Report

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This is a report about solving the wave equation.

I. INTRODUCTION

We solve the simple wave equation

$$u_{tt} - u_{xx} = 0 \tag{1}$$

where subscripts indicate derivatives $u_x = \partial u/\partial x$. We recast it to a first order system

$$\begin{aligned} \pi &:= \partial_t u \\ \xi &:= \partial_x u \\ \pi_t &= \partial_x \xi \\ \xi_t &= \partial_x \pi \end{aligned} \tag{2}$$

and evolve it in time with a Runge-Kutta integrator of 4th order.

The initial conditions are given as a Gaussian pulse travelling to the right.

$$u(x,0) = f(x) = A\exp[-(x-t)^2/\sigma] \tag{3}$$

$$u_t(x,0) = \pi(x,0) = f'(x) = \frac{2(x-t)}{\sigma} f(x) \tag{4}$$

We assume throught this report a Courant factor of 0.4, a domain $x \in [0,1]$ and the time step is then given by

$$dt = \mathcal{C} dx, \tag{5}$$

where $\mathcal{C}$ is the Courant factor.

II. BOUNDARY CONDITIONS

We use radiative/absorbing boundary conditions. Namely we use

$$u_{xt} - u_{xx} = 0. \tag{6}$$

Using Eq. 2 we can rewrite Eq. 6

$$\pi_t - \xi_t = 0, \tag{7}$$

$$\pi_t + \xi_t = \xi_x + \pi_x, \tag{8}$$

for the left boundary $x = 0$ and

$$\pi_t - \xi_t = \xi_x - \pi_x, \tag{9}$$

$$\pi_t + \xi_t = 0, \tag{10}$$

for the right boundary $x = L = 1$. Combining these formulas we derive the expressions for the boundary

$$\begin{aligned} \pi_t &= \frac{1}{2}(\xi_x + \pi_x), \\ \xi_t &= \frac{1}{2}(\xi_x + \pi_x), \end{aligned} \tag{11}$$

for the left boundary and

$$\begin{aligned} \pi_t &= \frac{1}{2}(\xi_x - \pi_x), \\ \xi_t &= \frac{1}{2}(\pi_x - \xi_x), \end{aligned} \tag{12}$$

for the right boundary. We apply these boundary conditions by modifying the right hand side function in our evolution scheme. This is known as imposing boundary conditions *strongly*.

Strong imposition of the boundary conditions leads to some high frequency "noise" so we opt to weak imposition as:

$$u = \pi - \xi, \tag{13}$$

$$v = \pi + \xi, \tag{14}$$

taking now the time derivatives we get

$$u_t = \pi_t - \xi_t = \xi_x - \pi_x = -u_x, \tag{15}$$

$$v_t = \pi_t + \xi_t = \xi_x + \pi_x = v_x. \tag{16}$$

On the Left Boundary u is the *incoming* variable and for the Right Boundary v is the *incoming* variable. We have to provide boundary conditions for the incoming variables only. For weak imposition the boundary condition for the Left Boundary we have

$$u_t = -u_x - \frac{a}{dx}(u - u*), \tag{17}$$

$$v_t = v_x. \tag{18}$$

For the Right Boundary we have

$$u_t = -u_x, \tag{19}$$

$$v_t = v_x - \frac{a}{dx}(v - v*). \tag{20}$$

Now using Eq. (13) and Eq. (14) to transform back to π and ξ we get for the Left Boundary

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[[Stamatis:: the 2 factor in dx stems from the transformation back. if I use the initial equations it doesnt appear.]]

$$\pi_t = \xi_x - \frac{a}{2dx}(\pi - \xi - u*), \tag{21}$$

$$\xi_t = \pi_x + \frac{a}{2dx}(\pi - \xi - u*), \tag{22}$$

and for the Right Boundary

$$\pi_t = \xi_x - \frac{a}{2dx}(\pi + \xi - v*), \tag{23}$$

$$\xi_t = \pi_x - \frac{a}{2dx}(\pi + \xi - v*), \tag{24}$$