

# Value Indefiniteness Is Almost Everywhere

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(Dated: December 4, 2013)

## Abstract

Kochen-Specker theorems assure the breakdown of certain types of non-contextual hidden variable theorems through the non-existence of global, holistic frame functions; alas they do not allow us to identify where this breakdown occurs, nor the extent of it. Here we show that this breakdown is maximal in that it occurs almost everywhere, and thus prove that quantum indeterminacy—often referred to as contextuality or value indefiniteness—is a global property as is often assumed. In contrast to the Kochen-Specker theorem, we assume only weak constraints on any definite values that may exist, in particular that any potential value assignments that may exist be locally non-contextual. Under this assumption, we prove that once a single arbitrary observable is fixed to be value definite, almost (i.e. with Lebesgue measure one) all remaining observables are indeterminate.

PACS numbers: 03.67.Lx, 05.40.-a, 03.65.Ta, 03.67.Ac, 03.65.Aa

Keywords: Kochen-Specker theorem, quantum value indefiniteness, quantum randomness, logical indeterminacy principle, quantum contextuality

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## I. INTRODUCTION

The Kochen-Specker theorem [1, 2] proves the impossibility of the existence of certain hidden variable theories for quantum mechanics by showing the existence of a finite set of observables  $O$  for which the following two assumptions cannot be simultaneously true:

(P1) every observable in  $O$  has a pre-assigned definite value,

(P2) the outcomes of measurements of observables are non-contextual.

Non-contextuality means that the outcomes of measurements of observables are independent of whatever other co-measurable observables are measured alongside them, along with the requirement that the relationship between hidden variables associated with sets of co-measurable observables behave quasi-classically, as expected from quantum mechanics. This requirement means that in any “complete, maximal” set of mutually co-measurable yes-no propositions (represented by mutually orthogonal projectors spanning the Hilbert space, or, equivalently, by a single “maximal” operator with a spectral decomposition containing these projectors) exactly one proposition should be assigned the value “yes.” Due to complementarity, the observables in  $O$  may not be all simultaneously co-measurable, that is, formally, commuting.

The Kochen-Specker theorem *does not explicitly identify* certain particular observables which violate one or both assumptions (P1) and (P2), but only proves their *existence*. This form of the theorem was amply sufficient for its intended scope, primarily metaphysical. The relation between value indefinite observables, that is, observables which do not have definite values before measurement, and quantum randomness in [1, 2], requires a more precise form of the Kochen-Specker theorem in which some value indefinite observables can be located (identified). A stronger form of the Kochen-Specker theorem providing this information was proved in [3].

In this paper we extend these results to show that indeed all observables must be value indefinite except for those corresponding to the context used to prepare the state. While it may seem intuitive that quantum indeterminism is this widespread, it does not follow from existing no-go theorems, so it is important that a theoretical grounding be given to this intuition. This not only helps provide an information theoretic certification of quantum random bits, but also develops our understanding of the origin of quantum indeterminism.

## II. LOGICAL INDETERMINACY PRINCIPLE

Pitowsky [4] (also in the subsequent paper [5] with Hrushovski) gave a constructive proof of Gleason's lemma in terms of orthogonality graphs which motivated the study of probability distributions on finite sets of rays. In this context he proved a result called "the logic indeterminacy principle" which has a striking similarity with the Kochen-Specker theorem and appears as if it could be used to locate value indefiniteness. However, as we discuss in this section this is not the case.

According to [5], a *frame function* on a set  $O \subset \mathbb{C}^n$  of projection observables (which we identify here with the unit vectors they project on to) in a dimension  $n \geq 3$  Hilbert space is a function  $p : O \rightarrow [0, 1]$  such that:

- (i) If  $\{|x_1\rangle, \dots, |x_n\rangle\}$  is an orthonormal basis,  $\sum_{i=1}^n p(|x_i\rangle) = 1$ , and for  $\{|x_1\rangle, \dots, |x_k\rangle\}$  orthonormal with  $k \leq n$ ,  $\sum_{i=1}^k p(|x_i\rangle) \leq 1$ .
- (ii) For all complex  $\alpha$  with  $|\alpha| = 1$  and all  $x \in O$ ,  $p(|x\rangle) = p(\alpha|x\rangle)$ .

A *Boolean frame function* is a frame function taking only 0, 1 values, i.e. for all  $|x\rangle \in O$ ,  $p(|x\rangle) \in \{0, 1\}$ .

**Theorem II.1** (Logical indeterminacy principle; Pitowsky [4]). *For all projectors  $a, b \in \mathbb{C}^3$  with  $0 < |\langle a|b \rangle| < 1$ , there exists a finite set of observables  $O$  with  $|a\rangle, |b\rangle \in O$  such that there is no Boolean frame function  $p$  on  $O$  unless  $p(|a\rangle) = p(|b\rangle) = 0$ .*

A consequence of this principle is that there is no Boolean frame function  $p$  on  $O$  such that  $p(|a\rangle) = 1$ . From the logical indeterminacy principle we can deduce the Kochen-Specker theorem because a Boolean frame function simply gives a non-contextual, value definite yes-no value assignment, so (P2) is satisfied.

As noted by Hrushovski and Pitowsky [5], the logical indeterminacy principle is stronger than the Kochen-Specker theorem because the result is true for arbitrary frame functions which can take every value in the unit interval  $[0, 1]$ , but which are restricted to Boolean values for  $|a\rangle, |b\rangle$ .

In fact, we may be tempted to use the logical indeterminacy principle to "locate" a value indefinite observable. Indeed, if we fix  $p$  and choose  $|a\rangle \in \mathbb{C}^3$  such that  $p(|a\rangle) = 1$ , then, by the logical indeterminacy principle, for every distinct non-orthogonal unit vector  $|b\rangle \in \mathbb{C}^3$  it is impossible to have  $p(|b\rangle) = 1$  and  $p(|b\rangle) = 0$ , hence one could be inclined to conclude that  $|b\rangle$  is value

indefinite. However, such reasoning would be incorrect because if  $p(|b\rangle)$  were 1, then the logical indeterminacy principle merely concludes that  $p$  *does not exist*; the same conclusion is obtained if  $p(|b\rangle)$  were 0. Hence, in both cases  $p$  *does not exist*, so it makes no sense to talk about its values, in particular, about  $p(|b\rangle)$ . (Pointedly stated from a physical viewpoint,  $p(|a\rangle)$  as well as  $p(|b\rangle)$  could take on any of the four combinations of definite values, provided that (P1) or (P2) is violated for some other observable in  $O$ . Nevertheless, as we shall demonstrate in Section V, using the formalism of [3], all observables in  $O$  except  $|a\rangle$  and those commuting with  $|a\rangle$  are indeed provable value indefinite.) This means that using the logical indeterminacy principle we get the same global information derived in the Kochen-Specker theorem, namely that some observable in  $O$  has to be value indefinite, and no more. The reason for this limitation is the use of frame functions, which by definition must be defined everywhere: they can model “local” value definiteness, but not “local” value indefiniteness, which, as in the Kochen-Specker theorem, “emerges” only as a global phenomenon.

### III. VALUE INDEFINITENESS AND CONTEXTUALITY

To remedy the above deficiency we will use the formalism proposed in [3] for pure quantum states. Specifically, we define value (in)definiteness and contextuality in the framework of quantum logic of Birkhoff and von Neumann [6, 7] and Kochen and Specker [8, 9].

Projection operators—projecting on to the linear subspace spanned by a non-zero vector  $|\psi\rangle$ —will be denoted by  $P_\psi = \frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle}$ .

We fix a positive integer  $n$ . Let  $O \subseteq \{P_\psi \mid |\psi\rangle \in \mathbb{C}^n\}$  be a non-empty set of *projection observables* in the Hilbert space  $\mathbb{C}^n$  and  $\mathcal{C} \subseteq \{\{P_1, P_2, \dots, P_n\} \mid P_i \in O \text{ and } \langle i|j\rangle = 0 \text{ for } i \neq j\}$  a set of measurement contexts over  $O$ . A *context*  $C \in \mathcal{C}$  is thus a maximal set of compatible (i.e. they can be simultaneous measured) projection observables. Let  $v : \{(o, C) \mid o \in O, C \in \mathcal{C} \text{ and } o \in C\} \xrightarrow{o} \{0, 1\}$  be a partial function (i.e.,  $v$  may be undefined for some values in its domain) called an *assignment function*. For some  $o, o' \in O$  and  $C, C' \in \mathcal{C}$  we say  $v(o, C) = v(o', C')$  if  $v(o, C), v(o', C')$  are both defined and have equal values.

Value definiteness corresponds to the notion of predictability in classical determinism: an observable is value definite if  $v$  assigns it a definite value—i.e. is able to predict in advance, independently of measurement, the value obtained via measurement. Here is the formal definition: an observable  $o \in C$  is *value definite* in the context  $C$  under  $v$  if  $v(o, C)$  is defined; otherwise  $o$  is *value*

*indefinite* in  $C$ . If  $o$  is value definite in all contexts  $C \in \mathcal{C}$  for which  $o \in C$  then we simply say that  $o$  is value definite under  $v$ . The set  $O$  is *value definite* under  $v$  if every observable  $o \in O$  is value definite under  $v$ . This fine-grained definition of value definiteness allows us to formalise located value (in)definiteness, which one cannot do with the global notion used in the Kochen-Specker theorem.

Non-contextuality corresponds to the classical notion that the value obtained via measurement is independent of other compatible observables measured alongside it. Formally, an observable  $o \in O$  is *non-contextual* under  $v$  if for all contexts  $C, C' \in \mathcal{C}$  with  $o \in C, C'$  we have  $v(o, C) = v(o, C')$ ; otherwise,  $v$  is *contextual*. The set of observables  $O$  is *non-contextual* under  $v$  if every observable  $o \in O$  which is not value indefinite (i.e. value definite in *some* context) is non-contextual under  $v$ ; otherwise, the set of observables  $O$  is *contextual*. Thus we require that *observables behave non-contextually only where definite values are defined*; this technicality is necessary in order to consider the possibility that some observables are value indefinite.

To be in agreement with quantum mechanics we restrict the assignment functions to admissible ones:  $v$  is *admissible* if the following hold for all  $C \in \mathcal{C}$ : a) if there exists an  $o \in C$  with  $v(o, C) = 1$ , then  $v(o', C) = 0$  for all  $o' \in C \setminus \{o\}$ , b) if there exists an  $o \in C$  such that  $v(o', C) = 0$  for all  $o' \in C \setminus \{o\}$ , then  $v(o, C) = 1$ .

#### IV. STRONG KOCHEN-SPECKER THEOREM

The incompatibility between the assumptions (P1) and (P2) is not *maximal* in the following sense: for any set of contexts over any set of observables, there exists an admissible assignment function under which the set of observables is value definite and at least one observable is non-contextual.

However, *there always exist pairs of observables such that, if one of them is assigned the value 1 by an admissible assignment function under which  $O$  is non-contextual, the other must be value indefinite*. The result is deduced in [3] using the weaker assumption that not all observables are assumed to be value definite: *An observable is deduced to be value definite only when the values of other commuting value-definite observables require it to be so*; this is expressed formally by the admissibility of  $v$ .

**Lemma IV.1** (Strong Kochen-Specker lemma; [3]). *Let  $|a\rangle, |b\rangle \in \mathbb{C}^3$  be unit vectors such that*

$0 < |\langle a|b \rangle| \leq \frac{3}{\sqrt{14}}$ . Then there exists a set of 24 projection observables  $O$  containing  $P_a$  and  $P_b$ , and a set of contexts  $C$  over  $O$ , such that there is no admissible assignment function under which  $O$  is non-contextual and  $P_a, P_b$  have the value 1.

The strong Kochen-Specker theorem can be used to “locate” a provable value indefinite observable which when measured “produces” a quantum random bit whose quality can be precisely evaluated under some physical assumptions:

**Theorem IV.2** (Strong Kochen-Specker theorem; [3]). *Let  $|a\rangle, |b\rangle \in \mathbb{C}^3$  be unit vectors such that  $\sqrt{\frac{5}{14}} \leq |\langle a|b \rangle| \leq \frac{3}{\sqrt{14}}$ . Then there exists a set of 24 projection observables  $O$  containing  $P_a$  and  $P_b$ , and a set of contexts  $C$  over  $O$ , such that there is no admissible assignment function under which  $O$  is non-contextual,  $P_a$  has the value 1 and  $P_b$  is value definite.*

## V. HOW WIDESPREAD IS VALUE INDEFINITENESS?

Assuming an observable  $P_a$  has the value 1 we know we can find an observable  $P_b$  that is value indefinite. In this section we address the following question: which of the remaining observables  $P_b$  can be proven to be value indefinite? We prove here the following answer, showing the extent of value indefiniteness: *only observables which commute with  $P_a$  can be value definite*. Specifically, we will prove the following, more general, theorem which extends the Strong Kochen-Specker theorem to cover the rest of the Bloch sphere.

**Theorem V.1.** *Let  $|a\rangle, |b\rangle \in \mathbb{C}^3$  be non orthogonal or parallel unit vectors, i.e.  $0 < |\langle a|b \rangle| < 1$ . Then there exists a set of projection observables  $O$  containing  $P_a$  and  $P_b$ , and a set of contexts  $C$  over  $O$ , such that there is no admissible assignment function under which  $O$  is non-contextual,  $P_a$  has the value 1 and  $P_b$  is value definite. The set  $O$  is finite and can be effectively constructed.*

While this result is similar in form to the original Kochen-Specker theorem, the subtle differences are critical. As mentioned previously, the Kochen-Specker theorem is unable to locate value definiteness: if  $P_a$  has the value 1, we cannot conclude that  $P_b$  is value indefinite, even if assigning it leads to a contradiction. This is due to the fact that this contradiction implies only that no *global* assignment function can exist; the Kochen-Specker theorem does not show that  $P_b$  could not be value definite, while some other  $P_c$  harbours the (necessary) value indefiniteness.

On the other hand, the sets of observables given in the proofs of the Strong Kochen-Specker theorems presented here are carefully constructed such that any attempt to place the value indef-

initeness on a  $P_c$  necessarily contradicts the admissibility of  $\nu$ . For example, it would require a context containing an observable assigned the value 1, and another observable being value indefinite. This contradicts both the the admissibility of  $\nu$ , and the physical understanding of what it means for that observable to be assigned the value 1—since we know measuring that observable will give the value 1, measuring the other observables *must* give the value 0, and hence the other observables are necessarily value definite. As a result, we are forced to conclude that  $P_b$  itself is value indefinite.

In order to prove Lemma IV.1, in [3] a specific proof for the case  $|\langle a|b \rangle| = \frac{3}{\sqrt{14}}$  was given, followed by a reduction to this proof for the case  $|\langle a|b \rangle| < \frac{3}{\sqrt{14}}$ . Here we prove the theorem holds for all cases by reducing the remaining case of  $|\langle a|b \rangle| > \frac{3}{\sqrt{14}}$  to the existing result. This reduction is more subtle and difficult than the first one.

For the purpose of illustrating the reduction technique, let us state the following lemma from [3], which will also turn out to be important for the reduction we will give here.

**Lemma V.2.** *Given any two unit vectors  $|a\rangle, |b\rangle$  with  $0 < |\langle a|b \rangle| < 1$  and an  $x$  such that  $|\langle a|b \rangle| < |x| < 1$ , there exist a unit vector  $|c\rangle$  with  $\langle a|c \rangle = x$ , a set of observables  $O$  containing  $P_a, P_b, P_c$  and a set of contexts  $C$  over  $O$  such that if  $P_a$  and  $P_b$  have the value 1, then  $P_c$  also has the value 1 under any admissible, non-contextual assignment function on  $O$ .*

Furthermore, if we choose our basis such that  $|a\rangle \equiv (1, 0, 0)$  and  $|b\rangle \equiv (p, q, 0)$ , where  $p = \langle a|b \rangle$  and  $q = \sqrt{1 - p^2}$ , then  $|c\rangle$  has the form  $|c\rangle = (x, y, \pm z)$ , where  $x = \langle a|c \rangle$ ,  $y = \frac{p(1-x^2)}{qx}$  and  $z = \sqrt{1 - x^2 - y^2}$ .

This lemma is illustrated in Fig. 2 and constitutes a simple “forcing” of value definiteness: given  $P_a$  and  $P_b$  both with the value 1, there is a  $P_c$  which is “closer” (i.e. at a smaller angle of our choosing) to  $P_a$  which forces  $P_c$  to also take the value 1.

This reduction, however, requires necessarily that  $|x| > |p|$ , and finding a reduction to “force” in the other direction (i.e. towards larger angles between  $P_a$  and  $P_c$ ) is difficult. Here we give an iterated reduction for this case in the following lemma.

**Lemma V.3.** *Given any two unit vectors  $|a\rangle, |b\rangle$  with  $\frac{3}{\sqrt{14}} < \langle a|b \rangle < 1$ , there exists a unit vector  $|c\rangle$  with  $\langle a|c \rangle \leq \frac{3}{\sqrt{14}}$ , a set of observables  $O$  containing  $P_a, P_b, P_c$  and a set of contexts  $C$  over  $O$  such that if  $P_a$  and  $P_b$  have the value 1, then  $P_c$  also has the value 1 under any admissible, non-contextual assignment function on  $O$ .*

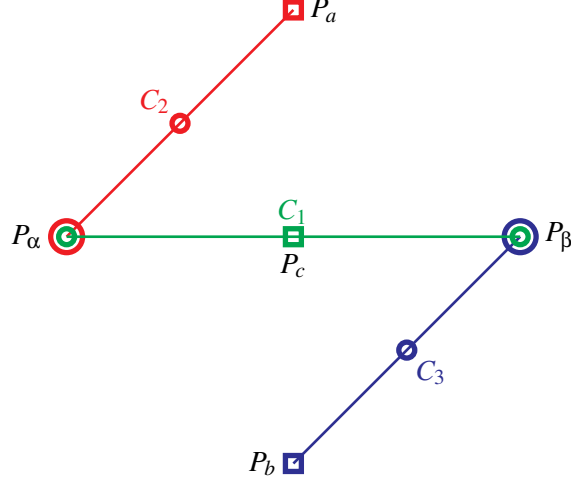


FIG. 1. (Color online) Greechie orthogonality diagram with an overlaid value assignment that illustrates the reduction in Lemma V.2. The circles and squares represent observables that will be given the values 0 and 1 respectively. They are joined by smooth lines which represent contexts.

The proof of this lemma is based on the generalisation of a specific reduction for the case of  $\langle a|b \rangle = \frac{1}{\sqrt{2}}$  to  $\langle a|c \rangle = \frac{1}{\sqrt{3}}$ ; i.e. it is a “forcing” argument in the required direction. The Greechie diagram for this is depicted in Fig. 2. In essence, this figure consists of three copies of the reduction shown in Fig. 1 glued together, ensuring that the Greechie diagram is indeed realisable. Specifically, the important relations are:  $\langle a|v_1 \rangle = \sqrt{\frac{2}{3}}$ ,  $\langle a|v_2 \rangle = \frac{2}{\sqrt{5}}$ ,  $\langle b|c \rangle = \sqrt{\frac{2}{3}}$  and  $\langle b|v_2 \rangle = \sqrt{\frac{2}{5}}$ . The angles between unit vectors in this proof are then scaled, in a way which we will soon make precise, to fit the required  $\langle a|b \rangle$  for the general case. However, since this doesn’t allow us to assert that an arbitrary  $|c \rangle$  must have the value 1 in the same way we could using Lemma V.2, this reduction is then iterated a finite number of times until a sufficiently small  $\langle a|c \rangle$  is obtained.

*Proof of Lemma V.3.* The constants which will be used for scaling, obtained from the reduction shown in Fig. 2, are as follows:

$$\alpha_1 = \frac{\arccos \sqrt{\frac{2}{3}}}{\arccos \frac{1}{\sqrt{2}}}, \quad \alpha_2 = \frac{\arccos \frac{2}{\sqrt{5}}}{\arccos \sqrt{\frac{2}{3}}}, \quad \alpha_3 = \frac{\arccos \sqrt{\frac{2}{3}}}{\arccos \sqrt{\frac{2}{5}}}.$$

Given the initial  $|a \rangle$ ,  $|b \rangle$  and the above constants, we thus make use of the following scaled angles between the relevant observables:

$$\theta_{a,b} = \arccos \langle a|b \rangle, \quad \theta_{a,v_1} = \alpha_1 \theta_{a,b}, \quad \theta_{a,v_2} = \alpha_2 \theta_{a,v_1}.$$



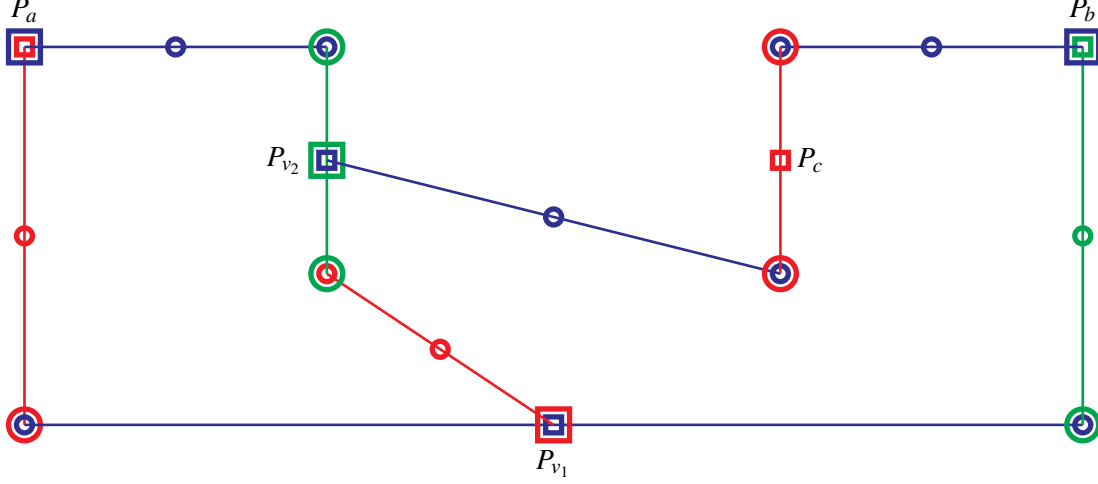


FIG. 2. (Color online) Greechie orthogonality diagram with an overlaid value assignment that illustrates the reduction in Lemma V.3.

Once  $|v_2\rangle$  is determined via the procedure to follow, we take the following:

$$\theta_{b,v_2} = \arccos\langle b|v_2\rangle, \quad \theta_{b,c} = \alpha_3\theta_{b,v_2}.$$

Without loss of generality, let  $|a\rangle = (1,0,0)$  and  $|b\rangle = (p_1, q_1, 0)$  where  $p_1 = \langle a|b\rangle$  and  $q_1 = \sqrt{1 - p_1^2}$ . This fixes our basis for the rest of the reduction. We want to have  $|v_1\rangle$  such that  $\langle a|v_1\rangle = x_1 = \cos\theta_{a,v_1}$ . From Lemma V.2 we know this is possible since  $x_1 > p_1$  (because  $\alpha_1 < 1$ ), and we have  $|v_1\rangle = (x_1, y_1, z_1)$ ,  $y_1 = \frac{p_1(1-x_1^2)}{q_1x_1}$  and  $z_1 = \sqrt{1 - x_1^2 - y_1^2}$ .

We now want  $|v_2\rangle$  such that  $\langle a|v_2\rangle = x_2 = \cos\theta_{a,v_2}$  (this is possible since  $\alpha_2 < 1$ ). In order to use the same general form (specified in Lemma V.2) as above, we perform a change of basis to bring  $|v_1\rangle$  into the  $xy$ -plane, describe  $|v_2\rangle$  in this basis using the above result, then perform the inverse change of basis. Our new basis vectors are given by  $|e_2\rangle = (1,0,0)$ ,

$$|f_2\rangle = (|v_1\rangle - x_1|e_2\rangle)/q_2 = (0, y_1/q_2, z_1/q_2)$$

where  $q_2 = \sqrt{1 - x_1^2}$ , and  $|g_2\rangle = |e_2\rangle \times |f_2\rangle = (0, z_1/q_2, -y_1/q_2)$ . We thus have the transformation matrix

$$T_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & y_1/q_2 & z_1/q_2 \\ 0 & z_1/q_2 & -y_1/q_2 \end{pmatrix}.$$

We can now put  $y_2 = \frac{x_1(1-x_2^2)}{q_2x_2}$  and  $z_2 = \sqrt{1-x_2^2-y_2^2}$  so that in our original basis we have

$$|v_2\rangle = T_1(x_2, y_2, -z_2)^t = (x_2, (y_1y_2 - z_1z_2)/q_2, (y_2z_1 + y_1z_2)/q_2).$$

We note at this point that the constant  $\theta_{b,v_2}$  is now determined, and we have

$$\langle b|v_2\rangle = p_1x_2 + \frac{q_1}{q_2}(y_1y_2 - z_1z_2).$$

For the last iteration of the reduction, we want to find  $|c\rangle$  such that  $\langle b|c\rangle = x_3 = \cos\theta_{b,c}$  (again this will be possible since  $\alpha_3 < 1$ ). Let  $p_3 = \langle b|v_2\rangle$  and  $q_3 = \sqrt{1-p_3^2}$ . Again we perform a basis transformation; we have  $|e_3\rangle = |b\rangle = (p_1, q_1, 0)$ ,

$$\begin{aligned} |f_3\rangle &= (|v_2\rangle - p_3|b\rangle)/k \\ &= (x_2 - p_3p_1, (y_1y_2 - z_1z_2)/q_2 - p_3q_1, (y_2z_1 + y_1z_2)/q_2)/k, \end{aligned}$$

where  $k$  is a constant such that  $|f_3\rangle$  is normalized, and

$$\begin{aligned} |g_3\rangle &= |e_3\rangle \times |f_3\rangle \\ &= \left( \frac{q_1}{q_2}(y_2z_1 + y_1z_2), \frac{-p_1}{q_2}(y_2z_1 + y_1z_2), \frac{p_1}{q_2}(y_1y_2 - z_1z_2) - q_1x_2 \right) / k. \end{aligned}$$

The transformation matrix is then given by

$$T_3 = \begin{pmatrix} p_1 & |f_3\rangle_x & |g_3\rangle_x \\ q_1 & |f_3\rangle_y & |g_3\rangle_y \\ 0 & |f_3\rangle_z & |g_3\rangle_z \end{pmatrix},$$

where the subscript indicates the component of the relative vector. We now put  $y_3 = \frac{p_3(1-x_3^2)}{q_3x_3}$  and  $z_3 = \sqrt{1-x_3^2-y_3^2}$  so that in the original basis we have

$$\begin{aligned} |c\rangle &= T_3(x_3, y_3, -z_3)^t \\ &= \left( x_3p_1 + \frac{y_3}{k}(x_2 - p_1p_3) - \frac{q_1z_3}{kq_2}(y_2z_1 + y_1z_2), \right. \\ &\quad x_3q_1 + \frac{y_3}{kq_2}(y_1y_2 - z_1z_2 - p_3q_1q_2) + \frac{z_3p_1}{kq_2}(y_2z_1 + y_1z_2), \\ &\quad \left. \frac{y_3}{kq_2}(y_2z_1 + y_1z_2) - \frac{z_3}{k} \left[ \frac{p_1}{q_2}(y_1y_2 - z_1z_2) - q_1x_2 \right] \right). \end{aligned}$$

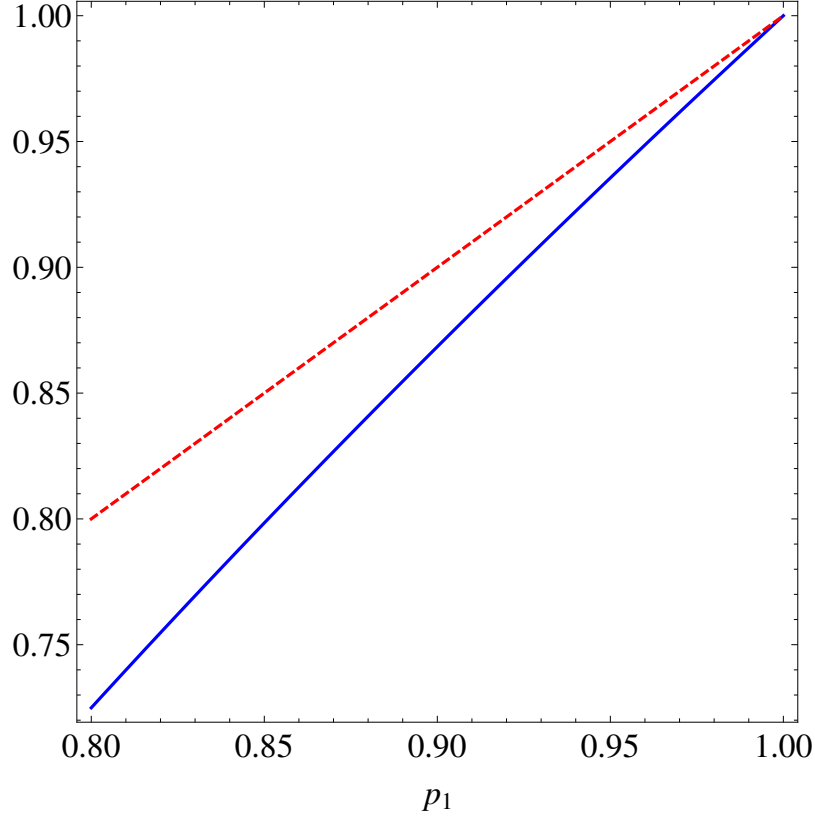


FIG. 3. (Color online) Plot of  $p_1$  (dashed in red) and  $f(p_1)$  (in blue) for  $p_1 \in (0.8, 1) \supset \left(\frac{3}{\sqrt{14}}, 1\right)$ .

Note that only the first term is of importance in the above expression. Specifically, we want to prove that  $\langle a|c \rangle < \langle a|b \rangle = p_1$ , where

$$\langle a|c \rangle = x_3 p_1 + \frac{y_3}{k}(x_2 - p_1 p_3) - \frac{q_1 z_3}{k q_2}(y_2 z_1 + y_1 z_2).$$

The product  $\langle a|c \rangle$  is, with appropriate substitutions, a function of one variable,  $p_1$ ; let us denote  $f(p_1) = \langle a|c \rangle$ . We thus need to determine if, for  $p_1 \in \left(\frac{3}{\sqrt{14}}, 1\right)$  the inequality  $f(p_1) < p_1$  holds.

The expanded form of  $f(p_1)$  is difficult to manipulate analytically, however  $f$  is well behaved and continuous on this domain, and although undefined for  $p_1 = 1$ ,  $\lim_{p_1 \rightarrow 1^-} f(p_1) = 1$ . With these facts we show, using a combination of direct analysis and Mathematica calculation and plots, that the inequality is indeed true.

Using Mathematica to take a Taylor series expansion around  $p_1 = 1$ , we find that for small  $|p_1 - 1|$ ,  $f(p_1) = 1 - m(1 - p_1)$ , where  $m \approx 1.27$  is a constant. Hence  $\lim_{p_1 \rightarrow 1^-} f(p_1) = 1$  as claimed and for some  $\varepsilon > 0$  we have  $f(p_1) < p_1$  for  $p_1 \in (1 - \varepsilon, 1)$ . Further, the continuity of  $f$  on this domain can be guaranteed by noting that  $f(p_1)$  is simply composed of trigonometric functions with arguments from  $(-1, 1) \setminus \{0\}$ ; since these are all continuous, so is  $f$ . From Fig. 3 and the

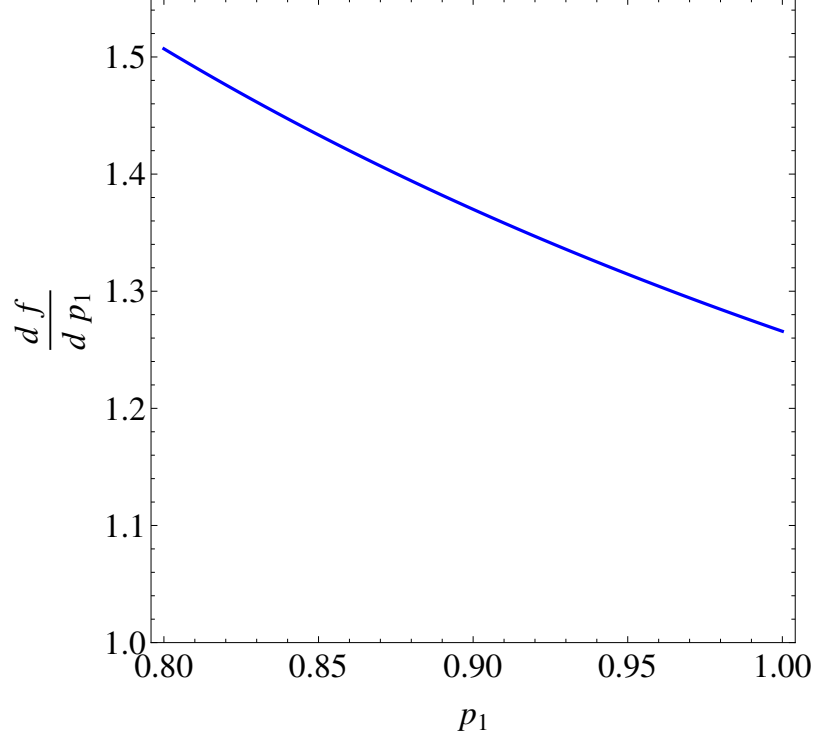


FIG. 4. (Color online) Plot of  $\frac{df}{dp_1}$  for  $p_1 \in (0.8, 1) \supset (\frac{3}{\sqrt{14}}, 1)$ .

above results it follows that to prove the inequality  $f(p_1) < p_1$  for all  $p_1 \in (\frac{3}{\sqrt{14}}, 1)$  we need to show that for no  $p_1 \rightarrow 1$  (which implies  $f(p_1) \rightarrow p_1$ ) we have  $f(p_1) > p_1$ .

Since we know from the Taylor series expansion that  $f(p_1) < p_1$  in the neighbourhood of  $p_1 = 1$ , if for some  $p_2 \in (\frac{3}{\sqrt{14}}, 1)$  we have  $f(p_2) > p_2$ , then  $\frac{df}{dp_2} < 1$ , which is false (see Fig. 4).

From Fig. 3 (and also the fact that the derivative of  $f(p_1) > 1$ ) it also follows that the difference  $p_1 - f(p_1)$  is strictly decreasing with  $p_1$  on  $(\frac{3}{\sqrt{14}}, 1) \subset (0.8, 1)$ . Thus, for large enough (but finite)  $k$ ,  $f^k(p_1) \leq \frac{3}{\sqrt{14}}$ , and the projector  $P_{c_k}$  must be assigned the value 1 by  $v$ .  $\square$

The proof of the main theorem follows rather straightforwardly from Lemma V.3.

*Proof of Theorem V.1.* If  $0 < |\langle a|b \rangle| < \frac{3}{\sqrt{14}}$ , we can appeal simply to Theorem IV.2, so let  $\frac{3}{\sqrt{14}} < |\langle a|b \rangle| < 1$ .

Without loss of generality, we can assume that  $\langle a|b \rangle \in (0, 1)$ , since  $P_b = P_{\alpha b}$  for  $\alpha \in \mathbb{C}$  with  $|\alpha| = 1$ , so the set of projection observables  $O$  obtained under this assumption will give the required result for the general case.

Let us assume, for the sake of contradiction, that such an admissible assignment function  $v$  exists for all  $O, C$ , i.e.  $v(P_a, C_a) = 1$  and  $v(P_b, C_b)$  is defined for all  $C_a, C_b \in C$  with  $P_a \in C_a$  and

$P_b \in C_a$ . (Since  $\nu$  is required to be non-contextual, we will omit the context and write  $\nu(P_a, \cdot)$  for simplicity.) Then, for all such  $C_a, C_b$ , if  $\nu(P_b, \cdot) = 1$ , then by Lemma V.3, there exists a  $|c\rangle$  with  $\langle a|c\rangle \leq \frac{3}{\sqrt{14}}$  such that  $\nu(P_c, \cdot) = 1$ . But this contradicts Theorem IV.2. Hence, if  $P_b$  is to be value definite we must have  $\nu(P_b, \cdot) = 0$ . However, we show that this also leads to a contradiction as follows.

Let  $p = \langle a|b\rangle$  and  $q = \sqrt{1-p^2}$ . We construct an orthonormal basis in which  $|a\rangle \equiv (1, 0, 0)$  and  $|b\rangle \equiv (p, q, 0)$ . Define  $|\alpha\rangle \equiv (0, 1, 0)$ ,  $|\beta\rangle \equiv (0, 0, 1)$  and  $|c\rangle \equiv (q, -p, 0)$ . Then  $\{|a\rangle, |\alpha\rangle, |\beta\rangle\}$  and  $\{|b\rangle, |c\rangle, |\beta\rangle\}$  are orthonormal bases for  $\mathbb{C}^3$ , so we can define the contexts  $C_1 = \{P_a, P_\alpha, P_\beta\}$  and  $C_2 = \{P_b, P_c, P_\beta\}$ . Since  $\nu(P_a, C_1) = 1$ , we must have  $\nu(P_\beta, C_1) = \nu(P_\beta, C_2) = 0$  by the admissibility of  $\nu$ . But since, by assumption,  $\nu(P_b, C_2) = 0$ , we must have  $\nu(P_c, C_2) = 1$ . However, this also contradicts Theorem IV.2, since it is easily seen that

$$0 < \langle a|c\rangle = q = \sqrt{1-p^2} < \sqrt{\frac{5}{14}} < \frac{3}{\sqrt{14}}.$$

Hence, we conclude that  $P_b$  must be value indefinite under  $\nu$ .  $\square$

**Theorem V.4.** *The set of value indefinite observables has Lebesgue measure one in  $\mathbb{C}^3$ .*

*Proof.* The set of value indefinite observables depends on an arbitrarily fixed single vector, say  $|a\rangle \in \mathbb{C}^3$ . Assume that  $P_a$  has a definite value (1 or 0). According to Theorem V.1, no observable outside the union of the linear subspaces (i) spanned by the single vector  $P_a$  (dimension one) and (ii) the plane orthogonal to this vector  $\{P_b \mid \langle P_a|P_b\rangle = 0\}$  (dimension two) is value definite. This set has Lebesgue measure zero in  $\mathbb{C}^3$  because any subset of  $\mathbb{C}^3$  whose dimension is smaller than 3 has Lebesgue measure zero in  $\mathbb{C}^3$ .  $\square$

In terms of unit vectors, the set in the above proof corresponds to the set  $\{(1, 0, 0), (0, 0, 0)\} \cup \{(0, x, y) \mid x^2 + y^2 = 1\}$  on the three dimensional unit sphere, consisting of (i) a single point of dimension zero, and (ii) a great circle of dimension one. Again this set has Lebesgue measure zero on the unit sphere.

## VI. FINAL COMMENTS

One immediate result of the above findings is that, if one insists on the type of non-contextuality formalized by admissible assignments, then value definiteness cannot exist outside of a star-shaped configuration in Greechie-type orthogonality diagrams. It is important to note that this form of

non-contextuality is *weak* in the sense that it is only required to apply locally when a definite value is assigned.

Let us be more specific what is meant by the “star(-shaped)” configuration of a quantum state  $|\psi\rangle$ . We consider a quantum system prepared in a state corresponding to the proposition that “a particular detector  $D_\psi$  clicks among, say, three mutual exclusive detectors” (corresponding to a three dimensional Hilbert space). Such a state can be formalized by a projector  $P_\psi = |\psi\rangle\langle\psi|$ , or, equivalently, by the linear subspace spanned by the normalized vector  $|\psi\rangle$  (together maybe with the other two orthonormal vectors to  $|\psi\rangle$  and to each other). Now, if a quantum state  $|\psi\rangle$  is prepared such that the detector  $D_\psi$  clicks, that corresponds to assigning  $|\psi\rangle$  the value  $v(P_\psi, \cdot) = 1$ .  $|\psi\rangle$ ’s star is formed by taking some or all vectors  $|\phi\rangle$  whose value assignments are consistent with  $v(P_\psi, \cdot) = 1$ . These are value assignments  $v(P_\phi, \cdot) = 0$ , with  $|\phi\rangle$  orthogonal to  $|\psi\rangle$ ; i.e.,  $\langle\phi|\psi\rangle = 0$ . Such potential observables  $|\phi\rangle\langle\phi|$  are thus value definite. As they correspond to vectors orthogonal to  $|\psi\rangle$ , they are, diagrammatically (i.e., in terms of Greechie orthogonality diagrams) speaking, “in  $|\psi\rangle$ ’s star.”

All other conceivable observables corresponding to vectors “outside of  $|\psi\rangle$ ’s star” remain value indefinite relative to our assumptions. The configuration can be represented by the Greechie orthogonality diagram depicted in Fig. 5(a). This finding is consistent with the Heisenberg uncertainty relations and quantum complementarity. Note that this still allows the value definite existence of a continuity of contexts (meaning that all observables therein are value definite) interlinked at  $|\psi\rangle$ , but on a set of Lebesgue measure zero.

One could be inclined to go one step further and conjecture that *there does not exist any value definite observable outside of a single context*. This context is defined by the preparation of the state: it consists of the observable corresponding to  $|\psi\rangle$ , as well as of the two other orthogonal projectors associated with the two idle detectors that do not click if  $D_\psi$  clicks. The configuration can be represented by the Greechie orthogonality diagram depicted in Fig. 5(b). This conjecture is strictly metaphysical with respect to quantum mechanics, because with our assumptions it seems that one cannot prove the sole existence of just one, unique context among the continuum of context forming “ $|\psi\rangle$ ’s star.”

Let us mention that one of the authors is inclined to believe in such an existence, another one is inclined to not believe therein, and the third author has no inclination towards either speculation.

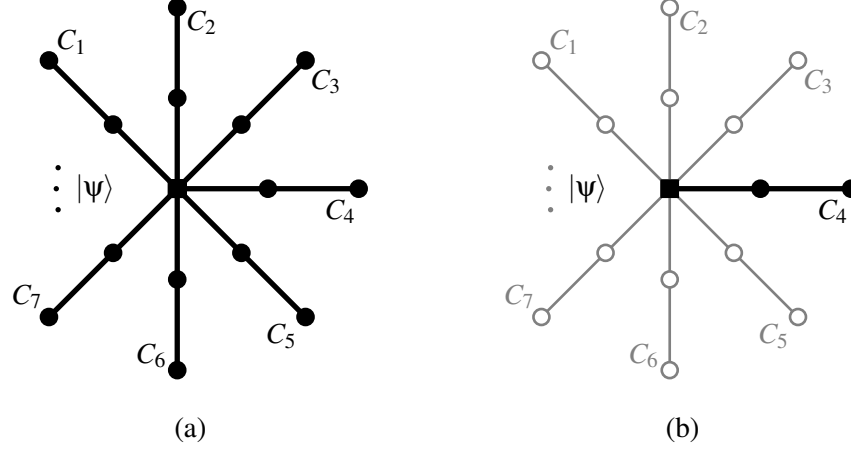


FIG. 5. (Color online) Greechie orthogonality diagram of a star-shaped configuration, representing a common detector observable  $|\psi\rangle\langle\psi|$  with an overlaid two-valued assignment reflecting  $v(P_\psi, \cdot) = 1$ . (a) all “branches” corresponding to contexts are assumed to be equally value definite; (b) it is assumed that, since the system is prepared in, say, context  $C_4$ , depicted by a block colored in thick filled black, only this context is value definite; all the other (continuity of) contexts are “phantom contexts” colored in gray.

## ACKNOWLEDGEMENT

This work was supported in part by Marie Curie FP7-PEOPLE-2010-IRSES Grant RANPHYS. Part of the work was done while Abbott and Calude were visiting the Institute for Theoretical Physics, Vienna University of Technology in September 2013. Calude was also supported by a University of Auckland Grant-in-Aid 2013. Svozil acknowledges Jeffrey Bub, William Demopoulos, and Christopher Fuchs pointing out similarities to Pitowsky’s indeterminacy principle.

## Appendix: Detailed Analysis of $f(p_1)$

The proof of Lemma V.3 relies critically on the analysis of the function  $f(p_1) = \langle a|c \rangle$  for  $p_1 \in \left(\frac{3}{\sqrt{14}}, 1\right)$ . Here we give further details of this analysis, which was carried out using Wolfram Mathematica 9.0.1.0.

Specifically, we have

$$f(p_1) = \langle a|c \rangle = x_3 p_1 + \frac{y_3}{k}(x_2 - p_1 p_3) - \frac{q_1 z_3}{k q_2}(y_2 z_1 + y_1 z_2),$$

where the constants are defined in terms of  $p_1$  as follows:

$$\alpha_1 = \frac{\arccos \sqrt{\frac{2}{3}}}{\arccos \frac{1}{\sqrt{2}}}, \quad \alpha_2 = \frac{\arccos \frac{2}{\sqrt{5}}}{\arccos \sqrt{\frac{2}{3}}}, \quad \alpha_3 = \frac{\arccos \sqrt{\frac{2}{3}}}{\arccos \sqrt{\frac{2}{5}}},$$

$$\theta_{a,b} = \arccos p_1, \quad \theta_{a,v_1} = \alpha_1 \theta_{a,b}, \quad \theta_{a,v_2} = \alpha_2 \theta_{a,v_1},$$

$$q_1 = \sqrt{1 - p_1^2}, \quad x_1 = \cos \theta_{a,v_1}, \quad y_1 = \frac{p_1(1 - x_1^2)}{q_1 x_1}, \quad z_1 = \sqrt{1 - x_1^2 - y_1^2},$$

$$q_2 = \sqrt{1 - x_1^2}, \quad x_2 = \cos \theta_{a,v_2}, \quad y_2 = \frac{x_1(1 - x_2^2)}{q_2 x_2}, \quad z_2 = \sqrt{1 - x_2^2 - y_2^2},$$

$$p_3 = p_1 x_2 + q_1 \frac{y_1 y_2 - z_1 z_2}{q_2}, \quad \theta_{b,v_2} = \arccos p_3, \quad \theta_{b,c} = \alpha_3 \theta_{b,v_2},$$

$$q_3 = \sqrt{1 - p_3^2}, \quad x_3 = \cos \theta_{b,c}, \quad y_3 = p_3 \frac{(1 - x_3^2)}{q_3 x_3}, \quad z_3 = \sqrt{1 - x_3^2 - y_3^2},$$

$$k = \sqrt{\left(\frac{q_1}{q_2}(y_2 z_1 + y_1 z_2)\right)^2 + \left(\frac{-p_1}{q_2}(y_2 z_1 + y_1 z_2)\right)^2 + \left(\frac{p_1}{q_2}(y_1 y_2 - z_1 z_2) - q_1 x_2\right)^2}.$$

Explicitly writing  $f(p_1)$  by completing the substitutions would take several pages, but using Mathematica, we find that the Taylor series expansion around  $p_1 = 1$  is

$$\begin{aligned} f(p_1) = & 1 + \frac{(p_1 - 1)}{\pi^2 \arccos^2 \sqrt{\frac{2}{5}}} \left( \pi^2 \left( \arccos^2 \sqrt{\frac{2}{5}} + \operatorname{arcosh}^2 \sqrt{\frac{2}{3}} \right) \right. \\ & + 8 \arccos \frac{2}{\sqrt{5}} \left( \arccos \frac{2}{\sqrt{5}} \left( 2 \arccos^2 \sqrt{\frac{2}{3}} \right. \right. \\ & + \sqrt{\left( \pi^2 + 16 \operatorname{arcosh}^2 \sqrt{\frac{2}{3}} \right) \left( \arccos^2 \sqrt{\frac{2}{5}} + \operatorname{arcosh}^2 \sqrt{\frac{2}{3}} \right)} \Bigg) + 4 \arccos \sqrt{\frac{2}{3}} \\ & \times \sqrt{\left( \arccos^2 \sqrt{\frac{2}{5}} + \operatorname{arcosh}^2 \sqrt{\frac{2}{3}} \right) \left( \arccos^2 \sqrt{\frac{2}{3}} + \operatorname{arcosh}^2 \frac{2}{\sqrt{5}} \right)} \Bigg) \\ & \left. + O((p_1 - 1)^2), \right. \end{aligned}$$

which numerically simplifies to

$$f(p_1) = 1 - 1.2658(1 - p_1) + O((p_1 - 1)^2).$$



We calculate  $\frac{df}{dp_1}$  from the full expression using Mathematica, but do not give here the resulting function as it is longer yet than the expression for  $f(p_1)$ .

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