

Report

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I. INTRODUCTION

We solve the simple wave equation

$$u_{tt} - u_{xx} = 0 \quad (1)$$

where subscripts indicate derivatives $u_x = \partial u / \partial x$. We recast it to a first order system

$$\begin{aligned} \pi &:= \partial_t u \\ \xi &:= \partial_x u \\ \pi_t &= \partial_x \xi \\ \xi_t &= \partial_x \pi \end{aligned} \quad (2)$$

and evolve it in time with a Runge-Kutta integrator of 4th order.

The initial conditions are given as a Gaussian pulse travelling to the right.

$$u(x, 0) = f(x) = A \exp[-(x - t)^2 / \sigma] \quad (3)$$

$$u_t(x, 0) = \pi(x, 0) = f'(x) = \frac{2(x - t)}{\sigma} f(x) \quad (4)$$

We assume throughout this report a Courant factor of 0.4, a domain $x \in [0, 1]$ and the time step is then given by

$$dt = \mathcal{C} dx, \quad (5)$$

where \mathcal{C} is the Courant factor.

II. BOUNDARY CONDITIONS

We use radiative/absorbing boundary conditions. Namely we use

$$u_{xt} - u_{xx} = 0. \quad (6)$$

Using Eq. 2 we can rewrite Eq. 6

$$\pi_t - \xi_t = 0, \quad (7)$$

$$\pi_t + \xi_t = \xi_x + \pi_x, \quad (8)$$

for the left boundary $x = 0$ and

$$\pi_t - \xi_t = \xi_x - \pi_x, \quad (9)$$

$$\pi_t + \xi_t = 0, \quad (10)$$

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for the right boundary $x = L = 1$. Combining these formulas we derive the expressions for the boundary

$$\begin{aligned} \pi_t &= \frac{1}{2}(\xi_x + \pi_x), \\ \xi_t &= \frac{1}{2}(\xi_x - \pi_x), \end{aligned} \quad (11)$$

for the left boundary and

$$\begin{aligned} \pi_t &= \frac{1}{2}(\xi_x - \pi_x), \\ \xi_t &= \frac{1}{2}(\pi_x - \xi_x), \end{aligned} \quad (12)$$

for the right boundary. We apply these boundary conditions by modifying the right hand side function in our evolution scheme. This is known as imposing boundary conditions *strongly*.

III. CONVERGENCE

We will denote the analytical solution as \tilde{u} and the numerical solution which depends on $h = dx$ as u_h . If the numerical method is of order p , then there exists a number C such that

$$|u_h - \tilde{u}| \leq Ch^p. \quad (13)$$

Assuming that the error $|u_h - \tilde{u}|$ depends smoothly on h

$$u_h - \tilde{u} = Ch^p + \mathcal{O}(h^{p+1}). \quad (14)$$

If the exact solution \tilde{u} is known, then check the sequence

$$\log |u_h - \tilde{u}| = \log |C| + p \log h + \mathcal{O}(h), \quad (15)$$

for different values of h and fit a linear functions of $\log h$ to approximate p . In order to study the convergence of our code we fit a linear function to the sequence of Eq. 15 for $h = [h/2, h/4, h/6, h/8, h/10]$ at an instance of time. For example, for $t=1$ we get the following convergence as seen in Fig. 1 We repeat this calculation of the convergence p , for all time instances and get a non constant convergence over time as seen in Fig. 2.

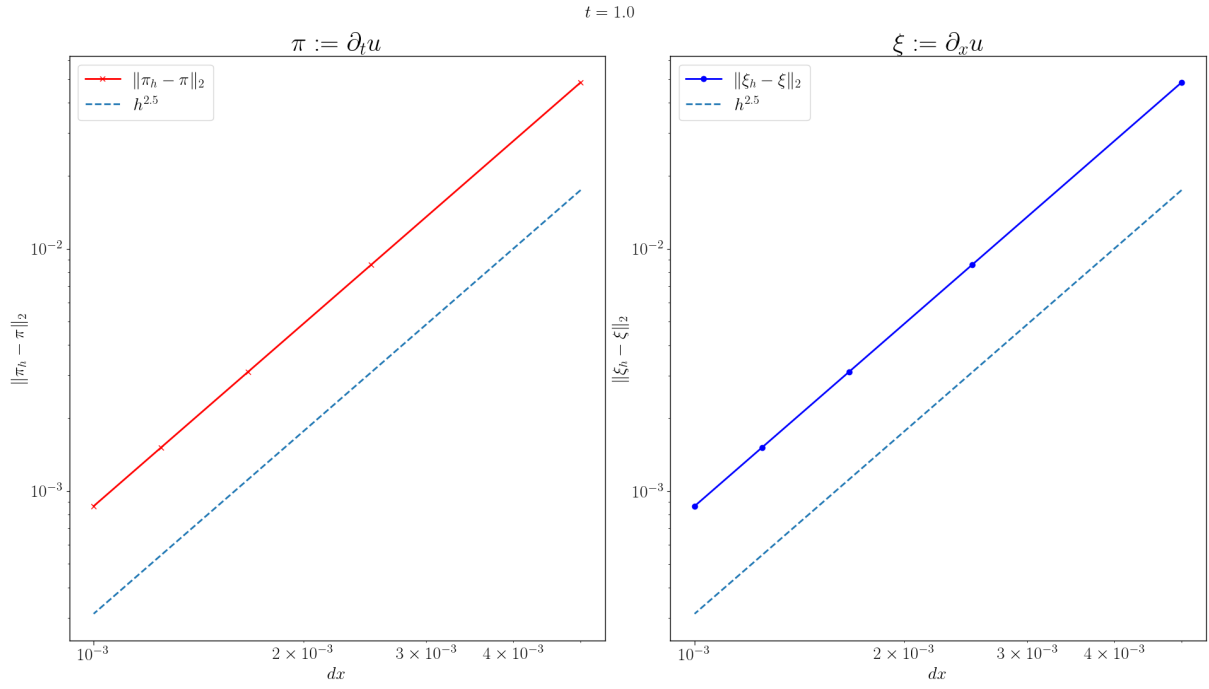


FIG. 1. Convergence of L_2 norm for $t=1$. The left subfigure shows the convergence of π and the right subfigure shows the convergence of ξ . We see here that the convergence is $p=2.5$.

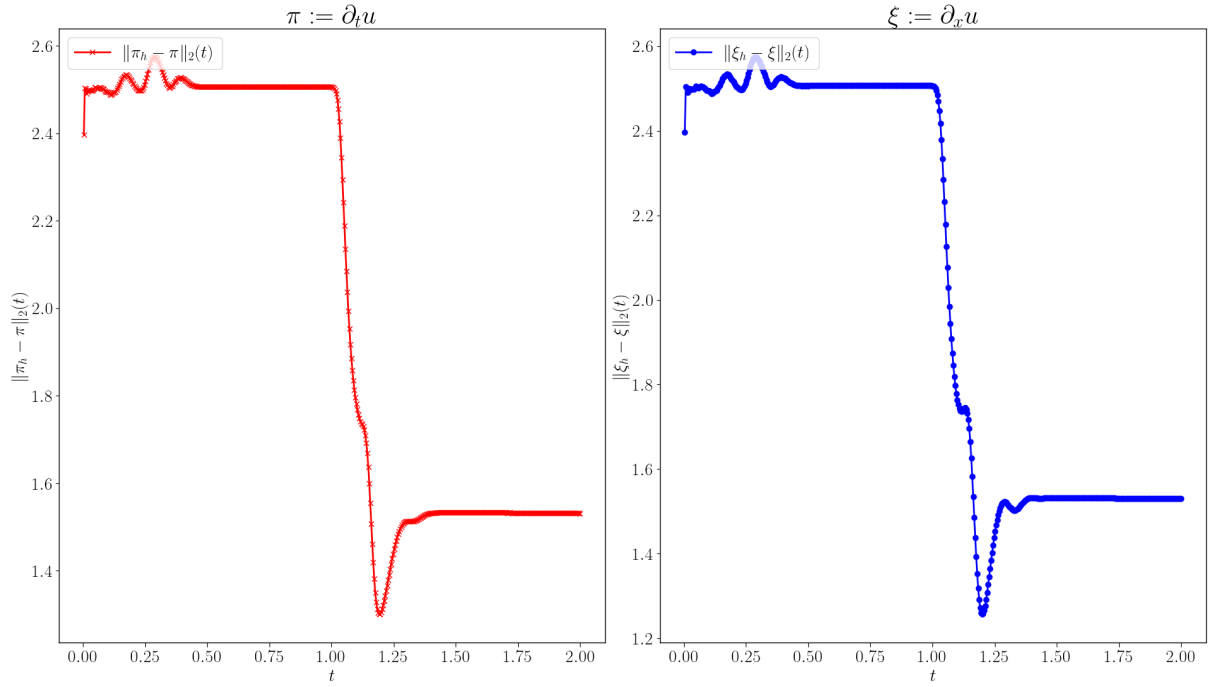


FIG. 2. Convergence of L_2 norm for $t=1$. The left subfigure shows the convergence over time of π and the right subfigure shows the convergence over time of ξ . We see that it is not constant.

