All the singlet states

Peter Kasperkowitz, Maria Schimpf, and Karl Svozil

Institute of Theoretical Physics, Vienna University of Technology,

*Wiedner Hauptstraße 8-10/136, A-1040 Vienna, Austria**

Abstract

Singlet states of multiple particles have the property that they are form invariant with respect to overall changes of measurement direction. We present a group theoretic method to construct all N-particle singlet states by recursion and iteration.

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*Electronic address: `svozil@tuwien.ac.at`

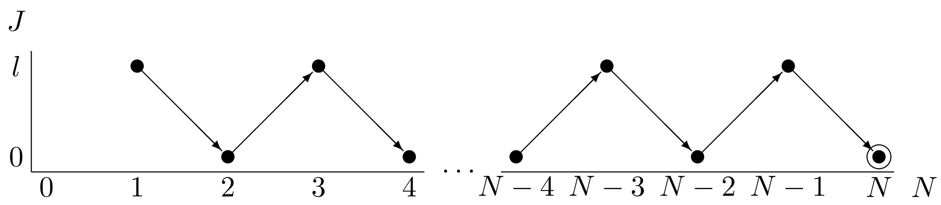

FIG. 1: Construction of the "zigzag" singlet state of N particles which effectively is a product state of $\frac{N}{2}$ spin-l particle states.

Singlet states are among the most useful states in quantum mechanics; yet their explicit structure—although well understood in general terms in group theory—has up to now neither been enumerated nor investigated beyond a few instances for spin-$\frac{1}{2}$ and spin-1 particles. Recent theoretical and experimental studies in multipartite production (e.g., Ref. [?]) elicit that a more systematic way to generate the complete set of arbitrary N-particle singlet states seems to be desirable.

In the present study we pursue an algorithmic generation strategy, and tabulate some of the first singlet states. The recursive method employed is based on the triangle relations and Clebsch-Gordan coefficients (e.g., Ch. 13, Sec. 27 of Ref. [?]). With this approach a complete table of all angular momentum states is created. The singlet states stem from the various pathways towards the $j = m = 0$ states. The procedure can best be illustrated in a triangular diagram where the states in ascending order of J are drawn against the number of particles. In such a diagram, the "lowest" states correspond to singlets.

There always exist "zigzag" singlet states, which are the product of r two-particle singlet states stemming from the rising and lowering of consecutive states. The situation is depicted in Fig. 1. For $J = 1$ and $N = 3r$ there exist "zigzag" singlet states, which are the product of r three-particle singlet states. For singlet states with $N = 2r + 3t$ (r, t integer) there exist singlet states being the product of r two-particle singlet states and t three-particle singlet states.

Here we present a method to construct all states for a given number of particles. They are the basis to construct non-trivial, e.g., non-zigzag singlet states, which are not just products of singlet states of a smaller number of particles.

We start by considering the spin state of a single spin-$\frac{1}{2}$ particle. A second spin-$\frac{1}{2}$ particle is added by combining two angular momenta $\frac{1}{2}$ to all possible angular momenta $j_{12} = 0, 1$. Next a third particle is introduced by coupling a third angular momentum $\frac{1}{2}$ to all previously derived states. Following the triangular equation, the resulting j-values for each j_{12} are: $|j_{12} - j_3| \leq j \leq j_{12} + j_3$.

In order to obtain all N-particle singlet states, we successively produce all states (not only singlets) of $\frac{N}{2}$ particles. From this point on, only certain states are necessary for the further procedure. For $\frac{N}{2} \leq h \leq N$ particles we only need angular momentum states with $0 \leq j \leq \frac{N-h}{2}$.

Angular momentum states will be written as $|h, j, m, i\rangle$, where h denotes the particle number, j the angular momentum, m the magnetic quantum number, i the number of state. The Clebsch-Gordan coefficient is denoted $\langle j_1, j_2, m_1, m_2 | j, m\rangle$. $f[j+1, h-1]$ denotes the number of states at h particles and angular momentum $\frac{j}{2}$.

Explicit procedure: In order to obtain the states $|h, j\rangle$ we first consider the states of $h-1$ particles and angular momentum $j + \frac{1}{2}$. To produce the concrete state $|h, j, m, i\rangle$ we multiply the Clebsch-Gordan coefficient $\langle j + \frac{1}{2}, m - \frac{1}{2}, \frac{1}{2}, \frac{1}{2} | j, m\rangle$ with the product state $|h-1, j+\frac{1}{2}, m-\frac{1}{2}, i\rangle \otimes |1, \frac{1}{2}, \frac{1}{2}, 1\rangle$. We take the state $|h-1, j+\frac{1}{2}, m+\frac{1}{2}, i\rangle$, build the product state $|h-1, j+\frac{1}{2}, m+\frac{1}{2}, i\rangle \otimes |1, \frac{1}{2}, -\frac{1}{2}, 1\rangle$ and multiply it with the Clebsch-Gordan coefficient $\langle j+\frac{1}{2}, m+\frac{1}{2}, \frac{1}{2}, -\frac{1}{2} | j, m\rangle$. Adding the two results we obtain the state $|h, j, m, i\rangle$. We do this for $m = -j, j$ and $i = 1, f[(2j+1)+1, h-1]$. If j is bigger than zero, we look at the states $|h-1, j-\frac{1}{2}, m-\frac{1}{2}, i\rangle$ and $|h-1, j-\frac{1}{2}, m+\frac{1}{2}, i\rangle$ and obtain the $|h, j, m, i\rangle$ particle state as the sum of $\langle j-\frac{1}{2}, m-\frac{1}{2}, \frac{1}{2}, \frac{1}{2} | j, m\rangle |h-1, j-\frac{1}{2}, m-\frac{1}{2}, i\rangle \otimes |1, \frac{1}{2}, \frac{1}{2}, 1\rangle$ and $\langle j-\frac{1}{2}, m+\frac{1}{2}, \frac{1}{2}, -\frac{1}{2} | j, m\rangle |h-1, j-\frac{1}{2}, m+\frac{1}{2}, i\rangle \otimes |1, \frac{1}{2}, -\frac{1}{2}, 1\rangle$. This procedure is done for $m = -j, j$ and $i = f[(2j+1)+1, h-1]+1, f[(2j+1)+1, h-1]+f[(2j+1)-1, h-1]$.

A concrete example is drawn in Fig. 2. It contains the pathways leading to the construction of both singlet states of four spin-$\frac{1}{2}$ particles. Another example is the construction of the single three spin-1 particle singlet state drawn in Fig. 3.

Next the singlet states of up to 6 spin-$\frac{1}{2}$ and 4 spin-1 particle are explicitly enumerated in Tables I and II.

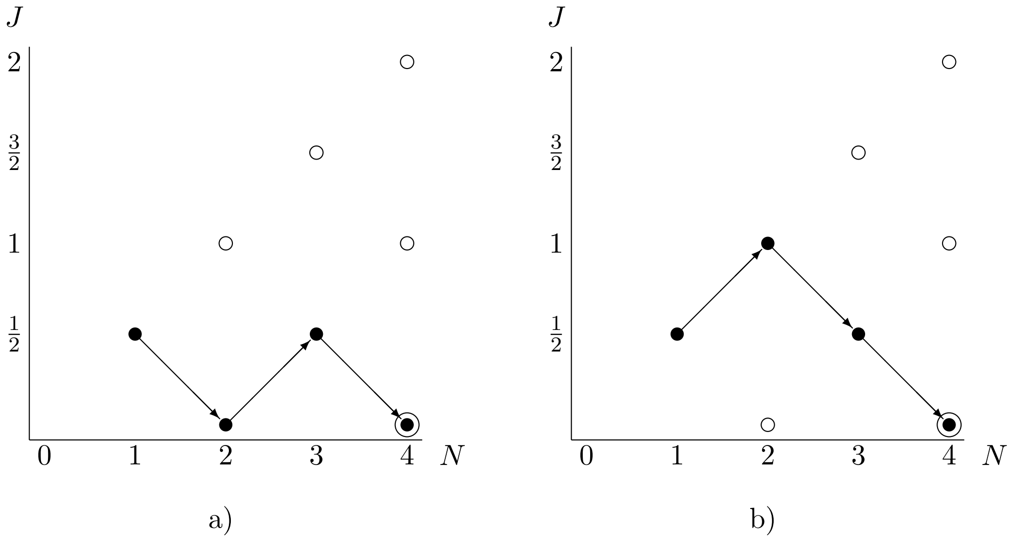

FIG. 2: Construction of both singlet states of four spin-$\frac{1}{2}$ particles. Concentric circles indicate the target states.

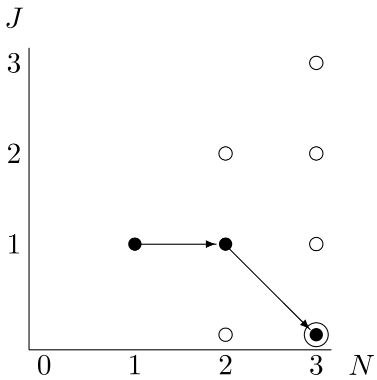

FIG. 3: Construction of the singlet state of three spin-1 particles.

N	#	
2	1	$\frac{1}{\sqrt{2}}\big(\|+,-\rangle - \|-,+\rangle\big);$
4	1	$-\frac{1}{2\sqrt{3}}\big(\|-,+,-,+\rangle + \|-,+,+,-\rangle + \|+,-,-,+\rangle + \|+,-,+,-\rangle\big)+$ $+\frac{1}{\sqrt{3}}\big(\|-,-,+,+\rangle + \|+,+,-,-\rangle\big);$
4	2	$\big(-\frac{1}{\sqrt{2}}\|-,+\rangle + \frac{1}{\sqrt{2}}\|+,-\rangle\big)^2;$
6	1	$-\frac{1}{2}\|-,-,-,+,+,+\rangle + -\frac{1}{6}\big(\|-,+,+,-,-,+\rangle + \|-,+,+,-,+,-\rangle+$ $+\|-,+,+,+,-,-\rangle + \|+,-,+,-,-,+\rangle + \|+,-,+,-,+,-\rangle+$ $+\|+,-,+,+,-,-\rangle + \|+,+,-,-,-,+\rangle + \|+,+,-,-,+,-\rangle+$ $+\|+,+,-,+,-,-\rangle\big) + \frac{1}{6}\big(\|-,-,+,-,+,+\rangle + \|-,-,+,+,-,+\rangle+$ $+\|-,-,+,+,+,-\rangle + \|-,+,-,-,+,+\rangle + \|-,+,-,+,-,+\rangle+$ $+\|-,+,-,+,+,-\rangle + \|+,-,-,-,+,+\rangle + \|+,-,-,+,-,+\rangle+$ $+\|+,-,-,+,+,-\rangle\big) + \frac{1}{2}\|+,+,+,-,-,-\rangle;$
6	2	$-\frac{\sqrt{2}}{3}\|-,-,+,-,+,+\rangle + -\frac{1}{3\sqrt{2}}\big(\|-,+,+,+,-,-\rangle + \|+,-,+,+,-,-\rangle+$ $+\|+,+,-,-,-,+\rangle + \|+,+,-,-,+,-\rangle\big) + -\frac{1}{6\sqrt{2}}\big(\|-,+,-,+,-,+\rangle+$ $+\|-,+,-,+,+,-\rangle + \|+,-,-,+,-,+\rangle + \|+,-,-,+,+,-\rangle\big)+$ $+\frac{1}{6\sqrt{2}}\big(\|-,+,+,-,-,+\rangle + \|-,+,+,-,+,-\rangle + \|+,-,+,-,-,+\rangle+$ $+\|+,-,+,-,+,-\rangle\big) + \frac{1}{3\sqrt{2}}\big(\|-,-,+,+,-,+\rangle + \|-,-,+,+,+,-\rangle+$ $+\|-,+,-,-,+,+\rangle + \|+,-,-,-,+,+\rangle\big) + \frac{\sqrt{2}}{3}\|+,+,-,+,-,-\rangle;$
6	3	$-\frac{1}{\sqrt{6}}\big(\|-,+,-,-,+,+\rangle + \|-,+,+,+,-,-\rangle\big) + -\frac{1}{2\sqrt{6}}\big(\|+,-,-,+,-,+\rangle+$ $+\|+,-,-,+,+,-\rangle + \|+,-,+,-,-,+\rangle + \|+,-,+,-,+,-\rangle\big)+$ $+\frac{1}{2\sqrt{6}}\big(\|-,+,-,+,-,+\rangle + \|-,+,-,+,+,-\rangle + \|-,+,+,-,-,+\rangle+$ $+\|-,+,+,-,+,-\rangle\big) + \frac{1}{\sqrt{6}}\big(\|+,-,-,-,+,+\rangle + \|+,-,+,+,-,-\rangle\big);$
6	4	$-\frac{1}{\sqrt{6}}\big(\|-,-,+,+,-,+\rangle + \|+,+,-,-,-,+\rangle\big) + -\frac{1}{2\sqrt{6}}\big(\|-,+,-,+,+,-\rangle+$ $+\|-,+,+,-,+,-\rangle + \|+,-,-,+,+,-\rangle + \|+,-,+,-,+,-\rangle\big)+$ $+\frac{1}{2\sqrt{6}}\big(\|-,+,-,+,-,+\rangle + \|-,+,+,-,-,+\rangle + \|+,-,-,+,-,+\rangle+$ $+\|+,-,+,-,-,+\rangle\big) + \frac{1}{\sqrt{6}}\big(\|-,-,+,+,+,-\rangle + \|+,+,-,-,+,-\rangle\big);$
6	5	$\big(-\frac{1}{\sqrt{2}}\|-,+\rangle + \frac{1}{\sqrt{2}}\|+,-\rangle\big)^3.$

TABLE I: First singlet states of N spin-$\frac{1}{2}$ particles.

N	#	
2	1	$\frac{1}{\sqrt{3}}\big(-\lvert 0,0\rangle+\lvert -1,1\rangle+\lvert 1,-1\rangle\big);$
3	1	$-\frac{1}{\sqrt{6}}\big(\lvert -1,0,1\rangle+\lvert 0,1,-1\rangle+\lvert 1,-1,0\rangle\big)+$ $+\frac{1}{\sqrt{6}}\big(\lvert -1,1,0\rangle+\lvert 0,-1,1\rangle+\lvert 1,0,-1\rangle\big);$
4	1	$-\frac{1}{2\sqrt{5}}\big(\lvert -1,0,0,1\rangle+\lvert -1,0,1,0\rangle+\lvert 0,-1,0,1\rangle+\lvert 0,-1,1,0\rangle+$ $+\lvert 0,1,-1,0\rangle+\lvert 0,1,0,-1\rangle+\lvert 1,0,-1,0\rangle+\lvert 1,0,0,-1\rangle\big)+$ $+\frac{1}{6\sqrt{5}}\big(\lvert -1,1,-1,1\rangle+\lvert -1,1,1,-1\rangle+\lvert 1,-1,-1,1\rangle+\lvert 1,-1,1,-1\rangle\big)+$ $+\frac{1}{3\sqrt{5}}\big(\lvert -1,1,0,0\rangle+\lvert 0,0,-1,1\rangle+\lvert 0,0,1,-1\rangle+\lvert 1,-1,0,0\rangle\big)+$ $+\frac{2}{3\sqrt{5}}\lvert 0,0,0,0\rangle+\frac{1}{\sqrt{5}}\big(\lvert -1,-1,1,1\rangle+\lvert 1,1,-1,-1\rangle\big);$
4	2	$-\frac{1}{2\sqrt{3}}\big(\lvert -1,0,1,0\rangle+\lvert -1,1,-1,1\rangle+\lvert 0,-1,0,1\rangle+\lvert 0,1,0,-1\rangle+$ $+\lvert 1,-1,1,-1\rangle+\lvert 1,0,-1,0\rangle\big)+\frac{1}{2\sqrt{3}}\big(\lvert -1,0,0,1\rangle+\lvert -1,1,1,-1\rangle+$ $+\lvert 0,-1,1,0\rangle+\lvert 0,1,-1,0\rangle+\lvert 1,-1,-1,1\rangle+\lvert 1,0,0,-1\rangle\big);$
4	3	$\Big(\frac{1}{\sqrt{3}}\big(-\lvert 0,0\rangle+\lvert -1,1\rangle+\lvert 1,-1\rangle\big)\Big)^2;$
5	1	$-\sqrt{\frac{2}{15}}\lvert -1,-1,0,1,1\rangle+-\frac{1}{\sqrt{30}}\big(\lvert -1,0,1,0,0\rangle+\lvert 0,-1,1,0,0\rangle+$ $+\lvert 0,0,-1,0,1\rangle+\lvert 0,0,-1,1,0\rangle+\lvert 0,1,1,-1,-1\rangle+$ $+\lvert 1,0,1,-1,-1\rangle+\lvert 1,1,-1,-1,0\rangle+\lvert 1,1,-1,0,-1\rangle\big)+$ $+-\frac{1}{2\sqrt{30}}\big(\lvert -1,0,1,-1,1\rangle+\lvert -1,0,1,1,-1\rangle+\lvert -1,1,-1,0,1\rangle+$ $+\lvert -1,1,-1,1,0\rangle+\lvert 0,-1,1,-1,1\rangle+\lvert 0,-1,1,1,-1\rangle+$ $+\lvert 0,1,0,-1,0\rangle+\lvert 0,1,0,0,-1\rangle+\lvert 1,-1,-1,0,1\rangle+$ $+\lvert 1,-1,-1,1,0\rangle+\lvert 1,0,0,-1,0\rangle+\lvert 1,0,0,0,-1\rangle\big)+$ $+\frac{1}{2\sqrt{30}}\big(\lvert -1,0,0,0,1\rangle+\lvert -1,0,0,1,0\rangle+\lvert -1,1,1,-1,0\rangle+$ $+\lvert -1,1,1,0,-1\rangle+\lvert 0,-1,0,0,1\rangle+\lvert 0,-1,0,1,0\rangle+$ $+\lvert 0,1,-1,-1,1\rangle+\lvert 0,1,-1,1,-1\rangle+\lvert 1,-1,1,-1,0\rangle+$ $+\lvert 1,-1,1,0,-1\rangle+\lvert 1,0,-1,-1,1\rangle+\lvert 1,0,-1,1,-1\rangle\big)+$ $+\frac{1}{\sqrt{30}}\big(\lvert -1,-1,1,0,1\rangle+\lvert -1,-1,1,1,0\rangle+\lvert -1,0,-1,1,1\rangle+$ $+\lvert 0,-1,-1,1,1\rangle+\lvert 0,0,1,-1,0\rangle+\lvert 0,0,1,0,-1\rangle+$ $+\lvert 0,1,-1,0,0\rangle+\lvert 1,0,-1,0,0\rangle\big)+\sqrt{\frac{2}{15}}\lvert 1,1,0,-1,-1\rangle;$
6	7	$-\frac{1}{\sqrt{15}}\big(\lvert -1,-1,0,1,1,0\rangle+\lvert 1,1,0,-1,-1,0\rangle\big)+-\frac{1}{2\sqrt{15}}\big(\lvert -1,-1,1,0,0,1\rangle+$ $+\lvert -1,-1,1,1,-1,1\rangle+\lvert -1,0,-1,1,0,1\rangle+\lvert -1,0,1,0,1,-1\rangle+$ $+\lvert 0,-1,-1,1,0,1\rangle+\lvert 0,-1,1,0,1,-1\rangle+\lvert 0,0,-1,0,1,0\rangle+$ $+\lvert 0,0,-1,1,1,-1\rangle+\lvert 0,0,1,-1,-1,1\rangle+\lvert 0,0,1,0,-1,0\rangle+$ $+\lvert 0,1,-1,0,-1,1\rangle+\lvert 0,1,1,-1,0,-1\rangle+\lvert 1,0,-1,0,-1,1\rangle+$ $+\lvert 1,0,1,-1,0,-1\rangle+\lvert 1,1,-1,-1,1,-1\rangle+\lvert 1,1,-1,0,0,-1\rangle\big)+$ $+-\frac{1}{4\sqrt{15}}\big(\lvert -1,0,0,0,0,1\rangle+\lvert -1,0,0,1,-1,1\rangle+\lvert -1,0,1,-1,1,0\rangle+$ $+\lvert -1,0,1,1,0,-1\rangle+\lvert -1,1,-1,0,1,0\rangle+\lvert -1,1,-1,1,1,-1\rangle+$ $+\lvert -1,1,1,-1,-1,1\rangle+\lvert -1,1,1,0,-1,0\rangle+\lvert 0,-1,0,0,0,1\rangle+$ $+\lvert 0,-1,0,1,-1,1\rangle+\lvert 0,-1,1,-1,1,0\rangle+\lvert 0,-1,1,1,0,-1\rangle+$ $+\lvert 0,1,-1,-1,0,1\rangle+\lvert 0,1,-1,1,-1,0\rangle+\lvert 0,1,0,-1,1,-1\rangle+$ $+\lvert 0,1,0,0,0,-1\rangle+\lvert 1,-1,-1,0,1,0\rangle+\lvert 1,-1,-1,1,1,-1\rangle+$ $+\lvert 1,-1,1,-1,-1,1\rangle+\lvert 1,-1,1,0,-1,0\rangle+\lvert 1,0,-1,-1,0,1\rangle+$ $+\lvert 1,0,-1,1,-1,0\rangle+\lvert 1,0,0,-1,1,-1\rangle+\lvert 1,0,0,0,0,-1\rangle\big)+$ $+\frac{1}{4\sqrt{15}}\big(\lvert -1,0,0,0,1,0\rangle+\lvert -1,0,0,1,1,-1\rangle+\lvert -1,0,1,-1,0,1\rangle+$ $+\lvert -1,0,1,1,-1,0\rangle+\lvert -1,1,-1,0,0,1\rangle+\lvert -1,1,-1,1,-1,1\rangle+$ $+\lvert -1,1,1,-1,1,-1\rangle+\lvert -1,1,1,0,0,-1\rangle+\lvert 0,-1,0,0,1,0\rangle+$ $+\lvert 0,-1,0,1,1,-1\rangle+\lvert 0,-1,1,-1,0,1\rangle+\lvert 0,-1,1,1,-1,0\rangle+$ $+\lvert 0,1,-1,-1,1,0\rangle+\lvert 0,1,-1,1,0,-1\rangle+\lvert 0,1,0,-1,-1,1\rangle+$ $+\lvert 0,1,0,0,-1,0\rangle+\lvert 1,-1,-1,0,0,1\rangle+\lvert 1,-1,-1,1,-1,1\rangle+$ $+\lvert 1,-1,1,-1,1,-1\rangle+\lvert 1,-1,1,0,0,-1\rangle+\lvert 1,0,-1,-1,1,0\rangle+$ $+\lvert 1,0,-1,1,0,-1\rangle+\lvert 1,0,0,-1,-1,1\rangle+\lvert 1,0,0,0,-1,0\rangle\big)+$

$$
\begin{gathered}
+\tfrac{1}{2\sqrt{15}}\big(|-1,-1,1,0,1,0\rangle+|-1,-1,1,1,1,-1\rangle+|-1,0,-1,1,1,0\rangle+\\
+|-1,0,1,0,-1,1\rangle+|0,-1,-1,1,1,0\rangle+|0,-1,1,0,-1,1\rangle+\\
+|0,0,-1,0,0,1\rangle+|0,0,-1,1,-1,1\rangle+|0,0,1,-1,1,-1\rangle+\\
+|0,0,1,0,0,-1\rangle+|0,1,-1,0,1,-1\rangle+|0,1,1,-1,-1,0\rangle+\\
+|1,0,-1,0,1,-1\rangle+|1,0,1,-1,-1,0\rangle+|1,1,-1,-1,-1,1\rangle+\\
+|1,1,-1,0,-1,0\rangle\big)+\tfrac{1}{\sqrt{15}}\big(|-1,-1,0,1,0,1\rangle+|1,1,0,-1,0,-1\rangle\big).
\end{gathered}
$$

TABLE II: First singlet states of N spin-1 particles.

In summary, we have present a detailed, algorithmic description of how to obtain all singlet states of spin-$\frac{1}{2}$ and spin-1 particles. The method can applied analogously for the construction of N-partite singlet states from particles with higher dimensional spin.