# Chromatic Quantum Contextuality

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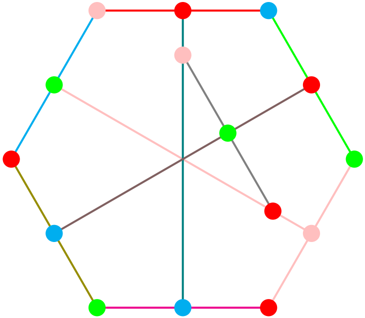
Gemini 2.5 Pro (final 6/27) enhanced version

http://tph.tuwien.ac.at/~svozil/publ/2025-IQSA-pres-Svozil.pdf

## Teaser: What's wrong with these color-assignments-as measurement-outcomes? If so, could this be improved?



# The General Strategy I: Physical Setup

- We begin with a finite collection of maximal quantum observables, also known as contexts, blocks, or subalgebras.
  - They are (with rare exceptions such as Bell's original or CHSH configurations) chosen to be intertwining (Gleason, 1957, doi:10.1512/iumj.1957.6.56050) or overlapping, existing in a Hilbert space of dimension $d \geq 3$.
  - Maximal observables have the finest spectral resolution and comprise "all information obtainable from a quantized system at a time" (von Neumann, 1931, Satz 8, doi:10.2307/1968185)
- This physical system is then represented by a $d$-uniform Greechie-type hypergraph—a graph-theoretic structure derived from the quantum model which has a Faithful Orthogonal Representation (FOR, Lovász, 1979, doi:10.1109/TIT.1979.1055985) in terms of (unit) vectors, or one-dimensional orthogonal projection operators. By its non-degenerate spectral decomposition, every such context or block can be identified with a maximal observable.

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# The General Strategy I (cntd): Analysis via Two-Valued States

We analyze the hypergraph's "classical performance" by assigning two-valued states (truth values $\{1, 0\}$ or $\{$true, false$\}$) to the observables. This method allows us to determine if the system exhibits:

- **Logical Classical Constraints:** Forced correlations like True-Implies-False (TIFS) or 'Hardy-type' True-Implies-True (TITS).
- **Classical Nonseparability:** Certain observables must be assigned the same truth value in all possible states (cf. Kochen & Specker, 1968, Separation Criterion in Theorem 0, doi:10.1512/iumj.1968.17.17004).
- **Kochen-Specker (KS) Contextuality:** Configurations where the set of admissible two-valued states is empty, proving a complete contradiction (by proof-by-contradiction) with classical assumptions.

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# The General Strategy I (cntd): Connection to Probability

- The complete set of two-valued states can be used to construct all classical probability distributions via convex summation.
- The convex hull of these states forms the classical correlation polytope.
- Bell-Type Inequalities From Hull Computation: The facets of this polytope mathematically define all possible Bell-type inequalities for the system (Froissart, 1981, doi:10.1007/BF02903286; Pitowski, 1986, doi:10.1063/1.527066).
- This remains true even for intertwining contexts/blocks (KS, 2001, doi:10.48550/arXiv.quant-ph/0012066); Klyachko, 2008, Can, Ali, Sinem & Shumovsky, doi:10.1103/PhysRevLett.101.020403).

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# An Alternative Strategy II: Analysis via Graph Colorings

- We again start with the same $d$-uniform hypergraph representing the quantum system.
- Instead of truth values, we analyze the system using graph colorings. A valid coloring assigns a "color" (a distinct outcome) to each observable such that all $d$ observables within any single context (hyperedge) receive different colors.
- This approach reveals contextual properties, such as:
  - **Forced Correlations:** Whether certain observables must have the same or different colors in every valid coloring.
  - **Generalized KS Theorem:** Contextuality is proven if the chromatic number $\chi(G)$ (the minimum number of colors needed) exceeds the dimension $d$: $\chi(G) > d$. This shows it is impossible to assign $d$ distinct outcomes per context consistently.

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# Postulate/Presumption of Chromatic Classicality, Aggregated Two-Valued States

Suppose a $d$-uniform hypergraph, that is, every hyperedge contains $d$ vertices (elements).

- **Chromatic Noncontextuality:** The color (value) of intertwining observables is independent of the (hyper)edge.
- **Chromatic Reality:** Existence of classical distinct $d$-ary elements of physical reality for all contexts/blocks of a $d$-uniform hypergraph.
- **Aggregated Two-Valued States:** Two-valued states derived from an irreversible (many-to-two) "collapse" or "reduction" or "condensation" of a $d$-coloring by identifying a single color with the value "1", and the remaining with the value "0" (Meyer, 1999, doi:10.1103/PhysRevLett.83.3751).
- Every $d$-coloring induces a canonical $d$-partitioning of all vertices (elements) of the hypergraph (Shekarriz & KS, 2022, doi:10.1063/5.0062801).

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# Information Loss from Asymmetric Aggregation

**Scenario:** An experiment with **n equiprobable outcomes** is aggregated into two groups: {1 outcome} → "1" and {d-1 outcomes} → "0".

## 1. Initial Information (in bits)

$H_{\text{initial}} = \log_2(d)$

## 2. Aggregated Information

$H_{\text{final}} = H\left(\frac{1}{d}, \frac{d-1}{d}\right)$

## 3. Information Loss (in bits)

The loss simplifies to:

$$\text{Loss} = \frac{d-1}{n} \log_2(d-1)$$

## 4. Examples (in bits)

| d | Initial H | Loss |
|---|---|---|
| **2** | 1.000 | **0.000** |
| **3** | 1.585 | **0.667** |
| **4** | 2.000 | **1.189** |

*Note: The loss for d=2 is zero because aggregation is just a one-to-one relabeling.*

# Asymptotic Behavior: Fraction of Information Lost

**Question:** What fraction of information is lost as the number of outcomes, $d$, tends to infinity?

## Limit of the Loss Ratio

The fraction of lost information is the ratio of the loss to the initial entropy:

$$\text{Fraction} = \frac{\text{Loss}}{H_{\text{initial}}} = \frac{\frac{d-1}{d}\log_2(d-1)}{\log_2(d)}$$

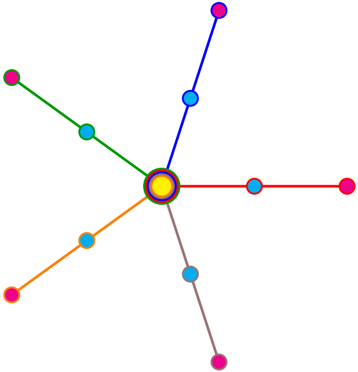
We evaluate the limit as $d \to \infty$:

$$\lim_{d\to\infty} \underbrace{\left(\frac{d-1}{d}\right)}_{\to 1} \cdot \underbrace{\left(\frac{\log_2(d-1)}{\log_2(d)}\right)}_{\to 1 \text{ (by L'Hôpital's Rule)}}$$

## Conclusion: A Total Information Loss

The limit of the fraction is $1 \times 1 = \mathbf{1}$. This means that as the number of initial outcomes grows infinitely large, we lose effectively **100%** of the initial information.

# One Coloring Instance of Three-coloring in 3 Dimensions: Star-like hypergraph



- The center yellow atom on the intertwining element maps to value 1.
- All other atoms with colors cyan and magenta (subtractive primary coloring scheme) map to two respective values 2 and 3.

In Hilbert space, this corresponds to projecting onto respective 1D subspaces—corresponding to maximal operators with non-degenerate maximal resolution; no degeneracy there.

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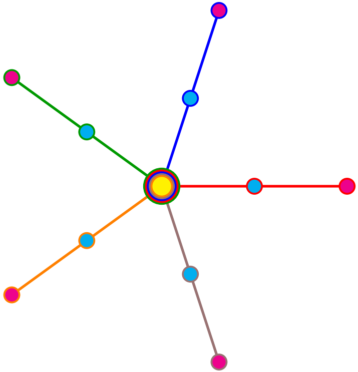

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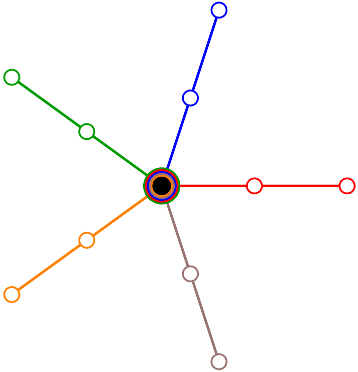
# Two-Valued State in 3 Dimensions Aggregated From Coloring: Star-like hypergraph



This (Greechie-type) hypergraph represents an aggregated system:

- The center atom, previously colored yellow, is now assigned the value 1, indicated by black.
- All non-center atoms, previously colored cyan or magenta, are now assigned the value 0, indicated by white.

In Hilbert space, this corresponds to projecting onto a 1D subspace vs. its orthogonal $(d-1)$D complement resulting in a "degeneracy of outcomes" and a non-maximal operator with degenerate spectrum.

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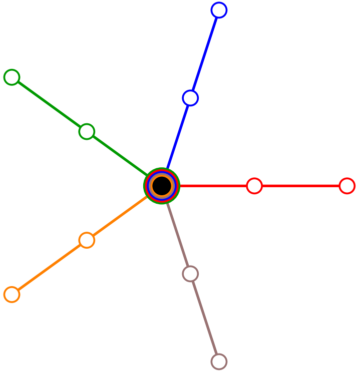

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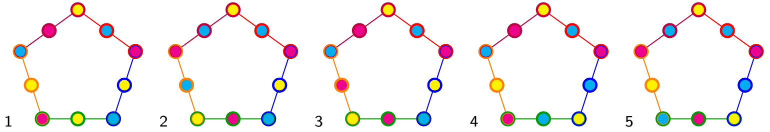
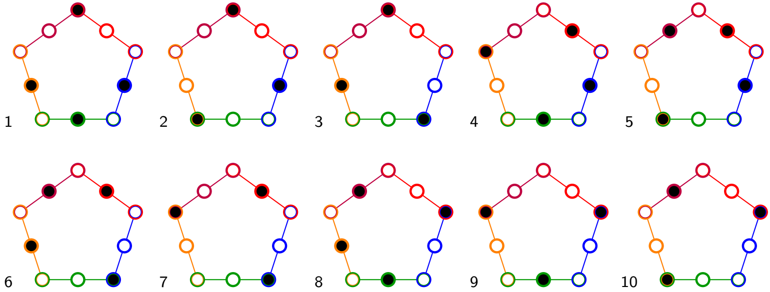
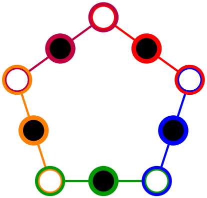
# House/Pentagon/Pentagram: 5 Non-Equivalent 3-Colorings Resulting In 10 Aggregated Two-Valued States




**Ten Aggregated (2 Colors $\rightarrow 0$, 1 Color $\rightarrow 1$) Two-Valued States (0=White, 1=Black)**



# An 11'th Non-Aggregated Two-Valued State



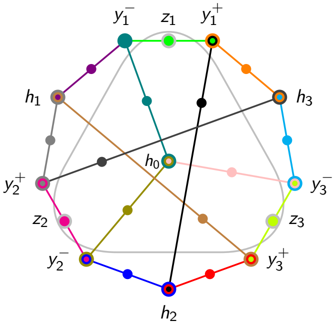
The intertwining atoms (vertices) are all assigned state 0 (white), while the non-intertwining atoms (midpoints) are all assigned state 1 (black). This state cannot be obtained by aggregating any of the five 3-colorings.

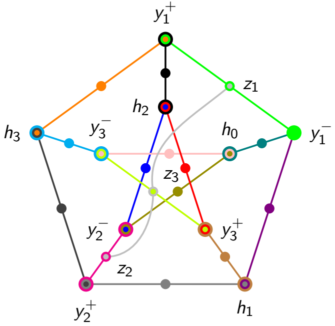
This is "dual" to a dispersionless state on the pentagon reported by Gerelle, Greechie & Miller (1974, Fig. V doi:10.1007/978-94-010-2274-3) as well as Wright (1978, doi:10.1016/B978-0-12-473250-6.50015-7), which has value 1/2 on the intertwining atoms and 0 on the non-intertwining ones.

Physically, non-aggregated two-valued states should not contribute, as they do not correspond to uniform outcomes (per context/block).

# Quantum (FOR) Chromatic Kochen-Specker Theorem

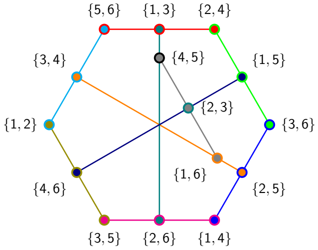
Two equivalent representations of the Yu-Oh 3-Uniform Hypergraph (2012, doi:10.1103/PhysRevLett.108.030402). Its chromatic number is 4, which is greater than the associated Hilbert space dimension $d = 3$ (and the clique number 3). Proof in KS (2025, doi:10.3390/e27040387).

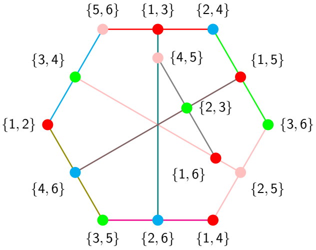


# Set Representable Chromatic Kochen-Specker Contextuality

Hypergraph representations of $G_{32}$ (Greechie, 1971, Figure 6, doi:10.1016/0097-3165(71)90015-X) which is set representable (with a separating set of two-valued states) as partition logic, as depicted. It has clique number 3 (3 elements per hyperedge/block), yet chromatic number $\chi(G_{32} = 4$. All 6 two-valued states are non-aggregated!
Proof in Shekarriz and KS (2022, doi:10.1063/5.0062801).



# Results on Chromatic Contextuality

Colorings—unlike two-valued states—represent the maximal information that is empirical available, as they fix the measurement context. They thus represent a formidable tool to identify and investigate quantum contextuality (as per differences in classical-versus-quantum predictions):

- Chromatic Kochen-Specker Theorem: There exist finite set as well as FOR (Hilbert space) representable collections of observables where the chromatic number of the associated $d$-uniform hypergraph $G$ exceeds the clique number or associated Hilbert space dimension, that is, $\chi(G) > d$.
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Thank you for your attention!