On the unpredictability of individual quantum measurement outcomes

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Abstract

Suppose we prepare a quantum in a pure state corresponding to a unit vector in Hilbert space. Choose an observable property of this quantum corresponding to a projector whose respective linear subspace is neither collinear nor orthogonal with respect to the pure state vector. Recent results can be then used to *prove* that this observable property has no predetermined value, and thus remains value indefinite.

As a consequence, we *prove* that the outcome of a measurement of such a property is *unpredictable* with respect to a very general model of prediction here developed.

These results are true relative to three assumptions, namely compatibility with quantum mechanical predictions, noncontextuality, and the value definiteness of observables corresponding to the preparation basis of a quantum state. This framework allows for the first time a rigorous proof of the unpredictability of some individual quantum measurement outcomes, a fact postulated or claimed for a long time.

Finally, unpredictability will be used to discuss quantum randomness—shown to be "maximally incomputable"—as well as *real* model hypercomputation whose computational power has yet to be determined.

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I. INTRODUCTION

Indeterminism has had a role at the heart of quantum mechanics since Born postulated that the modulus-squared of the wave function should be interpreted as a probability density [1] that, unlike in classical statistical physics [2], expresses fundamental, irreducible indeterminism. In Born's own words, "*I myself am inclined to give up determinism in the world of atoms.*" The nature of individual measurement outcomes in quantum mechanics was, for a period, a subject of much debate. Einstein famously dissented, stating his belief that [3, p. 204] "*He does not throw dice.*" Nonetheless, over time the conjecture that measurement outcomes are themselves fundamentally indeterministic became the quantum orthodoxy [4].

Much later, this assumption was in a certain sense vindicated by the theorems of Bell [5] and Kochen-Specker [6] on the impossibility of certain classes of deterministic theories. These results were further corroborated by the experimental verification of violation of Bell's inequalities [7].

Although these results have cemented the view that quantum mechanics is intrinsically indeterministic, it is crucial to recognise their limits. Firstly, such results nonetheless require belief in the validity of certain assumptions such as noncontextuality and locality. Secondly, even with these assumptions, such indeterminism need not entail complete unpredictability or randomness.

In this paper we systematically approach these issues aiming to deduce quantum indeterminism and unpredictability from more fundamental, assumptions, rather rely on their *ad hoc* postulation. Working directly from the results implied by strong versions of the Kochen-Specker theorem [8, 9] we propose a very general model of prediction of physical outcomes. We show that, relative to specific assumptions drawn from the Kochen-Specker theorem, no outcome of any single measurement of a value indefinite quantum observable is predictable.

II. UNPREDICTABILITY FROM VALUE INDEFINITENESS

A. A necessary physical basis

The generally accepted phenomenon of quantum indeterminism cannot be deduced from the Hilbert space formalism of quantum mechanics alone, as this specifies only the probability distribution for a given measurement which in itself need not indicate intrinsic indeterminism. While we could, in our effort to clarify and formalise this unpredictability, restrict ourselves to a particular interpretation it is preferable to work from a small set of physical assumptions and principles to arrive at a more general understanding of quantum unpredictability.

In [8] the physical assumptions needed to deduce quantum value indefiniteness—the formalisation of the intuitive notion of quantum indeterminism—were carefully analysed. Here we briefly review the key assumptions, as this will be important in the analysis of the link between value indefiniteness and unpredictability.

In the following we shall work in an idealised theoretical framework. That is, we consider perfect idealised measuring devices and experiments where the devices behave precisely as they should in theory. Further, any predictions or access to measurement results that are in principle possible are considered reasonable in this framework.

Since the central issue is that of value indefiniteness, it is crucial to have a clear understanding of when we should conclude that a physical quantity is indeed value definite. In [10, p. 777], Einstein, Podolsky and Rosen define *physical reality* in terms of certainty and predictability. Based on this accepted notion of an element of physical reality, we allow ourselves to be guided by the following "EPR principle" which identifies their notion of an "element of physical reality" with "value definiteness":

> *EPR principle*: If, without in any way disturbing a system, we can predict with certainty the value of a physical quantity, then there exists a *definite value* prior to observation corresponding to this physical quantity.

Our main assumptions based on value definiteness are the following. More technical descriptions and further discussion can be found in [8].

Admissibility: Definite values must not contradict the statistical quantum predictions for compatible observables on a single quantum. For example, given a set $\{P_1, \ldots, P_n\}$ of commuting projection observables, if P_1 were to have the definite value 1, all other observables in this set must have the value 0 as any other possibility would contradict the quantum prediction that one and only one such projector will yield the value 1 upon measurement.

Noncontextuality of definite values: If a measurement is made of a value definite observable, the outcome obtained (and thus the preexisting physical property) is *noncontextual*. That means it does not depend on other compatible (i.e. simultaneously co-measurable) observables which may be measured alongside the value definite observable.

Eigenstate principle: If a quantum system is prepared in the state $|\psi\rangle$, then the projection observable $P_\psi = |\psi\rangle\langle\psi|$ is value definite, as are (by the previous two assumptions) all observables which commute with P_ψ.

The eigenstate principle follows largely from the EPR principle, since by preparing a system in the state $|\psi\rangle$ we ensure that we can predict with certainty the value of the observable P_ψ. However, we make this explicit because of its importance in deducing the existence of value indefiniteness.

The further constraint that prediction acts "without in any way disturbing a system" is perhaps nontrivial [11], but nonetheless appears a reasonable requirement for prediction.

These assumptions give a generalised and formal base for the understanding of value indefiniteness in quantum physics. In particular, in [8, 9] the following result is proven from these assumptions:

Theorem 1. *Let there be a quantum system prepared in the state* $|\psi\rangle$ *in dimension* $n \geq 3$ *Hilbert space* $\mathbb{C}^n$*, and let* $|\phi\rangle$ *be any state neither orthogonal nor parallel to* $|\psi\rangle$*, i.e.* $0 < |\langle\psi|\phi\rangle| < 1$. *Then the projection observable* $P_\psi = |\phi\rangle\langle\phi|$ *is value indefinite.*

B. Unpredictability of individual measurements

The EPR principle renders a definition of value definiteness and physical reality based on the ability to predict. Conversely, *value indefiniteness* corresponds to the *absence of physical reality*; if no unique element of physical reality corresponding to a particular physical quantity exists, this is reflected by the physical quantity being value indefinite. If a physical property is value indefinite we cannot predict with certainty the outcome of any experiment measuring this property.

To elaborate on this point, suppose that we consider, as in Theorem 1, a quantum projection observable in dimension $n \geq 3$ Hilbert space projecting onto a linear subspace which is neither collinear nor orthogonal with respect to the pure state in which a single quantum system has been prepared.

According to Theorem 1 and the results of [8, 9], any such observable is *provable* (relative to the assumptions we have discussed) value indefinite. That is, by the noncontextuality of definite values, the result obtained upon its measurement cannot correspond to any deterministic function of the observable alone. More explicitly, neither of the two exclusive measurement outcomes $\{0,1\}$ can be consistent with the state preparation. In particular, neither one of these outcomes is certain to occur and therefore any kind of prediction of the outcome with certainty cannot exist. Stated in terms of the EPR principle we infer that, as there cannot be any certainty in predicting a value indefinite observable, there is no element of physical reality corresponding to this physical property.

Furthermore, suppose that observables—whose associated projectors are neither collinear nor orthogonal with respect to the pure state in which some quantum has been prepared—are measured. Then the resulting unpredictability (i.e. indeterminacy) of the outcome must be accepted for every such individual quantum measured. One possible interpretation of this unpredictability is that the physical property measured is logically independent of the information contained in the quantum system [12].

The "more conjugate" a measurement basis becomes relative to the state which has been used for preparing this quantum, the "more unpredictable" and thus "more indeterminate" in statistical terms the quantum behaves. In particular, if the state prepared is orthogonal to the projection observable measured (i.e. if there is a "maximal mismatch" between preparation and measurement), then the individual quantum not only cannot be predicted with certainty by any agent, but such an agent can do no better than blindly guessing the outcome of the measurement. In this sense the quantum behaves maximally unpredictably.

This, however, should not be understood as a claim that quantum randomness is "maximally random" in the sense that no correlations exist between successive measurement results. Such a notion of "perfect randomness" is in fact mathematically vacuous [13, 14]: there exist only degrees of randomness to which there is no upper limit.

III. A FORMAL MODEL OF PREDICTION

While we have argued from a purely physical standpoint that prediction of a single quantum is in general impossible, the argument needs a proper mathematical formalisation. In particular, to say more about the quality of a sequence of outcomes of quantum measurements, the notion of prediction needs to be given a more rigorous form.

Intuitively, a prediction must be a method of specifying in advance the result that will be obtained by a measurement. Such a prediction is correct if it indeed agrees with the measured value. However, even if a prediction turns out to be correct, how can we be sure it was not correct merely by luck?

Popper succinctly summarises this predicament in Ref. [15, 117–118]: "*If we assert of an observable event that it is unpredictable we do not mean, of course, that it is logically or physically* *impossible for anybody to give a correct description of the event in question before it has occurred; for it is clearly not impossible that somebody may hit upon such a description accidentally. What is asserted is that certain rational methods of prediction break down in certain cases—the methods of prediction which are practised in physical science.*"

One possibility to formalise predictability is then to demand a proof that the prediction will be correct—to formalise the "rational methods of prediction" that Popper refers to. However, this is notoriously difficult and must be made relative to the physical theory considered, which generally is not well axiomatised. Instead we demand that such predictions be *repeatable*, and not merely one-off events. This point of view is consistent with Popper's own framework of empirical falsification [16, 17]: an empirical theory (in our case, the prediction) can never be proven correct, but it can be falsified through decisive experiments (an incorrect prediction).

More formally, we consider a physical experiment E producing a single bit $x \in \{0,1\}$. An example of such an experiment is the measurement of a photon's polarisation after it has passed through a 50-50 beam splitter. As it will be seen later, the proposed framework can apply equally to other experiments. Further, with a particular instantiation or "trial" of E we associate the parameter λ, encoded as a real number, which fully describes the trial. While λ is not in its entirety an obtainable quantity, it contains any information that may be pertinent to prediction and we may have practical access to finite aspects of this information. In particular this information may be directly associated with the particular trial of E (e.g. initial conditions or hidden variables) and/or relevant external factors (e.g. the time, results of previous trials of E). Any such external factors should, however, be local in the sense of special relativity, as (even if we admit quantum nonlocality) any other information cannot be utilised for the purpose of prediction [11]. We can view λ as resource that one can extract finite information from in order to predict the outcome of the experiment E. We formalise this in the following.

An *extractor* is a function selecting a "finite" amount of information included in λ which can be used to make predictions of experiments performed with parameter λ. Formally, an extractor is a (deterministic) function $\lambda \mapsto \langle\lambda\rangle$ mapping reals to rationals. For example, $\langle\lambda\rangle$ may be an encoding of the result of the previous instantiation of E, or the time of day the experiment is performed.

A predictor for E is an algorithm (computable function) P_E which *halts* on every input and *outputs* either 0, 1 (cases in which P_E has made a prediction), or "prediction withheld". We interpret the last form of output as a refrain from making a prediction. The predictor P_E can utilise as input the information $\langle\lambda\rangle$ selected by an extractor encoding relevant information for a particular instantiation of E, but, *as required by* EPR, must not disturb or interact with E in any way; that is, it must be *passive*.

As we noted earlier, a certain predictor may give the correct output for a trial of E simply by chance. This may be due not only to a lucky choice of predictor, but also to the input being chosen by chance to produce the correct output. Thus, we rather consider the performance of a predictor P_E using, as input, information extracted by a particular fixed extractor. This way we ensure that P_E utilises in ernest information extracted from λ, and we avoid the complication of deciding under what input we should consider P_E's correctness.

A predictor P_E provides a *correct prediction* using the extractor $\langle\,\rangle$ for an instantiation of E with parameter λ if, when taking as input $\langle\lambda\rangle$, it outputs 0 or 1 (i.e. it does not refrain from making a prediction) and this output is equal to x, the result of the experiment.

Let us fix an extractor $\langle\,\rangle$. The predictor P_E is $k,\langle\,\rangle$*-correct* if there exists an $n \geq k$ such that when E is repeated n times with associated parameters $\lambda_1,\ldots,\lambda_n$ producing the outputs $x_1,x_2,\ldots,x_n$, P_E outputs the sequence $P_E(\langle\lambda_1\rangle),P_E(\langle\lambda_2\rangle),\ldots,P_E(\langle\lambda_n\rangle)$ with the following two properties: (i) no prediction in the sequence is incorrect, and (ii) in the sequence there are k correct predictions. The trials of E form a succession of events of the form "E is prepared, performed, the result recorded, E is reset", iterated n times in an algorithmic fashion.

If P_E is $k,\langle\,\rangle$-correct we can bound the probability that P_E is in fact operating by chance and may not continue to give correct predictions, and thus give a measure of our confidence in the predictions of P_E. Specifically, the sequence of n predictions made by P_E can be represented as a string of length n over the alphabet $\{T,F,W\}$, where T represents a correct prediction, F an incorrect prediction, and W a withheld prediction. Then, for a $k,\langle\,\rangle$-correct predictor there exists an $n \geq k$ such that the sequence of predictions contains k T's and $(n-k)$ W's. There are $\binom{n}{k}$ such possible prediction sequences out of 3^n possible strings of length n. Thus, the probability that such a correct sequence would be produced by chance is

$$\frac{\binom{n}{k}}{3^n} < \frac{2^n}{3^n} \leq \left(\frac{2}{3}\right)^k.$$

Clearly the confidence we have in a $k,\langle\,\rangle$-correct predictor increases as $k \to \infty$. If P_E is $k,\langle\,\rangle$-correct for all k, then P_E never makes an incorrect prediction and the number of correct predictions can be made arbitrarily large by repeating E enough times.

The definition of $k,\langle\,\rangle$-correctness allows P_E to refrain from predicting when it is unable to. A predictor P_E which is $k,\langle\,\rangle$-correct for all k, is, when using the extracted information $\langle\lambda\rangle$, guaranteed to always be capable of providing more correct predictions for E, so it will not output "prediction withheld" indefinitely. Furthermore, although P_E is technically used only a finite, but arbitrarily large, number of times, the definition guarantees that, in the hypothetical scenario where it is executed infinitely many times, P_E will provide infinitely many correct predictions and not a single incorrect one.

While a predictor's correctness is based on its performance in repeated trials, we can use the predictor to define the prediction of single bits produced by the experiment E. If P_E is not $k, \langle\,\rangle$-correct for all k, then we cannot exclude the possibility that any correct prediction P_E makes is simply due to chance. Hence, we propose the following definition: *the outcome x of a single trial of the experiment E performed with parameter λ is* predictable *(with certainty) if there exist an extractor $\langle\,\rangle$ and a predictor P_E which is $k, \langle\,\rangle$-correct for all k, and $P_E(\langle\lambda\rangle) = x$.*

IV. MAXIMAL INCOMPUTABILITY

The formal and physically motivated model of prediction we have presented can be applied to any physical experiment. However, let us turn our attention to using it to categorise more rigorously the unpredictability of quantum measurement outcomes discussed in Sec. II B.

We first show that experiments utilising quantum value indefinite observers cannot have a predictor which is $k, \langle\,\rangle$-correct for all k. More precisely: *if E is an experiment measuring a quantum value indefinite observer, then for every predictor P_E using any extractor $\langle\,\rangle$, P_E is not $k, \langle\,\rangle$-correct for all k.*

Throughout this section we will consider an experiment E performed in dimension $n \geq 3$ Hilbert space in which a quantum system is prepared in a state $|\psi\rangle$ and a value indefinite observable P_ϕ is measured producing a single bit x. By Theorem 1 such an observable is guaranteed to exist, and to identify one we need only a mismatch between preparation and observation contexts. The nature of the physical system in which this state is prepared and the experiment performed is not important, whether it be photons passing through generalised beam splitters [18], ions in an atomic trap, or any other quantum system in dimension $n \geq 3$ Hilbert space.

Let us fix an extractor $\langle\,\rangle$, and assume for the sake of contradiction that there exists a predictor P_E for E which is $k, \langle\,\rangle$-correct for all k. Consider the hypothetical situation where the experiment E is repeatedly initialised, performed and reset *ad infinitum* in an algorithmic "ritual" generating an infinite sequence of bits $\mathbf{x} = x_1 x_2 \dots$

Since P_E *never* makes an incorrect prediction, each of its predictions is correct with certainty. Then, according to the EPR principle we must conclude that each such prediction corresponds to a value definite property of the system measured in E. However, we chose E such that this is not the case: each x_i is the result of the measurement of a value indefinite observable, and thus we obtain a contradiction and conclude no such predictor P_E can exist.

Moreover, since there does not exist a predictor P_E which is $k, \langle\,\rangle$-correct using any extractor $\langle\,\rangle$ for all k, for such a quantum experiment E, no single outcome is predictable with certainty. Stated differently, in an infinite repetition of E as considered previously generating the infinite sequence $\mathbf{x} = x_1 x_2 \ldots$, *no single bit x_i can be predicted with certainty.*

A further consequence of this result is that the sequence $\mathbf{x}$ must be strongly incomputable, technically *bi-immune*.[19] This was shown in [8, 20], but follows directly and more naturally from this new formalism of prediction.

Let us assume for the sake of contradiction that $\mathbf{x} = x_1 x_2 \ldots$ is not bi-immune. Then, from the definition of bi-immunity, there exist an infinite computable set $I \subset \mathbb{N}^+$ and a partially computable function f whose domain is I and satisfies $f(i) = x_i$ for every $i \in I$. Consider the extractor $\langle \lambda_i \rangle = i$. Now we can use f to construct a predictor P_E which is $k, \langle\,\rangle$-correct for all $k > 0$. On the ith iteration of E with parameter λ_i,

$$P_E(\langle \lambda_i \rangle) = \begin{cases} f(i) = x_i, & \text{if } i \in I, \\ \text{“prediction withheld”}, & \text{if } i \notin I. \end{cases}$$

It is clear by the properties of f that P_E indeed satisfies the criteria to be $k, \langle\,\rangle$-correct for all k: each bit $x_{f(i)}$ for $i \in I$, for which there are infinitely many, is correctly predicted. Thus, since no such predictor can exist, the sequence $\mathbf{x}$ must be bi-immune; in particular, $\mathbf{x}$ is *incomputable*.

V. CONTEXTUAL ALTERNATIVES AND PREDICTABILITY

So far we have argued for the complete unpredictability of quantum bits literally created *“ex nihilo”*; that is, out of nowhere, thereby contradicting the *principle of sufficient reason “ex nihilo nihil fit”*. This is in accord with the orthodox viewpoint which associates irreducible indeterminism with certain single outcomes [1, 4].

It should be acknowledged, however, that the guarantee of such indeterminism relies on the assumptions made, in particular, on noncontextuality. If this assumption were abandoned the nature of unpredictability in quantum mechanics could be considerably different, and it is worth briefly considering this situation.

Perhaps the simplest alternative would be the explicit assumption of the context dependence of measurement results. The formal framework for such a theory is outlined in Refs. [8, 9], and most attempts to interpret quantum mechanics deterministically could be expressed in this framework. An important caveat is that, due to the experimental verification of Bell inequalities [7], any such deterministic hidden parameters must be explicitly nonlocal. The best-known such theory is Bohmian mechanics [21], although many others exist (see [11]). In such a theory unpredictability does not follow immediately as the *ex nihilo* results are sacrificed. However, predictability is still not an immediate consequence, as such hidden variables could potentially be "assigned" by a demon operating beyond the limits of computability.

A second alternative challenges the nontrivial assumption that a predetermined outcome (corresponding to a value definite property) needs to be a deterministic function of the observable alone. Instead one could insist that the *"... result of an observation may reasonably depend not only on the state of the system ... but also on the complete disposition of the apparatus"* [5, Sec. 5]. In particular this would apply to the situation of a mismatch between preparation and measurement for which the states prepared and measured are complementary (that is, neither collinear nor orthogonal). This position appears also to be in accord with Bohr's remarks [22, p. 210] on *"the impossibility of any sharp separation between the behaviour of atomic objects and the interaction with the measuring instruments which serve to define the conditions under which the phenomena appear."*

In this viewpoint, even when the macroscopic measurement apparatuses are still idealised as being perfect, their many degrees of freedom (which may by far exceed Avogadro's or Loschmidt's constants) contribute to any measurement of the single quantum. Most of these degrees of freedom might be totally uncontrollable by the experimenter, and may result in an *epistemic uncertainty* which is dominated by the combined complexities of interactions between the single quantum measured and the (macroscopic) measurement device producing the outcome.

In such a measurement, the pure single quantum and the apparatus would become entangled. In the absence of one-to-one uniqueness between the macroscopic states of the measurement apparatus and the quantum, any measurement would amount to a partial trace resulting in a mixed state of the apparatus, and thus to uncertainty and unpredictability of the readout.

In this minority view, just as for irreversibility in classical statistical mechanics [2], the indeterminism of single quantum measurements might not be irreducible at all, but an expression of, and relative to, the limited means available to analyse the situation. In Bell's terms, the outcome may be irreversible *for all practical purposes* [23].

VI. SUMMARY

The main thrust of our argument has been to formally certify the indeterminism of single quantum events and their consequential unpredictability, rather than rely on the *ad hoc* postulation of these properties. In particular, suppose that we prepare a quantum in a pure state corresponding to a unit vector in Hilbert space of dimension at least three. Then any complementary observable property of this quantum—corresponding to some projector whose respective linear subspace is neither collinear nor orthogonal with respect to the pure state vector—has no predetermined value, and thus remains value indefinite. Furthermore, we show that the outcome of a measurement of such a property is unpredictable with respect to a very general model of prediction. These results are true relative to the assumptions made, in particular, admissibility, noncontextuality, and the eigenstate principle.

In other terms the bit resulting from the measurement of such an observable property is "created from nowhere", and cannot be causally connected to any physical entity, whether it be knowable in practice or hidden. One might say that the quantum system acts like an *incomputable oracle.*

This irreducible indeterminacy "certifies" the use of quantum random number generators for various computational tasks in cryptography and elsewhere [24–26]. Our results can also be interpreted as justification for certain claims of *hypercomputation*, as no universal Turing machine will ever be able to produce in the limit an output that is identical with the sequence of bits generated by a quantum oracle [27]. More than that—no single bit of such sequences can ever be predicted.

As a concluding remark, we emphasise that the indeterminism and unpredictability of quantum measurement outcomes are based on the results of strong forms of the Kochen-Specker theorem, and hence require at minimum three-dimensional Hilbert space. This requirement is necessary to ensure the nontrivial interconnectedness of contexts (i.e. maximal sets of compatible observables) used to derive such results. We thus strongly recommend the use of at least three-dimensional Hilbert space in the construction of quantum random number generators based on quantised systems and quantum indeterminism.

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