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1 Extensions of Hardy-type true-implies-false gadgets to classically obtain indistinguishability

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In quantum logical terms, Hardy-type arguments can be uniformly presented and extended as collections of intertwined contexts and their observables. If interpreted classically, those structures serve as graph-theoretic "gadgets" that enforce correlations on the respective preselected and postselected observable terminal points. The method allows the generalization and extension to other types of relational properties, in particular, to novel joint properties predicting classical equality of quantum mechanically distinct observables. It also facilitates finding faithful orthogonal representations of quantum observables.

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I. CERTIFICATION OF NONCLASSICALITY

When it comes to certifying nonclassical observance of quantized systems, at least three types of approaches have been suggested: (i) Bell-type theorems, related to Boole's "conditions of possible experience" for observables in dis-
2 joint [1–3] or intertwined [4–7] contexts (also known as the maximal collection of compatible observables organized in a Boolean subalgebra), present empirical evidence involving statistical terms which (due to complementarity) cannot be obtained simultaneously, but are obtained from sequential "one term at a time" measurements: whatever the sampling, the events contributing to each term need to be temporally (mutually) apart from the other terms. (ii) Kochen-Specker-type theorems [8–12] are theoretical proofs by contradiction employing finite sets of intertwined quantum observables [13] which have no classical interpretation in terms of two-valued (truth) assignments. Indirect empirical corroborations of Kochen-Specker-type theorems (and, thereby, quantum contextuality) amount to violations of local Bell-type classical predictions [6]. (iii) A third, statistical method [7,14] is based on preselected (prepared) quantum states which are sequentially measured or postselected in terms of suitably chosen quantum observables: prepare one state, measure another. Thereby, pre- and postselection (and their respective observables) are imagined to be logically connected by suitable finite collections of hypothetical counterfactual [15] intertwined contexts, with their choice being motivated by their predictive capacities and yet remaining arbitrary [16]. In particular, particles are prepared in such states and observable properties and are logically connected to other observables in certain ways, such that their (non)occurrence is classically mandatory, but quantum mechanically unrestricted, and occasionally violates the respective classical predictions. By choosing different patterns of connection, this method could be strengthened to the point that any singular outcome contradicts the respective classical predictions. With this (counterfactual) adaptive modification, any such stochastic argument turns definite [16].

Common to all these cases is their reliance on complementary counterfactuals [15] because they suppose the simultaneous existence of more than one context: Except for common observables of intertwining [13] contexts, all observables in any such context are mutually complementary to all observables in a different context, and there is no physically feasible way of simultaneously measuring them.

For the sake of obtaining discrepancies between classical and quantum predictions, these conglomerates of contexts, and thereby the quantum observables they consist of, are interpreted "as if" they could have a classical interpretation. That is, a classical interpretation is forced upon such collections of (intertwining) observables. In quantum logic, a classical interpretation amounts to a two-valued (also known as $0-1$, false-true) state (frame function [13]), which is context independent; that is, its value on observables does not depend on the particular context (also known as maximal observable 3
[17], orthonormal basis) in which it occurs.

Kochen-Specker-type theorems employ configurations of intertwining contexts for which these classical interpretations (in terms of two-valued measures interpreted as dichotomic observables) fail: There does not exist any consistent classical interpretation for these conglomerates of contexts and the observables they hold.

Type-(iii) configurations exhibiting finite collections of observables (in intertwining contexts) may still allow classical interpretations, but the predictions based upon them statistically directly contradict the quantum predictions and without the need of type-(i) inequalities [18]. Indeed, Kochen and Specker used the latter [19] and constructed the former, stronger result [8] without explicitly mentioning the empirical opportunities of such configurations. Appendix B in Ref. [20], and Ref. [21], provide early discussions [7] of type-(iii) nonclassicality.

In what follows, we shall extend a particular type-(iii) instance involving two two-state particles, proposed by Hardy [18,22], to configurations that contain distinct quantum observables that cannot be "resolved" or distinguished

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classically. En route, we shall study configurations [14] of contexts enforcing classical predictions which are the "inverse" of the original relational properties resulting from Hardy-type arguments. The latter are often synonymously referred to as "Hardy's theorem" [10,23], "Hardy's proof" [14,24–26], or "Hardy's paradox" [27,28]. "Hardy's wonderful trick" [29], also called "Hardy's beautiful example" [30], and attempts to make it accessible to a wider audience abound [31,32]. Nevertheless, a detailed account in terms of the quantum logical structure of the counterfactual argument results in a better comprehension of the resources and assumptions involved; in particular, when it comes to related proposals. Such an account suggests extensions to other type-(iii) configurations with different relational properties, and also allows empirical predictions by yielding a systematic way of finding quantum realizations, in particular, in regard to desiderata such as (in)distinguishability of the associated quantized entities.

II. HYPERGRAPH NOMENCLATURE

The counterfactual arguments (ii) and (iii) mentioned earlier can be depicted structurally transparent by the use of hypergraphs [7,14,33,34] introduced by Greechie, drawing contexts as smooth lines. [In what follows, we shall use the following terms synonymously, thereby having in mind the different areas in which they occur: context, maximal observable, orthonormal basis, block, (Boolean) subalgebra, (maximal) clique, and complete graph.] In particular, Greechie has suggested to (amendments are indicated by square brackets) [35, p. 120] "[...] present [...] lattices as unions of [contexts] intertwined or pasted together in some fashion [...] by replacing, for example, the 2^n elements in the Hasse diagram of the power set of an n-element set with
4 the context complete graph [K_n] on n elements. The reduction in numbers of elements is considerable but the number of remaining 'links' or 'lines' is still too cumbersome for
5 our purposes. We replace the context complete graph on n elements by a single smooth curve (usually a straight line) containing n distinguished points. Thus we replace $n(n+1)/2$ 'links' with a single smooth curve. This representation is propitious and uncomplicated provided that the intersection of any pair of blocks contains at most one atom."

In what follows, we shall refer to such a general representation of observables as the (orthogonality) hypergraph [33]. The term should be understood in the broadest possible consistent sense. Most of our arguments will be in four-dimensional state space. An exception will be our mentioning
6 the gadget[1] [36–38], called "Käfer" (German for "bug") by Specker, which was introduced in 1965 [19] and used in the Kochen-Specker proof [8], serving as a true-implies-false configuration. The Specker bug is the three-dimensional analog of the four-dimensional Hardy configuration [14]. Its hypergraph is depicted in Fig. 1(b).

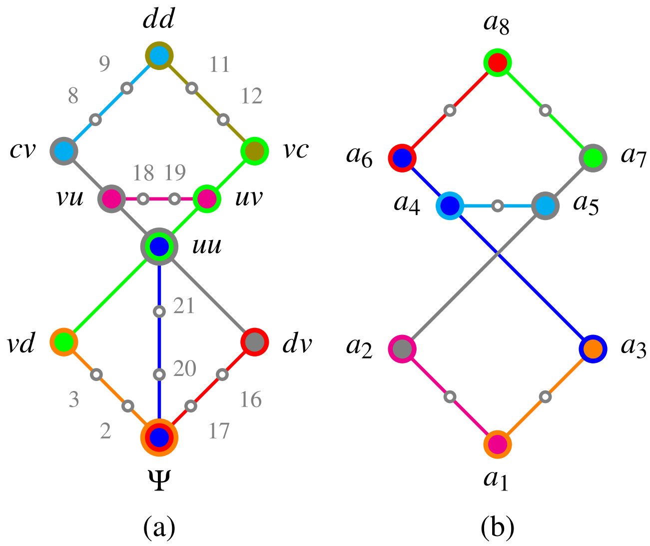

FIG. 1. Orthogonality hypergraphs of (a) the Hardy gadget with 8 contexts and 21 atoms $\{\{dd, 8, 9, cv\}, \{dd, 11, 12, vc\}, \{cv, vu, uu, dv\}, \{vc, uv, uu, vd\}, \{vu, 18, 19, uv\}, \{vd, 2, 3, \Psi\}, \{uu, 20, 21, \Psi\}, \{dv, 16, 17, \Psi\}\}$; (b) rendition of the true-implies-false Specker bug gadget with 7 contexts and 13 atoms $\{\{a_8, \dots, a_6\},$ 7
$\{a_8, \dots, a_7\}, \{a_6, a_4, a_3\}, \{a_7, a_5, a_2\}, \{a_4, \dots, a_5\}, \{a_2, \dots, a_1\}, \{a_3, \dots, a_1\}\}$. Small circles indicate "auxiliary" observables which can be chosen freely, subject to orthogonality constraints: all smooth lines indicate respective contexts representing orthonormal bases. Larger circles indicate observables common to two or more intertwining contexts.

We shall concentrate on orthogonality hypergraphs which are pasting [39] constructions [40] of a homogeneous single type of contexts K_n, where the (maximal) clique number n is fixed. (Note that other authors use related definitions for Greechie diagrams [41] and McKay-Megill-Pavicic (MMP) diagrams [42].) If interpreted as representing some configuration of mutually orthogonal vectors, the (maximal) clique number n equals the dimension of the Hilbert space.

III. FAITHFUL ORTHOGONAL REPRESENTATIONS OF (HYPER)GRAPHS

In what follows, ket vectors, which are usually represented by column vectors, will be represented by the respective transposed row vectors. In all our examples, hypergraphs have a faithful orthogonal representation [43–46] in terms of vectors which are mutually orthogonal within (maximal) cliques or contexts. The phrases faithful orthogonal representation, coordinatization [47], or vector encoding will be used synonymously.

When drawing hypergraphs, some of the atomic propositions will be omitted (or only drawn lightly) if they are not essential to the argument. In particular, in three and four dimensions, given two orthogonal (in general, noncollinear) vectors, it is always possible to "complete" this partially defined context by a Gram-Schmidt process [34,48]. Indeed, given two (orthogonal) noncollinear vectors, in three dimensions the span of the "missing" vector is uniquely determined by the span of the cross product of those two vectors. (A

[1] A clarification with regards to the use of the technical term "gadget" seems in order: this denomination is frequently used in graph theory to indicate "useful subgraphs." It is not meant to be polemic.

generalized cross product of $n-1$ vectors in n-dimensional space can be written as a determinant; that is, in the form of a
8 Levi-Civita symbol.) This "lack of freedom" is in one dimension; in particular, whenever the missing vector is collinear to some vector occurring in the faithful orthogonal representation of the incomplete hypergraph that one is attempting to complete. The most elementary such counterexample is a triangular hypergraph with three cyclically connected contexts $\{\{1, 2, 3\}, \{3, 4, 5\}, \{5, 6, 1\}\}$. Consider any incomplete faithful orthogonal of its intertwining atoms such as $1 = (0, 0, 1)$, $3 = (0, 1, 0)$, $5 = (1, 0, 0)$: any conceivable completion fails because the missing vectors would result in duplicities in the faithful orthogonal representation, that is, in $2 = 5, 4 = 1$, and $6 = 3$.

Nevertheless, in four dimensions, given at least two (orthogonal) noncollinear vectors, the two-dimensional orthogonal subspace is spanned by a continuity of (e.g., rotated) bases. Therefore, in such a case, there is always "enough room for breathing"; that is, for accommodating the basis vectors and thereby transforming them if necessary such that any hypergraph can be properly completed without duplicities. I encourage the reader to try to find a faithful orthogonal representation of the cyclic triangular shaped hypergraph $\{\{1, \dots, 4\}, \{4, \dots, 7\}, \{7, \dots, 1\}\}$ in four dimensions.

Whether or not such faithful orthogonal representations can be given in terms of decomposable or indecomposable vectors associated with factorizable or entangled states is an entirely different issue. In four dimensions, a careful analysis [49] yields a no-go theorem for four-dimensional coordinatizations of the triangle hypergraph by allowing a maximal number seven of nine decomposable vectors.

In general and for arbitrary dimensions, as long as there are two or more "free" (without any strings and intertwining contexts attached) vectors per context missing from a faithful orthogonal representation of a hypergraph, its completion is always possible. Stated differently, any faithful orthogonal representation of an incomplete hypergraph can be straightforwardly extended (without reshuffling of vector components) to a faithful orthogonal representation in a completed hypergraph (e.g., by a Gram-Schmidt process) if at least two or more nonintertwining vectors per context in that hypergraph are missing. Indeed, one may even drop an already existing vector "blocking" a faithful orthogonal representation of an entire (hyper)graph if the associated atom is not intertwining in two or more contexts, and if the new freedom facilitates continuous bases instead of a single vector whose addition may result in duplicities through collinear vectors (we shall mention such an instance later).

In the case of two or more "missing" vectors, any completion involves a two- or higher-dimensional subspace. Any such subspace $\mathbb{R}^{k\geqslant n-2}$ or $\mathbb{C}^{k\geqslant n-2}$ of the n-dimensional continua $\mathbb{R}^n$ or $\mathbb{C}^n$ is spanned by a continuity of (orthogonal) bases. A typical example is an incomplete faithful orthogonal representation of a basis of $\mathbb{R}^4$ rotated into a form $\{(1, 0, 0, 0), (0, 1, 0, 0)\}$. Its completion is then given by the continuity of bases $\{(0, 0, \cos\theta, \sin\theta), (0, 0, -\sin\theta, \cos\theta)\}$, with $0 \leqslant \theta < \pi$.

A completion should even be possible if one merely allows sets of bases which are denumerable—or even finitely but "sufficiently" many bases with respect to the hypergraph encoded. From this viewpoint, four dimensions offer a much wider variety of completions as compared to the three-dimensional case—indeed, the difference results from the abundance offered by a continuum of bases versus a single vector spanning the respective subspaces, a fact which is very convenient for all kinds of constructions. However, as has been mentioned earlier, the completion of coordinatizations of hypergraphs by (in)decomposable vectors—in particular, if one desires to maintain (non)decomposability—is an altogether different issue [49].

This possibility to complete incomplete contexts is also the reason why practically all papers introducing and reviewing Hardy's configuration operate not with the complete eight contexts including 21 atomic vertices, but merely with the nine vectors or vertices in which those eight contexts intertwine. 9 Nevertheless, for tasks such as determining whether a particular configuration of observables supports or does not allow a classical two-valued state, as well as for determining the set of two-valued states and their properties (e.g., separable, unital), the nonintertwining atomic propositions matter.

IV. QUANTUM LOGICAL FORMULATION OF HARDY's ARGUMENT

For the sake of being able to delineate Hardy's rather involved original derivation [22], let us stick to his nomenclature as much as possible. We shall, however, drop the particle index as it is redundant. So, for instance, Hardy's $|+\rangle_1|+\rangle_2$ will be written as $|+\rangle|+\rangle = |++\rangle$. Later, we shall be very explicit and identify the respective entities in terms of Hardy's Ansatz, but let us study Hardy's schematics in some generality first:

(i) It begins with a specific entangled state of two two-state particles $|\Psi\rangle$.

(ii) Then the argument suggests measuring two dichotomic (i.e., two-valued) observables $\hat{U}$ (exclusive) or $\hat{D}$ on each one of the two particles. This results in four measurement configurations $\hat{U}\otimes\hat{U}$, $\hat{U}\otimes\hat{D}\hat{D}\otimes\hat{U}\hat{D}\otimes\hat{D}$—that is, effectively, the two-particle observable $\hat{U}\otimes\hat{D}$ is measured "in Einstein-Podolsky-Rosen (EPR) terms of" $\hat{U}\otimes\mathbb{I}_2$ and $\mathbb{I}_2\otimes\hat{D}$.

(iii) As both of these dichotomic observables $\hat{U}$ and $\hat{D}$ have two possible outcomes called u and v for $\hat{U}$ and c and d for $\hat{D}$, respectively, there are $2^2 \times 2^2 = 2^4 = 16$ different outcomes that are denoted by the ordered pairs uu, uv, uc, ud, vu, vv, vc, vd, cu, cv, cc, cd, du, dv, dc, and dd.

(iv) From these 16 outcomes, 5 groups of (incomplete if not all atoms or vertices are specified—yet, as discussed earlier, a completion is straightforward if desired) contexts, which consist of simultaneously measurable and mutually exclusive observables, can be formed, namely, $\{dd, \dots, cv\}$, $\{dd, \dots, vc\}$, $\{cv, vu, uu, dv\}$, $\{vc, uv, uu, vd\}$, and $\{vu, \dots, uv\}$.

(v) Finally, this collection of five contexts are "bundled with" or "tied to" the (projection) observable corresponding to the original entangled state $|\Psi\rangle$ introduced in (i) by the three (incomplete) contexts $\{vd, \dots, \Psi\}$, $\{uu, \dots, \Psi\}$, and $\{dv, \dots, \Psi\}$.

As a result, these (incomplete) contexts, if pasted [39] together at their respective intertwining observables, result in

TABLE I. Partition logic representing classical probabilities of the Hardy configuration [22], whose intertwined contexts are enumerated in Eq. (1), obtained from the separating or distinguishing set of all 186 two-valued states it supports. Note that the intersection of $\Psi \cap dd = \{1, 2, 3, 4, 5, 6\} \cap \{11, 16, 21, 26, 55, 60, 73, 78, 83, 88, 117, 122, 135, 140, 145, 150, 155, 164, 173, 182\} = \emptyset$ is empty, yielding true-implies-false relations among Ψ and dd, and vice versa, respectively. 10

$\Psi = \{1, 2, 3, 4, 5, 6\}$,
$2 = \{$7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29,30,31,32,33,34,35,36,37,38,39, 40,41,42,43,44,45,46,47,48,49,50,51,52,53,54,55,56,57,58,59,60,61,62,63,64,65,66,67,68$\}$,
$3 = \{$69,70,71,72,73,74,75,76,77,78,79,80,81,82,83,84,85,86,87,88,89,90,91,92,93,94,95,96,97,98,99,100, 101,102,103,104,105,106, 107,108,109,110,111,112,113,114,115,116,117,118,119,120,121,122,123,124,125,126,127,128,129,130$\}$,
$vd = \{$131,132,133,134,135,136,137,138,139,140,141,142,143,144,145,146,147,148,149,150,151,152,153,154,155,156,157,158,159,160, 161,162,163,164,165,166,167,168,169,170,171,172,173,174,175,176,177,178,179,180,181,182,183,184,185,186$\}$,
$uu = \{$7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,69,70,71,72,73,74,75,76,77,78,79,80,81,82,83,84,85,86,87,88$\}$,
$vu = \{$1,2,35,36,39,40,43,44,47,48,97,98,101,102,105,106,109,110,151,152, 153,154,155,160,161,162,163,164,169,170,171,172,173,178,179,180,181,182$\}$,
$cv = \{$3,4,5,6,37,38,41,42,45,46,49,50,61,62,63,64,65,66,67,68,99,100,103,104,107,108,111,112,123,124,125,126, 127,128,129,130,156,157,158,159,165,166,167,168,174,175,176,177,183,184,185,186$\}$,
$8 = \{$1,7,8,12,13,17,18,22,23,27,29,31,33,35,39,43,47,51,52,56,57,69,70,74,75,79,80,84,85,89,91,93,95,97,101,105,109,113, 114,118,119,131,132,136,137,141,142,146,147,151,152,160,161,169,170,178,179$\}$,
$9 = \{$2,9,10,14,15,19,20,24,25,28,30,32,34,36,40,44,48,53,54,58,59,71,72,76,77,81,82,86,87,90,92,94,96,98,102,106,110,115, 116,120,121,133,134,138,139,143,144,148,149,153,154,162,163,171,172,180,181$\}$,
$dd = \{11, 16, 21, 26, 55, 60, 73, 78, 83, 88, 117, 122, 135, 140, 145, 150, 155, 164, 173, 182\}$,
$11 = \{$5,7,9,12,14,17,19,22,24,51,53,56,58,61,63,65,67,69,71,74,76,79,81,84,86,113,115,118,120,123,125,127,129,131,133,136,138, 141,143,146,148,151,153,156,158,160,162,165,167,169,171,174,176,178,180,183,185$\}$,
$12 = \{$6,8,10,13,15,18,20,23,25,52,54,57,59,62,64,66,68,70,72,75,77,80,82,85,87,114,116,119,121,124,126,128,130,132,134,137,139, 142,144,147,149,152,154,157,159,161,163,166,168,170,172,175,177,179,181,184,186$\}$,
$vc = \{$1,2,3,4,27,28,29,30,31,32,33,34,35,36,37,38,39,40,41,42,43,44,45,46,47,48,49,50,89,90,91,92,93,94,95,96,97, 98,99,100,101,102,103,104,105,106,107,108,109,110,111,112$\}$,
$uv = \{$5,6,51,52,53,54,55,56,57,58,59,60,61,62,63,64,65,66,67,68,113,114,115,116, 117,118,119,120,121,122,123,124,125,126,127,128,129,130$\}$,
$dv = \{$27,28,29,30,31,32,33,34,51,52,53,54,55,56,57,58,59,60,89,90,91,92,93,94,95,96,113,114,115,116,117,118,119,120,121,122, 131,132,133,134,135,136,137,138,139,140,141,142,143,144,145,146,147,148,149,150$\}$,
$16 = \{$7,8,9,10,11,12,13,14,15,16,35,36,37,38,39,40,41,42,61,62,63,64,69,70,71,72,73,74,75,76,77,78,97,98,99,100,101, 102,103,104,123,124,125,126,151,152,153,154,155,156,157,158,159,160,161,162,163,164,165,166,167,168$\}$,
$17 = \{$17,18,19,20,21,22,23,24,25,26,43,44,45,46,47,48,49,50,65,66,67,68,79,80,81,82,83,84,85,86,87,88,105,106,107,108,109, 110,111,112,127,128,129,130,169,170,171,172,173,174,175,176,177,178,179,180,181,182,183,184,185,186$\}$,
$18 = \{$3,7,8,9,10,11,17,18,19,20,21,27,28,31,32,37,41,45,49,69,70,71,72,73,79,80,81,82,83,89,90,93,94,99,103,107,111, 131,132,133,134,135,141,142,143,144,145,156,157,165,166,174,175,183,184$\}$

a collection of eight (incomplete) contexts,

$$\begin{aligned}
\{\{dd, \ldots, cv\} &= \{dd, 8, 9, cv\},\\
\{dd, \ldots, vc\} &= \{dd, 11, 12, vc\},\\
\{cv, vu, uu, dv\} &= \{cv, vu, uu, dv\},\\
\{vc, uv, uu, vd\} &= \{vc, uv, uu, vd\},\\
\{vu, \ldots, uv\} &= \{vu, 18, 19, uv\},\\
\{vd, \ldots, \Psi\} &= \{vd, 2, 3, \Psi\},\\
\{uu, \ldots, \Psi\} &= \{uu, 20, 21, \Psi\},\\
\{dv, \ldots, \Psi\} &= \{dv, 16, 17, \Psi\}\},
\end{aligned} \tag{1}$$

whose orthogonality hypergraph is depicted in Fig. 1(a).

A. Classical realization and predictions

In what follows, we shall prove the following:

(i) Hardy's configuration (1) allows a classical interpretation as it supports a distinguishing (often termed "separable") set of two-valued states. A "canonical" classical representation will be explicitly enumerated.

(ii) All classical interpretations of Hardy's configuration (1) enumerated in (i) predict that if the system is prepared in state Ψ, then the observable dd never occurs. That is, Hardy's setup is a gadget graph [36–38] with a "true-implies-false (classical) set of two-valued states" (TIFS). Indeed, it is one out of three minimal nonisomorphic true-implies-false configurations in four dimensions [14, Fig. 4(a)].

Hardy's configuration (1) allows a classical interpretation because it supports a set of 186 two-valued states that distinguishes different observables from one another. This means that the elements of every pair of distinct observables can be "separated" or "distinguished" by (at least) one two-valued state such that the respective state values of these elements are different. Therefore, by Kochen and Specker's Theorem 0 [8], the structure of its observables can be embedded in some Boolean algebra, which indicates classical representability.

An explicit construction of a classical model of a propositional structure corresponding to Hardy's 1993 configuration [22] is enumerated in Table I. Its realization is in terms of

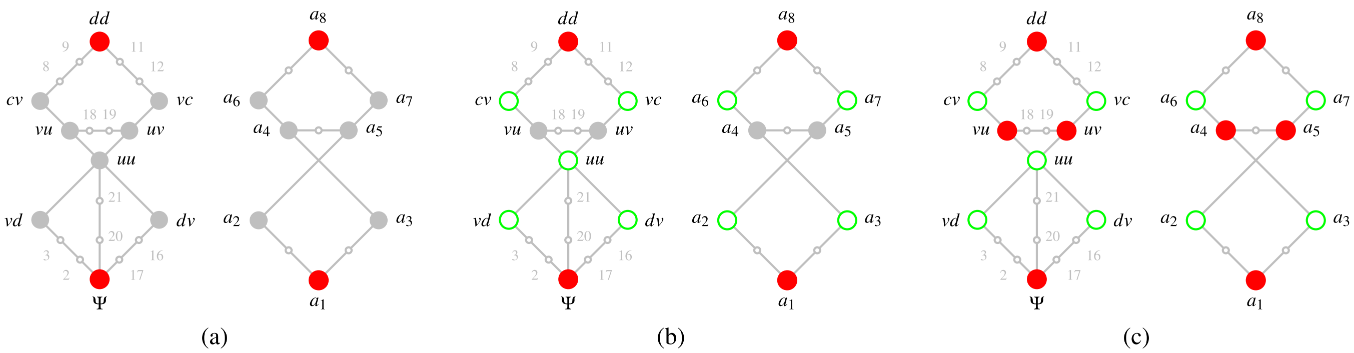

FIG. 2. Graphical presentation of a three-step proof by contradiction that from the pairs of observables $\{\Psi, dd\}$ and $\{a_1, a_8\}$, only one element can have assigned the classical value 1: (a) suppose otherwise; that is, $\Psi = dd = 1$ and $a_1 = a_8 = 1$; (b) then, by exclusivity, $vd = dv = uu = vc = cv = 0$ and $a_2 = a_3 = a_6 = a_7 = 0$; (c) then, by completeness, $vu = uv = 1$ and $a_4 = a5 = 1$, contradicting exclusivity. Small circles indicate "auxiliary" observables which can be chosen freely, subject to orthogonality constraints: all smooth lines indicate respective contexts representing orthonormal bases.

eight partitions (corresponding to the eight contexts) of the index set $\{1, 2, \dots, 185, 186\}$ of 186 two-valued states. The elements of the partitions corresponding to the 21 atomic propositions [which are obtained by "completing" the context as enumerated in Eq. (1)] are the index sets of all two-valued states which obtain the value "1" on the respective atoms. A detailed description of this construction can be found in Refs. [7,50,51].

Next, we shall elaborate on a classical prediction which is violated by quantum predictions: If Ψ is assumed to be true—that is, if a classical system is prepared (also known as preselected) in the state corresponding to observable Ψ—then the outcome corresponding to the observable dd cannot occur.

For a proof by contradiction depicted in Fig. 2, (wrongly) suppose that both Ψ as well as dd were both true simultaneously. Then, by the standard admissibility criteria for two-valued states [12,52] (also denoted as completeness and exclusivity [26,53,54]), $cv = vc = vd = dv = uu = 0$, enforcing $vu = uv = 1$, which contradicts admissibility (completeness and exclusivity).

The only remaining possibility is that ψ and dd have opposite values if one of them is true (they still may both be 0). Therefore, any two-valued state for which Ψ is 1—that is, in which the observable corresponding to Ψ occurs—must classically result in nonoccurrence of the outcome corresponding to the observable dd, and vice versa. This particular relation between the input and output ports of gadget graphs [16] has been called the 1-0 property [55], or one dominated by a true-implies-false set of two-valued states (TIFS) [14].

As mentioned earlier, the first true-implies-false gadget seems to have been introduced by Kochen and Specker ([19, Fig. 1] and used by them as a subgraph of Γ_1 [8] in three dimensions. Its orthogonality hypergraph is depicted in Fig. 1(b). As mentioned earlier, this gadget seems to have been independently discussed by, among others, Pitowsky, who called it the "cat's cradle" [56,57]. See, also, Fig. 1 in [52] (reprinted in Ref. [58]), a subgraph in Fig. 21 in Ref. [59], Fig. B.1 in [20], [21], Fig. 2 in [60], and Fig. 2.4.6 in [61] for early discussions of the true-implies-false prediction.

The full nuances of the predictions are revealed when the classical probabilities are computed. As the classical probability distributions are just the convex combinations of all two-valued states [62], it is easy to read them off from the canonical partition logic enumerated in Table I. In particular, the true-implies-false gadget behavior at the terminals Ψ and dd can be directly read off from

$$
\begin{aligned}
P_\Psi &= \sum_{i\in\Psi} \lambda_i = \lambda_1 + \lambda_2 + \cdots + \lambda_6,\\
P_{dd} &= \sum_{i\in dd} \lambda_i = \lambda_{11} + \lambda_{16} + \cdots + \lambda_{182},\\
\text{with}\quad & \lambda_i \geqslant 0,\ \text{and}\ \sum_{i=1}^{186} \lambda_i = 1.
\end{aligned} \tag{2}
$$

Since the intersection of the index sets Ψ and dd is empty, $P_{dd} = 0$ whenever $P_\Psi = 1$, and vice versa. For the sake of the example, all six two-valued measures assigning 1 to Ψ are depicted in Fig. 3.

One equivalent alternative way to characterize the classical probabilities completely would be to exploit the Minkowski-Weyl "main" representation theorem [63–69] and consider the classical convex polytope spanned by the 186 21-dimensional vectors whose components are the values in $\{0, 1\}$ of the two-valued states on the atomic propositions of the Hardy gadget. From these vertices (V-representation), the 35 half spaces that are the bounds of the polytope (H-representation) can be computed [1,62]. But due to space restrictions, we omit this discussion, although it might reveal quantum violations of Boole's (classical) "conditions of experience" [3].

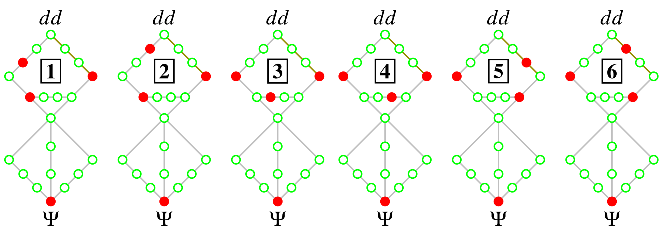

FIG. 3. Orthogonality hypergraphs of the Hardy gadget with overlaid six two-valued states which are 1 at Ψ.

B. Original quantum realization

Hardy's original quantum realization in terms of a particular type of faithful orthogonal representation is quite involved, but for the sake of delineating it, we shall mostly stick to the nomenclature of the 1993 paper [22]. Suppose two two-state particles and, for each one of the two particles, consider three orthonormal bases of its two-dimensional Hilbert space, namely,

$$\begin{aligned} B_1 &= \{|+\rangle, |-\rangle\} \equiv \{\mathbf{e}_1, \mathbf{e}_2\},\\ B_2 &= \{|u\rangle, |v\rangle\} \equiv \{\mathbf{f}_1, \mathbf{f}_2\},\\ B_3 &= \{|c\rangle, |d\rangle\} \equiv \{\mathbf{g}_1, \mathbf{g}_2\}. \end{aligned} \tag{3}$$

The components of the respective unitary transformations "rotating" these orthonormal bases into each other are defined by [70]

$$\begin{aligned} &B_1 \leftrightarrow B_2: \quad \mathbf{f}_j = \mathbf{U}^{12}_{ji}\mathbf{e}_i, \quad \text{and} \quad \mathbf{e}_j = (\mathbf{U}^{12})^{\dagger}{}_{ji}\mathbf{f}_i,\\ &B_2 \leftrightarrow B_3: \quad \mathbf{g}_k = \sum_{i=1}^{2} \mathbf{U}^{23}_{kj}\mathbf{f}_j, \quad \text{and} \quad \mathbf{f}_k = \sum_{i=1}^{2} (\mathbf{U}^{23})^{\dagger}{}_{kj}\mathbf{g}_j,\\ &B_1 \leftrightarrow B_3: \quad \mathbf{g}_k = \sum_{i=1}^{2} \mathbf{U}^{13}_{ki}\mathbf{e}_i = \sum_{i,j=1}^{2} \mathbf{U}^{23}_{kj}\mathbf{U}^{12}_{ji}\mathbf{e}_i,\\ &\qquad \text{and} \quad \mathbf{e}_k = \sum_{i,j=1}^{2} (\mathbf{U}^{12})^{\dagger}{}_{kj}(\mathbf{U}^{23})^{\dagger}{}_{ji}\mathbf{g}_i. \end{aligned} \tag{4}$$

One further ingredient of Hardy's configuration is a pure entangled state of two two-state particles which can be parameterized by [71] (the relative order of states matter, therefore, as pointed out earlier, we shall omit a subscript referring to the first and second particle, respectively)

$$\begin{aligned} |\Psi\rangle = \alpha|++\rangle - \beta|--\rangle \ \text{ with } \ \alpha, \beta \in \mathbb{R},\\ \alpha^2 + \beta^2 = \cos^2\phi + \sin^2\phi = 1 \ \text{ with } \ 0 \leqslant \phi \leqslant \pi/4. \end{aligned} \tag{5}$$

The minus sign (indicating a phase $\varphi = \pi$ for which $e^{i\varphi} = -1$) has been chosen for the sake of conforming to Hardy's conventions. Note that $0 \leqslant \alpha, \beta \leqslant 1$.

So far, the transformation matrices in (4) have not yet been specified, but in order for the argument to work, they should yield a faithful orthogonal representation of the orthogonality hypergraph depicted in Fig. 1(a). In particular, one needs to assure that

$$\langle\Psi|uu\rangle = \langle\Psi|vd\rangle = \langle\Psi|dv\rangle = 0. \tag{6}$$

At the same time, and in order to obtain a contradiction with the classical prediction "if Ψ is true then dd must be false" or, in physical terms, "if a system is prepared or (pre)selected in state Ψ then an event or outcome associated with dd cannot occur" (and vice versa), one needs to define those transformations such that, in addition to (6),

$$\langle\Psi|dd\rangle \neq 0 \ \text{ and "as great as possible}''. \tag{7}$$

In order to facilitate these desiderata (6) and (7), suppose *ad hoc* that

$$\begin{aligned} (\mathbf{U}^{12})^{\dagger} &= -\frac{i}{\sqrt{\alpha+\beta}}\begin{pmatrix} \sqrt{\beta} & \sqrt{\alpha}\\ \sqrt{\alpha} & -\sqrt{\beta}\end{pmatrix},\\ (\mathbf{U}^{23})^{\dagger} &= \frac{1}{\sqrt{1-\alpha\beta}}\begin{pmatrix} \sqrt{\alpha\beta} & -\alpha+\beta\\ \alpha-\beta & \sqrt{\alpha\beta}\end{pmatrix},\\ (\mathbf{U}^{13})^{\dagger} &= (\mathbf{U}^{12})^{\dagger} \cdot (\mathbf{U}^{23})^{\dagger}. \end{aligned} \tag{8}$$

Assume further, without loss of generality, that the first basis B_1 in (3) is identified with the Cartesian basis; that is, $|+\rangle = (1, 0)$ and $|-\rangle = (0, 1)$. Consequently, the vectors of the other bases B_2 and B_3 are obtained by applying the respective transformations (4) and (8),

$$\begin{aligned} |u\rangle &= \frac{i\left(\sqrt[4]{1-\alpha^2}, \sqrt{\alpha}\right)}{\sqrt{\sqrt{1-\alpha^2}+\alpha}},\\ |v\rangle &= \frac{i\left(\sqrt{\alpha}, -\sqrt[4]{1-\alpha^2}\right)}{\sqrt{\sqrt{1-\alpha^2}+\alpha}},\\ |c\rangle &= \frac{i(\alpha^{3/2}, (1-\alpha^2)^{3/4})}{\sqrt{\alpha^3 - \sqrt{1-\alpha^2}\alpha^2 + \sqrt{1-\alpha^2}}},\\ |d\rangle &= \frac{i((1-\alpha^2)^{3/4}, -\alpha^{3/2})}{\sqrt{\alpha^3 - \sqrt{1-\alpha^2}\alpha^2 + \sqrt{1-\alpha^2}}}. \end{aligned} \tag{9}$$

We are only dealing with pure states represented as normalized vectors which are (the sum of) the Kronecker (that is, "delineated" outer or tensor) products [72], e.g., $|\Psi\rangle = \alpha(1, 0) \otimes (1, 0) - \sqrt{1-\alpha^2}(0, 1) \otimes (0, 1) = (\alpha, 0, 0, -\sqrt{1-\alpha^2})$. The associated propositional observables can then be written in terms of the orthogonal projections formed as dyadic products $|x\rangle\langle x|$ of the unit (state) vectors $|x\rangle$.

By applying the transformations (4), $|\Psi\rangle$ can be rewritten in terms of either (i) the second basis B_2 for the first particle and the second basis B_2 for the second particle, (ii) the second basis B_2 for the first particle and the third basis B_3 for the second particle, (iii) the third basis B_3 for the first particle and the second basis B_2 for the second particle, or (iv) the third basis B_3 for the first particle and the third basis B_3 for the second particle. That is,

$$|\Psi\rangle = -|uv\rangle\sqrt{\alpha\beta} - |v\rangle[|u\rangle\sqrt{\alpha\beta} + |v\rangle(\alpha-\beta)] \tag{10}$$

$$= \frac{1}{\sqrt{1-\alpha\beta}}[|uc\rangle(\sqrt{\alpha\beta^3} - \sqrt{\alpha^3\beta}) - |vc\rangle(\alpha^2 - \alpha\beta + \beta^2) - |ud\rangle\alpha\beta] \tag{11}$$

$$= \frac{1}{\sqrt{1-\alpha\beta}}[|cu\rangle(\sqrt{\alpha\beta^3} - \sqrt{\alpha^3\beta}) - |cv\rangle(\alpha^2 - \alpha\beta + \beta^2) - |du\rangle\alpha\beta] \tag{12}$$

$$= -\frac{1}{1-\alpha\beta}[|cc\rangle(\alpha-\beta)(\alpha^2+\beta^2) + (|cd\rangle + |dc\rangle)(\alpha\beta)^{3/2} + |dd\rangle\alpha\beta(\beta-\alpha)]. \tag{13}$$

As can be readily read off from these representations of $|\Psi\rangle$, the conditions (6) and desideratum (7) are satisfied: (10) has no term proportional to $|uu\rangle$, (11) has no term proportional to $|vd\rangle$, (12) has no term proportional to $|dv\rangle$, and (13) has a term proportional to $|dd\rangle$.

To complete Hardy's original argument, we compare the classical prediction of "zero outcome" (nonoccurrence) for observable dd to the quantum prediction probability,

$$|\langle dd|\Psi\rangle|^2 = \left\{ \frac{\alpha\left[\alpha\left(\sqrt{1-\alpha^2}+\alpha\right)-1\right]}{\alpha\sqrt{1-\alpha^2}-1} \right\}^2, \qquad (14)$$

obtained from preparing (also known as preselecting) two entangled particles in state $|\Psi\rangle$ and measuring (e.g., by postselection) the nonvanishing probability to find them in state $|dd\rangle$ (thus contradicting the aforementioned classical predictions). $|\langle dd|\Psi\rangle|^2$ acquires its maximal value $\frac{1}{2}(5\sqrt{5}-11) \approx 0.09$ at $\alpha_\pm = \sqrt{1 \pm \sqrt{6\sqrt{5}-13}}/\sqrt{2}$. This is slightly below the maximal violation of the three-dimensional "minimal" true-implies-false case [the Specker bug [7,14,19] depicted in Fig. 1(b)] with probability $1/9 \approx 0.1$ [20,34,59,73,74].

V. VARIETIES OF COORDINATIZATION

13 In what follows, we shall enumerate a few faithful orthogonal representations of the Hardy gadget. Presently, no general analytic construction for finding even a single faithful orthogonal representation of a (hyper)graph (if any) exists, let alone a method for finding all such coordinatizations. Nevertheless, *ad hoc* faithful orthogonal representations can be generated in extenso by heuristic algorithms. With regards to (in)decomposability, the Hardy gadget allows almost all types of faithful orthogonal representations: "mixed" ones which have entangled as well as factorizable states, and ones which use entangled states. Entirely decomposable configurations are prohibited for geometric reasons.

From now on, the observables need not be formed by some
14 sort of composition, and therefore two symbols such as "uv" should only be understood as a label. Note that indecomposable vectors can be interpreted as pure entangled states. Likewise, decomposable vectors represent pure factorizable states. The coordinatizations will not be enumerated completely, as only the intertwining vertices will be explicitly mentioned. Nevertheless, completions are straightforward and have been discussed earlier. Reference [49] contains a careful categorization with respect to (in)decomposability.

A. Mixed (in)decomposability

Previous parametrizations [10,18,22,24,27] of Hardy's (minimal with respect to the number of vertices in four dimensions [14]) true-implies-false gadget, depicted in Fig. 1(a), in terms of four-dimensional vectors appear to be motivated by high yield—that is, by maximizing the quantum predictions of occurrence of the output (postselection) port dd, as well as by (in)decomposability of the associated vectors. This is motivated by what is sometimes referred to as "demonstrations of nonlocal contextuality"; that is, the "spread" of the relational information [75] among pairs of (spacelike) separated particles.

The first, mixed with respect to (in)decomposability of the vectors, type of coordinatization can almost directly be read off from the orthogonality hypergraph of the Hardy gadget depicted in Fig. 1(a). Note that the two "central full contexts" $\{|cv\rangle, |vu\rangle, |uu\rangle, |dv\rangle\}$ and $\{|vc\rangle, |uv\rangle, |uu\rangle, |vd\rangle\}$ intertwine at one common element $|uu\rangle$ and are actually "generated" by the flattened tensor products of two nonidentical two-dimensional contexts representable by the two orthonormal bases $\{|c\rangle, |d\rangle\}$ and $\{|u\rangle, |v\rangle\}$, respectively. Hence all that is necessary is to make sure that $|\Psi\rangle$ is orthogonal to three vectors $|vd\rangle$, $|uu\rangle$, and $|dv\rangle$ of four-dimensional space (and no multiplicities occur), as already encoded in Eqs. (6):

$$\begin{aligned} |\Psi\rangle \propto \big(&d_2u_2^2v_1 - 2d_2u_1u_2v_2 + d_1u_2^2v_2, \\ &d_2u_1^2v_2 - d_1u_2^2v_1, \\ &d_2u_1^2v_2 - d_1u_2^2v_1, \\ &2d_1u_1u_2v_1 - d_2u_1^2v_1 - d_1u_1^2v_2\big), \end{aligned} \qquad (15)$$

where x_i stands for the ith component of the vector x with respect to some basis common to all vectors.

In order to be able to claim nonlocality, additional constraints can be required from the components of $|\Psi\rangle$. Suppose one desires $|\Psi\rangle$ to be entangled. Then the product of its outer components should not be equal to the product of its inner components; that is, $\Psi_1\Psi_4 \neq \Psi_2\Psi_3$ because every decomposable product state of two vectors with components (a, b) and (c, d) is of the (delineated) form $(x_1 = ac, x_2 = ad, x_3 = bc, x_4 = bd)$, so that because of commutativity of scalars, $x_1x_4 = (ac)(bd) = abcd = (ad)(bc) = x_2x_3$. If one prefers the tensor product in matrix notation, then $\begin{pmatrix} x_1 = ac & x_2 = ad \\ x_3 = bc & x_4 = bd \end{pmatrix}$ and the criterion for factorizability and decomposability is a vanishing determinant, $x_1x_4 - x_2x_3 = 0$. Applying this constraint to Eq. (15) 15
results in

$$(d_2u_1 - d_1u_2)(u_1v_2 - u_2v_1) \neq 0. \qquad (16)$$

The third and the fourth rows of Table II containing vectors present two *ad hoc* configurations satisfying this "indecomposability" constraint.

B. Indecomposable configurations

The last two rows of Table II contain faithful orthogonal representations of the Hardy gadget in which all intertwining vectors are entangled (e.g., in the complex realization because the number of components of any vector with imaginary units i and $-i$ is odd; that is, either one or three). These coordinatizations have been obtained with a heuristic algorithm developed by McKay *et al.* [47].

C. Impossibility of complete decomposability

Conversely, it might be desirable to keep $|\Psi\rangle$ decompos- 16
able and factorizable; that is, all entities should be in a product state. In this case, the product of the outer components of $|\Psi\rangle$ should be equal to the product of its inner components; that is, $\Psi_1\Psi_4 = \Psi_2\Psi_3$. This results in the constraint $d_1 = d_2u_1/u_2$, with $u_2 \neq 0$ from Eq. (15), and consequently in a multiplicity of vectors; more explicitly, $d \propto u$. The resulting

TABLE II. Tabulation of some faithful orthogonal representations of the Hardy gadget. Two missing vectors per context as well as normalizations can be completed with a little effort. Labels of the form "*ab*" should not be understood as product states, but have been used merely to conform to Hardy's original nomenclature.

	ψ	dv	vd	uu	uv	vu	cv	vc	dd
CEG-A 1996 [10]	$(1,-1,-1,0)$	$(1,0,1,0)$	$(1,1,0,0)$	$(0,0,0,1)$	$(0,0,1,0)$	$(0,1,0,0)$	$(1,0,-1,0)$	$(1,-1,0,0)$	$(1,1,1,1)$
Cabello 1997 [24]	AB	β_+	$\beta-$	α	δ_-	δ_+	γ_+	γ_-	aB
$A=B=(1,0)$, $a=(\frac{1}{\sqrt{3}},\frac{-2\sqrt{2}}{\sqrt{3}})$	$(1,0,0,0)$	$(0,\frac{1}{2},\frac{-\sqrt{3}}{2},0)$	$(0,\frac{1}{2},\frac{\sqrt{3}}{2},0)$	$(0,0,0,1)$	$(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{2}},\frac{-1}{\sqrt{6}},0)$	$(\frac{-1}{\sqrt{3}},\frac{1}{\sqrt{2}},\frac{1}{\sqrt{6}},0)$	$(\sqrt{\frac{2}{3}},\frac{1}{2},\frac{1}{2\sqrt{3}},0)$	$(\sqrt{\frac{2}{3}},\frac{-1}{2},\frac{1}{2\sqrt{3}},0)$	$(\frac{1}{3},0,\frac{-2\sqrt{2}}{3},0)$
BBCGL 2011 [27]	Ψ	$\bar{a}_2b_1$	$a_1\bar{b}_2$	a_2b_2	$\bar{a}_2b_2$	$a_2\bar{b}_2$	$\bar{a}_2\bar{b}_1$	$\bar{a}_1\bar{b}_2$	a_1b_1
$u=(1,0)$, $c=(1,1)$	$(0,-1,-1,-1)$	$(0,1,0,-1)$	$(0,0,1,-1)$	$(1,0,0,0)$	$(0,1,0,0)$	$(0,0,1,0)$	$(0,1,0,1)$	$(0,0,1,1)$	$(1,-1,-1,1)$
$u=(1,0)$, $c=(2,3)$	$(0,-2,-2,-3)$	$(0,3,0,-2)$	$(0,0,3,-2)$	$(1,0,0,0)$	$(0,1,0,0)$	$(0,0,1,0)$	$(0,2,0,3)$	$(0,0,2,3)$	$(9,-6,-6,4)$
VECFIND [47] $\{0,1,-2,\sqrt{2}\}$	$(1,-2,\sqrt{2},0)$	$(-2,0,\sqrt{2},0)$	$(0,1,\sqrt{2},0)$	$(0,0,0,1)$	$(1,0,0,0)$	$(0,1,0,0)$	$(1,0,\sqrt{2},0)$	$(0,-2,\sqrt{2},0)$	$(-2,1,\sqrt{2},0)$
VECFIND $\{1,2,\frac{-1}{2},3,5,\pm i\}$	$(i,3,3,5)$	$(i,\frac{-1}{2},1,\frac{-1}{2})$	$(i,1,\frac{-1}{2},\frac{-1}{2})$	$(1,i,i,-i)$	$(1,i,-i,i)$	$(1,-i,i,i)$	$(i,2,1,2)$	$(i,1,2,2)$	$(5,i,i,i)$
VECFIND $\{1,-1,2,\frac{1}{2},3,5\}$	$(-1,3,3,5)$	$(1,-1,\frac{1}{2},\frac{1}{2})$	$(1,1,1,-1)$	$(1,-1,1,1)$	$(-1,2,1,2)$	$(1,-1,-1,2)$	$(-1,1,2,2)$	$(1,1,-1,1)$	$(1,\frac{1}{2},-1,\frac{1}{2})$

unattainability of a coordinatization with purely decomposable vectors should come as no surprise as the hypergraph of the Hardy gadget depicted in Fig. 1(a) contains three triangular subgraphs, namely, the cyclically intertwining sets of contexts $\{\{\Psi, 2, 3, vd\}, \{vd, uv, vc, uu\}, \{uu, 20, 21, \Psi\}\}$, $\{\{\Psi, 16, 17, dv\}, \{dv, vu, cv, uu\}, \{uu, 20, 21, \Psi\}\}$, as well as $\{\{vu, dv, cv, uu\}, \{uu, vd, vc, uv\}, \{vu, 18, 19, uv\}\}$. Triangular hypergraphs have no faithful orthogonal representation by purely decomposable vectors [49].

VI. EXTENSIONS TO TRUE-IMPLIES-TRUE GADGETS

We now turn to important extensions of the Hardy gadget which have a classical true-implies-true structure, as already employed in Kochen and Specker's Γ_1 [8] and discussed in Ref. [14]. A further escalation is a combination of these true-implies-true gadgets, similar to Kochen and Specker's Γ_3, which delivers a truly nonclassical performance on the

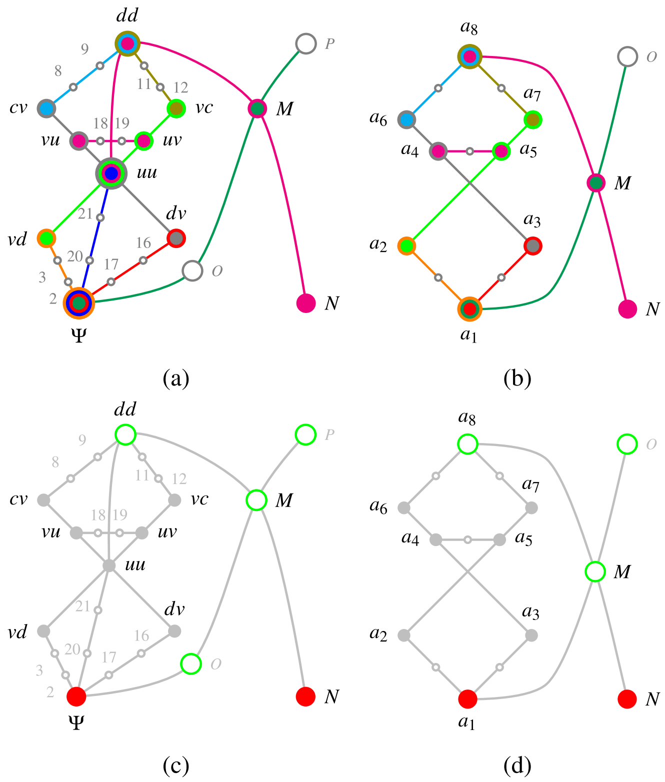

FIG. 4. Orthogonality hypergraphs of (a) the Hardy gadget extended to a true-implies-true gadget [14], as enumerated in (c), with $8+2=10$ contexts and $21+4=25$ atoms $\{\{dd, 8, 9, cv\}, \{dd, 11, 12, vc\}, \{cv, vu, uu, dv\}, \{vc, uv, uu, vd\}, \{vu, 18, 19, uv\}, \{vd, 2, 3, \Psi\}, \{uu, 20, 21, \Psi\}, \{dv, 16, 17, \Psi\}, \{5, dd, M, N\}, \{1, O, P, M\}\}$; (b) rendition based on the true-implies-true extended Specker bug or cat's cradle gadget [7,8], as enumerated in (c), with $7+2=9$ contexts and $13+3=17$ atoms $\{\{a_8, \ldots, a_6\}, \{a_8, \ldots, a_7\}, \{a_6, a_4, a_3\}, \{a_7, a_5, a_2\}, \{a_4, \ldots, a_5\}, \{a_2, \ldots, a_1\}, \{a_3, \ldots, a_1\}, \{a_8, M, N\}, \{a_1, \ldots, M\}\}$. Due to the way these true-implies-false gadgets are constructed, N always turns out to be 1 if Ψ or a_1 are supposed to be 1. Small circles indicate "auxiliary" observables, which can be chosen freely, subject to orthogonality constraints: all smooth lines indicate respective contexts representing orthonormal bases. 17

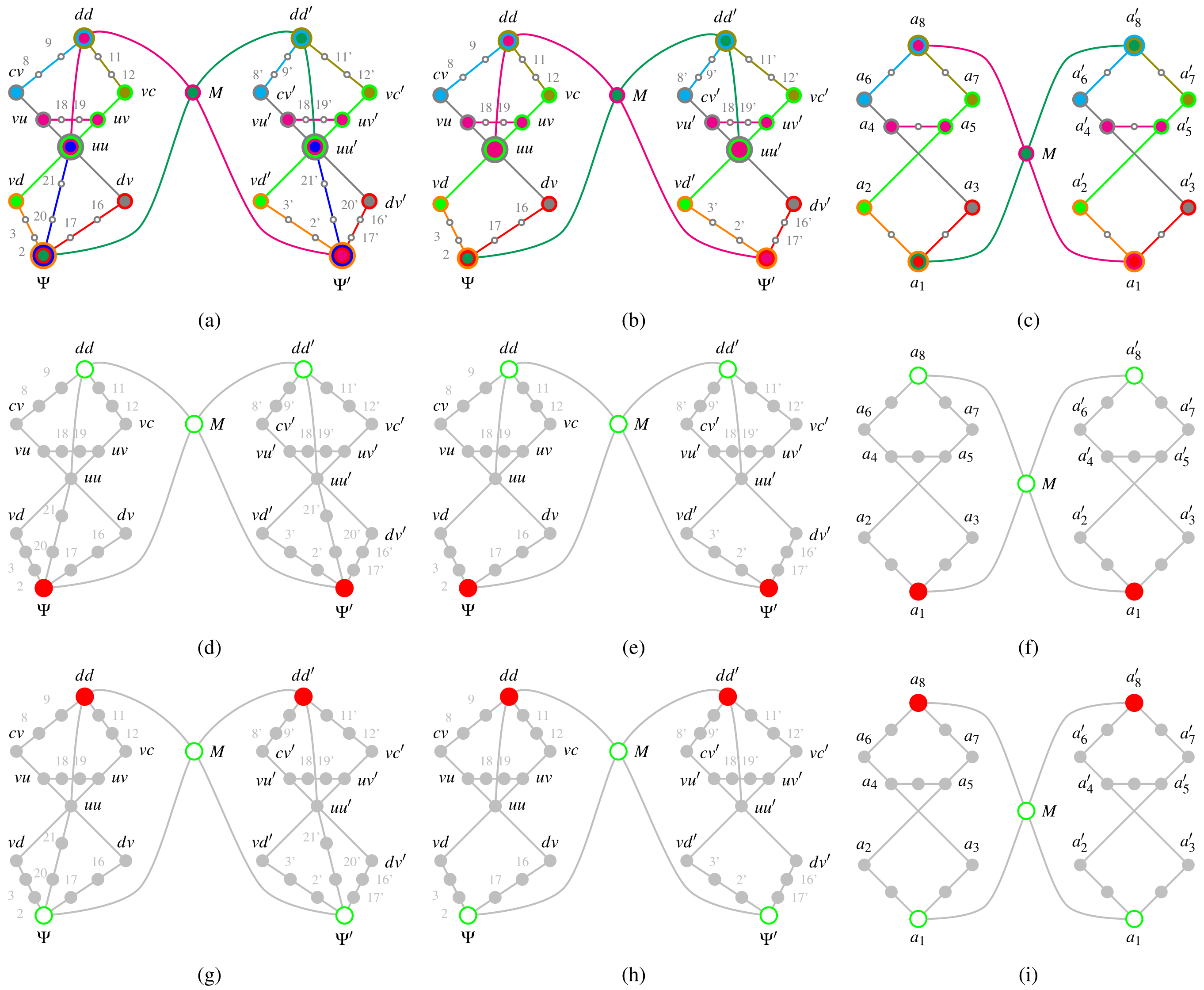

FIG. 5. Orthogonality hypergraphs of (a) a combination of two extended Hardy gadgets form a structure of quantum observables which cannot be classically embedded with $2 \times 8 + 2 = 18$ contexts and $2 \times 21 + 1 = 43$ atoms that cannot be classically embedded because of the indistinguishability with classical means (two-valued states) of the two pairs of atoms $\{\psi_1, \psi_2\}$ as well as $\{dd_1, dd_2\}$, respectively; (b) a combination of two extended Hardy-like gadgets first introduced in Fig. 4(b) of Ref. [14] form a structure of quantum observables with $2 \times 7 = 14$ contexts and $2 \times 19 + 1 = 39$ atoms that cannot be classically embedded because of the indistinguishability with classical means
18 (two-valued states) of the two pairs of atoms $\{\psi_1, \psi_2\}$ as well as $\{dd_1, dd_2\}$, respectively; (c) a combination of two extended Specker bug or cat's cradle gadgets [7,8] with $2 \times 7 + 2 = 16$ contexts and $2 \times 13 + 1 = 27$ atoms that cannot be classically embedded because of the indistinguishability with classical means (two-valued states) of the two pairs of atoms $\{a_1, a_1'\}$ as well as $\{a_8, a_8'\}$, respectively. (d)–(i) depict the associated two-valued states which are not 0 on all four observables $\{\psi_1, \psi_2, dd_1, dd_2\}$ as well as $\{a_1, a_1', a_8, a_8'\}$, respectively. Only valuations that are relevant for the proof are drawn. Small circles indicate "auxiliary" observables which can be chosen freely, subject to orthogonality constraints: all smooth lines indicate respective contexts representing orthonormal bases.

algebraic level of the propositional observables (and not just probabilistic predictions based upon classical probabilities): Unlike the Hardy and its extended true-implies-true gadgets, those observables can no longer be faithfully embedded into any Boolean algebra ([8], Theorem 0).

Figure 4 depicts the extension of the Hardy gadget which delivers a classical true-implies-true prediction at its terminal points Ψ and N. A faithful orthogonal representation of the extended Hardy gadget can be obtained *ad hoc* by the heuristic algorithm VECFIND [47] in the coordinate basis $\{0, \pm 1, \pm 2, 3\}$ and the -nk option, which is capable of finding "almost all" vectors, including the true-implies-true terminal points Ψ and N *ex machina*, and (for this coordinate basis) needs a little helping hand (or the additional component basis elements $\{-3, 5, 7, \sin\theta, \cos\theta\}$ with $\theta \neq n\pi/4$, $n \in \mathbb{Z}$) to find the complete set, given by $\Psi = (0, 1, 1, -1)$, $2 = (2, 2, -1, 1)$, $3 = (3, -2, 1, -1)$, $vd = (0, 0, 1, 1)$, $uu = (1, 0, 0, 0)$, $vu = (0, 0, 0, 1)$, $cv = (0, 1, 1, 0)$, $8 = (3, 1, -1, 2)$, $9 = (-2, 1, -1, 2)$, $dd = (0, -1, 1, 1)$, $11 = (3, -2, -1, -1)$, $12 = (2, 2, 1, 1)$, $vc = (0, 0, 1, -1)$, $uv = (0, 1, 0, 0)$, $dv = (0, 1, -1, 0)$, $16 = (-2, 1, 1, 2)$, $17 = (3, 1, 1, 2)$,

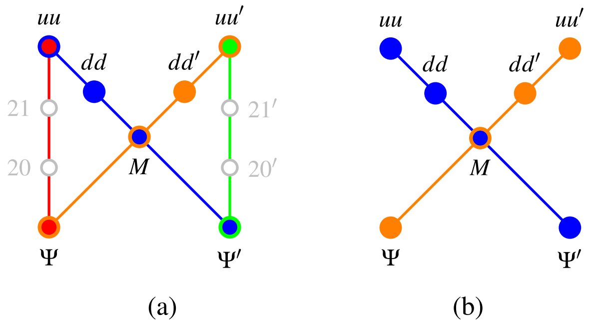

FIG. 6. Hypergraphs of the "orthogonality backbones" of (a) Fig. 5(a) and (b) Fig. 5(b) supporting the two-valued states depicted in Figs. 5(c) and 5(d), respectively.

$18 = (\cos\theta, 0, \sin\theta, 0)$, $19 = (-\sin\theta, 0, \cos\theta, 0)$, $20 = (0, 4, -3, 1)$, $21 = (0, 2, 5, 7)$, $M = (0, 1, 0, 1)$, $N = (0, 1, 2, -1)$, $O = (2, -1, 2, 1)$, $P = (3, 1, -2, -1)$, where $\theta \neq n\pi/4$, $n \in \mathbb{Z}$. [The original coordinatization suggested for atom 20 was $(0, 1, -1, 0)$, but a completion would have resulted in duplicities, namely, $21 = (0, 1, 1, 2) = dv$; and therefore the original suggestion had to be dropped.] Although we do not concentrate on maximal violations of classical predictions by quantum probabilities, for reasons mentioned later, it is worth noting that as $|\langle\psi|N\rangle|^2 = 8/9$, the quantum violation of the classical predictions will, in this particular configuration, occur in one out of nine times; that is, with probability $0.\bar{1}$ (proper normalization is always assumed). Fortuitously, if one concentrates on the quantum signal for observable $|dd\rangle\langle dd|$, then one obtains the same quantum prediction $|\langle\psi|dd\rangle|^2 = 1/9$ for this outcome—although classically it should never occur.

VII. EXTENSIONS TO GADGETS WITH INDISTINGUISHABLE CLASSICAL TRUTH ASSIGNMENTS

One way to proceed would be what Kochen and Specker did with their true-implies-true gadget Γ_1, and serially compose them at their respective (properly parametrized) terminal points often enough to obtain Γ_2, which renders a complete contradiction with exclusivity [8]. Instead of this head-on strategy for obtaining complete contradictions with classical noncontextual hidden-variable models, we shall use a more subtle approach and consider a hypergraph which, again in analogy with Kochen and Specker's Γ_3 in three dimensions, cannot be classically embedded in a Boolean algebra. The construction uses two true-implies-true extended Hardy gadgets to construct two pairs of observable propositions which cannot be differentiated by classical two-valued measures—and thus by any classical probability distributions—although "plenty" such two-valued states still exist (but their set is "too meager" to allow mutual distinguishability of all pairs of distinct propositions).

Searches for a faithful orthogonal representation of an extensions using two (a combination) of the "original" version of the Hardy gadget, as depicted in Fig. 6(a), have been inconclusive so far. Nevertheless, as it turns out, this task can be completed by using (a combination of) a slight modification of Hardy's gadget introduced in Fig. 4(b) of Ref. [14], in which the original context $\{\Psi, \ldots, uu\}$ is "relocated" or "reshuffled" into the context $\{uu, \ldots, dd\}$. The resulting gadget, depicted in Fig. 5(b), not only has less atoms but, most importantly, has a less tight "orthogonality backbone" structure, depicted in Fig. 6(b), of just two contexts intertwined in a single atom M, namely, $\{\{uu_1, dd_1, M, \Psi_2\}, \{uu_2, dd_2, M, \Psi_1\}\}$, as compared to the tight configuration resulting from a composition of two of Hardy's original gadgets $\{\{Psi_1, \ldots, uu_1\}, \{uu_1, dd_1, M, \Psi_2\}, \{uu_2, dd_2, M, \Psi_1\}, \{Psi_2, \ldots, uu_2\}\}$ depicted in Fig. 6(a).

More explicitly, VECFIND [47], with the component basis $\{0, \pm 1, 2, -3, 4, 5\}$, yields an *ad hoc* coordinatization of the intertwine atoms $\psi = (1, 0, 0, 0)$, $vd = (0, 2, -1, 1)$, $uu = (0, 0, 1, 1)$, $vu = (1, -1, 1, -1)$, $cv = (-3, -1, 1, -1)$, $dd = (1, -3, 0, 0)$, $vc = (-3, -1, -1, 1)$, $uv = (1, -1, -1, 1)$, $dv = (0, 2, 1, -1)$, $\psi' = (-3, -1, 0, 0)$, $vd' = (1, -3, 4, -1)$, $uu' = (0, 1, 1, 1)$, $vu' = (-3, -1, 2, -1)$, $cv' = (1, 0, 1, -1)$, $dd' = (0, 2, -1, -1)$, $vc' = (5, 0, -1, 1)$, $uv' = (-1, -3, -1, 4)$, $dv' = (1, -3, 1, 2)$, and $M = (0, 0, 1, -1)$, which can be readily completed into a faithful orthogonal representation of the hypergraph depicted in Fig. 5(b). Note that in this particular configuration, because of indistinguishability, the classical prediction to find a particle prepared in a state Ψ in the state Ψ' is one (certainty), whereas quantum mechanics predicts nonoccurrence of the elementary propositional observable $|\Psi'\rangle\langle\Psi'|$ given a preselected, prepared state $|\Psi\rangle$ with probability $|\langle\Psi|\Psi'\rangle|^2 = 9/10$; that is, the violation of the classical prediction by quantum mechanics occurs in this case in one out of ten experimental runs.

VIII. SUMMARY AND CAUTIONARY REMARKS

Hardy-type configurations have been extended to configurations of contexts which show a different nonclassical 19 performance: they contain distinct quantum observables that cannot be distinguished from one another by any classical (noncontextual) means. To appreciate the difference of this as- 20 pect beyond the realization of just another relational property among some prepared state and its measurement, it is important to keep in mind that according to a finding by Kochen and Specker [8, Theorem 0], indistinguishability serves a demarcation criterion for strong forms of nonclassicality: The absence of classical distinguishability indicates a stronger contraindication of hidden-variable theories (relative to the assumptions) than, say, exploitation of true-implies-{true, false} properties [14] which still allow faithful classical embeddability of the quantum observables (albeit with different statistical predictions), and merely requires complementarity.

Indistinguishability by classical means indicates a rather strong form of nonclassicality—that is, the impossibility to faithfully embed the quantum mechanical observables in classical Boolean structures—while still allowing the direct experimental falsification of the respective quantum and classical predictions. Therefore, it is not affected by questions related to the empirical pertinence of Kochen-Specker proofs (by contradiction) of the absence of any classical interpretation, stated pointedly by Clifton, "how can you measure a contradiction?" [60]. 21

(quoted from my memory \cite{clifton})

@misc{clifton, Author = {Robert K. Clifton},
Note = {private communication at the occation of the second Biennial International Quantum Structures Association Conference ``Quantum Structures'' in Prague, Czech Republic, September 1994}, Year = {1994}}

Further efforts could advance by "improving the nonclassical performance" of gadgets not in terms of the number of (counterfactual, complementary) observables, but in terms of quantum-to-classical discords in four dimensions. The hypergraph method developed earlier might suggest such advancements by their emphasis on the logico-algebraic structure, thereby making possible a more systematic exploitation of feasible configurations of observables. There already exist true-implies-{true, false} gadgets which yield high performance in three dimensions [38,76,77].

Nevertheless, once the vectors corresponding to pre- and postselected states are fixed, it is always possible to find any kind of conforming or disagreeing classical-versus-quantum behavior. As I have pointed out elsewhere [16], these kind of statements are contingent on the chosen gadget consisting of mostly counterfactual observables "in the mind" of the observer [78,79]. Nevertheless, any such considerations raise fascinating, challenging issues in a variety of fields which might have been perceived unrelated so far: graph theory, (linear) algebra, functional analysis, geometry, automated theorem proving, and—last but not least—quantum physics and quantum information (processing) technology.

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The logic of propositions which are not simultaneously decidable

English translation

[1] M. Froissart, Constructive generalization of Bell's inequalities, Il Nuovo Cimento B (1971–1996) **64**, 241 (1981).

Tsirelson

[2] B. S. ~~Cirel'son (=Tsirel'son),~~ Some results and problems on quantum Bell-type inequalities, Hadronic J. Suppl. **8**, 329 (1993).

23

[3] I. Pitowsky, George Boole's 'conditions of possible experience' and the quantum puzzle, British J. Philos. Sci. **45**, 95 (1994).

[4] K. Svozil, On generalized probabilities: Correlation polytopes for automaton logic and generalized urn models, extensions of quantum mechanics and parameter cheats, arXiv:quant-ph/0012066.

24

no capitalizations etc...

[5] A. A. Klyachko, M. A. Can, S. Binicioğlu, and A. S. Shumovsky, Simple Test For Hidden Variables In Spin-1 Systems, Phys. Rev. Lett. **101**, 020403 (2008).

[6] A. Cabello, Experimentally Testable State-Independent Quantum Contextuality, Phys. Rev. Lett. **101**, 210401 (2008).

[7] K. Svozil, What is so special about quantum clicks? Entropy **22**, 602 (2020); see Sec. 5.5.3. arXiv:1707.08915.

[8] S. Kochen and E. P. Specker, The problem of hidden variables in quantum mechanics, J. Math. Mech. (now Indiana Univ. Math. J.) **17**, 59 (1967); see p. 68.

[9] I. Pitowsky, Infinite and finite Gleason's theorems and the logic of indeterminacy, J. Math. Phys. **39**, 218 (1998).

[10] A. Cabello, J. M. Estebaranz, and G. García-Alcaine, Bell-Kochen-Specker theorem: A proof with 18 vectors, Phys. Lett. A **212**, 183 (1996).

[11] A. A. Abbott, C. S. Calude, and K. Svozil, Value-indefinite observables are almost everywhere, Phys. Rev. A **89**, 032109 (2014).

[12] A. A. Abbott, C. S. Calude, and K. Svozil, A variant of the Kochen-Specker theorem localising value indefiniteness, J. Math. Phys. **56**, 102201 (2015).

[13] A. M. Gleason, Measures on the closed subspaces of a Hilbert space, J. Math. Mech. (now Indiana Univ. Math. J.) **6**, 885 (1957).

[14] A. Cabello, J. R. Portillo, A. Solís, and K. Svozil, Minimal true-implies-false and true-implies-true sets of propositions in noncontextual hidden-variable theories, Phys. Rev. A **98**, 012106 (2018).

[15] E. Specker, Die Logik nicht gleichzeitig entscheidbarer Aussagen, Dialectica **14**, 239 (1960), English translation at arXiv:1103.4537. 25

[16] K. Svozil, Classical predictions for intertwined quantum observables are contingent and thus inconclusive, Quantum Rep. **2**, 278 (2020).

[17] P. R. Halmos, *Finite-Dimensional Vector Spaces*, Undergraduate Texts in Mathematics (Springer, New York, 1958); see Sec. 84, pp. 172, 173.

[18] L. Hardy, Quantum Mechanics, Local Realistic Theories, And Lorentz-Invariant Realistic Theories, Phys. Rev. Lett. **68**, 2981 (1992).

Editors = {J. W. Addison and Leon Henkin and Alfred Tarski}

[19] S. Kochen and E. P. Specker, Logical structures arising in quantum theory, in *The Theory of Models, Proceedings of the 1963 International Symposium at Berkeley* (North Holland, Amsterdam, NY, 1965), pp. 177–189, reprinted in Ref. [80, pp. 209–221]; see Fig. 1, p. 182. 26

[20] F. J. Belinfante, *A Survey of Hidden-variables Theories* (Pergamon, Elsevier, Oxford, 1973); see Appendix B, pp. 64, 65 and Fig. B.1, p. 64.

[21] A. Stairs, Quantum logic, realism, and value definiteness, Philos. Sci. **50**, 578 (1983); see pp. 588, 589.

[22] L. Hardy, Nonlocality For Two Particles Without Inequalities For Almost All Entangled States, Phys. Rev. Lett. **71**, 1665 (1993).

[23] S. Goldstein, Nonlocality Without Inequalities For Almost All Entangled States For Two Particles, Phys. Rev. Lett. **72**, 1951 (1994).

[24] A. Cabello, No-hidden-variables proof for two spin-$\frac{1}{2}$ particles preselected and postselected in unentangled states, Phys. Rev. A **55**, 4109 (1997).

[25] A. Cabello, P. Badziag, M. Terra Cunha, and M. Bourennane, Simple Hardy-Like Proof Of Quantum Contextuality, Phys. Rev. Lett. **111**, 180404 (2013).

[26] Z.-P. Xu, J.-L. Chen, and O. Gühne, Proof of the Peres Conjecture for Contextuality, Phys. Rev. Lett. **124**, 230401 (2020).

[27] P. Badziaąg, I. Bengtsson, A. Cabello, H. Granström, and J.-A. Larsson, Pentagrams and paradoxes, Found. Phys. **41**, 414 (2011).

[28] J.-L. Chen, A. Cabello, Z.-P. Xu, H.-Yi. Su, C. Wu, and L. C. Kwek, Hardy's paradox for high-dimensional systems, Phys. Rev. A **88**, 062116 (2013).

[29] N. D. Mermin, The best version of Bell's theorem, Ann. NY Acad. Sci. **755**, 616 (1995).

[30] R. Penrose, *The Road to Reality: A Complete Guide to the Laws of the Universe* (Cape and Knopf, London, 2004/2005), see Sec. 23.5, p. 589ff.

[31] N. D. Mermin, Quantum mysteries refined, Am. J. Phys. **62**, 880 (1994).

[32] P. G. Kwiat and L. Hardy, The mystery of the quantum cakes, Am. J. Phys. **68**, 33 (2000).

[33] G. Kalmbach, *Orthomodular Lattices*, London Mathematical Society Monographs, Vol. 18 (Academic Press, London, 1983).

[34] K. Svozil and J. Tkadlec, Greechie diagrams, nonexistence of measures in quantum logics and Kochen–Specker type constructions, J. Math. Phys. **37**, 5380 (1996).

[35] R. J. Greechie, Orthomodular lattices admitting no states, J. Combin. Theory, Ser. A **10**, 119 (1971).

[36] W. T. Tutte, A short proof of the factor theorem for finite graphs, Can. J. Math. **6**, 347 (1954).

[37] J. Szabó, Good characterizations for some degree constrained subgraphs, J. Combin. Theory, Ser. B **99**, 436 (2009).

[38] R. Ramanathan, M. Rosicka, K. Horodecki, S. Pironio, M. Horodecki, and P. Horodecki, Gadget structures in proofs of the Kochen-Specker theorem, Quantum **4**, 308 (2020).

[39] R. J. Greechie, On the structure of orthomodular lattices satisfying the chain condition, J. Combin. Theory **4**, 210 (1968).

[40] R. J. Greechie, Orthomodular lattices, Ph.D. thesis, University of Florida, FL, 1996; see Chap. 2.

[41] B. D. McKay, N. D. Megill, and M. Pavičić, Algorithms for Greechie diagrams, Int. J. Theor. Phys. **39**, 2381 (2000).

[42] M. Pavičić, J.-P. Merlet, B. McKay, and N. D. Megill, Kochen-Specker vectors, J. Phys. A: Math. Gen. **38**, 1577 (2005).

[43] L. Lovász, On the Shannon capacity of a graph, IEEE Trans. Inf. Theory **25**, 1 (1979).

[44] L. Lovász, M. Saks, and A. Schrijver, Orthogonal representations and connectivity of graphs, Linear Algebra Appl. **114–115**, 439 (1989), Special Issue Dedicated to Alan J. Hoffman.

[45] M. Grötschel, L. Lovász, and A. Schrijver, *Geometric Algorithms and Combinatorial Optimization*, 2nd ed., Algorithms and Combinatorics, Vol. 2 (Springer, Berlin, 1993).

[46] A. Solís-Encina and J. Portillo, Orthogonal representation of graphs, arXiv:1504.03662.

[47] M. Pavičić and N. D. Megill, Vector generation of quantum contextual sets in even dimensional Hilbert spaces,
928 Entropy 20, 928 (2018), doi:10.3390/e20120928, program code at
27 https://puh.srce.hr/s/Qegixzz2BdjYwFL arXiv:1905.01567.

[48] M. Pavičić, Hypergraph contextuality, Entropy **21**, 1107 (2019).

[49] H. Havlicek and K. Svozil, Decomposability versus entanglement of quantum states in four dimensions bound by pre-selection of two-dimensional subspaces, arXiv:2010.09506.

[50] K. Svozil, Logical equivalence between generalized urn models and finite automata, Int. J. Theor. Phys. **44**, 745 (2005).

[51] K. Svozil, Faithful orthogonal representations of graphs from partition logics, Soft Comput. **24**, 10239 (2020).

[52] R. J. Greechie, Some results from the combinatorial approach to quantum logic, Synthese **29**, 113 (1974); see Fig. 1 (reprinted in Ref. [58]), p. 123.

[53] A. Cabello, L. E. Danielsen, A. J. López-Tarrida, and J. R. Portillo, Basic exclusivity graphs in quantum correlations, Phys. Rev. A **88**, 032104 (2013).

[54] A. Cabello, S. Severini, and A. Winter, Graph-Theoretic Approach To Quantum Correlations, Phys. Rev. Lett. **112**, 040401 (2014).

[55] K. Svozil, Quantum scholasticism: On quantum contexts, counterfactuals, and the absurdities of quantum omniscience, Inf. Sci. **179**, 535 (2009).

[56] I. Pitowsky, Betting on the outcomes of measurements: A Bayesian theory of quantum probability, Studies in History and Philosophy of Science Part B. Studies Hist. Philos. Modern Phys. **34**, 395 (2003); Quantum Inf. Comput., arXiv:quant-ph/0208121.

[57] I. Pitowsky, Quantum mechanics as a theory of probability, in *Physical Theory and its Interpretation*, The Western Ontario Series in Philosophy of Science, Vol. 72, edited by W. Demopoulos and I. Pitowsky (Springer, Netherlands, 2006), pp. 213–240.

[58] R. J. Greechie, Some results from the combinatorial approach to quantum logic, in *Logic and Probability in Quantum Mechanics*, edited by Patrick Suppes (Springer, Netherlands, 1976), pp. 105–119.

[59] M. Redhead, *Incompleteness, Nonlocality, and Realism: A Prolegomenon to the Philosophy of Quantum Mechanics* (Clarendon, Oxford, 1987); see subgraph in Fig. 21, pp. 126, 127.

[60] R. K. Clifton, Getting contextual and nonlocal elements-of-reality the easy way, Am. J. Phys. **61**, 443 (1993); see Fig. 2, p. 446.

[61] P. Pták and S. Pulmannová, *Orthomodular Structures as Quantum Logics. Intrinsic Properties, State Space and Probabilistic Topics*, Fundamental Theories of Physics, Vol. 44 (Kluwer Academic, Springer, Netherlands, 1991); see Fig. 2.4.6, p. 39.

[62] I. Pitowsky, *Quantum Probability—Quantum Logic*, Lecture Notes in Physics, Vol. 321 (Springer-Verlag, Berlin, 1989); see Chap. 2.

[63] G. M. Ziegler, *Lectures on Polytopes* (Springer, New York, 1994).

[64] M. Henk, J. Richter-Gebert, and G. M. Ziegler, Basic properties of convex polytopes, in *Handbook of Discrete and Computational Geometry*, 2nd ed., edited by J. E. Goodman and J. O'Rourke (Chapman and Hall/CRC Press, Boca Raton, FL, 2004), pp. 355–383.

[65] D. Avis, D. Bremner, and R. Seidel, How good are convex hull algorithms? Comput. Geometry: Theory Applic. **7**, 265 (1997).

[66] P. McMullen and G. C. Shephard, *Convex Polytopes and the Upper Bound Conjecture*, London Mathematical Society Lec-

Algebraic tests for the impossibility of hidden variables in quantum mechanics

English translation

ture Notes Series 3 (Cambridge University Press, Cambridge, 1971).

[67] A. Schrijver, *Theory of Linear and Integer Programming*, Wiley Series in Discrete Mathematics & Optimization (Wiley, New York, 1998).

[68] B. Grünbaum, *Convex Polytopes*, 2nd ed., Graduate Texts in Mathematics, Vol. 221 (Springer, New York, 2003).

[69] K. Fukuda, Frequently asked questions in polyhedral computa-
28 tion (unpublished).

[70] J. Schwinger, Unitary operators bases, Proc. Natl. Acad. Sci. (PNAS) **46**, 570 (1960).

[71] A. Acín, A. Andrianov, L. Costa, E. Jané, J. I. Latorre, and R. Tarrach, Generalized Schmidt Decomposition And Classification ~~Of~~ Three-Quantum-Bit States, Phys. Rev. Lett. **85**, 1560 (2000).

[72] D. N. Mermin, *Quantum Computer Science* (Cambridge University Press, Cambridge, 2007); see Chap. 1.

[73] A. Cabello, A simple proof of the Kochen-Specker theorem, Eur. J. Phys. **15**, 179 (1994).

[74] A. Cabello, Pruebas algebraicas de imposibilidad de variables ocultas en mecánica cuántica, Ph.D. thesis, Universidad Com-
plutense de Madrid, Madrid, Spain (1996). 29

[75] A. Zeilinger, A foundational principle for quantum mechanics, Found. Phys. **29**, 631 (1999).

[76] K. Svozil, New forms of quantum value indefiniteness suggest that incompatible views on contexts are epistemic, Entropy **20**,
406 ~~(22)~~ (2018). 30

[77] K. Svozil, Roots and (re)sources of value (in)definiteness versus contextuality, in *Quantum, Probability, Logic: The Work and Influence of Itamar Pitowsky*, Jerusalem Studies in Philosophy and History of Science (JSPS), Vol. 1, edited by M. Hemmo and O. Shenker (Springer International, Cham, 2020), pp. 521–544.

[78] G. Berkeley, *A Treatise Concerning the Principles of Human Knowledge* (Aaron Rhames, for Jeremy Pepyat, Bookseller, Skinner-Row, Dublin, 1710).

[79] W. T. Stace, The refutation of realism, Mind **43**, 145 (1934).

[80] E. Specker, *Selecta* (Birkhäuser Verlag, Basel, 1990). 31

English translation: Selections