True-implies-false, true-implies-true and nonseparable Kochen-Specker sets

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The essential ingredient in Hardy-like proofs of Bell and Kochen-Specker (KS) theorems is a set of propositions about the outcomes of quantum measurements such that, when outcome noncontextuality is assumed, if proposition A is true then, due to exclusiveness and completeness, a nonexclusive proposition B must be false. We call such a set a true-implies-false set (TIFS). The KS theorem can be proved for any quantum state of a system of a given dimension by concatenating a set of propositions such that, if proposition A is true then a nonexclusive proposition C must be true. We call such a set a true-implies-true set (TITS). A nonseparting set (NSS) is one which contains propositions A and C which cannot be separated by any valid truth assignment. Here we identify the TIFSes, TITSes, and NSSes that have the smallest number of propositions for every d-dimensional quantum system with $d \geq 3$.

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I. INTRODUCTION

Every noncontextuality (NC) [? ?] and Boole-Bell type inequality [?] can be represented by a graph in which vertices represent the propositions appearing when the correlations are expressed as a positive linear combination of probabilities of propositions, and exclusive propositions are represented by adjacent nodes [? ?]. Quantum theory (QT) violates NC and Bell inequalities if and only if their exclusivity graphs are imperfect, i.e., contain, as induced subgraphs, odd cycles of length five or more (i.e., pentagons, heptagons, etc.), or their complements [? ? ?]. Moreover, every proof, with or without inequalities, of the Kochen-Specker (KS) [? ? ?] and Bell [?] theorems can be associated to an imperfect graph. Reciprocally, every imperfect graph can be used to prove that QT cannot be explained with noncontextual hidden variable theories [? ?].

Some imperfect graphs allow us to present the conflict between QT and hidden variables in very appealing way: by pointing out a contradiction between a prediction with certainty of hidden variables and the corresponding prediction of QT. Proofs of this type are the ones by Stairs [?], Hardy, and others []. In addition, these imperfect graphs play a fundamental role in quantum state-independent proofs of the KS theorem. See the proof of Bell [?] and Kochen and Specker [?]). In this article we will obtain the simplest of these imperfect graphs.

Hereafter by propositions we will mean statements the type "outcomes a and b will be respectively obtained when observables A and B will be jointly measured on the same physical system", where A and B are assumed to be observables represented in QT by rank-one projectors that commute. We will say that two propositions are compatible if and only if all the observables involved in both propositions are jointly measurable, i.e., represented in QT by operators that commute. Two propositions are exclusive when both cannot simultaneously true. A set of mutually exclusive compatible propositions is complete when one of the propositions must be true. A context is a complete set ~~(complete or not)~~ of mutually exclusive compatible propositions. (Greechie) orthogonality diagrams [?] compactly represent contexts as single smooth lines (such as circles or straight unbroken lines) connecting mutually compatible (atomic) propositions, which are represented as small circles; contexts intertwining at a single proposition are represented as nonsmoothly connected lines, broken at that proposition.

The assumption of outcome noncontextuality assigns the same truth value (true or false) to any proposition with independence of which other compatible propositions are considered simultaneously. QT is in conflict with the assumption of outcome noncontextuality [1–4]. This conflict can be manifested in many ways, and in the following we shall investigate some which are particularly appealing as they involve smaller sets of propositions than configurations for proofs of the KS theorem.

We define (i) a true-implies-false set (TIFS), (ii) a true-implies-true set (TITS), and (iii) a nonseparating

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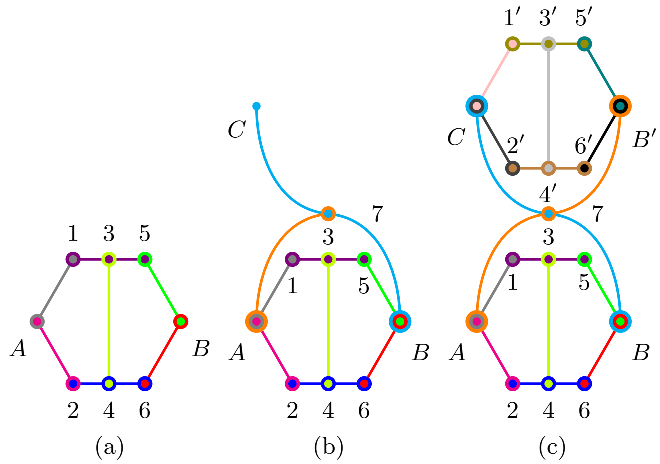


FIG. 1. (Color online) Orthogonality diagram of the simplest known [? ? ? ? ? ?] (a) TIFS, (b) TITS, and (c) NSS in $d=3$. Points represent propositions, smooth lines represent complete sets (i.e., one and only one of the propositions must be true); in particular, they indicate that any pair of propositions connected by a smooth line cannot both be true (exclusiveness). (a) If A is true then B is false [2, Fig. 1, p. 182]. (b) If A is true then C is true [? , Γ_1]. (c) A and C can only be both true or both false [? , Γ_3]. These sets are realizable in S^2 by taking, for instance [? , p. 206, Fig. 1], $A=(1,\sqrt{2},0)/\sqrt{3}$, $v_1=(\sqrt{2},-1,1)/2$, $v_2=(\sqrt{2},-1,-1)/2$, $v_3=(0,1,1)/\sqrt{2}$, $v_4=(0,1,-1)/\sqrt{2}$, $v_5=(\sqrt{2},1,-1)/2$, $v_6=(\sqrt{2},1,1)/2$, $B=(-1,\sqrt{2},0)/\sqrt{3}$, $v_7=(0,0,1)$, $C=(\sqrt{2},1,0)/\sqrt{3}$, $v_{1'}=(-1,\sqrt{2},-1)/2$, $v_{2'}=(-1,\sqrt{2},1)/2$, $v_{3'}=(1,0,-1)/\sqrt{2}$, $v_{4'}=(1,0,1)/\sqrt{2}$, $v_{5'}=(1,\sqrt{2},1)/2$, $v_{6'}=(1,\sqrt{2},-1)/2$. In QT, the proposition v_i is represented by the projector $|v_i\rangle\langle v_i|$. To obtain a TIFS, TITS and NSS in $d=4$ it is enough to add $\langle v|=(0,0,0,1)$, and similarly to obtain TITSes in higher dimensions.

set (NSS) as a set S of propositions represented in QT by rank-one projection operators (or the vectors onto which they project) such that, due to the exclusiveness and completeness of some of the elements of S, when outcome noncontextuality is assumed, if proposition $A\in S$ is true, then a nonexclusive proposition (i) $B\in S$ must be false, (ii) $C\in S$ must be true as well, and (iii) $C\in S$ must be true, as well as *vice versa*, whenever $C\in S$ is true, then A must be true; i.e., A and C cannot be separated with classical probabilities. This is not the same as in (ii), as for (ii) $v(C)=1$ does not imply $v(A)=1$ (in particular, $v(A)=0$ is allowed). Explicit examples for the cases (i–iii) are shown in Fig. 1(a–c), respectively. A TIFS, TITS, or NSS S is said to be critical if the set resulting from removing any element of S is not a TIFS, TITS, or NSS, respectively.

Any TIFS, TITS, or NSS proves quantum contextuality (i.e., the impossibility of explaining QT with models satisfying outcome noncontextuality) because for a system prepared in the quantum state in which proposition A is true there is a nonzero probability of finding proposition B or C true, false, and different, respectively. This is the method followed in the proofs of quantum contextuality proposed by Stairs [5], Clifton [6], and Cabello et al. [7, 8]). All these proofs can be converted into experimental tests of whether nature can be described with noncontextual hidden variable theories [9].

TITSes also serve to prove state-independent quantum contextuality, i.e., quantum contextuality for any initial quantum state of a quantum system of a given dimension $d\geq 3$. A proof of this can be obtained by concatenating several TITSes. This is the method followed by Bell [3] and Kochen and Specker (KS) [4] to prove state-independent quantum contextuality in $d=3$. The same method can be extended to any $d\geq 3$ [10]. State-independent proofs of quantum contextuality have been recently converted into quantum state-independent experimental tests of contextuality [11–17].

TITS in which proposition A corresponds to an entangled state and the rest corresponds to product states can be used to prove quantum nonlocality (i.e., the impossibility of explaining QT with local hidden variable theories). This is exactly what is behind Hardy-like proofs of quantum nonlocality [18, 19] (for a detailed explanation, see Refs. [20, 21]).

These sets, and, in particular, NSS, are interesting because they demonstrate a growing deviation from nonclassicality below the KS theorem: there still exist classical valuations and truth tables, but they get "more pathologic" up to the point where propositional structures with NSS cannot be "embedded" [?] into any kind of hidden parameter model, such as partition logics [?], and their model realizations as Wright's generalized urn model [?] or automaton logic [?] (still allowing logics with TIFS or TITS).

TITSes are known for any physical system described by a Hilbert space of dimension $d\geq 2$ [2–4]. In $d=3$, Bell found one with $n=13$ propositions [3], and KS found one with $n=10$ [2–4], which is illustrated in Fig. 1. Both Bell's and KS's sets belong to a broader family with $n=10+3m$, with $m=0,1,\ldots$ [7]. For $d>3$, TITS with $n=7+d$ are easy to construct from the set of Fig. 1 by adding the vector with all components zero but the one corresponding to the new dimension [10].

An important question is which are the simplest TIFSes, TITSes and NSSes for any $d\geq 3$, i.e., those with a minimum number of propositions. This is the problem we address in this paper.

II. METHOD TO OBTAIN TITS WITH MINIMUM NUMBER OF PROPOSITIONS

A TITS can be represented by a graph in which vertices represent propositions, edges connect exclusive propositions, and d mutually connected vertices represent complete contexts. A graph is said to be nonrealizable in dimension d if it represents a set of rays (i.e., unit vectors) which is not realizable in S^{d-1}.

Lemma 1: The simplest nonrealizable graph in $d = 1$ consists of two vertices. The simplest nonrealizable graph in $d = 2$ has three vertices with one of them connected to the other two. From these to nonrealizable graphs one can recursively construct nonrealizable graphs in any dimension d by starting from the nonrealizable graph in dimension $d-2$ and adding to it to vertices connected with all vertices of the nonrealizable graph in $d-2$.

Proof: One cannot have two different ways in S^0 because there is only one ray in S^0. In S^1, if a ray is orthogonal to a second ray, and the second is orthogonal to the third, then the first and third ray should be the same. If we add two dimensions and two different rays, both orthogonal to those of previous set I, then these two new rays span a two-dimensional subspace orthogonal to I. Therefore, if I was nonrealizable, then the resulting set is also nonrealizable.

Lemma 2: Every n-vertex graph corresponding to a critical TITS in dimension d contains an $(n+1-d)$-vertex graph corresponding to a TIFS.

Proof: Let be G a graph corresponding to a TITS, then $b \in K_d$ is true and any adjacent v must be false. Then the subgraph taken from the other true a to false v forms a TIFS. Moreover, any $v_i \in K_d - 1$ can be taken as false to get a TIFS.

Lemma 3: The graph of a critical TIFS must be biconnected (i.e., when removing any vertex the resulting graph must be still connected).

Proof: Suppose it is not. Cut vertex split graph in two connected components, if true and false vertices are in the same component, then the graph is not critical and if true and no vertices are in different components then cut vertex can substitute anyone of them getting a smaller TIFS.

Corollary 1: Every vertex of a graph corresponding to a TIFS must be connected to, at least, two vertices (i.e., the graph must have minimal valency two).

Proof: Directly from Lemma 3.

Lemma 4: Every TIFS graph in dimension d contains, at least, two K_d graphs.

Proof: Let be G a TIFS graph with true vertex a and false vertex b. G is not trivial TIFS, so there is another true vertex x linked to b and other $d-1$ vertices. Let be C the rest of no vertices in G. We consider two different cases. a) Every vertex in $C - \{x\}$ belongs to $N(a)$ and b is linked to C. Note that in this case $a \cup N(a)$ and $b \cup N(b)$ form two different K_d. b) Only several vertices in $C - \{x\}$ belongs to $N(a)$. We have a set of vertices $v_i \in C - \{x\}$ with value no, but such vertices are not linked directly to a, so their value must be forced through another true vertex out C. This vertex is not linked to a, so it has to belong to a context, i.e. a K_d.

Lemma 5: The graph of any 8-vertex TIFS in $d = 3$ must contain at least two triangles.

Proof: From Lemma 4. Otherwise the graph can be colored as follows: first vertex a true and everything else false.

Therefore, we have a method to obtain the minimal TITS in $d = 3$.

Step 1: We generate all nonisomorphic n-vertex biconnected graphs of minimal valence two, not containing cycles on length four and containing at least two triangles. This can be efficiently done using the computer program nauty [22]. We obtain that there are 2 graphs for $n = 7$, and 8 graphs for $n = 8$. All of them are illustrated in Fig. 2.

Step 2: For every graph obtained after step 1, consider all possible pairs of vertices (v_i, v_j). If, for one (v_i, v_j), the graph does not admit a noncontextual assignment when $v_i = 1$ and $v_j = 1$ (i.e., then both are true), then the graph is a TIFS in which $A = v_i$ and $B = v_j$. The test of whether or not a graph admits a noncontextual assignment can be done using a simple computer program (e.g., [23]).

After step 2, we find that only the last graph in Fig. 2 corresponds to a TIFS. This graph, also depicted in Fig. 1(a), was first introduced by Kochen and Specker [2, Fig. 1, p. 182]; and later used as a subgraph of the graph Γ_1 of Kochen and Specker [**?**], as depicted in Fig. 1(b).

This proves that, in $d = 3$, there is no TIFS with less than or equal number of propositions than the one introduced by Kochen and Specker in 1965 [2, Fig. 1, p. 182] (and then used in Ref. [4]), which means that there is no TITS with less than or equal number of propositions than the one in Fig. 1.

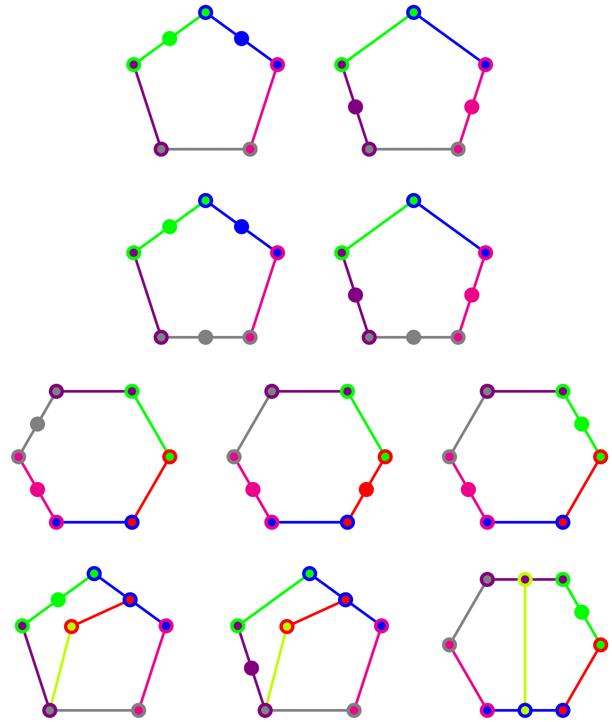


FIG. 2. (Color online) Orthogonality diagrams of all nonisomorphic 7-vertex (first row) and 8-vertex (the rest) biconnected graphs of minimal valence two, not containing cycles on length four, and containing at least two triangles.

Theorem 1: Minimal TITS in dimension $d = 3$ have 10 vertices.

Proof: This show gets a construction method from the TIFS to obtain a minimal TITS in the same dimension. It is sufficient to add two vertices, one true vertex b and other support vertex v_3 with value no. Both vertices must belong to K_d by the constraints of the problem, as much as the rest of vertices connected to b have value no by TIFS included, simply we must force the value of v_3 to be no, which is achieved by simply adding and edge from the true vertex a in the TIFS to vertex v_3.

It will be seen later that this method is applicable to higher dimensions.

Do these aruments also apply for NSS?

III. HIGHER DIMENSIONS

In this section we generalize the results of the previous section to higher dimensions than 3.

Theorem 2: Let be G a critical TIFS graph in dimension d with two K_d, then $V(G) \geq d + 5$.

Proof: For this proof we use the notion of forbidden graph in dimension d for the orthogonal rank of a graph [28]. This concept is related to the impossible representation of a graph with several structure.

For a non trivial TIFS we have to use a context, so the first valid linked structure in dimension d is K_{d-1} linked to true vertex y and false vertex x such that $N(y) = N(x)$. But this graph ($|V(G)| = d + 1$) is not realizable because it form a forbbiden graph in dimension d.

Suppose that we have a graph with one more vertex c ($|V(G)| = d + 2$). Is evident that $y \nsim c$, so c have to belong to a context with other vertex from K_{d-1} (for now call Q). We consider two cases: a) $N(x) = \{v/v \in \mathrm{Q}\}$, Then we have another forbbiden graph between c and x. b) Choose $N(x) \cap N(y) = \emptyset$, in that case we have a forbbiden structure between y and x.

We must consider a new vertex w ($|V(G)| = d + 3$). Previous results shows that neither y or x can belong to the two K_d necessary for the lemma 5, so the structure between y and x must have two K_d, but we only have c and w as internal vertex different of Q. Is not enough to evade a forbidden structure again (both will create forbbiden graph).

We introduce another vertex t ($|V(G)| = d + 4$). This new vertex will support to anyone internal vertex w or c. w.l.o.g. We consider that t support w, i.e. They belong to the same context. Now we have two K_d from $c \cup N(c) = Q$ and $w \cup t \cup Q - q$ where $q \in N(c)$. Here we can observe that again appear forbbiden structure in dimension d between t and q because $N(t) = N(q)$.

Then we must consider $|V(G)| \geq d+5$ to get a minimal TIFS. In fact, we know a structure of TIFS with $d + 5$ vertices obtained from the Fig. 7.

Example: The graph G of Fig. 1, with $V(G) = \{A, v_1, v_2, v_4, v_5, v_6, v_7, v_8\}$, is a critical TIFS graph in dimension 3.

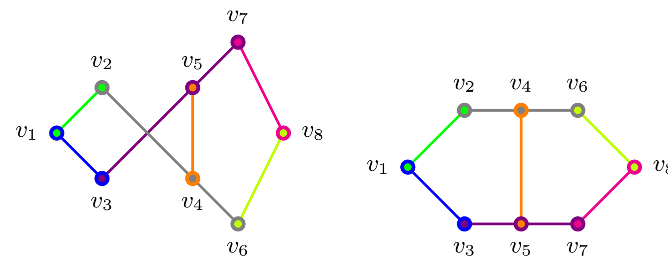


FIG. 3. (Color online) Two equivalent, isomorphic representations of the orthogonality diagram of the the simplest known [**?** , Fig. 1, p. 182] TIFS in $d = 3$, with 7 contexts and 8 atomic propositions. It is realizable in S^2 by taking, for instance, $v_1 = (0, -1, \sqrt{2})/\sqrt{3}$, $v_2 = (1, 0, 0)$, $v_3 = (1, \sqrt{2}, 1)/2$, $v_4 = (0, 1, 0)$, $v_5 = (1, 0, -1)/\sqrt{2}$, $v_6 = (0, 0, 1)$, $v_7 = (-1, \sqrt{2}, -1)/2$, $v_8 = (\sqrt{2}, 1, 0)/\sqrt{3}$. Values on v_1 and v_8 cannot both be 1, as this would imply the values on $v_2 = v_3 = v_6 = v_7 = v_9$ to vanish, which in turn would require both values v_4 as well as v_5 to be 1; which contradicts exclusivity because those observables belong to the same context and are orthogonal. In QT, the proposition v_i is represented by the projector $|v_i\rangle\langle v_i|$.

Theorem 3: Let be G a critical TITS that contain a TIFS graph with $d + k$ vertices, then G have, at least, $d + k + 2$ vertices.

Proof: We dismiss two different cases: a) Suppose the critical TITS graph has $d + k$ vertices with yes vertices a and c. Then the vertex c must belong to one of two K_d in TIFS with false vertex b, in that case reaches vertex b not be affected by the obtained colored TITS with vertices a and c that, for the Lemma 2, contains a smaller TIFS than the original. b) Suppose the critical TITS with $d + k + 1$ vertices and true vertex a, c. We add a vertex x to TIFS from true vertex a to false vertex b, is clear that the new vertex must be yes, i.e., it must be equal to vertex c, because if c was an internal vertex of TIFS with a yes and b no then we have the previous case.

Theorem 4: Minimal TITS have $d + 7$ vertices in dimension $d \geq 3$.

Proof: The addition of two vertices of the TIFS in dimension d with true vertex a and false vertex b by addition of necessary edges in the TITS with yes vertices a and c can not be a forbidden graph (you can check that the structure with 3 independent triangles full linked to K_{d-3} has dimension d), so this obtained a TITS realizable in dimension d. The general structure of the graph is in Fig. 7.

IV. DIMENSION 3 - SPECKER BUG

The number of TIFSes in dimension 3,4,5,6... is 1,3,4,8,13,19...

In general, the number of TITFSes in dimension d is $(d^2 - d - 4)/3$.

V. DIMENSION 4 - HARDY BUG & OTHER BUGS

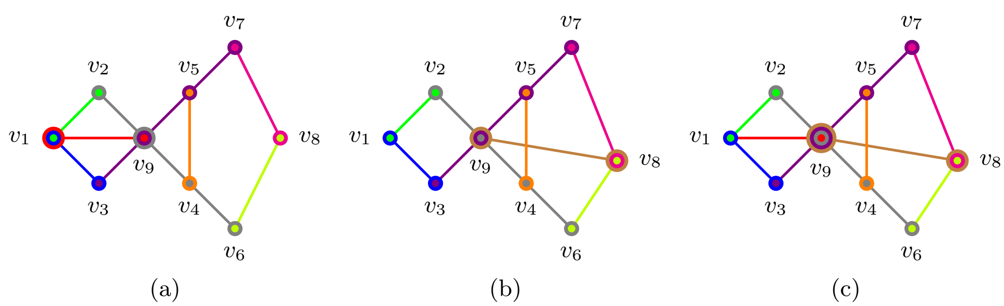
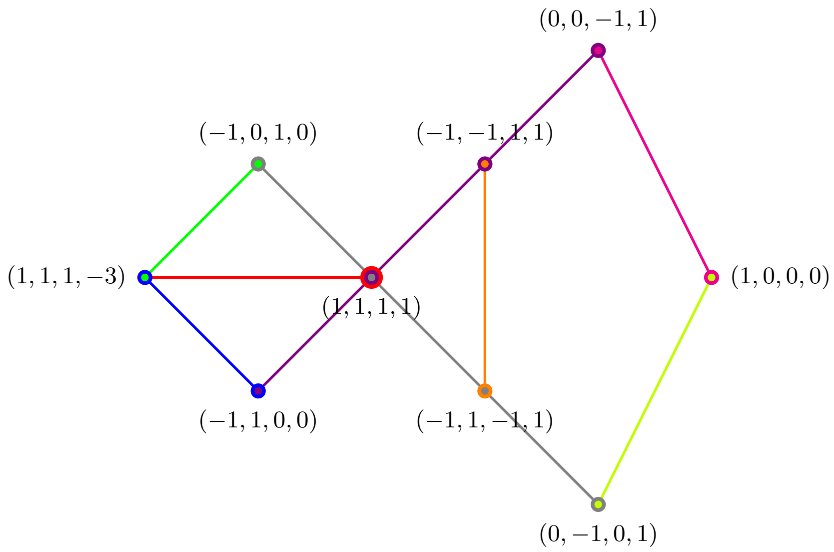


FIG. 4. (Color online) Orthogonality diagram of the the three simplest (partially known [**?** **?** **?**]) TIFS in $d = 4$, with 8(9) contexts and 9 atomic propositions. It is realizable in S^3 by taking, for instance, (a) $v_1 = (0, -1, \sqrt{2}, 0)/\sqrt{3}$, $v_2 = (1, 0, 0, 0)$, $v_3 = (1, \sqrt{2}, 1, 0)/2$, $v_4 = (0, 1, 0, 0)$, $v_5 = (1, 0, -1, 0)/\sqrt{2}$, $v_6 = (0, 0, 1, 0)$, $v_7 = (-1, \sqrt{2}, -1, 0)/2$, $v_8 = (\sqrt{1-\varepsilon^2}/\sqrt{3})(\sqrt{2}, 1, 0, 0) + \varepsilon(0, 0, 0, 1)$, $v_9 = (0, 0, 0, 1)$, (b) $v_1 = (\sqrt{1-\varepsilon^2}/\sqrt{3})(0, -1, \sqrt{2}, 0) + \varepsilon(0, 0, 0, 1)$, $v_8 = (\sqrt{2}, 1, 0, 0)/\sqrt{3}$, (c) $v_1 = (0, -1, \sqrt{2}, 0)/\sqrt{3}$, $v_8 = (\sqrt{2}, 1, 0, 0)/\sqrt{3}$. Values on v_1 and v_8 cannot both be 1, as this would imply the values on $v_2 = v_3 = v_6 = v_7 = v_9$ to vanish, which in turn would require both values v_4 as well as v_5 to be 1; which contradicts exclusivity because those observables belong to the same context and are orthogonal. In QT, the proposition v_i is represented by the projector $|v_i\rangle\langle v_i|$.



A. Specific realizations

1. Pentagrams and Paradoxes

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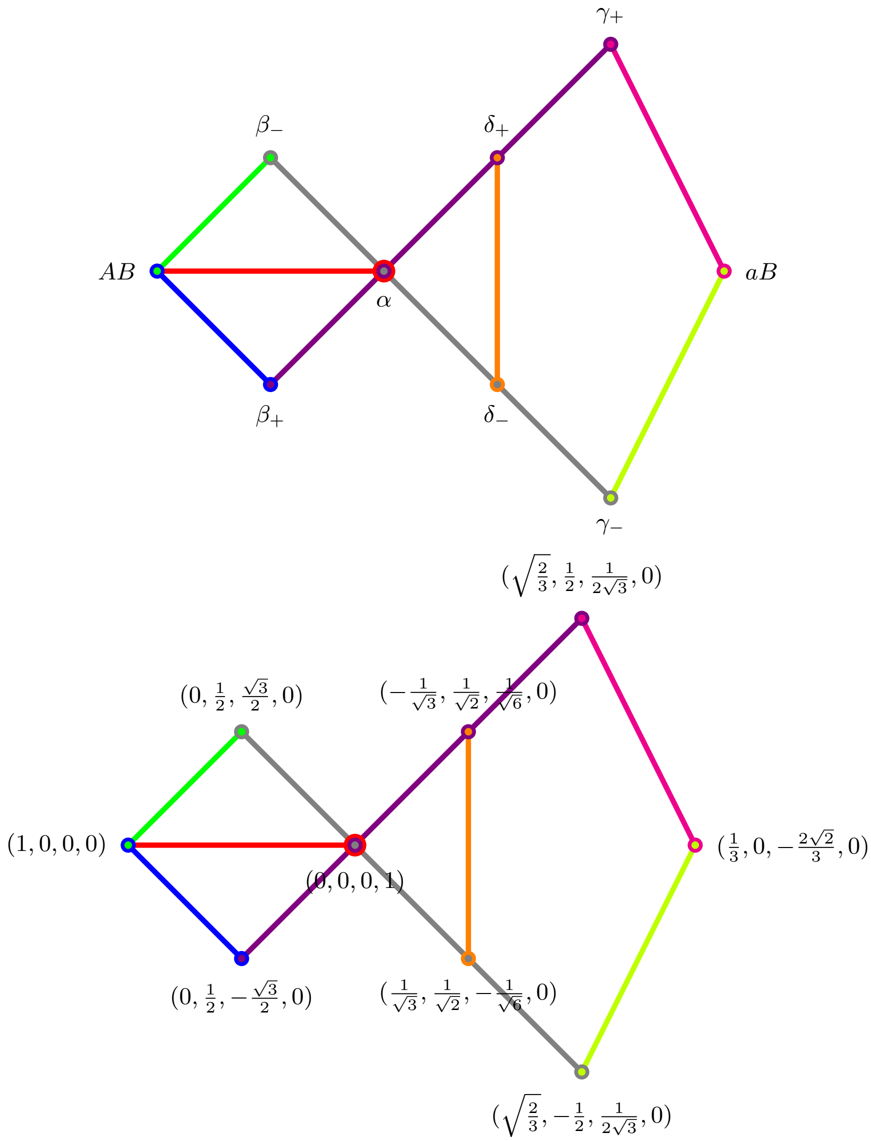

2. No-hidden-variables proof for two spin- particles preselected and postselected in unentangled states

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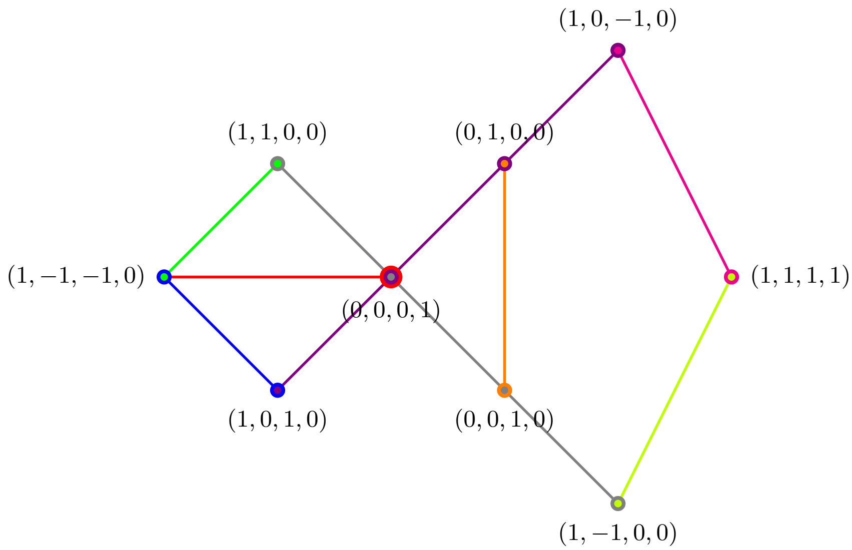

3. Bell-Kochen-Specker theorem: A proof with 18 vectors

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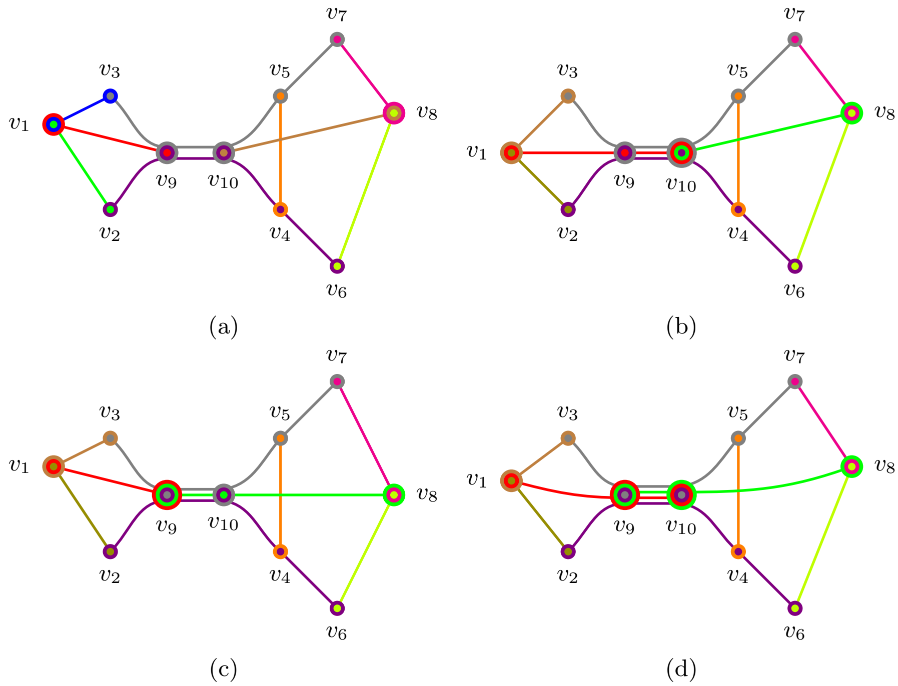

FIG. 5. (Color online) Orthogonality diagram of the the four simplest known [] TIFS in $d = 5$, with 9 contexts and 10 atomic propositions. It is realizable in S^4 by taking, for instance, (a) $v_1 = (\sqrt{1-\varepsilon^2}/\sqrt{3})(0,-1,\sqrt{2},0,0) + \varepsilon(0,0,0,0,1)$, $v_2 = (1,0,0,0,0)$, $v_3 = (1,\sqrt{2},1,0,0)/2$, $v_4 = (0,1,0,0,0)$, $v_5 = (1,0,-1,0,0)/\sqrt{2}$, $v_6 = (0,0,1,0,0)$, $v_7 = (-1,\sqrt{2},-1,0,0)/2$, $v_8 = (\sqrt{1-\varepsilon^2}/\sqrt{3})(\sqrt{2},1,0,0,0) + \varepsilon(0,0,0,1,0)$, $v_9 = (0,0,0,1,0)$, $v_{10} = (0,0,0,0,1)$, (b) $v_1 = (0,-1,\sqrt{2},0,0,)/\sqrt{3}$, $v_8 = (\sqrt{1-\varepsilon^2}/\sqrt{3})(\sqrt{2},1,0,0,0) + \varepsilon(0,0,0,1,0)$, (c) $v_1 = (\sqrt{1-\varepsilon^2}/\sqrt{3})(0,-1,\sqrt{2},0,0) + \varepsilon(0,0,0,1,0)$, $v_8 = (\sqrt{2},1,0,0,0)/\sqrt{3}$, (d) $v_1 = (0,-1,\sqrt{2},0,0)/\sqrt{3}$, $v_8 = (\sqrt{2},1,0,0,0)/\sqrt{3}$. Values on v_1 and v_8 cannot both be 1, as this would imply the values on $v_2 = v_3 = v_6 = v_7 = v_9 = v_{10} = v_{11}$ to vanish, which in turn would require both values $v_4 = v_5$ to be 1; which contradicts exclusivity because those observables belong to the same context and are orthogonal. In QT, the proposition v_i is represented by the projector $|v_i\rangle\langle v_i|$.

VI. DIMENSION 5 - GENERALIZED HARDY BUGS

VII. DIMENSION 6

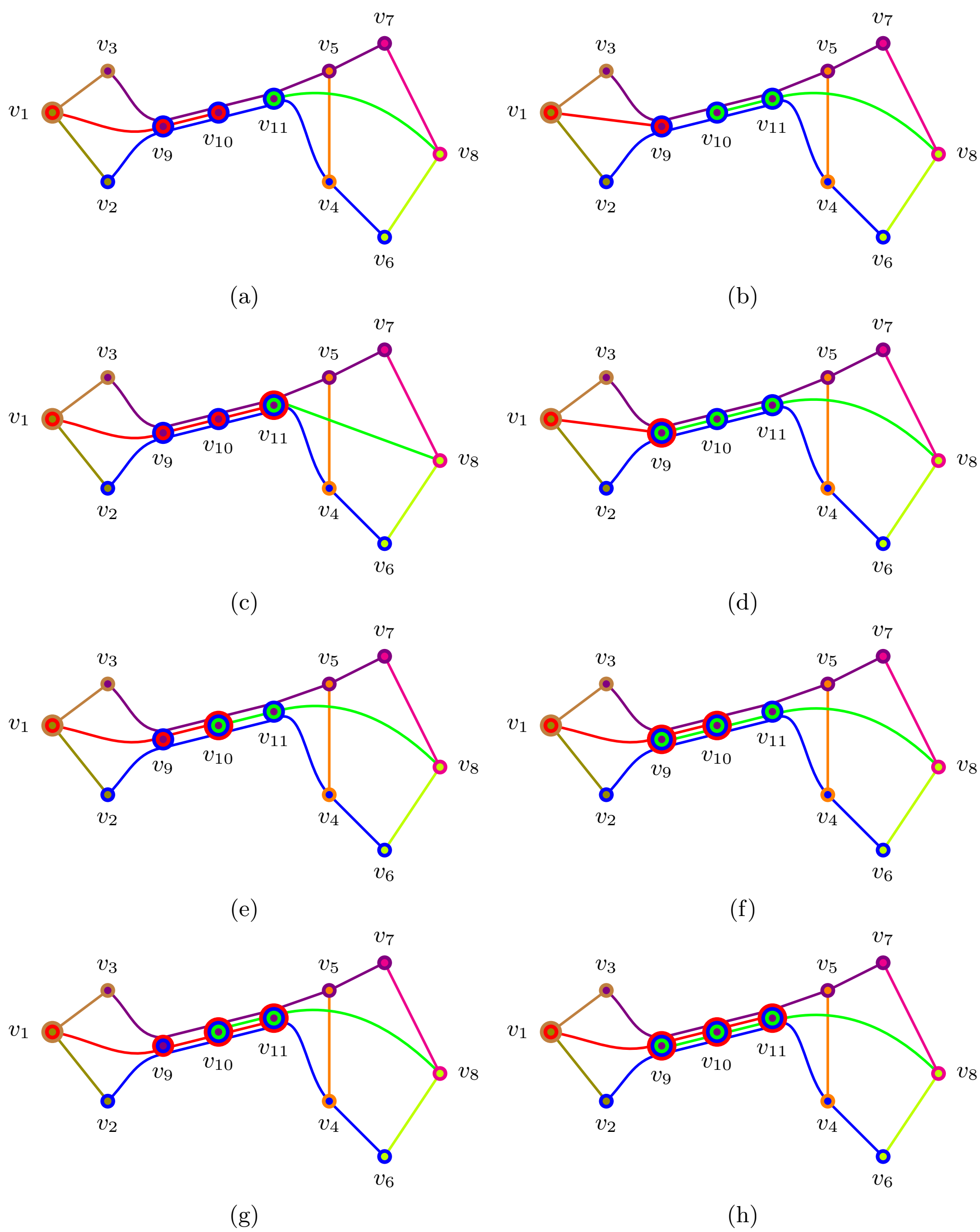

FIG. 6. (Color online) see text

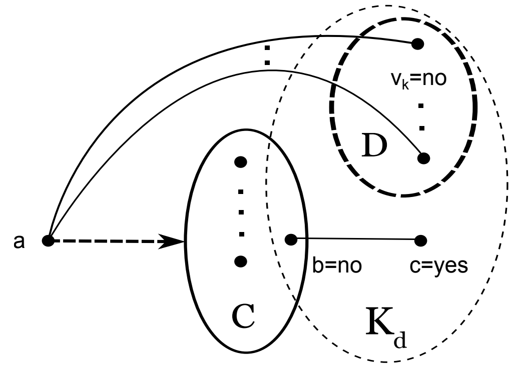

FIG. 7. Schema for minimal TITS with $d+k+2$ vertices in dimension d. The arrow represents connection between a and some vertices in C such that they form a minimal TIFS, also vertex a is linked with all vertices in D.

Orthogonality diagram of the the four simplest known [] TIFS in $d=6$, with 9 contexts and 11 atomic propositions. It is realizable in S^5 by taking, for instance, (a) $v_1 = (\sqrt{1-\varepsilon^2}/\sqrt{3})(0,-1,\sqrt{2},0,0,0) + \varepsilon(0,0,0,0,0.1)$, $v_2 = (1,0,0,0,0,0)$, $v_3 = (1,\sqrt{2},1,0,0,0)/2$, $v_4 = (0,1,0,0,0,0)$, $v_5 = (1,0,-1,0,0,0)/\sqrt{2}$, $v_6 = (0,0,1,0,0,0)$, $v_7 = (-1,\sqrt{2},-1,0,0,0)/2$, $v_8 = (\sqrt{1-\varepsilon^2}/\sqrt{3})(\sqrt{2},1,0,0,0,0) + (\varepsilon/\sqrt{2})(0,0,0,1,1,0)$, $v_9 = (0,0,0,1,0,0)$, $v_{10} = (0,0,0,0,1,0)$, $v_{10} = (0,0,0,0,0,1)$, (b) $v_1 = (\sqrt{1-\varepsilon^2}/\sqrt{3})(0,-1,\sqrt{2},0,0,0) + (\varepsilon/\sqrt{2})(0,0,0,0,1,1)$, $v_8 = (\sqrt{1-\varepsilon^2}/\sqrt{3})(\sqrt{2},1,0,0,0,0) + \varepsilon(0,0,0,1,0,0)$, (c) $v_1 = (0,-1,\sqrt{2},0,0,0)/\sqrt{3}$, $v_8 = (\sqrt{1-\varepsilon^2}/\sqrt{3})(\sqrt{2},1,0,0,0,0) + (\varepsilon/\sqrt{2})(0,0,0,1,1,0)$, (d) $v_1 = (\sqrt{1-\varepsilon^2}/\sqrt{3})(0,-1,\sqrt{2},0,0,0) + (\varepsilon/\sqrt{2})(0,0,0,0,1,1)$, $v_8 = (\sqrt{2},1,0,0,0,0)/\sqrt{3}$, (e) $v_1 = (\sqrt{1-\varepsilon^2}/\sqrt{3})(0,-1,\sqrt{2},0,0,0) + \varepsilon(0,0,0,0,0,1)$, $v_8 = (\sqrt{1-\varepsilon^2}/\sqrt{3})(\sqrt{2},1,0,0,0,0) + \varepsilon(0,0,0,1,0,0)$, (f) $v_1 = (\sqrt{1-\varepsilon^2}/\sqrt{3})(0,-1,\sqrt{2},0,0,0) + \varepsilon(0,0,0,0,0,1)$, $v_8 = (\sqrt{2},1,0,0,0,0)/\sqrt{3}$, (g) $v_1 = (0,-1,\sqrt{2},0,0,0)/\sqrt{3}$, $v_8 = (\sqrt{1-\varepsilon^2}/\sqrt{3})(\sqrt{2},1,0,0,0,0) + \varepsilon(0,0,0,1,0,0)$, (h) $v_1 = (0,-1,\sqrt{2},0,0,0)/\sqrt{3}$, $v_8 = (\sqrt{2},1,0,0,0,0)/\sqrt{3}$. Values on v_1 and v_8 cannot both be 1, as this would imply the values on $v_2 = v_3 = v_6 = v_7 = v_9 = v_{10} = v_{11}$ to vanish, which in turn would require both values $v_4 = v_5$ to be 1; which contradicts exclusivity because those observables belong to the same context and are orthogonal. In QT, the proposition v_i is represented by the projector $|v_i\rangle\langle v_i|$.

Corollary 2: The family of minimal TITSes with $d+7$ vertices contains exactly 3 k_d.

Proof: Consider the structure of minimal TITS (Fig. 7), where $\{a\} \cup C$ is TIFS in dimension d and $|D| = d-2$. There exist some edges between $C \subset TIFS_{d=3}$ and D that form one more K_d. Furthermore, $\forall v \in D$ verifies that $v \in N(a)$.

VIII. CONCLUSIONS AND OPEN PROBLEMS

Peres conjetured that the simplest posible state-independet proof is one in [24], which requires 18 propositions in $d=4$ [26]. The method we have developed in this article, and the obtained minimal TITS, can be helped to prove Peres' conjecture [27]. Other related open problem which can benefit from these results is which is the simplest state-independent proof in $d=3$.

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