# Report


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This is a report about solving the wave equation.


## I. INTRODUCTION

We solve the simple wave equation

$$u_{tt} - u_{xx} = 0 \tag{1}$$

where subscripts indicate derivatives $u_x = \partial u/\partial x$. We recast it to a first order system

$$\begin{aligned} \pi &:= \partial_t u \\ \xi &:= \partial_x u \\ \pi_t &= \partial_x \xi \\ \xi_t &= \partial_x \pi \end{aligned} \tag{2}$$

and evolve it in time with a Runge-Kutta integrator of 4th order.

The initial conditions are given as a Gaussian pulse travelling to the right.

$$u(x,0) = f(x) = A \exp[-(x-t)^2/\sigma] \tag{3}$$

$$u_t(x,0) = \pi(x,0) = f'(x) = \frac{2(x-t)}{\sigma} f(x) \tag{4}$$

We assume throught this report a Courant factor of 0.4, a domain $x \in [0,1]$ and the time step is then given by

$$dt = \mathcal{C} dx, \tag{5}$$

where $\mathcal{C}$ is the Courant factor.

## II. BOUNDARY CONDITIONS

We use radiative/absorbing boundary conditions. Namely we use

$$u_{xt} - u_{xx} = 0. \tag{6}$$

Using Eq. 2 we can rewrite Eq. 6

$$\pi_t - \xi_t = 0, \tag{7}$$

$$\pi_t + \xi_t = \xi_x + \pi_x, \tag{8}$$

for the left boundary $x = 0$ and

$$\pi_t - \xi_t = \xi_x - \pi_x, \tag{9}$$

$$\pi_t + \xi_t = 0, \tag{10}$$

for the right boundary $x = L = 1$. Combining these formulas we derive the expressions for the boundary

$$\begin{aligned} \pi_t &= \frac{1}{2}(\xi_x + \pi_x), \\ \xi_t &= \frac{1}{2}(\xi_x + \pi_x), \end{aligned} \tag{11}$$

for the left boundary and

$$\begin{aligned} \pi_t &= \frac{1}{2}(\xi_x - \pi_x), \\ \xi_t &= \frac{1}{2}(\pi_x - \xi_x), \end{aligned} \tag{12}$$

for the right boundary. We apply these boundary conditions by modifying the right hand side function in our evolution scheme. This is known as imposing boundary conditions *strongly*.

## III. CONVERGENCE

We will denote the analytical solution as $\tilde{u}$ and the numerical solution which depends on $h = dx$ as $u_h$. If the numerical method is of order $p$, then there exists a number $C$ such that

$$|u_h - \tilde{u}| \leq Ch^p. \tag{13}$$

Assuming that the error $|u_h - \tilde{u}|$ depends smoothly on $h$

$$u_h - \tilde{u} = Ch^p + \mathcal{O}(h^{p+1}). \tag{14}$$

If the exact solution $\tilde{u}$ is known, then check the sequence

$$\log |u_h - \tilde{u}| = \log |C| + p \log h + \mathcal{O}(h), \tag{15}$$

for different values of $h$ and fit a linear functions of $\log h$ to approximate $p$. In order to study the convergence of our code we fit a linear function to the sequence of Eq. 15 for $h = [h/2, h/4, h/6, h/8, h/10]$ at an instance of time. For example, for t=1 we get the following convergence as seen in Fig. 1 We repeat this calculation of the convergece $p$, for all time instances and get a non constant convergence over time as seen in Fig. 2.


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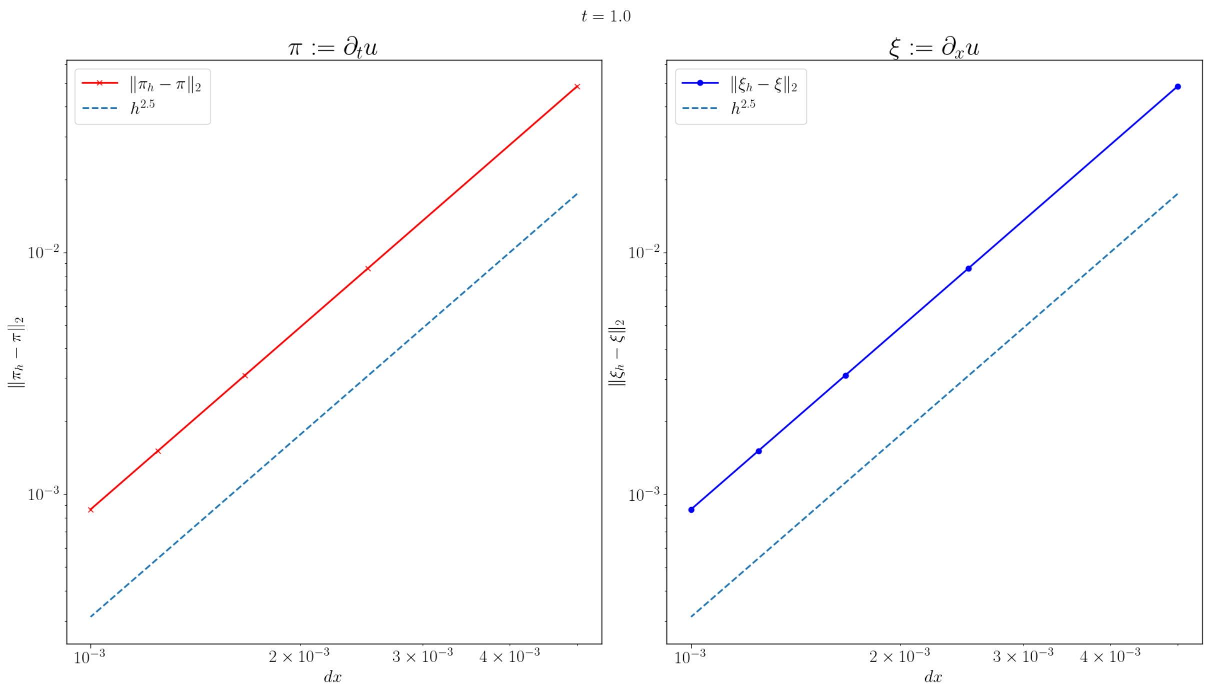


FIG. 1. Convergence of $L_2$ norm for t=1. The left subfigure shows the convergence of $\pi$ and the right subfigure shows the convergence of $\xi$. We see here that the convergence is p=2.5.

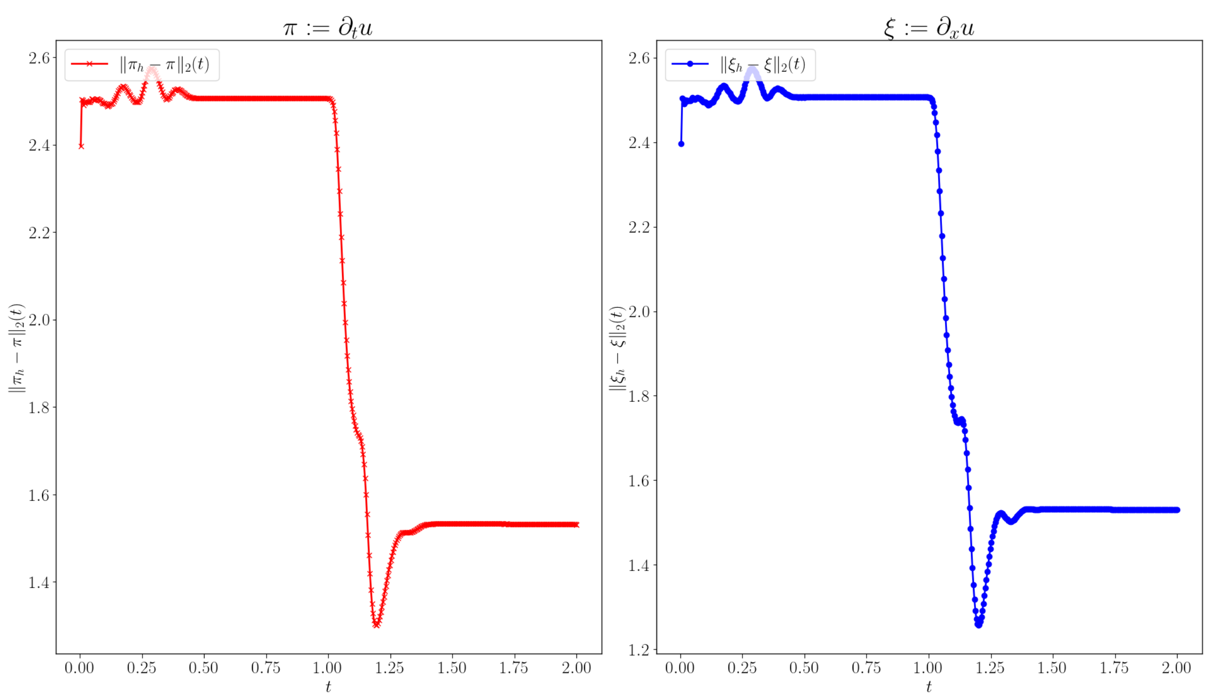


FIG. 2. Convergence of $L_2$ norm for t=1. The left subfigure shows the convergence over time of $\pi$ and the right subfigure shows the convergence over time of $\xi$. We see that it is not constant.