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MATHEMATICAL METHODS OF THEORETICAL PHYSICS

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1.0.1 Functional spaces

That complex continuous waveforms or functions are comprised of a number of harmonics seems to be an idea at least as old as the Pythagoreans. In physical terms, Fourier analysis [1] attempts to decompose a function into its constituent frequencies, known as a frequency spectrum. Thereby the goal is the expansion of periodic and aperiodic functions into sine and cosine functions. Fourier's observation or conjecture is, informally speaking, that any "suitable" function $f(x)$ can be expressed as a possibly infinite sum (i.e. linear combination), of sines and cosines of the form

$$f(x)=\sum_{k=-\infty}^{\infty}\left[A_k\cos(Ckx)+B_k\sin(Ckx)\right]. \tag{1.1}$$

[1] Kenneth B. Howell. *Principles of Fourier analysis*. Chapman & Hall/CRC, Boca Raton, London, New York, Washington, D.C., 2001; and Russell Herman. *Introduction to Fourier and Complex Analysis with Applications to the Spectral Analysis of Signals*. University of North Carolina Wilmington, Wilmington, NC, 2010. URL `http://people.uncw.edu/hermanr/mat367/FCABook/Book2010/FTCA-book.pdf`. Creative Commons Attribution-NoncommercialShare Alike 3.0 United States License

Moreover, it is conjectured that any "suitable" function $f(x)$ can be expressed as a possibly infinite sum (i.e. linear combination), of exponentials; that is,

$$f(x)=\sum_{k=-\infty}^{\infty}D_k e^{ikx}. \tag{1.2}$$

More generally, it is conjectured that any "suitable" function $f(x)$ can be expressed as a possibly infinite sum (i.e. linear combination), of other (possibly orthonormal) functions $g_k(x)$; that is,

$$f(x)=\sum_{k=-\infty}^{\infty}\gamma_k g_k(x). \tag{1.3}$$

The bigger picture can then be viewed in terms of *functional (vector) spaces*: these are spanned by the elementary functions g_k, which serve as elements of a *functional basis* of a possibly infinite-dimensional vector space. Suppose, in further analogy of the set of all such functions $\mathfrak{G}=\bigcup_k g_k(x)$ to the (Cartesian) standard basis, we can consider these elementary functions g_k to be *orthonormal* in the sense of a *generalized functional scalar product* [cf. also Section **??** on page **??**; in particular Eq. (**??**)]

$$\langle g_k \mid g_l\rangle=\int_a^b g_k(x)g_l(x)dx=\delta_{kl}. \tag{1.4}$$

One could arrange the coefficients γ_k into a tuple (an ordered list of elements) $(\gamma_1, \gamma_2, \ldots)$ and consider them as components or coordinates of a vector with respect to the linear orthonormal functional basis $\mathfrak{G}$.

1.0.2 Fourier series

Suppose that a function $f(x)$ is periodic in the interval $[-\frac{L}{2}, \frac{L}{2}]$ with period L. (Alternatively, the function may be only defined in this interval.) A function $f(x)$ is *periodic* if there exist a period $L \in \mathbb{R}$ such that, for all x in the domain of f,

$$f(L+x) = f(x). \tag{1.5}$$

Then, under certain "mild" conditions – that is, f must be piecewise continuous and have only a finite number of maxima and minima – f can be decomposed into a *Fourier series*

$$\begin{aligned}
&f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_k \cos\left(\frac{k\pi x}{L}\right) + b_k \sin\left(\frac{k\pi x}{L}\right) \right], \text{ with} \\
&a_k = \frac{2}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} f(x) \cos\left(\frac{k\pi x}{L}\right) dx \text{ for } k \geq 0 \\
&b_k = \frac{2}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} f(x) \sin\left(\frac{k\pi x}{L}\right) dx \text{ for } k > 0.
\end{aligned} \tag{1.6}$$

For proofs and additional information see §8.1 in

Kenneth B. Howell. *Principles of Fourier analysis*. Chapman & Hall/CRC, Boca Raton, London, New York, Washington, D.C., 2001

For a (heuristic) proof, consider the Fourier conjecture (1.1), and compute the coefficients A_k, B_k, and C.

First, observe that we have assumed that f is periodic in the interval $[-\frac{L}{2}, \frac{L}{2}]$ with period L. This should be reflected in the sine and cosine terms of (1.1), which themselves are periodic functions in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ with period 2π. Thus in order to map the functional period of f into the sines and cosines, we can "stretch/shrink" L into 2π; that is, C in Eq. (1.1) is identified with

$$C = \frac{2\pi}{L}. \tag{1.7}$$

Thus we obtain

$$f(x) = \sum_{k=-\infty}^{\infty} \left[A_k \cos(\frac{2\pi}{L} kx) + B_k \sin(\frac{2\pi}{L} kx) \right]. \tag{1.8}$$

Now use the following properties: (i) for $k = 0$, $\cos(0) = 1$ and $\sin(0) = 0$. Thus, by comparing the coefficient a_0 in (1.6) with A_0 in (1.1) we obtain $A_0 = \frac{a_0}{2}$.

(ii) Since $\cos(x) = \cos(-x)$ is an *even function* of x, we can rearrange the summation by combining identical functions $\cos(-\frac{2\pi}{L} kx) = \cos(\frac{2\pi}{L} kx)$, thus obtaining $a_k = A_{-k} + A_k$ for $k > 0$.

(iii) Since $\sin(x) = -\sin(-x)$ is an *odd function* of x, we can rearrange the summation by combining identical functions $\sin(-\frac{2\pi}{L} kx) = -\sin(\frac{2\pi}{L} kx)$, thus obtaining $b_k = -B_{-k} + B_k$ for $k > 0$.

Having obtained the same form of the Fourier series of $f(x)$ as exposed in (1.6), we now turn to the derivation of the coefficients a_k and b_k. a_0 can be derived by just considering the functional scalar product exposedin Eq. (1.4) of $f(x)$ with the constant identity function $g(x)=1$; that is,

$$\begin{aligned}
\langle g \mid f\rangle &= \int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)dx \\
&= \int_{-\frac{L}{2}}^{\frac{L}{2}} \left\{\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{L}\right)+b_n\sin\left(\frac{n\pi x}{L}\right)\right]\right\}dx \\
&= a_0\frac{L}{2},
\end{aligned} \tag{1.9}$$

and hence

$$a_0=\frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)dx \tag{1.10}$$

In a very similar manner, the other coefficients can be computed by considering $\left\langle \cos\left(\frac{k\pi x}{L}\right) \mid f(x)\right\rangle$ $\left\langle \sin\left(\frac{k\pi x}{L}\right) \mid f(x)\right\rangle$ and exploiting the *orthogonality relations for sines and cosines*

$$\begin{aligned}
&\int_{-\frac{L}{2}}^{\frac{L}{2}} \sin\left(\frac{k\pi x}{L}\right)\cos\left(\frac{l\pi x}{L}\right)dx=0, \\
&\int_{-\frac{L}{2}}^{\frac{L}{2}} \cos\left(\frac{k\pi x}{L}\right)\cos\left(\frac{l\pi x}{L}\right)dx=\int_{-\frac{L}{2}}^{\frac{L}{2}} \sin\left(\frac{k\pi x}{L}\right)\sin\left(\frac{l\pi x}{L}\right)dx=\frac{L}{2}\delta_{kl}.
\end{aligned} \tag{1.11}$$

For the sake of an example, let us compute the Fourier series of

$$f(x)=|x|=\begin{cases}-x, & \text{für } -\pi\le x<0;\\ +x, & \text{für } 0\le x\le \pi.\end{cases}$$

First observe that $L=2\pi$, and that $f(x)=f(-x)$; that is, f is an *even* function of x; hence $b_n=0$, and the coefficients a_n can be obtained by considering only the integration between 0 and π.

For $n=0$,

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} dxf(x)=\frac{2}{\pi}\int_0^{\pi} xdx=\pi.$$

For $n>0$,

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx=\frac{2}{\pi}\int_0^{\pi} x\cos(nx)dx= \\
&= \frac{2}{\pi}\left[\frac{\sin(nx)}{n}x\Big|_0^{\pi}-\int_0^{\pi}\frac{\sin(nx)}{n}dx\right]=\frac{2}{\pi}\frac{\cos(nx)}{n^2}\Big|_0^{\pi}= \\
&= \frac{2}{\pi}\frac{\cos(n\pi)-1}{n^2}=-\frac{4}{\pi n^2}\sin^2\frac{n\pi}{2}=\begin{cases}0 & \text{for even } n\\ -\frac{4}{\pi n^2} & \text{for odd } n\end{cases}
\end{aligned}$$

Thus,

$$\begin{aligned}
f(x) &= \frac{\pi}{2}-\frac{4}{\pi}\left(\cos x+\frac{\cos 3x}{9}+\frac{\cos 5x}{25}+\cdots\right)= \\
&= \frac{\pi}{2}-\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\cos[(2n+1)x]}{(2n+1)^2}.
\end{aligned}$$

One could arrange the coefficients $(a_0, a_1, b_1, a_2, b_2, \ldots)$ into a tuple (an ordered list of elements) and consider them as components or coordinats of a vector spanned by the linear independent sine and cosine functions which serve as a basis of an infinite dimensional vector space.

1.0.3 Exponential Fourier series

Suppose again that a function is periodic in the interval $[-\frac{L}{2}, \frac{L}{2}]$ with period L. Then, under certain "mild" conditions – that is, f must be piecewise continuous and have only a finite number of maxima and minima – f can be decomposed into an *exponential Fourier series*

$$\begin{aligned} &f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}, \text{ with} \\ &c_k = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x')e^{-ikx'}dx'. \end{aligned} \tag{1.12}$$

The expontial form of the Fourier series can be derived from the Fourier series (1.6) by Euler's formula (**??**), in particular, $e^{ik\varphi} = \cos(k\varphi) + i\sin(k\varphi)$, and thus

$$\cos(k\varphi) = \frac{1}{2}\left(e^{ik\varphi} + e^{-ik\varphi}\right), \text{ as well as } \sin(k\varphi) = \frac{1}{2i}\left(e^{ik\varphi} - e^{-ik\varphi}\right).$$

By comparing the coefficients of (1.6) with the coefficients of (1.12), we obtain

$$\begin{aligned} &a_k = c_k + c_{-k} \text{ for } k \geq 0, \\ &b_k = i(c_k - c_{-k}) \text{ for } k > 0, \end{aligned} \tag{1.13}$$

or

$$c_k = \begin{cases} \frac{1}{2}(a_k - ib_k) \text{ for } k > 0, \\ \frac{a_0}{2} \text{ for } k = 0, \\ \frac{1}{2}(a_{-k} + ib_{-k}) \text{ for } k < 0. \end{cases} \tag{1.14}$$

Eqs. (1.12) can be combined into

$$f(x) = \frac{1}{L}\sum_{\check{k}=-\infty}^{\infty}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x')e^{-i\check{k}(x'-x)}dx'. \tag{1.15}$$

1.0.4 Fourier transformation

Suppose we define $\Delta k = 2\pi/L$, or $1/L = \Delta k/2\pi$. Then Eq. (1.15) can be rewritten as

$$f(x) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x')e^{-ik(x'-x)}dx'\Delta k. \tag{1.16}$$

Now, in the "aperiodic" limit $L \to \infty$ we obtain the *Fourier transformation*

$$\begin{aligned} &f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x')e^{-ik(x'-x)}dx'dk, \text{ whereby} \\ &f(x) = \alpha\int_{-\infty}^{\infty}\tilde{f}(k)e^{\pm ikx}dk, \text{ and} \\ &\tilde{f}(k) = \beta\int_{-\infty}^{\infty} f(x')e^{\mp ikx'}dx'. \end{aligned} \tag{1.17}$$

$\tilde{f}(k)$ is called the *Fourier transform* of $f(x)$. *Per* convention, either one of the two sign pairs $+-$ or $-+$ must be chosen. The factors α and β must be chosen such that

$$\alpha\beta = \frac{1}{2\pi}; \tag{1.18}$$

that is, the factorization can be "spread evenly among α and β," such that $\alpha = \beta = 1/\sqrt{2\pi}$, or "unevenly," such as, for instance, $\alpha = 1$ and $\beta = 1/2\pi$, or $\alpha = 1/2\pi$ and $\beta = 1$.

Let us compute the Fourier transform of the Gaussian

$$f(x) = e^{-x^2} \quad .$$

Hint: e^{-t^2} is analytic in the region $-k \le \operatorname{Im} t \le 0$; also, as will be shown in Eq. (**??**),

$$\int_{-\infty}^{\infty} e^{-t^2}\,dt = \pi^{1/2} \quad .$$

$$\begin{aligned}
\tilde{f}(k) &= \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} e^{-x^2} e^{ikx}\,dx = \text{ (completing the exponent)} \\
&= \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} e^{-\frac{k^2}{4}} e^{-\left(x-\frac{i}{2}k\right)^2}\,dx
\end{aligned}$$

The variable transformation $t = x - \dfrac{i}{2}k$ yields $dt = dx$ and

$$\tilde{f}(k) = \frac{e^{-\frac{k^2}{4}}}{\sqrt{2\pi}} \int\limits_{-\infty-i\frac{k}{2}}^{+\infty-i\frac{k}{2}} e^{-t^2}\,dt$$

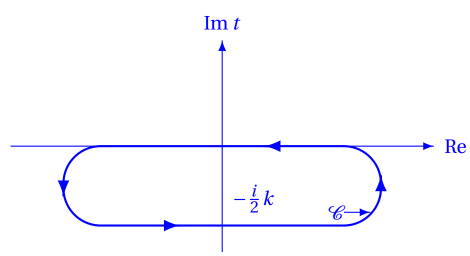

Figure 1.1: Integration path to compute the Fourier transform of the Gaussian.

$$\oint\limits_{\mathscr{C}} dt e^{-t^2} = \int\limits_{+\infty}^{-\infty} e^{-t^2}\,dt + \int\limits_{-\infty-\frac{i}{2}k}^{+\infty-\frac{i}{2}k} e^{-t^2}\,dt = 0,$$

because e^{-t^2} is analytic in the region $-k \le \operatorname{Im} t \le 0$. Thus,

$$\int\limits_{-\infty-\frac{i}{2}k}^{+\infty-\frac{i}{2}k} e^{-t^2}\,dt = \int\limits_{-\infty}^{+\infty} e^{-t^2}\,dt,$$

and

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} e^{-\frac{k^2}{4}} \underbrace{\int\limits_{-\infty}^{+\infty} dt e^{-t^2}}_{\sqrt{\pi}} = \frac{e^{-\frac{k^2}{4}}}{\sqrt{2}}.$$

Bibliography

Russell Herman. *Introduction to Fourier and Complex Analysis with Applications to the Spectral Analysis of Signals*. University of North Carolina Wilmington, Wilmington, NC, 2010. URL `http://people.uncw.edu/hermanr/mat367/FCABook/Book2010/FTCA-book.pdf`. Creative Commons Attribution-NoncommercialShare Alike 3.0 United States License.

Kenneth B. Howell. *Principles of Fourier analysis*. Chapman & Hall/CRC, Boca Raton, London, New York, Washington, D.C., 2001.

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