Chromatic Contextuality

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Shannon Entropy: Quantifying Information

- Shannon Entropy (H) measures the average uncertainty or information content of a random variable.
- The more uncertain an outcome (i.e., the more equally likely the possibilities), the higher its entropy, and thus, the more information we gain when observing it.
- It's typically measured in bits (when using $\log_2$).

Formula for Shannon Entropy

For a discrete random variable X with possible outcomes $x_1, x_2, \ldots, x_n$ and probabilities $P(x_1), P(x_2), \ldots, P(x_n)$:

$$H(X) = -\sum_{i=1}^{n} P(x_i) \log_2 P(x_i) \quad \text{bits}$$

- Assumption: For these examples, we assume the initial underlying states are **equiprobable**.

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3-State System: Full Resolution

- **System:** A system with 3 distinct states.
- **Observed Outcomes:** $\{0, 1, 2\}$
- **Probabilities (Equiprobable):**

$$P(0) = P(1) = P(2) = 1/3$$

Calculating Information (H_{full})

$$H_{full} = -\left[\frac{1}{3}\log_2\left(\frac{1}{3}\right) + \frac{1}{3}\log_2\left(\frac{1}{3}\right) + \frac{1}{3}\log_2\left(\frac{1}{3}\right)\right]$$
$$= -3 \times \frac{1}{3}\log_2\left(\frac{1}{3}\right) = -\log_2\left(\frac{1}{3}\right) = \log_2 3$$

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- **Mapping (Collapse):** $0 \to 0_{obs}$, $1 \to 1_{obs}$ and $2 \to 1_{obs}$
- **New Observed Outcomes:** $\{0_{obs}, 1_{obs}\}$
- **New Probabilities:**

$$P(0_{obs}) = P(0) = 1/3$$
$$P(1_{obs}) = P(1) + P(2) = 1/3 + 1/3 = 2/3$$

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$$H_{collapsed} = -\left[P(0_{obs})\log_2 P(0_{obs}) + P(1_{obs})\log_2 P(1_{obs})\right]$$
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Information Loss

- The act of collapsing states (losing the ability to distinguish between original 1 and 2) reduces the information obtained.
- Information Loss $= H_{full} - H_{collapsed} = 1.585 - 0.918 = \mathbf{0.667}$ bits

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$$= -4 \times \frac{1}{4}\log_2\left(\frac{1}{4}\right) = -\log_2\left(\frac{1}{4}\right) = -(-2)$$

$$\boldsymbol{H_{full} = 2}\ \text{bits}$$

A 4-state equiprobable system inherently provides 2 bits of information.

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- **New Probabilities:**

$$P(0'_{obs}) = P(0) + P(1) = 1/4 + 1/4 = 2/4 = 1/2$$
$$P(1'_{obs}) = P(2) + P(3) = 1/4 + 1/4 = 2/4 = 1/2$$

$$H = -\left[P(0'_{obs})\log_2 P(0'_{obs}) + P(1'_{obs})\log_2 P(1'_{obs})\right]$$
$$= -\left[\frac{1}{2}\log_2\left(\frac{1}{2}\right) + \frac{1}{2}\log_2\left(\frac{1}{2}\right)\right] = -\left[\frac{1}{2}(-1) + \frac{1}{2}(-1)\right] = 1$$

$\boldsymbol{H = 1}$ bit — This scenario behaves like a fair coin flip.

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4-State System: More Collapsed Case

- **Original States:** $\{0, 1, 2, 3\}$ (equiprobable)
- **Mapping (More Collapse):** $\{0, 1, 2\} \rightarrow 0''_{obs}$, $3 \rightarrow 1''_{obs}$
- **New Observed Outcomes:** $\{0''_{obs}, 1''_{obs}\}$
- **New Probabilities:**

$$P(0''_{obs}) = P(0) + P(1) + P(2) = 1/4 + 1/4 + 1/4 = 3/4$$
$$P(1''_{obs}) = P(3) = 1/4$$

Calculating Information (H)

$$\begin{aligned} H &= -\left[P(0''_{obs}) \log_2 P(0''_{obs}) + P(1''_{obs}) \log_2 P(1''_{obs})\right] \\ &= -\left[\frac{3}{4} \log_2 \left(\frac{3}{4}\right) + \frac{1}{4} \log_2 \left(\frac{1}{4}\right)\right] \approx 0.81125 \end{aligned}$$

$$H \approx \mathbf{0.811} \text{ bits}$$

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Calculating Information (H)

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$$= -\left[\frac{3}{4} \log_2 \left(\frac{3}{4}\right) + \frac{1}{4} \log_2 \left(\frac{1}{4}\right)\right] \approx 0.81125$$

$$\boldsymbol{H} \approx \mathbf{0.811} \text{ bits}$$

4-State System: Summary

- **Full Resolution:** $H_{full} = 2$ bits
- **Collapsed Case (ii):** $H_{collapsed\_ii} = 1$ bit
- **More Collapsed Case (iii):** $H_{collapsed\_iii} \approx 0.811$ bits

Observation

- Each step of collapsing states leads to a reduction in the measurable information content (entropy).
- The more states are merged, and the more skewed the resulting probabilities become, the lower the entropy.

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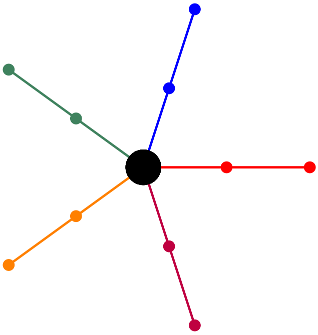
Key Takeaway

- The amount of information obtained from a system is directly related to its uncertainty or entropy.
- Collapsing or grouping outcomes reduces the resolution of our observation, leading to:
  - A decrease in the system's entropy.
  - A loss of information about the original, finer-grained states.
- This demonstrates how information is lost when we approximate or simplify a more complex system.

2-valued state in 3 dimensions - spectral composition in terms of 1- and $d-1$ dimensional subspaces



This is essentially a 3-state collapsed system: two states are mapped into 0 (painted in nonblack color), and one into 1 (painted black). In Hilbert space this represents a two-dimensional subspace, spanned by the eigenvectors of a continuum of orthonormal bases (here only 5 such bases are drawn).

Spectral Decomposition of Maximal Versus Degenerate Operators

Let $\{\mathbf{e}_i | 1 \leq i \leq d\}$ be an orthonormal basis.

- **Maximal operator (von Neumann, 1931):** Let λ_i be mutually distinct "colors", and $\sum_{i=1}^{d} \lambda_i |\mathbf{e}_i\rangle\langle\mathbf{e}_i|$
- **Degenerate Operator:** let $1 \leq j \leq d$ be fixed, and $\sum_{i=1}^{d} \delta_{ij} |\mathbf{e}_i\rangle\langle\mathbf{e}_i|$

Postulate of Classicality

Existence of classical d-ary elements of physical reality for certain finite quantum-inspired "chromatic Kochen-Specker" sets. Again, chromatic noncontextuality is assumed: "color is independent of the (hyper)edge".

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Some Results on Chromatic contextuality

- Whenever a (d-uniform hyper)graph has chromatic number d it has at least d two-valued states (by contraction) (M. H. Shekarriz and KS, JMP 63 (3), 032104 (2022) DOI: 10.1063/5.0062801).
- The Yu and Oh 3-uniform (hyper)graph has clique (element per hyperedge)number 3 but chromatic number 4 (and yet its set representable; same for Greechies G_{32}). "Chromatic Kochen Specker theorem" (KS, Entropy 27(4), 387 (2025) DOI: 10.3390/e27040387).
- The house/pentagon/pentagram d-uniform hypergraph has one "exotic" 2-valued state that cannot be obtained from contracting one of its 5 nonequivalent (modulo permuttations) colorings (KS, Entropy 27(4), 387 (2025) DOI: 10.3390/e27040387).

Summary

Colorings might be a formidable tool to investigate quantum contextuality and for classical probabilities.