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Value Indefiniteness Is Almost Everywhere

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Abstract

Kochen-Specker type theorems assure the breakdown of certain non-contextual hidden variable theorems through the non-existence of global, holistic frame functions; alas they do not target the location and the extent of this phenomenon. Here we show the strongest conceivable form of quantum indeterminacy—sometimes also referred to as contextuality or value indefiniteness—by proving that, once a single arbitrary observable is fixed to be value definite, almost (i.e. with Lebesgue measure one) all remaining observables are indeterminate relative to the assumptions, in particular that any value assignments must necessarily be non-contextual.

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I. INTRODUCTION

The Kochen-Specker theorem [1, 2] proves the impossibility of the existence of a hidden variable theory for quantum mechanics by showing the existence of a finite set of observables O for which the following two assumptions cannot be simultaneously true:

(P1) every observable in O has a pre-assigned definite value,

(P2) the outcomes of measurements of observables are non-contextual.

Non-contextuality means that the outcomes of measurements of observables are independent of whatever other co-measurable observables are measured alongside them, along with the requirement that the relationship between hidden variables associated with sets of co-measurable observables behave quasi-classically, as expected from quantum mechanics. This requirement means that in any "complete, maximal" set of mutually co-measurable yes-no propositions (represented by mutually orthogonal projectors spanning the Hilbert space, or, equivalently, by a single "maximal" operator with a spectral decomposition containing these projectors) exactly one proposition should be assigned the value "yes." Due to complementarity, the observables in O may not be all simultaneously co-measurable, that is, formally, commuting.

The Kochen-Specker theorem *does not explicitly identify* certain particular observables which violate one or both assumptions (P1) and (P2), but only proves their *existence*. This form of the theorem was amply sufficient for its intended scope, primarily metaphysical. The relation between value indefinite observables, that is, observables which do not have definite values before measurement, and quantum randomness in [1, 2], requires a more precise form of the Kochen-Specker theorem in which some value indefinite observables can be located (identified). A stronger form of the Kochen-Specker theorem providing this information was proved in [3].

II. LOGICAL INDETERMINACY PRINCIPLE

Pitowsky [4] (also in the subsequent paper [5] with Hrushovski) gave a constructive proof of Gleason's lemma in terms of orthogonality graphs which motivated the study of probability distributions on finite sets of rays. In this context he proved a result called "the logic indeterminacy principle" which has a striking similarity with the Kochen-Specker theorem.

According to [4], a *frame function* on a set $O \subset \mathbb{C}^n$ of projection observables (which we identify here with the unit vectors they project on to) in a dimension $n \geq 3$ Hilbert space is a function $p : O \to [0,1]$ such that:

(i) If $\{|x_1\rangle, \ldots, |x_n\rangle\}$ is an orthonormal basis, $\sum_{i=1}^{n} p(|x_i\rangle) = 1$, and for $\{|x_1\rangle, \ldots, |x_k\rangle\}$ orthonormal with $k \leq n$, $\sum_{i=1}^{k} p(|x_i\rangle) \leq 1$.

(ii) For all complex α with $|\alpha| = 1$ and all $x \in O$, $p(|x\rangle) = p(\alpha |x\rangle)$.

A *Boolean frame function* is a frame function taking only $0,1$ values, i.e. for all $|x\rangle \in O$, $p(|x\rangle) \in \{0,1\}$.

Theorem II.1 (Logical indeterminacy principle). *Let p be a Boolean frame function. For all projectors $a,b \in \mathbb{C}^3$ with $0 < |\langle a|b\rangle| < 1$, there exists a finite set of observables O with $|a\rangle, |b\rangle \in O$ such that there is no Boolean frame function p on O unless $p(|a\rangle) = p(|b\rangle) = 0$.*

The theorem states that for every two distinct non-orthogonal unit vectors $|a\rangle, |b\rangle \in \mathbb{C}^3$ there exists a finite set of observables O such that there is a Boolean frame function p on O only if $p(|a\rangle) = p(|b\rangle) = 0$. Consequently, there is no Boolean frame function p on O such that $p(|a\rangle) = 1$. From the logical indeterminacy principle we can deduce the Kochen-Specker theorem because a Boolean frame function simply gives a non-contextual, value definite yes-no value assignment, so (P2) is satisfied.

As noted by Hrushovski and Pitowsky [5], the logical indeterminacy principle is stronger than the Kochen-Specker theorem because the result is true for arbitrary frame functions which can take every value in the unit interval $[0,1]$, but which are restricted to Boolean values for $|a\rangle, |b\rangle$.

In fact, we may be tempted to use the logical indeterminacy principle to "locate" a value indefinite observable. Indeed, if we fix p and choose $|a\rangle \in \mathbb{C}^3$ such that $p(|a\rangle) = 1$, then, by the logical indeterminacy principle, for every distinct non-orthogonal unit vector $|b\rangle \in \mathbb{C}^3$ it is impossible to have $p(|b\rangle) = 1$ and $p(|b\rangle) = 0$, hence one could be inclined to conclude that $|b\rangle$ is value indefinite. However, such reasoning would be incorrect because if $p(|b\rangle)$ were 1, then the logical indeterminacy principle merely concludes that *p does not exist*; the same conclusion is obtained if $p(|b\rangle)$ were 0. Hence, in both cases *p does not exist*, so it makes no sense to talk about its values, in particular, about $p(|b\rangle)$. (Pointedly stated from a physical viewpoint, $p(|a\rangle)$ as well as $p(|b\rangle)$ could take on any of the four combinations of definite values, provided that (P1) or (P2) is violated for some other observable in O. Nevertheless, as we shall demonstrate in Section V, using the formalism of [3], all observables in O except $|a\rangle$ and those commuting with $|a\rangle$ are indeed provable value indefinite.) This means that using the logical indeterminacy principle we get the same global information derived in the Kochen-Specker theorem, namely that some observable in O has to be value indefinite, and no more. The reason for this limitation is the use of frame functions, which are necessarily defined everywhere: they can model "local" value definiteness, but not "local" value indefiniteness, which, as in the Kochen-Specker theorem, "emerges" only as a global phenomenon.

III. VALUE INDEFINITENESS AND CONTEXTUALITY

To remedy the above deficiency we will use the formalism proposed in [3] for pure quantum states. Specifically, we define value (in)definiteness and contextuality in the framework of quantum logic of Birkhoff and von Neumann [6, 7] and Kochen and Specker [8, 9].

Projection operators—projecting on to the linear subspace spanned by a non-zero vector $|\psi\rangle$—will be denoted by $P_\psi = \frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle}$.

We fix a positive integer n. Let $O \subseteq \{P_\psi \mid |\psi\rangle \in \mathbb{C}^n\}$ be a non-empty set of *projection observables* in the Hilbert space $\mathbb{C}^n$ and $\mathcal{C} \subseteq \{\{P_1, P_2, \ldots P_n\} \mid P_i \in O \text{ and } \langle i|j\rangle = 0 \text{ for } i \neq j\}$ a set of measurement contexts over O. A *context* $C \in \mathcal{C}$ is thus a maximal set of compatible (i.e. they can be simultaneous measured) projection observables. Let $v : \{(o,C) \mid o \in O, C \in \mathcal{C} \text{ and } o \in C\} \xrightarrow{o} B$ be a partial function (i.e., it may be undefined for some values in its domain) called an *assignment function*. For some $o, o' \in O$ and $C, C' \in \mathcal{C}$ we say $v(o,C) = v(o',C')$ if $v(o,C), v(o',C')$ are both defined and have equal values.

Value definiteness corresponds to the notion of predictability in classical determinism: an observable is value definite if v assigns it a definite value—i.e. is able to predict in advance, independently of measurement, the value obtained via measurement. Here is the formal definition: an observable $o \in C$ is *value definite* in the context C under v if $v(o,C)$ is defined; otherwise o is *value indefinite* in C. If o is value definite in all contexts $C \in \mathcal{C}$ for which $o \in C$ then we simply say that o is value definite under v. The set O is *value definite* under v if every observable $o \in O$ is value definite under v.

Non-contextuality corresponds to the classical notion that the value obtained via measurement is independent of other compatible observables measured alongside it. Formally, an observable $o \in$

O is *non-contextual* under v if for all contexts $C, C' \in \mathcal{C}$ with $o \in C, C'$ we have $v(o,C) = v(o,C')$; otherwise, v is *contextual*. The set of observables O is *non-contextual* under v if every observable $o \in O$ which is not value indefinite (i.e. value definite in *some* context) is non-contextual under v; otherwise, the set of observables O is *contextual*. Thus we require that observables behave non-contextually only where definite values are defined; this technicality is necessary in order to consider the possibility that some observables are value indefinite..

To be in agreement with quantum mechanics we restrict the assignment functions to admissible ones: v is *admissible* if the following hold for all $C \in \mathcal{C}$: a) if there exists an $o \in C$ with $v(o,C) = 1$, then $v(o',C) = 0$ for all $o' \in C \setminus \{o\}$, b) if there exists an $o \in C$ such that $v(o',C) = 0$ for all $o' \in C \setminus \{o\}$, then $v(o,C) = 1$.

IV. STRONG KOCHEN-SPECKER THEOREM

The incompatibility between the assumptions (P1) and (P2) is not *maximal*: for any set of contexts over any set of observables, there exists an admissible assignment function under which the set of observables is value definite and at least one observable is non-contextual.

However, *there always exist pairs of observables such that, if one of them is assigned the value* 1 *by an admissible assignment function under which O is non-contextual, the other must be* value indefinite. The result is deduced in [3] using the weaker assumption that not all observables are assumed to be value definite: *An observable is deduced to be value definite only where the specific requirements of the admissibility of v requires it to be so.*

Lemma IV.1 (Strong Kochen-Specker lemma)**.** *Let $|a\rangle, |b\rangle \in \mathbb{C}^3$ be unit vectors such that $0 < |\langle a|b\rangle| \leq \frac{3}{\sqrt{14}}$. Then there exists a set of 24 projection observables O containing P_a and P_b, and a set of contexts $\mathcal{C}$ over O, such that there is no admissible assignment function under which O is non-contextual and P_a, P_b have the value* 1.

The strong Kochen-Specker theorem can be used to "locate" a provable value indefinite observable which when measured "produces" a quantum random bit whose quality can be precisely evaluated under some physical assumptions [3]:

Theorem IV.2 (Strong Kochen-Specker theorem)**.** *Let $|a\rangle, |b\rangle \in \mathbb{C}^3$ be unit vectors such that $\sqrt{\frac{5}{14}} \leq |\langle a|b\rangle| \leq \frac{3}{\sqrt{14}}$. Then there exists a set of 24 projection observables O containing P_a and P_b,* *and a set of contexts $\mathcal{C}$ over O, such that there is no admissible assignment function under which O is non-contextual, P_a has the value* 1 *and P_b is value definite.*

V. HOW WIDESPREAD IS VALUE INDEFINITENESS?

The natural question which remains to be answered is the following. Assuming that P_a has the value 1, we know that P_b is value indefinite: which of the remaining 22 projection observables in O can be value definite? The answer is: *only those which commute with P_a.* Specifically, we will prove the following, more general, theorem which extends the Strong Kochen-Specker theorem to cover the rest of the Bloch sphere.

Theorem V.1. *Let $|a\rangle, |b\rangle \in \mathbb{C}^3$ be non orthogonal or parallel unit vectors, i.e.* $0 < |\langle a|b\rangle| < 1$. *Then there exists a set of projection observables O containing P_a and P_b, and a set of contexts $\mathcal{C}$ over O, such that there is no admissible assignment function under which O is non-contextual, P_a has the value 1 and P_b is value definite. The set O is finite and can be effectively constructed.*

While this result is similar in form to the original Kochen-Specker theorem, the subtle differences are critical. As mentioned previously, the Kochen-Specker theorem is unable to locate value definiteness: if P_a has the value 1, we cannot conclude that P_b is value indefinite, even if assigning it leads to a contradiction. This is due to the fact that this contradiction implies only that no *global* assignment function can exist; the Kochen-Specker theorem does not show that P_b could not be value definite, while some other P_c harbours the (necessary) value indefiniteness.

On the other hand, the sets of observables given in the proofs of the Strong Kochen-Specker theorems presented here are carefully constructed such that any attempt to place the value indefiniteness on a P_c necessarily contradicts the admissibility of v. For example, it would require a context containing an observable assigned the value 1, and another observable being value indefinite. This contradicts both the the admissibility of v, and the physical understanding of what it means for that observable to be assigned the value 1—since we know measuring that observable will give the value 1, measuring the other observables *must* give the value 0, and hence the other observables are necessarily value definite. As a result, we are forced to conclude that P_b itself is value indefinite.

In order to prove Lemma IV.1, a specific proof for the case of $|\langle a|b\rangle| = \frac{3}{\sqrt{14}}$ was given, followed by a reduction to this proof for the case $|\langle a|b\rangle| < \frac{3}{\sqrt{14}}$. Here we show a reduction for the remaining

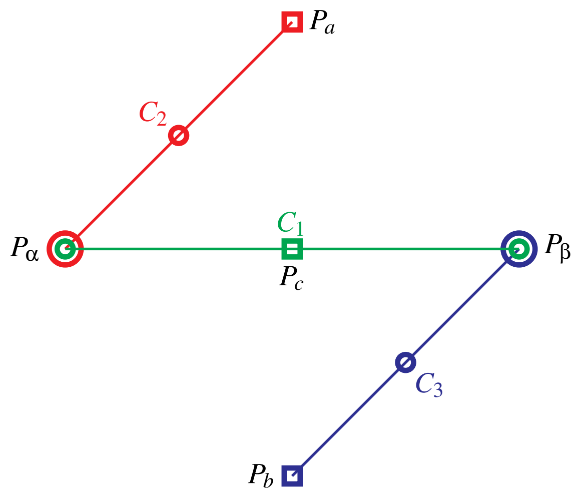

FIG. 1. (Color online) Greechie orthogonality diagram with an overlaid value assignment that illustrates the reduction in Lemma V.2. The circles and squares represent observables that will be given the values 0 and 1 respectively. They are joined by smooth lines which represent contexts.

case of $|\langle a|b\rangle| > \frac{3}{\sqrt{14}}$ to the existing result. This reduction turns out the be somewhat more subtle and difficult than for the case proven in [3].

For the purpose of illustrating the reduction technique, let us recall the following lemma from [3], which will also turn out to be important for the reduction we will give here.

Lemma V.2. *Given any two unit vectors* $|a\rangle, |b\rangle$ *with* $0 < |\langle a|b\rangle| < 1$, *there exists for every* x *such that* $|\langle a|b\rangle| < |x| < 1$ *a unit vector* $|c\rangle$ *with* $\langle a|c\rangle = x$, *a set of observables* O *containing* P_a, P_b, P_c *and a set of contexts* $\mathcal{C}$ *over* O *such that if* P_a *and* P_b *have the value 1, then* P_c *also has the value 1 under any admissible, non-contextual assignment function on* O.

Furthermore, if we choose our basis such that $|a\rangle \equiv (1,0,0)$ *and* $|b\rangle \equiv (p,q,0)$, *where* $p = \langle a|b\rangle$ *and* $q = \sqrt{1-p^2}$, *then* $|c\rangle$ *has the form* $|c\rangle = (x,y,\pm z)$, *where* $x = \langle a|c\rangle$, $y = \frac{p(1-x^2)}{qx}$ *and* $z = \sqrt{1-x^2-y^2}$.

This lemma is illustrated in Fig. 2 and constitutes a simple "forcing" of value definiteness: given P_a and P_b both with the value 1, there is a P_c which is "closer" (i.e. at a smaller angle of our choosing) to P_a which forces P_c to also take the value 1.

This reduction, however, requires necessarily that $|x| > |p|$, and finding a reduction to "force" in the other direction (i.e. towards larger angles between P_a and P_c) proves difficult. Here we give an iterated reduction for this case in the following lemma.

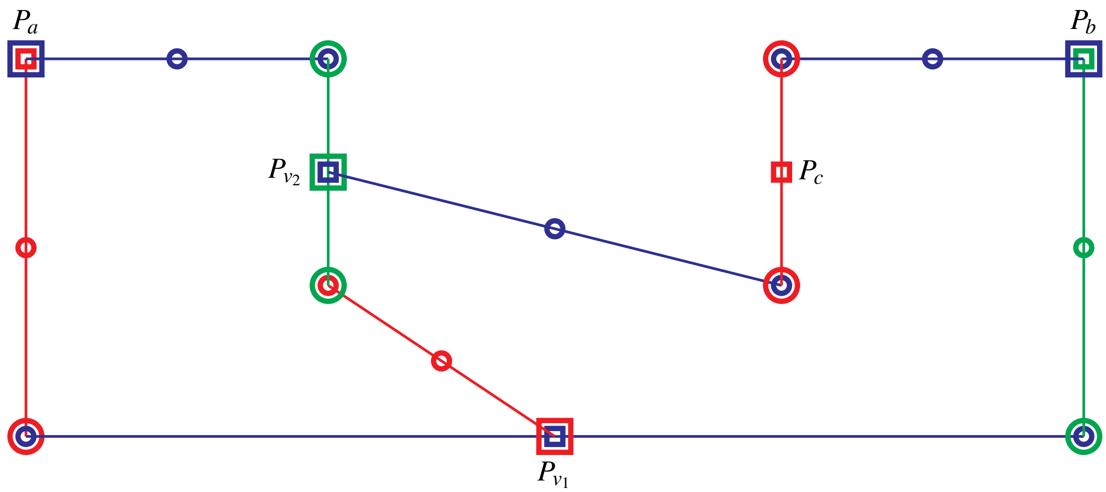

FIG. 2. (Color online) Greechie orthogonality diagram with an overlaid value assignment that illustrates the reduction in Lemma V.3.

Lemma V.3. *Given any two unit vectors $|a\rangle, |b\rangle$ with $\frac{3}{\sqrt{14}} < \langle a|b\rangle < 1$, there exists a unit vector $|c\rangle$ with $\langle a|c\rangle \leq \frac{3}{\sqrt{14}}$, a set of observables $\mathcal{O}$ containing P_a, P_b, P_c and a set of contexts $\mathcal{C}$ over $\mathcal{O}$ such that if P_a and P_b have the value 1, then P_c also has the value 1 under any admissible, non-contextual assignment function on $\mathcal{O}$.*

The proof of this lemma is based on the generalisation of a specific reduction for the case of $\langle a|b\rangle = \frac{1}{\sqrt{2}}$ to $\langle a|c\rangle = \frac{1}{\sqrt{3}}$; i.e. it is a "forcing" argument in the required direction. The Greechie diagram for this is depicted in Fig. 2. In essence, this figure consists of three copies of the reduction shown in Fig. 1 glued together, ensuring that the Greechie diagram is indeed realisable. Specifically, the important relations are: $\langle a|v_1\rangle = \sqrt{\frac{2}{3}}$, $\langle a|v_2\rangle = \frac{2}{\sqrt{5}}$, $\langle b|c\rangle = \sqrt{\frac{2}{3}}$ and $\langle b|v_2\rangle = \sqrt{\frac{2}{5}}$. The angles between unit vectors in this proof are then scaled, in a way which we will soon make precise, to fit the required $\langle a|b\rangle$ for the general case. However, since this doesn't allow us to assert that an arbitrary $|c\rangle$ must have the value 1 in the same way we could using Lemma V.2, this reduction is then iterated a finite number of times until a sufficiently small $\langle a|c\rangle$ is obtained.

Proof of Lemma V.3. The constants which will be used for scaling, obtained from the reduction shown in Fig. 2, are as follows:

$$\alpha_1 = \frac{\arccos\sqrt{\frac{2}{3}}}{\arccos\frac{1}{\sqrt{2}}}, \quad \alpha_2 = \frac{\arccos\frac{2}{\sqrt{5}}}{\arccos\sqrt{\frac{2}{3}}}, \quad \alpha_3 = \frac{\arccos\sqrt{\frac{2}{3}}}{\arccos\sqrt{\frac{2}{5}}}.$$

Given the initial $|a\rangle$, $|b\rangle$ and the above constants, we thus make use of the following scaled angles between the relevant observables:

$$\theta_{a,b} = \arccos\langle a|b\rangle, \quad \theta_{a,v_1} = \alpha_1\theta_{a,b}, \quad \theta_{a,v_2} = \alpha_2\theta_{a,v_1}.$$

Once $|v_2\rangle$ is determined via the procedure to follow, we take the following:

$$\theta_{b,v_2} = \arccos\langle b|v_2\rangle, \quad \theta_{b,c} = \alpha_3\theta_{b,v_2}.$$

Without loss of generality, let $|a\rangle = (1,0,0)$ and $|b\rangle = (p_1,q_1,0)$ where $p_1 = \langle a|b\rangle$ and $q_1 = \sqrt{1-p_1^2}$. This fixes our basis for the rest of the reduction. We want to have $|v_1\rangle$ such that $\langle a|v_1\rangle = x_1 = \cos\theta_{a,v_1}$. From Lemma V.2 we know this is possible since $x_1 > p_1$ (because $\alpha_1 < 1$), and we have $|v_1\rangle = (x_1,y_1,z_1)$, $y_1 = \frac{p_1(1-x_1^2)}{q_1x_1}$ and $z_1 = \sqrt{1-x_1^2-y_1^2}$.

We now want $|v_2\rangle$ such that $\langle a|v_2\rangle = x_2 = \cos\theta_{a,v_2}$ (this is possible since $\alpha_2 < 1$). In order to use the same general form (specified in Lemma V.2) as above, we perform a change of basis to bring $|v_1\rangle$ into the xy-plane, describe $|v_2\rangle$ in this basis using the above result, then perform the inverse change of basis. Our new basis vectors are given by $|e_2\rangle = (1,0,0)$,

$$|f_2\rangle = (|v_1\rangle - x_1\,|e_2\rangle)/q_2 = (0,y_1/q_2,z_1/q_2)$$

where $q_2 = \sqrt{1-x_1^2}$, and $|g_2\rangle = |e_2\rangle \times |f_2\rangle = (0,z_1/q_2,-y_1/q_2)$. We thus have the transformation matrix

$$T_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & y_1/q_2 & z_1/q_2 \\ 0 & z_1/q_2 & -y_1/q_2 \end{pmatrix}.$$

We can now put $y_2 = \frac{x_1(1-x_2^2)}{q_2x_2}$ and $z_2 = \sqrt{1-x_2^2-y_2^2}$ so that in our original basis we have

$$|v_2\rangle = T_1(x_2,y_2,-z_2)^t = (x_2,(y_1y_2-z_1z_2)/q_2,(y_2z_1+y_1z_2)/q_2).$$

We note at this point that the constant θ_{b,v_2} is now determined, and we have

$$\langle b|v_2\rangle = p_1x_2 + \frac{q_1}{q_2}(y_1y_2-z_1z_2).$$

For the last iteration of the reduction, we want to find $|c\rangle$ such that $\langle b|c\rangle = x_3 = \cos\theta_{b,c}$ (again this will be possible since $\alpha_3 < 1$). Let $p_3 = \langle b|v_2\rangle$ and $q_3 = \sqrt{1-p_3^2}$. Again we perform a basis transformation; we have $|e_3\rangle = |b\rangle = (p_1,q_1,0)$,

$$\begin{aligned} |f_3\rangle &= (|v_2\rangle - p_3\,|b\rangle)/k \\ &= (x_2-p_3p_1,(y_1y_1-z_1z_2)/q_2-p_3q_1,(y_2z_1+y_1z_2)/q_2)/k, \end{aligned}$$

where k is a constant such that $|f_3\rangle$ is normalized, and

$$\begin{aligned}|g_3\rangle &= |e_3\rangle \times |f_3\rangle \\ &= \left(\frac{q_1}{q_2}(y_2z_1+y_1z_2), \frac{-p_1}{q_2}(y_2z_1+y_1z_2), \frac{p_1}{q_2}(y_1y_2-z_1z_2)-q_1x_2\right)/k.\end{aligned}$$

The transformation matrix is then given by

$$T_3 = \begin{pmatrix} p_1 & |f_3\rangle_x & |g_3\rangle_x \\ q_1 & |f_3\rangle_y & |g_3\rangle_y \\ 0 & |f_3\rangle_z & |g_3\rangle_z \end{pmatrix},$$

where the subscript indicates the component of the relative vector. We now put $y_3 = \frac{p_3(1-x_3^2)}{q_3x_3}$ and $z_3 = \sqrt{1-x_3^2-y_3^2}$ so that in the original basis we have

$$\begin{aligned}|c\rangle =& T_3(x_3, y_3, -z_3)^t \\ =& \Big(x_3p_1 + \frac{y_3}{k}(x_2-p_1p_3) - \frac{q_1z_3}{kq_2}(y_2z_1+y_1z_2), \\ & x_3q_1 + \frac{y_3}{kq_2}(y_1y_2-z_1z_2-p_3q_1q_2) + \frac{z_3p_1}{kq_2}(y_2z_1+y_1z_2), \\ & \frac{y_3}{kq_2}(y_2z_1+y_1z_2) - \frac{z_3}{k}\left[\frac{p_1}{q_2}(y_1y_2-z_1z_2)-q_1x_2\right]\Big).\end{aligned}$$

While this is particularly messy, only the first term is of importance. Specifically, we want to prove that $\langle a|c\rangle < \langle a|b\rangle = p_1$, where

$$\langle a|c\rangle = x_3p_1 + \frac{y_3}{k}(x_2-p_1p_3) - \frac{q_1z_3}{kq_2}(y_2z_1+y_1z_2).$$

The product $\langle a|c\rangle$ is, with appropriate substitutions, a function of one variable, p_1; let us denote $f(p_1) = \langle a|c\rangle$. We thus need to determine if, for $p_1 \in \left(\frac{3}{\sqrt{14}}, 1\right)$ the inequality $f(p_1) < p_1$ holds.

Unfortunately, the expanded form of $f(p_1)$ is difficult to manipulate analytically. However, f is well behaved and continuous on this domain; although undefined for $p_1 = 1$, $\lim_{p_1\to 1^-} f(p_1) = 1$. With these facts we show, using a combination of direct analysis and Mathematica calculation and plots, that the inequality is indeed true.

Using Mathematica to take a Taylor series expansion around $p_1 = 1$, we find that for small $|p_1-1|$, $f(p_1) = 1-m(1-p_1)$, where $m \approx 1.27$ is a constant. Hence $\lim_{p_1\to 1^-} f(p_1) = 1$ as claimed and for some $\varepsilon > 0$ we have $f(p_1) < p_1$ for $p_1 \in (1-\varepsilon, 1)$. Further, the continuity of f on this domain can be guaranteed by noting that $f(p_1)$ is simply composed of trigonometric

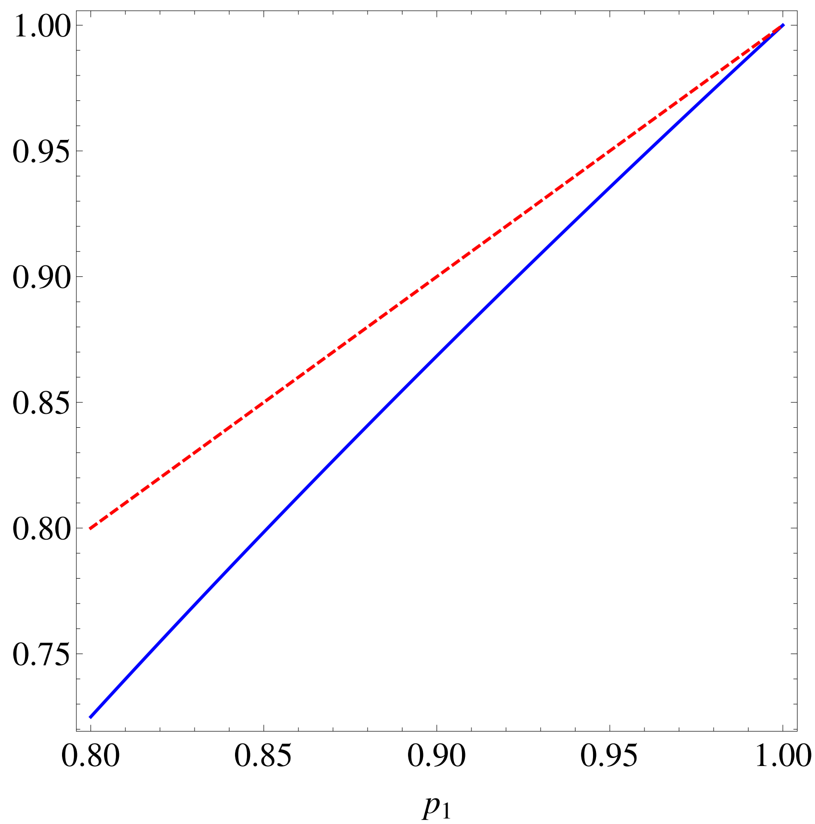

FIG. 3. (Color online) Plot of p_1 (dashed in red) and $f(p_1)$ (in blue) for $p_1 \in (0.8,1) \supset \left(\frac{3}{\sqrt{14}},1\right)$.

functions with arguments from $(-1,1)\setminus\{0\}$; since these are all continuous, so is f. Hence, from Fig. 3 and these results it would seem evident that $f(p_1) < p_1$ for all p_1 in this domain. However, since $f(p_1)$ approaches p_1 as $p_1 \to 1$, there remains the possibility that $f(p_1) > p_1$ for some p_1 close to 1.

Since we know from the Taylor series expansion that $f(p_1) < p_1$ in the neighbourhood of $p_1 = 1$, for it to not be the case that $f(p_1) < p_1$ for all $p_1 \in \left(\frac{3}{\sqrt{14}},1\right)$, we would have to have $\frac{df}{dp_1} < 1$, for some $p_1 \in \left(\frac{3}{\sqrt{14}},1\right)$. However, it is clear from Fig. 4 that this is not the case. We emphasize that this is *not* an analytic proof, but, given the continuity and analytical properties we have mentioned, there is ample evidence for the fact the $f(p_1)$ is indeed less than p_1 on the relevant domain.

From Fig. 3 (and also the fact that the derivative of $f(p_1) > 1$) it also follows that the difference $p_1 - f(p_1)$ is strictly decreasing with p_1 on $\left(\frac{3}{\sqrt{14}},1\right) \subset (0.8,1)$. Thus, for large enough (but finite) k, $f^k(p_1) \leq \frac{3}{\sqrt{14}}$, and this $|c_k\rangle$ must be assigned the value 1 by the v. $\square$

The proof of the main theorem follows rather straightforwardly from Lemma V.3.

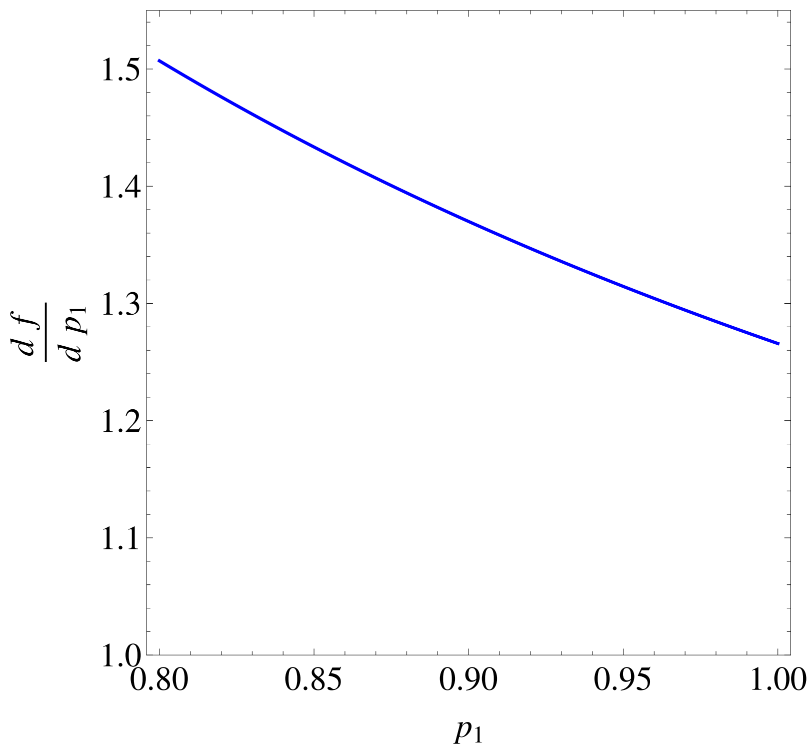

FIG. 4. (Color online) Plot of $\frac{df}{dp_1}$ for $p_1 \in (0.8,1) \supset (\frac{3}{\sqrt{14}},1)$.

Proof of Theorem V.1. If $0 < |\langle a|b\rangle| < \frac{3}{\sqrt{14}}$, we can appeal simply to Theorem IV.2, so let $\frac{3}{\sqrt{14}} < |\langle a|b\rangle| < 1$.

Without loss of generality, we can assume that $\langle a|b\rangle \in (0,1)$, since $P_b = P_{\alpha b}$ for $\alpha \in \mathbb{C}$ with $|\alpha| = 1$, so the set of projection observables O obtained under this assumption will give the required result for the general case.

Let us assume, for the sake of contradiction, that such an admissible assignment function v exists for all $O, \mathcal{C}$, i.e. $v(P_a, C_a) = 1$ and $v(P_b, C_b)$ is defined for all $C_a, C_b \in \mathcal{C}$ with $P_a \in C_a$ and $P_b \in C_a$. (Since v is required to be non-contextual, we will omit the context and write $v(P_a, \cdot)$ for simplicity.) Then, for all such C_a, C_b the following holds. If $v(P_b, \cdot) = 1$, then by Lemma V.3, there exists a $|c\rangle$ with $\langle a|c\rangle \leq \frac{3}{\sqrt{14}}$ such that $v(P_c, \cdot) = 1$.But this contradicts Theorem IV.2. Hence, if P_b is to be value definite we must have $v(P_b, \cdot) = 0$. However, we show that this also leads to a contradiction as follows.

Let $p = \langle a|b\rangle$ and $q = \sqrt{1-p^2}$. We construct an orthonormal basis in which $|a\rangle \equiv (1,0,0)$ and $|b\rangle \equiv (p,q,0)$. Define $|\alpha\rangle \equiv (0,1,0)$, $|\beta\rangle \equiv (0,0,1)$ and $|c\rangle \equiv (q,-p,0)$. Then $\{|a\rangle, |\alpha\rangle, |\beta\rangle\}$ and $\{|b\rangle, |c\rangle, |\beta\rangle\}$ are orthonormal bases for $\mathbb{C}^3$, so we can define the contexts $C_1 = \{P_a, P_\alpha, P_\beta\}$ and $C_2 = \{P_b, P_c, P_\beta\}$. Since $v(P_a, C_1) = 1$, we must have $v(P_\beta, C_1) = v(P_\beta, C_2) = 0$ by the admissibility of v. But since, by assumption, $v(P_b, C_2) = 0$, we must have $v(P_c, C_2) = 1$. However, this also contradicts Theorem IV.2, since it is easily seen that

$$0 < \langle a|c\rangle = q = \sqrt{1-p^2} < \sqrt{\frac{5}{14}} < \frac{3}{\sqrt{14}}.$$

Hence, we conclude that P_b must be value indefinite under v. □

Theorem V.4. *The set of value indefinite observables has Lebesgue measure one in* $\mathbb{C}^3$.

Proof. The set of value indefinite observables depends on an arbitrarily fixed single vector, say $|a\rangle \in \mathbb{C}^3$. Assume that P_a has a definite value (1 or 0). According to Theorem V.1, no observable outside the union of the linear subspaces (i) spanned by the single vector P_a (dimension one) and (ii) the plane orthogonal to this vector $\{P_b \mid \langle P_a|P_b\rangle = 0\}$ (dimension two) is value definite. This set has Lebesgue measure zero in $\mathbb{C}^3$ because any subset of $\mathbb{C}^3$ whose dimension is smaller than 3 has Lebesgue measure zero in $\mathbb{C}^3$. □

In terms of unit vectors, the set in the above proof corresponds to the set $\{(1,0,0),(0,0,0)\} \cup \{(0,x,y) \mid x^2+y^2 = 1\}$ on the three dimensional unit sphere, consisting of (i) a single point of dimension zero, and (ii) a great circle of dimension one. Again this set has Lebesgue measure zero on the unit sphere.

VI. FINAL COMMENTS

One immediate result of the above findings is that, if one insists on the type of non-contextuality formalized by admissible assignments, then value definiteness cannot exist outside of a star-shaped configuration in Greechie-type orthogonality diagrams.

Let us be more specific what is meant by the "star(-shaped)" configuration of a quantum state $|\psi\rangle$. We consider a a quantum system prepared in a state corresponding to the proposition that "a particular detector D_ψ clicks among, say, three mutual exclusive detectors" (corresponding to a three dimensional Hilbert space). Such a state can be formalized by a projector $P_\psi = |\psi\rangle\langle\psi|$, or, equivalently, by the linear subspace spanned by the normalized vector $|\psi\rangle$ (together maybe with the other two orthonormal vectors to $|\psi\rangle$ and to each other). Now, if a quantum state $|\psi\rangle$ is prepared such that the detector D_ψ clicks, that corresponds to assigning $|\psi\rangle$ the value $v(P_\psi,\cdot) = 1$. $|\psi\rangle$'s star is formed by taking some or all vectors $|\varphi\rangle$ whose value assignements are consistent with $v(P_\psi,\cdot) = 1$. These are value assignments $v(P_\varphi,\cdot) = 0$, with $|\varphi\rangle$ orthogonal to $|\psi\rangle$; i.e., $\langle\varphi|\psi\rangle = 0$.

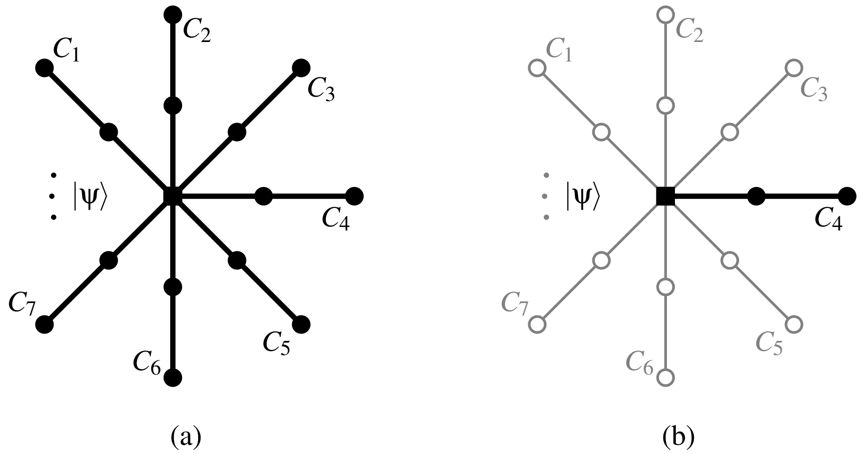

FIG. 5. (Color online) Greechie orthogonality diagram of a star-shaped configuration, representing a common detector observable $|\psi\rangle\langle\psi|$ with an overlaid two-valued assignment reflecting $v(P_\psi,\cdot) = 1$. (a) all "branches" corresponding to contects are assumed to be equally value definite; (b) it is assumed that, since the system is prepared in, say, context C_4, depicted by a block colored in thick filled black, only this context is value definite; all the other (continuity of) contexts are "phantom contexts" colored in gray.

Such potential observables $|\varphi\rangle\langle\varphi|$ are thus value definite. As they correspond to vectors orthogonal to $|\psi\rangle$, they are, diagrammatically (i.e., in terms of Greechie orthogonality diagrams) speaking, "in $|\psi\rangle$'s star."

All other conceivable observables corresponding to vectors "outside of $|\psi\rangle$'s star" remain value indefinite relative to our assumptions. The configuration can be represented by the Greechie orthogonality diagram depicted in Fig. 5(a). This finding is consistent with the Heisenberg uncertainty relations and quantum complementarity. Note that this still allows the value definite existence of a continuity of contexts interlinked at $|\psi\rangle$, but on a set of Lebesgue measure zero.

One could be inclined to go one step further and conjecture that *there does not exist any value definite observable outside of a single context.* This context is defined by the preparation of the state: it consists of the observable corresponding to $|\psi\rangle$, as well as of the two other orthogonal projectors associated with the two idle detectors that do not click if D_ψ clicks. The configuration can be represented by the Greechie orthogonality diagram depicted in Fig. 5(b). This conjecture is strictly metaphysical with respect to quantum mechanics, because even with our assumptions it seems that one cannot prove the sole existence of just one, unique context among the continuity of context forming "$|\psi\rangle$'s star."

Let us mention that one of the authors is inclined to believe in such an existence, another one is inclined to not believe therein, and the third author has no inclination towards either speculation.

ACKNOWLEDGEMENT

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Appendix: Detailed Analysis of $f(p_1)$

The proof of Lemma V.3 relies critically on the analysis of the function $f(p_1) = \langle a|c\rangle$ for $p_1 \in \left(\frac{3}{\sqrt{14}}, 1\right)$. Here we give further details of this analysis, which was carried out using Wolfram Mathematica 9.0.1.0.

Specifically, we have

$$f(p_1) = \langle a|c\rangle = x_3 p_1 + \frac{y_3}{k}(x_2 - p_1 p_3) - \frac{q_1 z_3}{k q_2}(y_2 z_1 + y_1 z_2),$$

where the constants are defined in terms of p_1 as follows:

$$\alpha_1 = \frac{\arccos\sqrt{\frac{2}{3}}}{\arccos\frac{1}{\sqrt{2}}}, \quad \alpha_2 = \frac{\arccos\frac{2}{\sqrt{5}}}{\arccos\sqrt{\frac{2}{3}}}, \quad \alpha_3 = \frac{\arccos\sqrt{\frac{2}{3}}}{\arccos\sqrt{\frac{2}{5}}},$$

$$\theta_{a,b} = \arccos p_1, \quad \theta_{a,v_1} = \alpha_1\theta_{a,b}, \quad \theta_{a,v_2} = \alpha_2\theta_{a,v_1},$$

$$q_1 = \sqrt{1-p_1^2}, \quad x_1 = \cos\theta_{a,v_1}, \quad y_1 = \frac{p_1(1-x_1^2)}{q_1 x_1}, \quad z_1 = \sqrt{1-x_1^2-y_1^2},$$

$$q_2 = \sqrt{1-x_1^2}, \quad x_2 = \cos\theta_{a,v_2}, \quad y2 = \frac{x_1(1-x_2^2)}{q_2 x_2}, \quad z_2 = \sqrt{1-x_2^2-y_2^2},$$

$$p_3 = p_1 x_2 + q_1\frac{y_1 y_2 - z_1 z_2}{q_2}, \quad \theta_{b,v_2} = \arccos p_3, \quad \theta_{b,c} = \alpha_3\theta_{b,v_2},$$

$$q_3 = \sqrt{1-p_3^2}, \quad x_3 = \cos\theta_{b,c}, \quad y_3 = p_3\frac{(1-x_3^2)}{q_3 x_3}, \quad z_3 = \sqrt{1-x_3^2-y_3^2},$$

$$k = \sqrt{\left(\frac{q_1}{q_2}(y_2 z_1 + y_1 z_2)\right)^2 + \left(\frac{-p_1}{q_2}(y_2 z_1 + y_1 z_2)\right)^2 + \left(\frac{p_1}{q_2}(y_1 y_2 - z_1 z_2) - q_1 x_2\right)^2}.$$

Explicitly writing $f(p_1)$ by completing the substitutions would take several pages, but using Mathematica, we find that the Taylor series expansion around $p_1 = 1$ is

$$\begin{aligned}
f(p_1) = & 1 + \frac{(p_1-1)}{\pi^2 \arccos^2\sqrt{\frac{2}{5}}}\left(\pi^2\left(\arccos^2\sqrt{\frac{2}{5}} + \operatorname{arcosh}^2\sqrt{\frac{2}{3}}\right)\right. \\
& + 8\arccos\frac{2}{\sqrt{5}}\left(\arccos\frac{2}{\sqrt{5}}\left(2\arccos^2\sqrt{\frac{2}{3}}\right.\right. \\
& + \left.\sqrt{\left(\pi^2 + 16\operatorname{arcosh}^2\sqrt{\frac{2}{3}}\right)\left(\arccos^2\sqrt{\frac{2}{5}} + \operatorname{arcosh}^2\sqrt{\frac{2}{3}}\right)}\right) + 4\arccos\sqrt{\frac{2}{3}} \\
& \times \left.\left.\sqrt{\left(\arccos^2\sqrt{\frac{2}{5}} + \operatorname{arcosh}^2\sqrt{\frac{2}{3}}\right)\left(\arccos^2\sqrt{\frac{2}{3}} + \operatorname{arcosh}^2\frac{2}{\sqrt{5}}\right)}\right)\right) \\
& + O((p_1-1)^2),
\end{aligned}$$

which numerically simplifies to

$$f(p_1) = 1 - 1.2658(1-p_1) + O((p_1-1)^2).$$

We calculate $\frac{df}{dp_1}$ from the full expression using Mathematica, but do not give here the resulting function as it is longer yet than the expression for $f(p_1)$.

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