Assumptions: $p \geq \ell$

$$\| \delta_x Q_\gamma^{\ell,p} - \pi_\gamma \|_{V^\alpha} \lesssim \rho^{\Gamma_{\ell,p}} V^\alpha(x) \qquad \alpha \in (0,1). \quad \Gamma_{\ell,p} = \sum_{q=\ell}^{p} \gamma_q.$$

$$\Rightarrow Q_\gamma^{\ell,p} V^\alpha(x) \lesssim \rho^{\Gamma_{\ell,p}} V^\alpha(x) + \pi_\gamma(V^\alpha)$$

$$\mathbb{E}[f(X_p) \mid X_{p-1}] = R_{\gamma_p} f(X_{p-1}) \qquad \mathbb{E}[f(X_p) \mid X_{\ell-1}] = R_{\gamma_\ell} \cdots R_{\gamma_p} f(X_{\ell-1})$$
$$= Q_\gamma^{\ell,p} f(X_{\ell-1})$$

I have doubt that Lemma 12 holds true without a the summability $\sum \gamma_p^2 < \infty$.

$\hookrightarrow$ error in the definition p.9

$$\mathrm{Var}\left(\sum_{p=\ell}^{N+n} \gamma_{p+1} \varphi(X_p) \,\middle|\, X_{\ell-1} \right) = \sum_{p=\ell}^{N+n} \gamma_{p+1} \gamma_{q+1} \mathrm{Cov}(\varphi(X_p), \varphi(X_q) \mid \ell-1).$$

$$= \sum_{p=\ell}^{N+n} \gamma_{p+1}^2 \mathrm{Var}(\varphi(X_p) \mid X_{\ell-1}) + 2 \sum_{p=\ell}^{N+n} \gamma_{p+1} \sum_{s=1}^{N+n-p} \gamma_{p+s+1} \mathrm{Cov}(\varphi(X_p), \varphi(X_{p+s}) \mid X_{\ell-1}).$$

$$\begin{aligned}
\mathrm{Var}(\varphi(X_p) \mid X_{\ell-1}) &= \mathbb{E}\left[\{\varphi(X_p) - \mathbb{E}[\varphi(X_p) \mid X_{\ell-1}]\}^2 \mid X_{\ell-1}\right] \\
&= \mathbb{E}\left[\{\varphi(X_p) - Q_\gamma^{\ell,p} \varphi(X_{\ell-1})\}^2 \mid X_{\ell-1}\right] \\
&\leq \mathbb{E}\left[\{\varphi(X_p) - \pi_\gamma(\varphi)\}^2 \mid X_{\ell-1}\right] \leq \mathbb{E}[\varphi^2(X_p) \mid X_{\ell-1}] \\
&\lesssim \rho^{\Gamma_{\ell,p}} V(X_{\ell-1}) \|\varphi\|_{V^{1/2}}^2 + \pi_\gamma(V)
\end{aligned}$$

Hence:

$$\sum_{p=\ell}^{N+n} \gamma_{p+1}^2 \mathrm{Var}(\varphi(X_p) \mid X_{\ell-1}) \lesssim \pi_\gamma(V) \sum_{p=\ell}^{N+n} \gamma_{p+1}^2 + \sum_{p=\ell}^{N+n} \gamma_{p+1}^2 \rho^{\Gamma_{\ell,p}}.$$
$$\lesssim \sum_{p=\ell}^{N+n} \gamma_{p+1}^2$$