SBP-Compatible Spherical-Symmetry Operators by Folding a Mirrored Cartesian Grid

SphericalSBPOperators Development Team[1]

[1] *Computational Science and Engineering*

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We present a discrete construction of spherical-symmetry differential operators on $[0, R]$ based on folding a Cartesian summation-by-parts (SBP) operator built on the mirrored interval $[-R, R]$. The method enforces parity at the origin by construction, uses a metric-weighted mass matrix, and derives the divergence operator from the SBP identity with an outer-boundary-only boundary operator. The origin row of the divergence is not fixed by SBP when $p > 0$ because the metric weight vanishes at $r = 0$; we close this degree of freedom with the removable-singularity condition $(Du)(0) = (p+1)u'(0)$ computed from the odd-folded derivative. We provide a complete algorithm, variable definitions, diagnostic tests, and closure-aware validation metrics. Numerical diagnostics demonstrate that the observed polynomial test errors are boundary-closure dominated while safe-interior errors are near machine precision.

I. PROBLEM SETTING AND GOALS

We target radially symmetric operators on $r \in [0, R]$ for

$$\mathrm{Div}_p(u) = \frac{1}{r^p}\frac{d}{dr}\left(r^p u\right), \quad p \in \{0, 1, 2\}, \tag{1}$$

with $p = 2$ for 3D spherical symmetry, $p = 1$ for cylindrical symmetry, and $p = 0$ for Cartesian symmetry.

The design requirements are:

1. one half-grid vector size for all field types (no parity-specific DOF removal);
2. parity enforcement through extension maps from $[0, R]$ to $[-R, R]$;
3. metric-consistent SBP relation
$$HD + G^T H = B; \tag{2}$$
4. regular origin treatment consistent with the continuum limit.

II. PARITY AND REGULARITY AT THE ORIGIN

At $r = 0$, parity is a symmetry condition, not a boundary condition:

$$\phi(-r) = \phi(r) \quad \text{(even scalar-like field)}, \tag{3}$$
$$u(-r) = -u(r) \quad \text{(odd radial flux-like field)}. \tag{4}$$

Hence $u(0) = 0$ for odd fields.

For an odd regular flux $u(r) = a_1 r + a_3 r^3 + \cdots$, the divergence has a removable singularity:

$$\lim_{r\to 0} \mathrm{Div}_p(u) = (p+1)a_1 = (p+1)u'(0). \tag{5}$$

The discrete scheme imposes this condition exactly in the first row of D.

III. FULL-GRID SBP INGREDIENTS

Let the full mirrored grid be

$$\boldsymbol{x}^{\mathrm{full}} = (x_1, \ldots, x_M), \qquad x_1 = -R, \quad x_M = R, \quad M = 2N+1. \tag{6}$$

From `SummationByPartsOperators.jl`, we construct:

- first-derivative matrix/operator G^{full} on $[-R, R]$,
- diagonal-norm mass matrix H^{full}.

Matrix extraction supports:

1. direct matrix path when `Matrix(Dfull)` is square and consistent,
2. probing path: column j is $D^{\mathrm{full}} e_j$.

IV. FOLDING MAPS AND HALF-GRID OPERATORS

Define half-grid indices as nonnegative full-grid nodes:

$$\mathcal{I}_+ = \{j \mid x_j \geq 0\}, \qquad \boldsymbol{r} = (r_1, \ldots, r_{N_h}), \qquad r_1 = 0,\ r_{N_h} = R. \tag{7}$$

We define sparse maps:

$$R \in \mathbb{R}^{N_h \times M} \quad \text{(restriction full} \to \text{half)}, \tag{8}$$
$$E_{\mathrm{even}} \in \mathbb{R}^{M \times N_h} \quad \text{(even extension half} \to \text{full)}, \tag{9}$$
$$E_{\mathrm{odd}} \in \mathbb{R}^{M \times N_h} \quad \text{(odd extension half} \to \text{full)}. \tag{10}$$

For each full node x_j, let $\sigma(j)$ be the paired half-node index satisfying $r_{\sigma(j)} = |x_j|$. Then

$$(E_{\mathrm{even}})_{j,\sigma(j)} = 1, \tag{11}$$
$$(E_{\mathrm{odd}})_{j,\sigma(j)} = \mathrm{sign}(x_j), \tag{12}$$

with $\mathrm{sign}(0) = 0$ so that odd extensions vanish at the origin.

Folded derivatives on $[0, R]$ are

$$G_{\text{even}} = R\, G^{\text{full}}\, E_{\text{even}}, \tag{13}$$
$$G_{\text{odd}} = R\, G^{\text{full}}\, E_{\text{odd}}. \tag{14}$$

G_{even} applies to even data and returns odd derivatives; G_{odd} applies to odd data and is used for $u'(0)$ in the origin closure.

V. MASS MATRIX, BOUNDARY OPERATOR, AND DIVERGENCE CONSTRUCTION

For even integrands on the mirrored domain,

$$\int_{-R}^{R} f(x)\, dx = 2 \int_{0}^{R} f(r)\, dr, \tag{15}$$

which yields the half-grid Cartesian mass

$$H_{\text{cart,half}} = \frac{1}{2} E_{\text{even}}^{T} H^{\text{full}} E_{\text{even}}. \tag{16}$$

The metric-weighted mass is

$$H = H_{\text{cart,half}}\, \text{diag}\big(r_1^p, \dots, r_{N_h}^p\big)\,. \tag{17}$$

Only the outer boundary contributes:

$$B = \text{diag}(0, \dots, 0, R^p). \tag{18}$$

The divergence operator is defined by

$$HD = B - G_{\text{even}}^{T} H. \tag{19}$$

For diagonal-norm H, rows $i \geq 2$ are obtained by diagonal scaling:

$$D_{ij} = \frac{\big(B - G_{\text{even}}^{T} H\big)_{ij}}{H_{ii}}, \qquad i \geq 2. \tag{20}$$

At $r = 0$ and $p > 0$, $H_{11} = 0$, so SBP does not constrain row 1. We impose

$$D_{1,:} = (p+1)\, G_{\text{odd},1,:}, \tag{21}$$

which enforces $(Du)(0) = (p+1)u'(0)$ for odd u.

VI. VARIABLE AND SYMBOL DEFINITIONS

VII. CLOSURE-AWARE DIAGNOSTICS AND VALIDATION

A. Why closure detection matters

For optimized SBP operators, boundary closure rows may have the same number of nonzeros as interior rows, so row-nonzero counts are insufficient to detect closures. We therefore detect closure rows by row pattern and optionally coefficients.

B. Pattern and coefficient closure detection

Let $G = G_{\text{even}}$, $N_h = \text{size}(G, 1)$, and choose reference row $i_0 = \lfloor N_h/2 \rfloor$. Define row pattern

$$\pi_i = \text{sort}\{j - i \mid G_{ij} \neq 0\}. \tag{22}$$

Right closure width is the number of trailing rows from N_h backward with $\pi_i \neq \pi_{i_0}$. Left closure is analogous from row 1 forward.

Coefficient-sensitive refinement marks a row as closure when

$$\|G_{i,:} - G_{i_0,:}\|_\infty > \tau_{\text{coeff}}, \quad \tau_{\text{coeff}} = 10^3 \epsilon_{\text{mach}} \max(1, \|G_{i_0,:}\|_\infty)\,. \tag{23}$$

An additional right-closure hint from SBP metadata is

$$n_{\text{rw}} = \texttt{length}(D^{\text{full}}\texttt{.coefficients.right\_weights}). \tag{24}$$

The effective right closure is chosen conservatively as

$$w_{\text{right}} = \max(w_{\text{right,pattern}}, n_{\text{rw}}). \tag{25}$$

C. Safe interior and reported errors

Safe interior indices are

$$\mathcal{I}_{\text{safe}} = \{1 + w_{\text{left}}, \dots, N_h - w_{\text{right}}\}. \tag{26}$$

For each tested polynomial moment, we report:

- global maximum error,
- safe-interior maximum error on $\mathcal{I}_{\text{safe}}$,
- near-boundary maximum error on the last n_b nodes (default $n_b = 8$).

VIII. REPRESENTATIVE NUMERICAL REPORT

For a typical run (`accuracy_order`=4, $N = 64$, $R = 1$, $p = 2$), the implementation reports:

- closure widths: $w_{\text{left}} = 2$, $w_{\text{right}} = 4$;
- quartic gradient test ($\phi = r^4$):
$$\max |e| = 2.4351 \times 10^{-3},$$
$$\max_{\mathcal{I}_{\text{safe}}} |e| = 3.55 \times 10^{-15},$$
$$\max_{\text{near boundary}} |e| = 2.4351 \times 10^{-3};$$
- linear odd-divergence test ($u = r$, exact $\text{Div}_2(u) = 3$):
$$\max |e| = 6.1753 \times 10^{-4},$$
$$\max_{\mathcal{I}_{\text{safe}}} |e| = 1.24 \times 10^{-14},$$
$$\max_{\text{near boundary}} |e| = 6.1753 \times 10^{-4}.$$

TABLE I. Symbols used in the folding-SBP construction.

Symbol	Type / size	Meaning
R	scalar	outer radius of half-domain $[0, R]$
p	integer	metric power in Div_p $(0, 1, 2$ typical)
N	integer	half-grid resolution parameter ($N_h = N + 1$ for uniform mirrored grid)
M	integer	full-grid size, $M = 2N + 1$
N_h	integer	half-grid size (number of nonnegative nodes)
$\boldsymbol{x}^{\text{full}}$	$\mathbb{R}^M$	full mirrored grid coordinates on $[-R, R]$
$\boldsymbol{r}$	$\mathbb{R}^{N_h}$	half-grid coordinates on $[0, R]$
G^{full}	$\mathbb{R}^{M\times M}$	full-grid Cartesian SBP first-derivative matrix
H^{full}	$\mathbb{R}^{M\times M}$ diagonal	full-grid Cartesian SBP mass matrix
R (matrix)	$\mathbb{R}^{N_h\times M}$	restriction matrix (full $\to$ half)
E_{even}	$\mathbb{R}^{M\times N_h}$	even extension matrix (half $\to$ full)
E_{odd}	$\mathbb{R}^{M\times N_h}$	odd extension matrix (half $\to$ full)
G_{even}	$\mathbb{R}^{N_h\times N_h}$	folded gradient for even inputs
G_{odd}	$\mathbb{R}^{N_h\times N_h}$	folded derivative for odd inputs
$H_{\text{cart,half}}$	$\mathbb{R}^{N_h\times N_h}$ diagonal	folded Cartesian half-mass
H	$\mathbb{R}^{N_h\times N_h}$ diagonal	metric-weighted half-mass
B	$\mathbb{R}^{N_h\times N_h}$ diagonal	boundary operator, only $B_{N_h N_h} = R^p$ nonzero
D	$\mathbb{R}^{N_h\times N_h}$	folded divergence operator
$\mathcal{I}_{\text{safe}}$	index set	safe interior rows after closure exclusion

These data show that the dominant polynomial errors are closure-localized near $r = R$, while interior consistency is near machine precision.

IX. ALGORITHM SUMMARY

1. Build full-grid SBP derivative and mass on $[-R, R]$.
2. Construct R, E_{even}, E_{odd} from full-grid coordinates.
3. Fold derivatives: $G_{\text{even}} = RG^{\text{full}}E_{\text{even}}$ and $G_{\text{odd}} = RG^{\text{full}}E_{\text{odd}}$.
4. Fold mass and apply metric: $H = H_{\text{cart,half}}\,\mathrm{diag}(r^p)$.
5. Set $B = \mathrm{diag}(0, \dots, 0, R^p)$.
6. Compute D from SBP for rows $i \geq 2$ and enforce $D_{1,:} = (p + 1)G_{\text{odd},1,:}$.
7. Run diagnostics and closure-aware validation with safe interior reporting.

X. CONCLUSIONS

The mirrored-grid folding framework yields parity-consistent spherical operators that retain SBP structure with metric weighting and correct origin regularity. The key practical point is closure-aware interpretation: global polynomial errors can be dominated by a few outer closure rows even when interior behavior is near round-off. Pattern-plus-coefficient closure detection and right-weight metadata from the SBP operator provide a robust basis for defining safe interior metrics.

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This report documents the current implementation in `SphericalSBPOperators.jl` and its validation diagnostics.

[1] B. Strand, Summation by parts for finite difference approximations for d/dx, *J. Comput. Phys.* **110**, 47–67 (1994).

[2] K. Mattsson and J. Nordstrom, Summation by parts operators for finite difference approximations of second derivatives, *J. Comput. Phys.* **199**, 503–540 (2004).