All the singlet states

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Abstract

We present a group theoretic method to construct all N-particle singlet states by recursion and iteration and we derive the symmetries of N-particle singlet states.

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I. INTRODUCTION

Singlet states are among the most useful states in quantum mechanics; yet their explicit structure—although well understood in general terms in group theory—has up to now neither been enumerated nor investigated beyond a few instances for spin-$\frac{1}{2}$ and spin-1 particles. Recent theoretical and experimental studies in multi-particle production (e.g., Ref. [3]) elicit that a more systematic way to generate the complete set of arbitrary N-particle singlet states is desirable.

In the present study we pursue an algorithmic generation strategy, and tabulate some of the first singlet states. The recursive method employed is based on triangle relations and Clebsch-Gordan coefficients (e.g., Ch. 13, Sec. 27 of Ref. [4]). With this approach, a complete table of all angular momentum states is created. The singlet states stem from the various pathways towards the $j = m = 0$ states. The procedure can best be illustrated in a triangular diagram where the states in ascending order of angular momentum are drawn against the number of particles. In such a diagram, the "lowest" states correspond to singlets.

A. Description of the algorithm for obtaining spin-$\frac{1}{2}$ and spin-1 N-particle singlet states

There always exist "zigzag" singlet states, which are the product of r two-particle singlet states stemming from the rising and lowering of consecutive states. The situation is depicted in Fig. 1. For $j = 1$ and $N = 3r$ there exist "zigzag" singlet states, which are the product of r three-particle singlet states. For singlet states with $N = 2r + 3t$ (r, t integer) there exist singlet states being the product of r two-particle singlet states and t three-particle singlet states.

Here we present a method to construct all states for a given number of particles. They are the basis to construct non-trivial, e.g., non-zigzag singlet states, which are not just products of singlet states of a smaller number of particles.

We start by considering the spin state of a single spin-$\frac{1}{2}$ particle. A second spin-$\frac{1}{2}$ particle is added by combining two angular momenta $\frac{1}{2}$ to all possible angular momenta $j_{12} = 0, 1$. Next a third particle is introduced by coupling a third angular momentum $\frac{1}{2}$ to all previously derived states. Following the triangle equation, the resulting j-values for each j_{12} are:

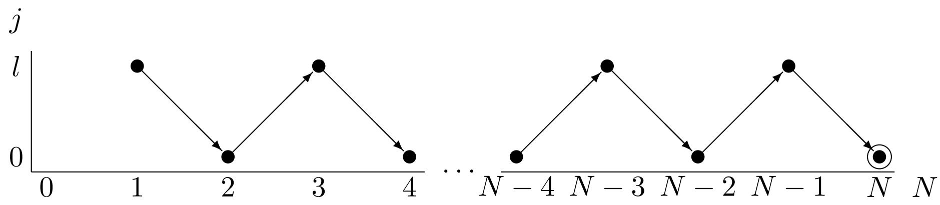

FIG. 1: Construction of the "zigzag" singlet state of N particles which effectively is a product state of $\frac{N}{2}$ spin-l particle states.

$|j_{12} - j_3| \leq j \leq j_{12} + j_3$.

In order to obtain all N-particle singlet states, we successively produce all states (not only singlets) of $\frac{N}{2}$ particles. From this point on, only certain states are necessary for the further procedure. For $\frac{N}{2} \leq h \leq N$ particles we only need angular momentum states with $0 \leq j \leq \frac{N-h}{2}$.

Angular momentum states will be written as $|h, j, m, i\rangle$, where h denotes the particle number, j the angular momentum, m the magnetic quantum number, i the number of state. The Clebsch-Gordan coefficient is denoted $\langle j_1, j_2, m_1, m_2 | j, m\rangle$. $f[j+1, h-1]$ denotes the number of states at h particles and angular momentum $\frac{j}{2}$.

For the sake of demonstration of the above method, we consider the explicit procedure for obtaining the states $|h, j, m, i\rangle$. We first consider the states of $h-1$ particles and angular momentum $j + \frac{1}{2}$. To produce the concrete state $|h, j, m, i\rangle$ we multiply the Clebsch-Gordan coefficient $\langle j+\frac{1}{2}, m-\frac{1}{2}, \frac{1}{2}, \frac{1}{2} | j, m\rangle$ with the product state $|h-1, j+\frac{1}{2}, m-\frac{1}{2}, i\rangle \otimes |1, \frac{1}{2}, \frac{1}{2}, 1\rangle$. We take the state $|h-1, j+\frac{1}{2}, m+\frac{1}{2}, i\rangle$, build the product state $|h-1, j+\frac{1}{2}, m+\frac{1}{2}, i\rangle \otimes |1, \frac{1}{2}, -\frac{1}{2}, 1\rangle$ and multiply it with the Clebsch-Gordan coefficient $\langle j+\frac{1}{2}, m+\frac{1}{2}, \frac{1}{2}, -\frac{1}{2} | j, m\rangle$. Adding the two results we obtain the state $|h, j, m, i\rangle$. We do this for $m = -j, j$ and $i = 1, f[(2j+1)+1, h-1]$. If j is greater than zero, we look at the states $|h-1, j-\frac{1}{2}, m-\frac{1}{2}, i\rangle$ and $|h-1, j-\frac{1}{2}, m+\frac{1}{2}, i\rangle$ and obtain the $|h, j, m, i\rangle$ particle state as the sum of $\langle j-\frac{1}{2}, m-\frac{1}{2}, \frac{1}{2}, \frac{1}{2} | j, m\rangle |h-1, j-\frac{1}{2}, m-\frac{1}{2}, i\rangle \otimes |1, \frac{1}{2}, \frac{1}{2}, 1\rangle$ and $\langle j-\frac{1}{2}, m+\frac{1}{2}, \frac{1}{2}, -\frac{1}{2} | j, m\rangle |h-1, j-\frac{1}{2}, m+\frac{1}{2}, i\rangle \otimes |1, \frac{1}{2}, -\frac{1}{2}, 1\rangle$. This

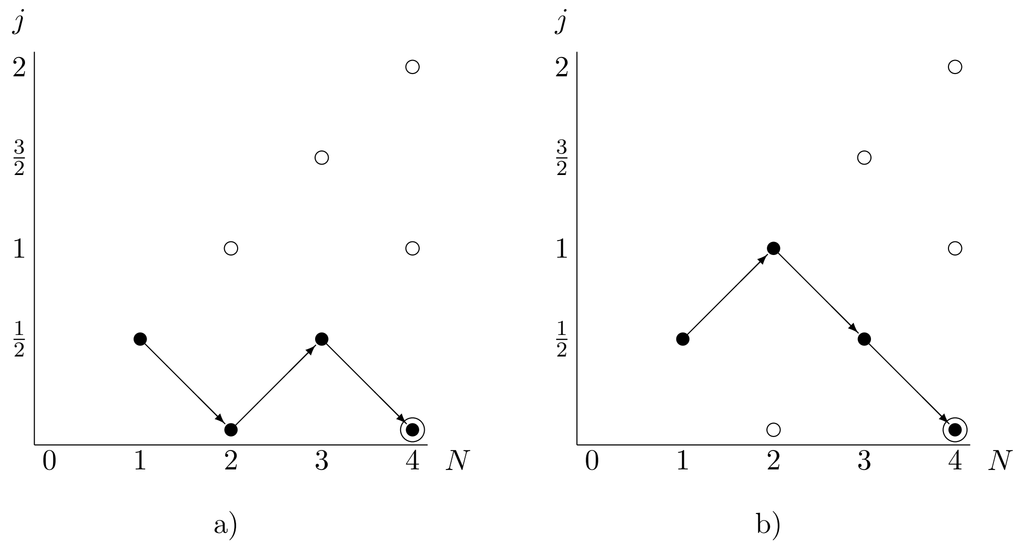

FIG. 2: Construction of both singlet states of four spin-$\frac{1}{2}$ particles. Concentric circles indicate the target states.

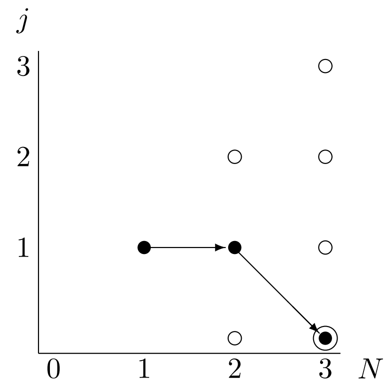

FIG. 3: Construction of the singlet state of three spin-1 particles.

procedure is carried out for $m = -j, j$ and $i = f[(2j+1)+1, h-1]+1, f[(2j+1)+1, h-1]+f[(2j+1)-1, h-1]$.

A concrete example is drawn in Fig. 2. It contains the pathways leading to the construction of both singlet states of four spin-$\frac{1}{2}$ particles. Another example is the construction of the three spin-1 particle singlet state drawn in Fig. 3. The singlet states of up to 6 spin-$\frac{1}{2}$ and 4 spin-1 particle are explicitly enumerated in Tables I and II.

N	#	
2	1	$\frac{1}{\sqrt{2}}\big(\lvert+,-\rangle - \lvert-,+\rangle\big);$
4	1	$-\frac{1}{2\sqrt{3}}\big(\lvert-,+,-,+\rangle + \lvert-,+,+,-\rangle + \lvert+,-,-,+\rangle + \lvert+,-,+,-\rangle\big)+$ $+\frac{1}{\sqrt{3}}\big(\lvert-,-,+,+\rangle + \lvert+,+,-,-\rangle\big);$
4	2	$\big(-\frac{1}{\sqrt{2}}\lvert-,+\rangle + \frac{1}{\sqrt{2}}\lvert+,-\rangle\big)^2;$
6	1	$-\frac{1}{2}\lvert-,-,-,+,+,+\rangle + -\frac{1}{6}\big(\lvert-,+,+,-,-,+\rangle + \lvert-,+,+,-,+,-\rangle+$ $+\lvert-,+,+,+,-,-\rangle + \lvert+,-,+,-,-,+\rangle + \lvert+,-,+,-,+,-\rangle+$ $+\lvert+,-,+,+,-,-\rangle + \lvert+,+,-,-,-,+\rangle + \lvert+,+,-,-,+,-\rangle+$ $+\lvert+,+,-,+,-,-\rangle\big) + \frac{1}{6}\big(\lvert-,-,+,-,+,+\rangle + \lvert-,-,+,+,-,+\rangle+$ $+\lvert-,-,+,+,+,-\rangle + \lvert-,+,-,-,+,+\rangle + \lvert-,+,-,+,-,+\rangle+$ $+\lvert-,+,-,+,+,-\rangle + \lvert+,-,-,-,+,+\rangle + \lvert+,-,-,+,-,+\rangle+$ $+\lvert+,-,-,+,+,-\rangle\big) + \frac{1}{2}\lvert+,+,+,-,-,-\rangle;$
6	2	$-\frac{\sqrt{2}}{3}\lvert-,-,+,-,+,+\rangle + -\frac{1}{3\sqrt{2}}\big(\lvert-,+,+,+,-,-\rangle + \lvert+,-,+,+,-,-\rangle+$ $+\lvert+,+,-,-,-,+\rangle + \lvert+,+,-,-,+,-\rangle\big) + -\frac{1}{6\sqrt{2}}\big(\lvert-,+,-,+,-,+\rangle+$ $+\lvert-,+,-,+,+,-\rangle + \lvert+,-,-,+,-,+\rangle + \lvert+,-,-,+,+,-\rangle\big)+$ $+\frac{1}{6\sqrt{2}}\big(\lvert-,+,+,-,-,+\rangle + \lvert-,+,+,-,+,-\rangle + \lvert+,-,+,-,-,+\rangle+$ $+\lvert+,-,+,-,+,-\rangle\big) + \frac{1}{3\sqrt{2}}\big(\lvert-,-,+,+,-,+\rangle + \lvert-,-,+,+,+,-\rangle+$ $+\lvert-,+,-,-,+,+\rangle + \lvert+,-,-,-,+,+\rangle\big) + \frac{\sqrt{2}}{3}\lvert+,+,-,+,-,-\rangle;$
6	3	$-\frac{1}{\sqrt{6}}\big(\lvert-,+,-,-,+,+\rangle + \lvert-,+,+,+,-,-\rangle\big) + -\frac{1}{2\sqrt{6}}\big(\lvert+,-,-,+,-,+\rangle+$ $+\lvert+,-,-,+,+,-\rangle + \lvert+,-,+,-,-,+\rangle + \lvert+,-,+,-,+,-\rangle\big)+$ $+\frac{1}{2\sqrt{6}}\big(\lvert-,+,-,+,-,+\rangle + \lvert-,+,-,+,+,-\rangle + \lvert-,+,+,-,-,+\rangle+$ $+\lvert-,+,+,-,+,-\rangle\big) + \frac{1}{\sqrt{6}}\big(\lvert+,-,-,-,+,+\rangle + \lvert+,-,+,+,-,-\rangle\big);$
6	4	$-\frac{1}{\sqrt{6}}\big(\lvert-,-,+,+,-,+\rangle + \lvert+,+,-,-,-,+\rangle\big) + -\frac{1}{2\sqrt{6}}\big(\lvert-,+,-,+,+,-\rangle+$ $+\lvert-,+,+,-,+,-\rangle + \lvert+,-,-,+,+,-\rangle + \lvert+,-,+,-,+,-\rangle\big)+$ $+\frac{1}{2\sqrt{6}}\big(\lvert-,+,-,+,-,+\rangle + \lvert-,+,+,-,-,+\rangle + \lvert+,-,-,+,-,+\rangle+$ $+\lvert+,-,+,-,-,+\rangle\big) + \frac{1}{\sqrt{6}}\big(\lvert-,-,+,+,+,-\rangle + \lvert+,+,-,-,+,-\rangle\big);$
6	5	$\big(-\frac{1}{\sqrt{2}}\lvert-,+\rangle + \frac{1}{\sqrt{2}}\lvert+,-\rangle\big)^3.$

TABLE I: First singlet states of N spin-$\frac{1}{2}$ particles.

N	#	
2	1	$\frac{1}{\sqrt{3}}\big(-\vert 0,0\rangle+\vert -1,1\rangle+\vert 1,-1\rangle\big);$
3	1	$-\frac{1}{\sqrt{6}}\big(\vert -1,0,1\rangle+\vert 0,1,-1\rangle+\vert 1,-1,0\rangle\big)+ +\frac{1}{\sqrt{6}}\big(\vert -1,1,0\rangle+\vert 0,-1,1\rangle+\vert 1,0,-1\rangle\big);$
4	1	$-\frac{1}{2\sqrt{5}}\big(\vert -1,0,0,1\rangle+\vert -1,0,1,0\rangle+\vert 0,-1,0,1\rangle+\vert 0,-1,1,0\rangle+ +\vert 0,1,-1,0\rangle+\vert 0,1,0,-1\rangle+\vert 1,0,-1,0\rangle+\vert 1,0,0,-1\rangle\big)+ +\frac{1}{6\sqrt{5}}\big(\vert -1,1,-1,1\rangle+\vert -1,1,1,-1\rangle+\vert 1,-1,-1,1\rangle+\vert 1,-1,1,-1\rangle\big)+ +\frac{1}{3\sqrt{5}}\big(\vert -1,1,0,0\rangle+\vert 0,0,-1,1\rangle+\vert 0,0,1,-1\rangle+\vert 1,-1,0,0\rangle\big)+ +\frac{2}{3\sqrt{5}}\vert 0,0,0,0\rangle+\frac{1}{\sqrt{5}}\big(\vert -1,-1,1,1\rangle+\vert 1,1,-1,-1\rangle\big);$
4	2	$-\frac{1}{2\sqrt{3}}\big(\vert -1,0,1,0\rangle+\vert -1,1,-1,1\rangle+\vert 0,-1,0,1\rangle+\vert 0,1,0,-1\rangle+ +\vert 1,-1,1,-1\rangle+\vert 1,0,-1,0\rangle\big)+\frac{1}{2\sqrt{3}}\big(\vert -1,0,0,1\rangle+\vert -1,1,1,-1\rangle+ +\vert 0,-1,1,0\rangle+\vert 0,1,-1,0\rangle+\vert 1,-1,-1,1\rangle+\vert 1,0,0,-1\rangle\big);$
4	3	$\big(\frac{1}{\sqrt{3}}\big(-\vert 0,0\rangle+\vert -1,1\rangle+\vert 1,-1\rangle\big)\big)^2;$
5	1	$-\sqrt{\frac{2}{15}}\vert -1,-1,0,1,1\rangle+-\frac{1}{\sqrt{30}}\big(\vert -1,0,1,0,0\rangle+\vert 0,-1,1,0,0\rangle+ +\vert 0,0,-1,0,1\rangle+\vert 0,0,-1,1,0\rangle+\vert 0,1,1,-1,-1\rangle+ +\vert 1,0,1,-1,-1\rangle+\vert 1,1,-1,-1,0\rangle+\vert 1,1,-1,0,-1\rangle\big)+ +-\frac{1}{2\sqrt{30}}\big(\vert -1,0,1,-1,1\rangle+\vert -1,0,1,1,-1\rangle+\vert -1,1,-1,0,1\rangle+ +\vert -1,1,-1,1,0\rangle+\vert 0,-1,1,-1,1\rangle+\vert 0,-1,1,1,-1\rangle+ +\vert 0,1,0,-1,0\rangle+\vert 0,1,0,0,-1\rangle+\vert 1,-1,-1,0,1\rangle+ +\vert 1,-1,-1,1,0\rangle+\vert 1,0,0,-1,0\rangle+\vert 1,0,0,0,-1\rangle\big)+ +\frac{1}{2\sqrt{30}}\big(\vert -1,0,0,0,1\rangle+\vert -1,0,0,1,0\rangle+\vert -1,1,1,-1,0\rangle+ +\vert -1,1,1,0,-1\rangle+\vert 0,-1,0,0,1\rangle+\vert 0,-1,0,1,0\rangle+ +\vert 0,1,-1,-1,1\rangle+\vert 0,1,-1,1,-1\rangle+\vert 1,-1,1,-1,0\rangle+ +\vert 1,-1,1,0,-1\rangle+\vert 1,0,-1,-1,1\rangle+\vert 1,0,-1,1,-1\rangle\big)+ +\frac{1}{\sqrt{30}}\big(\vert -1,-1,1,0,1\rangle+\vert -1,-1,1,1,0\rangle+\vert -1,0,-1,1,1\rangle+ +\vert 0,-1,-1,1,1\rangle+\vert 0,0,1,-1,0\rangle+\vert 0,0,1,0,-1\rangle+ +\vert 0,1,-1,0,0\rangle+\vert 1,0,-1,0,0\rangle\big)+\sqrt{\frac{2}{15}}\vert 1,1,0,-1,-1\rangle;$
6	7	$-\frac{1}{\sqrt{15}}\big(\vert -1,-1,0,1,1,0\rangle+\vert 1,1,0,-1,-1,0\rangle\big)+-\frac{1}{2\sqrt{15}}\big(\vert -1,-1,1,0,0,1\rangle+ +\vert -1,-1,1,1,-1,1\rangle+\vert -1,0,-1,1,0,1\rangle+\vert -1,0,1,0,1,-1\rangle+ +\vert 0,-1,-1,1,0,1\rangle+\vert 0,-1,1,0,1,-1\rangle+\vert 0,0,-1,0,1,0\rangle+ +\vert 0,0,-1,1,1,-1\rangle+\vert 0,0,1,-1,-1,1\rangle+\vert 0,0,1,0,-1,0\rangle+ +\vert 0,1,-1,0,-1,1\rangle+\vert 0,1,1,-1,0,-1\rangle+\vert 1,0,-1,0,-1,1\rangle+ +\vert 1,0,1,-1,0,-1\rangle+\vert 1,1,-1,-1,1,-1\rangle+\vert 1,1,-1,0,0,-1\rangle\big)+ +-\frac{1}{4\sqrt{15}}\big(\vert -1,0,0,0,0,1\rangle+\vert -1,0,0,1,-1,1\rangle+\vert -1,0,1,-1,1,0\rangle+ +\vert -1,0,1,1,0,-1\rangle+\vert -1,1,-1,0,1,0\rangle+\vert -1,1,-1,1,1,-1\rangle+ +\vert -1,1,1,-1,-1,1\rangle+\vert -1,1,1,0,-1,0\rangle+\vert 0,-1,0,0,0,1\rangle+ +\vert 0,-1,0,1,-1,1\rangle+\vert 0,-1,1,-1,1,0\rangle+\vert 0,-1,1,1,0,-1\rangle+ +\vert 0,1,-1,-1,0,1\rangle+\vert 0,1,-1,1,-1,0\rangle+\vert 0,1,0,-1,1,-1\rangle+ +\vert 0,1,0,0,0,-1\rangle+\vert 1,-1,-1,0,1,0\rangle+\vert 1,-1,-1,1,1,-1\rangle+ +\vert 1,-1,1,-1,-1,1\rangle+\vert 1,-1,1,0,-1,0\rangle+\vert 1,0,-1,-1,0,1\rangle+ +\vert 1,0,-1,1,-1,0\rangle+\vert 1,0,0,-1,1,-1\rangle+\vert 1,0,0,0,0,-1\rangle\big)+ +\frac{1}{4\sqrt{15}}\big(\vert -1,0,0,0,1,0\rangle+\vert -1,0,0,1,1,-1\rangle+\vert -1,0,1,-1,0,1\rangle+ +\vert -1,0,1,1,-1,0\rangle+\vert -1,1,-1,0,0,1\rangle+\vert -1,1,-1,1,-1,1\rangle+ +\vert -1,1,1,-1,1,-1\rangle+\vert -1,1,1,0,0,-1\rangle+\vert 0,-1,0,0,1,0\rangle+ +\vert 0,-1,0,1,1,-1\rangle+\vert 0,-1,1,-1,0,1\rangle+\vert 0,-1,1,1,-1,0\rangle+ +\vert 0,1,-1,-1,1,0\rangle+\vert 0,1,-1,1,0,-1\rangle+\vert 0,1,0,-1,-1,1\rangle+ +\vert 0,1,0,0,-1,0\rangle+\vert 1,-1,-1,0,0,1\rangle+\vert 1,-1,-1,1,-1,1\rangle+ +\vert 1,-1,1,-1,1,-1\rangle+\vert 1,-1,1,0,0,-1\rangle+\vert 1,0,-1,-1,1,0\rangle+ +\vert 1,0,-1,1,0,-1\rangle+\vert 1,0,0,-1,-1,1\rangle+\vert 1,0,0,0,-1,0\rangle\big)+$

$$+\frac{1}{2\sqrt{15}}\big(|-1,-1,1,0,1,0\rangle+|-1,-1,1,1,1,-1\rangle+|-1,0,-1,1,1,0\rangle+$$
$$+|-1,0,1,0,-1,1\rangle+|0,-1,-1,1,1,0\rangle+|0,-1,1,0,-1,1\rangle+$$
$$+|0,0,-1,0,0,1\rangle+|0,0,-1,1,-1,1\rangle+|0,0,1,-1,1,-1\rangle+$$
$$+|0,0,1,0,0,-1\rangle+|0,1,-1,0,1,-1\rangle+|0,1,1,-1,-1,0\rangle+$$
$$+|1,0,-1,0,1,-1\rangle+|1,0,1,-1,-1,0\rangle+|1,1,-1,-1,-1,1\rangle+$$
$$+|1,1,-1,0,-1,0\rangle\big)+\frac{1}{\sqrt{15}}\big(|-1,-1,0,1,0,1\rangle+|1,1,0,-1,0,-1\rangle\big).$$

TABLE II: First singlet states of N spin-1 particles.

II. SYMMETRIES

In what follows we shall discuss the symmetry behavior of singlet states. In our approach the singlet states are orthogonal to each other. This can be demonstrated by considering the formula

$$\begin{aligned}\langle(j_1'j_2')jm|(j_1j_2)jm\rangle =& \sum_{m_1'+m_2'=m,m_1+m_2=m}\langle(j_1'j_2')jm|j_1'm_1'j_2'm_2'\rangle\times\\ &\langle j_1'm_1'j_2'm_2'|j_1m_1j_2m_2\rangle\langle j_1m_1j_2m_2|(j_1j_2)jm\rangle\\ =&\delta_{j_1j_1'}\delta_{j_2j_2'}\delta_{m_1m_1'}\delta_{m_2m_2'}.\end{aligned} \tag{1}$$

States stemming from different j_1 values are orthogonal to each other. Hence, also the singlet states derived from them are orthogonal. By iteration it follows that even singlet states stemming from the same j_1 are orthogonal. The method allows us to construct the full basis for each singlet space which has the appropriate dimension.

A. Symmetries of the states with respect to changing all magnetic quantum numbers into their negative

For the Clebsch-Gordan coefficients the following formula holds

$$\langle j_1,-m_1,j_2,-m_2|j,-m\rangle=(-1)^{j_1+j_2-j}\langle j_1m_1j_2m_2|jm\rangle. \tag{2}$$

Let us consider the $j=1$ case first. The symmetry described above implies for the coupling of j to $j+1$:

$$\begin{aligned}\langle j,-m-1,1,1|j+1,-m\rangle =&(-1)^0\langle j,m+1,1,-1|j+1,m\rangle\\ \langle j,-m,1,0|j+1,-m\rangle =&(-1)^0\langle j,m,1,0|j+1,m\rangle,\end{aligned} \tag{3}$$

i.e. the Clebsch-Gordan coefficients are the same. For the coupling of j to j,

$$\begin{aligned}\langle j,-m-1,1,1|j,-m\rangle =&(-1)^1\langle j,m+1,1,-1|j,m\rangle\\ \langle j,-m,1,0|j,-m\rangle =&(-1)^1\langle j,m,1,0|j,m\rangle,\end{aligned} \tag{4}$$

i.e. all Clebsch-Gordan coefficients change their sign. Similarly for the coupling of $j+1$ to j,

$$\begin{aligned}\langle j+1,m,1,1|j,m+1\rangle =&(-1)^2\langle j+1,-m,1,-1|j,-m-1\rangle\\ \langle j+1,m,1,0|j,m\rangle =&(-1)^2\langle j+1,-m,1,0|j,-m\rangle,\end{aligned} \tag{5}$$

i.e. they all stay the same.

Using these symmetries, we conclude that the symmetry behavior stays the same if one goes from the angular momentum subspace $|N,j\rangle$ to the angular momentum subspace $|N+1,j+1\rangle$. The symmetry behavior does not change for coupling $|N,j+1\rangle$ to $|N+1,j\rangle$. Coupling $|N,j\rangle$ to $|N+1,j\rangle$ changes the symmetry behaviour from even to odd and from odd to even. The situation is depicted in Fig. 4. N-particle singlet states with N even are even, whereas N-particle singlet states with N odd are odd.

Next we turn to the $j=\frac{1}{2}$ case. For the coupling of j to $j+\frac{1}{2}$, the Clebsch-Gordan coefficients are

$$\begin{aligned}\langle j,-m-\frac{1}{2},\frac{1}{2},\frac{1}{2}|j+\frac{1}{2},-m\rangle =&(-1)^0\langle j,m+\frac{1}{2},\frac{1}{2},-\frac{1}{2}|j+\frac{1}{2},m\rangle\\ \langle j,m+\frac{1}{2},\frac{1}{2},-\frac{1}{2}|j+\frac{1}{2},m\rangle =&(-1)^0\langle j,-m-\frac{1}{2},\frac{1}{2},\frac{1}{2}|j+\frac{1}{2},-m\rangle.\end{aligned} \tag{6}$$

If we negate all the magnetic quantum numbers, all these Clebsch-Gordan coefficients stay the same. For the coupling of $j+\frac{1}{2}$ to j,

$$\begin{aligned}\langle j+\frac{1}{2},m,\frac{1}{2},\frac{1}{2}|j,m+\frac{1}{2}\rangle =&(-1)^1\langle j+\frac{1}{2},-m,\frac{1}{2},-\frac{1}{2}|j,-m-\frac{1}{2}\rangle\\ \langle j+\frac{1}{2},-m,\frac{1}{2},-\frac{1}{2}|j,-m-\frac{1}{2}\rangle =&(-1)^1\langle j+\frac{1}{2},m,\frac{1}{2},\frac{1}{2}|j,m+\frac{1}{2}\rangle.\end{aligned} \tag{7}$$

Here all the Clebsch-Gordan coefficients change their signs.

We conclude that the symmetry behavior stays the same if one goes from the angular momentum subspace $|N,j\rangle$ to the angular momentum subspace $|N+1,j+\frac{1}{2}\rangle$. Going from the subspace $|N,j\rangle$ to the subspace $|N+1,j-\frac{1}{2}\rangle$, the symmetry behavior changes from even to odd and from odd to even, respectively. The situation is depicted in Fig. 5. In particular, singlet states where N is $k\cdot 2\cdot 2$ (k is an integer) are even, and singlet states where N is $k\cdot 2\cdot(2+1)$ are odd.

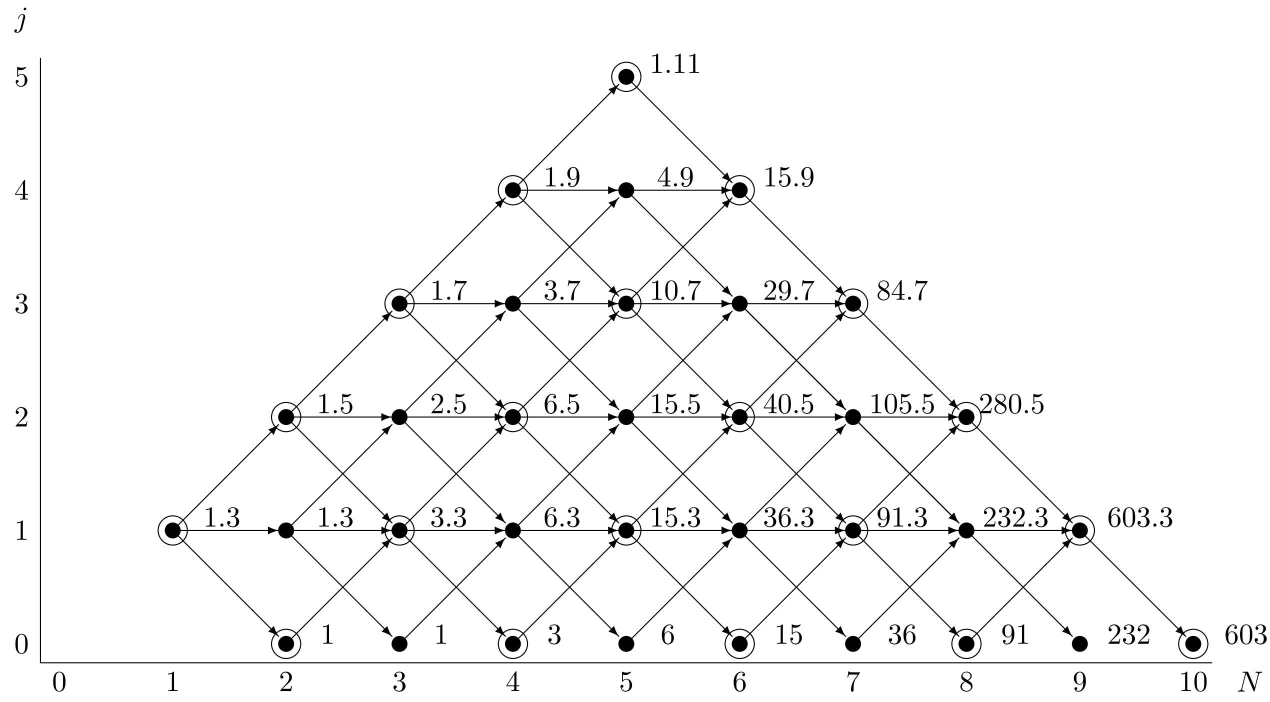

FIG. 4: Symmetries of spin-1 particle states. Even subspaces are denoted by concentric circles, odd subspaces are denoted by filled circles. The numbers denote the dimension of the subspaces. The first number stands for the number of states $|h, j\rangle$ and the second stands for the $2j + 1$ projections. Arrows represent the way of coupling.

B. Symmetries of the states obtained from consideration of the symmetric group

We want to assign the appropriate representation to irreducible singlet spaces. Therefore we consider the symmetric group. In every product state of every N-particle state we permute the N magnetic quantum numbers, more explicitly, we apply $(N-1)$ transpositions, since every permutation of N particles can be written as the product of (N-1) transpositions. We analyse $(N-1)$ transpositions of the form $(j, j+1)$, the transposition of j and $j+1$, which generate the whole symmetric group and in particular all the $N(N-1)/2$ transpositions, since $(j, k+1) = (k, k+1)(j, k)(k, k+1)$. Hence we consider the class (21^{N-2}). Each irreducible representation can be labelled by an ordered partition of integers which corresponds to a specific Young diagram.

1. *Application of the symmetries to the $j = \frac{1}{2}$ case*

As stated in App. D, Sec. 14 of Ref. [4], the space spanned by the vectors of total spins (SM) formed by N identical spins $\frac{1}{2}$ is associated with an irreducible representation of S_N,

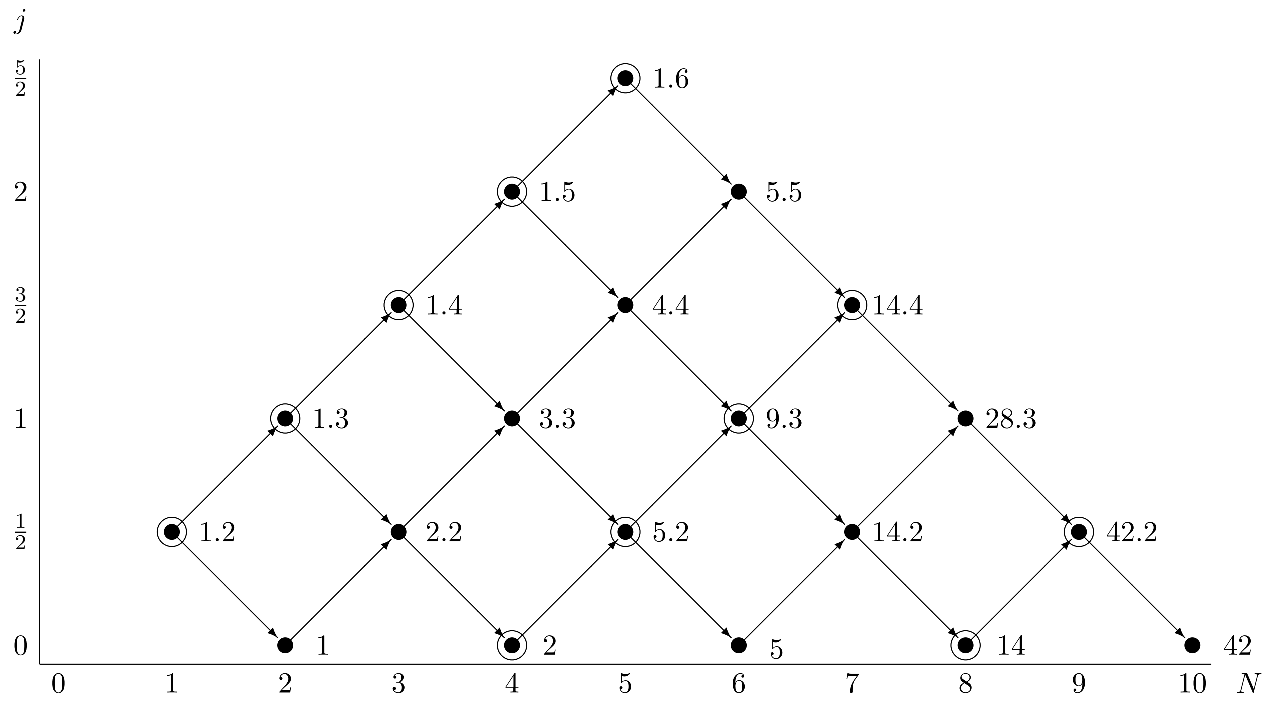

FIG. 5: Symmetries of spin-$\frac{1}{2}$ particles. Even subspaces are denoted by concentric circles, odd subspaces are denoted by filled circles. The numbers denote the dimension of the subspaces. The first number stands for the number of states $|h, j\rangle$ and the second stands for the $2j+1$ projections. Arrows represent the way of coupling.

the representation whose Young diagram corresponds to the partition $[\frac{1}{2}N + S, \frac{1}{2}N - S]$ of the integer N. It is apparent that the Young tableaux for the irreducible components of the representation of S_N have at most two lines. For $N > 2$, any state contains at least two individual spins in the same state. Suppose the state contains the factor $u_+^{(i)} u_+^{(j)}$, i.e. $m_i, m_j = \frac{1}{2}$. Since $A = \frac{1}{2}(1 - (i,j))$ is the antisymmetrizer and $\frac{1}{2}(1-(i,j))u_+^{(i)} u_+^{(j)} = 0$, it follows that $A|jm\rangle = 0$.

Using the theorem mentioned above, the Young diagrams of the irreducible spaces of the N-particle singlet states correspond to the partitions $[\frac{1}{2}N, \frac{1}{2}N]$. Hence the two-particle singlet state is an antisymmetric one dimensional space. The four- and six-particle singlet spaces form a two and a five dimensional irreducible space whose Young diagrams are of the form $[2,2]$ and $[3,3]$. Using the formula for the dimension of an irreducible representation having the partition [λ] (e.g., Ref. [5])

$$f^\lambda = n! \frac{\Pi_{i<j\leq k}(\lambda_i - \lambda_j + j - i)}{\Pi_{i=1}^k(\lambda_i + k - i)!}, \tag{8}$$

we can verify the dimension.

2. Application of the symmetries to the $j = 1$ case

Here, the Young tableaux for the irreducible components of the representation of S_N have at most three lines. For $N > 3$, any state contains at least two individual spins in the same state, hence we can argue as above. We calculate the matrix representations of the (N-1) transpositions.

Next we use Schur's Lemma. If a matrix S commutes with all the matrices of an irreducible representation G of a group, then it is a multiple of the unit matrix (e.g., App. D, Sec. 8 of Ref. [4]).

Furthermore, we can calculate the characteristic of each element. The trace of the matrix representing the element S_i which belongs to the irreducible representation $\Gamma^{(j)}$ is called the characteristic of S_i in $\Gamma^{(j)}$ and is denoted by $\chi^{(j)}(S_i)$. The set of characteristics of all elements S of the group as represented in $\Gamma^{(j)}$ is called a group character $\chi^{(j)}$. All elements of the same class ρ have the same characteristic $\chi_\rho^{(j)}$ [5].

The tables of characters can be used to find the appropriate partition. The results can be verified by calculating the characteristic of one element per class. Moreover, consideration of the outer product restricts the possible irreducible representations.

The two-particle singlet state is a one dimensional symmetric space. The three-particle singlet state is an antisymmetric one dimensional space. Hence their Young diagrams are of the form [2] and [1^3], respectively. Again we can check the dimension using Eq. (8). For four particles the three dimensional space (see Table II) of singlets is reducible and the class (21^2) has in this representation characteristic 1.

The character of a reducible representation $\Gamma^{(j)}$ is the sum of the characters of the irreducible parts of $\Gamma^{(j)}$. Hence we build the class sum, i.e., the sum of all the matrix representations of the transpositions. The calculation of the eigenvalues and the eigenvectors yields the decomposition of the space into a one and a two dimensional subspace. The one dimensional state is symmetric, denoted by [4]. The two dimensional irreducible representation has characteristic 0 which, using the table of characters, yields the partition [2,2]. We consider the outer product of two irreducible representations whose Young diagrams correspond to the partition [2], the product of the representations of two two-particle singlet states; i.e.,

$$[2] \times [2] = [4] + [3,1] + [2,2]. \tag{9}$$

This also yields the appropriate partitions. There are further restrictions, such as $m_1 = m_3$. Hence only some of the partitions are realized.

For five particles, the six dimensional space is irreducible and has the partition $[3, 1^2]$. This can be derived from the table of characters. Looking at the outer product of $[1,1,1]\times[2]$, the product of the partitions of the three- and the two-particle singlet state yields

$$[1,1,1] \times [2] = [3, 1^2] + [2, 1^3]. \tag{10}$$

Here the first partition has the required dimension.

The 15 dimensional space of singlets for six particles is reducible. In this representation the class (21^4) has characteristic three. The space decomposes into a one, a five and a nine dimensional irreducible subspace. Using the outer product yields for $[2] \times [2] \times [2]$, the product of the partitions of the two-particle singlet state to the cube

$$[4] \times [2] = [4,2] + [5,1] + [6] \qquad \text{and} \qquad [2,2] \times [2] = [4,2] + [3,2,1] + [2,2,2], \tag{11}$$

and for the product of the partitions of the two three-particle singlet states

$$[1,1,1] \times [1,1,1] = [1,1,1,1,1,1] + [2,1,1,1,1] + [2,2,1,1] + [2,2,2]. \tag{12}$$

The nine dimensional representation has characteristic three. By using the Eq. (8) and the table of characters, one finds that the appropriate partition is [4,2]. For the five dimensional irreducible representation we find characteristic -1, which leads to the partition $[2^3]$. Crosschecking with Eq. (11) and Eq. (12) we see that, as required, these partitions are contained in the outer products. Finally, the one dimensional state has characteristic one which leads to the partition [6]; hence it is a symmetric state. The one dimensional irreducible six-particle singlet state is enumerated in Table III.

$$
\begin{aligned}
&-\tfrac{3\sqrt{3}}{7}|0,0,0,0,0,0\rangle + \tfrac{3\sqrt{3}}{35}\big(|-1,0,0,0,0,1\rangle + |-1,0,0,0,1,0\rangle + |-1,0,0,1,0,0\rangle +\\
&|-1,0,1,0,0,0\rangle + |-1,1,0,0,0,0\rangle + |0,-1,0,0,0,1\rangle + |0,-1,0,0,1,0\rangle +\\
&|0,-1,0,1,0,0\rangle + |0,-1,1,0,0,0\rangle + |0,0,-1,0,0,1\rangle + |0,0,-1,0,1,0\rangle +\\
&|0,0,-1,1,0,0\rangle + |0,0,0,-1,0,1\rangle + |0,0,0,-1,1,0\rangle + |0,0,0,0,-1,1\rangle +\\
&|0,0,0,0,1,-1\rangle + |0,0,0,1,-1,0\rangle + |0,0,0,1,0,-1\rangle + |0,0,1,-1,0,0\rangle +\\
&|0,0,1,0,-1,0\rangle + |0,0,1,0,0,-1\rangle + |0,1,-1,0,0,0\rangle + |0,1,0,-1,0,0\rangle +\\
&|0,1,0,0,-1,0\rangle + |0,1,0,0,0,-1\rangle + |1,-1,0,0,0,0\rangle + |1,0,-1,0,0,0\rangle +\\
&|1,0,0,-1,0,0\rangle + |1,0,0,0,-1,0\rangle + |1,0,0,0,0,-1\rangle\big) + -\tfrac{2\sqrt{3}}{35}\big(|-1,-1,0,0,1,1\rangle +\\
&|-1,-1,0,1,0,1\rangle + |-1,-1,0,1,1,0\rangle + |-1,-1,1,0,0,1\rangle + |-1,-1,1,0,1,0\rangle +\\
&|-1,-1,1,1,0,0\rangle + |-1,0,-1,0,1,1\rangle + |-1,0,-1,1,0,1\rangle + |-1,0,-1,1,1,0\rangle +\\
&|-1,0,0,-1,1,1\rangle + |-1,0,0,1,-1,1\rangle + |-1,0,0,1,1,-1\rangle + |-1,0,1,-1,0,1\rangle +\\
&|-1,0,1,-1,1,0\rangle + |-1,0,1,0,-1,1\rangle + |-1,0,1,0,1,-1\rangle + |-1,0,1,1,-1,0\rangle +\\
&|-1,0,1,1,0,-1\rangle + |-1,1,-1,0,0,1\rangle + |-1,1,-1,0,1,0\rangle + |-1,1,-1,1,0,0\rangle +\\
&|-1,1,0,-1,0,1\rangle + |-1,1,0,-1,1,0\rangle + |-1,1,0,0,-1,1\rangle + |-1,1,0,0,1,-1\rangle +\\
&|-1,1,0,1,-1,0\rangle + |-1,1,0,1,0,-1\rangle + |-1,1,1,-1,0,0\rangle + |-1,1,1,0,-1,0\rangle +\\
&|-1,1,1,0,0,-1\rangle + |0,-1,-1,0,1,1\rangle + |0,-1,-1,1,0,1\rangle + |0,-1,-1,1,1,0\rangle +\\
&|0,-1,0,-1,1,1\rangle + |0,-1,0,1,-1,1\rangle + |0,-1,0,1,1,-1\rangle + |0,-1,1,-1,0,1\rangle +\\
&|0,-1,1,-1,1,0\rangle + |0,-1,1,0,-1,1\rangle + |0,-1,1,0,1,-1\rangle + |0,-1,1,1,-1,0\rangle +\\
&|0,-1,1,1,0,-1\rangle + |0,0,-1,-1,1,1\rangle + |0,0,-1,1,-1,1\rangle + |0,0,-1,1,1,-1\rangle +\\
&|0,0,1,-1,-1,1\rangle + |0,0,1,-1,1,-1\rangle + |0,0,1,1,-1,-1\rangle + |0,1,-1,-1,0,1\rangle +\\
&|0,1,-1,-1,1,0\rangle + |0,1,-1,0,-1,1\rangle + |0,1,-1,0,1,-1\rangle + |0,1,-1,1,-1,0\rangle +\\
&|0,1,-1,1,0,-1\rangle + |0,1,0,-1,-1,1\rangle + |0,1,0,-1,1,-1\rangle + |0,1,0,1,-1,-1\rangle +\\
&|0,1,1,-1,-1,0\rangle + |0,1,1,-1,0,-1\rangle + |0,1,1,0,-1,-1\rangle + |1,-1,-1,0,0,1\rangle +\\
&|1,-1,-1,0,1,0\rangle + |1,-1,-1,1,0,0\rangle + |1,-1,0,-1,0,1\rangle + |1,-1,0,-1,1,0\rangle +\\
&|1,-1,0,0,-1,1\rangle + |1,-1,0,0,1,-1\rangle + |1,-1,0,1,-1,0\rangle + |1,-1,0,1,0,-1\rangle +\\
&|1,-1,1,-1,0,0\rangle + |1,-1,1,0,-1,0\rangle + |1,-1,1,0,0,-1\rangle + |1,0,-1,-1,0,1\rangle +\\
&|1,0,-1,-1,1,0\rangle + |1,0,-1,0,-1,1\rangle + |1,0,-1,0,1,-1\rangle + |1,0,-1,1,-1,0\rangle +\\
&|1,0,-1,1,0,-1\rangle + |1,0,0,-1,-1,1\rangle + |1,0,0,-1,1,-1\rangle + |1,0,0,1,-1,-1\rangle +\\
&|1,0,1,-1,-1,0\rangle + |1,0,1,-1,0,-1\rangle + |1,0,1,0,-1,-1\rangle + |1,1,-1,-1,0,0\rangle +\\
&|1,1,-1,0,-1,0\rangle + |1,1,-1,0,0,-1\rangle + |1,1,0,-1,-1,0\rangle + |1,1,0,-1,0,-1\rangle +\\
&|1,1,0,0,-1,-1\rangle\big) + \tfrac{6\sqrt{3}}{35}\big(|-1,-1,-1,1,1,1\rangle + |-1,-1,1,-1,1,1\rangle + |-1,-1,1,1,-1,1\rangle +\\
&|-1,-1,1,1,1,-1\rangle + |-1,1,-1,-1,1,1\rangle + |-1,1,-1,1,-1,1\rangle + |-1,1,-1,1,1,-1\rangle +\\
&|-1,1,1,-1,-1,1\rangle + |-1,1,1,-1,1,-1\rangle + |-1,1,1,1,-1,-1\rangle + |1,-1,-1,-1,1,1\rangle +\\
&|1,-1,-1,1,-1,1\rangle + |1,-1,-1,1,1,-1\rangle + |1,-1,1,-1,-1,1\rangle + |1,-1,1,-1,1,-1\rangle +\\
&|1,-1,1,1,-1,-1\rangle + |1,1,-1,-1,-1,1\rangle + |1,1,-1,-1,1,-1\rangle + |1,1,-1,1,-1,-1\rangle +\\
&|1,1,1,-1,-1,-1\rangle\big)
\end{aligned}
$$

TABLE III: The one dimensional irreducible six-particle singlet state.

The partitions for up to ten particles are obtained by using the outer product and neglecting diagrams with more then three lines. The are enumerated in Table IV.

We conjecture that for N even one obtains all possible diagrams from the diagram [N] by splitting the partition [2] as often as possible and adding them again to all possible diagrams with at most three lines. Furthermore, we conjecture that for odd N, all diagrams can be obtained by splitting [2] as often as possible from the diagram [N-2,1,1] and adding them again to the diagram to all possible regular diagrams with at most three lines.

N	partition
7	[3,3,1],[5,1,1]
8	[8],[6,2],[4,4],[4,2,2]
9	[7,1,1],[5,3,1],[3,3,3]
10	[10],[8,2],[6,4],[4,4,2],[6,2,2]

TABLE IV: The partitions for seven to ten particles.

III. SUMMARY

In summary, we have present a detailed, algorithmic description of how to obtain all singlet states of spin-$\frac{1}{2}$ and spin-1 particles. The method can applied analogously for the construction of N-particle singlet states from particles with higher dimensional spin. We have also investigated the behaviour of these states under symmetry transformations.

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