Noncontextual coloring of orthogonality hypergraphs

Mohammad H. Shekarriz[*]

Department of Mathematics, Yazd University, 89195-741, Yazd, Iran

Karl Svozil[†]

Institute for Theoretical Physics, Vienna University of Technology,

Wiedner Hauptstrasse 8-10/136, 1040 Vienna, Austria

(Dated: September 22, 2021)

We discuss chromatic constructions on orthogonality hypergraphs which are classical set representable or have a faithful orthogonal representation. The latter ones have a quantum mechanical realization in terms of intertwined contexts or maximal observables. Structure reconstruction of these hypergraphs from their table of two-valued states is possible for a class of hypergraphs, namely perfectly separable hypergraphs. Some examples from exempt categories that either cannot be reconstructed by two-valued states or whose set of two-valued states does not yield a coloring are presented.

I.   COLORINGS OF QUANTUM CONTEXTS

Graph and hypergraph representations of (intertwining) configurations of physical observables facilitate a reliable, compact identification and recognition of important properties of these collections. For instance, proofs of (extended) Kochen-Specker theorem, in its various forms [1–4], can be structurally represented compactly by (hyper)graphs.

The chromatic properties of such (hyper)graphs are directly related to their (non)classical aspects. For instance, in quantum logic [5] certain colorings—with the number of colors equal to the clique number that can be identified with the Hilbert space dimension—can be "reduced" to

---
[*] mhshekarriz@yazd.ac.ir

[†] svozil@tuwien.ac.at; http://tph.tuwien.ac.at/~svozil

or "collapsed" into two-valued dispersionless states, which in turn are interpretable as noncontextual classical truth assignments of the elementary propositions represented by vertices of such graphs. If the chromatic number exceeds the Hilbert space dimension then no uniform, global two-valued state and also no corresponding uniform, global classical truth value assignment exists. Thereby, the chromatic number signifies important properties and (non)existence of classical interpretations of the aforementioned configurations of observables; in particular, arrangements of observables realizable by quantum means.

One example of the usefulness of these reduction techniques is the construction of a dense yet discontinuous coloring of the rational sphere [6]—with the resulting states and propositions represented by unit vectors with rational coordinates—based on Pythagorean triples [7]. Thereby, a two-valued dispersionless state can be straightforwardly obtained by identifying all but one colors of colorings of the sphere [8].

The usefulness of colorings is not restricted to reductions of the colors but can be extended to certain properties of operators. The chromaticity of observables within a given context—which in quantum mechanics can be essentially identified with an orthonormal basis of the associated finite-dimensional Hilbert space—can be associated with particular types of spectral forms of maximal operators [9, § 84] formed by the elements of the contexts. Thereby, unit vectors are interpreted as the orthogonal projection operators formed by the respective dyadic products. Those orthogonal projection operators within any given context are mutually orthogonal and can be inserted into the spectral sum of a (normal, or, more specifically) self-adjoint operator. The respective (real) eigenvalues can be identified with some real-valued encoding of the color or chromaticity of the element of the context. Thereby a uniform way of defining (intertwining) context identifiable with maximal quantum operators representing quantum observables is obtained.

If the chromatic number equals the dimension of the Hilbert space—which is, at the same time, identical to the clique number of the graph—this yields a uniform way of defining (maximal) observables even among complementary observables and physical properties (associated with nonidentical contexts). However, if the chromatic number exceeds the dimension of Hilbert space, the construction yields entirely new potential features of observables: if one insists in the uniform global simultaneous (yet counterfactual [10]) existence of observables within such structures, then consistency dictates the abandonment of uniform eigenvalues associated with observables within such contexts. This holds even though the chromatic number of a (hyper)graph and its respective encoding by real values for the spectral sum within a single such context cannot exceed the

dimension of the pertinent Hilbert space. As of today, neither such "omni-realistic" escape or generalization of the Kochen-Specker theorem has been discussed nor observed; so we might assume that it is an inapplicable option. And yet, it is feasible in principle.

In what follows we shall first develop the necessary nomenclature and then present some criteria and results with regards to the reconstruction of (hypergraphs) representing logics—aka, collections of (intertwined) contexts containing a uniform number of observables. We shall also find criteria for the algorithmic (constructive) generation of colorings by the set of two-valued states on such (hyper)graphs or logics.

II. NOMENCLATURE

The following terms are used by authors synonymuously: context, block, (maximal Boolean) subalgebra, (maximal) clique, complete subgraph and *hyperedge*. The same is applied to atom, element and *vertex*.

Greechie has suggested [11] to (amendments are indicated by square brackets "[...]")

> [...] present [...] lattices as unions of [contexts] intertwined or pasted together in some fashion [...] by replacing, for example, the 2^n elements in the Hasse diagram of the power set of an n-element set with the [context aka] complete [sub]graph [K_n] on n elements. The reduction in numbers of elements is considerable but the number of remaining "links" or "lines" is still too cumbersome for our purposes. We replace the [context aka] complete [sub]graph on n elements by a single smooth curve (usually a straight line) containing n distinguished points. Thus we replace $n(n+1)/2$ "links" with a single smooth curve. This representation is propitious and uncomplicated provided that the intersection of any pair of blocks contains at most one atom.

In what follows, we shall refer to such a structure as Greechie diagram [12] (References [13–18] contain variants thereof). The Greechie diagrammatical representation of such, possibly intertwined, collection of blocks, is a *hypergraph*, which is a well-known structure in discrete mathematics. We shall briefly introduce the required terms here, but an interested reader might take a look at 7 [17] for further theory of hypergraphs.

A hypergraph H is an ordered pair $H = (V; E)$ where $V = V(H)$ is the set of vertices and $E = E(H)$ is a family of subsets of V called the hyperedges. It depicts a collection of quantum contexts

faithfully represented in an *n*-dimensional Hilbert space, whereby the vertices are identified with elementary quantum propositions whose label assignments are in terms of vectors (or with their respective one-dimensional orthogonal projection operators), and the hyperedges are identified with quantum contexts. An *n*-subset of atoms forms a hyperedge if its elements are mutually orthogonal.

Let $H = (V; E)$ be a hypergraph. The *2-section* of H is a graph, denoted by $[H]_2$, whose vertices are the same as $V(H)$, and two distinct vertices form an edge if and only if they are on the same hyperedge of H. One may think of the 2-section of a hypergraph as the graph associated with the hypergraph. A hypergraph H is called *conformal* if any maximal clique (with respect to the inclusion) of its 2-section $[H]_2$ is on a hyperedge of H. If all the hyperedges of a hypergraph H consist of exactly *n* elements, then H is called *n-uniform* [17].

Quantum orthogonality hypergraphs are conformal *n*-uniform. Some authors referred to them by their associated graphs, i. e., their 2-sections. Quantum contexts are represented by hyperedges of these quantum hypergraphs, that is, by the maximal cliques of their 2-sections. We also reserve the letter *n* for the clique number of those 2-sections, so we always mean $n = \omega([H]_2)$, which is a constant integer. Now, to formally state a constraint that has to be met by an orthogonality hypergraph H on *k* vertices, we remind the reader that a *vertex labeling* of a (hyper)graph H is a function $f : V(H) \longrightarrow L$ that assigns labels from a set L to vertices of H.

Criterion 1. There is a vertex labeling $\boldsymbol{x} : V(H) \longrightarrow L$ that assigns a set of *k* mutually non-colinear vectors in an *n*-dimensional Hilbert space L to vertices of H such that any pair of vertices *a* and *b* are adjacent if and only if $\boldsymbol{x}(a)$ is orthogonal to $\boldsymbol{x}(b)$.

Such a vertex labeling is called an *n-dimensional faithful orthogonal representation* [19–21] (FOR) for H.

To represent an orthogonality hypergraphs, we shall concentrate on Greechie diagrams which are pasting [13] constructions [22, Chapter 2] of a homogeneous single type of contexts K_n where the clique number *n* is fixed. This means that every hyperedge is shown by a straight line segment or, more general, by a smooth curve which has exactly *n* elements as nodes on it.

A hypergraph coloring of H is a *proper vertex coloring* which associates colors to vertices of H so that every two vertices lying on a hyperedge receive different colors. That is, the *n* distinguished points of any single smooth curve in the hypergraph have *n* different colors. The coloring is noncontextual; that is, the coloring of atomic elements common to two or more contexts

(intertwining there) is independent of the context.

For a hypergraph H, the *chromatic number* $m = \chi(H)$ is the minimum number of colors required for a proper coloring of vertices of H. Obviously the clique number $n = \omega([H]_2)$ is a lower bound. If these numbers are the same, that is, if $m = \chi(H) = \omega([H]_2) = n$, then one could obtain two-valued measures from colorings by "projecting" one of the colors into the value 1, and all the other $n - 1$ colors into the value 0 [6–8]. A hypergraph H, whose chromatic and clique numbers are equal, is called here *semi-perfect*.

Finite examples for which the chromatic number exceeds the clique number, that is, $m > n$, are the logical structures involved in proofs of the Kochen-Specker theorem. Explicit constructions are, for instance, Γ_2 of Ref. [1], as well as the configurations enumerated in Figure 9 of [14], Figure 1–3 of [23], Ref. [2], as well as Table I, Figure 2 of Ref. [4], among numerous others which have a faithful orthogonal representation [19–21] in "small dimensions" greater than two.

A logic has a *separable set of two-valued states* if for any pair a_i and a_j, $i \neq j$, of different propositions, there is at least one two-valued state, say t, such that $t(a_i) \neq t(a_j)$ [14]. Such a logic is called *unitary* if for each proposition a there is a state t for which we have $t(a) = 1$. A hypergraph H is said to be *separable* (respectively unitary) if there is a logic with a separable (resp. unitary) set of two-valued states on its vertices.

A "true implies false set" (a, b)-TIFS (gadget [24–26]) is a conformal n-uniform orthogonality hypergraph H containing two vertices a and b such that for all two-valued states of H, we have b is true only if a is false, and conversely, a is true only if b is false. That is, the relation is symmetrical: a exclusive or b true implies b exclusive or a false, respectively.

Similarly, a "true implies true set" (c, d)-TITS (gadget) is a conformal n-uniform orthogonality hypergraph H' containing two vertices c and d such that for all two-valued states of H', we have d is true whenever c is true [18] (in this case the converse need not be true as both c and d could be false, or d could be true and c false). That is, c true implies d true.

The hypergraphs considered here are assumed to be on k vertices, and all their hyperedges (contexts) uniformly contain exactly n vertices. Equivalently, we can assume that our objects are connected graphs like G on k vertices with clique number $\omega(G) = n$, with the assumption that every vertex $v \in V(G)$ lies on at least one maximal clique (context) of size n. This assumption is not always necessary, but it is not harmful to our argument, as we can always add vertices to those contexts that have less number of vertices. Therefore, we can state *the completion criterion* as follows:

Criterion 2. If *a* and *b* are adjacent in an orthogonality hypergraph *H*, then there is a hyperedge that contains *a* and *b* along with $n-2$ other vertices.

In other words, the completion criterion says that there must be no hyperedge with less than *n* vertices. The importance of this simple criterion becomes clear in Section IV B.

III.   PRELIMINARY CONJECTURES REGARDING THE CHROMATIC CONSTRUCTION FROM TWO-VALUED STATES

In this section we consider two statements. The first one is about the relation between a hypergraph's semi-perfectness and existence of a certain number of two-valued states that induce a partition logic on its vertex set.

Conjecture 3. *The following statements are equivalent:*

(i) The hypergraph H is semi-perfect, i.e., its chromatic number and 2-section clique number are equal.

(ii) The set of two-valued states contains n members which correspond to, or induce, a partitioning of all elements of the partition logic; the equivalence relation defined by each one of these n states evaluating to 1 on some element of every context. That is, those n states are 1 on different atoms of every context.

This does not exclude the existence of partition logics which are not semi-perfect. Indeed, in general, their chromatic number can exceed their 2-section's clique number. A concrete example is Greechie's G_{32} [11, Figure 6, p. 121] mentioned in the Appendix, Section B, and depicted in Figure 11.

Another statement is about the possibility of reconstructing the hypergraph from its table of two-valued states—its Travis [27] matrix [22]—if it is separable. For two-valued states, whenever a vertex is assigned true, all other vertices adjacent to it have to be assigned false. This, which is called later the adjacency criterion, is the most evident property that every pair of adjacent vertices has. For a nonseparable hypergraph, however, this does not let us to reconstruct the hypergraph because for two distinct vertices *a* and *b* with $t(a) = t(b)$ for every two-valued state *t*, it cannot be understood whether or not *a* is adjacent to all the neighbors of *b*.

Conjecture 4. *A hypergraph H is reconstructable from the table of its two-valued states if and only if H is separable.*

Knowing that an orthogonality hypergraph is reconstructable from its logical states gives us the opportunity to find perpendicular pairs of atoms without actually measuring angles between them. We examine these two conjectures in sections IV and V.

IV. STRUCTURE RECONSTRUCTION FROM TWO-VALUED STATES

Suppose we have a table of all two-valued states of a hypergraph H. Under what condition(s) can we reconstruct the graph theoretical structure of H? We saw that Conjecture 4 claims that whenever H is separable, we may be able to reconstruct its structure from the set of its two-valued states. This is what we are going to examine here.

A. Perfectly separable hypergraphs

In this section, we introduce a type of hypergraphs for which Conjecture 4 holds.

Let H be a hypergraph with $V(H) = \{a_1, a_2, \ldots, a_k\}$, and its set of two-valued states contains s elements (which can be shown by saying that $nTS(H) = s$). Its Travis matrix $T(H) = [t_{ij}]_{s \times k}$ enumerates all two-valued states which are represented by row vectors on the vertices of H, arranged in the columns, such that each vertex corresponds to one column. Then, *separability* of H can be extended to the following statement:

Definition 5. The hypergraph H is *perfectly separable* if and only if for all pairs a_i and a_j of vertices of H we get:

1. There is an $r_1 \in \{1, \ldots, s\}$ such that $t_{r_1 i} = 0$ and $t_{r_1 j} = 1$.

2. There is an $r_2 \in \{1, \ldots, s\}$ such that $t_{r_2 i} = 1$ and $t_{r_2 j} = 0$.

3. If a_i and a_j are not adjacent in H, then there is an $r_3 \in \{1, \ldots, s\}$ such that $t_{r_3 i} = 1$ and $t_{r_3 j} = 1$.

4. If a_i and a_j are not adjacent in H, then there is an $r_4 \in \{1, \ldots, s\}$ such that $t_{r_4 i} = 0$ and $t_{r_4 j} = 0$.

Note that a hypergraph is separable if item 1 or 2 of Definition 5 holds. Therefore, every perfectly separable orthogonality hypergraph is also separable, but the converse is not always true. Elementary counterexamples are true-implies-false gadgets such as the Specker bug depicted in Figure 8.

B. Reconstruction of perfectly separable hypergraphs

We know that whenever a_i and a_j are adjacent in H (or, equivalently, they are on the same context), then $t_{ri} = 1$ implies $t_{rj} = 0$ for $r = 1, \ldots, s$. We want to know under what conditions the converse is true as well, i.e., the following *adjacency criterion* holds:

Criterion 6. for all $r = 1, \ldots, s$, if $t_{ri} = 1$ implies $t_{rj} = 0$ (which means its contraposition, $t_{rj} = 1$ implies $t_{ri} = 0$, is also true) then a_i and a_j are adjacent.

The next theorem asserts that when H is a perfectly separable hypergraph, we can always reconstruct H from the information presented in its Travis matrix $T(H)$ and the adjacency criterion. And conversely, a reconstructable hypergraph using Criterion 6 has to be a perfectly separable hypergraph. In other words, Conjecture 4 is true for perfectly separable hypergraphs.

Theorem 7. *Let H be a hypergraph on $\{a_1, a_2, \ldots, a_k\}$ whose Travis matrix $T(H)$ is available and $\omega([H]_2) = n \geq 3$. Moreover, suppose that H' is the hypergraph on $\{a_1, a_2, \ldots, a_k\}$ whose adjacency is defined by Criterion 6. Then $H = H'$ if and only if H is perfectly separable.*

Proof. First suppose that H is perfectly separable. Then, because of item 3 of Definition 5, for all a_i and a_j that are not adjacent in H there is a row r in $T(H)$ such that $t_{ri} = 1$ and $t_{rj} = 1$. Therefore, Criterion 6 in H' can only be satisfied for those vertices that are already adjacent in H. Consequently, $H = H'$.

Conversely, suppose that $H = H'$ and a_i and a_j are two distinct non-adjacent vertices. Then the adjacency criterion does not meet for a_i and a_j. As a result, there is an r, $1 \leq r \leq s$ such that $t_{ri} = 1$ and $t_{rj} = 1$, i.e., item 3 of Definition 5 already holds for a_i and a_j. Moreover, we have the following two cases:

Case 1. *Item 1 (or 2) of Definition 5 does not hold for i and j.* Therefore, the adjacency criterion makes a_i adjacent to all vertices already adjacent to a_j, which cannot happen unless $a_i = a_j$, or a_i is colinear with a_j, which is a contradiction.

Case 2. *Item 4 of Definition 5 does not hold of i and j.* Therefore, whenever $t_{ri} = 0$ we have $t_{rj} = 1$ and when $t_{rj} = 0$ we have $t_{ri} = 1$. We distinguish the following two cases:

Case 2.1. *Both t_{ri} and t_{rj} are 1 for all $r = 1, \ldots, t$.* Therefore, a_i is adjacent to all $n - 1 \geq 2$ vertices of a context a_j lies on, which in its turn means that a_i and a_j are colinear, a contradiction.

Case 2.2. *At least one of t_{ri} or t_{rj} is 0 for some $r = 1, \ldots, t$.* Without loss of generality, let it be t_{ri} that is sometimes 0. Therefore, there is an a_l on a context containing a_i that is assigned 1 in a two-valued state t_r while it is not on any contexts containing a_j which is also assigned 1 by t_r. As a result, all other vertices on a context containing a_j must be adjacent to a_l and consequently, a_l and a_j must be colinear, a contradiction.

As we see, both these cases lead to complete contradictions. Therefore, for every pair of non-adjacent vertices of H, items 1 to 4 of Definition 5 hold. If for any pair of adjacent vertices in H, namely a_i and a_j, also both items 1 and 2 of Definition 5 are true, then it can be inferred that H must be perfectly separable.

To see this, first notice that if one of items 1 or 2 does not hold for adjacent vertices a_i and a_j, then one them, say a_j, is always assigned 0. Therefore, because $H = H'$, and H' is constructed via Criterion 6, a_j must be adjacent to any vertex that is assigned 1 at least once. Hence, we can distinguish the following two cases:

Case 1. *The vertex a_j belongs to all contexts of H.* Then there is a two-valued state that assigns 1 to a_j and 0 to all other vertices, a contradiction to our assumption that a_j is always assigned 0.

Case 2. *There is at least one context, $\mathscr{A}$, which does not contain a_j.* Since $H = H'$, only one vertex of $\mathscr{A}$, say a_p, can be assigned always 0. Then a_j is adjacent to at least $n - 1$ vertices of $\mathscr{A}$ other than a_p, and consequently, a_j and a_p are colinear, a contradiction.

Therefore, items 1 and 2 of Definition 5 also hold for every pair of adjacent vertices of H. Now the conclusion is evident. □

From the proof of Theorem 7 we see that, for a reconstructable hypergraph H, the most important factor is that at least item 1 or 2 is true for every pair of non-adjacent vertices. It should be noted that item 2 is actually the contraposition for item 1. Yet conversely, if we only want to imply Criterion 6, we see that for separable hypergraphs which are not perfectly separable, the
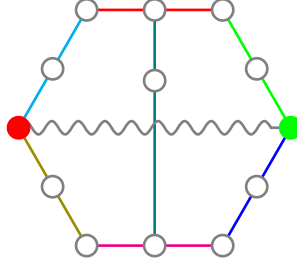
Figure 1. Reconstruction of the Specker bug using Criterion 6 which gives the gray snake-shaped extra hyperedge that contains only two (the red and green) vertices. This extra hyperedge can be easily eliminated if we use Criterion 2.

reconstruction produces some extra hyperedges for H' and makes it different from H. For example, when we try to reconstruct the Specker bug from its two-valued states using the adjacency criterion, we produce an extra hyperedge on two vertices, as depicted in Figure 1.

However, one might think that we may be able to reconstruct H from H' by finding the extra hyperedge(s) that must be eliminated. To do this, suppose that $\omega([H]_2) = n \geq 3$ and every vertex is on a hyperedge of size n. Then the reconstructed H' must have the same property, i.e., we must have Criterion 2 for all hyperedges of H', and if a hyperedge does not meet this criterion, then it has to be eliminated. These "unwanted" adjacency between a_i and a_j in H' appear only when these two are not on the same context but whenever one is assigned 1 the other has to be 0 and vice versa—a typical situation in a TIFS gadget—like what is happening when we try to reconstruct the Specker bug from its two-valued states. However, we show in the next section that this method cannot always guarantee that we can identify these unwanted hyperedges from the table of two-valued states.

C. Examples of non-reconstructable hypergraphs using Criteria 2 and 6

In this section we want to give a counterexample to Conjecture 4: a hypergraph that is separable but not perfectly separable. We construct the hypergraph $B(H)$ of the dimension $n = 3$, but for higher dimensions the construction should be analogous.

A counterexample to Conjecture 4, say H, is an orthogonality hypergraph that has some non-adjacent vertices but Criteria 2 and 6 detect them as they are on a hyperedge in the reconstructed hypergraph H'; and consequently these two hypergraphs are not the same. Note that, up to per-
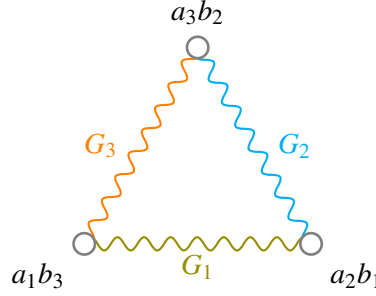
Figure 2. A layer hypergraph serving as a quasi-block for the construction of a counterexample to Conjecture 4. For $i = 1, 2, 3$, the snake-shaped edges are distinct copies of an arbitrary separable (a_i, b_i)-TIFS gadget G such that whenever a_i is assigned true by a two-valued state, b_i has to be false: their respective ends have been "cyclically folded" on each other, eg., a_1 from G_1 is identified with b_3 from G_3. Note that vertices a_1b_3, a_2b_1 and a_3b_2 are not adjacent.

mutations of rows and columns, their Travis matrices are the same. Since our dimension is 3, we must have at least 3 vertices a, b and c which are not adjacent in H, but whenever one is assigned true by a two-valued state, the other two have to be assigned false. Moreover, there must not be a two-valued state that simultaneously assigns a, b and c false.

Suppose G is an (a, b)-TIFS gadget such that whenever a is assigned true by a two-valued state, b has to be false. As mentioned earlier, the reverse is also true; that is, whenever b is assigned true by a two-valued state, a has to be false. This means that a and b cannot be assigned true at the same time (but they can both be false). Therefore, if we make a triangle, using a true implies false set (TIFS) as its edges, then we have the desired property that whenever one end is true, the other two are false. However, this hypergraph might have several two-valued states that assign false to all its three ends a, b and c. Figure 2 depicted this layer hypergraph, with $a = a_1b_3$, $b = a_2b_1$ and $c = a_3b_2$.

For an (a_i, b_i)-TIFS gadget G, Criteria 2 and 6 do not detect a hyperedge containing a_1b_3, a_2b_1 and a_3b_2 from the two-valued states of a hypergraph like Figure 2. This is because usually there are some two-valued states that assign false to all of these three vertices. Therefore, the layer hypergraph of Figure 2 must become a part of a larger hypergraph so that in every two-valued state, exactly one of the vertices a, b or c be assigned true.

To do so, one possibility is to use three copies of the layer hypergraph of Figure 2 and use three extra contexts to bind them together; a configuration drawn in Figure 3. When we use G as an
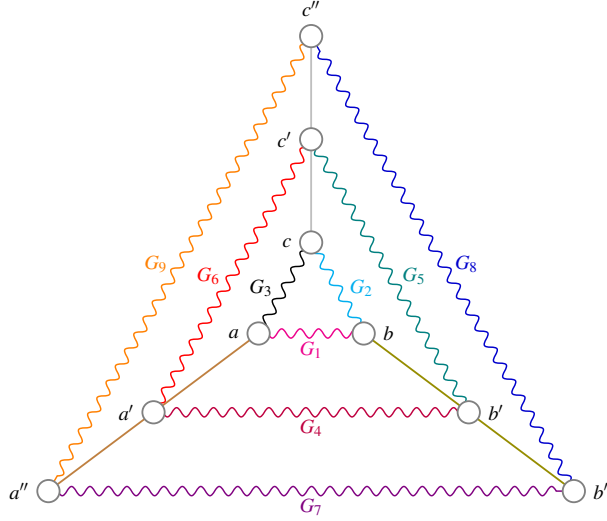
Figure 3. A hypergraph $B(G)$ depicting a counterexample to Conjecture 4. For $i = 1, \ldots, 9$, the snake-shaped curves indicate different copies of an (a_i, b_i)-TIFS gadget G with their terminals suitably identified, that is, a_1 from G_1 is identified with b_3 from G_3 as the vertex a. Straight lines are ordinary hyperedges, i. e., vertices a, a' and a'' are on a context, drawn in brown.

(a_i, b_i)-TIFS gadget the resulting hypergraph of this construction is denoted by $B(G)$.

Since G is a TIFS gadget, in every two-valued state of $B(G)$, exactly one of a, b or c has to be assigned true while the other two have to be false. This is because $\{a, a', a''\}$, $\{b, b', b''\}$ and $\{c, c', c''\}$ are contexts, and has to have exactly one true value in every two-valued states. Consequently, if for example b and c are assigned false by a two-valued state t_r, then one of b' and b'', and one of c' and c'' has to be assigned true by t_r. Without loss of generality, let b' be the true one and b'' be false. then since G_4 and G_5 are TIFS gadgets, it can be inferred that a' and c' have to be assigned false by t_r. Therefore, c'' has also to be assigned true because c and c' are false. Again, since G_9 is a TIFS gadget, a'' has to be false. Now, a has to be assigned true by t_r because a' and a'' are both false.

Therefore, for nonadjacent vertices a, b and c we have exactly one of them has to be assigned true, and the other two are false. This is just like if these three vertices are on the same context. It means that if one tries to reconstruct $B(G)$ from its table of two-valued states using criteria 2 and 6, he finds extra hyperedges of $\{a, b, c\}$, $\{a', b', c'\}$ and $\{a'', b'', c''\}$. This implies that $B(G)$ is not reconstructable using these criteria.

One question that arises in this regard is the following: suppose we know that G is separable,

then does this imply that $B(G)$ is also separable? This is not too hard to answer, and it is always "yes" if it does not contain a true implies true set (TITS) gadget.

To proceed, we need one more thing. A function $f : S \longrightarrow T$ defined on a subset $S \subsetneq X$ is said to be *lifted* to $\tilde{f} : X \longrightarrow T$ if $\tilde{f}(a) = f(a)$ for each $a \in S$. It must be mentioned that when f possesses a property, like if f is a proper coloring [28] or a two-valued state, there might not always be a lift with the same property.

Lemma 8. *Let G be a separable unitary TIFS gadget which does not contain a TITS gadget and* $\omega([G]_2) = 3$. *Then B(G) is also separable.*

Proof. Since G is unitary, there is no vertex that has to be assigned 0 by all the two-valued states of G. On the other hand, there is no vertex that is given 1 by all the two-valued states of G because else it must lie on at least one hyperedge with two other vertices, those that have to be always assigned 0, a contradiction to separability (and being unitary) of G. Thus, it can be inferred that for a vertex u of G, there are states φ and ψ such that $\varphi(u) = 0$ and $\psi(u) = 1$.

For an (a, b)-TIFS gadget G, the set of two-valued states are nonempty. Let s_1 be the set of those states in which a is true, s_2 be the set of those states that b is true and s_3 be the set of those states in which both a and b are false. Then these sets s_1, s_2 and s_3 partition the set of two-valued states of G. For an (a, b)-TIFS gadget copy G_i, these sets are shown here by s_{i1}, s_{i2} and s_{i3}. Therefore, if for example a two-valued state is in s_1, it is a two-valued state of G so that its head, i. e. a, is assigned 1.

To show that $B(G)$ is separable, we show that for any pair of distinct vertices x and y, there is a two-valued state of it that gives them different values. There are the following cases:

Case 1. *x and y belong to the same copy of G, say G_i.* Because of the symmetry, we can assume without loss of generality that $i = 1$. Since G is separable, there must be a two-valued state, say t on G, such that $t(x) \neq t(y)$. We know that $t \in s_1 \cup s_2 \cup s_3$. If $t \in s_l$ for $l = 1, 2, 3$, then there is a two-valued state for the underlying hypergraph of $B(G)$ in Section A 4 such that it agrees with the values of $t(a)$ and $t(b)$. Now, using this two-valued state we define $\tilde{t}$ for the end vertices a, b, c, a', b', c', a'', b'' and c''. Then using appropriate two-valued states of G, we can find a suitable two-valued state for internal vertices of $G_2, \ldots, G_9$. Therefore, there are two-valued states such as $\tilde{t}$ of $B(G)$ which is a lifting for t and $\tilde{t}(x) \neq \tilde{t}(y)$.

Case 2. *x and y lie on different copies of G, say G_i and G_j respectively.* Then there are another three cases:

Case 2.1. *G_i and G_j lie on the same layer of $B(G)$.* Without loss of generality, suppose that it is the layer consisting of the vertices a, b and c. Then G_i and G_j has a common vertex that again without loss of generality, we can assume that it is a. Let G_i's head and tail be a and b, and G_j's head and tail be c and a, respectively (see Figure 4 (a)). Therefore, every two-valued state of the induced subhypergraph $G_i \cup G_j$ in $B(G)$ is a member of $s_{i1} \cup s_{j2}$, $s_{i2} \cup s_{j3}$ or $s_{i3} \cup s_{j1}$. Suppose on contrary that x and y receive the same value by all two-valued states of $B(G)$. Then, since there is a two-valued state of $B(G)$ that assigns 1 to x, at least one of the following statements holds:

1. If x is assigned 1 by a two-valued state of s_{i1}, then y has to be assigned 1 by all two-valued states of s_{j2}. This means that G_j (and therefore G) is a (a,y)-TITS gadget, a contradiction to our assumption.

2. If x is assigned 1 by a two-valued state of s_{i2}, then y has to be assigned 1 by all two-valued states of s_{j3}. Consequently, x cannot be assigned 0 by a s_{i2}, because else, it can be lifted to the required separation of x and y. Hence, G_i (and therefore G) is a (b,x)-TITS gadget, a contradiction to our assumption.

3. If x is assigned 1 by a two-valued state of s_{i3}, then y has to be assigned 1 by all two-valued states of s_{j1}. This means that G_j (and therefore G) is a (c,y)-TITS gadget, again a contradiction to our assumption.

Case 2.2 *G_i and G_j lie on different layers of $B(G)$, and both ends of G_i and G_j lie on the same contexts of $B(G)$.* Without loss of generality suppose that G_i has a and b and G_j has a' and b' as their heads and tails, respectively (see Figure 4 (b)). Therefore, every two-valued state of the induced subhypergraph $G_i \cup G_j$ in $B(G)$ is a member of $s_{i1} \cup s_{j2} \cup s_{j3}$, $s_{i2} \cup s_{j1} \cup s_{j3}$ or $s_{i3} \cup s_{j1} \cup s_{j2}$ (or $s_{j1} \cup s_{i2} \cup s_{i3}$, $s_{j2} \cup s_{i1} \cup s_{i3}$ or $s_{j3} \cup s_{i1} \cup s_{i2}$ which are completely similar). Suppose on contrary that x and y receive the same value by all two-valued states of $B(G)$. Then, since there is a two-valued state of $B(G)$ that assigns 1 to x, at least one of the following statements holds:

1. If x is assigned 1 by a two-valued state of s_{i1}, then y has to be assigned 1 by all two-valued states of $s_{j2} \cup s_{j3}$. This means that G_j (and therefore G) is a (b',y)-TITS gadget, a contradiction to our assumption.

2. If x is assigned 1 by a two-valued state of s_{i2}, then y has to be assigned 1 by all two-valued states of $s_{j1} \cup s_{j3}$. Consequently, x cannot be assigned 0 by a s_{i2},

because else, it can be lifted to the required separation of x and y. Hence, G_i (and therefore G) is a (b,x)-TITS gadget, a contradiction to our assumption.

3. If x is assigned 1 by a two-valued state of s_{i3}, then y has to be assigned 1 by all two-valued states of $s_{j1} \cup s_{j2}$. This means that G_j (and therefore G) is a (b',y)-TITS gadget (and also a (c',y)-TITS gadget), again a contradiction to our assumption.

Case 2.3 G_i *and* G_j *lie on different layers of* $B(G)$, *but only one end from* G_i *and one end from* G_j *lie on the same context.* Again without loss of generality, suppose that G_i has a and b and G_j has b' and c' as their heads and tails, respectively (see Figure 4 (c)). Therefore, every two-valued state of the induced subhypergraph $G_i \cup G_j$ in $B(G)$ are members of $s_{i1} \cup s_{j1} \cup s_{j2} \cup s_{j3}$, $s_{i2} \cup s_{j2} \cup s_{j3}$, or $s_{i3} \cup s_{j1} \cup s_{j2} \cup s_{j3}$ (or $s_{j1} \cup s_{i1} \cup s_{i2} \cup s_{i3}$, $s_{j2} \cup s_{i2} \cup s_{i3}$, or $s_{j3} \cup s_{i1} \cup s_{i2} \cup s_{i3}$ which can be treated similarly). Suppose on contrary that x and y receive the same value by all two-valued states of $B(G)$. Then, since there is a two-valued state of $B(G)$ that assigns 1 to x, at least one of the following statements holds:

1. If x is assigned 1 by a two-valued state of s_{i1}, then y has to be assigned 1 by all two-valued states of $s_{j1} \cup s_{j2} \cup s_{j3}$, a contradiction to the fact that no vertex of G can be assigned 1 by all the two-valued states.

2. If x is assigned 1 by a two-valued state of s_{i2}, then y has to be assigned 1 by all two-valued states of $s_{j2} \cup s_{j3}$. Consequently, G_j (and therefore G) is a (c',y)-TITS gadget, a contradiction to our assumption.

3. If x is assigned 1 by a two-valued state of s_{i3}, then y has to be assigned 1 by all two-valued states of $s_{j1} \cup s_{j2} \cup s_{j3}$, a contradiction to the fact that no vertex of G can be assigned 1 by all the two-valued states.

We showed that in any case, x and y can be separated by a two-valued state of $B(G)$ (or else there is a contradiction) which concludes the proof. $\square$

We can go further by calculating the number of two-valued states of $B(G)$ based on G's. Suppose that G is an (a,b)-TIFS and has respectively n_a, n_b and n_n two-valued states that give a true, b true and none of a and b true. In other words, $n_a = |s_1|$, $n_b = |s_2|$ and $n_n = |s_3|$. Then, by the elementary methods of counting, the number of two-valued states of $B(G)$ is

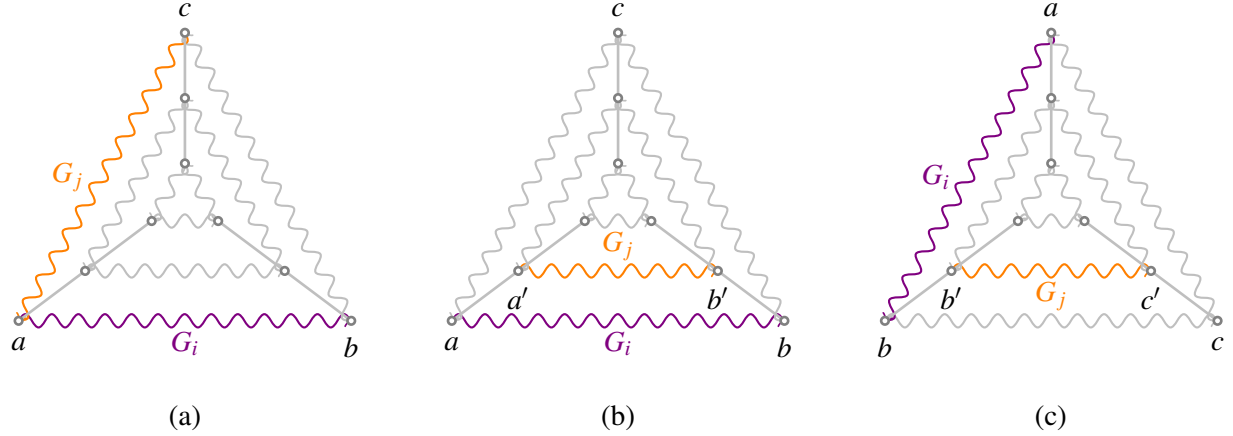$$nTS(B(H)) = (3+2+1) \cdot n_a^3 \cdot n_b^3 \cdot n_n^3. \tag{1}$$

Figure 4. Illustrations of cases 2.1 (a), 2.2 (b) and 2.3 (c) in the proof of Lemma 8.

For the first layer we have three choices for an end vertex to be true, for the second layer we only have two such choices and for the third layer only one such a choice we have got. That is why we have 3+2+1=6 choices.

For the sake of an example take the Specker bug G discussed in the Appendix Section A 3. From its Travis matrix A3 we know that $n_a = 3$, $n_b = 3$ and $n_n = 8$. Therefore, Formula 1 implies that $B(G)$, which is a separable hypergraph on 108 vertices and 66 contexts, has

$$6 \cdot 3^3 \cdot 3^3 \cdot 8^3 = 2,239,488$$

two-valued states, a number that can easily be checked via an ordinary computer.

However, it is not difficult to show that this hypergraph $B(G)$, when G is the Specker bug, does not meet our requirements. This is because the two ends of the Specker bug cannot be orthogonal in the 3-space [29]. It can also be discussed using graph theoretical terminology; one orthogonality hypergraph cannot have a cycle of length 4 because else any pair of antipodal vertices of the cycle of length 4 have to be colinear. Therefore, even if $B(G)$ is an orthogonality hypergraph, the reconstructed hypergraph with an extra context of $\{a,b,c\}$ is certainly not.

We have to find a TIFS gadget H, other than the Specker bug, in which the distance between its two ends are not 3 (so the two ends can be orthogonal in the reconstructed hypergraph). A candidate for such a hypergraph is shown in Figure 5, which is a TIFS on 43 vertices whose end-points (say a and b) are far enough, so that it is not only separable, but also probably a FOR. This hypergraph has 2589 two-valued states, 45 of which assign a true, 504 give b true and 2040 give both a and b false. In other words, for the hypergraph H we have $n_a = 45$, $n_b = 504$ and $n_n = 2048$.
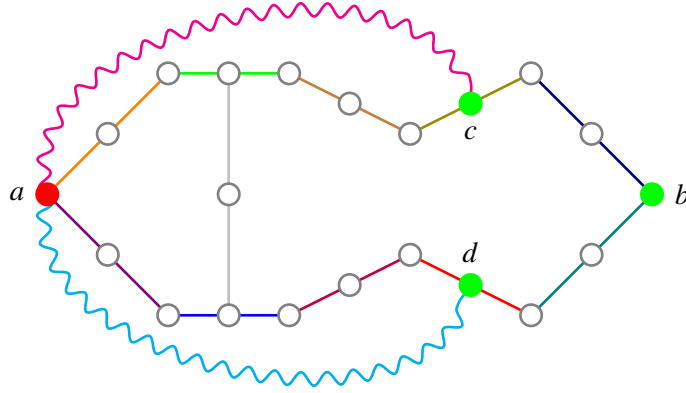
Figure 5.   An (a,b)-TIFS gadget whose distance between its two terminal points a and b is at least five contexts. The snake-like decorated curves indicate Specker bugs, so this hypergraph has 43 vertices.

If we construct $B(H)$, then Lemma 8 implies that the resulting hypergraph on 378 vertices is separable and there would be enough space for the three end vertices to be perpendicular.

The hypergraph $B(H)$ is on 378 vertices and 228 contexts, and by Formula 1 it has

$$6 \cdot 45^3 \cdot 504^3 \cdot 2040^3 = 594,252,343,817,330,688,000,000$$

two-valued states. This huge number makes it hard to quickly check by an ordinary computer.

Another pertinent problem is to show that these hypergraphs—namely $B(H)$ and also its counterpart, the reconstructed hypergraph $B(H)'$ from $B(H)$'s table of two-valued states—have a faithful orthogonal representation; and to enumerate an explicit example of such a representation.

However, it seems that Criterion 1 is independent from Criteria 2 and 6, having an example such as $B(H)$ on 378 vertices raises the possibility that Conjecture 4 is false for separable hypergraphs that are not perfectly-separable.

V.   TWO-VALUED STATES VS COLORING

A.   Coloring vs independent partitions

A hypergraph (whose vertices all lie on at least one hyperedge of size $\omega([H]_2) = n$) is called to have "an n-partition system" if there is a vertex partition $\mathscr{S} = \{S_1, \ldots, S_n\}$ of H (into exactly n

cells) with the following properties:

1. whenever $v, w \in S_i$ for an $i = 1, \ldots, n$, we have v and w are not adjacent, and

2. for each $i = 1, \ldots, n$ and every $v \in V(G)$, there is a vertex $w \in S_i$ such that either $v = w$ or v and w are adjacent.

When a hypergraph H has an n-partition system, we might simply say that H is n-partitionable. While $\mathscr{S}$ is a partition, every S_i is non-empty and $\bigcup_{i=1}^{n} S_i = V(G)$. With Property 1, we can be sure that if we assign the true value to all the vertices in S_i and the false value to the rest of vertices, then no two true vertices are adjacent, and consequently, every context has at most one true valued vertex. Moreover, Property 2 assures us that there is no context without a true-valued vertex. (Using graph theoretical terminology, these two properties mean that every S_i is an *independent dominating set* for H.) Therefore, an n-partition system actually induces n two-valued states on H.

We have the following theorem.

Theorem 9. *A hypergraph H is n-colorable if and only if it is n-partitionable.*

Proof. It is evident that if σ is a proper vertex coloring of G with $\{1, \ldots, n\}$, we can easily find $\mathscr{S} = \{S_1, \ldots, S_k\}$ by putting

$$S_i = \{v \in V(G) \ : \ \sigma(v) = i\}.$$

It is also clear that property 1 holds because σ is a proper coloring. Moreover, we also have property 2 because of the assumption we made that, in $[H]_2$, every vertex is on a maximal clique of size n.

To prove the converse, suppose that we have a partition $\mathscr{S} = \{S_1, \ldots, S_n\}$ of vertices of H which satisfies properties 1 and 2 above. Define, $\sigma : V(G) \longrightarrow \{1, \ldots, n\}$ such that

$$\sigma(v) = i \text{ if } v \in S_i.$$

While property 2 implies that every S_i is non-empty, for $i = 1 \ldots, n$, it shows also that every hyperedge contains a vertex v such that $\sigma(v) = i$. In other words, every color $i = 1, \ldots, k$ is used in each hyperedge. Furthermore, property 1 implies that every S_i is an independent set. Hence σ is a proper coloring of H and consequently, H is n-colorable. $\qquad\qquad\square$

As a result, we can say that for each proper n-coloring of H we have n different two-valued states on vertices of H. Conversely we can construct exactly $n!$ proper n-colorings for H from an available n-partition system on vertices of H. Therefore, Conjecture 3 is a corollary of Theorem 9.

B. Reconstructing coloring from logical assignments

From Theorem 9 we know that when there is an n-coloring for an orthogonality hypergraph H, there are n two-valued states corresponding to it so that they induce a partition logic. In other words, there are n rows in the Travis matrix of H such that when one of them assigns 1 to a vertex u, the rest of them assign 0 to u. Consequently, if $t_{s_1}, \ldots, t_{s_n}$ are the rows of the Travis matrix $T(H)$ corresponding to an n-coloring and $t_{s_i} + t_{s_j}$ be the element-wise sum of t_{s_i} and t_{s_j} in $\mathbb{Z}_2$, then this yields the $|V(H)|$–tuple whose entries are one:

$$\sum_{i=1}^{n} t_{s_i} = \Big(\underbrace{1,1,\ldots,1}_{|V(H)| \text{ times}} \Big) = \Big(\mathbf{1}_{1 \times |V(H)|} \Big). \tag{2}$$

Therefore, when H is n-colorable, there is at least one set of n two-valued states that induce a partition logic on H.

Algorithm 1 searches for such states when $T(H)$ is available. It

(1) takes the Travis matrix $T(H)$ and the clique number n, and

(2) gives a list of rows, A, from which we can retrieve an n coloring for H.

Variables of this algorithm are

(1) *AvailableRows* which is a list of active rows in $T(H)$, with each such row representing a two-valued state of H,

(2) i which runs from 1 to the clique number n,

(3) j which runs from 1 to the number of two-valued states $nTS(H)$ which is the number of rows of the Travis matrix, and

(4) *RemovedRows* which is a list of lists, whose ith element is the rows of $T(H)$ that become inactive at the ith step of filling A.

Algorithm1 Finding an n-coloring for H from its set of two-valued states encoded by the Travis matrix

---

    Input: $T(H), n$ $\triangleright$ Travis matrix, clique number

    Output: A $\triangleright$ a list of n rows of $T(H)$

1: $i \leftarrow 1$ $\triangleright$ start of variable initialization

2: $AvailableRows \leftarrow (1, \ldots, nTS(H))$

3: $A \leftarrow (\ )$

4: $RemovedRows \leftarrow (\ )$ $\triangleright$ end of variable initialization

5: **while** $i \leq n$ and $(i \neq 1$ or $AvailableRows \neq \emptyset)$ **do** $\triangleright$ try all colors

6:     **if** $AvailableRows = \emptyset$ **then** $\triangleright$ start over again if all two-valued states are exhausted

7:         $Append\ RemovedRows[i]$ to $AvailableRows$

8:         $Remove\ RemovedRows[i]$ from $RemovedRows$

9:         $Remove\ A[i]$ from A

10:       $i \leftarrow i - 1$

11:     **else** $\triangleright$ try to identify a new color assignment by the next available two-valued state

12:         $j \leftarrow$ first available cell in $AvailableRows$

13:         $A[i] \leftarrow AvailableRows[j]$

14:         $Append\ AvailableRows[j]$ to $RemovedRows[i]$

15:         $Remove\ AvailableRows[j]$ from $AvailableRows$

16:         $i \leftarrow i + 1$

17:         $Append$ to $RemovedRows[i]$ all $AvailableRows[s]$ for which there is a vertex u

                such that the state of rows $AvailableRows[s]$ and $AvailableRows[j]$ both assign 1 to u

18:         $Remove$ all elements of $RemovedRows[i]$ from $AvailableRows$

19:     **end if**

20: **end while**

If the output of Algorithm 1 has less than n elements, then H has no admissible n-coloring—because else Theorem 9 guarantees that there are n two-valued states partitioning the logic, in which case Algorithm 1 would have given $|A| = n$. If $|A| = n$, then A is a list of n rows in $T(H)$,

each of which corresponds to a color class of an n-coloring of H. In other words, when $s \in A$, the two-valued state t_s presents the color class consisting all the vertices it assigns, or maps to, 1. It is evident that the resulting color classes are independent sets while Formula 2 implies that they cover all the vertices. Consequently, A induces a proper n-coloring on vertices of H.

Algorithm 1 is not highly efficient in finding an n-coloring for H. The main reason is that in the worst case study it has to check all the two-valued states of H whose number, i.e. $nTS(H)$, can grow exponentially in terms of the clique number, number of vertices and hyperedges.

Moreover, one could conjecture that Algorithm 1 could be modified to render a coloring even if the (hyper)graph is not n-partitionable, in which case Theorem 9 does not apply. Because even if one has exhausted all combinations of two-valued states one could still attempt to "complete" the coloring by identifying the missing colors with "suitable segments" of the remaining two-valued states (if there are any leftovers). Of course in this way the column sums of all the respective two valued states cannot be 1; and hence formula (2) is no longer valid. In any case, Brooks' theorem [30, 31]—stating that for any connected undirected graph G the chromatic number of G is at most its maximum degree (the maximal number of edges that are incident to some vertex) Δ unless G is a complete graph or an odd cycle, in which case the chromatic number is $\Delta + 1$—and its generalization to hypergraphs [17, page 45, Theorem 3.2] yield an upper bound for the chromatic number of such (hyper)graphs.

VI.   SUMMARY AND CONCLUDING REMARKS

We have presented a constructive, algorithmic way to generate a coloring of a (hyper)graph from its set of two-valued states. By this, we have been able to find a "compact" partition logic within the logical states of the hypergraph. The only criterion for a success of this approach is by ascertaining that the respective hypergraph is semi-perfect, that is, its chromatic number equals to the clique number of its 2-section.

With regard to representing and reconstructing (hyper)graphs or logics in terms of their two-valued states, in particular, regarding separability of vertices or elementary propositions, we conjecture that there exist quantum logics with a separable set of two-valued states that cannot be reconstructed from these states. We have presented a hypergraph, namely $B(G)$ depicted in Figure 3 in Section IV C with a TIFS gadget such as the one depicted in Figure 5, that has this characteristic but we could not find a faithful orthogonal representation in a Hilbert space.

Yet, stronger forms of separability, in particular, perfect separability, can be identified that allow (hyper)graphs or logics to be represented (by their Travis matrices) and reconstructed in terms of their two-valued states. And while the conditions on perfectly separable (hyper)graphs are rather strong, one can be certain that such a reconstruction exists.

Indeed, such a reconstruction helps to directly identify mutually perpendicular elementary propositions, and thus the contexts corresponding to the maximal operators they form: if an orthogonality (hyper)graph is reconstructable from its set of two-valued states we can deduce the mutual orthogonality of the elementary quantum propositions by just looking at these two-valued states. This facilitates the construction of the (mutually perpendicular) orthogonal operators in the spectral sums associated with the contexts, and thus supports finding a global faithful orthogonal representation, i. e., the assignment of vectors to vertices, of (hyper)graphs.

So while for perfectly separable (hyper)graphs we can be certain that they can be reconstructed; and for Kochen-Specker type (hyper)graphs that they cannot be reconstructed, because there is no two-valued state associated with any classical value assignment, for the remaining (hyper)graphs reconstructability remains an open question.

Appendix A: Examples

1.   Triangle logic

The coloring procedure of the triangle hypergraph is depicted in Figure 6. Consider the set of all four two-valued states on the six atoms which can be tabulated by a (compactified) Travis matrix T_{ij} whose rows indicate the ith state s_i and whose columns indicate the atoms a_j, respectively; that is, $T_{ij} = s_i(a_j)$:

$$T_{ij} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}. \tag{A1}$$

It is not too difficult to see that the first three measures, represented by the first three row vectors of the Travis matrix, add up to $\left(1,1,1,1,1,1\right)$. They can thus be taken as the basis of a coloring.
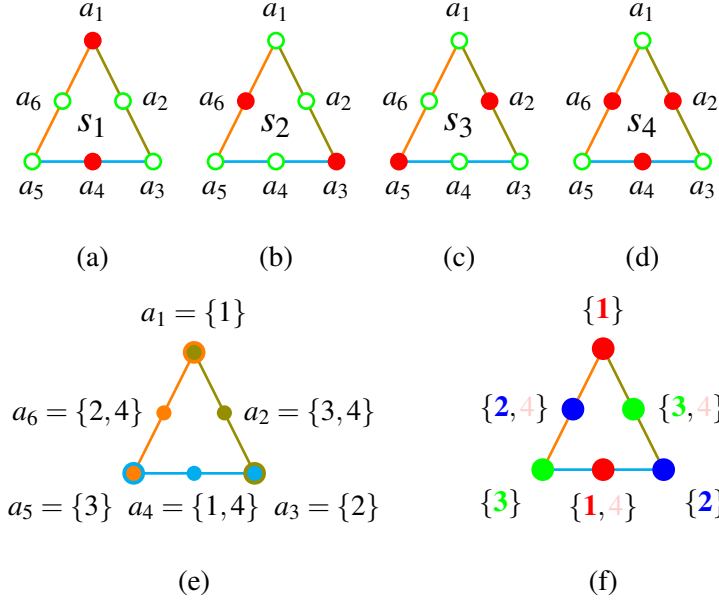
Figure 6. One (nonunique) coloring (f) construction of the triangle hypergraph of the logic: first compose a (nonunique) canonical partition logic (e) from enumerating the set of all 4 two-valued states depicted in (a)–(d). Then choose the context $\{a_1, a_2, a_3\}$, and from this context choose the atom $a_1 = \{1\}$. Now identify the first color (red) with the index 1, thereby identifying $a_1 = \{1\}$ as well as $a_4 = \{1,4\}$ with red. Then delete the index number 4 from every atom; that is, $a_2 = \{3,4\} \to \{3\}$ and $a_6 = \{2,4\} \to \{2\}$. Finally identify 3 with the second color (green) and 2 with the third color (blue), thereby identifying a_2 and a_5 with green, and a_3 and a_6 with blue, respectively. Note that s_1, s_2, and s_3 "generate" a 3-partitioning of the set of atoms $\{a_1, \ldots, a_6\}$ of this logic.
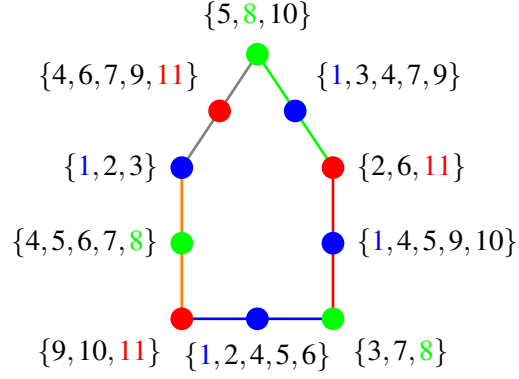
Figure 7. Coloring scheme of the house or pentagon or pentagram logic from the set of two-valued states.

2. House or pentagon or pentagram logic

The Travis matrix of the house or pentagon or pentagram logic is a matrix representation of its 11 dispersion free states [32]

$$
T_{ij} = \begin{pmatrix}
1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1
\end{pmatrix}. \tag{A2}
$$

A coloring can be constructed with the earlier mentioned construction which results in three states partitioning all 10 atoms. The associated 1st, the 8th and the 11th row vectors of T_{ij} are partitioning the 10 atoms.

3. Specker bug gadget

The hypergraph depicted in Figure 8 is a minimal [18] true-implies false gadget introduced by Kochen and Specker [33, Fig. 1, p. 182] (see also [34, Figure 1, p. 123], among others). It is a subgraph of G_{32} introduced later in Figure 11. Its Travis matrix is

$$T_{ij} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}. \tag{A3}$$

4. The underlying hypergraph of $B(G)$

All the two-valued states of the structure introduced in Figure 3 have to assign the vertices a, b, c, a', b', c' and a'', b'', c'' the same combination of two-valued states as for the hypergraph of Figure 9.

The Travis matrix of this tightly bi-connected hypergraph is as follows (columns from left to
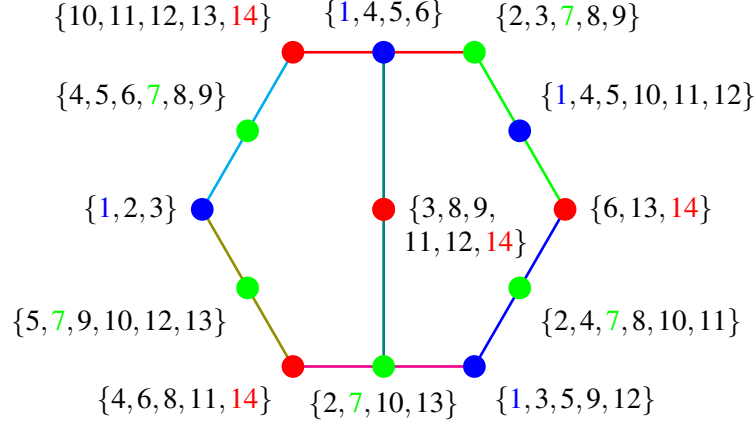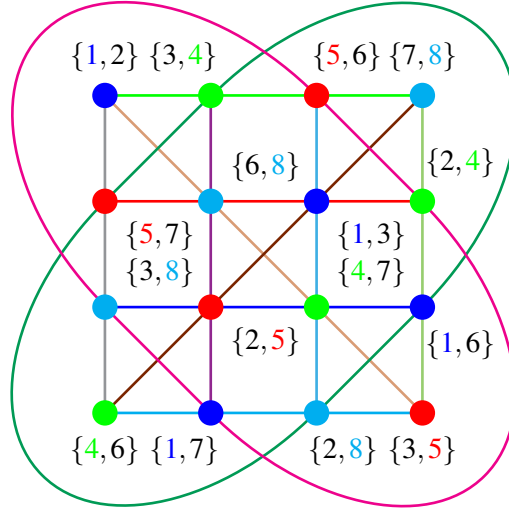
Figure 8. Coloring scheme of the "Specker bug" gadget [33, 34] from two-valued states. The set-theoretic representation is in terms of the canonical partition logic as an equipartitioning of the set $\{1, 2, \ldots, 14\}$ obtained from all 14 two-valued states on this gadget.
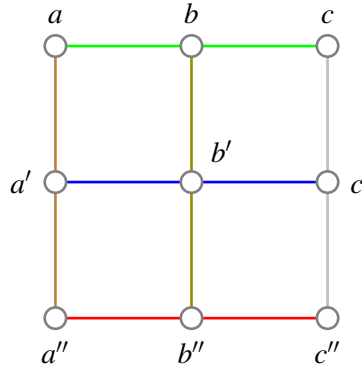
Figure 9. From the point of view of two-valued states, for which all the nine G_i's, $1 \le i \le 9$ are TIFS, the three "new" contexts $\{a, b, c\}$, $\{a', b', c'\}$ and $\{a'', b'', c''\}$ are formed through three triples of TIFS $\{G_1, G_2, G_3\}$, $\{G_4, G_5, G_6\}$, and $\{G_7, G_8, G_9\}$, respectively; thereby rendering a tightly bi-connected hypergraph underlying the one depicted in Figure 3. Note that the vertices of each row in the original graph of $B(G)$ do not lie on a context, but here in the underlying hypergraph they are.

right correspond to a, b, c, a', b', c', a'', b'' and c''):

$$T_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}. \tag{A4}$$

5. "Tight GHZ" logic

The hypergraph depicted in Figure 10 is a sublogic of the observables in the Greenberger-Horn-Zeilinger setup [35]. Its Travis matrix is

$$T_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{A5}$$

The coloring is depicted in Figure 10.

Appendix B: A counterexample: Greechie's G_{32}

It is quite straightforward to demonstrate that the logic G_{32} introduced by Greechie [11, Figure 6, p. 121] (see also Refs. [34, 36–38]) whose hypergraph is depicted in Figure 11(a) has a chromatic number larger than three; and, in particular, while having a separating and a unital set of two-valued states, cannot be colored by two-valued states in the algorithmic way proposed

Figure 10.   Coloring scheme of the "tight GHZ" logic [35] from two-valued states. The set-theoretic representation is in terms of the canonical partition logic as an equipartitioning of the set $\{1, 2, \ldots, 8\}$ obtained from all eight two-valued states on this gadget.

earlier. Consider the set of all six two-valued states which can be tabulated by the Travis matrix

$$T_{ij} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}. \tag{B1}$$

There is no way how three of these six row vectors add up to a vector whose components are all one; that is, $\left(1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1 \right)$. "Completing" the partition logic and "extending" G_{32} by adding five more contexts $\{\{1, 2\}, \{3, 6\}, \{4, 5\}\}$, $\{\{1, 4\}, \{2, 3\}, \{5, 6\}\}$, $\{\{1, 3\}, \{2, 5\}, \{4, 6\}\}$, $\{\{1, 5\}, \{2, 6\}, \{3, 4\}\}$, and $\{\{1, 6\}, \{2, 4\}, \{3, 5\}\}$ does not change the set of two-valued states and thus the Travis matrix.

Another way of seeing this is to associate a color to, say, the first state. As a consequence, all other states, namely states number 2, 3, 4, 5, and 6, need to be eliminated, leaving no state which can be associated with another color.

One possibility for finding a proper coloring is to drop "exclusivity", or rather, the unique asso-
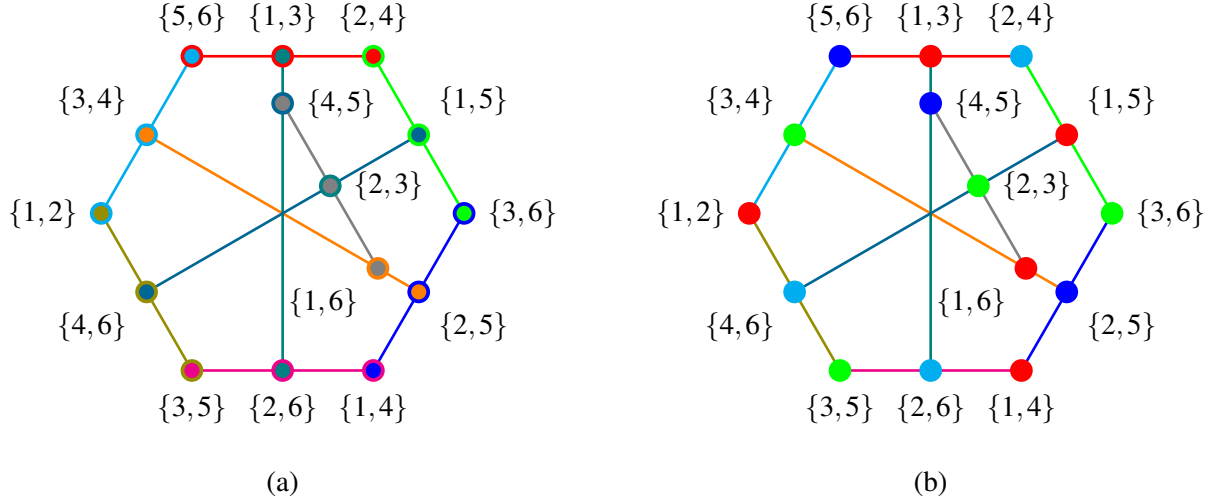
Figure 11. (a) Greechie diagram of G_{32} introduced by Greechie [11, Figure 6, p. 121]. The overlaid set theoretic representation is in terms of the canonical partition logic as an equipartitioning of the set $\{1,2,3,4,5,6\}$ obtained from all 6 two-valued states on G_{32}; (b) a nonunique coloring by four colors.

ciation of two-valued states with colors; but not entirely. This can be achieved by not eliminating two-valued states if they appear in association with previous colors. A construction identifying state numbers 1 with red, 3 with green, 5 with blue, and then 2 or four with cyan is depicted in Figure 11(b).

G_{32} cannot have a faithful orthogonal representation because of the following proof by contradiction: Suppose G_{32} has a faithful orthogonal representation. Then each one of the nine biconnected contexts of G_{32} can be uniformly represented by a maximal operator [9, § 84, p. 171,172]. In order for a faithful orthogonal representation to exist the spectral decomposition of two "intertwining" maximal operators must have (i) (at least) one common projector (ii) with identical eigenvalues which can be identified with identical colors. If the logic can be consistently "covered" or colored by three colors then the eigenvalues associated with the maximal operators can be the same–that is, these three values (or colors) would occur uniformly in all nine contexts. But this is not the case for G_{32}. Therefore, a uniform representation cannot be be given in terms of nine maximal operators with just three eigenvalues per context (that is, maximal operator). This is a form of nonclassicality based on a chromatic number exceeding the 2-section's clique number. In this case Brooks' theorem [30, 31] yields an upper bound of 4 for the chromatic number of G_{32}.

ACKNOWLEDGMENTS

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

---

[1] Simon Kochen and Ernst P. Specker, "The problem of hidden variables in quantum mechanics," Journal of Mathematics and Mechanics (now Indiana University Mathematics Journal) **17**, 59–87 (1967).

[2] Adán Cabello, José M. Estebaranz, and G. García-Alcaine, "Bell-Kochen-Specker theorem: A proof with 18 vectors," Physics Letters A **212**, 183–187 (1996), arXiv:quant-ph/9706009.

[3] Ehud Hrushovski and Itamar Pitowsky, "Generalizations of Kochen and Specker's theorem and the effectiveness of Gleason's theorem," Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics **35**, 177–194 (2004), arXiv:quant-ph/0307139.

[4] Alastair A. Abbott, Cristian S. Calude, and Karl Svozil, "A variant of the Kochen-Specker theorem localising value indefiniteness," Journal of Mathematical Physics **56**, 102201 (2015), arXiv:1503.01985.

[5] Garrett Birkhoff and John von Neumann, "The logic of quantum mechanics," Annals of Mathematics **37**, 823–843 (1936).

[6] David A. Meyer, "Finite precision measurement nullifies the Kochen-Specker theorem," Physical Review Letters **83**, 3751–3754 (1999), arXiv:quant-ph/9905080.

[7] Hans Havlicek, Günther Krenn, Johann Summhammer, and Karl Svozil, "Colouring the rational quantum sphere and the Kochen-Specker theorem," Journal of Physics A: Mathematical and General **34**, 3071–3077 (2001), arXiv:quant-ph/9911040.

[8] Chris D. Godsil and J. Zaks, "Coloring the sphere," (1988, 2012), University of Waterloo research report CORR 88-12, arXiv:1201.0486.

[9] Paul Richard Halmos, *Finite-Dimensional Vector Spaces*, Undergraduate Texts in Mathematics (Springer, New York, 1958).

[10] Ernst Specker, "Die Logik nicht gleichzeitig entscheidbarer Aussagen," Dialectica **14**, 239–246 (1960), english translation at https://arxiv.org/abs/1103.4537, arXiv:1103.4537.

[11] Richard Joseph Greechie, "Orthomodular lattices admitting no states," Journal of Combinatorial Theory. Series A **10**, 119–132 (1971).

[12] Gudrun Kalmbach, *Orthomodular Lattices*, London Mathematical Society Monographs, Vol. 18 (Academic Press, London and New York, 1983).

[13] Richard Joseph Greechie, "On the structure of orthomodular lattices satisfying the chain condition," Journal of Combinatorial Theory **4**, 210–218 (1968).

[14] Karl Svozil and Josef Tkadlec, "Greechie diagrams, nonexistence of measures in quantum logics and Kochen–Specker type constructions," Journal of Mathematical Physics **37**, 5380–5401 (1996).

[15] Brendan D. McKay, Norman D. Megill, and Mladen Pavičić, "Algorithms for Greechie diagrams," International Journal of Theoretical Physics **39**, 2381–2406 (2000), quant-ph/0009039.

[16] Mladen Pavičić, Jean-Pierre Merlet, Brendan McKay, and Norman D Megill, "Kochen-Specker vectors," Journal of Physics A: Mathematical and General **38**, 1577–1592 (2005), arXiv:quant-ph/0409014.

[17] Alain Bretto, *Hypergraph theory*, Mathematical Engineering (Springer, Cham, Heidelberg, New York, Dordrecht, London, 2013) pp. xiv+119.

[18] Adán Cabello, José R. Portillo, Alberto Solís, and Karl Svozil, "Minimal true-implies-false and true-implies-true sets of propositions in noncontextual hidden-variable theories," Physical Review A **98**, 012106 (2018), arXiv:1805.00796.

[19] László Lovász, "On the Shannon capacity of a graph," IEEE Transactions on Information Theory **25**, 1–7 (1979).

[20] László Lovász, M. Saks, and Alexander Schrijver, "Orthogonal representations and connectivity of graphs," Linear Algebra and its Applications **114-115**, 439–454 (1989), special Issue Dedicated to Alan J. Hoffman.

[21] Alberto Solís-Encina and José Ramón Portillo, "Orthogonal representation of graphs," (2015), arXiv:1504.03662.

[22] Richard Joseph Greechie, *Orthomodular Lattices*, Ph.D. thesis, University of Florida, Florida, USA (1966).

[23] Josef Tkadlec, "Diagrams of Kochen-Specker type constructions," International Journal of Theoretical Physics **39**, 921–926 (2000).

[24] W. T. Tutte, "A short proof of the factor theorem for finite graphs," Canadian Journal of Mathematics **6**, 347–352 (1954).

[25] Jácint Szabó, "Good characterizations for some degree constrained subgraphs," Journal of Combinatorial Theory, Series B **99**, 436–446 (2009).

[26] Ravishankar Ramanathan, Monika Rosicka, Karol Horodecki, Stefano Pironio, Michał Horodecki, and Paweł Horodecki, "Gadget structures in proofs of the Kochen-Specker theorem," (2020), arXiv:1807.00113.

[27] Raymond David Travis, *The Logic of a Physical Theory*, Master's thesis, Wayne State University, Detroit, Michigan, USA (1962), Master's Thesis under the supervision of David J. Foulis.

[28] Michael O. Albertson, "You can't paint yourself into a corner," Journal of Combinatorial Theory, Series B **73**, 189–194 (1998).

[29] Adán Cabello, *Pruebas algebraicas de imposibilidad de variables ocultas en mecánica cuántica*, Ph.D. thesis, Universidad Complutense de Madrid, Madrid, Spain (1996).

[30] R. L. Brooks, "On colouring the nodes of a network," Mathematical Proceedings of the Cambridge Philosophical Society **37**, 194–197 (1941).

[31] László Lovász, "Three short proofs in graph theory," Journal of Combinatorial Theory, Series B **19**, 269–271 (1975).

[32] Ron Wright, "The state of the pentagon. A nonclassical example," in *Mathematical Foundations of Quantum Theory*, edited by A. R. Marlow (Academic Press, New York, 1978) pp. 255–274.

[33] Simon Kochen and Ernst P. Specker, "Logical structures arising in quantum theory," in *The Theory of Models, Proceedings of the 1963 International Symposium at Berkeley* (North Holland, Amsterdam, New York, Oxford, 1965) pp. 177–189, reprinted in Ref. [39, pp. 209-221].

[34] Richard Joseph Greechie, "Some results from the combinatorial approach to quantum logic," Synthese **29**, 113–127 (1974).

[35] Karl Svozil, "Revisiting the Greenberger-Horne-Zeilinger argument in terms of its logical structure, orthogonality, and probabilities," (2020), arXiv:2006.14623.

[36] Samuel S. Holland, "The current interest in orthomodular lattices," in *The Logico-Algebraic Approach to Quantum Mechanics: Volume I: Historical Evolution*, edited by C. A. Hooker (Springer Netherlands, Dordrecht, 1975) pp. 437–496.

[37] Mary Katherine Bennett, "A finite orthomodular lattice which does not admit a full set of states," SIAM Review **12**, 267–271 (1970).

[38] Richard Joseph Greechie, "Some results from the combinatorial approach to quantum logic," in *Logic and Probability in Quantum Mechanics*, edited by Patrick Suppes (Springer Netherlands, Dordrecht, 1976) pp. 105–119.

[39] Ernst Specker, *Selecta* (Birkhäuser Verlag, Basel, 1990).