for a larger audience.

To repeat the gist: it is suggested that the signalling velocity occurring in the Lorentz transformation is purely conventional. This effectively turns the interpretation of relativity theory upside down and splits it into two parts, one geometric and one physical, as will be discussed next.

8

So where is the physics gone? The claim of conventionality arouses suspicions. The proper space-time transformations cannot be purely conventional or even a matter of epistemology! After all, the Michelson-Morley experiment and most of its various pre- and successors actual yielded null results, which are valid physical observations as can be. The experimenters never explicitly acknowledged the conventionality of the constancy of the speed of light and approved their instruments according to these specifications. Just on the contrary, they first assumed to measure the inequality and anisotropy of the speed of light. And what if Alice and Bob assume, say, the constancy of the speed of sound instead of light? Would the mere assumption change the reading of the instruments in a Michelson-Morley experiment using sound instead of light? This seems to be against all intuitions and interpretations and the huge accumulated body of evidence.

The answer to these issues can be sketched as follows. First of all, the physics is in the form invariance of the electromagnetic equations under a particular type of Lorentz transformations: those which contain the speed of electromagnetic signals; i.e., light, as the invariant speed. Thus, merely the convention of the constancy of the speed of light in all reference frames yields the desirable relativistic covariance of the theory of electromagnetism. This

is a preference which cannot be motivated by geometry or epistemology; it is purely physical.

However, any such Lorentz transformation will result in a non-invariance of the theory of sound or any other phenomena which are not directly dominated by electromagnetism. Operationally, the length of rods which are stabilized by the interactions represented by the non-invariant laws would depend on the inertial frame where they are at rest [12]. Furthermore, an asymmetry will appear, singling out a particular frame of reference from all the other ones.

Thus, one may speculate that the most efficient "symmetric" representation of the physical laws is by transformations which assume the convention of the invariant signalling velocity which directly reflects the phenomena involved. For electromagnetic phenomena it is the speed of electromagnetic waves; i.e., light. For sound phenomena it is the speed of sound. For gravity it is the speed of gravitational waves. Thus the conventionality of the relativity theory not only relativizes simultaneity but must also reflect the particular class of phenomena; in particular their transmission speed(s). In that way, a general form invariance or covariance is achieved, satisfying Poicaré's 1904 "principle of relativity" mentioned above, which is not only limited to

electromagnetism but is valid also for a wider class of systems.

Secondly, it is not unreasonable to assume that in the particular context of the Michelson-Morley and similar experiments, all relevant physical system parameters and instruments are governed by electromagnetic phenomena and not by sound, gravity or something else. Thus, although not explicitly intended, the experiments are implicitly implementing the conventionality of the constancy of light. Of course, the experimenter could decide to counteract the most natural way to gauge the instruments and measure space and time differently than as suggested by the instruments. For instance, one may adopt scales to measure space which are anisotropic and velocity dependent. But this would be a highly unreasonable, inconvenient and complicating thing to do.

One may neverthelss speculate that, if all phenomena could be explained by a single unified interaction, then the speed of this interaction should be taken as the fundamental speed which should be invariant in all reference frames. Thus such an interaction would make a strong suggestion towards this particular choice of conventions. But then, one may ask whether this could still be considered a free choice, or simply would reflect a distinguished feature of the physical world.

Hence, from a system theoretic standpoint, the proper convention suggests itself by the dominating type of interaction, and only in this way corresponds to a physical proposition. The result is a generalized principle of relativity.

3   Conventionality of quantum measurements

In what follows the idea is put forward and reviewed that measurements in quantum mechanics, or indeed in any reversible system, are (at least in principle) purely conventional. More precisely, it is purely conventional and subjective what exactly an "observer" calls "measurement." There is no distinction between "measurement" and ordinary (unitary) quantum evolution other that the particular interpretation some (conscious?) agent ascribes to a particular process [26]. Indeed, the mere distinction between an "observer" and the "object" measured is purely conventional. Stated pointedly. measurement is a subjective illusion. We shall call this the "no measurement" interpretation of quantum mechanics.

The idea that measurements, when compared to other processes (involving entanglement), are nothing special, seems to be quite widespread among

the quantum physics community; but it is seldom spelled out publicly [27, 28]. Indeed, the possibility to "undo" a quantum measurement has been experimentally documented [29], while it is widely acknowledged that practical bounds to maintain quantum coherence pose an effective upper limit on the possibility to reconstruct a quantum state. We shall not be concerned with this upper bounds, which does not seem to reflect some deep physical truth but rather depends on technology, financial commitments and cleverness on the experimenter's side.

Rather, we shall discuss the differences between the two types of time evolution which are usually postulated in quantum mechanics: (i) unitary, one-to-one, isometric time evolution between two measurements and (ii) many-to-one state reduction at the measurement.

Between two measurements, the quantum state undergoes a deterministic, unitary time evolution, which is reversible and one-to-one. This amounts to arbitrary generalized isometries—distance-preserving maps—in complex Hilbert space. A discrete analogue of this situation is the permutation of states. An "initial message" is constantly being re-encoded. As the evolution is reversible, there is no principle reason why it should not be possible to "undo" it. (There may be insurmountable practical obstacles, though.)

Any irreversible measurement is formally accompanied by a state reduction or "wave function collapse" which is many-to-one. Indeed, this many-to-oneness is the formal mathematical expression of irreversibility.

What is a measurement? Besides all else, it is associated with a some sort of "information" transfer through a fictitious boundary between some "measurement apparatus" and the "object." In the following we shall call this fictitious boundary the "interface." The interface has no absolute physical relevance but is purely conventional. It serves as a scaffolding to mediate and model the exchange. In principle, it can be everywhere. It is symmetric: the role of the observer and the observed system is interchangeable and a distinction is again purely conventional.

In more practical terms, it is mostly rather obvious what is the observer's side. It is usually inhabited by a conscious experimenter and his measurement device. It should be also in most cases quite reasonable to define the interface as the location where some agent serving as the experimenter looses control of one-to-onenness. This is the point where "the quantum turns classical." But from the previous discussion it should already be quite clear that any irreversibility in no way reflects a general physical principle but rather the experimenter's ability to reconstruct previous states. Another "observer"

14

equipped with another technology (or just more money) may draw very different interface lines.

In this line of thought, one pragmatic argument against the conventionality of measurements can be put forward as follows. In a typical idealized experiment, an observer registers some click in a counter, or looks at a lamp which flashes if a particle has been detected. An observer looking at the lamp will become aware of this new fact, which may be seen as an elementary act of measurement. Any such observation may then be interpreted as a gain of knowledge about the physical world and thus, it may be argued, could not be perceived as being a subjective illusion. In this line of thought, the cut between observer and observed object could not be shifted arbitrarily, and therefore measurements are not merely conventional.

Such view assumes that the awareness of a conscious observer is something beyond the domain of the reversible system in which the observer is embedded. Pointedly stated, the conscious observer cannot "unthink" a measurement, since consciousness resides in a domain which is metaphysical and beyond the reach of the reversible physical system. (If it were not so, there would be no obstacle to (at least in principle) "unthink" any elementary act of observation on the observer's side.)

15

Yet even if no transcendence on the observer's side is involved, "unthinking" a measurement for all practical purposes will never happen, as the task of reconstruction of the observer's state turns out to be too hard. The "objective physical reality" emerges from a consensus of such "macroscopic" intrinsic observers. "Macroscopic" here means "impossible to reconstruct." In this scenario, although principally acknowledging the conventionality of measurement, the cut between observer and observed object emerges due to effective inabilities to reconstruct previous states of reversible physical systems.

Let me add here one particular aspect of quantum information. Assume as an axiom that a physical system always consists of a natural number of n quanta which are in a single pure state among q others. Any single such particle is thus the carrier of exactly one q-it, henceforth called "quit." (In the spin-one half case, this reduces to the bit.) That is, encoded in such a quantum system are always n quits of information. The quit is an irreducible amount of classical and quantum information. The quits need not be located at any single particle (i.e., one quit per particle), but they may be spread over the n particles [30]. In this case one calls the state of the particles "entangled." According to Schrödinger's own interpretation [31], the quantum wave

16

function (or quantum state) is a "catalogue of expectation values" about this state; and in particular about the quits. Since an experimenter's knowledge about a quantum system may be very limited, the experimenter might not have operational access to the "true" pure state out there. (In particular, it need not be clear which questions have to be asked to sort out the valid pure state from other ones.) This ignorance on the experimenter's side is characterized by a nonpure state. Thus one should differentiate between the "true" quantum state out there and the experimenter's "poor man's version" of it. Both type of states undergo a unitary time evolution, but their ontological status is different.

Why has the no-measurement interpretation of quantum mechanics been not wider accepted and has attracted so little attention so far? One can only speculate about the reasons.

For one thing, the interpretation seems to have no operational, testable consequences. Indeed, hardly any interpretation does. So, what is any kind of interpretation of some formalism good for if it cannot be operationalized?

Think of the Everett interpretation of quantum mechanics, which is nevertheless highly appreciated among some circles, mainly in the quantum computation community. It has to offer no operationalizable consequences, just

mindboggling scenarios.

Or consider Bohr's "Copenhagen" interpretation, whatever that means to its successors or to Bohr himself. It is the canonical interpretation of quantum mechanics, a formalism co-created by people, most notably Einstein, Schrödinger and De Broglie, who totally disagreed with that interpretation. This does not seem to be the case for Heisenberg and von Neumann. The latter genius even attempted to state an inapplicable theorem directed against hidden parameters to support some of Bohr's tendencies. Nowadays, many eminent researchers in the foundations of quantum mechanics still stick with Bohr's interpretation or whatever sense they have made out of it. But does Bohr's "Copenhagen" interpretation have any operational consequences?

With the advent of quantum information theory, the notion of information seems to be the main interpretational concept. Consequently, information interpretations of quantum mechanics begin to be widespread. Yet, despite the heavy use of the term "information," the community does not seem to have settled upon an unambiguous meaning of the term "information." And also in this case, the interpretations do not seem to have operational consequences.

Many recent developments in quantum information theory are consistent

with the no-measurement interpretation. Unitarity and the associated one-to-onenness even for one quantum events seems to be the guiding principle. Take, for example, the no-cloning theorem, quantum teleportation, entanglement swapping, purification and so on [32, 33]. Actually, the no-measurement interpretation seems to promote the search for new phenomena in this regime, and might thus contribute to a progressive research program.

Indeed, it is the author's belief that being helpful in developing novel theories and testing phenomena is all one can ever hope for a good interpretation. Any "understanding" of or "giving meaning" to the formalism is desirable only to the effect that it yields new predictions, phenomena and technologies. And in this sense, the no-measurement interpretation claiming the conventionality of quantum measurements should be perceived. It too cannot offer direct operationalizable consequences, yet may facilitate thoughts in new, prosperous directions.

4 Summary

We have reviewed conventions in two of the dominating theories of contemporary physics, the theory of relativity and quantum mechanics. In relativity

theory we suggest to accept the constancy of one particular speed as a convention. Lorentz-type transformation laws can then be geometrically derived under mild side assumptions. In order for a generalzied principle of relativity and thus generalized form invariance to hold, the particular signalling type entering the transformations should correspond to the dominating type of physical interactions.

The no-measurement interpretation of quantum mechanics suggests that there is no such thing as an irreversible measurement. In fact, there is no measurement at all, never. This kind of irreversibility associated with the measurement process is just an idealistic, subjective construction on the experimenter's side to express the for-all-practical-purposes impossibility to undo a measurement.

Appendix. Proof of Alexandrov's theorem

Alexandrov's theorem states that, for $\mathbb{R}^n$, $n \geq 3$ with the metric $\mathrm{diag}(+, +, +, \cdots, +, -)$ and a one-to-one map $r \mapsto r'$ preserving light cones (i.e., zero distance) such that for all $r, s \in \mathbb{R}^n$,

$$(r - s, r - s) = 0 \iff (r' - s', r' - s') = 0,$$

$r \mapsto r'$ is essentially a Lorentz transformation; i.e., it has the form $r \mapsto r' = \alpha Lr + a$ for some nonzero $\alpha \in \mathbb{R}$, $a \in \mathbb{R}^n$, and a linear one-to-one map $L : \mathbb{R}^n \mapsto \mathbb{R}^n$ satisfying $(Lr, Ls) = (r, s)$ for all r and s in $\mathbb{R}^n$.

In what follows we shall review a complete proof of Alexandrov's theorem very similar to the one sketched by Lester [23]. The proof consists of three stages:

(I) a proof that, given $\mathbb{R}^n$, $n \geq 3$ with the metric $\text{diag}(+, +, \cdots, +, -)$ and a one-to-one map preserving light cones (i.e., zero distance), all lines are mapped onto lines;

(II) a proof of the *fundamental theorem of affine geometry* stating that a one-to-one map from $\mathbb{R}^n$, $n \geq 2$ onto itself which maps all lines onto lines must be *affine*; i.e., must be a *linear* map followed by a translation;

(III) a proof that, given $\mathbb{R}^n$, $n \geq 2$ with the metric $\text{diag}(+, \cdots, +, -)$ and a *linear* one-to-one map preserving a single light cone (i.e., zero distance) must be essentially a Lorentz transformation (up to a translation and a dilatation); i.e., it has the form $r \mapsto \alpha Lr + a$ for some nonzero $\alpha \in \mathbb{R}$, $a \in \mathbb{R}^n$, and a linear one-to-one map $L : \mathbb{R}^n \mapsto \mathbb{R}^n$ satisfying $(Lr, Ls) = (r, s)$ for all r and s in $\mathbb{R}^n$.

21

In what follows, a constant translation is taken account of by addition of a vector $a \in \mathbb{R}^n$. The remaining transformation preserves the origin; i.e., $0 \mapsto 0'$. We shall often refer to this remaining transformation (after the constant parallel shift moving the map of the origin back into the origin) simply as (homogeneous) transformation. (Note that if $f : r \mapsto \alpha L r + a$, then the homogeneous part is obtained by subtracting $a = f(0)$.) This constant shift a has to be added to the final mapping.

The geometric proof of (I) proceeds in five steps, covering the mapping of (i) lightlike (null) lines onto lightlike lines; (ii) lightlike (null) planes onto null planes; (iii) spacelike lines onto spacelike lines; (iv) timelike planes onto planes; and finally (v) timelike lines onto lines.

In what follows, the configurations are demonstrated for $\mathbb{R}^3$ with the metric $(r, s) = r_1 s_1 + r_2 s_2 - (1/c^2) r_3 s_3$. For arbitrary dimensions we refer to [22]. In this section, the velocity of light c will be set to unity; i.e., $c = 1$. The terms "null" and "lightlike" will be used synonymously.

To show (i) let us first define a *null cone* with *vertex a* by

$$\mathcal{C}(a) = \left\{ r \in \mathbb{R}^3 \mid (r - a, r - a) = 0 \right\}.$$

By assumption, light cones are preserved, i.e., $\mathcal{C}(a) \leftrightarrow \mathcal{C}(a')$.
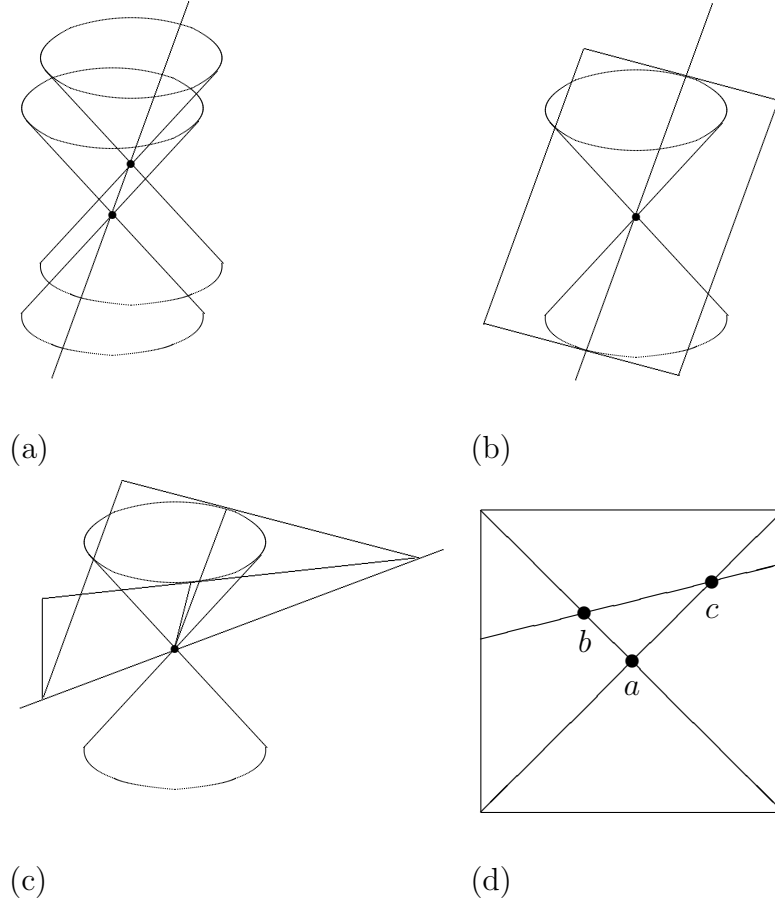
(a)

(b)

(c)

(d)

Figure 1: Illustrations of the proof that (a) lightlike (null) lines into lightlike lines; (b) lightlike (null) planes into null planes; (c) spacelike lines into spacelike lines; (d) timelike planes map into planes.

As illustrated in Fig. 1(a), any null (lightlike) line is the intersection of two tangent null cones. Since null cones are preserved, so are null (lightlike) lines. Thus, null lines are mapped into null lines. The same is true for the inverse map. Hence, null lines are mapped *onto* null lines. (The same is true for the other proof steps as well but will not be mentioned explicitly.)

To show (ii), notice that, as illustrated in Fig. 1(b), a null cone with vertex on some null plane is tangent to that plane along a null line. Points of $\mathbb{R}^3$ are on the null plane if and only if they either lie on this null line or on no null cone with vertex on this line. The latter sentence could be understood as follows. Imagine any point of $\mathbb{R}^3$ outside of the null plane (either "below" or "above"). Any such point is element of some null cone with vertex on the null line mentioned. On the contrary, any point on the null plane cannot be reached by such null cones (except the cones located on the null line mentioned), but by other null cones whose vertex is not on that null line. Null lines and cones are preserved; thus null planes are preserved as well.

To show (iii), notice that, as illustrated in Fig. 1(c), any spacelike line is the intersection of two null planes. Since null planes are preserved, spacelike lines are preserved.

To show (iv), notice that, as illustrated in Fig. 1(d), the points in a timelike plane are covered by infinitely many intersecting null and spacelike lines in that plane. By fixing, for instance, a triangle formed by the vertices a, b, c of three such lines (e.g., two null and one spacelike line) "spans" the timelike plane. Because of the one-to-oneness of the mapping, the image of the triangle with the vertices a', b', c' "spans" the transformed plane (different points are mapped onto different points). Therefore, the three lines forming the transformed triangle must be coplanar. In general, the images of all lines lying in the original timelight plane must be coplanar. Thus, timelike planes map into planes.

To show (v), notice that any timelight line is the intersection of two timelight planes. Since timelike planes are mapped onto planes, they intersect into a line. Thus, any timelike line is mapped into a line.

In summary, all three types of lines—lightlike (null), spacelike and timelike lines—are mapped onto lines. (Recall that the same arguments apply for the inverse transformation as well.)

The geometric proof of (II), in particular the linearity of the transformation proceeds from the preservation of lines essentially by utilizing the preservation of *parallelism* among lines. As will be demonstrated below, the

25

preservation of parallelism implies that the transformation is additive. The associated transformation of the field $\mathbb{R}$ is an automorphism. It then only remains to be proven that the only automorphism of $\mathbb{R}$ is the identity function.

Let us first introduce some notation. For a much more comprehensive approach the reader is referred to the literature (e.g., the book by Gruenberg & Weir [34]). Let a be a fixed "translation" vector of $\mathbb{R}^n$ and M be a linear subspace of $\mathbb{R}^n$. [Recall that a subset $S \subset \mathbb{R}^n$ is called a (linear) subspace if S is a vector space in its own right with respect to the same vector addition and scalar multiplication than $\mathbb{R}^n$.] Then $a + M$ denotes the set of all vectors $a + M = \{a + m \mid m \in M\}$. It is called *translated subspace* or *coset* or *affine subspace* of $\mathbb{R}^n$. The dimension of a translated subspace $a + M$ is the dimension of the linear subspace M; i.e., $\dim(a + M) = \dim(M)$. Translated subspaces of dimensions $0, 1, 2$ are called *points, lines* and *planes*, respectively. Let the *join* $S_1 \circ S_2$ of two translated subspaces S_1, S_2 be the intersection of all translated subspaces in $\mathbb{R}^n$ which contain both S_1 and S_2. (The join is again a translated subspace.) Furthermore, if $S \subset \mathbb{R}^n$ is any set of vectors in $\mathbb{R}^n$, we denote by the (linear) *span* $\mathrm{span}(S)$ the intersection of all the subspaces of $\mathbb{R}^n$ which contain S.

We shall call an *automorphism* a one-to-one mapping of $\mathbb{R}^n$ *onto itself* preserving all translated subspaces. The *fundamental theorem of affine geometry* (e.g., ref. [34, Theorem 5]) states that, for $\mathbb{R}^n$, $n \geq 2$, any automorphism induces a *linear* transformation L and a translation vector a such that $r \mapsto r' = Lr + a$.

In what follows, a proof of the fundamental theorem of affine geometry will be given for the case of the vector space $\mathbb{R}^n$, $n \geq 2$ with field $\mathbb{R}$. First, a proof will be given that any such automorphism of $\mathbb{R}^n$ implies an automorphism on the field of reals $\mathbb{R}$ (a definition will be given below). By invoking the preservation of parallelism one obtains both the uniqueness of the associated mapping of the field $\mathbb{R}$ onto itself and furthermore the additivity of the transformation as a whole.

Note that the automorphism preserves parallelism. This can be seen by "fixing" appropriate four points a, b, c, d on two lines which are originally parallel, drawing two nonparallel lines through them which meet in another point e. Since by assumption all lines are preserved, so are their meeting points a', b', c', d'. Furthermore, because of bijectivity, two parallel lines have no point in common. Thus, the two lines which are originally parallel are mapped onto copanar lines which are disjoint; i.e., they are again parallel.
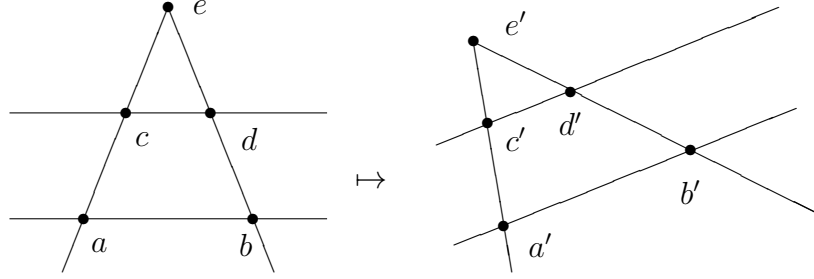
Figure 2: Geometrical proof of the preservation of parallelism due to the preservation of lines.

Hence, parallelism is conserved. A concrete configuration illustrating this geometrical argument is drawn in Fig. 2.

Consider an arbitrary nonzero vector $a \in \mathbb{R}^n$. According to the assumptions, any line $0 \circ a = \mathrm{span}(a)$ is transformed into a line $0' \circ a' = \mathrm{span}(a')$, thereby inducing a one-to-one mapping of all points of $\mathrm{span}(a)$ onto the points of $\mathrm{span}(a')$. That is, the transformation defines a one-to-one mapping

$$\zeta : x \mapsto x' \tag{1}$$

of the field of real numbers onto itself by the definition

$$(xa)' = x'a'. \tag{2}$$

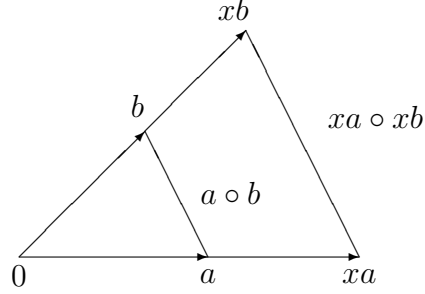It immediately follows that $\zeta : 0 \mapsto 0'$ as well as $\zeta : 1 \mapsto 1'$. It will be shown

28

Figure 3: Rescaling does not effect parallelism.

that ζ is an automorphism; i.e., a one-to-one mapping of $\mathbb{R}$ onto itself with the properties that $\zeta(x + y) = \zeta(x) + \zeta(y)$, as well as $\zeta(xy) = \zeta(x)\zeta(y)$.

First it is shown that ζ does not depend on the particular choice of $a \in \mathbb{R}^n$. (i) Case 1: Consider two linearly independent vectors a, b of $\mathbb{R}^n$, $(xa)' = x'a'$ and $(xb)' = x''b'$, $x' \neq x''$. Since $0 = 0' = 0''$, one can assume that $x \neq 0$. The join $xa \circ xb$ is the intersection of all the subspaces of $\mathbb{R}^n$ containing both xa and xb. Since xa and xb are vectors, this is just the line joining them. $xa \circ xb$ is parallel to $a \circ b$. (Rescaling does not affect parallelism; cf. Fig. 3.) The transformation preserves parallelism, and therefore $a' \circ b'$ must also be parallel to $x'a \circ x'b'$ and $x'a \circ x''b'$, the lines connecting $x'a$ with $x'b'$ and $x'a$ with $x''b'$. This can only be satisfied for $x' = x''$. Hence, ζ is independent of the argument and only depends on the transformation.

29

(ii) Case 2: Consider two linearly dependent vectors a, b of $\mathbb{R}^n$. In this case, choose a third vector c which does not lie in the linear subspace $\text{span}(a)$ spanned by a and b. Then, by the argument used in case 1, ζ is the same for a, c and b, c; thus ζ is also the same for a, b. Hence, to sum up the finding in the two cases, ζ is independent of the argument vector and only depends on the transformation.

We shall pursue the proof that, given the preservation of lines, the associated mapping is additive (up to translations). A geometric interpretation of this proof is drawn in Fig. 4. (i) Case 1: If a and b are linearly independent nonzero vectors (the zero vector case is trivial) of $\mathbb{R}^n$, then the parallelogram $a, 0, b, a + b$ is mapped into the parallelogram $a', 0', b', a' + b'$ and

$$(a + b)' = a' + b'. \tag{3}$$

This is also true if a or b is the zero vector.

(ii) Case 2: If a and b are linearly dependent and nonzero, choose a third vector $c \notin \text{span}(a)$ (so that c is linearly independent of a and b), and apply the above considerations for the pairs $a + b$ & c rendering $a + b + c \mapsto (a+b)' + c'$, a & $b+c$ rendering $a + b + c \mapsto a' + (b+c)'$, b & c rendering $b + c \mapsto b' + c'$, such that $(a+b)' + c' = a' + b' + c'$, which is satisfied only if again $(a+b)' = a' + b'$.
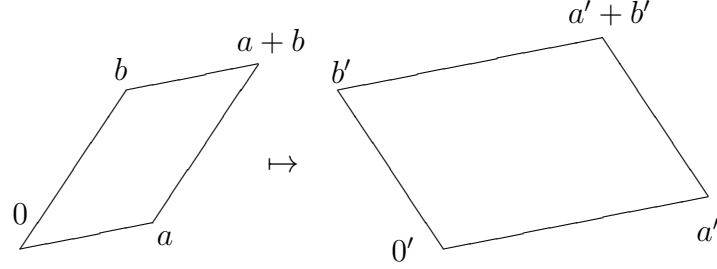
30

Figure 4: Geometrical proof of the preservation of any parallelogram and of additivity due to the conservation of parallelism.

Two further properties assuring that ζ is an automorphism can be deduced from the uniqueness of Eq. (2) and Eq. (3) and the usual axioms of linear vector spaces. (i) Automorphism property 1: Let $a' \neq 0$, then for all $x, y \in \mathbb{R}$,

$$(x + y)'a' = [(x + y)a]' = (xa + ya)' = (xa)' + (ya)' = x'a' + y'a' = (x' + y')a'$$

(4)

and thus

$$(x + y)' = x' + y'.$$

(5)

(ii) Automorphism property 2: By the assumption of vector spaces, $(xy)a =$

31

$x(ya)$ for all $x, y \in \mathbb{R}$ and $a \in \mathbb{R}^n$. Therefore,

$$(xy)'a' = x'(ya)' = x'(y'a') = (x'y')a' \tag{6}$$

and thus

$$(xy)' = x'y'. \tag{7}$$

In order to complete the proof of linearity, it will be shown that the only automorphism of the field $\mathbb{R}$ into itself is the identity function id : $x \mapsto x$. This can be demonstrated by realizing that the algebraic properties of neutral elements $0, 1$ with regard to addition and multiplication have to be preserved; i.e., $0 \mapsto 0$ and $1 \mapsto 1$. Furthermore, since 1 has to be preserved and any natural number $n \in \mathbb{N}$ is the sum of n 1's, $\mathbb{N}$ has only a single automorphism– the identity function id. A very similar argument holds for $\mathbb{Z}$. Since any element of the positive rationals can be represented by the quotient n/m with $n, m \in \mathbb{N}$, again $\mathbb{Q}$ has only a single automorphism–the identity function id. In order to be able to obtain the same result for $\mathbb{R}$, one has to make sure the Dedekind construction of the reals works; in particular the preservation of Dedekind sections. This requires the preservation of the order relation "$<$" in $\mathbb{R}$, which is equivalent to the preservation of positivity, since $x < y$ can always be rewritten into $0 < y - x$. Notice that every positive $0 < x \in \mathbb{R}$

can be written as $x = y^2$, $y \in \mathbb{R}$, $y \neq 0$. Since $x = y^2$ is mapped onto $x' = (y^2)' = (y')^2$ with $y' \neq 0$ (recall that $0 \mapsto 0$), $x' > 0$. This allows the Dedekind construction of the reals using the rationals, which in turn yields the desired fact that $\mathbb{R}$ has only a single automorphism–the identity function id. (This is not true for example for $\mathbb{C}$, since for example $x + iy \mapsto x - iy$ is an automorphism but not the identity.)

We shall now concentrate on a proof of (III). Let us first note that, in the case of a linear map, the preservation of a *single* light cone is a sufficient condition for the preservation of all of them. For, given the transformation $x \mapsto \alpha Lx + a$, any shift of the null cone $\mathcal{C}(p)$ with vertex p by a vector $s = q - p$ results in a null cone $\mathcal{C}(q) = \mathcal{C}(p) + s$ with vertex $q = p + s$. The latter null cone $\mathcal{C}(q)$ is mapped onto the null cone

$$
\begin{aligned}
\mathcal{C}(q) \mapsto \alpha L \mathcal{C}(q) + a &= \alpha L(\mathcal{C}(p) + s) + a \\
&= \alpha L \mathcal{C}(p) + \alpha Ls + a \\
&= (\alpha L \mathcal{C}(p) + a) + \alpha Ls \\
&= \mathcal{C}(p') + \alpha Ls,
\end{aligned}
$$

which again is a null cone.

A way to get rid of the factor α is by considering the tangent hyperboloid

33

$x^2 + y^2 - z^2 = 1$ of the null cone $x^2 + y^2 - z^2 = 0$, translating it once and then back to the original figure. The requirement that this should result in the same hyperboloid fixes α [35].

Recall that in Einstein's original work [7, par 3], linearity was never derived but was assumed for physical reasons. *"In the first place it is clear that the equations must be* linear *on account of the properties of homogeneity which we attribute to space and time."* [["Zunächst ist klar, daß die Gleichungen* linear *sein müssen wegen der Homogenitätseigenschaften, welche wir Raum und Zeit beilegen."]]* That is, straight particle trajectories should transform into straight particle trajectories.

Let us first prove the invariance of the scalar product $(Ls, Ls) = (s, s)$ for arbitrary $s = (x, y, z, t)$ under the assumption of linearity and preservation of zero distances for the metric $\text{diag}(1, 1, 1, -c)$ in $\mathbb{R}^4$ (c is often set to 1 for convenience). Because of linearity of the transformations, (Ls, Ls) must have the following form:

$$
\begin{aligned}
(s')^2 = (Ls, Ls) &= (x')^2 + (y')^2 + (z')^2 + (ct')^2 \\
&= Ac^2t^2 + Bx^2 + Cy^2 + Dz^2 \\
&\quad + Ictx + Jcty + Kctz + Lyz + Mzx + Nxy \quad (8)
\end{aligned}
$$

Because of preservation of zero distances, a necessary condition for $(s')^2$ to be zero is

$$s^2 = x^2 + y^2 + z^2 - c^2t^2 = 0. \tag{9}$$

Eq. (9) is satisfied in the following cases: (i) $s = (\pm 1, 0, 0, 1)$, (ii) $s = (0, \pm 1, 0, 1)$, (iii) $s = (0, 0, \pm 1, 1)$, (iv) $s = (1, 1, 0, \sqrt{2})$, (v) $s = (1, 0, 1, \sqrt{2})$, and (vi) $s = (0, 1, 1, \sqrt{2})$. From (i), we obtain

$$A + B \pm I = 0 \quad \text{and thus} \quad I = 0, \quad B = -A. \tag{10}$$

by insertion into Eq. (8). From (ii)–(vi) we obtain

$$C = D = -A, \qquad J = K = L = M = N = 0, \tag{11}$$

respectively.

Insertion into Eq. (8) yields

$$(s')^2 = (x')^2 + (y')^2 + (z')^2 + (ct')^2 = A\left[(x)^2 + (y)^2 + (z)^2 + (ct)^2\right] = As^2. \tag{12}$$

Because of linearity, A could only depend on the velocity v. By a back-transformation one obtains $A(v)A(-v) = 1$. The assumption that $A(v)$ does not depend on the direction of v (isotropy assumption, see also below) yields

$A(s) = \pm 1$. Finally, for zero velocity $v = 0$, the case with the minus sign can be eliminated, yielding $A(v) = 1$ and thus $(Ls, Ls) = (s, s)$ for arbitrary s.

Preservation of (r, s) follows immediately from the linearity and symmetry of the metric. Take $q = r + s$ with arbitrary r. Then,

$$
\begin{aligned}
(L[r+s], L[r+s]) &= (r+s, r+s), \\
2(Lr, Ls) + (Ls, Ls) + (Lr, Lr) &= 2(r, s) + (s, s) + (r, r), \\
2(Lr, Ls) + (s, s) + (r, r) &= 2(r, s) + (s, s) + (r, r), \\
(Lr, Ls) &= (r, s).
\end{aligned}
$$

In what follows we shall closely follow Einstein's original argument rendering L to be the Lorentz transformations. Take the standard four dimensional space-time case $\mathbb{R}^4$, and consider, for the sake of simplicity, the quasi-twodimensional case (one space and the time coordinate) of the constant motion along the x-axis of K with velocity v of a coordinate frame K' with the components (x', y', z', t') against another coordinate frame "at rest" K with the components (x, y, z, t). (Otherwise, K can be rotated such that the direction of motion is along the x axis.) Again, c stands for the velocity of light.

Now define a particular series (in time) of points $\bar{x} = x - vt$. Notice that

36

the "worldlines" $(x = vt, 0, 0, t)$ just mark the parametrization by the time parameter t of all points at rest with respect to the moving frame K'. That is, any such point has constant $\bar{x}, y, z$ throughout all times t. It is sometimes convenient (cf. below) to write the parameters of events in terms of $(\bar{x}, y, z, t)$ instead of (x, y, z, t).

Let us construct "radar coordinates" of K' by utilizing a light clock starting at some arbitrary point $\bar{x} = 0$ at t'_0, traveling some distance $\Delta \bar{x}$ to a mirror, where it arrives and is instantly reflected at K'-time t'_1 towards the original source mirror and arrives there at K'-time t'_2 [cf. Fig. 5(a)]. If one adopts the usual conventions for synchronization, t'_1 is just the arithmetic mean of the two times t'_0 and t'_2; i.e.,

$$t_1 = \frac{1}{2}(t'_0 + t'_2). \tag{13}$$

In order to find the transformation mapping K onto K', rewrite the transformed coordinates as functions of the original system; e.g., $t' = t'(\bar{x}, y, z, t)$. In this parametrization, the coordinates are given by

$$t'_0 \;=\; t'(0, 0, 0, t), \tag{14}$$

$$t'_1 \;=\; t'\left(\Delta\bar{x}, 0, 0, t + \frac{\Delta\bar{x}}{c^2 - v^2}\right), \tag{15}$$

$$t'_2 \;=\; t'\left(0, 0, 0, t + \frac{\Delta\bar{x}}{c^2 - v^2} + \frac{\Delta\bar{x}}{c^2 + v^2}\right), \tag{16}$$
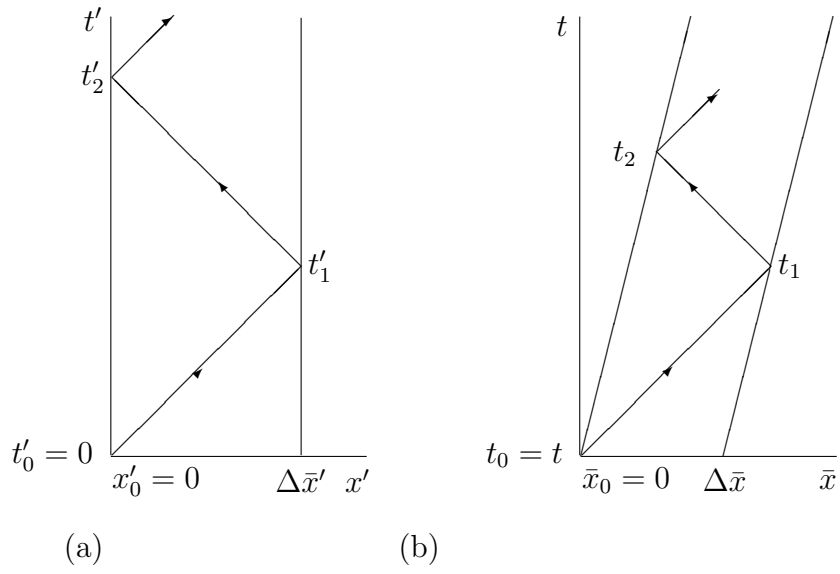
Figure 5: Generation of radar coordinates by a light clock following Einstein's procedures and conventions (a) from within the system K'; (b) the same procedure seen from the system K.

where

$$t_0 = t,$$

$$t_1 = t + \frac{\Delta \bar{x}}{c^2 - v^2},$$

$$t_2 = t + \frac{\Delta \bar{x}}{c^2 - v^2} + \frac{\Delta \bar{x}}{c^2 + v^2}$$

results from the following consideration. The K-time $\Delta t_1 = t_1 - t_0$ it takes

for light to arrive at the first mirror is given by the total distance it takes for

light to travel to it, divided by the velocity of light. Since the mirror travels

with velocity v,

$$\Delta t_1 = \frac{\Delta \bar{x} + v \Delta t_1}{c} = \frac{\Delta \bar{x}}{c - v}. \tag{17}$$

A similar argument yields

$$\Delta t_1 = t_2 - t_1 = \frac{\Delta \bar{x} - v \Delta t_1}{c} = \frac{\Delta \bar{x}}{c + v}.$$

Inserting Eqs. (14)–(16) into (13) yields

$$t' \left(\Delta \bar{x}, 0, 0, t + \frac{\Delta \bar{x}}{c^2 - v^2} \right) = \frac{1}{2} \left[t'(0, 0, 0, t) + t' \left(0, 0, 0, t + \frac{\Delta \bar{x}}{c^2 - v^2} + \frac{\Delta \bar{x}}{c^2 + v^2} \right) \right].$$

$$\tag{18}$$

$\Delta \bar{x}$ can be arbitrarily small. A partial derivation of (18) by $\frac{\partial t'}{\partial \Delta \bar{x}}$ in the limit

of infinitesimal $\Delta \bar{x}$ yields

$$\frac{\partial t'}{\partial \bar{x}} + \frac{1}{c - v} \frac{\partial t'}{\partial t} = \frac{1}{2} \left(\frac{1}{c - v} + \frac{1}{c + v} \right) \frac{\partial t'}{\partial t}, \tag{19}$$

and thus

$$\frac{\partial t'}{\partial \bar{x}} + \frac{1}{c^2 - v^2} \frac{\partial t'}{\partial t} = 0. \tag{20}$$

Likewise, $(\partial t'/\partial y) = (\partial t'/\partial z) = 0$. As a result of this and Eq. (20), t' must be a linear function of t and $\bar{x}$ of the form

$$t'(\bar{x}, y, z, t) = \alpha(v) \left(t - \frac{v}{c^2 - v^2} \bar{x} \right). \tag{21}$$

$\alpha(v)$ is a yet arbitrary scale factor depending only on v. Note that, without loss of generality, the origins of K and K' has been chosen such that $t = t' = 0$. By substituting the explicit parameters for $\bar{x} = x - vt$ one obtains

$$t'(\bar{x}, y, z, t) = \alpha(v) \frac{1}{1 - \frac{v^2}{c^2}} \left(t - \frac{v}{c^2} x \right). \tag{22}$$

The transformation rules of the x' parameter can be obtained by considering the propagation of a light ray in K' which starts at the origin of K and K' (same origins) and moves along the x- and x'-axes. The convention of the constancy of the speed of light requires

$$x' = ct' = \alpha(v)c \left(t - \frac{v}{c^2 - v^2} \bar{x} \right). \tag{23}$$

Now recall that, in terms of the K-parameters, this propagation of this light ray is given by Eq. (17); i.e., by $t = \bar{x}/(c - v)$ (the differences Δ can be omitted because of the ray starting at the coordinate origins). By substituting t
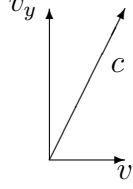
Figure 6: Velocity v_y of a light ray propagating along the positive y' axis of a system traveling with velocity v along the x- and x'-axes.

in (23) one obtains

$$x' = \alpha(v)\frac{c^2}{c^2 - v^2}\,\bar{x} = \alpha(v)\frac{1}{1 - \frac{v^2}{c^2}}\,\bar{x} = \alpha(v)\frac{1}{1 - \frac{v^2}{c^2}}\,(x - vt)\,. \qquad (24)$$

Let us now turn to the transformation of coordinates y, z perpendicular to the direction of motion x. Consider a light ray propagating along the y'-axis, and hence $\bar{x} = 0$. Inside the system K, the y-component of the light propagation follows from the Pythagorean theorem, which is illustrated in Fig. 6; i.e., $v_y = \sqrt{c^2 - v^2}$. Hence,

$$y' = ct' = \alpha(v)c\left(t - \frac{v}{c^2 - v^2}\,\bar{x}\right)\,, \qquad (25)$$

for $\bar{x} = 0$ and $t = y/v_y = y/\sqrt{c^2 - v^2}$

$$y' = \alpha(v)\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}\,y\,. \qquad (26)$$

41

The same consideration applies to the transformation of the z- and z'-axes. Summing up, we obtain a transformation of the coordinates $x \mapsto x' = Lx$ given by

$$L(v) = \alpha(v)\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}\begin{pmatrix} \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} & 0 & 0 & -\frac{v}{\sqrt{1 - \frac{v^2}{c^2}}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{v}{c^2\sqrt{1 - \frac{v^2}{c^2}}} & 0 & 0 & \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \end{pmatrix} \tag{27}$$

We now fix the factor $\alpha(v)$ by the conventional requirement that a back-transformation should recover the original coordinates. For this purpose we invent a third coordinate frame K'' which propagates with the reverse (relative to K') velocity $-v$ (measured in K) along the x-, x'-, and its x''-axes. The successive application of the transformation (27) with $L(v)$ and $L(-v)$ should bring back the coordinates to their original form; i.e.,

$$L(v)L(-v) = \mathbb{I}_4, \tag{28}$$

where $\mathbb{I}_4 = \operatorname{diag}(1, 1, 1, 1)$ stands for the four-dimensional unit matrix. After evaluating the matrix product and comparing the coefficients, one obtains

$$\alpha(v)\alpha(-v) = 1 - \frac{v^2}{c^2}. \tag{29}$$

42

That $\alpha(v) = \alpha(|v|)$ only depends on the absolute value of the velocity can be seen by symmetry and isotropy arguments. For the length l' of a rod $\{p' \in \mathbb{R}^4 \mid x' = 0,\ 0 \leq y' \leq l,\ z' = 0\}$ which is at rest along the y'-axis with respect to the system K' traveling along the x-axis should not depend on the direction of motion; i.e., should only depend on the absolute magnitude of the velocity. If this is granted, one obtains

$$\alpha(v) = \sqrt{1 - \frac{v^2}{c^2}}, \tag{30}$$

and finally the transformation laws $x \mapsto x' = Lx$ with

$$L(v) = \begin{pmatrix} \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} & 0 & 0 & -\frac{v}{\sqrt{1-\frac{v^2}{c^2}}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{v}{c^2\sqrt{1-\frac{v^2}{c^2}}} & 0 & 0 & \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \end{pmatrix} \tag{31}$$

up to constant translations $a \in \mathbb{R}^4$. As can be easily checked, L preserves the distance of any two points; i.e., $(Lr, Ls) = (r, s)$ for all r and s in $\mathbb{R}^4$.

It would be nice to have a more general result using a more general metric and/or relaxation of bijectivity. For instance, by maintaining the metric but requiring bijectivity only for finite space-time points, one may speculate to be able to recover space-time transformations among accelerated frames.

Acknowledgments

References

[1] P. W. Bridgman. *The Logic of Modern Physics*. New York, 1927.

[2] P. W. Bridgman. A physicists second reaction to Mengenlehre. *Scripta Mathematica*, 2:101–117, 224–234, 1934. Cf. R. Landauer [36].

[3] P. W. Bridgman. *The Nature of Physical Theory*. Princeton, 1936.

[4] P. W. Bridgman. *Reflections of a Physicist*. Philosophical Library, New York, 1950.

[5] P. W. Bridgman. *The Nature of Some of Our Physical Concepts*. Philosophical Library, New York, 1952.

[6] Arthur I. Miller. *Albert Einstein's special theory of relativity*. Springer, New York, 1998.

[7] Albert Einstein. Zur Elektrodynamik bewegter Körper. *Annalen der Physik*, 17:891–921, 1905. English translation in the Appendix of [6].

[8] D. Malament. Causal theories of time and the conventionality of simultaniety. *Noûs*, 11:293–300, 1977.

[9] Michael Redhead. The conventionality of simultaneity. In J. Earman, A. Janis, G. Massey, and N. Rescher, editors, *Philosophical Problems of the Internal and External Worlds*, pages 103–128. University of Pittsburgh Press and Universitätsverlag Konstanz, Pittsburgh and Konstanz, 1993.

[10] Allen I. Janis. Conventionality of simultaneity. In *Stanford Encyclopedia of Philosophy. Winter 1998 Edition*. The Metaphysics Research

Lab Center for the Study of Language and Information. Ventura Hall Stanford University Stanford, CA 94305-4115, 1998. Internet address: http://plato.stanford.edu/entries/spacetime-convensimul/.

[11] Karl Svozil. Relativizing relativity. *Foundations of Physics*, 30(7):1001–1016, 2000. e-print `arXiv:quant-ph/0001064`.

[12] Asher Peres. Defining length. *Nature*, 312:10, 1984.

[13] R. J. Boskovich. *De spacio et tempore, ut a nobis cognoscuntur*. Vienna, 1755. English translation in [37].

[14] George Francis FitzGerald. In J. Larmor, editor, *The Scientific Writings of the Late George Francis FitzGerald*. Dublin, 1902.

[15] John S. Bell. How to teach special relativity. *Progress in Scientific Culture*, 1(2), 1976. Reprinted in [38, pp. 67-80].

[16] John S. Bell. George Francis FitzGerald. *Physics World*, 5(9):31–35, September 1992. Abridged version by Denis Weaire.

[17] A. D. Alexandrov. On Lorentz transformations. *Uspehi Mat. Nauk.*, 5(3):187, 1950.

[18] A. D. Alexandrov. A contribution to chronogeometry. *Canadian Journal of Math.*, 19:1119–1128, 1967.

[19] A. D. Alexandrov. Mappings of spaces with families of cones and space-time transformations. *Annali die Matematica Pura ed Applicata*, 103:229–257, 1967.

[20] A. D. Alexandrov. On the principles of relativity theory. In *Classics of Soviet Mathemotics. Volume 4. A. D. Alexandrov. Selected Works*, pages 289–318. 1996.

[21] H. J. Borchers and G. C. Hegerfeldt. The structure of space-time transformations. *Communications in Mathematical Physics*, 28:259–266, 1972.

[22] Walter Benz. *Geometrische Transformationen*. BI Wissenschaftsverlag, Mannheim, 1992.

[23] June A. Lester. Distance preserving transformations. In Francis Buekenhout, editor, *Handbook of Incidence Geometry*. Elsevier, Amsterdam, 1995.

[24] Karl Svozil. Connections between deviations from lorentz transformation and relativistic energy-momentum relation. *Europhysics Letters*, 2:83–85, 1986. excerpts from [39].

[25] Karl Svozil. Operational perception of space-time coordinates in a quantum medium. *Il Nuovo Cimento*, 96B:127–139, 1986.

[26] Karl Svozil. Quantum interfaces. In *Sciences of the Interface*, pages 76–88, Tübingen, Germany, 2001. Genista Verlag. e-print `arXiv:quant-ph/0001064` available http://arxiv.org/abs/quant-ph/0001064.

[27] Daniel B. Greenberger and A. YaSin. "Haunted" measurements in quantum theory. *Foundation of Physics*, 19(6):679–704, 1989.

[28] Daniel M. Greenberger. Solar Eclipse Workshop, Vienna, August 1999.

[29] Thomas J. Herzog, Paul G. Kwiat, Harald Weinfurter, and Anton Zeilinger. Complementarity and the quantum eraser. *Physical Review Letters*, 75(17):3034–3037, 1995.

[30] Niko Donath and Karl Svozil. Finding a state in a haystack. `arXiv:quant-ph/0105046`, 2001.

[31] Erwin Schrödinger. Die gegenwärtige Situation in der Quantenmechanik. *Naturwissenschaften*, 23:807–812, 823–828, 844–849, 1935. English translation in [40] and [41, pp. 152-167]; a copy can be found at www.emr.hibu.no/lars/eng/cat/.

[32] Josef Gruska. *Quantum Computing*. McGraw-Hill, London, 1999.

[33] M. A. Nielsen and I. L. Chuang. *Quantum Computation and Quantum Information*. Cambridge University Press, Cambridge, 2000.

[34] K. W. Gruenberg and A. J. Weir. *Linear Geometry (2nd Edition)*. Springer Verlag, New,York,Heidelberg,Berlin, 1977.

[35] Hans Havlicek. private communication.

[36] Rolf Landauer. Advertisement for a paper I like. In John L. Casti and J. F. Traub, editors, *On Limits*, page 39. Santa Fe Institute Report 94-10-056, Santa Fe, NM, 1994.

[37] R. J. Boskovich. De spacio et tempore, ut a nobis cognoscuntur. In J. M. Child, editor, *A Theory of Natural Philosophy*, pages 203–205. Open Court (1922) and MIT Press, Cambridge, MA, 1966.

[38] John S. Bell. *Speakable and Unspeakable in Quantum Mechanics.* Cambridge University Press, Cambridge, 1987.

[39] Karl Svozil. On the setting of scales for space and time in arbitrary quantized media. *Lawrence Berkeley Laboratory preprint*, LBL-16097, May 1983.

[40] J. D. Trimmer. The present situation in quantum mechanics: a translation of Schrödinger's "cat paradox". *Proc. Am. Phil. Soc.*, 124:323–338, 1980. Reprinted in [41, pp. 152-167].

[41] John Archibald Wheeler and Wojciech Hubert Zurek. *Quantum Theory and Measurement.* Princeton University Press, Princeton, 1983.