Convergence of diffusions and their discretizations: from continuous to discrete processes and back

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Abstract

In this paper, we establish new quantitative convergence bounds for a class of functional autoregressive models in weighted total variation metrics. To derive this result, we show that under mild assumptions explicit minorization and Foster-Lyapunov drift conditions hold. Our bounds are then obtained adapting classical results from Markov chain theory. To illustrate our results we study the geometric ergodicity of Euler-Maruyama discretizations of diffusion with covariance matrix identity. Second, we provide a new approach to establish quantitative convergence of these diffusion processes by applying our conclusions in the discrete-time setting to a well-suited sequence of discretizations whose associated stepsizes decrease towards zero.

1 Introduction

The study of the convergence of Markov processes in general state space is a very attractive and active field of research motivated by applications in mathematics, physics and statistics [42]. Among the many works on the subject, we can mention the pioneering results from [57, 55, 56] using the renewal approach. Then, the work

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of [62, 49] paved the way for the use of *Foster-Lyapunov drift conditions* [34, 5] which, in combination of an appropriate *minorization condition*, implies (f, r)-ergodicity on general state space, drawing links with control theory, see [71, 18, 40]. This approach was successively applied to the study of Markov chains in numerous papers [10, 14, 65] and was later extended and used in the case of continuous-time Markov processes in [46, 50, 51, 21, 35, 33, 17, 72]. However, most of these results establish convergence in total variation or in V-norm and are non-quantitative. Explicit convergence bounds in the same metrics for Markov chains have been established in [68, 31, 20, 69, 67, 44, 48, 52, 32]. One of the main motivation for this work was to obtain quantitative bounds for Markov Chain Monte Carlo (MCMC) algorithms to derive stopping rules for simulations. To the authors' knowledge, the techniques developed in these papers have not been adapted to continuous-time Markov processes, expect in [66]. One of the main reason is that deriving quantitative minorization conditions for continuous-time process seems to be even more difficult than for their discrete counterpart [29]. Indeed, in most cases, the constants which appear in minorization conditions are either really pessimistic or hard to quantify precisely [43, 63].

Since the last decade, in order to avoid the use of minorization conditions, attention has been paid to consider other metrics than the total variation distance or V-norm. In particular, Wasserstein metrics have shown to be very interesting in the study of Markov processes and to derive quantitative bounds of convergence as well as in the study of perturbation bounds for Markov chains [70, 61]. For example, [58, 45, 60] introduced the notion of Ricci curvature of Markov chains and its use to derive precise bounds on variance and concentration inequalities for additive functionals. Following [36], [37] generalizes the Harris' theorem for V-norms to handle more general Wasserstein type metrics. In the same spirit, [9] establishes conditions which imply subgeometric convergence in Wasserstein distance of Markov processes. In addition, the use of Wasserstein distance has been successively applied to the study of diffusion processes and MCMC algorithms. In particular, [28, 29] establish explicit convergence rates for diffusions and McKean Vlasov processes. Regarding analysis of MCMC methods, [38] establishes geometric convergence of the pre-conditioned Crank-Nicolson algorithm. Besides, [25, 15, 11, 1] study the computational complexity in Wasserstein distance to sample from a log-concave density on $\mathbb{R}^d$ using appropriate discretizations of the overdamped Langevin diffusion. One key idea introduced in [37] and [28] is the construction of an appropriate metric designed specifically for the Markov process under consideration. While this approach leads to quantitative results in the case of diffusions or their discretization, we can still wonder if appropriate minorization conditions can be found to derive similar bounds using classical results cited above.

In this paper, we show that for a class of functional auto-regressive models, sharp minorization conditions hold using an iterated Markov coupling kernel. As a result new quantitative convergence bounds can be obtained combining this conclusion and drift inequalities for well-suited Lyapunov functionals. We apply them to the study of the Euler-Maruyama discretization of diffusions with identity covariance matrix under various curvature assumptions on the drift. The rates of convergence we derive in weighted total variation metric in this case improve the one recently established in [30]. Note that this study is significant to be able to bound the computational complexity of this scheme when it is applied to the overdamped Langevin diffusion to sample from a target density π on $\mathbb{R}^d$. Indeed, while recent papers have established precise bounds between the n-th iterate of the Euler-Maruyama scheme and π in different metrics (e.g. total variation or Wasserstein distances), the convergence of the associated Markov kernel is in general needed to obtain quantitative bounds on the mean square error or concentration inequalities for additive functionals, see [25, 45].

In a second part, we show how the results we derive for functional auto-regressive models can be used to establish explicit convergence rates for diffusion processes. First, we show that, under proper conditions on a sequence of discretizations, the distance in some metric between the distributions of the diffusion at time t with different starting points can be upper bounded by the limit of the distance between the corresponding discretizations, when the discretization stepsize decreases towards zero. Second, we design appropriate discretizations satisfying the necessary conditions we obtain and which belong to the class of functional autoregressive models we study. Therefore, under the same curvature conditions as in the discrete case, we get quantitative convergence rates for diffusions by taking the limit in the bounds we derived for the Euler-Maruyama discretizations. Finally, the rates we obtain scale similarly with respect to the parameters of the problem under consideration to the ones given in [28, 29] for the Kantorovitch-Rubinstein distance, and improve them in the case of the total variation norm. Note that in diffusion case, earlier results were derived in [13, 12, 74].

The paper is organized as follows. First, in Section 2.1, we present our main convergence results for a class of functional autoregressive models. We then specialize them to the Euler-Maruyama discretization of diffusions under various assumptions on the drift function in Section 2.2. We derive sufficient conditions for the convergence of diffusion processes with identity covariance matrix in Section 3.1, based on a sequence of well-suited discretizations. In Section 3.2, we apply our results to the continuous counterpart of the situations considered in Section 2.2. Some proofs and derivations are postponed to Section 4. Finally technical and additional results are presented in appendix. They are not the main contribution and focus of this paper but are nevertheless given for completeness.

Notations

Let A, B and C three sets with $\mathsf{C} \subset \mathsf{B}$ and $f : \mathsf{A} \to \mathsf{B}$, we set $f^{\leftarrow}(\mathsf{C}) = \{x \in \mathsf{A} : f(x) \in \mathsf{C}\}$. Let $d \in \mathbb{N}^\star$ and $\langle \cdot, \cdot \rangle$ be a scalar product over $\mathbb{R}^d$, and $\| \cdot \|$ be the corresponding norm. Let $\mathsf{A} \subset \mathbb{R}^d$ and $R \geq 0$, we denote $\operatorname{diam}(\mathsf{A}) = \sup_{(x,y) \in \mathsf{A}} \|x - y\|$ and $\Delta_{\mathsf{A},R} = \{(x, y) \in \mathsf{A} : \|x - y\| \leq R\} \subset \mathbb{R}^{2d}$ and $\Delta_{\mathsf{A}} = \Delta_{\mathsf{A},0} = \{(x, x) : x \in \mathsf{A}\}$. In this paper, we consider that $\mathbb{R}^d$ is endowed with the topology of the norm $\|\cdot\|$. $\mathcal{B}(\mathbb{R}^d)$ denotes the Borel σ-field of $\mathbb{R}^d$. Let U be an open set of $\mathbb{R}^d$, $n \in \mathbb{N}^\star$ and set $\mathrm{C}^n(\mathsf{U})$ be the set of the n-differentiable functions defined over U. Let $f \in \mathrm{C}^1(\mathsf{U})$, we denote by ∇f its gradient. Furthermore, if $f \in \mathrm{C}^2(\mathsf{U})$ we denote $\nabla^2 f$ its Hessian and Δ its Laplacian. We also denote $\mathrm{C}(\mathsf{U})$ the set of continuous functions defined over U. Let $f : \mathsf{A} \to \mathbb{R}^p$ with $p \in \mathbb{N}^\star$. The function f is said to be L-Lipschitz with $L \geq 0$ if for any $x, y \in \mathsf{A}$, $\|f(x) - f(y)\| \leq L \|x - y\|$.

Let $\mathsf{X} \in \mathcal{B}(\mathbb{R}^d)$, X is equipped with the trace of $\mathcal{B}(\mathbb{R}^d)$ over X defined by $\mathcal{X} = \{\mathsf{A} \cap \mathsf{X} : \mathsf{A} \in \mathcal{B}(\mathbb{R}^d)\}$. Let $(\mathsf{Y}, \mathcal{Y})$ be some measurable space, we denote by $\mathbb{F}(\mathsf{X}, \mathsf{Y})$ the set of the $\mathcal{X}$-measurable functions over X. For any $f \in \mathbb{F}(\mathsf{X}, \mathbb{R})$ we define its essential supremum by $\operatorname{esssup}(f) = \inf\{a \geq 0 : \lambda(|f|^{\leftarrow}(a, +\infty)) = 0\}$. Let $\mathbb{M}(\mathcal{X})$ be the set of finite signed measures over $\mathcal{X}$ and $\mu \in \mathbb{M}(\mathcal{X})$. For $f \in \mathbb{F}(\mathsf{X}, \mathbb{R})$ a μ-integrable function we denote by $\mu(f)$ the integral of f w.r.t. to μ. Let $V \in \mathbb{F}(\mathbb{R}^d, [1, +\infty))$. We define the V-norm for any $f \in \mathbb{F}(\mathsf{X}, \mathbb{R})$ and the V-total variation norm for any $\mu \in \mathbb{M}(\mathcal{X})$ as follows

$$\|f\|_V = \operatorname{esssup}(|f| / V) , \qquad \|\mu\|_V = (1/2) \sup_{f \in \mathbb{F}(\mathsf{X},\mathbb{R}), \|f\|_V \leq 1} \left| \int_{\mathbb{R}^d} f(x) \mathrm{d}\mu(x) \right| .$$

In the case where $V = 1$ this norm is called the total variation norm of μ. Let μ, ν be two probability measures over $\mathcal{X}$, *i.e.* two elements of $\mathbb{M}(\mathcal{X})$ such that $\mu(\mathsf{X}) = \nu(\mathsf{X}) = 1$. A probability measure ζ over $\mathcal{X}^{\otimes 2}$ is said to be a transference plan between μ and ν if for any $\mathsf{A} \in \mathcal{X}$, $\zeta(\mathsf{A} \times \mathcal{X}) = \mu(\mathsf{A})$ and $\zeta(\mathcal{X} \times \mathsf{A}) = \nu(\mathsf{A})$. We denote by $\mathbf{T}(\mu, \nu)$ the set of all transference plans between μ and ν. Let $\mathbf{c} \in \mathbb{F}(\mathsf{X} \times \mathsf{X}, [0, +\infty))$. We define the Wasserstein metric/distance $\mathbf{W_c}(\mu, \nu)$ between μ and ν by

$$\mathbf{W_c}(\mu, \nu) = \inf_{\zeta \in \mathbf{T}(\mu,\nu)} \int_{\mathsf{X}^2} \mathbf{c}(x, y) \mathrm{d}\zeta(x, y) .$$

Note that the term Wasserstein metric/distance is an abuse of terminology since $\mathbf{W_c}$ is only a real metric on a subspace of probability measures on X under additional conditions on $\mathbf{c}$, e.g. if $\mathbf{c}$ is a metric on $\mathbb{R}^d$, see [73, Definition 6.1]. If $\mathbf{c}(x,y) = \|x-y\|^p$ for $p \geq 1$, the Wasserstein distance of order p is defined by $\mathbf{W}_p = \mathbf{W}_{\mathbf{c}}^{1/p}$. Assume that $\mathbf{c}(x,y) = \mathbb{1}_{\Delta_{\mathsf{X}}^{\mathrm{c}}}(x,y) W(x,y)$ with $W \in \mathbb{F}(\mathsf{X} \times \mathsf{X}, [0,+\infty))$ such that W is symmetric, satisfies the triangle inequality, *i.e.* for any $x,y,z \in \mathsf{X}$, $W(x,z) \leq W(x,y) + W(y,z)$, and for any $x,y \in \mathsf{X}$, $W(x,y) = 0$ implies $x = y$. Then $\mathbf{c}$ is a metric over X^2 and the associated Wasserstein cost, denoted by $\mathbf{W}_{\mathbf{c}}$, is an extended metric. Note that if $W(x,y) = \{V(x) + V(y)\}/2$ then $\mathbf{W}_{\mathbf{c}}(\mu,\nu) = \|\mu - \nu\|_V$, see [19, Theorem 19.1.7].

Assume that $\mu \ll \nu$ and denote by $\frac{\mathrm{d}\mu}{\mathrm{d}\nu}$ its Radon-Nikodyn derivative. We define the Kullback-Leibler divergence, $\mathrm{KL}(\mu|\nu)$, between μ and ν, by

$$\mathrm{KL}(\mu|\nu) = \int_{\mathsf{X}} \log\left(\frac{\mathrm{d}\mu}{\mathrm{d}\nu}(x)\right) \mathrm{d}\mu(x) .$$

Let $\mathcal{Z}$ be a σ-field. We say that $\mathrm{P} : \mathsf{X} \times \mathcal{Z} \to [0,+\infty)$ is a Markov kernel if for any $x \in \mathsf{X}$, $\mathrm{P}(x,\cdot)$ is a probability measure over $\mathcal{Z}$ and for any $\mathsf{A} \in \mathcal{Z}$, $\mathrm{P}(\cdot,\mathsf{A}) \in \mathbb{F}(\mathsf{X},[0,+\infty))$. Let $\mathsf{Y} \in \mathcal{B}(\mathbb{R}^d)$ be equipped with $\mathcal{Y}$ the trace of $\mathcal{B}(\mathbb{R}^d)$ over $\mathcal{Y}$, $\mathrm{P} : \mathsf{X} \times \mathcal{Z}$ and $\mathrm{Q} : \mathsf{Y} \times \mathcal{Z}$ be two Markov kernels. We say that $\mathrm{K} : \mathsf{X} \times \mathsf{Y} \to \mathcal{Z}^{\otimes 2}$ is a Markov coupling kernel if for any $(x,y) \in \mathsf{X} \times \mathsf{Y}$, $\mathrm{K}((x,y),\cdot)$ is a transference plan between $\mathrm{P}(x,\cdot)$ and $\mathrm{Q}(y,\cdot)$.

2 Quantitative convergence bounds for a class of functional autoregressive models

2.1 Main results

Let $\mathsf{X} \in \mathcal{B}(\mathbb{R}^d)$ endowed with the trace of $\mathcal{B}(\mathbb{R}^d)$ on X denoted by $\mathcal{X} = \{\mathsf{A} \cap \mathsf{X} : \mathsf{A} \in \mathcal{B}(\mathbb{R}^d)\}$. In this section we consider the Markov chain $(X_k)_{k \in \mathbb{N}}$ defined by $X_0 \in \mathsf{X}$ and the following recursion for any $k \in \mathbb{N}$

$$X_{k+1} = \Pi\left(\mathcal{T}_\gamma(X_k) + \sqrt{\gamma}\, Z_{k+1}\right) , \tag{1}$$

where $\{\mathcal{T}_{\tilde{\gamma}} : \tilde{\gamma} \in (0,\bar{\gamma}]\}$ is a family of measurable functions from X to $\mathbb{R}^d$ with $\bar{\gamma} > 0$, $\gamma \in (0,\bar{\gamma}]$ is a stepsize, $(Z_k)_{k \in \mathbb{N}^\star}$ is a family of i.i.d d-dimensional zero mean Gaussian random variables with covariance identity and $\Pi : \mathbb{R}^d \to \mathsf{X}$ is a measurable function. The Markov chain $(X_k)_{k \in \mathbb{N}}$ defined by (1) is associated with the Markov kernel R_γ defined on $\mathsf{X} \times \mathcal{B}(\mathbb{R}^d)$ for any $\gamma \in (0,\bar{\gamma}]$, $x \in \mathbb{R}^d$ and $\mathsf{A} \in \mathcal{B}(\mathbb{R}^d)$ by

$$\mathrm{R}_\gamma(x,\mathsf{A}) = (2\pi\gamma)^{-d/2} \int_{\Pi^{\leftarrow}(\mathsf{A})} \exp\left[-(2\gamma)^{-1}\|y - \mathcal{T}_\gamma(x)\|^2\right] \mathrm{d}y . \tag{2}$$

Note that for any $x \in \mathsf{X}$, $\mathrm{R}_\gamma(x, \mathsf{X}) = 1$ and therefore, R_γ given in (2) is also a Markov kernel over $\mathsf{X} \times \mathcal{X}$.

In this section we state explicit convergence results for R_γ for some Wasserstein distances and discuss the rates we obtain. These results rely on appropriate minorization and Foster-Lyapunov drift conditions. We first derive the minorization condition for the n-th iterate of R_γ. To do so, we consider a Markov coupling kernel K_γ for R_γ for any $\gamma \in (0, \bar{\gamma}]$, *i.e.* for any $x, y \in \mathbb{R}^d$, $\mathrm{K}_\gamma((x,y), \cdot)$ is a transference plan between $\mathrm{R}_\gamma(x, \cdot)$ and $\mathrm{R}_\gamma(y, \cdot)$. Indeed, in that case, by [19, Theorem 19.1.6], we have for any $x, y \in \mathsf{X}$, $\gamma \in (0, \bar{\gamma}]$ and $n \in \mathbb{N}^\star$,

$$\|\delta_x \mathrm{R}_\gamma^n - \delta_y \mathrm{R}_\gamma^n\|_{\mathrm{TV}} \leq \mathrm{K}_\gamma^n((x,y), \Delta_{\mathsf{X}}^{\mathrm{c}}) \;, \tag{3}$$

where $\Delta_{\mathsf{X}} = \{(x,x) \,:\, x \in \mathsf{X}\}$. For any $x, y, z \in \mathbb{R}^d$, $\gamma \in (0, \bar{\gamma}]$, define

$$\mathrm{e}(x,y) = \begin{cases} \mathrm{E}(x,y)/\|\mathrm{E}(x,y)\| & \text{if } x \neq y \\ 0 & \text{otherwise} \end{cases} \;, \qquad \mathrm{E}(x,y) = \mathcal{T}_\gamma(y) - \mathcal{T}_\gamma(x) \;,$$

and

$$\begin{aligned} \mathcal{S}_\gamma(x,y,z) &= \mathcal{T}_\gamma(y) + (\mathrm{Id} - 2\mathrm{e}(x,y)\mathrm{e}(x,y)^\top) z \;, \\ p_\gamma(x,y,z) &= 1 \wedge \frac{\boldsymbol{\varphi}_\gamma(\|\mathrm{E}(x,y)\| - \langle \mathrm{e}(x,y), z\rangle)}{\boldsymbol{\varphi}_\gamma(\langle \mathrm{e}(x,y), z\rangle)} \;, \end{aligned}$$

where $\boldsymbol{\varphi}_\gamma$ is the one dimensional zero mean Gaussian distribution function with variance γ. Let $(U_k)_{k \in \mathbb{N}^\star}$ be a sequence of i.i.d. uniform random variables on $[0,1]$ independent of $(Z_k)_{k \in \mathbb{N}^\star}$. Define the Markov chain $(X_k, Y_k)_{k \in \mathbb{N}}$ starting from $(X_0, Y_0) \in \mathsf{X}^2$ by the recursion: for any $k \in \mathbb{N}$,

$$\begin{aligned} \tilde{X}_{k+1} &= \mathcal{T}_\gamma(X_k) + \sqrt{\gamma} Z_{k+1} \;, \\ \tilde{Y}_{k+1} &= \begin{cases} \tilde{X}_{k+1} & \text{if } \mathcal{T}_\gamma(X_k) = \mathcal{T}_\gamma(Y_k) \;, \\ W_{k+1}\tilde{X}_{k+1} + (1 - W_{k+1}) \mathcal{S}_\gamma(X_k, Y_k, \sqrt{\gamma} Z_{k+1}) & \text{otherwise} \;, \end{cases} \end{aligned}$$

where $W_{k+1} = \mathbb{1}_{(-\infty,0]}(U_{k+1} - p(X_k, Y_k, \sqrt{\gamma} Z_{k+1}))$ and finally set

$$(X_{k+1}, Y_{k+1}) = (\Pi(\tilde{X}_{k+1}), \Pi(\tilde{Y}_{k+1})) \;. \tag{4}$$

The Markov chain $(X_k, Y_k)_{k \in \mathbb{N}}$ is associated with the Markov kernel K_γ on $\mathsf{X}^2 \times \mathcal{X}^{\otimes 2}$

given for all $\gamma \in (0, \bar{\gamma}]$, $x, y \in \mathsf{X}$ and $\mathsf{A} \in \mathcal{X}^{\otimes 2}$ by

$$\begin{aligned}\mathrm{K}_\gamma((x,y),\mathsf{A}) = \frac{\mathbb{1}_{\Delta_{\mathbb{R}^d}}(\mathcal{T}_\gamma(x),\mathcal{T}_\gamma(y))}{(2\pi\gamma)^{d/2}} \int_{\mathbb{R}^d} \mathbb{1}_{\Pi_\mathsf{A}}(\tilde{x},\tilde{x}) \mathrm{e}^{-\frac{\|\tilde{x}-\mathcal{T}_\gamma(x)\|^2}{2\gamma}} \mathrm{d}\tilde{x} \\ + \frac{\mathbb{1}_{\Delta^{\mathrm{c}}_{\mathbb{R}^d}}(\mathcal{T}_\gamma(x),\mathcal{T}_\gamma(y))}{(2\pi\gamma)^{d/2}} \left[\int_{\mathbb{R}^d} \mathbb{1}_{\Pi_\mathsf{A}}(\tilde{x},\tilde{x}) p_\gamma\left(x,y,\tilde{x}-\mathcal{T}_\gamma(x)\right) \mathrm{e}^{-\frac{\|\tilde{x}-\mathcal{T}_\gamma(x)\|^2}{2\gamma}} \mathrm{d}\tilde{x} \right. \\ \left. + \int_{\mathbb{R}^d} \mathbb{1}_{\Pi_\mathsf{A}}\left(\tilde{x}, \mathcal{S}_\gamma\left(x,y,\tilde{x}-\mathcal{T}_\gamma(x)\right)\right) \left\{1 - p_\gamma\left(x,y,\tilde{x}-\mathcal{T}_\gamma(x)\right)\right\} \mathrm{e}^{-\frac{\|\tilde{x}-\mathcal{T}_\gamma(x)\|^2}{2\gamma}} \mathrm{d}\tilde{x} \right] , \end{aligned} \quad (5)$$

where $\Pi_\mathsf{A} = (\Pi, \Pi)^{\leftarrow}(\mathsf{A})$ and $\Delta_{\mathbb{R}^d} = \left\{(\tilde{x},\tilde{x}) \, : \, \tilde{x} \in \mathbb{R}^d\right\}$. Note that marginally, by definition, the distribution of X_{k+1} given X_k is $\mathrm{R}_\gamma(X_k, \cdot)$. It is well-know (see e.g. [8, Section 3.3]) that $\tilde{Y}_{k+1}$ and $\mathcal{T}_\gamma(Y_k) + \sqrt{\gamma} Z_{k+1}$ have the same distribution given Y_k, and therefore the distribution of Y_{k+1} given Y_k is $\mathrm{R}_\gamma(Y_k, \cdot)$. As a result, for any $\gamma \in (0, \bar{\gamma}]$, $x, y \in \mathsf{X}$, $\mathrm{K}_\gamma((x,y), \cdot)$ is a transference plan between $\mathrm{R}_\gamma(x, \cdot)$ and $\mathrm{R}_\gamma(y, \cdot)$.

As emphasized previously, based on (3), to study convergence of R_γ for $\gamma \in (0, \bar{\gamma}]$ (in total variation or V-norm), we first give upper bounds for $\mathrm{K}^n_\gamma((x,y), \Delta^{\mathrm{c}}_\mathsf{X})$ for any $x, y \in \mathsf{X}$ and $n \in \mathbb{N}^\star$ under appropriate conditions on $\mathcal{T}_\gamma$ and Π.

A1. *The function $\Pi : \mathbb{R}^d \to \mathsf{X}$ is non expansive: i.e. for any $x, y \in \mathbb{R}^d$, $\|\Pi(x) - \Pi(y)\| \leq \|x - y\|$.*

Note that **A1** is satisfied if Π is the proximal operator [2, Proposition 12.27] associated with a convex lower semi-continuous function $\mathrm{f} : \mathbb{R}^d \to (-\infty, +\infty]$. For example, if $\mathrm{f}(x) = \sum_{i=1}^d |x_i|$, the associated proximal operator is the soft thresholding operator [59, Section 6.5.2]. For f the convex indicator of a convex closed set $\mathsf{C} \subset \mathbb{R}^d$, defined by $\mathrm{f}(x) = 0$ for $x \in \mathsf{C}$, $\mathrm{f}(x) = +\infty$ otherwise, the proximal operator is simply the orthogonal projection onto C by [2, Example 12.21]:

$$\Pi_\mathsf{C}(x) = \operatorname*{arg\,min}_{y \in \mathsf{C}} \|y - x\| \ . \quad (6)$$

First, the class of Markov chains defined by (1) contains Euler-Maruyama discretizations of diffusion processes with identity diffusion matrix and for which $\Pi = \mathrm{Id}$. Our results will be specified for this particular case in Section 2.2. Second, for the applications that we have in mind, the use of Markov chains defined by (1) with $\Pi \neq \mathrm{Id}$ satisfying **A1**, has been proposed based on optimization literature to sample non-smooth log-concave densities [27, 7, 23, 3]. Finally, we will also make use of (1) with $\Pi = \Pi_{\mathsf{K}_n}$, where Π_{K_n} is defined by (6) with $\mathsf{C} \leftarrow \mathsf{K}_n$, and $(\mathsf{K}_n)_{n \in \mathbb{N}^\star}$ is a sequence of compact sets of $\mathbb{R}^d$, to derive our results on diffusion processes in Section 3.2.

We now consider the following assumption on $\{\mathcal{T}_{\tilde{\gamma}} \, : \, \tilde{\gamma} \in (0, \bar{\gamma}]\}$ where $\mathsf{A} \in \mathcal{B}(\mathbb{R}^{2d})$.

A 2 (A)**.** *There exists $\kappa : (0, \bar{\gamma}] \to \mathbb{R}$ such that for any $\gamma \in (0, \bar{\gamma}]$, $\gamma\kappa(\gamma) \in (-1, +\infty)$ and for any $(x, y) \in \mathsf{A} \cap \mathsf{X}^2$*

$$\|\mathcal{T}_\gamma(x) - \mathcal{T}_\gamma(y)\|^2 \leq (1 + \gamma\kappa(\gamma))\|x - y\|^2 . \tag{7}$$

Further, one of the following conditions holds for any $\gamma \in (0, \bar{\gamma}]$: (i) $\kappa(\gamma) < 0$; (ii) $\kappa(\gamma) \leq 0$; (iii) $\kappa(\gamma) > 0$.

Note that **A**2(X^2)-(i) or **A**2(X^2)-(ii) imply that for any $\gamma \in (0, \bar{\gamma}]$, $\mathcal{T}_\gamma$ is non-expansive itself (see **A**1). For $\mathsf{\kappa} : (0, \bar{\gamma}] \to \mathbb{R}$ and $\ell \in \mathbb{N}^\star$, $\gamma \in (0, \bar{\gamma}]$ such that $\gamma\mathsf{\kappa}(\gamma) \in (-1, +\infty)$, define

$$\Xi_n(\mathsf{\kappa}) = \gamma \sum_{k=1}^{n} (1 + \gamma\mathsf{\kappa}(\gamma))^{-k} . \tag{8}$$

The following theorem gives a generalization of a minorization condition on autoregressive models [25, Section 6].

Theorem 1. *Let $\mathsf{A} \in \mathcal{B}(\mathbb{R}^{2d})$ and assume* **A**1 *and* **A**2(A). *Let $(X_k, Y_k)_{k\in\mathbb{N}}$ be defined by (4) with $(X_0, Y_0) = (x, y) \in \mathsf{A} \cap \mathsf{X}^2$. Then for any $n \in \mathbb{N}^\star$*

$$\begin{aligned}\mathbb{P}\,&(X_n \neq Y_n \text{ and for any } k \in \{1, \ldots, n-1\}, (X_k, Y_k) \in \mathsf{A}) \\ &\leq \mathbb{1}_{\Delta_{\mathsf{X}}^{\mathrm{c}}}(x, y) \left\{1 - 2\mathbf{\Phi}\left(-\frac{\|x - y\|}{2\Xi_n^{1/2}(\kappa)}\right)\right\} ,\end{aligned}$$

where $\mathbf{\Phi}$ is the cumulative distribution function of the standard normal distribution on $\mathbb{R}$.

Proof. The proof is a simple application of Theorem 30 in Appendix B. □

Based on Theorem 1, since $\mathbb{P}(X_n \neq Y_n) = \mathrm{K}^n((x, y), \Delta_{\mathsf{X}}^{\mathrm{c}})$ where $(X_k, Y_k)_{k\in\mathbb{N}}$ is defined by (4) with $(X_0, Y_0) = (x, y) \in \mathsf{X}^2$, we can derive minorization conditions for the Markov kernel R_γ^n with $n \in \mathbb{N}^\star$ for any $\gamma \in (0, \bar{\gamma}]$ depending on the assumption we make on κ in **A**2(X^2). More precisely, these minorization conditions are derived using $\mathrm{K}_\gamma^{\ell\lceil 1/\gamma\rceil}$ with $\ell \in \mathbb{N}^\star$. This is a requirement to obtain sharp bounds in the limit $\gamma \to 0$. Indeed, for any $x, y \in \mathsf{X}$, based only on the results of Theorem 1, we get that for any $\ell \in \mathbb{N}^\star$, $\lim_{\gamma\to 0} \|\delta_x \mathrm{R}_\gamma^\ell - \delta_y \mathrm{R}_\gamma^\ell\|_{\mathrm{TV}} \leq 1$, whereas the following lemma implies that for any $\ell \in \mathbb{N}^\star$, $\lim_{\gamma\to 0} \|\delta_x \mathrm{R}_\gamma^{\ell\lceil 1/\gamma\rceil} - \delta_y \mathrm{R}_\gamma^{\ell\lceil 1/\gamma\rceil}\|_{\mathrm{TV}} < 1$.

Lemma 2. *Let $\bar{\gamma} > 0$ and $\mathsf{\kappa} : (0, \bar{\gamma}] \to \mathbb{R}$, with $\mathsf{\kappa}(\gamma)\gamma \in (-1, +\infty)$ for any $\gamma \in (0, \bar{\gamma}]$. We have for any $\gamma \in (0, \bar{\gamma}]$ such that $\mathsf{\kappa}(\gamma) \neq 0$ and $\ell \in \mathbb{N}^\star$*

$$\Xi_{\ell\lceil 1/\gamma\rceil}(\mathsf{\kappa}) = -\mathsf{\kappa}^{-1}(\gamma) \left\{\exp\left[-\ell \lceil 1/\gamma\rceil \log\{1 + \gamma\mathsf{\kappa}(\gamma)\}\right] - 1\right\} , \tag{9}$$

where $\Xi_{\ell\lceil 1/\gamma\rceil}(\mathsf{\kappa})$ is defined by (8). In addition, for any $\ell \in \mathbb{N}^\star$ and $\gamma \in (0, \bar{\gamma}]$

(a) $\Xi_{\ell\lceil 1/\gamma\rceil}(\mathsf{\kappa}) \geq \alpha_-(\mathsf{\kappa},\gamma,\ell) = -\mathsf{\kappa}^{-1}(\gamma)\left[\exp(-\ell\mathsf{\kappa}(\gamma))-1\right]$ *if for any* $\gamma \in (0,\bar{\gamma}]$, $\mathsf{\kappa}(\gamma) < 0$;

(b) $\Xi_{\ell\lceil 1/\gamma\rceil}(\mathsf{\kappa}) \geq \alpha_0(\mathsf{\kappa},\gamma,\ell) = \ell$ *if for any* $\gamma \in (0,\bar{\gamma}]$, $\mathsf{\kappa}(\gamma) \leq 0$;

(c) $\Xi_{\ell\lceil 1/\gamma\rceil}(\mathsf{\kappa}) \geq \alpha_+(\mathsf{\kappa},\gamma,\ell) = \mathsf{\kappa}^{-1}(\gamma)\left[1-\exp\left\{-\frac{\ell\mathsf{\kappa}(\gamma)}{1+\gamma\mathsf{\kappa}(\gamma)}\right\}\right]$ *if for any* $\gamma \in (0,\bar{\gamma}]$, $\mathsf{\kappa}(\gamma) > 0$.

Proof. The proof is postponed to Section 4.1. □

Proposition 3. *Let* $\mathsf{A} \in \mathcal{B}(\mathbb{R}^{2d})$ *and assume* **A**1 *and* **A**2(A) *hold. Let* $(X_k, Y_k)_{k\in\mathbb{N}}$ *be defined by* (4) *with* $(X_0, Y_0) = (x, y) \in \mathsf{A}\cap\mathsf{X}^2$. *Then for any* $\ell \in \mathbb{N}^*$ *and* $\gamma \in (0,\bar{\gamma}]$,

$$\mathbb{P}\left(X_{\ell\lceil\gamma\rceil} \neq Y_{\ell\lceil\gamma\rceil} \text{ and for any } k \in \{1,\ldots,n-1\}, (X_k,Y_k) \in \mathsf{A}^2\right) \\ \leq 1 - 2\mathbf{\Phi}\left(-\alpha^{-1/2}(\kappa,\gamma,\ell)\|x-y\|/2\right) , \tag{10}$$

where

(a) $\alpha = \alpha_-$ *is given in Lemma 2-(a) if* **A**2(A)-(i) *holds ;*

(b) $\alpha = \alpha_0$ *is given in Lemma 2-(b) if* **A**2(A)-(ii) *holds ;*

(c) $\alpha = \alpha_+$ *is given in Lemma 2-(c) if* **A**2(A)-(iii) *holds.*

Proof. The proof is a direct application of Theorem 1 and Lemma 2 with $\mathsf{\kappa}(\gamma) = \kappa(\gamma)$. □

Depending on the conditions imposed on κ defined in **A**2(X^2), we obtain the following consequences of Proposition 3 which establish, either an explicit convergence bound in total variation for R_γ, or a quantitative minorization condition satisfied by this kernel.

Corollary 4. *Assume* **A**1 *and* **A**2(X^2).

(a) *If* **A**2(X^2)-(i) *holds and* $\kappa_- = \sup_{\gamma\in(0,\bar{\gamma}]}\kappa(\gamma) < 0$. *Then, for any* $\gamma \in (0,\bar{\gamma}]$, R_γ *admits a unique invariant probability measure* π_γ *and we have for any* $x \in \mathbb{R}^d$ *and* $\ell \in \mathbb{N}^\star$,

$$\|\delta_x \mathrm{R}_\gamma^{\ell\lceil 1/\gamma\rceil} - \pi_\gamma\|_{\mathrm{TV}} \\ \leq 1 - 2\int_{\mathbb{R}^d} \mathbf{\Phi}\left\{-(-\kappa_-)^{1/2}\|x-y\|/\{2(\exp(-\ell\kappa_-)-1)^{1/2}\}\right\} \mathrm{d}\pi_\gamma(y) .$$

(b) If $\mathbf{A}2(\mathsf{X}^2)$*-(ii) holds and for any* $\gamma \in (0, \bar{\gamma}]$, R_γ *admits an invariant probability measure* π_γ*, then we have for any* $\gamma \in (0, \bar{\gamma}]$, $x \in \mathbb{R}^d$ *and* $\ell \in \mathbb{N}^\star$,

$$\|\delta_x \mathrm{R}_\gamma^{\ell\lceil 1/\gamma \rceil} - \pi_\gamma\|_{\mathrm{TV}} \leq 1 - 2\int_{\mathbb{R}^d} \mathbf{\Phi}\left\{-\|x-y\|/(2\ell^{1/2})\right\} \mathrm{d}\pi_\gamma(y) .$$

Proof. The proof is postponed to Section 4.2. □

In other words, if $\mathcal{T}_\gamma$ is a contractive mapping, see $\mathbf{A}2(\mathsf{X}^2)$-(i), then for $x \in \mathbb{R}^d$ the convergence of $(\delta_x \mathrm{R}_\gamma^{\ell\lceil 1/\gamma \rceil})_{\ell \in \mathbb{N}^\star}$ to π_γ in total variation is exponential in $\ell^{1/2}$. If $\mathcal{T}_\gamma$ is non expansive, see $\mathbf{A}2(\mathsf{X}^2)$-(ii), and R_γ admits an invariant probability measure, for $x \in \mathbb{R}^d$, the convergence of $(\delta_x \mathrm{R}_\gamma^{\ell\lceil 1/\gamma \rceil})_{\ell \in \mathbb{N}^\star}$ to π_γ in total variation is linear in $\ell^{1/2}$. In the case where $\mathcal{T}_\gamma$ is non expansive, see $\mathbf{A}2(\mathsf{X}^2)$-(ii), or simply Lipschitz, see $\mathbf{A}2(\mathsf{X}^2)$-(iii) and no additional assumption is made, we do not directly obtain contraction in total variation but only minorization conditions.

Corollary 5. *Assume* $\mathbf{A}1$ *and* $\mathbf{A}2(\mathsf{X}^2)$*. Then, for any* $\gamma \in (0, \bar{\gamma}]$,

(a) if $\mathbf{A}2(\mathsf{X}^2)$*-(ii) holds, for any* $x, y \in \mathsf{X}$ *with* $\|x - y\| \leq M$ *with* $M \geq 0$ *and* $\ell \in \mathbb{N}^\star$ *with* $\ell \geq \lceil M^2 \rceil$,

$$\mathrm{K}_\gamma^{\ell\lceil 1/\gamma \rceil}((x,y), \Delta_{\mathsf{X}}^{\mathrm{c}}) \leq 1 - 2\mathbf{\Phi}(-1/2) \ ; \tag{11}$$

(b) if $\mathbf{A}2(\mathsf{X}^2)$*-(iii) holds, for any* $x, y \in \mathsf{X}$ *and* $\ell \in \mathbb{N}^\star$,

$$\mathrm{K}_\gamma^{\ell\lceil 1/\gamma \rceil}((x,y), \Delta_{\mathsf{X}}^{\mathrm{c}}) \leq 1 - 2\mathbf{\Phi}\left\{-(1+\bar{\gamma})^{1/2}(1+\kappa_+)^{1/2}\|x-y\|/2\right\} , \tag{12}$$

where $\kappa_+ = \sup_{\gamma \in (0,\bar{\gamma}]} \kappa(\gamma)$.

Proof. (a) The proof is a direct application of Proposition 3-(b), the fact that $(X_k, Y_k) \in \mathsf{X}^2$ for any $k \in \mathbb{N}$ and that K_γ is the Markov kernel associated with $(X_k, Y_k)_{k \in \mathbb{N}}$.

(b) Consider the case where $\mathbf{A}2(\mathsf{X}^2)$-(iii) holds. Using that for any $t \geq 0$, $1 - \mathrm{e}^{-t} \geq t/(t+1)$ we obtain that for any $\gamma \in (0, \bar{\gamma}]$ and $\ell \in \mathbb{N}^\star$

$$\alpha_+(\kappa, \gamma, \ell) \geq \ell/(1 + (\ell + \bar{\gamma})\kappa(\gamma)) \geq (1 + (1+\bar{\gamma})\kappa_+)^{-1} \geq (1+\bar{\gamma})^{-1}(1+\kappa_+)^{-1} ,$$

where α_+ is given in Lemma 2-(c). Then, combining this result and Proposition 3-(c) complete the proof.

□

Under $\mathbf{A}2(\mathsf{X}^2)$-(ii) or $\mathbf{A}2(\mathsf{X}^2)$-(iii), to conclude that R_γ admits a unique invariant probability measure π_γ, for $\gamma \in (0, \bar{\gamma}]$, we need to impose some additional conditions.

In our application below, we are mainly interested in the case where R_γ satisfies a geometric drift condition. Let $(\mathsf{Y}, \mathcal{Y})$ be a measurable space, $\lambda \in (0,1)$, $A \geq 0$ and $V : \mathbb{R}^d \to [1, +\infty)$ be a measurable function.

D 1 ($\mathbf{D}_{\mathrm{d}}(V, \lambda, A, \mathsf{C})$)**.** *A Markov kernel* R *on* $\mathsf{Y} \times \mathcal{Y}$ *satisfies the discrete Foster-Lyapunov drift condition if for all* $y \in \mathsf{Y}$

$$\mathrm{R}V(y) \leq \lambda V(y) + A \mathbb{1}_{\mathsf{C}}(y) \; .$$

Note that this drift condition implies the existence of an invariant probability measure if R is a Feller kernel and the level sets of V are compact, see [19, Theorem 12.3.3]. In the sequel, we implicitly assume that R_γ admits a unique invariant distribution π_γ for any $\gamma \in (0, \bar{\gamma}]$, and are interested in establishing a contraction for the Wasserstein metric $\mathbf{W}_{\mathbf{c}}$ associated with the cost

$$\mathbf{c} : (x, y) \mapsto \mathbb{1}_{\Delta_{\mathsf{X}}^{\mathrm{c}}}(x, y) W(x, y) \tag{13}$$

where $W : \mathsf{X} \times \mathsf{X} \to [0, +\infty)$ satisfies for any $x, y, z \in \mathsf{X}$, $W(x, y) = W(y, x)$, $W(x, z) \leq W(x, y) + W(y, z)$ and $W(x, y) = 0$ implies that $x = y$. Note that under these conditions on W, $\mathbf{c}$ defines an extended metric on $\mathbb{R}^d$. Let μ, ν be two probability measures over $\mathcal{X}$, we highlight three cases.

- total variation: if $W = 1$ then $\mathbf{W}_{\mathbf{c}}(\mu, \nu) = \|\mu - \nu\|_{\mathrm{TV}}$;
- V-norm: if $W(x, y) = \{\tilde{V}(x) + \tilde{V}(y)\}/2$ where $\tilde{V} : \mathbb{R}^d \to [1, +\infty)$ is measurable then $\mathbf{W}_{\mathbf{c}}(\mu, \nu) = \|\mu - \nu\|_{\tilde{V}}$;
- total variation + Kantorovitch-Rubinstein metric: if $W(x, y) = 1 + \vartheta \|x - y\|$ with $\vartheta > 0$, then $\mathbf{W}_{\mathbf{c}}(\mu, \nu) \geq \|\mu - \nu\|_{\mathrm{TV}} + \vartheta \mathbf{W}_1(\mu, \nu)$.

We now state convergence bounds for Markov kernels which satisfy one of the conclusions of Corollary 5. Indeed, in order to deal with the two assumptions $\mathbf{A}2(\mathsf{X}^2)$-(ii) and $\mathbf{A}2(\mathsf{X}^2)$-(iii) together, we provide a general result regarding the contraction of R_γ in the metric $\mathbf{W}_{\mathbf{c}}$ for some function W on X^2. This result is based on an abstract condition on $\tilde{\mathrm{K}}_\gamma \mathbb{1}_{\Delta_{\mathsf{X}}^{\mathrm{c}}}$, which is satisfied under $\mathbf{A}2(\mathsf{X}^2)$-(ii) or $\mathbf{A}2(\mathsf{X}^2)$-(iii) by Corollary 5 with $\tilde{\mathrm{K}}_\gamma \leftarrow \mathrm{K}_\gamma$, and a drift condition for $\tilde{\mathrm{K}}_\gamma$, where $\tilde{\mathrm{K}}_\gamma$ is a Markov coupling kernel for R_γ. Its proof is an application of Theorem 32 in Appendix C which essentially follows from [19, Lemma 19.4.2], see also [20]. We recall that for any $M \geq 0$,

$$\Delta_{\mathsf{X}, M} = \{(x, y) \in \mathsf{X} \; : \; \|x - y\| \leq M\} \; . \tag{14}$$

Theorem 6. *Assume that there exists a measurable function* $W : \mathsf{X} \times \mathsf{X} \to [1, +\infty)$ *such that for any* $C \geq 0$,

$$\operatorname{diam}\left\{(x, y) \in \mathsf{X}^2 \ : \ W(x, y) \leq C\right\} < +\infty \ .$$

Assume in addition that there exist $\lambda \in (0, 1)$, $A \geq 0$ *such that for any* $\gamma \in (0, \bar{\gamma}]$, *there exists* $\tilde{\mathrm{K}}_\gamma$, *a Markov coupling kernel for* R_γ, *satisfying* $\mathbf{D}_{\mathrm{d}}(W, \lambda^\gamma, A\gamma, \mathsf{X}^2)$. *Further, assume that there exists* $\mathbf{\Psi} : (0, \bar{\gamma}] \times \mathbb{N}^\star \times \mathbb{R}_+ \to [0, 1]$ *such that for any* $\gamma \in (0, \bar{\gamma}]$, $\ell \in \mathbb{N}^\star$ *and* $x, y \in \mathsf{X}$

$$\begin{aligned} &\tilde{\mathrm{K}}_\gamma^{\ell\lceil 1/\gamma \rceil}((x, y), \Delta_{\mathsf{X}}^{\mathrm{c}}) \leq 1 - \mathbf{\Psi}(\gamma, \ell, \|x - y\|) \ , \qquad \mathbf{\Psi}(\gamma, \ell, 0) = 1 \ , \\ &\text{and for any } M \geq 0, \ \inf_{(x,y) \in \Delta_{\mathsf{X},M}} \mathbf{\Psi}(\gamma, \ell, \|x - y\|) > 0 \ . \end{aligned} \tag{15}$$

Then the following results hold.

(a) For any $\gamma \in (0, \bar{\gamma}]$, $M_{\mathrm{d}} \geq \operatorname{diam}\left(\{(x, y) \in \mathsf{X}^2 : W(x, y) \leq K_{\mathrm{d}}\}\right)$ *with* $K_{\mathrm{d}} = 2A(1 + \bar{\gamma})\{1 + \log^{-1}(1/\lambda)\}$, $\ell \in \mathbb{N}^\star$, $x, y \in \mathsf{X}$ *and* $k \in \mathbb{N}$

$$\mathbf{W}_{\mathbf{c}}(\delta_x \mathrm{R}_\gamma^k, \delta_y \mathrm{R}_\gamma^k) \leq C_\gamma \rho_\gamma^{\left\lfloor k(\ell\lceil 1/\gamma \rceil)^{-1} \right\rfloor} W(x, y) \ , \tag{16}$$

where $\mathbf{W}_{\mathbf{c}}$ *is the Wasserstein metric associated with* $\mathbf{c}$ *defined by* (13),

$$\begin{aligned} C_\gamma &= 2[1 + A_\gamma][1 + c_\gamma/\{(1 - \bar{\varepsilon}_{\mathrm{d},1})(1 - \lambda_\gamma)\}] \ , \\ \log(\rho_\gamma) &= \{\log(1 - \bar{\varepsilon}_{\mathrm{d},1}) \log(\lambda_\gamma)\}/\{\log(1 - \bar{\varepsilon}_{\mathrm{d},1}) + \log(\lambda_\gamma) - \log(c_\gamma)\} < 0 \ , \\ A_\gamma &= A\gamma(1 - \lambda^{\gamma\ell\lceil 1/\gamma \rceil})/(1 - \lambda^\gamma) \ , \quad c_\gamma = \lambda^{\gamma\ell\lceil 1/\gamma \rceil} A_\gamma + K_{\mathrm{d}} \ , \\ \bar{\varepsilon}_{\mathrm{d},1} &= \inf_{\gamma \in (0,\bar{\gamma}], \ (x,y) \in \Delta_{\mathsf{X},M_{\mathrm{d}}}} \mathbf{\Psi}(\gamma, \ell, \|x - y\|) \ , \quad \lambda_\gamma = (\lambda^{\gamma\ell\lceil 1/\gamma \rceil} + 1)/2 \ . \end{aligned}$$

(b) For any $\gamma \in (0, \bar{\gamma}]$, $M_{\mathrm{d}} \geq \operatorname{diam}\left(\{(x, y) \in \mathsf{X}^2 : W(x, y) \leq K_{\mathrm{d}}\}\right)$ *with* $K_{\mathrm{d}} = 2A(1 + \bar{\gamma})\{1 + \log^{-1}(1/\lambda)\}$ *and* $\ell \in \mathbb{N}^\star$, *it holds that*

$$\begin{aligned} &C_\gamma \leq \bar{C}_1 \ , \quad \log(\rho_\gamma) \leq \log(\bar{\rho}_1) \leq 0 \ , \\ &\bar{C}_1 = 2[1 + \bar{A}_1][1 + \bar{c}_1 \Big/ \left\{(1 - \bar{\varepsilon}_{\mathrm{d},1})(1 - \bar{\lambda}_1)\right\}] \ , \\ \log(\bar{\rho}_1) &= \left\{\log(1 - \bar{\varepsilon}_{\mathrm{d},1}) \log(\bar{\lambda}_1)\right\} \Big/ \left\{\log(1 - \bar{\varepsilon}_{\mathrm{d},1}) + \log(\bar{\lambda}_1) - \log(\bar{c}_1)\right\} < 0 \ , \\ &\bar{A}_1 = A(1 + \bar{\gamma}) \min(\ell, 1 + \log^{-1}(1/\lambda)) \ , \quad \bar{c}_1 = \bar{A}_1 + K_{\mathrm{d}} \ , \quad \bar{\lambda}_1 = (\lambda + 1)/2 \ , \end{aligned}$$

(c) In addition, if $\bar{\gamma} \leq 1$, $-\log(\lambda) \in [0, \log(2)]$, $A \geq 1$ *and* $0 < \bar{\varepsilon}_{\mathrm{d},1} \leq 1 - \mathrm{e}^{-1}$, *then*

$$\log^{-1}(1/\bar{\rho}_1) \leq 12 \log(2) \log\left[6A\left\{1 + \log^{-1}(1/\lambda)\right\}\right] \Big/ (\log(1/\lambda)\bar{\varepsilon}_{\mathrm{d},\bar{\gamma}}) \ . \tag{17}$$

Proof. First, note that $1-\lambda^t = -\int_0^t \log(\lambda)\mathrm{e}^{s\log(\lambda)}\mathrm{d}s \geq -\log(\lambda)\, t\, \mathrm{e}^{t\log(\lambda)}$ for any $t \in (0, \bar{t}]$, for $\bar{t} > 0$, and therefore

$$t/(1-\lambda^t) = t + t\lambda^t/(1-\lambda^t) \leq \bar{t} + \log^{-1}(\lambda^{-1}) \; . \tag{18}$$

(a) To establish (16), we apply an extension of [19, Theorem 19.4.1], given for completeness in Theorem 32. For any $x, y \in \mathsf{X}$ such that $W(x,y) \leq K_{\mathrm{d}}$ we have

$$\tilde{\mathrm{K}}_\gamma^{\ell\lceil 1/\gamma\rceil}((x,y), \Delta_{\mathsf{X}}^{\mathrm{c}}) \leq 1 - \bar{\varepsilon}_{\mathrm{d},1} \; .$$

Using that $\tilde{\mathrm{K}}_\gamma$ satisfies $\mathbf{D}_{\mathrm{d}}(W, \lambda^\gamma, A\gamma, \mathsf{X}^2)$, we can apply Theorem 32 with $M \leftarrow K_{\mathrm{d}} \geq 2A\gamma/(1-\lambda^\gamma)$ by (18), which completes the proof of (a).

(b) We now provide upper bounds for C_γ and ρ_γ. These constants are non-decreasing in c_γ and λ_γ. Therefore it suffices to give upper bounds on $c_\gamma, \varepsilon_{\mathrm{d},\gamma}$ and λ_γ. The result is then straightforward using that $(1-\lambda^{\gamma\ell\lceil 1/\gamma\rceil})/(1-\lambda^\gamma) \leq \ell\lceil 1/\gamma\rceil$, $\gamma(1-\lambda^{\gamma\ell\lceil 1/\gamma\rceil})/(1-\lambda^\gamma) \leq \bar{\gamma} + \log^{-1}(1/\lambda)$ and $\lambda^{\gamma\ell\lceil 1/\gamma\rceil} \leq \lambda$.

(c) By assumption on $\bar{\gamma}$, λ and $\bar{\varepsilon}_{\mathrm{d},1}$ we have that $\log((1-\bar{\varepsilon}_{\mathrm{d},1})^{-1}) \leq 1$ and

$$\log(\bar{\lambda}_1^{-1}) \leq \log(\lambda^{-1}) \leq \log(2) \; , \quad \mathrm{e} \leq 2(1 + 1/\log(2)) \leq K_{\mathrm{d}} \leq \bar{c}_1 \; .$$

As a result, we obtain that $\log(\bar{\lambda}_1^{-1})/\log(\bar{c}_1) \leq 1$, $\log((1-\bar{\varepsilon}_{\mathrm{d},1})^{-1})/\log(\bar{c}_1) \leq 1$. Therefore we have

$$\begin{aligned}
\log^{-1}(1/\bar{\rho}_1) &= \left[\log(\bar{\lambda}_1^{-1}) + \log((1-\bar{\varepsilon}_{\mathrm{d},1})^{-1}) + \log(\bar{c}_1)\right] \\
&\qquad\qquad / \left[\log(\bar{\lambda}_1^{-1})\log((1-\bar{\varepsilon}_{\mathrm{d},1})^{-1})\right] \\
&\leq 3\log[6A(1+\log^{-1}(1/\lambda))] / \left[\log(\bar{\lambda}_1^{-1})\log((1-\bar{\varepsilon}_{\mathrm{d},1})^{-1})\right] \; .
\end{aligned}$$

Using that $\log(1-t) \leq -t$ for any $t \in (0,1]$ and the definition of $\bar{\lambda}_1$, we obtain that

$$\log^{-1}(\bar{\rho}_1^{-1}) \leq 6\bar{\varepsilon}_{\mathrm{d},1}^{-1}(1-\lambda)^{-1}\log[6A(1+\log^{-1}(1/\lambda))] \; .$$

Finally, we get (17) using that for any $t \in [0, \log(2)]$, $1 - e^{-t} \geq (2\log(2))^{-1} t$.

□

Note that Theorem 6-(c) gives an upper bound on the rate of convergence ρ_1 in the worst case scenario for which the minorization constant $\bar{\varepsilon}_{\mathrm{d},1}$ is small and the constant λ in $\mathbf{D}_{\mathrm{d}}(V, \lambda^\gamma, A\gamma, \mathsf{X}^2)$ is close to one.

The following theorem gives the same conclusion as Theorem 6 but the dependency in the problem constants in the convergence bounds is different. We compare these bounds at the end of this section. Note however that Theorem 7 requires a stronger drift condition than Theorem 6.

Theorem 7. *Assume that there exist* $\lambda \in (0,1)$, $A \geq 0$, $\tilde{M}_{\mathrm{d}} > 0$, *a measurable function* $W : \mathsf{X} \times \mathsf{X} \to [1, +\infty)$ *and for any* $\gamma \in (0, \bar{\gamma}]$, $\tilde{\mathrm{K}}_\gamma$ *a Markov coupling kernel for* R_γ *satisfying* $\mathbf{D}_{\mathrm{d}}(W, \lambda^\gamma, A\gamma, \Delta_{\mathsf{X}, \tilde{M}_{\mathrm{d}}})$, *where* $\Delta_{\mathsf{X}, \tilde{M}_{\mathrm{d}}}$ *is defined by* (14). *Further, assume that there exists* $\mathbf{\Psi} : (0, \bar{\gamma}] \times \mathbb{N}^\star \times \mathbb{R}_+ \to [0,1]$ *such that for any* $\gamma \in (0, \bar{\gamma}]$, $\ell \in \mathbb{N}^\star$ *and* $x, y \in \mathsf{X}$, (15) *is satisfied. Then the following results holds.*

(a) For any $\gamma \in (0, \bar{\gamma}]$, $x, y \in \mathsf{X}$ *and* $k \in \mathbb{N}$

$$\mathbf{W}_{\mathbf{c}}(\delta_x \mathrm{R}_\gamma^k, \delta_y \mathrm{R}_\gamma^k) \leq \lambda^{k\gamma/4}[D_{\gamma,1} W(x,y) + D_{\gamma,2}] + \tilde{C}_\gamma \tilde{\rho}_\gamma^{k\gamma/4} , \tag{19}$$

where $\mathbf{W}_{\mathbf{c}}$ *is the Wasserstein metric associated with* $\mathbf{c}$ *defined by* (13),

$$\begin{aligned}
D_{\gamma,1} &= 1 + 4A[\log(1/\lambda)\lambda^\gamma]^{-1} , \qquad D_{\gamma,2} = D_{\gamma,1}\left[A\lambda^{-\gamma\lceil 1/\gamma \rceil \ell}\gamma \lceil 1/\gamma \rceil \ell\right] , \\
\tilde{C}_\gamma &= 8A\log^{-1}(1/\tilde{\rho}_\gamma)/\tilde{\rho}_\gamma^\gamma , \\
\log(\tilde{\rho}_\gamma) &= \{\log(\lambda)\log(1 - \tilde{\varepsilon}_{\mathrm{d},\gamma})\} / \{-\log(\tilde{c}_\gamma) + \log(1 - \tilde{\varepsilon}_{\mathrm{d},\gamma})\} , \\
\tilde{K}_{\mathrm{d}} &= \sup_{(x,y) \in \Delta_{\mathsf{X}, \tilde{M}_{\mathrm{d}}}} [W(x,y)] , \quad \tilde{c}_\gamma = \tilde{K}_{\mathrm{d}} + A\lambda^{-\gamma\lceil 1/\gamma \rceil \ell}\gamma \lceil 1/\gamma \rceil \ell , \\
\tilde{\varepsilon}_{\mathrm{d},\gamma} &= \inf_{(x,y) \in \Delta_{\mathsf{X}, \tilde{M}_{\mathrm{d}}}} \mathbf{\Psi}(\gamma, \ell, \|x - y\|) .
\end{aligned}$$

(b) For any $\gamma \in (0, \bar{\gamma}]$ *and* $\ell \in \mathbb{N}^\star$, *it holds that*

$$\begin{aligned}
D_{\gamma,1} &\leq \bar{D}_1 = 1 + 4A\log^{-1}(1/\lambda)/\lambda^{\bar{\gamma}} , \; D_{\gamma,2} \leq \bar{D}_2 = \bar{D}_1 A\lambda^{-(1+\bar{\gamma})\ell}(1+\bar{\gamma})\ell , \\
\tilde{C}_\gamma &\leq \bar{C}_2 = 8A\log^{-1}(1/\bar{\rho}_2)/\bar{\rho}_2^{\bar{\gamma}} , \\
\log(\tilde{\rho}_\gamma) &\leq \log(\bar{\rho}_2) = \{\log(\lambda)\log(1 - \bar{\varepsilon}_{\mathrm{d},2})\} / \{-\log(\bar{c}_2) + \log(1 - \bar{\varepsilon}_{\mathrm{d},2})\} , \\
\bar{c}_2 &= \tilde{K}_{\mathrm{d}} + A\lambda^{-(1+\bar{\gamma})\ell}(1+\bar{\gamma})\ell , \\
\bar{\varepsilon}_{\mathrm{d},2} &= \inf_{\gamma \in (0,\bar{\gamma}], \; (x,y) \in \Delta_{\mathsf{X}, \tilde{M}_{\mathrm{d}}}} \mathbf{\Psi}(\gamma, \ell, \|x - y\|) .
\end{aligned}$$

(c) In addition, if $\bar{\gamma} \leq 1$ *and* $\bar{\varepsilon}_{\mathrm{d},2} \leq 1 - \mathrm{e}^{-1}$, *then*

$$\log^{-1}(\bar{\rho}_2^{-1}) \leq \left[1 + \log(\tilde{K}_{\mathrm{d}}) + \log(1 + 2A\ell) + 2\ell\log(\lambda^{-1})\right] \Big/ \left[\log(\lambda^{-1})\bar{\varepsilon}_{\mathrm{d},2}\right] .$$

Proof. (a) The proof of this theorem is an application of Theorem 27 in Appendix A. Let $\gamma \in (0, \bar{\gamma}]$. Consider $\mathbf{d}(x,y) = \mathbb{1}_{\Delta_{\mathsf{X}}^{\mathrm{c}}}(x,y)$ which satisfies **H** 1. Then, since $\tilde{\mathrm{K}}_\gamma$ and $\mathbf{\Psi}$ satisfy $\mathbf{D}_{\mathrm{d}}(W, \lambda^\gamma, A\gamma, \Delta_{\mathsf{X}, \tilde{M}_{\mathrm{d}}})$ and (15) respectively, $\mathbf{H}2(\mathrm{K}_\gamma)$ and $\mathbf{H}3(\mathrm{K}_\gamma)$ are satisfied. More precisely, for any $\gamma \in (0, \bar{\gamma}]$ let $\tilde{\varepsilon}_{\mathrm{d},\gamma} = \inf_{(x,y)\in\Delta_{\mathsf{X},\tilde{M}_{\mathrm{d}}}} \mathbf{\Psi}(\gamma, \ell, \|x-y\|)$, then $\mathbf{H}2(\mathrm{K}_\gamma)$-(i) is satisfied since for any $x, y \in \Delta_{\mathsf{X},\tilde{M}_{\mathrm{d}}}$,

$$\begin{aligned}
\tilde{\mathrm{K}}_\gamma^{\lceil 1/\gamma \rceil \ell} \mathbb{1}_{\Delta_{\mathsf{X}}^{\mathrm{c}}}(x,y) &\leq \left\{ 1 - \inf_{(x,y)\in\Delta_{\mathsf{X},\tilde{M}_{\mathrm{d}}}} \mathbf{\Psi}(\gamma, \ell, \|x-y\|) \right\} \mathbb{1}_{\Delta_{\mathsf{X}}^{\mathrm{c}}}(x,y) \\
&\leq (1 - \tilde{\varepsilon}_{\mathrm{d},\gamma}) \mathbb{1}_{\Delta_{\mathsf{X}}^{\mathrm{c}}}(x,y) \; .
\end{aligned}$$

$\mathbf{H}2(\mathrm{K}_\gamma)$-(ii) is satisfied since for any $\gamma \in (0, \bar{\gamma}]$ and $x, y \in \mathsf{X}$, $\mathrm{K}_\gamma \mathbb{1}_{\Delta_{\mathsf{X}}^{\mathrm{c}}}(x,y) \leq \mathbb{1}_{\Delta_{\mathsf{X}}^{\mathrm{c}}}(x,y)$. Finally, the conditions $\mathbf{H}\,2(\mathrm{K}_\gamma)$-(iii) and $\mathbf{H}\,3(\mathrm{K}_\gamma)$ hold using $\mathbf{D}_{\mathrm{d}}(W, \lambda^\gamma, A\gamma, \Delta_{\mathsf{X},\tilde{M}_{\mathrm{d}}})$ with $W_1 \leftarrow W$, $W_2 \leftarrow W\mathbf{d}$, $\lambda_1 = \lambda_2 \leftarrow \lambda^\gamma$, $A_1 = A_2 \leftarrow A\gamma$, $\mathtt{n}_0 \leftarrow \ell \lceil 1/\gamma \rceil$ and $\mathsf{C} \leftarrow \Delta_{\mathsf{X},\tilde{M}_{\mathrm{d}}}$. Applying Theorem 27, we obtain that for any $k \in \mathbb{N}$, $\gamma \in (0, \bar{\gamma}]$ and $x, y \in \mathsf{X}$

$$\begin{aligned}
&\mathbf{W}_{\mathbf{c}}(\delta_x \mathrm{R}_\gamma^k, \delta_y \mathrm{R}_\gamma^k) \\
&\leq \lambda^{k\gamma} W(x,y) + A\gamma \left[\tilde{\rho}_\gamma^{k\gamma/4} r_{\tilde{\rho}_\gamma^\gamma} (1 + \mathbb{1}_{\Delta_{\mathsf{X}}^{\mathrm{c}}}(x,y)) + \lambda^{k\gamma/4} r_{\lambda^\gamma} \mathbf{\Xi}(x, y, \ell \lceil 1/\gamma \rceil) \right] \\
&\leq \lambda^{k\gamma/4} W(x,y) + 2 r_{\tilde{\rho}_\gamma^\gamma} A\gamma \tilde{\rho}_\gamma^{k\gamma/4} + A\gamma r_{\lambda^\gamma} \lambda^{k\gamma/4} \mathbf{\Xi}(x, y, \ell \lceil 1/\gamma \rceil) \\
&\leq \lambda^{k\gamma/4} (1 + A\gamma r_{\lambda^\gamma}) \left[W(x,y) + A\gamma \lambda^{-\ell \lceil 1/\gamma \rceil \gamma} \ell \lceil 1/\gamma \rceil \gamma \right] + 2 r_{\tilde{\rho}_\gamma^\gamma} A\gamma \tilde{\rho}_\gamma^{k\gamma/4} \; ,
\end{aligned}$$

where

$$r_{\tilde{\rho}_\gamma^\gamma} = 4 \log^{-1}(1/\tilde{\rho}_\gamma)/(\gamma \tilde{\rho}_\gamma^\gamma) \; , \quad r_{\lambda^\gamma} = 4 \log^{-1}(1/\lambda)/(\gamma \lambda^\gamma) \; .$$

This concludes the proof of (19).

(b) The proof is straightforward using that $\lambda^\gamma \geq \lambda^{\bar{\gamma}}$.

(c) First, by assumption on $\bar{\gamma}$ and λ, we have $\lambda^{-\gamma \lceil 1/\gamma \rceil \ell} \gamma \lceil 1/\gamma \rceil \ell \leq \lambda^{-(1+\bar{\gamma})\ell}(1+\bar{\gamma})\ell$. As a result and using the fact that $\log(1-t) \leq -t$ for any $t \in (0,1)$, $\log((1-\bar{\varepsilon}_{\mathrm{d},2})^{-1}) \leq 1$ and $W(x,y) \geq 1$ for any $x, y \in \mathsf{X}$, we obtain that

$$\begin{aligned}
\log^{-1}(\bar{\rho}_2^{-1}) &\leq [\log(\lambda^{-1}) \log((1 - \bar{\varepsilon}_{\mathrm{d},\bar{\gamma}})^{-1})]^{-1} [1 + \log(\bar{c}_2)] \\
&\leq \left[\log(\lambda^{-1}) \bar{\varepsilon}_{\mathrm{d},2} \right]^{-1} \left[1 + \log(\tilde{K}_{\mathrm{d}}) + \log(1 + 2A\ell\lambda^{-2\ell}) \right] \; , \\
&\leq \left[\log(\lambda^{-1}) \bar{\varepsilon}_{\mathrm{d},2} \right]^{-1} \left[1 + \log(\tilde{K}_{\mathrm{d}}) + \log(1 + 2A\ell) + 2\ell \log(\lambda^{-1}) \right] \; ,
\end{aligned}$$

which completes the proof.

□

First, note that in (19), the leading term, $\tilde{C}_\gamma \tilde{\rho}_\gamma^{k\gamma/2}$, does not depend on $x, y \in \mathsf{Y}$. Indeed, the rate in front of the initial conditions $W(x,y)$ is given by $\lambda^{\gamma/4}$ which is smaller than $\tilde{\rho}_\gamma^{\gamma/4}$.

Some remarks are in order here concerning the bounds obtained in Theorem 6-(c) and Theorem 7-(c). Assume that $\ell = 1$, we will see in Section 2.2 that the leading term in the upper bound in Theorem 6-(c), respectively Theorem 7-(c), is given by $\log(A)/(\log(\lambda^{-1})\bar{\varepsilon}_{\mathrm{d},1})$, respectively $\log(A)/(\log(\lambda^{-1})\bar{\varepsilon}_{\mathrm{d},2})$. In addition, in some of our applications below, $\bar{\varepsilon}_{\mathrm{d},1}$ is smaller than $\bar{\varepsilon}_{\mathrm{d},2}$. Therefore, in these cases the bounds provided in Theorem 7-(c) yield better rates than the ones in Theorem 6-(c). The main difference between the two results is that in the proof of Theorem 6 a drift condition on the *iterated* coupling kernel $\tilde{\mathrm{K}}_{\gamma}^{\lceil 1/\gamma \rceil}$ is required. However, even if such drift conditions can be derived from a drift condition on $\tilde{\mathrm{K}}_{\gamma}$, the constants obtained by this technique are not sharp in general. On the contrary, the proof of Theorem 7 uses the iterated minorization condition and a drift condition on the *original* coupling $\tilde{\mathrm{K}}_{\gamma}$.

2.2 Application to the projected Euler-Maruyama discretization

Here we consider the case in which the operator $\mathcal{T}_\gamma$ in (1) is given by the discretization of a diffusion. For $b : \mathbb{R}^d \to \mathbb{R}^d$, we study the projected Euler-Maruyama discretization associated to the diffusion with drift function b and diffusion coefficient Id, *i.e.* we consider the following assumption for $\mathsf{X} \subset \mathbb{R}^d$.

B 1 (X). X *is assumed to be a convex closed subset of* $\mathbb{R}^d$, $\Pi = \Pi_{\mathsf{X}}$ *is the orthogonal projection onto* X *defined in* (6) *and*

$$\mathcal{T}_\gamma(x) = x + \gamma b(x) \text{ for any } \gamma > 0 \text{ and } x \in \mathsf{X} , \tag{20}$$

where $b : \mathbb{R}^d \to \mathbb{R}^d$ *is continuous.*

Note that if $\mathsf{X} = \mathbb{R}^d$ and $\Pi = \mathrm{Id}$, then this scheme is the classical Euler-Maruyama discretization of a diffusion with drift b and diffusion coefficient Id. The application to the tamed Euler-Maruyama discretization of the results of Section 2.1 is given in Appendix D. In what follows, we show the convergence in weighted total variation for the projected Euler-Maruyama discretization and discuss the dependency of the constants appearing in the bounds we obtain with respect to the properties we assume on the drift b.

First, we show that some regularity/curvature conditions on the drift b imply condition **A**2(X^2) for $\mathcal{T}_\gamma$ given by (20). Let $\mathtt{m} \in \mathbb{R}$.

B2. *There exists* $\mathtt{L} \geq 0$ *such that* b *is* $\mathtt{L}$*-Lipschitz, i.e. for any* $x, y \in \mathsf{X}$, $\|b(x) - b(y)\| \leq \mathtt{L}\|x - y\|$ *and* $b(0) = 0$.

B 3 ($\mathtt{m}$). *For any $x, y \in \mathsf{X}$,*

$$\langle b(x) - b(y), x - y \rangle \leq -\mathtt{m} \|x - y\|^2 .$$

If there exists $U \in \mathrm{C}^1(\mathsf{X}, \mathbb{R}^d)$ such that for any $x \in \mathsf{X}$, $b(x) = -\nabla U(x)$ and **B**3($\mathtt{m}$) holds with $\mathtt{m} = 0$, respectively $\mathtt{m} > 0$ then U is convex, respectively strongly convex. Note that **B**3(0) does not imply that $\mathcal{T}_\gamma$ given by (20) is non-expansive, therefore we consider the following assumption.

B 4. *There exists $\mathtt{m}_b > 0$ such that for any $x, y \in \mathsf{X}$,*

$$\langle b(x) - b(y), x - y \rangle \leq -\mathtt{m}_b \|b(x) - b(y)\|^2 . \tag{21}$$

Note that **B** 4 implies that **B** 2 with $\mathtt{L} = \mathtt{m}_b^{-1}$ and **B** 3(0) hold. In the case where $\mathsf{X} = \mathbb{R}^d$ and there exists $U \in \mathrm{C}^1(\mathbb{R}^d, \mathbb{R})$ such that for any $x \in \mathbb{R}^d$, $b(x) = -\nabla U(x)$, [53, Theorem 2.1.5] implies that under **B** 2 and **B** 3(0), **B** 4 holds with $\mathtt{m}_b = \mathtt{L}^{-1}$. Based on Proposition 8 and assuming **B** 1, we obtain the following results on the Markov kernel R_γ defined by (2) with $\gamma > 0$.

Proposition 8. *Assume* **B**1(X) *holds for* $\mathsf{X} \subset \mathbb{R}^d$.

(a) If **B**2 *and* **B**3($\mathtt{m}$) *hold with* $\mathtt{m} \in \mathbb{R}$. *Then* (7) *in* **A**2(X^2) *holds for any* $\gamma > 0$ *with* $\kappa(\gamma) = -2\mathtt{m} + \mathtt{L}^2\gamma$. *In particular, if* $\mathtt{m} > 0$ *then* **A**2(X^2)-(i) *holds for any* $\bar{\gamma} < 2\mathtt{m}/\mathtt{L}^2$ *and if* $\mathtt{m} \leq 0$ *then* **A**2(X^2)-(iii) *holds for any* $\bar{\gamma} > 0$ *;*

(b) If **B**4 *holds, then* **A**2(X^2)-(ii) *holds with* $\kappa(\gamma) = 0$ *for any* $\bar{\gamma} \leq 2\mathtt{m}_b$.

Proof. See Section 4.3 □

Combining Proposition 8 and Proposition 3 and/or Corollary 4, we can draw the following conclusions.

• If **B**2 and **B**3($\mathtt{m}$) hold with $\mathtt{m} > 0$, then we obtain, by Proposition 8-(a), Proposition 3-(a) and Lemma 2-(a), that for any $\gamma \in (0, 2\mathtt{m}/\mathtt{L}^2)$ and $\ell \in \mathbb{N}^\star$, (10) holds with $\alpha = \alpha_-$ given by

$$\alpha_-(\kappa, \gamma, \ell) = -\frac{\exp(-\ell(-2\mathtt{m} + \mathtt{L}^2\gamma)) - 1}{-2\mathtt{m} + \mathtt{L}^2\gamma} .$$

In addition, Corollary 4-(a) implies that for any $\gamma \in (0, 2\mathtt{m}/\mathtt{L}^2]$ and $x \in \mathsf{X}$, $(\delta_x \mathrm{R}_\gamma^{\lceil 1/\gamma \rceil \ell})_{\ell \in \mathbb{N}}$ converges exponentially fast to its invariant probability measure π_γ in total variation, with a rate which does not depend on γ, but only on $\mathtt{m}$ and $\mathtt{L}$.

- Under **B** 4, combining Proposition 8-(b) and Corollary 5-(a) we obtain that on any compact set $\mathsf{K} \subset \mathsf{X}$, $\mathrm{R}_\gamma^{\lceil 1/\gamma \rceil \ell}$ satisfies the minorization condition (11) with universal constants for $\ell \geq \mathrm{diam}(\mathsf{K})^2$. In addition, if R_γ admits an invariant probability measure π_γ, then Corollary 4-(b) implies that for any $\gamma \in (0, 2\mathtt{m}_b]$ and $x \in \mathsf{X}$, $(\delta_x \mathrm{R}_\gamma^{\lceil 1/\gamma \rceil \ell})_{\ell \in \mathbb{N}}$ converges linearly in $\ell^{1/2}$ to π_γ in total variation.

- In the case where **B**2 and **B**3($\mathtt{m}$) are satisfied with $\mathtt{m} \in \mathbb{R}_-$, we obtain that for any $\gamma > 0$ and $\ell \in \mathbb{N}^\star$, (10) holds with $\alpha = \alpha_+$ given by

$$\begin{aligned}\alpha_+(\kappa, \gamma, \ell) &= (-2\mathtt{m} + \mathtt{L}^2\gamma)^{-1}\left\{1 - \exp\left[-\ell(-2\mathtt{m} + \mathtt{L}^2\gamma)/(1 + \gamma(-2\mathtt{m} + \mathtt{L}^2\gamma))\right]\right\} \\ &\leq (-2\mathtt{m} + \mathtt{L}^2\gamma)^{-1} ,\end{aligned}$$

which implies that the bound given by Proposition 3-(c) does not go to 0 when ℓ goes to infinity. Therefore we cannot directly conclude that the Markov chain converges in total variation. However, by Proposition 8-(a), Corollary 5-(b) shows that for any $\gamma \in (0, \bar{\gamma}]$ with $\bar{\gamma} > 0$, $\mathrm{R}_\gamma^{\lceil 1/\gamma \rceil}$ satisfies the minorization condition (12), with constants which only depend on $\mathtt{m}$ and $\mathtt{L}$.

We consider in the sequel of this section several assumptions on the drift function b which imply drift conditions on the Markov coupling kernel K_γ for R_γ with $\gamma \in (0, \bar{\gamma}]$. These results in combination with Proposition 8 will allow us to use Theorem 6 or Theorem 7. First, we consider conditions on b which imply that R_γ for $\gamma \in (0, \bar{\gamma}]$, is geometrically convergent in a metric which dominates the total variation distance and the Wasserstein distance of order 1. This result will be an application of Theorem 7 and the constants we end up with are independent of the dimension d. To do so, we establish that there exists a Lyapunov function W for which K_γ satisfies for $\gamma \in (0, \bar{\gamma}]$, $\mathbf{D}_{\mathrm{d}}(W, \lambda^\gamma, A\gamma, \Delta_{\mathsf{X}, M_{\mathrm{d}}})$ where $\Delta_{\mathsf{X}, M_{\mathrm{d}}}$ is given by (14) and $M_{\mathrm{d}} \geq 0$ which do not depend on the dimension.

C 1. *There exist $R_1 > 0$ and $\mathtt{m}_1^+ > 0$ such that for any $x, y \in \mathsf{X}$ with $\|x - y\| \geq R_1$,*

$$\langle b(x) - b(y), x - y \rangle \leq -\mathtt{m}_1^+ \|x - y\|^2 \ .$$

This assumption has been considered in [30, 28] and is sometimes referred to as strong convexity of the drift b outside of the ball $\mathrm{B}(0, R_1)$. For example, this condition is met if $b = -\nabla U$, with $U = U_1 + U_2$ with $U_1, U_2 \in \mathrm{C}^2(\mathbb{R}^d, \mathbb{R})$, $\sup_{\mathbb{R}^d} \|\nabla U_1\| < +\infty$ and U_2 strongly convex. In the next proposition, we derive the announced drift for $W : \mathsf{X}^2 \to [1, +\infty)$ defined for any $x, y \in \mathsf{X}$ by

$$W_1(x, y) = 1 + \|x - y\| / R_1 \ . \tag{22}$$

Proposition 9. *Assume* $\mathbf{B}1(\mathsf{X})$ *for* $\mathsf{X} \subset \mathbb{R}^d$*,* $\mathbf{B}2$*,* $\mathbf{B}3(\mathtt{m})$ *for* $\mathtt{m} \in \mathbb{R}_-$ *and* $\mathbf{C}1$*. Then* K_γ *defined by* (5) *satisfies* $\mathbf{D}_{\mathrm{d}}(W_1, \lambda^\gamma, A\gamma, \Delta_{\mathsf{X},R_1})$ *for any* $\gamma \in (0, \bar{\gamma}]$ *where* Δ_{X,R_1} *is given by* (14)*,* $\bar{\gamma} \in (0, 2\mathtt{m}_1^+/\mathtt{L}^2)$ *and*

$$\lambda = \exp\left[-(\mathtt{m}_1^+/2 - \bar{\gamma}\mathtt{L}^2/4)\right] \;, \quad A = \mathtt{m}_1^+ - \mathtt{m} \;. \tag{23}$$

Proof. Let $x, y \in \mathsf{X}$ and set $\mathrm{E} = \mathcal{T}_\gamma(y) - \mathcal{T}_\gamma(x)$. If $\mathrm{E} = 0$ then the proposition is trivial, therefore we suppose that $\mathrm{E} \neq 0$ and let $\mathrm{e} = \mathrm{E}/\|\mathrm{E}\|$. Consider Z_1, a d-dimensional Gaussian random variable with zero mean and covariance identity. By (5) and the fact that Π_{X} is non expansive, we have for any $\gamma \in (0, \bar{\gamma}]$

$$\begin{aligned}
&\mathrm{K}_\gamma \|x - y\| \qquad\qquad (24)\\
&\leq \mathbb{E}\left[(1 - p_\gamma(x, y, \sqrt{\gamma}Z_1))\left\|(\mathcal{T}_\gamma(x) + \sqrt{\gamma}Z_1) - (\mathcal{T}_\gamma(y) + \sqrt{\gamma}(\mathrm{Id} - 2\mathrm{e}\mathrm{e}^\top)Z_1)\right\|\right]\\
&= \mathbb{E}\left[\left\|\mathrm{E} - 2\sqrt{\gamma}\mathrm{e}\mathrm{e}^\top Z_1\right\|(1 - p_\gamma(x, y, \sqrt{\gamma}Z_1))\right]\\
&= \int_{\mathbb{R}} \|\mathrm{E} - 2z\mathrm{e}\| \left\{\boldsymbol{\varphi}_\gamma(z) - (\boldsymbol{\varphi}_\gamma(z) \wedge \boldsymbol{\varphi}_\gamma(\|\mathrm{E}\| - z))\right\} \mathrm{d}z\\
&= \int_{-\infty}^{\|\mathrm{E}\|/2} (\|\mathrm{E}\| - 2z)\left\{\boldsymbol{\varphi}_\gamma(z) - \boldsymbol{\varphi}_\gamma(\|\mathrm{E}\| - z)\right\} \mathrm{d}z \leq \|\mathrm{E}\| \;,
\end{aligned}$$

where we have used the change of variable $z \mapsto \|\mathrm{E}\| - z$ for the last line. Consider now the case $(x, y) \in \Delta^{\mathrm{c}}_{\mathsf{X},R_1}$. By $\mathbf{B}2$, $\mathbf{C}1$, and since for any $t \in [-1, +\infty)$, $\sqrt{1+t} \leq 1 + t/2$, we have that

$$\|\mathcal{T}_\gamma(x) - \mathcal{T}_\gamma(y)\| \leq (1 - 2\gamma\mathtt{m}_1^+ + \gamma^2\mathtt{L}^2)^{1/2}\|x - y\| \leq (1 - \gamma\mathtt{m}_1^+ + \gamma^2\mathtt{L}^2/2)\|x - y\| \;. \tag{25}$$

Combining (24) and (25) and since $\gamma < 2\mathtt{m}_1^+/\mathtt{L}^2$, we obtain that for any $(x, y) \in \Delta^{\mathrm{c}}_{\mathsf{X},R_1}$,

$$\begin{aligned}
\mathrm{K}_\gamma W_1(x, y) &\leq (1 - \gamma\mathtt{m}_1^+ + \gamma^2\mathtt{L}^2/2)\|x - y\|/R_1 + 1\\
&\leq (1 - \gamma\mathtt{m}_1^+/2 + \gamma^2\mathtt{L}^2/4)(1 + \|x - y\|/R_1) \leq \lambda^\gamma W_1(x, y) \;.
\end{aligned} \tag{26}$$

Similarly, we obtain using Proposition 8-(a) that for any $(x, y) \in \Delta_{\mathsf{X},R_1}$

$$\begin{aligned}
\mathrm{K}_\gamma W_1 &\leq (1 - \gamma\mathtt{m} + \gamma^2\mathtt{L}^2/2)\|x - y\|/R_1 + 1\\
&\leq (1 - \gamma\mathtt{m}_1^+/2 + \gamma^2\mathtt{L}^2/4)\|x - y\|/R_1 + 1 + \gamma\left\{\mathtt{m}_1^+/2 - \mathtt{m} + \gamma\mathtt{L}^2/4\right\}\\
&\leq (1 - \gamma\mathtt{m}_1^+/2 + \gamma^2\mathtt{L}^2/4)W_1(x, y) + \gamma\left[\mathtt{m}_1^+ - \mathtt{m}\right] \leq \lambda^\gamma W_1(x, y) + A\gamma \;.
\end{aligned} \tag{27}$$

We conclude the proof upon combining (26) and (27). $\square$

Theorem 10. *Assume* $\mathbf{B}1(\mathsf{X})$ *for* $\mathsf{X} \subset \mathbb{R}^d$, $\mathbf{B}2$ *and* $\mathbf{C}1$. *Assume in addition either* $\mathbf{B}3(\mathtt{m})$ *for* $\mathtt{m} \in \mathbb{R}_-$ *or* $\mathbf{B}4$. *Then the conclusions of Theorem 7 hold with* $\bar{\gamma}$, λ *and* A *given by Proposition 9*, $W = W_1$ *defined in (22), and for any* $\gamma \in (0, \bar{\gamma}]$, $\ell \in \mathbb{N}^\star$ *and* $t > 0$,

$$\text{under } \mathbf{B}3(\mathtt{m}) , \boldsymbol{\Psi}(\gamma, \ell, t) = 2\boldsymbol{\Phi}\{-t/(2\Xi_{\ell\lceil 1/\gamma\rceil}^{1/2}(\kappa))\} , \tag{28}$$

$$\text{under } \mathbf{B}4 \ , \boldsymbol{\Psi}(\gamma, \ell, t) = \begin{cases} 2\boldsymbol{\Phi}\{-1/2\} & \text{if } \ell \geq \lceil R_1 \rceil^2 \text{ and } t \leq R_1 , \\ 2\boldsymbol{\Phi}\{-t/(2\Xi_{\ell\lceil 1/\gamma\rceil}^{1/2}(\kappa))\} & \text{otherwise} , \end{cases} \tag{29}$$

where κ *is given in Proposition 8-(a) and* $\Xi_{\ell\lceil 1/\gamma\rceil}$ *in (9).*

Proof. We first assume that $\mathbf{B}3(\mathtt{m})$ holds. Let $\bar{\gamma} \in (0, 2\mathtt{m}_1^+/\mathtt{L}^2)$. Using Proposition 9 we obtain that W_1 given by (22) satisfies $\mathbf{D}_{\mathrm{d}}(W_1, \lambda^\gamma, A\gamma, \Delta_{\mathsf{X},R_1})$ for any $\gamma \in (0, \bar{\gamma}]$ with λ and A given in (23). Using Theorem 1, Proposition 8-(a) and Lemma 2, we have for any $\gamma \in (0, \bar{\gamma}]$, $\ell \in \mathbb{N}^\star$ and $x, y \in \mathsf{X}$

$$\mathrm{K}_\gamma((x,y), \Delta_{\mathsf{X}}^{\mathrm{c}}) \leq 1 - 2\boldsymbol{\Phi}(-\Xi_{\ell\lceil 1/\gamma\rceil}^{-1/2}(\kappa) \|x - y\| /2) ,$$

where $\kappa(\gamma) = -2\mathtt{m} + \gamma\mathtt{L}^2$, which concludes the proof.

The proof under $\mathbf{B}4$ follows the same lines upon noting that $\mathbf{B}4$ implies that $\mathbf{B}3(0)$ holds and using Proposition 8-(b) instead of Proposition 8-(a). □

Let $\bar{\gamma} \in (0, \max(2\mathtt{m}_1^+/\mathtt{L}^2, 1))$, $\ell \in \mathbb{N}^\star$ specified below, $\lambda_{\bar{\gamma},a}, \rho_{\bar{\gamma},a} \in (0,1)$ and $D_{\bar{\gamma},1,a}, D_{\bar{\gamma},2,a}, C_{\bar{\gamma},a} \geq 0$ the constants given by Theorem 10 such that for any $k \in \mathbb{N}$, $\gamma \in (0, \bar{\gamma}]$ and $x, y \in \mathsf{X}$

$$\mathbf{W}_{\mathbf{c}_1}(\updelta_x \mathrm{R}_\gamma^k, \updelta_y \mathrm{R}_\gamma^k) \leq \lambda_{\bar{\gamma},a}^{k\gamma/4}[D_{\bar{\gamma},1,a} W_1(x,y) + D_{\bar{\gamma},2,a}] + C_{\bar{\gamma},a}\rho_{\bar{\gamma},a}^{k\gamma/4} , \tag{30}$$

with $\mathbf{c}_1(x,y) = \mathbb{1}_{\Delta_{\mathsf{X}}^{\mathrm{c}}}(x,y)(1 + \|x-y\|/R_1)$ for any $x, y \in \mathsf{X}$. Note that using Theorem 7-(b), we obtain that the following limits exist and do not depend on $\mathtt{L}$

$$\begin{cases} D_{1,a} = \lim_{\bar{\gamma}\to 0} D_{\bar{\gamma},1,a} , \quad D_{2,a} = \lim_{\bar{\gamma}\to 0} D_{\bar{\gamma},2,a} , \quad C_a = \lim_{\bar{\gamma}\to 0} C_{\bar{\gamma},a} , \\ \lambda_a = \lim_{\bar{\gamma}\to 0} \lambda_{\bar{\gamma},a} , \quad \rho_a = \lim_{\bar{\gamma}\to 0} \rho_{\bar{\gamma},a} . \end{cases} \tag{31}$$

We now give upper bounds on the rate $\rho_{\bar{\gamma},a}$ and ρ_a using Theorem 7-(c) depending on the assumptions in Theorem 10.

(a) If $\mathbf{B}4$ holds, set $\ell = \lceil R_1^2 \rceil$. Using that $2\boldsymbol{\Phi}(-1/2) \leq 1 - \mathrm{e}^{-1}$ and choosing $\mathtt{m}_1^+$ sufficiently small such that the conditions of Theorem 7-(c) hold, we have

$$\log^{-1}(\rho_{\bar{\gamma},a}^{-1}) \leq \Big[1 + \log(2) + \log\Big(1 + 2(1 + R_1^2)\mathtt{m}_1^+\Big) \\ + 2(1 + R_1^2)(\mathtt{m}_1^+ - \bar{\gamma}\mathtt{L}^2/2)\Big] \Big/ \Big[(\mathtt{m}_1^+ - \bar{\gamma}\mathtt{L}^2/2)\boldsymbol{\Phi}\{-1/2\}\Big] . \tag{32}$$

Taking the limit $\bar{\gamma} \to 0$ in (32) and using that for any $t \geq 0$, $\log(1+t) \leq t$, we get that

$$\log^{-1}(\rho_a^{-1}) \leq (1 + \log(2))/(\mathtt{m}_1^+ \boldsymbol{\Phi}\{-1/2\}) + 4(1 + R_1^2)/\boldsymbol{\Phi}\{-1/2\} \ . \tag{33}$$

The leading term in (33) is of order $\max(R_1^2, 1/\mathtt{m}_1^+)$, which corresponds to the one identified in [30, Theorem 2.8] and is optimal, see [28, Remark 2.10].

(b) If **B**3($\mathtt{m}$) holds with $\mathtt{m} \in \mathbb{R}_-$, set $\ell = 1$. Choosing $\mathtt{m}_1^+ > 0$ sufficiently small and $R_1, |\mathtt{m}|$ sufficiently large such that the conditions of Theorem 7-(c) hold, we have

$$\begin{aligned}\log^{-1}(\rho_{\bar{\gamma},a}^{-1}) \leq \Big[1 + \log(2) + \log\Big(1 + 2\{\mathtt{m}_1^+ - \mathtt{m}\}\Big) \\ +2(\mathtt{m}_1^+ - \bar{\gamma}\mathtt{L}^2/2)\Big]\Big/\Big[(\mathtt{m}_1^+ - \bar{\gamma}\mathtt{L}^2/2)\boldsymbol{\Phi}\{-\Xi_{\lceil 1/\bar{\gamma}\rceil}^{-1/2}(\kappa)R_1/2\}\Big] \ .\end{aligned}$$

Taking the limit $\bar{\gamma} \to 0$ in this result and using (9), we get that

$$\begin{aligned}\log^{-1}(\rho_a^{-1}) \leq \Big[1 + \log(2) + \log(1 + 2\{\mathtt{m}_1^+ - \mathtt{m}\}) + 2\mathtt{m}_1^+\Big] \\ \Big/\Big[\mathtt{m}_1^+ \boldsymbol{\Phi}\{-(-\mathtt{m})^{1/2}R_1/(2 - 2\mathrm{e}^{2\mathtt{m}})^{1/2}\}\Big] \ .\end{aligned} \tag{34}$$

The leading term on the right hand side is then $\{\boldsymbol{\Phi}\{-(-\mathtt{m})^{1/2}R_1/(2-2\mathrm{e}^{2\mathtt{m}})\}\}^{-1}$ which is bounded by $\exp\{\mathtt{m}R_1^2/(4(1-\mathrm{e}^{2\mathtt{m}}))\}$. Then, we can compare our result with [30, Theorem 2.10, Equation (2.64)]. Denoting by ρ_E the rate obtained taking the limit $h \to 0$ in [30, Theorem 2.10, Equation (2.64)], we get that

$$\log^{-1}(\rho_E^{-1}) = \psi(\mathtt{m}_1^+, R_1)\exp(-\tilde{c}_0^{-1}\mathtt{m}R_1^2) \ ,$$

with $\psi : \mathbb{R}^2 \to \mathbb{R}_+$ a rational function in both variables and $\tilde{c}_0 \leq 0.00051$. Therefore, the leading term in $\log^{-1}(\rho_E^{-1})$ is lower bounded by $\exp(1960\mathtt{m}R_1^2)$ whereas for $\mathtt{m} \geq 1$, the leading term in the upper bound for $\log^{-1}(\rho_a^{-1})$ given in (34) is $\exp(0.29\mathtt{m}R_1^2)$.

We now derive uniform ergodic convergence in V-norm under weaker conditions. The following assumption enforces that the radial part of b decreases faster than a linear function with slope $-\mathtt{m}_2^+ < 0$.

C2. *There exist $R_2 \geq 0$ and $\mathtt{m}_2^+ > 0$ such that for any $x \in \bar{\mathrm{B}}(0, R_2)^{\mathrm{c}} \cap \mathsf{X}$,*

$$\langle b(x), x\rangle \leq -\mathtt{m}_2^+ \|x\|^2 \ .$$

In the next proposition we derive a Foster-Lyapunov drift condition for $W_2 : \mathsf{X}^2 \to [1, +\infty)$ defined for any $x, y \in \mathsf{X}$ by

$$W_2(x, y) = 1 + \|x\|^2 + \|y\|^2 \ . \tag{35}$$

Note that for any $x, y \in \mathsf{X}$, $W_2(x, y) = \{V(x) + V(y)\}/2$ with $V(x) = 1 + \|x\|^2$.

Proposition 11. *Assume* **B** 1(X) *for* $\mathsf{X} \subset \mathbb{R}^d$*,* **B** 2*,* **B** 3$(\mathtt{m})$ *for* $\mathtt{m} \in \mathbb{R}_-$ *and* **C** 2*. Then* K_γ *defined by* (5) *satisfies* $\mathbf{D}_{\mathrm{d}}(W_2, \lambda^\gamma, A\gamma, \mathsf{X}^2)$ *for any* $\gamma \in (0, \bar{\gamma}]$ *where* $\bar{\gamma} \in (0, 2\mathtt{m}_2^+/\mathtt{L}^2)$ *and*

$$\lambda = \exp[-(2\mathtt{m}_2^+ - \bar{\gamma}\mathtt{L}^2)] \ , \quad A = d + 2R_2^2(\mathtt{m}_2^+ - \mathtt{m}) + 2\mathtt{m}_2^+ \ .$$

Proof. See Section 4.4. □

Theorem 12. *Assume* **B**1(X) *for* $\mathsf{X} \subset \mathbb{R}^d$*,* **B**2 *and* **C**2*. Assume in addition either* **B**3$(\mathtt{m})$ *for* $\mathtt{m} \in \mathbb{R}_-$ *or* **B**4*. Then the conclusions of Theorem 6 hold with* $W = W_2$ *defined in* (35)*,* $\bar{\gamma}$*,* λ *and* A *given by Proposition 11,*

$$M_{\mathrm{d}} = 2\sqrt{2K_{\mathrm{d}}} \quad \text{with } K_{\mathrm{d}} = 2A(1 + \bar{\gamma})(1 + \log^{-1}(1/\lambda)) \ ,$$

and $\boldsymbol{\Psi}$ *is given by* (28) *or* (29)*.*

Proof. The proof is postponed to Section 4.5. □

Let $\bar{\gamma} \in (0, 2\mathtt{m}_2^+/\mathtt{L}^2)$, $\ell \in \mathbb{N}^\star$ specified below, $\tilde{\rho}_{\bar{\gamma},b} \in (0,1)$ and $\tilde{C}_{\bar{\gamma},b} \geq 0$ the constants given by Theorem 12 such that for any $\gamma \in (0, \bar{\gamma}]$, $k \in \mathbb{N}$ and $x, y \in \mathsf{X}$

$$\mathbf{W}_{\mathbf{c}_2}(\delta_x \mathrm{R}_\gamma^k, \delta_y \mathrm{R}_\gamma^k) \leq \tilde{C}_{\bar{\gamma},b} \tilde{\rho}_{\bar{\gamma},b}^{\lfloor k(\ell\lceil 1/\gamma\rceil)^{-1}\rfloor} W(x, y) \ ,$$

with $\mathbf{c}_2(x, y) = \mathbb{1}_{\Delta_{\mathsf{X}}^{\mathrm{c}}}(x, y)(1 + \|x\|^2 + \|y\|^2)$ for any $x, y \in \mathsf{X}$. Using the fact that $W_2(x, y) = \{V(x) + V(y)\}/2$, that $\lfloor k/(\ell \lceil 1/\gamma \rceil)\rfloor \geq k\gamma/(\ell(1+\bar{\gamma})) - 1$, letting $C_{\bar{\gamma},b} = \tilde{C}_{\bar{\gamma},b}\tilde{\rho}_{\bar{\gamma},b}^{-1}/2$ and $\rho_{\bar{\gamma},b} = \tilde{\rho}_{\bar{\gamma},b}^{1/(\ell(1+\bar{\gamma}))}$, we obtain that for any $\gamma \in (0, \bar{\gamma}]$, $k \in \mathbb{N}$ and $x, y \in \mathsf{X}$

$$\|\delta_x \mathrm{R}_\gamma^k - \delta_y \mathrm{R}_\gamma^k\|_V \leq C_{\bar{\gamma},b} \rho_{\bar{\gamma},b}^{k\gamma} \{V(x) + V(y)\} \ .$$

Note that by Theorem 6-(b), we obtain that the following limits exist and do not depend on $\mathtt{L}$

$$\rho_b = \lim_{\bar{\gamma} \to 0} \rho_{\bar{\gamma},b} \ , \qquad C_b = \lim_{\bar{\gamma} \to 0} C_{\bar{\gamma},b} \ . \tag{36}$$

We now discuss the dependency of ρ_b with respect to the problem constants, depending on the sign of $\mathtt{m}$, based on Theorem 6-(c).

(a) If $\mathbf{B}4$ holds, set $\ell = \lceil M_{\mathrm{d}}^2 \rceil$. Then, if we consider $\mathtt{m}_2^+$ sufficiently small and $\mathtt{m}$ and R_2 sufficiently large such that the conditions of Theorem 6-(c) hold, we have

$$\log^{-1}(\rho_b^{-1}) \leq 12\log(2)\left(1 + 16\{d + 2\mathtt{m}_2^+(1+R_2^2)\}\{1 + 1/(2\mathtt{m}_2^+)\}\right) \times \log\left(6\{d + 2\mathtt{m}_2^+(1+R_2^2)\}\{1 + 1/(2\mathtt{m}_2^+)\}\right) \Big/ \left[\mathtt{m}_2^+ \boldsymbol{\Phi}(-1/2)\right] . \tag{37}$$

Note that the leading term on the right hand side of this equation is of order $\max(d, R_2^2/\mathtt{m}_2^+)$.

(b) If $\mathbf{B}3(\mathtt{m})$ with $\mathtt{m} \in \mathbb{R}_-$, set $\ell = 1$. Then, if we consider $\mathtt{m}_2^+$ sufficiently small and $\mathtt{m}$ and R_2 sufficiently large such that the conditions of Theorem 6-(c) hold, we have

$$\log^{-1}(\rho_b^{-1}) \leq 12\log(2)\log(6D_b) \Big/ \left(\mathtt{m}_2^+ \boldsymbol{\Phi}\left\{-2\left[-\mathtt{m}D_b/(1-\mathrm{e}^{2\mathtt{m}})\right]^{1/2}\right\}\right) , \tag{38}$$

with

$$D_b = \{d + 2(\mathtt{m}_2^+ - \mathtt{m})R_2^2 + 2\mathtt{m}_2^+\}\{1 + 1/(2\mathtt{m}_2^+)\} .$$

Note that the right hand side of (38) is exponential in $\mathtt{m}d$, $\mathtt{m}R_2^2$ and $\mathtt{m}/\mathtt{m}_2^+$.

We now consider a condition which enforces weak curvature outside of a compact set.

C 3. *There exist $R_3, \mathtt{a} \geq 0$, $\mathtt{k}_1, \mathtt{k}_2 > 0$, such that for any $x \in \mathbb{R}^d$*

$$\langle b(x), x \rangle \leq -\mathtt{k}_1\|x\|\mathbb{1}_{\bar{\mathrm{B}}(0,R_3)^{\mathrm{c}}}(x) - \mathtt{k}_2\|b(x)\|^2 + \mathtt{a}/2 .$$

In the case where $\mathsf{X} = \mathbb{R}^d$, $\Pi_{\mathsf{X}} = \mathrm{Id}$ and there exists $U \in \mathrm{C}^1(\mathbb{R}^d, \mathbb{R})$ such that $\mathbf{B}2$ and $\mathbf{B}3(0)$ hold with $b = -\nabla U$ and $\int_{\mathbb{R}^d} \mathrm{e}^{-U(x)}\mathrm{d}x < +\infty$, then there exist $R_3, \mathtt{a} \geq 0$ and $\mathtt{k}_1 > 0$ such that $\mathbf{C}3$ holds with $\mathtt{k}_2 = (4\mathtt{L})^{-1}$, see [16, Proposition 5]. Define $V : \mathsf{X} \to [1, +\infty)$ for any $x \in \mathsf{X}$ by

$$V(x) = \exp(\mathtt{m}_3^+\phi(x)) , \qquad \phi(x) = \sqrt{1 + \|x\|^2} , \qquad \mathtt{m}_3^+ \in (0, \mathtt{k}_1/2] .$$

We also define for any $x, y \in \mathsf{X}$, $W_3(x,y) = \{V(x) + V(y)\}/2$.

Proposition 13. *Assume $\mathbf{B}1(\mathsf{X})$ for $\mathsf{X} \subset \mathbb{R}^d$ and $\mathbf{C}3$. Then for any $\gamma \in (0, \bar{\gamma}]$, K_γ defined by (5) satisfies $\mathbf{D}_{\mathrm{d}}(W_3, \lambda^\gamma, A\gamma, \mathsf{X}^2)$ where $\bar{\gamma} \in (0, 2\mathtt{k}_2)$, $R_4 = \max(1, R_3, (d + \mathtt{a})/\mathtt{k}_1)$ and*

$$\begin{aligned} &\lambda = \mathrm{e}^{-(\mathtt{m}_3^+)^2/2} , \\ &A = \exp\left[\bar{\gamma}(\mathtt{m}_3^+(d+\mathtt{a}) + (\mathtt{m}_3^+)^2)/2 + \mathtt{m}_3^+(1+R_4^2)^{1/2}\right](\mathtt{m}_3^+(d+\mathtt{a})/2 + (\mathtt{m}_3^+)^2) . \end{aligned} \tag{39}$$

Proof. The proof is postponed to Section 4.6. □

Theorem 14. *Assume* $\mathbf{B}1(\mathsf{X})$ *for* $\mathsf{X} \subset \mathbb{R}^d$, $\mathbf{B}2$ *and* $\mathbf{C}3$*. Assume in addition either* $\mathbf{B}3(\mathtt{m})$ *for* $\mathtt{m} \in \mathbb{R}_-$ *or* $\mathbf{B}4$*. Then the conclusions of Theorem 6 hold with* $W = W_3$*,* $\bar{\gamma}$*,* λ *and* A *given by Proposition 13,*

$$M_{\mathrm{d}} = 2\log(2K_{\mathrm{d}})/\mathtt{m}_3^+ \quad \text{with } K_{\mathrm{d}} = 2A(1+\bar{\gamma})(1+\log^{-1}(1/\lambda)) \;,$$

and $\mathbf{\Psi}$ *is given by* (28) *or* (29)*.*

Proof. The proof is postponed to Section 4.7. □

The dependency of the rate given by Theorem 14 with respect to the constants is discussed in Appendix E.

3 Quantitative convergence bounds for diffusions

3.1 Main results

In this section, we aim at deriving quantitative convergence bounds with respect to some Wasserstein metrics for diffusion processes under regularity and curvature assumptions on the drift b. Consider the following Stochastic Differential Equation

$$\mathrm{d}\mathbf{X}_t = b(\mathbf{X}_t)\mathrm{d}t + \mathrm{d}\mathbf{B}_t \;, \tag{40}$$

where $(\mathbf{B}_s)_{s\geq 0}$ is a d-dimensional Brownian motion and $b : \mathbb{R}^d \to \mathbb{R}^d$ is a continuous drift. In the sequel we will always assume the following assumption.

L 1. *There exists a unique strong solution of* (40) *for any starting point* $\mathbf{X}_0 = x$*, with* $x \in \mathbb{R}^d$*.*

We define the semi-group $(\mathrm{P}_t)_{t\geq 0}$ for any $\mathsf{A} \in \mathcal{B}(\mathbb{R}^d)$, $x \in \mathbb{R}^d$ and $t \geq 0$ by $\mathrm{P}_t(x, \mathsf{A}) = \mathbb{P}\left(\mathbf{X}_t \in \mathsf{A}\right)$ where $(\mathbf{X}_t)_{t\geq 0}$ is the solution of (40) starting from $\mathbf{X}_0 = x \in \mathbb{R}^d$. Consider the extended infinitesimal generator $\mathcal{A}$ associated with $(\mathrm{P}_t)_{t\geq 0}$ defined for any $f \in \mathrm{C}^2(\mathbb{R}^d, \mathbb{R})$ by

$$\mathcal{A}f = (1/2)\Delta f + \langle \nabla f, b \rangle \;.$$

Let $V \in \mathrm{C}^2(\mathbb{R}^d, [1, +\infty))$, $\zeta \in \mathbb{R}$ and $B \geq 0$

D 2 $(\mathbf{D}_{\mathrm{c}}(V, \zeta, B))$**.** *The extended infinitesimal generator* $\mathcal{A}$ *satisfies the continuous Foster-Lyapunov drift condition if for all* $x \in \mathbb{R}^d$

$$\mathcal{A}V(x) \leq -\zeta V(x) + B \;.$$

This assumption is the continuous counterpart of $\mathbf{D}_{\mathrm{d}}(V, \lambda, A, \mathbb{R}^d)$.

We now turn to establishing that $(\mathrm{P}_t)_{t\geq 0}$ converges for some Wasserstein metrics. In order to prove these results we will rely on discretizations of the Stochastic Differential Equation (40). If the hypotheses of Theorem 6 or Theorem 7 are satisfied, these discretized processes are uniformly geometrically ergodic.

First, we draw a link between the continuous drift condition $\mathbf{D}_{\mathrm{c}}(V, \zeta, B)$ and the discrete drift condition $\mathbf{D}_{\mathrm{d}}(V, \lambda, A, \mathbb{R}^d)$. The result and its proof are standard [51, Theorem 2.1] but are given here for completeness. Denote by $(\mathcal{F}_t)_{t\geq 0}$ the filtration associated to $(\mathbf{B}_t)_{t\geq 0}$ satisfying the usual conditions [39, Chapter I, Section 5]

Lemma 15. *Let $\zeta \in \mathbb{R}$, $B \geq 0$ and $V \in \mathrm{C}^2(\mathbb{R}^d, [1, +\infty))$ such that $\lim_{\|x\|\to+\infty} V(x) = +\infty$. Assume* **L**1 *and $\mathbf{D}_{\mathrm{c}}(V, \zeta, B)$.*

(a) If $B = 0$, then for any $x \in \mathbb{R}^d$, $(V(\mathbf{X}_t)\mathrm{e}^{\zeta t})_{t\geq 0}$ is a $(\mathcal{F}_t)_{t\geq 0}$-supermartingale where $(\mathbf{X}_t)_{t\geq 0}$ is the solution of (40) starting from $\mathbf{X}_0 = x$.

(b) For any $t_0 > 0$, P_{t_0} satisfies $\mathbf{D}_{\mathrm{d}}(V, \exp(-\zeta t_0), B(1 - \exp(-\zeta t_0))/\zeta, \mathbb{R}^d)$.

Proof. The proof is given in Appendix F. □

Consider a family of drifts $\left\{b_{\gamma,n} : \mathbb{R}^d \to \mathbb{R}^d \,:\, \gamma \in (0, \bar{\gamma}], n \in \mathbb{N}\right\}$ for some $\bar{\gamma} > 0$. For all $\gamma \in (0, \bar{\gamma}]$ and $n \in \mathbb{N}$, we denote by $\tilde{\mathrm{R}}_{\gamma,n}$ the Markov kernel associated with (1) where $\mathcal{T}_\gamma(x) = x + \gamma b_{\gamma,n}(x)$, $\mathsf{X} = \mathbb{R}^d$ and $\Pi = \mathrm{Id}$. We will show that under the following assumptions the family $\{\mathrm{R}_{\gamma,n}^{\lceil T/\gamma \rceil} : \mathbb{R}^d \to \mathbb{R}^d \,:\, \gamma \in (0, \bar{\gamma}], n \in \mathbb{N}\}$ approximates P_T for $T \geq 0$ as $\gamma \to 0$ and $n \to +\infty$.

L 2. *There exist $\beta > 0$ and $C_1 \geq 0$ such that for any $\gamma \in (0, \bar{\gamma}]$, $n \in \mathbb{N}$, $b_{\gamma,n} \in \mathrm{C}(\mathbb{R}^d, \mathbb{R}^d)$ and for any $x \in \mathbb{R}^d$,*

$$\|b(x) - b_{\gamma,n}(x)\|^2 \leq C_1 \gamma^\beta \|b(x)\|^2 \ .$$

The following assumption is mainly technical and is satisfied in our applications.

L 3. *There exists $\varepsilon_b > 0$ such that $\sup_{s\in[0,T]} \left\{\mathbb{E}\left[\|b(\mathbf{X}_s)\|^{2(1+\varepsilon_b)}\right]\right\} < +\infty$, for any $x \in \mathbb{R}^d$ and $T \geq 0$, where $(\mathbf{X}_t)_{t\geq 0}$ is the solution of (40) starting from $\mathbf{X}_0 = x$.*

By Lemma 15-(a), if $\mathbf{D}_{\mathrm{c}}(V, \zeta, 0)$ is satisfied with $\zeta \in \mathbb{R}$, it holds that for any starting point $x \in \mathbb{R}^d$, $\sup_{t\in[0,T]} \mathbb{E}\left[V(\mathbf{X}_t)\right] \leq \mathrm{e}^{-\zeta T} V(x)$, where $(\mathbf{X}_t)_{t\geq 0}$ is solution of (40) starting from x. Therefore, if $\|b(x)\|^{2(1+\varepsilon_b)} \leq V(x)$ for any $x \in \mathbb{R}^d$, **L** 3 is satisfied.

Proposition 16. *Let $V : \mathbb{R}^d \to [1, +\infty)$. Assume* **L** 1*,* **L** 2 *and* **L** 3*. In addition, assume that for any $n \in \mathbb{N}$, $T \geq 0$ and $x \in \mathbb{R}^d$*

$$\mathrm{P}_T V^2(x) < +\infty \ , \qquad \limsup_{m \to +\infty} \tilde{\mathrm{R}}^m_{T/m,n} V^2(x) < +\infty \ . \tag{41}$$

Then for any $n \in \mathbb{N}$, $T \geq 0$ and $x \in \mathbb{R}^d$

$$\lim_{m \to +\infty} \| \updelta_x \mathrm{P}_T - \updelta_x \tilde{\mathrm{R}}^m_{T/m,n} \|_V = 0 \ ,$$

where $(\mathrm{P}_t)_{t \geq 0}$ is the semigroup associated with (40) and for any $\gamma \in (0, \bar{\gamma}]$ and $n \in \mathbb{N}$, $\tilde{\mathrm{R}}_{\gamma,n}$ is the Markov kernel associated with (1) where $\mathcal{T}_\gamma(x) = x + \gamma b_{\gamma,n}(x)$ and $\Pi = \mathrm{Id}$.

Proof. Let $T \geq 0$, $x \in \mathbb{R}^d$, $n \in \mathbb{N}$ and $m \in \mathbb{N}^\star$ with $T/m \leq \bar{\gamma}$. Using [26, Lemma 24], we obtain

$$\begin{aligned} &\| \updelta_x \mathrm{P}_T - \updelta_x \tilde{\mathrm{R}}^m_{T/m,n} \|_V \\ &\qquad \leq (1/\sqrt{2}) \left(\updelta_x \mathrm{P}_T V^2(x) + \updelta_x \tilde{\mathrm{R}}^m_{T/m,n} V^2(x) \right)^{1/2} \mathrm{KL} \left(\updelta_x \mathrm{P}_T | \updelta_x \tilde{\mathrm{R}}^m_{T/m,n} \right)^{1/2} \ . \end{aligned}$$

Let $M \geq 0$, $n \in \mathbb{N}^\star$ with $n^{-1} < \bar{\gamma}$, $x \in \mathbb{R}^d$ and $k \in \mathbb{N}$. Therefore by (41), we only need to show that $\lim_{m \to +\infty} \mathrm{KL}(\updelta_x \mathrm{P}_T | \updelta_x \tilde{\mathrm{R}}^m_{T/m,n}) = 0$. Consider the two processes $(\mathbf{X}_t)_{t \in [0,T]}$ and $(\tilde{\mathbf{X}}_t)_{t \in [0,T]}$ defined by (40) with $\mathbf{X}_0 = \tilde{\mathbf{X}}_0 = x$ and

$$\mathrm{d}\tilde{\mathbf{X}}_t = \tilde{b}_{T/m,n}(t, (\tilde{\mathbf{X}}_s)_{s \in [0,T]}) \mathrm{d}t + \mathrm{d}\mathbf{B}_t \ , \qquad \tilde{\mathbf{X}}_0 = x \ ,$$

where for any $(\mathrm{w}_s)_{s \in [0,T]} \in \mathrm{C}([0,T], \mathbb{R}^d)$, $t \in [0,T]$,

$$\tilde{b}_{T/m,n}(t, (\mathrm{w}_s)_{s \in [0,T]}) = \sum_{i=0}^{m-1} b_{T/m,n}(\mathrm{w}_{iT/n}) \mathbb{1}_{[iT/m,(i+1)T/m)}(t) \ . \tag{42}$$

Note for any $i \in \{0, \ldots, m\}$, the distribution of $\tilde{\mathbf{X}}_{iT/m}$ is $\updelta_x \tilde{\mathrm{R}}^i_{T/m,n}$. Using that b and $b_{T/m,n}$ are continuous and that $(\mathbf{X}_t)_{t \in [0,T]}$ and $(\tilde{\mathbf{X}}_t)_{t \in [0,T]}$ take their values in $\mathrm{C}([0,T], \mathbb{R}^d)$, we obtain that

$$\begin{aligned} &\mathbb{P} \left(\int_0^T \| b(\mathbf{X}_t) \|^2 \mathrm{d}t < +\infty \right) = 1 \ , \\ &\mathbb{P} \left(\int_0^T \| \tilde{b}_{T/m,n}(t, (\tilde{\mathbf{X}}_s)_{s \in [0,T]}) \|^2 \mathrm{d}t < +\infty \right) = 1 \ , \end{aligned}$$

and

$$\mathbb{P}\left(\int_0^T \|b(\mathbf{B}_t)\|^2 \mathrm{d}t < +\infty\right) = 1 \; ,$$
$$\mathbb{P}\left(\int_0^T \|\tilde{b}_{T/m,n}(t, (\mathbf{B}_s)_{s\in[0,T]})\|^2 \mathrm{d}t < +\infty\right) = 1 \; ,$$

where $(\mathbf{B}_t)_{t\in[0,T]}$ is the d-dimensional Brownian motion associated with (40). Therefore by [47, Theorem 7.7] the distributions of $(\mathbf{X}_t)_{t\in[0,T]}$ and $(\tilde{\mathbf{X}}_t)_{t\in[0,T]}$, denoted by μ^x and $\tilde{\mu}^x$ respectively, are equivalent to the distribution of the Brownian motion μ_B^x starting at x. In addition, μ^x admits a Radon-Nikodyn density w.r.t. to μ_B^x and μ_B^x admits a Radon-Nikodyn density w.r.t. to $\tilde{\mu}^x$, given μ_B^x-almost surely for any $(\mathrm{w}_t)_{t\in[0,T]} \in \mathrm{C}([0,T], \mathbb{R}^d)$ by

$$\frac{\mathrm{d}\mu^x}{\mathrm{d}\mu_B^x}((\mathrm{w}_t)_{t\in[0,T]}) = \exp\left((1/2)\int_0^T \langle b(\mathrm{w}_s), \mathrm{d}\mathrm{w}_s\rangle - (1/4)\int_0^T \|b(\mathrm{w}_s)\|^2 \mathrm{d}s\right) \; ,$$
$$\frac{\mathrm{d}\mu_B^x}{\mathrm{d}\tilde{\mu}^x}((\mathrm{w}_t)_{t\in[0,T]}) = \exp\left(-(1/2)\int_0^T \langle \tilde{b}_{T/m,n}(s, (\mathrm{w}_u)_{u\in[0,T]}), \mathrm{d}\mathrm{w}_s\rangle \right.$$
$$\left. +(1/4)\int_0^T \|\tilde{b}_{T/m,n}(s, (\mathrm{w}_u)_{u\in[0,T]})\|^2 \mathrm{d}s\right) \; .$$

Finally we obtain that μ_B^x-almost surely for any $(\mathrm{w}_s)_{s\in[0,T]} \in \mathrm{C}([0,T], \mathbb{R}^d)$

$$\frac{\mathrm{d}\mu^x}{\mathrm{d}\tilde{\mu}^x}((\mathrm{w}_t)_{t\in[0,T]}) = \exp\left((1/2)\int_0^T \langle b(\mathrm{w}_s) - \tilde{b}_{T/m,n}(s, (\mathrm{w}_u)_{u\in[0,T]}), \mathrm{d}\mathrm{w}_s\rangle \right.$$
$$\left. +(1/4)\int_0^T \|\tilde{b}_{T/m,n}(s, (\mathrm{w}_u)_{u\in[0,T]})\|^2 - \|b(\mathrm{w}_s)\|^2 \mathrm{d}s\right) \; . \quad (43)$$

Now define for any $(\mathrm{w}_s)_{s\in[0,T]} \in \mathrm{C}([0,T], \mathbb{R}^d)$ and $t \in [0,T]$

$$b_{T/m}(t, (\mathrm{w}_s)_{s\in[0,T]}) = \sum_{i=0}^{m-1} b(\mathrm{w}_{iT/m}) \mathbb{1}_{[iT/m,(i+1)T/m)}(t) \; . \quad (44)$$

Using (40), (42), (43), **L** 2, and for any $a_1, a_1 \in \mathbb{R}^d$, $\|a_1 - a_2\|^2 \leq 2(\|a_1\|^2 + \|a_2\|^2)$, we obtain that

$$2\mathrm{KL}\left(\updelta_x \mathrm{P}_T | \updelta_x \tilde{\mathrm{R}}^m_{T/m,n}\right) \leq 2^{-1}\mathbb{E}\left[\int_0^T \|b(\mathbf{X}_s) - \tilde{b}_{T/m,n}(s, (\mathbf{X}_u)_{u\in[0,T]})\|^2 \mathrm{d}s\right] \quad (45)$$

$$\leq \mathbb{E}\left[\int_0^T \|b(\mathbf{X}_s) - b_{T/m}(s, (\mathbf{X}_u)_{u\in[0,T]})\|^2 \mathrm{d}s\right]$$
$$+ \sum_{i=0}^{m-1} \mathbb{E}\left[\int_{iT/m}^{(i+1)T/m} \|b(\mathbf{X}_{iT/m}) - b_{T/m,n}(\mathbf{X}_{iT/m})\|^2 \mathrm{d}s\right]$$
$$\leq \mathbb{E}\left[\int_0^T \|b(\mathbf{X}_s) - b_{T/m}(s, (\mathbf{X}_u)_{u\in[0,T]})\|^2 \mathrm{d}s\right]$$
$$+ C_1 T^{1+\beta} m^{-\beta} \sup_{s\in[0,T]} \mathbb{E}\left[\|b(\mathbf{X}_s)\|^2\right] .$$

It only remains to show that the first term goes to 0 as $m \to +\infty$. Note that since $(\mathbf{X}_s)_{s\in[0,T]}$ is almost surely continuous and b is continuous on $\mathbb{R}^d$, $\lim_{m\to+\infty} \|b(\mathbf{X}_s) - b_{T/m}(s, (\mathbf{X}_u)_{u\in[0,T]})\|^2 = 0$ for any $s \in [0,T]$ almost surely. Then, using the Lebesgue dominated convergence theorem and the continuity of b, we obtain that for any $M \geq 0$,

$$\lim_{m\to+\infty} \mathbb{E}\left[\mathbb{1}_{[0,M]}\left(\sup_{s\in[0,T]} \|\mathbf{X}_s\|\right) \int_0^T \|b(\mathbf{X}_s) - b_{T/m}(s, (\mathbf{X}_u)_{u\in[0,T]})\|^2 \mathrm{d}s\right] = 0 . \quad (46)$$

On the other hand, using Hölder's inequality and the definition of $b_{T/m}$ (44), we obtain that for any $M \geq 0$,

$$\mathbb{E}\left[\mathbb{1}_{(M,+\infty)}\left(\sup_{s\in[0,T]} \|\mathbf{X}_s\|\right) \int_0^T \|b(\mathbf{X}_s) - b_{T/m}(s, (\mathbf{X}_s)_{s\in[0,T]})\|^2 \mathrm{d}s\right]$$
$$\leq 2\left(\mathbb{P}\left(\sup_{s\in[0,T]} \|\mathbf{X}_s\| > M\right)\right)^{\varepsilon_b/(1+\varepsilon_b)}$$
$$\int_0^T \left(\mathbb{E}^{1/(1+\varepsilon_b)}\left[\|b(\mathbf{X}_s)\|^{2(1+\varepsilon_b)}\right] + \mathbb{E}^{1/(1+\varepsilon_b)}\left[\left\|b_{T/m}(s, (\mathbf{X}_u)_{u\in[0,T]})\right\|^{2(1+\varepsilon_b)}\right]\right) \mathrm{d}s$$
$$\leq 4T\left(\mathbb{P}\left(\sup_{s\in[0,T]} \|\mathbf{X}_s\| > M\right)\right)^{\varepsilon_b/(1+\varepsilon_b)} \left(\sup_{s\in[0,T]} \mathbb{E}\left[\|b(\mathbf{X}_s)\|^{2(1+\varepsilon_b)}\right]\right)^{1/(1+\varepsilon_b)} .$$

Combining this result, **L**3, and (46) in (45), we obtain that for any $M \geq 0$,

$$\limsup_{m\to+\infty} \mathrm{KL}\left(\updelta_x \mathrm{P}_T | \updelta_x \tilde{\mathrm{R}}^m_{T/m,n}\right)$$
$$\leq 2T\left(\mathbb{P}\left(\sup_{s\in[0,T]} \|\mathbf{X}_s\| > M\right)\right)^{\varepsilon_b/(1+\varepsilon_b)} \left(\sup_{s\in[0,T]} \mathbb{E}\left[\|b(\mathbf{X}_s)\|^{2(1+\varepsilon_b)}\right]\right)^{1/(1+\varepsilon_b)} .$$

Since $(\mathbf{X}_s)_{s\in[0,T]}$ is a.s. continuous, we get by the monotone convergence theorem and **L** 3, taking $M \to +\infty$, that $\lim_{m\to+\infty} \mathrm{KL}(\delta_x \mathrm{P}_T | \delta_x \tilde{\mathrm{R}}^m_{T/m,n}) = 0$, which concludes the proof. □

If $V = 1$, Proposition 16 implies that $\lim_{m\to+\infty} \|\delta_x \mathrm{P}_T - \delta_x \tilde{\mathrm{R}}^m_{T/m,n}\|_{\mathrm{TV}} = 0$. Let $V : \mathbb{R}^d \to [1, +\infty)$ and $\mathbf{c} : \mathbb{R}^d \times \mathbb{R}^d \to [1, +\infty)$ such that for any $x, y \in \mathbb{R}^d$, $\mathbf{c}(x, y) \leq \{V(x) + V(y)\}/2$. Then, under the conditions of Proposition 16, we obtain that for any $T \geq 0$, $n \in \mathbb{N}$ and $x, y \in \mathbb{R}^d$

$$\mathbf{W}_{\mathbf{c}}(\delta_x \mathrm{P}_T, \delta_y \mathrm{P}_T) \leq \lim_{m\to+\infty} \mathbf{W}_{\mathbf{c}}(\delta_x \tilde{\mathrm{R}}^m_{T/m,n}, \delta_y \tilde{\mathrm{R}}^m_{T/m,n}) , \tag{47}$$

Therefore, if for any $T \geq 0$, $\mathbf{W}_{\mathbf{c}}(\delta_x \tilde{\mathrm{R}}^m_{T/m,n}, \delta_y \tilde{\mathrm{R}}^m_{T/m,n})$ can be bounded using Theorem 6 or Theorem 7 uniformly in m, we obtain an explicit bound for $\mathbf{W}_{\mathbf{c}}(\delta_x \mathrm{P}_T, \delta_y \mathrm{P}_T)$ for any $T \geq 0$. Then, this result easily implies non-asymptotic convergence bounds of $(\mathrm{P}_t)_{t\geq 0}$ to its invariant measure if it exists. However, in our applications, global Lipschitz regularity on $b_{T/m,n} : \mathbb{R}^d \to \mathbb{R}^d$ is needed in order to apply Theorem 6 or Theorem 7 to $\tilde{\mathrm{R}}_{T/m,n}$ for $T \geq 0$, $m \in \mathbb{N}^\star$ and $n \in \mathbb{N}$. To be able to deal with the fact that $b_{T/m,n}$ is non necessarily globally Lipschitz, we consider an appropriate sequence of projected Euler Maruyama schemes associated to a sequence of subsets of $\mathbb{R}^d$, $(\mathsf{K}_n)_{n\in\mathbb{N}}$ satisfying the following assumption.

L 4. *For any $n \in \mathbb{N}$, K_n is convex and closed, and $\bar{\mathrm{B}}(0, n) \subset \mathsf{K}_n$.*

Consider for any $\gamma \in (0, \bar{\gamma}]$ and $n \in \mathbb{N}$ the Markov chain associated (1), where for any $x \in \mathbb{R}^d$, $\mathcal{T}_\gamma(x) = x + \gamma b_{\gamma,n}(x)$, $\mathsf{X} = \mathsf{K}_n$ and $\Pi = \Pi_{\mathsf{K}_n}$, the projection on K_n. The Markov kernel associated with this chain is denoted $\mathrm{R}_{\gamma,n}$ for any $\gamma \in (0, \bar{\gamma}]$ and $n \in \mathbb{N}$. Assuming only local Lipschitz regularity we can apply these theorems to the projected version of the Markov chain associated with $\mathrm{R}_{T/m,n}$. Therefore we want to replace $\tilde{\mathrm{R}}_{T/m,n}$ by $\mathrm{R}_{T/m,n}$ in (47). In order to do so we consider the following assumption on the family of drifts $\{b_{\gamma,n} \,;\, \gamma \in (0, \bar{\gamma}], n \in \mathbb{N}\}$.

L 5. *There exist $\tilde{A} > 0$ and $\tilde{V} : \mathbb{R}^d \to [1, +\infty)$ such that for any $n \in \mathbb{N}$ there exist $\tilde{E}_n \geq 0$, $\tilde{\varepsilon}_n > 0$ and $\bar{\gamma}_n \in (0, \bar{\gamma}]$ satisfying for any $\gamma \in (0, \bar{\gamma}_n]$ and $x \in \mathbb{R}^d$,*

$$\tilde{\mathrm{R}}_{\gamma,n} \tilde{V}(x) \leq \exp\left[\log(\tilde{A})\gamma(1 + \tilde{E}_n \gamma^{\tilde{\varepsilon}_n})\right] \tilde{V}(x) , \qquad \sup_{x\in\mathbb{R}^d} \left\{\|x\| / \tilde{V}(x)\right\} \leq 1 ,$$

where for any $\gamma \in (0, \bar{\gamma}]$ and $n \in \mathbb{N}$, $\tilde{\mathrm{R}}_{\gamma,n}$ is the Markov kernel associated with (1) where $\mathcal{T}_\gamma(x) = x + \gamma b_{\gamma,n}(x)$ and $\Pi = \mathrm{Id}$.

Proposition 17. *Let $V : \mathbb{R}^d \to [1, +\infty)$. Assume* **L** 1*,* **L** 4*,* **L** 5 *and that for any $T \geq 0$, $x \in \mathbb{R}^d$*

$$\limsup_{n\to+\infty} \limsup_{m\to+\infty} \left(\mathrm{R}^m_{T/m,n} + \tilde{\mathrm{R}}^m_{T/m,n} \right) V^2(x) < +\infty \ .$$

Then for any $T \geq 0$ and $x \in \mathbb{R}^d$

$$\lim_{n\to+\infty} \limsup_{m\to+\infty} \| \delta_x \mathrm{R}^{mk}_{T/m,n} - \delta_x \tilde{\mathrm{R}}^{mk}_{T/m,n} \|_V = 0 \ ,$$

Proof. For any $n \in \mathbb{N}$ and $\gamma \in (0, \bar{\gamma}]$, we consider the synchronous Markov coupling $\mathrm{Q}_{\gamma,n}$ for $\mathrm{R}_{\gamma,n}$ and $\tilde{\mathrm{R}}_{\gamma,n}$ defined for any $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$ and $\mathsf{A} \in \mathcal{B}(\mathbb{R}^d)$ by

$$\begin{aligned} &\mathrm{Q}_{\gamma,n}((x, y), \mathsf{A}) \\ &= \frac{1}{(2\pi\gamma)^{d/2}} \int_{\mathbb{R}^d} \mathbb{1}_{(\mathrm{Id},\Pi_{\mathsf{K}_n})^{\leftarrow}(\mathsf{A})} (\mathcal{T}_\gamma(x) + \sqrt{\gamma} z, \mathcal{T}_\gamma(y) + \sqrt{\gamma} z) \mathrm{e}^{-\|z\|^2/2} \mathrm{d}z \ . \end{aligned} \tag{48}$$

Let $T \geq 0$, $n \in \mathbb{N}$, $m \in \mathbb{N}^\star$ such that $T/m \leq \bar{\gamma}$. Consider $(X_j, \tilde{X}_j)_{j\in\mathbb{N}}$ a Markov chain with Markov kernel $\mathrm{Q}_{T/m,n}$ and started from $X_0 = \tilde{X}_0 = x$ for a fixed $x \in \mathbb{R}^d$. Note that by definition and **L** 4, we have that for $k < \tau$, $X_k = \tilde{X}_k$ where $\tau = \inf\{j \in \mathbb{N} : \tilde{X}_j \notin \bar{\mathrm{B}}(0, n)\}$ and using **L** 5, $\left(\tilde{V}(\tilde{X}_j) \exp\left[-j \log(\tilde{A})(T/m)(1 + \tilde{E}_n(T/m)^{\tilde{\varepsilon}_n}) \right] \right)_{j\in\mathbb{N}}$ is a positive supermartingale. Using (48), the Cauchy-Schwarz inequality, **L** 5 and the Doob maximal inequality for positive supermartingale [54, Proposition II-2-7], we get for any $x \in \mathbb{R}^d$

$$\begin{aligned} &\| \delta_x \mathrm{R}^m_{T/m,n} - \delta_x \tilde{\mathrm{R}}^m_{T/m,n} \|_V \leq \mathbb{E}\left[\mathbb{1}_{\Delta^{\mathrm{c}}_{\mathbb{R}^d}} (X_m, \tilde{X}_m)(V(X_m) + V(\tilde{X}_m))/2 \right] \\ &\leq (1/2)\mathbb{P}\left(\sup_{j\in\{0,\ldots,m\}} \left\| \tilde{X}_j \right\| \geq n \right) \left(\mathbb{E}\left[V^2(X_m) \right]^{1/2} + \mathbb{E}\left[V^2(\tilde{X}_m) \right]^{1/2} \right) \\ &\leq (1/2)\mathbb{P}\left(\sup_{j\in\{0,\ldots,m\}} \tilde{V}(\tilde{X}_j) \geq n \right) \left(\mathbb{E}\left[V^2(X_m) \right]^{1/2} + \mathbb{E}\left[V^2(\tilde{X}_m) \right]^{1/2} \right) \\ &\leq (2n)^{-1} \exp\left[\log(\tilde{A})(T/m)(1 + \tilde{E}_n(T/m)^{\varepsilon_n}) \right] \tilde{V}(x) \\ &\qquad\qquad \times \left((\mathrm{R}^m_{T/m,n} V^2(x))^{1/2} + (\tilde{\mathrm{R}}^m_{T/m,n} V^2(x))^{1/2} \right) \ , \end{aligned}$$

which concludes the proof upon taking $m \to +\infty$ then $n \to +\infty$. □

Based on Proposition 16 and Proposition 17, we have the following result which establishes a clear link between the convergence of the family of the projected Euler-Maruyama scheme $\{\mathrm{R}_{\gamma,n} : \gamma \in (0, \bar{\gamma}], n \in \mathbb{N}\}$ and the semigroup $(\mathrm{P}_t)_{t\geq 0}$ associated with (40).

Theorem 18. *Let $W : \mathbb{R}^d \times \mathbb{R}^d \to [1, +\infty)$ and $V : \mathbb{R}^d \to [1, +\infty)$ satisfying for any $x, y \in \mathbb{R}^d$, $\sup_{(x,y)\in\mathbb{R}^d\times\mathbb{R}^d} W(x, y) \{V(x) + V(y)\}^{-1} < +\infty$. Assume* **L**1*,* **L**2*,* **L**3*,* **L**4 *and* **L**5*. In addition, assume that for any $T \geq 0$ and $x \in \mathbb{R}^d$*

$$\mathrm{P}_T V^2(x) < +\infty \ , \quad \limsup_{n\to+\infty} \limsup_{m\to+\infty} \left(\mathrm{R}^m_{T/m,n} + \tilde{\mathrm{R}}^m_{T/m,n}\right) V^2(x) < +\infty \ . \tag{49}$$

Then,

$$\mathbf{W_c}(\delta_x \mathrm{P}_T, \delta_y \mathrm{P}_T) \leq \limsup_{n\to+\infty} \limsup_{m\to+\infty} \mathbf{W_c}(\delta_x \mathrm{R}^m_{T/m,n}, \delta_y \mathrm{R}^m_{T/m,n}) \ ,$$

where for any $x, y \in \mathbb{R}^d$, $\mathbf{c}(x, y) = \mathbb{1}_{\Delta^c_{\mathsf{X}}}(x, y)W(x, y)$, $(\mathrm{P}_t)_{t\geq 0}$ is the semigroup associated with (40) and for any $\gamma \in (0, \bar{\gamma}]$, $n \in \mathbb{N}$, $\mathrm{R}_{\gamma,n}$ is the Markov kernel associated with (1) where $\mathcal{T}_\gamma(x) = x+\gamma b_{\gamma,n}(x)$, $\mathsf{X} = \mathsf{K}_n$ and $\Pi = \Pi_{\mathsf{K}_n}$, $\tilde{\mathrm{R}}_{\gamma,n}$ is the Markov kernel associated with (1) where $\mathcal{T}_\gamma(x) = x + \gamma b_{\gamma,n}(x)$, $\mathsf{X} = \mathbb{R}^d$ and $\Pi = \mathrm{Id}$.

Proof. Let $T \geq 0$, $x, y \in \mathbb{R}^d$ and

$$C_V = 2 \sup_{(x,y)\in\mathbb{R}^d\times\mathbb{R}^d} W(x, y) \{V(x) + V(y)\}^{-1} < +\infty \ .$$

We have the following inequality for any $n \in \mathbb{N}$ and $m \in \mathbb{N}^\star$ such that $x, y \in \mathsf{K}_n$ and $T/m \leq \bar{\gamma}$

$$\begin{aligned}
\mathbf{W_c}(\delta_x \mathrm{P}_T, \delta_y \mathrm{P}_T) &\leq C_V \|\delta_x \mathrm{P}_T - \delta_x \tilde{\mathrm{R}}^m_{T/m,n}\|_V \\
&\quad + C_V \|\delta_x \mathrm{R}^m_{T/m,n} - \delta_x \tilde{\mathrm{R}}^m_{T/m,n}\|_V + \mathbf{W_c}(\delta_x \mathrm{R}^m_{T/m,n}, \delta_y \mathrm{R}^m_{T/m,n}) \\
&\qquad + C_V \|\delta_y \mathrm{P}_T - \delta_y \tilde{\mathrm{R}}^m_{T/m,n}\|_V + C_V \|\delta_y \mathrm{R}^m_{T/m,n} - \delta_y \tilde{\mathrm{R}}^m_{T/m,n}\|_V \ ,
\end{aligned}$$

which concludes the proof upon combining Proposition 16 and Proposition 17. □

3.2 Applications

In this section, we combine the results of Theorem 18 with the convergence bounds for discrete processes derived in Section 2.2, in order to obtain convergence bounds for continuous processes solution of (40). Consider the following condition on the drift b.

B5. *b is locally Lipschitz, i.e. for any $M \geq 0$, there exists $\mathtt{L}_M \geq 0$ such that for any $x, y \in \bar{\mathrm{B}}(0, M)$, $\|b(x) - b(y)\| \leq \mathtt{L}_M \|x - y\|$ and $b(0) = 0$.*

If **B**5 holds, by [39, Chapter 4, Theorem 2.3], (40) admits a unique solution $(\mathbf{X}_t)_{t\in[0,+\infty)}$ with $\mathbf{X}_0 = x \in \mathbb{R}^d$ and let $e = \inf\{s \geq 0 : \|\mathbf{X}_s\| = +\infty\}$. In particular, the condition $e = +\infty$ is met a.s. if we assume that b is sub-linear [39, Chapter 4, Theorem 2.3] or the condition $\mathbf{D}_{\mathrm{c}}(V, \zeta, 0)$ holds with $\zeta \in \mathbb{R}$ and $\lim_{\|x\| \to +\infty} V(x) = +\infty$ [46, Theorem 3.5]. It will be the case in the examples that we consider in this section as stated by the following result.

Theorem 19. *Assume* $\mathbf{B}3(\mathtt{m})$ *for* $\mathtt{m} \in \mathbb{R}$ *and* $\mathbf{B}5$*, then* $\mathbf{L}1$ *holds. In addition:*

(a) if there exists $\varepsilon_b > 0$ *and* $p \in \mathbb{N}^\star$ *such that* $\sup_{x \in \mathbb{R}^d}\{\|b(x)\|^{2(1+\varepsilon_b)} (1 + \|x\|^{2p})^{-1}\} < +\infty$ *then* $\mathbf{L}3$ *holds ;*

(b) assume that $\mathbf{C}2$ *holds and* $\sup_{x \in \mathbb{R}^d}\{\|b(x)\|^{2(1+\varepsilon_b)} \mathrm{e}^{-\mathtt{m}_2^+ \|x\|^2}\} < +\infty$ *for some* $\varepsilon_b > 0$ *satisfying then* $\mathbf{L}3$ *holds.*

Proof. The proof is postponed to Section 4.8. □

We also show that under general conditions on b, $\mathrm{P}_T V(x) < +\infty$ which is a necessary condition in Theorem 18

Theorem 20. *Assume* $\mathbf{L}1$ *and that* $\sup_{x \in \mathbb{R}^d} \langle b(x), x \rangle < +\infty$*. Then for any* $M \geq 0$*, there exists* $\zeta \in \mathbb{R}$ *such that* $\mathbf{D}_{\mathrm{c}}(V_M, \zeta, 0)$ *holds with* $V_M(x) = \exp[M\phi(x)]$ *and* $\phi(x) = (1 + \|x\|^2)^{1/2}$*. In particular, for any* $T, M \geq 0$*,* $\mathrm{P}_T V_M(x) < +\infty$*.*

Proof. The proof is postponed to Section 4.9. □

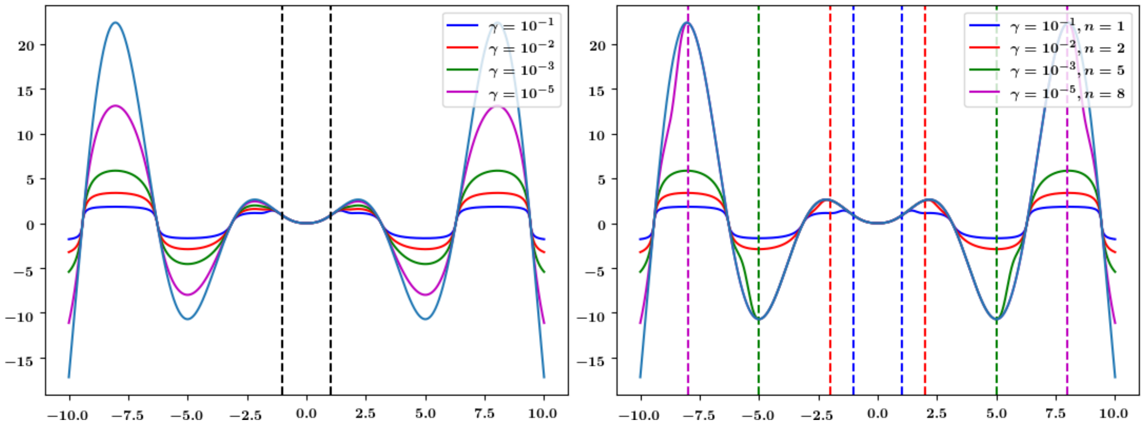
We now aim to apply Theorem 18 to the diffusion process defined by (40) and combine this result with bounds from Section 2.2. Theorem 19 gives conditions upon which $\mathbf{L}1$ and $\mathbf{L}3$ hold. In addition, $\mathbf{L}4$ is satisfied if we take for any $n \in \mathbb{N}$, $\mathsf{K}_n = \bar{\mathrm{B}}(0, n)$. Therefore, it only remains to: 1) find a family of drifts which satisfies $\mathbf{L}2$, $\mathbf{L}5$ and (49) ; 2) take the limit when $m \to +\infty$, *i.e.* when the discretization step goes to zero, in the bounds found in Section 2.2. To this end, consider the following families of drift $\{\mathrm{b}_{\gamma,n} : \gamma \in (0, \bar{\gamma}], n \in \mathbb{N}\}$ defined for any $\gamma > 0$, $n \in \mathbb{N}$ and $x \in \mathbb{R}^d$ by

$$\mathrm{b}_{\gamma,n}(x) = \varphi_n(x) b(x) + (1 - \varphi_n(x)) \frac{b(x)}{1 + \gamma^\alpha \|b(x)\|} \ , \tag{50}$$

with $\alpha < 1/2$ and $\varphi_n \in \mathrm{C}(\mathbb{R}^d, \mathbb{R})$ such that for any $n \in \mathbb{N}$ and $x \in \mathbb{R}^d$,

$$\varphi_n(x) \in [0,1] \quad \text{and} \quad \varphi_n(x) = \begin{cases} 1 & \text{if } x \in \bar{\mathrm{B}}(0,n), \\ 0 & \text{if } x \in \bar{\mathrm{B}}(0,n+1)^{\mathrm{c}} \end{cases} . \tag{51}$$

An example of such a family is displayed in Figure 1.



(a) (b)

Figure 1: In this figure we illustrate the approximation properties of the family of drifts defined by (50). Let $b(x) = |x|^{1.5} \sin(x)$ and for any $n \in \mathbb{N}$, $\varphi_n(x) = d(x, \bar{\mathrm{B}}(0,+1)^{\mathrm{c}})^2/(d(x, \bar{\mathrm{B}}(0,n))^2 + d(x, \bar{\mathrm{B}}(0,n+1)^{\mathrm{c}})^2)$. In both figures the original drift is displayed in cyan and we fix $\alpha = 0.3$. In (a), we fix $n = 1$, represented by the black dashed lines, and observe the behavior of the drifts for different values of $\gamma > 0$. In (b), we plot the drift for different $\gamma > 0$ and $n \in \mathbb{N}$.

Theorem 21. *Assume* $\mathbf{B}3(\mathtt{m})$ *for* $\mathtt{m} \in \mathbb{R}$ *and* $\mathbf{B}5$*, then*

(a) $\mathbf{L}2$ *and* $\mathbf{L}5$ *hold for the family* $\{\mathrm{b}_{\gamma,n} : \gamma \in (0,\bar{\gamma}], n \in \mathbb{N}\}$ *defined by* (50) *;*

(b) for any $T, M \geq 0$ *and* $x \in \mathbb{R}^d$

$$\limsup_{n \to +\infty} \limsup_{m \to +\infty} \left(\mathrm{R}^m_{T/m,n} + \tilde{\mathrm{R}}^m_{T/m,n}\right) V_M(x) < +\infty ,$$

with $V_M(x) = \exp[M\phi(x)]$, $\phi(x) = (1 + \|x\|^2)^{1/2}$ *and where for any* $\gamma \in (0,\bar{\gamma}]$, $n \in \mathbb{N}$, $\mathrm{R}_{\gamma,n}$ *is the Markov kernel associated with* (1) *where* $\mathcal{T}_\gamma(x) = x + \gamma \mathrm{b}_{\gamma,n}(x)$, $\mathsf{X} = \bar{\mathrm{B}}(0,n)$ *and* $\Pi = \Pi_{\bar{\mathrm{B}}(0,n)}$, $\tilde{\mathrm{R}}_{\gamma,n}$ *is the Markov kernel associated with* (1) *where* $\mathcal{T}_\gamma(x) = x + \gamma \mathrm{b}_{\gamma,n}(x)$, $\mathsf{X} = \mathbb{R}^d$ *and* $\Pi = \mathrm{Id}$.

Proof. The proof is postponed to Section 4.10. □

We now have all the tools to apply Theorem 18. As in Section 2.2, we consider three different curvature assumptions on the drift b and derive convergence of the associated continuous process.

Theorem 22. *Assume either* $\mathbf{B}3(\mathtt{m})$ *for* $\mathtt{m} \in \mathbb{R}_-$ *or* $\mathbf{B}4$*. Assume in addition* $\mathbf{B}5$*,* $\mathbf{C}1$ *and* $\sup_{x\in\mathbb{R}^d}\{\|b(x)\|^{2(1+\varepsilon_b)} \mathrm{e}^{-\mathtt{m}_1^+\|x\|^2}\} < +\infty$ *for some* $\varepsilon_b > 0$*. Then, for any* $T \geq 0$*, and* $x, y \in \mathbb{R}^d$

$$\mathbf{W}_{\mathbf{c}_1}(\delta_x \mathrm{P}_T, \delta_y \mathrm{P}_T) \leq \lambda_a^{T/4}(D_{1,a}W_1(x,y) + D_{2,a}) + C_a \rho_a^{T/4} \ ,$$

with $D_{1,a}, D_{2,a}, C_a \geq 0$*,* $\lambda_a, \rho_a \in (0,1)$ *given by* (31) *and for any* $x, y \in \mathbb{R}^d$*,* $\mathbf{c}_1(x,y) = \mathbb{1}_{\Delta_{\mathsf{X}}^{\mathrm{c}}}(x,y)W_1(x,y)$ *with* $W_1(x,y) = 1 + \|x-y\|/R_1$*.*

Proof. Let $T \geq 0$. Using $\mathbf{B}5$, that $\mathbf{C}1$ implies $\mathbf{C}2$ and the fact that $\sup_{x\in\mathbb{R}^d}\{\|b(x)\| \mathrm{e}^{-\mathtt{m}_1^+\|x\|}\} < +\infty$, we obtain that $\mathbf{L}1$ and $\mathbf{L}3$ hold using Theorem 19. Let $V(x) = \exp[\phi(x)]$, with $\phi(x) = (1+\|x\|^2)^{1/2}$. Using Theorem 20 we obtain that $\mathrm{P}_T V^2(x) < +\infty$. Consider for any $n \in \mathbb{N}$, $\mathsf{K}_n = \bar{\mathrm{B}}(0,n)$ which satisfies $\mathbf{L}4$. Applying Theorem 21 to the family $\{\mathrm{b}_{\gamma,n} : \gamma \in (0,\bar{\gamma}], n \in \mathbb{N}\}$ defined by (50) we get that $\mathbf{L}2$, $\mathbf{L}5$ hold and for any $x \in \mathbb{R}^d$

$$\limsup_{n\to+\infty}\limsup_{m\to+\infty}\left(\mathrm{R}_{T/m,n}^m + \tilde{\mathrm{R}}_{T/m,n}^m\right)V^2(x) < +\infty \ ,$$

with $\mathrm{R}_{\gamma,n}$ the Markov kernel associated with (1) where $\mathcal{T}_\gamma(x) = x + \gamma \mathrm{b}_{\gamma,n}(x)$, $\mathsf{X} = \bar{\mathrm{B}}(0,n)$ and $\Pi = \Pi_{\bar{\mathrm{B}}(0,n)}$ and $\tilde{\mathrm{R}}_{\gamma,n}$ the Markov kernel associated with (1) where $\mathcal{T}_\gamma(x) = x + \gamma \mathrm{b}_{\gamma,n}(x)$, $\mathsf{X} = \mathbb{R}^d$ and $\Pi = \mathrm{Id}$. Hence, $\mathbf{L}1$, $\mathbf{L}2$, $\mathbf{L}3$, $\mathbf{L}4$ and $\mathbf{L}5$ hold. We have $\sup_{(x,y)\in\mathbb{R}^d\times\mathbb{R}^d} W_1(x,y)\{V(x)+V(y)\}^{-1} < +\infty$ and applying Theorem 18, we obtain that for any $x, y \in \mathbb{R}^d$

$$\mathbf{W}_{\mathbf{c}_1}(\delta_x \mathrm{P}_T, \delta_y \mathrm{P}_T) \leq \limsup_{n\to+\infty}\limsup_{m\to+\infty}\mathbf{W}_{\mathbf{c}_1}(\delta_x \mathrm{R}_{T/m,n}^m, \delta_y \mathrm{R}_{T/m,n}^m) \ .$$

Let $n \in \mathbb{N}$ and $m \in \mathbb{N}^\star$ such that $x, y \in \bar{\mathrm{B}}(0,n)$ and $T/\mathtt{m} \leq 2\mathtt{m}_1^+/\mathtt{L}_n^2$. Since $\mathbf{B}1(\bar{\mathrm{B}}(0,n))$ holds and $\mathbf{B}5$ implies $\mathbf{B}2$ on $\bar{\mathrm{B}}(0,n)$, we can apply Theorem 10 and we get

$$\begin{aligned}\mathbf{W}_{\mathbf{c}_1}&(\delta_x \mathrm{R}_{T/m,n}^m, \delta_y \mathrm{R}_{T/m,n}^m) \\ &\leq \lambda_{T/m,a}^{T/4}(D_{T/m,1,a}W_1(x,y) + D_{T/m,2,a}) + C_{T/m,a}\rho_{T/m,a}^{T/4} \ ,\end{aligned}$$

where $D_{T/m,1,a}, D_{T/m,2,a}, C_{T/m,a}, \lambda_{T/m,a}$ and $\rho_{T/m,a}$ are given in (30). In addition, these quantities admit limits $D_{1,a}, D_{2,a}, C_a \geq 0$ and $\lambda_a, \rho_a \in (0,1)$ when $m \to +\infty$ which do not depend on $\mathtt{L}_n$, hence on n, see (31). □

The discussion on the dependency of ρ_a with respect to the parameters of the problem conducted in Section 2.2 still holds. We distinguish the following cases, assuming that the conditions of Theorem 7-(c) are satisfied.

(a) If $\mathbf{B}4$ holds, we have

$$\log^{-1}(\rho_a^{-1}) \le (1+\log(2))/(\mathbf{\Phi}\{-1/2\}\mathtt{m}_1^+) + 4R_1^2/\mathbf{\Phi}\{-1/2\} \; . \tag{52}$$

The leading term in (52) is of order $\max(R_1^2, 1/\mathtt{m}_1^+)$, which corresponds to the one identified in [30, Theorem 2.8] and is optimal, see [28, Remark 2.10].

(b) If $\mathbf{B}3(\mathtt{m})$ holds with $\mathtt{m} \in \mathbb{R}_-$, we have

$$\log^{-1}(\rho_a^{-1}) \le \left[1+\log(2)+\log(1+2\{\mathtt{m}_1^+ - \mathtt{m}\}) + 2\mathtt{m}_1^+\right] \Big/ \left[\mathtt{m}_1^+ \mathbf{\Phi}\{-(-\mathtt{m})^{1/2} R_1/(2-2\mathrm{e}^{2\mathtt{m}})^{1/2}\}\right] \; .$$

For any $t \ge C$ with $C \ge 0$ we have

$$\mathbf{\Phi}(-t)^{-1} \le \sqrt{2\pi}(1+C^{-2}) t \mathrm{e}^{t^2/2} \; . \tag{53}$$

As a consequence if we also have $R_1 \ge 2$, $1 \le -\mathtt{m}$ and using that for any $t \in (0,1)$, $-\log(1-t) \le t$ as well as (53) we get that $\log^{-1}(\rho_a^{-1}) \le \log^{-1}(\rho_{\max}^{-1})$

$$\log^{-1}(\rho_{\max}^{-1}) = C\left[1+\log(1+2\{\mathtt{m}_1^+ - \mathtt{m}\}) + 2\mathtt{m}_1^+\right] R_1(-\mathtt{m})^{1/2} \times \exp\left[-\mathtt{m}R_1^2/(4-4\mathrm{e}^{2\mathtt{m}})\right] \Big/ \left[\mathtt{m}_1^+(1-\mathrm{e}^{2\mathtt{m}})^{1/2}\right] \; ,$$

with $C = 2(1+\log(2))\sqrt{\pi} \approx 6.00$. Under the same assumption than Theorem 22, [28, Equation (2.18)] identifies a convergence rate for the diffusion (40) for the Wasserstein metric of order 1 given by

$$\log^{-1}(\rho_E^{-1}) = 4\sqrt{\pi} R_1^{-1}(-\mathtt{m})^{-1/2}(-1/\mathtt{m} + 1/\mathtt{m}_1^+)\exp[-\mathtt{m}R_1^2/4] + 8/(R_1\mathtt{m}_1^+)^2 \; .$$

The rate $\rho_{\max}$ we obtain is always smaller than ρ_E but taking the limit we get that

$$\lim_{\mathtt{m}\to-\infty} \frac{\log(\log(\rho_{\max}^{-1}))}{\log(\log(\rho_E^{-1}))} = 1 \; .$$

Note in addition that [28, Lemma 2.9, Equation (2.18)] ensures a contraction of $(\mathrm{P}_t)_{t\ge 0}$ for the Wasserstein distance of order 1 whereas our result ensures the convergence of $(\mathrm{P}_t)_{t\ge 0}$ for the Wasserstein distance of order 1 as well as in total variation.

Theorem 23. *Assume either $\mathbf{B}3(\mathtt{m})$ for $\mathtt{m} \in \mathbb{R}_-$ or $\mathbf{B}4$. Assume in addition $\mathbf{B}5$, $\mathbf{C}2$ and $\sup_{x\in\mathbb{R}^d}\{\|b(x)\|^{2(1+\varepsilon_b)} \mathrm{e}^{-\mathtt{m}_2^+\|x\|^2}\} < +\infty$ for some $\varepsilon_b > 0$. Then for any $T \ge 0$ and $x, y \in \mathbb{R}^d$*

$$\|\delta_x \mathrm{P}_T - \delta_y \mathrm{P}_T\|_V \le C_b \rho_b^T \{V(x) + V(y)\} \; ,$$

with $C_b \ge 0$ and $\rho_b \in (0,1)$ given by (36) and for any $x \in \mathbb{R}^d$, $V(x) = 1 + \|x\|^2$.

Proof. The proof is identical to the one of Theorem 22 upon replacing Theorem 10 by Theorem 12. $\square$

The rates we obtain are identical to the ones derived taking the limit $\bar{\gamma} \to 0$ in Theorem 12. An upper bound on ρ_b depending on the curvature condition on b is provided in (37) and (38).

Theorem 24. *Assume either* **B**3($\mathtt{m}$) *for* $\mathtt{m} \in \mathbb{R}_-$ *or* **B**4*. Assume in addition* **B**5*,* **C** 3 *and* $\sup_{x\in\mathbb{R}^d} \|b(x)\|^{2(1+\varepsilon_b)} \mathrm{e}^{-\mathtt{k}_1(1+\|x\|)^{1/2}} < +\infty$ *for some* $\varepsilon_b > 0$*. Then*

$$\|\delta_x \mathrm{P}_T - \delta_y \mathrm{P}_T\|_V \leq C_c \rho_c^T \{V(x) + V(y)\} \ ,$$

with $C_c \geq 0$ *and* $\rho_c \in (0,1)$ *given by Appendix E and for any* $x \in \mathbb{R}^d$*,* $V(x) = \exp(\mathtt{m}_3^+(1+\|x\|^2)^{1/2})$ *and* $\mathtt{m}_3 \in (0, \mathtt{k}_1/4)$*.*

Proof. The proof is postponed to Section 4.11. $\square$

The rates we obtain are identical to the ones derived taking the limit $\bar{\gamma} \to 0$ in Theorem 14. An upper bound on ρ_c depending on the curvature condition on b is given in Section 4.6.

4 Postponed Proofs

4.1 Proof of Lemma 2

Let $\ell \in \mathbb{N}^\star$ and $\gamma \in (0, \bar{\gamma}]$. First note that the following equalities hold if $\kappa(\gamma) \neq 0$

$$\begin{aligned}
\Xi_{\ell\lceil 1/\gamma\rceil}(\kappa) &= \gamma \sum_{i=1}^{\ell\lceil 1/\gamma\rceil} (1+\gamma\kappa(\gamma))^{-i} \\
&= \gamma(1+\gamma\kappa(\gamma))^{-1} \frac{1-(1+\gamma\kappa(\gamma))^{-\ell\lceil 1/\gamma\rceil}}{1-(1+\gamma\kappa(\gamma))^{-1}} \\
&= -\kappa^{-1}(\gamma)\left\{[1+\gamma\kappa(\gamma)]^{-\ell\lceil 1/\gamma\rceil} - 1\right\} \\
&= -\kappa^{-1}(\gamma)\left\{\exp\left[-\ell\lceil 1/\gamma\rceil \log\{1+\gamma\kappa(\gamma)\}\right] - 1\right\} \ . \qquad (54)
\end{aligned}$$

We now give a lower-bound on $\Xi_{\ell\lceil 1/\gamma\rceil}(\kappa)$ depending on the condition satisfied by $\gamma \mapsto \kappa(\gamma)$.

(a) Assume that for any $\tilde{\gamma} \in (0, \bar{\gamma}]$, $\mathsf{\kappa}(\tilde{\gamma}) < 0$. Using that $\log(1-t) \leq -t$ for $t \in (0,1)$, we obtain that

$$\exp\left[-\ell \lceil 1/\gamma \rceil \log\left\{1 + \gamma \mathsf{\kappa}(\gamma)\right\}\right] \geq \exp(-\ell \lceil 1/\gamma \rceil \gamma \mathsf{\kappa}(\gamma)) \geq \exp(-\ell \mathsf{\kappa}(\gamma)) \; ,$$

which together with (54) concludes the proof for (a).

(b) Assume that for any $\tilde{\gamma} \in (0, \bar{\gamma}]$, $\mathsf{\kappa}(\tilde{\gamma}) \leq 0$. Then,

$$\Xi_{\ell\lceil 1/\gamma \rceil}(\mathsf{\kappa}) = \gamma \sum_{i=1}^{\ell\lceil 1/\gamma \rceil} (1 + \gamma \mathsf{\kappa}(\gamma))^{-i} \geq \gamma \lceil 1/\gamma \rceil \ell \geq \ell \; .$$

(c) Assume that for any $\tilde{\gamma} \in (0, \bar{\gamma}]$, $\mathsf{\kappa}(\tilde{\gamma}) > 0$. Using that $\log(1+t) \geq t/(1+t)$ for $t > 0$, we obtain that

$$\begin{aligned} \exp\left[-\ell \lceil 1/\gamma \rceil \log\left\{1 + \gamma \mathsf{\kappa}(\gamma)\right\}\right] &\leq \exp\left[-(\ell/\gamma) \log\left\{1 + \gamma \mathsf{\kappa}(\gamma)\right\}\right] \\ &\leq \exp\left[-\ell \mathsf{\kappa}(\gamma)/(1 + \gamma \mathsf{\kappa}(\gamma))\right] \; , \end{aligned}$$

which concludes the proof for (c).

4.2 Proof of Corollary 4

(a) Consider $V : \mathsf{X} \to [1, +\infty]$ given for any $x \in \mathsf{X}$ by $V(x) = 1 + \|x\|$. Then since **A**2(X^2) with $\sup_{\gamma \in (0, \bar{\gamma}]} \kappa(\gamma) \leq \kappa_- < 0$ holds, using the triangle inequality and the Cauchy-Schwarz inequality, we have for any $\gamma \in (0, \bar{\gamma}]$ and $x \in \mathsf{X}$

$$\mathrm{R}_\gamma V(x) \leq \|\mathcal{T}_\gamma(x)\| + \sqrt{\gamma d} \leq (1 + \kappa_- \gamma) \|x\| + \|\mathcal{T}_\gamma(0)\| + \sqrt{\gamma d} + 1 \leq \lambda V(x) + A \; ,$$

with $\lambda \in (0,1)$ and $A \geq 0$. As a result, since for any $\gamma \in (0, \bar{\gamma}]$, R_γ is a Feller kernel and the level sets of V are compact, R_γ admits a unique invariant probability measure π_γ for any $\gamma \in (0, \bar{\gamma}]$ by [19, Theorem 12.3.3]. Then the last result is a straightforward consequence of Proposition 3-(a), (3) and the fact that for any $\ell \in \mathbb{N}^\star$ and $\gamma \in (0, \bar{\gamma}]$, $\alpha_-(\kappa, \gamma, \ell) \geq -(\exp(-\ell \kappa_-) - 1)/\kappa_-$ since $t \mapsto (\exp(\ell t) - 1)/t$ is increasing on $\mathbb{R}$.

(b) This result is a direct consequence of Proposition 3-(b), (3) and the fact that R_γ admits an invariant probability measure π_γ.

4.3 Proof of Proposition 8

(a) By **B** 2 and **B** 3($\mathtt{m}$) we have for any $\gamma > 0$ and $x, y \in \mathsf{X}$, $\|\mathcal{T}_\gamma(x) - \mathcal{T}_\gamma(y)\|^2 \leq (1 - 2\gamma\mathtt{m} + \gamma^2\mathtt{L}^2)\|x - y\|^2 \leq (1 + \gamma\kappa(\gamma))\|x - y\|^2$, which concludes the proof.

(b) By (21), we have for any $\gamma > 0$ and $x, y \in \mathsf{X}$, $\|\mathcal{T}_\gamma(x) - \mathcal{T}_\gamma(y)\|^2 \leq \|x - y\|^2 + \gamma(-2\mathtt{m}_b + \gamma)\|b(x) - b(y)\|^2$. Then if $\gamma \leq 2\mathtt{m}_b$, $\|\mathcal{T}_\gamma(x) - \mathcal{T}_\gamma(y)\|^2 \leq \|x - y\|^2$, which concludes the proof.

4.4 Proof of Proposition 11

We preface the proof by a technical result.

Lemma 25. *Let $\bar{\gamma} > 0$, such that for any $\gamma \in (0, \bar{\gamma}]$, P_γ is a Markov kernel and Q_γ is a Markov coupling kernel for P_γ. In addition, let $V : \mathsf{X} \to [1, +\infty)$, $\lambda \in (0, 1)$ and $A \geq 0$ such that for any $\gamma \in (0, \bar{\gamma}]$, P_γ satisfies $\mathbf{D}_{\mathrm{d}}(V, \lambda^\gamma, A\gamma, \mathsf{X})$ then Q_γ satisfies $\mathbf{D}_{\mathrm{d}}(W, \lambda^\gamma, A\gamma, \mathsf{X}^2)$, where for all $x, y \in \mathsf{X}$, $W(x, y) = \{V(x) + V(y)\}/2$.*

Proof. Let $\gamma \in (0, \bar{\gamma}]$ and $x, y \in \mathsf{X}$. Since $\delta_{(x,y)}\mathrm{Q}_\gamma$ is a transference plan between $\delta_x\mathrm{P}_\gamma$ and $\delta_y\mathrm{P}_\gamma$ we have

$$\mathrm{Q}_\gamma W(x, y) = \mathrm{Q}_\gamma \{V(x) + V(y)\}/2 = \mathrm{P}_\gamma V(x)/2 + \mathrm{P}_\gamma V(y)/2 \leq \lambda^\gamma W(x, y) + A\gamma\,.$$

□

Proof of Proposition 11. Let $\gamma \in (0, \bar{\gamma}]$ and $x \in \mathsf{X}$. Using (1), **B** 2, **B** 3($\mathtt{m}$), **C** 2, that the projection Π_{X} is non expansive and $\gamma < 2\mathtt{m}_2^+/\mathtt{L}^2$, we obtain the following inequalities

$$\begin{aligned} \mathrm{R}_\gamma V(x) &\leq 1 + \|x + \gamma b(x)\|^2 + \gamma d \\ &\leq 1 + \|x\|^2 + 2\gamma\langle x, b(x)\rangle + \gamma^2\|b(x)\|^2 + \gamma d \\ &\leq (1 + \|x\|^2)\left[1 - \gamma(2\mathtt{m}_2^+ - \bar{\gamma}\mathtt{L}^2)\right] + \gamma\left(d + 2R_2^2(\mathtt{m}_2^+ - \mathtt{m})_+ + 2\mathtt{m}_2^+\right)\,, \end{aligned}$$

which concludes the proof using Lemma 25.

□

4.5 Proof of Theorem 12

Let $\bar{\gamma} \in (0, 2\mathtt{m}^+/\mathtt{L}^2)$. Using Proposition 11 we obtain that for any $\gamma \in (0, \bar{\gamma}]$, K_γ satisfies $\mathbf{D}_{\mathrm{d}}(W_2, \lambda^\gamma, A\gamma, \mathsf{X})$, with λ and A explicitly given by in Proposition 11. For any $x, y \in \mathsf{X}$ such that $\|x-y\| \geq M_{\mathrm{d}}$, either $V(x) \geq 2K_{\mathrm{d}}$ or $V(y) \geq 2K_{\mathrm{d}}$, and therefore $W_2(x,y) > K_{\mathrm{d}}$. Note also that (15) is satisfied for any $\gamma \in (0, \bar{\gamma}]$ and $x, y \in \mathsf{X}$ and Theorem 1 and Lemma 2. Therefore we can apply Theorem 6, which concludes the proof.

4.6 Proof of Proposition 13

Let $\gamma \in (0, \bar{\gamma}]$. Using the fact that Π_{X} is non expansive, the Log-Sobolev inequality, the fact that π is 1-Lipschitz, [4, Theorem 5.5] and the Jensen inequality we obtain for any $x \in \mathbb{R}^d$

$$
\begin{aligned}
\mathrm{R}_\gamma V(x) &\leq \exp\left[\mathtt{m}_3^+ \mathrm{R}_\gamma \phi(x) + \gamma(\mathtt{m}_3^+)^2/2\right] \leq \exp\left[\mathtt{m}_3^+ \sqrt{1 + \mathrm{R}_\gamma \|x\|^2} + \gamma(\mathtt{m}_3^+)^2/2\right] \\
&\leq \exp\left[\mathtt{m}_3^+ \sqrt{1 + \|\mathcal{T}_\gamma(x)\|^2 + \gamma d} + \gamma(\mathtt{m}_3^+)^2/2\right] . \qquad (55)
\end{aligned}
$$

Let $x \in \mathbb{R}^d$. The rest of the proof is divided in two parts.

(a) In the first case, $\|x\| \geq R_4$. Since $\|x\| \geq R_3$ and $\gamma \leq 2\mathtt{k}_2$, we have using **C**3

$$
\|\mathcal{T}_\gamma(x)\|^2 \leq \|x\|^2 - 2\gamma\mathtt{k}_1 \|x\| + \gamma(\gamma - 2\mathtt{k}_2) \|b(x)\|^2 + \gamma\mathtt{a} \leq \|x\|^2 - 2\gamma\mathtt{k}_1 \|x\| + \gamma\mathtt{a} . \qquad (56)
$$

Since $\|x\| \geq 1$ we have $2\|x\| \geq \phi(x)$ and therefore, using that $\|x\| \geq (d+\mathtt{a})/\mathtt{k}_1$, $2\mathtt{k}_1 \|x\| \geq 2\mathtt{m}_3^+ \phi(x) + d + \mathtt{a}$. This inequality, combined with the fact that for any $t \in (-1, +\infty)$, $\sqrt{1+t} \leq 1 + t/2$, yields

$$
\begin{aligned}
&\sqrt{1 + \|x\|^2 + \gamma(-2\mathtt{k}_1 \|x\| + d + \mathtt{a})} - \phi(x) \\
&\qquad \leq \gamma(-2\mathtt{k}_1 \|x\| + d + \mathtt{a})/(2\phi(x)) \leq -\gamma\mathtt{m}_3^+ . \qquad (57)
\end{aligned}
$$

Combining (55), (56) and (57) we get

$$
\mathrm{R}_\gamma V(x) \leq \lambda^\gamma V(x) .
$$

(b) In the second case $\|x\| \leq R_4$. We have the following inequality using **C**3 and that $\gamma \leq 2\mathtt{k}_2$

$$
\|\mathcal{T}_\gamma(x)\|^2 \leq \|x\|^2 + \gamma(\gamma - 2\mathtt{k}_2) \|b(x)\|^2 + \gamma c \leq \|x\|^2 + \gamma\mathtt{a} . \qquad (58)
$$

Combining (55), (58) and the fact that for any $t \in (-1, +\infty)$, $\sqrt{1+t} \leq 1 + t/2$ we get

$$
\begin{aligned}
\mathrm{R}_\gamma V(x) &\leq \exp\left[\gamma\mathtt{m}_3^+ (d+\mathtt{a})/(2\phi(x)) + \gamma(\mathtt{m}_3^+)^2/2\right] V(x) \qquad (59) \\
&\leq \exp\left[\gamma(\mathtt{m}_3^+(d+\mathtt{a}) + (\mathtt{m}_3^+)^2)/2\right] V(x) .
\end{aligned}
$$

Note that for any $c_1 \geq c_2$ and $t \in [0, \bar{t}]$ we have the following inequality

$$\mathrm{e}^{c_1 t} \leq \mathrm{e}^{c_2 t} + \mathrm{e}^{c_1 \bar{t}}(c_1 - c_2)t \; . \tag{60}$$

Combining (59) and (60) we get

$$\mathrm{R}_\gamma V(x) \leq \lambda^\gamma V(x) + \exp\left[\bar{\gamma}(\mathtt{m}_3^+(d+\mathtt{a}) + (\mathtt{m}_3^+)^2)/2\right] C_{\mathtt{a}}\gamma \; ,$$

with $C_{\mathtt{a}} = (\mathtt{m}_3^+(d+\mathtt{a})/2 + (\mathtt{m}_3^+)^2)\exp(\mathtt{m}_3^+(1+R_4^2)^{1/2})$, which concludes the proof using Lemma 25.

4.7 Proof of Theorem 14

Let $\bar{\gamma} \in (0, 2\mathtt{k}_2]$. Using Proposition 11 we obtain that for any $\gamma \in (0, \bar{\gamma}]$, K_γ satisfies $\mathbf{D}_{\mathrm{d}}(W_3, \lambda^\gamma, A\gamma, \mathsf{X})$, with λ and A explicitly given in Proposition 11. Let

$$K_{\mathrm{d}} = 2A(1+\bar{\gamma})(1+\log^{-1}(1/\lambda)) \quad \text{and } M_{\mathrm{d}} = 2\log(2K_{\mathrm{d}})/\mathtt{m}_3^+ \; .$$

Note that for any $x, y \in \mathsf{X}$ such that $\|x-y\| \geq M_{\mathrm{d}}$, either $V(x) \geq 2K_{\mathrm{d}}$ or $V(y) \geq 2K_{\mathrm{d}}$, and therefore $W_3(x,y) > K_{\mathrm{d}}$. The rest of the proof is similar to the one of Theorem 12.

4.8 Proof of Theorem 19

Let $p \in \mathbb{N}^\star$ and $V \in \mathrm{C}^2(\mathbb{R}^d, [1, +\infty))$ be defined for any $x \in \mathbb{R}^d$ by $V(x) = 1 + \|x\|^{2p}$. For any $x \in \mathbb{R}^d$, $\nabla V(x) = 2p\|x\|^{2(p-1)} x$ and $\Delta V(x) = (4p(p-1) + 2pd)\|x\|^{2(p-1)}$. Therefore, using $\mathbf{B}3(\mathtt{m})$ and the definition of $\mathcal{A}$ we obtain that for any $x \in \mathbb{R}^d$

$$\mathcal{A}V(x) \leq [2p(p-1) + p(d - 2\mathtt{m})]\, V(x) \; . \tag{61}$$

Hence, using (61) and [46, Theorem 3.5], we obtain that $(\mathbf{X}_t)_{t\geq 0}$ is defined for any $t \geq 0$, for any starting point $\mathbf{X}_0 = x \in \mathbb{R}^d$.

(a) If there exists $\varepsilon_b > 0$ such that $\sup_{x\in\mathbb{R}^d} \|b(x)\|^{2(1+\varepsilon_b)} (1 + \|x\|^{2p})^{-1} < +\infty$, using (61) and Lemma 15-(a) we obtain that $\mathbf{L}3$ holds.

(b) If there exists $\varepsilon_b > 0$ such that $\sup_{x\in\mathbb{R}^d} \|b(x)\|^{2(1+\varepsilon_b)} \mathrm{e}^{-\mathtt{m}_2^+\|x\|^2} < +\infty$, and $\mathbf{C}$ 2 holds, then consider for any $x \in \mathbb{R}^d$, $V(x) = \mathrm{e}^{\mathtt{m}_2^+\|x\|^2}$. We have for any $x \in \mathbb{R}^d$, $\nabla V(x) = 2\mathtt{m}_2^+\mathrm{e}^{\mathtt{m}_2^+\|x\|^2} x$ and $\Delta V(x) = 4\mathtt{m}_2^{+2}\mathrm{e}^{\mathtt{m}_2^+\|x\|^2}\|x\|^2 + 2\mathtt{m}_2^+\mathrm{e}^{\mathtt{m}_2^+\|x\|^2} d$. Therefore, using $\mathbf{C}2$ we have for any $x \in \bar{\mathrm{B}}(0, R_2)^{\mathrm{c}}$

$$\mathcal{A}V(x) \leq \mathtt{m}_2^+\left[d + (4\mathtt{m}_2^+/2 - 2\mathtt{m}_2^+)\|x\|^2\right] V(x) \leq \mathtt{m}_2^+ d V(x) \; . \tag{62}$$

Setting $\zeta = (\mathtt{m}_2^+ d) \vee \sup_{x\in\bar{\mathrm{B}}(0,R_2)} \mathcal{A}V(x)$, we obtain that V satisfies $\mathbf{D}_{\mathrm{c}}(V, \zeta, 0)$. Therefore using (62) and Lemma 15-(a) , we obtain that $\mathbf{L}3$ holds.

4.9 Proof of Theorem 20

We have for any $x \in \mathbb{R}^d$,

$$\nabla\phi(x) = x/\phi(x) \; , \qquad \nabla^2\phi(x) = \mathrm{Id}/\phi(x) - xx^\top/\phi^2(x) \; ,$$

and therefore since $V_M(x) = \exp(M\phi(x))$,

$$\begin{aligned} &\nabla V_M(x) = M\nabla\phi(x)V_M(x) \; , \\ &\nabla^2 V_M(x) = \left\{M^2\nabla\phi(x)(\nabla\phi(x))^\top + M\nabla^2\phi(x)\right\} V_M(x) \; . \end{aligned}$$

Therefore, for any $x \in \mathbb{R}^d$,

$$\begin{aligned} &(\mathcal{A}V_M(x))/V_M(x) \\ &\quad \le \left[M^2\|x\|^2/\phi^2(x) + M\left\{d/\phi(x) - \|x\|^2/\phi^2(x)\right\}\right]\Big/2 + M\sup_{x\in\mathbb{R}^d}\langle b(x), x\rangle_+ \; . \end{aligned}$$

Hence, for any $x \in \mathbb{R}^d$, $\mathcal{A}V_M(x) \le \zeta V_M(x)$ with $\zeta = M\{\sup_{x\in\mathbb{R}^d}\langle b(x), x\rangle_+ + d/2\} + M^2$. We conclude using Lemma 15-(a) and the Doob maximal inequality.

4.10 Proof of Theorem 21

We preface the proof by a preliminary computation. Let $n \in \mathbb{N}$, $\gamma \in (0, \bar{\gamma}]$, $x \in \mathbb{R}^d$ and $X = x + \gamma \mathrm{b}_{\gamma,n}(x) + \sqrt{\gamma}Z$, where Z is a d-dimensional Gaussian random variable with zero mean and covariance identity. We have using **B**3($\mathtt{m}$) and (50)

$$\mathbb{E}\left[\|X\|^2\right] \le \|x\|^2 - 2\gamma\mathtt{m}\,\Phi_n(x)\|x\|^2 + \gamma^2\Phi_n(x)^2\|b(x)\|^2 + \gamma d \; , \tag{63}$$

with $\Phi_n(x) = \varphi_n(x) + (1-\varphi_n(x))(1+\gamma^\alpha\|b(x)\|)^{-1}$. In addition by **B**5 and (51), we have

$$\Phi_n(x)\|b(x)\| \le \mathtt{L}_{n+1}\|x\| + \gamma^{-\alpha} \; . \tag{64}$$

Combining (63) and (64) and since $\Phi_n(x) \le 1$ by (51), we obtain

$$\mathbb{E}\left[1 + \|X\|^2\right] \le (1 + \|x\|^2)\left[1 + 2\gamma|\mathtt{m}| + 2\gamma^2\mathtt{L}_{n+1}^2\right] + 2\gamma^{2-2\alpha} + \gamma d \; . \tag{65}$$

We are now able to complete the proof of Theorem 21.

(a) It is easy to check that **L**2 holds with $\beta = 2\alpha$. It only remains to show that **L**5 holds. Consider for any $x \in \mathbb{R}^d$, $\tilde{V}(x) = 1 + \|x\|$. By (65), for any $\gamma \in (0, \bar{\gamma}]$, $n \in \mathbb{N}$

and $x \in \mathbb{R}^d$, we have using for any $s \geq \mathbb{R}$, $1 + s \leq \mathrm{e}^s$ we obtain

$$\begin{aligned}
\mathrm{R}_{\gamma,n}\tilde{V}(x) &\leq \tilde{V}(x)\left[1 + 2\gamma\,|\mathtt{m}| + 2\gamma^2\mathtt{L}_{n+1}^2 + 2\gamma^{2-2\alpha} + \gamma d\right] \\
&\leq \tilde{V}(x)\exp\left[\gamma\left\{2\,|\mathtt{m}| + d + 2\gamma^{1-2\alpha}(\gamma^{2\alpha}\mathtt{L}_{n+1} + 1)\right\}\right] \\
&\leq \tilde{V}(x)\exp\left[2\gamma\left\{2\,|\mathtt{m}| + d\right\}\left\{1 + \gamma^{1-2\alpha}(\gamma^{2\alpha}\mathtt{L}_{n+1} + 1)\right\}\right] \; .
\end{aligned}$$

As a result using that $d \geq 1$, **L**5 holds upon taking $\tilde{A} = \exp(4\,|\mathtt{m}| + 2d)$, $\tilde{\varepsilon}_n = 1 - 2\alpha$ and $\tilde{E}_n = 2(\mathtt{L}_{n+1}\bar{\gamma}^{2\alpha} + 1)$.

(b) Let $M \geq 0$, $n \in \mathbb{N}$ and $p \geq 1$. Using the Log-Sobolev inequality [4, Theorem 5.5], the fact that ϕ is 1-Lipschitz and that $\Pi_{\bar{\mathrm{B}}(0,n)}$ is non expansive, as well as the Jensen inequality we obtain for any $\gamma \in (0, \bar{\gamma}]$ and $x \in \mathbb{R}^d$,

$$\begin{aligned}
\mathrm{R}_{\gamma,n}V_M^p(x) &\leq \exp\left[pM\tilde{\mathrm{R}}_{\gamma,n}\phi(x) + (pM)^2\gamma/2\right] \\
&\leq \exp\left[pM\sqrt{\tilde{\mathrm{R}}_{\gamma,n}\phi^2(x)} + (pM)^2\gamma/2\right] \; .
\end{aligned}$$

Using (65) and that $\sqrt{1+t} \leq 1 + t/2$ for any $t \in (-1, +\infty)$ we get for any $\gamma \in (0, \bar{\gamma}]$ and $x \in \bar{\mathrm{B}}(0, n)$

$$\begin{aligned}
&\mathrm{R}_{\gamma,n}V_M^p(x) \\
&\leq \exp\left[pM\left\{\phi(x)^2(1 + 2\gamma\,|\mathtt{m}| + 2\gamma^2\mathtt{L}_{n+1}^2) + 2\gamma^{2-2\alpha} + \gamma d\right\}^{1/2} + (pM)^2\gamma/2\right] \\
&\leq \exp\left[(1 + \gamma\,|\mathtt{m}| + \gamma^2\mathtt{L}_{n+1}^2)pM\phi(x)\right]\exp\left[(1 + pM)^2\left\{\gamma(d+1)/2 + \gamma^{2-2a}\right\}\right] \\
&\leq V_M^{p(1 + C_1\gamma + C_{2,n}\gamma^2)}(x)\exp\left[p^2C_3\gamma\right] \; ,
\end{aligned}$$

with $C_1 = |\mathtt{m}|$, $C_{2,n} = \mathtt{L}_{n+1}^2$ and $C_3 = (1 + M)^2(d + 3)/2$. By recursion, we obtain that for any $m, n \in \mathbb{N}$ with $m^{-1} \in (0, \bar{\gamma}]$, $T \geq 0$ and $x \in \bar{\mathrm{B}}(0, n)$

$$\begin{aligned}
&\mathrm{R}_{T/m,n}^m V_M(x) \\
&\leq V_M(x)^{a_m}\exp\left[TC_3\sum_{j=0}^{m-1}(1 + TC_1/m + C_{2,n}(T/m)^2)^{2j}/m\right] \\
&\leq V_M(x)^{a_m}\exp\left[TC_3(1 + TC_1/m + C_{2,n}(T/m)^2)^{2m}\right] \; ,
\end{aligned}$$

with $a_m = (1 + TC_1/m + C_{2,n}(T/m)^2)^m$. Since $\lim_{m\to+\infty}(1 + TC_1/m + C_{2,n}(T/m)^2)^{tm} = \exp(tTC_1)$ for any $t, T \geq 0$, we get that for any $n \in \mathbb{N}$, $T \geq 0$ and $x \in \bar{\mathrm{B}}(0, n)$

$$\limsup_{m\to+\infty}\mathrm{R}_{T/m,n}^m V_M(x) \leq \exp(TC_3\exp(2TC_1))V_M^{\exp(TC_1)}(x) \; . \tag{66}$$

We conclude the proof upon remarking that the right-hand side quantity in (66) does not depend on n and that the same inequality holds replacing $\mathrm{R}_{T/m,n}$ by $\tilde{\mathrm{R}}_{T/m,n}$ in (66).

4.11 Proof of Theorem 24

Let $T \geq 0$, and for any $x \in \mathbb{R}^d$, $V_1(x) = \exp(\mathtt{k}_1 \phi(x))$ with $\phi(x) = (1 + \|x\|^2)^{1/2}$. Since **B**3($\mathtt{m}$) holds we obtain using Theorem 19 that **L**1 holds. By **C**3, $\sup_{x \in \mathbb{R}^d} \langle x, b(x) \rangle < +\infty$, therefore there exists $\zeta \in \mathbb{R}$ such that $\mathbf{D}_{\mathrm{c}}(V_1, \zeta, 0)$ holds. Combining this result with Lemma 15 and the fact that $\sup_{x \in \mathbb{R}^d} \|b(x)\|^{2(1+\varepsilon_b)} \mathrm{e}^{-\mathtt{k}_1 \phi(x)} < +\infty$ for $\varepsilon_b > 0$, we get that **L**3 holds. Since for any $x \in \mathbb{R}^d$, $V^2(x) \leq V_1(x)$, we have that $\mathrm{P}_T V^2(x) < +\infty$. Consider for any $n \in \mathbb{N}$, $\mathsf{K}_n = \bar{\mathrm{B}}(0,n)$ which satisfies **L**4. Using Theorem 21-(a) we get that **L**2 and **L**5 hold. Hence, **L**1, **L**2, **L**3, **L**4 and **L**5 are satisfied. In the proof of Proposition 13, see Section 4.6, we show, since $2\mathtt{m}_3^+ \in (0, \mathtt{k}_1/2]$, that there exist $\lambda \in (0,1)$ and $A \geq 0$ given by (84) upon replacing $\mathtt{m}_3^+$ by $2\mathtt{m}_3^+$, such that $\mathbf{D}_{\mathrm{d}}(V^2, \lambda^{T/m}, AT/m, \bar{\mathrm{B}}(0,n))$ holds for $\mathrm{R}_{T/m,n}$ and $\tilde{\mathrm{R}}_{T/m,n}$ with $m, n \in \mathbb{N}$ and m large enough and where $\mathrm{R}_{\gamma,n}$ is the Markov kernel associated with (1) where $\mathcal{T}_\gamma(x) = x + \gamma \mathrm{b}_{\gamma,n}(x)$, $\mathsf{X} = \bar{\mathrm{B}}(0,n)$ and $\Pi = \Pi_{\bar{\mathrm{B}}(0,n)}$, $\tilde{\mathrm{R}}_{\gamma,n}$ is the Markov kernel associated with (1) where $\mathcal{T}_\gamma(x) = x + \gamma \mathrm{b}_{\gamma,n}(x)$, $\mathsf{X} = \mathbb{R}^d$ and $\Pi = \mathrm{Id}$. We get that for any $x \in \mathbb{R}^d$

$$\limsup_{n \to +\infty} \limsup_{m \to +\infty} \left\{ \updelta_x \mathrm{R}_{T/m,n}^m V^2(x) + \updelta_y \mathrm{R}_{T/m,n}^m V^2(x) \right\} \leq 2V^2(x) + 2AT < +\infty \ ,$$

We can apply Theorem 18 and we obtain that for any $x, y \in \mathbb{R}^d$

$$\|\updelta_x \mathrm{P}_T - \updelta_y \mathrm{P}_T\|_V \leq \limsup_{n \to +\infty} \limsup_{m \to +\infty} \|\updelta_x \mathrm{R}_{T/m,n}^m - \updelta_y \mathrm{R}_{T/m,n}^m\|_V \ .$$

Since **B**1($\bar{\mathrm{B}}(0,n)$) holds and **B**5 implies **B**2 on $\bar{\mathrm{B}}(0,n)$, we can apply Theorem 14 and we get that for any $m, n \in \mathbb{N}$ with $x, y \in \bar{\mathrm{B}}(0,n)$ and $T/m \in (0, 2\mathtt{k}_2)$

$$\|\updelta_x \mathrm{R}_{T/m,n}^m - \updelta_y \mathrm{R}_{T/m,n}^m\|_V \leq C_{1/m,c} \rho_{1/m,c}^T \left\{ V(x) + V(y) \right\} \ ,$$

where $C_{1/m,c} \geq 0$ and $\rho_{1/m,c} \in (0,1)$, see Appendix E. We conclude upon noting that $C_{1/m,c}$ and $\rho_{1/m,c}$ admit limits C_c and ρ_c when $m \to +\infty$ which do not depend on n.

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A Quantitative bounds for geometric convergence of Markov chains in Wasserstein distance

In this section, we establish new quantitative bounds for Markov chains in Wasserstein distance. We consider a Markov kernel P on the measurable space $(\mathsf{Y}, \mathcal{Y})$ equipped with the bounded semi-metric $\mathbf{d} : \mathsf{Y} \times \mathsf{Y} \to \mathbb{R}_+$, *i.e.* which satisfies the following condition.

H 1. *For any $x, y \in \mathsf{Y}$, $\mathbf{d}(x, y) \leq 1$, $\mathbf{d}(x, y) = \mathbf{d}(y, x)$ and $\mathbf{d}(x, y) = 0$ if and only if $x = y$.*

Let K be a Markov coupling kernel for P. We assume in this section the following condition on K.

H 2 (K)**.** *There exists $\mathsf{C} \in \mathcal{Y}^{\otimes 2}$ such that*

(i) there exist $\mathtt{n}_0 \in \mathbb{N}^\star$ and $\varepsilon > 0$ such that for any $x, y \in \mathsf{C}$, $\mathrm{K}^{\mathtt{n}_0}\mathbf{d}(x, y) \leq (1 - \varepsilon)\mathbf{d}(x, y)$;

(ii) for any $x, y \in \mathsf{Y}$, $\mathrm{K}\mathbf{d}(x, y) \leq \mathbf{d}(x, y)$;

(iii) there exist $W_1 : \mathsf{Y}^2 \to [1, +\infty)$ measurable, $\lambda_1 \in (0, 1)$ and $A_1 \geq 0$ such that K satisfies $\mathbf{D}_{\mathrm{d}}(W_1, \lambda_1, A_1, \mathsf{C})$.

We consider the Markov chain $(\mathrm{X}_n, \mathrm{Y}_n)_{n\in\mathbb{N}}$ associated with the Markov kernel K defined on the canonical space $((\mathsf{Y} \times \mathsf{Y})^{\mathbb{N}}, (\mathcal{Y}^{\otimes 2})^{\mathbb{N}})$ and denote by $\mathbb{P}_{(x,y)}$ and $\mathbb{E}_{(x,y)}$ the corresponding probability distribution and expectation respectively when $(\mathrm{X}_0, \mathrm{Y}_0) = (x, y)$. Denote by $(\mathcal{G}_n)_{n\in\mathbb{N}}$ the canonical filtration associated with $(\mathrm{X}_n, \mathrm{Y}_n)_{n\in\mathbb{N}}$. Note that for any $n \in \mathbb{N}$ and $x, y \in \mathsf{Y}$, under $\mathbb{P}_{(x,y)}$, $(\mathrm{X}_n, \mathrm{Y}_n)$ is by definition a coupling of $\delta_x\mathrm{P}^n$ and $\delta_y\mathrm{P}^n$. The main result of this section is the following which by the previous observation implies quantitative bounds on $\mathbf{W}_{\mathbf{d}}(\delta_x\mathrm{P}^n, \delta_y\mathrm{P}^n)$.

Theorem 26. *Let K be a Markov coupling kernel for P and assume **H**1 and **H**2(K). Then for any $n \in \mathbb{N}$ and $x, y \in \mathsf{Y}$,*

$$\begin{aligned}
&\mathbb{E}_{(x,y)}\left[\mathbf{d}(\mathrm{X}_n, \mathrm{Y}_n)\right] \\
&\qquad \leq \min\left[\rho^n(M_{\mathsf{C},\mathtt{n}_0}\Xi(x, y, \mathtt{n}_0) + \mathbf{d}(x, y)), \rho^{n/2}(1 + \mathbf{d}(x, y)) + \lambda_1^{n/2}\Xi(x, y, \mathtt{n}_0)\right] ,
\end{aligned}$$

where

$$\begin{aligned}
\Xi(x, y, \mathtt{n}_0) &= W_1(x, y) + A_1\lambda_1^{-\mathtt{n}_0}\mathtt{n}_0 \\
\log(\rho) &= \log(1 - \varepsilon)\log(\lambda_1)/[-\log(M_{\mathsf{C},\mathtt{n}_0}) + \log(1 - \varepsilon)] \;, \\
M_{\mathsf{C},\mathtt{n}_0} &= \sup_{(x,y)\in\mathsf{C}} \Xi(x, y, \mathtt{n}_0) = \sup_{(x,y)\in\mathsf{C}} [W_1(x, y)] + A_1\lambda_1^{-\mathtt{n}_0}\mathtt{n}_0 \;.
\end{aligned} \tag{67}$$

In Theorem 26, we obtain geometric contraction for P in bounded Wasserstein metric $\mathbf{W}_{\mathbf{d}}$ since $\mathbf{d}$ is assumed to be bounded. To obtain convergence associated with unbounded Wasserstein metric associated with $W_2 : \mathsf{Y}^2 \to [0, +\infty)$, we consider the next assumption which is a generalized drift condition linking W_2 and the bounded semi-metric $\mathbf{d}$.

H 3 (K). *There exist $W_2 : \mathsf{Y}^2 \to [0, +\infty)$ measurable, $\lambda_2 \in (0, 1)$ and $A_2 \geq 0$ such that for any $x, y \in \mathsf{Y}$,*

$$\mathrm{K}W_2(x, y) \leq \lambda_2 W_2(x, y) + A_2 \mathbf{d}(x, y) \ .$$

In the special case where $\mathbf{d}(x, y) = \mathbb{1}_{\Delta_{\mathsf{Y}}^{\mathrm{c}}}(x, y)$, $W_2(x, y) = \mathbb{1}_{\Delta_{\mathsf{Y}}^{\mathrm{c}}}(x, y) W_1(x, y)$ and for any $x \in \mathsf{Y}$, $\mathrm{K}((x, x), \Delta_{\mathsf{Y}}) = 1$, we obtain that $\mathbf{D}_{\mathrm{d}}(W_1, \lambda_1, A_1, \mathsf{Y})$ implies **H** 3(K). The following result implies quantitative bounds on the Wasserstein distance $\mathbf{d}_{W_2}(\delta_x \mathrm{P}^n, \delta_y \mathrm{P}^n)$ for any $x, y \in \mathsf{Y}$ and $n \in \mathbb{N}^\star$.

Theorem 27. *Let K be a Markov coupling kernel for P and assume* **H** 1*,* **H** 2(K) *and* **H** 3(K)*. Then for any $n \in \mathbb{N}$ and $x, y \in \mathsf{Y}$,*

$$\begin{aligned} &\mathbb{E}_{(x,y)} \left[W_2(\mathrm{X}_n, \mathrm{Y}_n) \right] \leq \lambda_2^n W_2(x, y) \\ &+ A_2 \min \left[\tilde{\rho}^{n/4} r_\rho (\mathbf{d}(x, y) + \Xi(x, y, \mathtt{n}_0)), \tilde{\rho}^{n/4} r_\rho (1 + \mathbf{d}(x, y)) + \tilde{\lambda}^{n/4} r_\lambda \Xi(x, y, \mathtt{n}_0) \right] \ , \end{aligned}$$

where

$$\begin{aligned} \tilde{\rho} &= \max(\lambda_2, \rho) \in (0, 1) \ , \qquad & \tilde{\lambda} &= \max(\lambda_1, \lambda_2) \in (0, 1) \ , \\ r_\rho &= 4 \log^{-1}(1/\tilde{\rho}) / \tilde{\rho} \ , & r_\lambda &= 4 \log^{-1}(1/\tilde{\lambda}) / \tilde{\lambda} \ , \end{aligned}$$

and $\Xi(x, y, \mathtt{n}_0)$, $M_{\mathsf{C}, \mathtt{n}_0}$ and ρ are given in (67)*.*

Theorem 26 and Theorem 27 share some connections with [68, Theorem 5], [37] and [24] but hold under milder assumptions than the ones considered in these works. Compared to [37] and [24], the main difference is that we allow here only a contraction for the $\mathtt{n}_0$-th iterate of the Markov chain (condition **H** 2-(i)) which is necessary if we want to use Theorem 1 to obtain sharp quantitative convergence bounds for (1). Finally, [68, Theorem 5] also considers minorization condition for the the $\mathtt{n}_0$-th iterate, however our results compared favourably for large $\mathtt{n}_0$. Indeed, Theorem 26 implies that the rate of convergence $\min(\rho, \lambda_1)$ is of the form $C\mathtt{n}_0^{-1}$ for $C \geq 0$ independent of $\mathtt{n}_0$. Applying [68, Theorem 5], we found a rate of convergence of the form $C\mathtt{n}_0^{-2}$. Finally, a recent work [64] has established new results based on the technique used in [37]. However, we were not able to apply them since they assume as in [37], a contraction for $\mathtt{n}_0 = 1$ which does not imply sharp bounds on the situations we consider.

The rest of this section is devoted to the proof of Theorem 26 and Theorem 27. Denote by $\theta : (\mathsf{Y}\times\mathsf{Y})^{\mathbb{N}} \to (\mathsf{Y}\times\mathsf{Y})^{\mathbb{N}}$ the shift operator defined for any $(x_n, y_n)_{n\in\mathbb{N}} \in (\mathsf{Y}\times\mathsf{Y})^{\mathbb{N}}$ by $\theta((x_n,y_n)_{n\in\mathbb{N}}) = (x_{n+1}, y_{n+1})_{n\in\mathbb{N}}$. Define by induction, for any $m\in\mathbb{N}$, the sequence of $(\mathcal{G}_n)_{n\in\mathbb{N}}$-stopping times $(T^{(m)}_{\mathsf{C},\mathtt{n}_0})_{m\in\mathbb{N}}$, with $T^{(0)}_{\mathsf{C},\mathtt{n}_0} = 0$ and for any $m \in \mathbb{N}^\star$

$$\begin{aligned} T^{(m)}_{\mathsf{C},\mathtt{n}_0} &= \inf\left\{k \geq T^{(m-1)}_{\mathsf{C},\mathtt{n}_0} + \mathtt{n}_0 \,:\, (\mathrm{X}_k, \mathrm{Y}_k) \in \mathsf{C}\right\} \\ &= T^{(m-1)}_{\mathsf{C},\mathtt{n}_0} + \mathtt{n}_0 + \tilde{T}_{\mathsf{C}} \circ \theta^{T^{(m-1)}_{\mathsf{C}} + \mathtt{n}_0} \\ &= T^{(1)}_{\mathsf{C},\mathtt{n}_0} + \sum_{i=1}^{(m-1)} T^{(1)}_{\mathsf{C},\mathtt{n}_0} \circ \theta^{T^{(i)}_{\mathsf{C},\mathtt{n}_0}} \;, \\ \tilde{T}_{\mathsf{C}} &= \inf\left\{k \geq 0 \,:\, (\mathrm{X}_k, \mathrm{Y}_k) \in \mathsf{C}\right\} \;. \end{aligned} \tag{68}$$

Note that $(T^{(m)}_{\mathsf{C},\mathtt{n}_0})_{m\in\mathbb{N}^\star}$ are the successive return times to C delayed by $\mathtt{n}_0 - 1$ and $\tilde{T}_{\mathsf{C}}$ is the first hitting time to C. We will use the following lemma which borrows from [22] and [41, Lemma 3.1].

Lemma 28 ([22, Proposition 14])**.** *Let* K *be a Markov coupling kernel for* P *and assume* $\mathbf{H}2$(K)-(i)-(ii)*. Then for any $n, m \in \mathbb{N}$, $x, y \in \mathsf{Y}$,*

$$\mathbb{E}_{(x,y)}\left[\mathbf{d}(\mathrm{X}_n, \mathrm{Y}_n)\right] \leq (1-\varepsilon)^m \mathbf{d}(x,y) + \mathbb{E}_{(x,y)}\left[\mathbf{d}(\mathrm{X}_n, \mathrm{Y}_n)\mathbb{1}_{[n,+\infty]}(T^{(m)}_{\mathsf{C},\mathtt{n}_0})\right] \;.$$

Proof. Using $\mathbf{H}2$(K)-(ii), we have that $(\mathbf{d}(X_n, Y_n))_{n\in\mathbb{N}}$ is a $(\mathcal{G}_n)_{n\in\mathbb{N}}$-supermartingale and therefore using the strong Markov property and $\mathbf{H}2$(K)-(i) we obtain for any $m\in\mathbb{N}$ and $x,y\in\mathsf{Y}$ that

$$\begin{aligned} \mathbb{E}_{(x,y)}\left[\mathbf{d}(\mathrm{X}_{T^{(m+1)}_{\mathsf{C},\mathtt{n}_0}}, \mathrm{Y}_{T^{(m+1)}_{\mathsf{C},\mathtt{n}_0}})\right] &\leq \mathbb{E}_{(x,y)}\left[\mathbb{E}\left[\mathbf{d}(\mathrm{X}_{T^{(m)}_{\mathsf{C},\mathtt{n}_0}+\mathtt{n}_0}, \mathrm{Y}_{T^{(m)}_{\mathsf{C},\mathtt{n}_0}+\mathtt{n}_0})\middle|\mathcal{G}_{T^{(m)}_{\mathsf{C},\mathtt{n}_0}}\right]\right] \\ &\leq (1-\varepsilon)\mathbb{E}_{(x,y)}\left[\mathbf{d}(\mathrm{X}_{T^{(m)}_{\mathsf{C},\mathtt{n}_0}}, \mathrm{Y}_{T^{(m)}_{\mathsf{C},\mathtt{n}_0}})\right] \;. \end{aligned} \tag{69}$$

Therefore by recursion and using (69) we obtain that for any $m\in\mathbb{N}$ and $x,y\in\mathsf{Y}$

$$\mathbb{E}_{(x,y)}\left[\mathbf{d}(\mathrm{X}_{T^{(m)}_{\mathsf{C},\mathtt{n}_0}}, \mathrm{Y}_{T^{(m)}_{\mathsf{C},\mathtt{n}_0}})\right] \leq (1-\varepsilon)^m \mathbf{d}(x,y) \;. \tag{70}$$

For any $n,m\in\mathbb{N}$ we have using (70) and that $(\mathbf{d}(X_n, Y_n))_{n\in\mathbb{N}}$ is a supermartingale,

$$\begin{aligned} &\mathbb{E}_{(x,y)}\left[\mathbf{d}(\mathrm{X}_n, \mathrm{Y}_n)\right] \leq \mathbb{E}_{(x,y)}\left[\mathbf{d}(\mathrm{X}_{n\wedge T^{(m)}_{\mathsf{C},\mathtt{n}_0}}, \mathrm{Y}_{n\wedge T^{(m)}_{\mathsf{C},\mathtt{n}_0}})\right] \\ &\leq \mathbb{E}_{(x,y)}\left[\mathbf{d}(\mathrm{X}_{T^{(m)}_{\mathsf{C},\mathtt{n}_0}}, \mathrm{Y}_{T^{(m)}_{\mathsf{C},\mathtt{n}_0}})\mathbb{1}_{[0,n]}(T^{(m)}_{\mathsf{C},\mathtt{n}_0})\right] + \mathbb{E}_{(x,y)}\left[\mathbf{d}(\mathrm{X}_n, \mathrm{Y}_n)\mathbb{1}_{[n,+\infty]}(T^{(m)}_{\mathsf{C},\mathtt{n}_0})\right] \\ &\leq (1-\varepsilon)^m \mathbf{d}(x,y) + \mathbb{E}_{(x,y)}\left[\mathbf{d}(\mathrm{X}_n, \mathrm{Y}_n)\mathbb{1}_{[n,+\infty]}(T^{(m)}_{\mathsf{C},\mathtt{n}_0})\right] \;. \end{aligned}$$

$\square$

By Lemma 28 and since $\mathbf{d}$ is bounded by 1, we need to obtain a bound on $\mathbb{P}_{(x,y)}(T^{(m)}_{\mathsf{C},\mathtt{n}_0} \geq n)$ for $x, y \in \mathsf{Y}$ and $m, n \in \mathbb{N}^\star$. To this end, we will use the following proposition which gives an upper bound on exponential moment of the hitting times $(T^{(m)}_{\mathsf{C},\mathtt{n}_0})_{m\in\mathbb{N}^\star}$.

Lemma 29. *Let* K *be a Markov coupling kernel for* P *and assume* $\mathbf{H}2(\mathrm{K})$-(iii)*. Then for any* $x, y \in \mathsf{Y}$ *and* $m \in \mathbb{N}^\star$*,*

$$\mathbb{E}_{(x,y)}\left[\lambda_1^{-T^{(1)}_{\mathsf{C},\mathtt{n}_0}}\right] \leq \mathbf{\Xi}(x, y, \mathtt{n}_0) \; ,$$
$$\mathbb{E}_{(x,y)}\left[\lambda_1^{-T^{(m)}_{\mathsf{C},\mathtt{n}_0}+T^{(1)}_{\mathsf{C},\mathtt{n}_0}}\right] \leq M^{m-1}_{\mathsf{C},\mathtt{n}_0} \; ,$$
$$\mathbb{E}_{(x,y)}\left[\lambda_1^{-T^{(m)}_{\mathsf{C},\mathtt{n}_0}}\right] \leq \mathbf{\Xi}(x, y, \mathtt{n}_0) M^{m-1}_{\mathsf{C},\mathtt{n}_0} \; ,$$

where $\mathbf{\Xi}(x, y, \mathtt{n}_0)$ *and* $M_{\mathsf{C},\mathtt{n}_0}$ *are defined in* (67)*.*

Proof. We first show that for any $x, y \in \mathsf{Y}$ we have that $\mathbb{P}_{(x,y)}(\tilde{T}_\mathsf{C}) < +\infty$. Let $x, y \in \mathsf{Y}$. For any $n \in \mathbb{N}$ we have using $\mathbf{H}2(\mathrm{K})$-(iii) and the Markov property

$$\mathbb{E}_{(x,y)}\left[W_1(\mathrm{X}_{n+1}, \mathrm{Y}_{n+1}) | \mathcal{G}_n\right] \leq \lambda_1 W_1(\mathrm{X}_n, \mathrm{Y}_n) + A_1 \mathbb{1}_\mathsf{C}(\mathrm{X}_n, \mathrm{Y}_n) \; .$$

Therefore applying the comparison theorem [19, Theorem 4.3.1] we get that

$$(1-\lambda_1)\mathbb{E}_{(x,y)}\left[\sum_{k=0}^{\tilde{T}_\mathsf{C}-1} W_1(\mathrm{X}_k, \mathrm{Y}_k)\right] + \mathbb{E}_{(x,y)}\left[\mathbb{1}_{[0,+\infty)}(\tilde{T}_\mathsf{C}) W(\mathrm{X}_{\tilde{T}_\mathsf{C}}, \mathrm{Y}_{\tilde{T}_\mathsf{C}})\right] \leq W(x, y) \; .$$

Since for any $\tilde{x}, \tilde{y} \in \mathsf{Y}$, $1 \leq W_1(\tilde{x}, \tilde{y}) < +\infty$ we obtain that $(1-\lambda_1)\mathbb{E}_{(x,y)}[\tilde{T}_\mathsf{C}] \leq W(x, y)$ which implies $\mathbb{P}_{(x,y)}(\tilde{T}_\mathsf{C}) < +\infty$ since $\lambda_1 \in (0, 1)$. We now show the stated result. Let $x, y \in \mathsf{Y}$ and $(\mathrm{S}_n)_{n\in\mathbb{N}}$ be defined for any $n \in \mathbb{N}$ by $\mathrm{S}_n = \lambda_1^{-n} W_1(\mathrm{X}_n, \mathrm{Y}_n)$. For any $n \in \mathbb{N}$ we have using $\mathbf{H}2(\mathrm{K})$-(iii) and the Markov property

$$\begin{aligned} \mathbb{E}\left[\mathrm{S}_{n+1} | \mathcal{G}_n\right] &\leq \lambda_1^{-n} W_1(\mathrm{X}_n, \mathrm{Y}_n) + A_1 \lambda_1^{-(n+1)} \mathbb{1}_\mathsf{C}(\mathrm{X}_n, \mathrm{Y}_n) \\ &\leq \mathrm{S}_n + A_1 \lambda_1^{-(n+1)} \mathbb{1}_\mathsf{C}(\mathrm{X}_n, \mathrm{Y}_n) \; . \end{aligned} \tag{71}$$

Using the Markov property, the definition of $T^{(1)}_{\mathsf{C},\mathtt{n}_0}$ given in (68), the comparison theorem [19, Theorem 4.3.1], (71) and $\mathbf{H}2$(K)-(iii) we obtain that

$$
\begin{aligned}
&\mathbb{E}_{(x,y)}\left[\mathrm{S}_{T^{(1)}_{\mathsf{C},\mathtt{n}_0}}\right] = \mathbb{E}_{(x,y)}\left[\mathbb{E}_{(x,y)}\left[\mathrm{S}_{T^{(1)}_{\mathsf{C},\mathtt{n}_0}}\middle|\mathcal{G}_{\mathtt{n}_0}\right]\right] = \mathbb{E}_{(x,y)}\left[\mathbb{E}_{(x,y)}\left[\mathrm{S}_{\mathtt{n}_0+\tilde{T}_{\mathsf{C}}\circ\theta^{\mathtt{n}_0}}\middle|\mathcal{G}_{\mathtt{n}_0}\right]\right] \\
&= \mathbb{E}_{(x,y)}\left[\lambda_1^{-\mathtt{n}_0}\ \mathbb{E}_{(x,y)}\left[W_1(X_{\mathtt{n}_0+T^{(1)}_{\mathsf{C},\mathtt{n}_0}}, Y_{\mathtt{n}_0+T^{(1)}_{\mathsf{C},\mathtt{n}_0}})\lambda_1^{-T^{(1)}_{\mathsf{C},\mathtt{n}_0}}\middle|\mathcal{G}_{\mathtt{n}_0}\right]\right] \\
&\leq \mathbb{E}_{(x,y)}\left[\lambda_1^{-\mathtt{n}_0}\ \mathbb{E}_{(X_{\mathtt{n}_0},Y_{\mathtt{n}_0})}\left[\mathrm{S}_{\tilde{T}_{\mathsf{C}}}\right]\right] \\
&\leq \mathbb{E}_{(x,y)}\left[\lambda_1^{-\mathtt{n}_0}\ \mathbb{E}_{(X_{\mathtt{n}_0},Y_{\mathtt{n}_0})}\left[\mathrm{S}_{\tilde{T}_{\mathsf{C}}}\mathbb{1}_{[0,+\infty)}(\tilde{T}_{\mathsf{C}})\right]\right] \\
&\leq \mathbb{E}_{(x,y)}\left[\lambda_1^{-\mathtt{n}_0}\ \mathbb{E}_{(X_{\mathtt{n}_0},Y_{\mathtt{n}_0})}\left[\mathrm{S}_0 + A_1\sum_{k=0}^{\tilde{T}_{\mathsf{C}}-1}\lambda_1^{-(k+1)}\mathbb{1}_{\mathsf{C}}(X_k,Y_k)\right]\right] \\
&\leq \mathbb{E}_{(x,y)}\left[\lambda_1^{-\mathtt{n}_0}W_1(X_{\mathtt{n}_0},Y_{\mathtt{n}_0})\right] \leq W_1(x,y) + A_1\lambda_1^{-\mathtt{n}_0}\mathtt{n}_0\ . \qquad (72)
\end{aligned}
$$

Combining (72) and the fact that for any $x, y \in \mathsf{Y}$, $W_1(x,y) \geq 1$, we obtain that

$$
\mathbb{E}_{(x,y)}\left[\lambda_1^{-T^{(1)}_{\mathsf{C},\mathtt{n}_0}}\right] \leq W_1(x,y) + A_1\lambda_1^{-\mathtt{n}_0}\mathtt{n}_0\ . \qquad (73)
$$

We conclude by a straightforward recursion and using (73), the definition of $T^{(m)}_{\mathsf{C},\mathtt{n}_0}$ (68) for $m \geq 1$, the strong Markov property and the fact that for any $m \in \mathbb{N}^\star$, $(X_{T^{(m)}_{\mathsf{C},\mathtt{n}_0}}, Y_{T^{(m)}_{\mathsf{C},\mathtt{n}_0}}) \in \mathsf{C}$. □

Proof of Theorem 26. Let $x, y \in \mathsf{Y}$ and $n \in \mathbb{N}$. By Lemma 28, Lemma 29, $\mathbf{H}1$, the fact that $M_{\mathsf{C},\mathtt{n}_0} \geq 1$ and the Markov inequality, we have for any $m \in \mathbb{N}$,

$$
\begin{aligned}
\mathbb{E}_{(x,y)}\left[\mathbf{d}(\mathrm{X}_n, \mathrm{Y}_n)\right] &\leq (1-\varepsilon)^m\mathbf{d}(x,y) + \mathbb{P}_{(x,y)}\left[T^{(m)}_{\mathsf{C},\mathtt{n}_0} \geq n\right] \\
&\leq (1-\varepsilon)^m\mathbf{d}(x,y) + \lambda_1^n\,\mathbb{E}_{(x,y)}\left[\lambda_1^{-T^{(m)}_{\mathsf{C},\mathtt{n}_0}}\right] \\
&\leq (1-\varepsilon)^m\mathbf{d}(x,y) + \lambda_1^n M^m_{\mathsf{C},\mathtt{n}_0}\boldsymbol{\Xi}(x,y,\mathtt{n}_0)\ ,
\end{aligned}
$$

where $\boldsymbol{\Xi}(x,y,\mathtt{n}_0)$ is given in Theorem 26. Combining this result and Lemma 29, we can conclude that $\mathbb{E}_{(x,y)}\left[\mathbf{d}(\mathrm{X}_n, \mathrm{Y}_n)\right] \leq \rho^n(M_{\mathsf{C},\mathtt{n}_0}\boldsymbol{\Xi}(x,y,\mathtt{n}_0) + \mathbf{d}(x,y))$ setting

$$
m = \lceil n\log(\lambda_1)/\{\log(1-\varepsilon) - \log(M_{\mathsf{C},\mathtt{n}_0})\}\rceil\ .
$$

To show that $\mathbb{E}_{(x,y)}[\mathbf{d}(\mathrm{X}_n, \mathrm{Y}_n)] \leq \rho^{n/2}(1+\mathbf{d}(x,y)) + \lambda_1^{n/2}\boldsymbol{\Xi}(x,y,\mathtt{n}_0)$, first note that Lemma 28 and **H**1 imply that for any $m \in \mathbb{N}$,

$$\begin{aligned}
&\mathbb{E}_{(x,y)}[\mathbf{d}(\mathrm{X}_n, \mathrm{Y}_n)] \\
&\leq (1-\varepsilon)^m \mathbf{d}(x,y) + \mathbb{P}_{(x,y)}\left[T^{(m)}_{\mathsf{C},\mathtt{n}_0} - T^{(1)}_{\mathsf{C},\mathtt{n}_0} \geq n/2\right] + \mathbb{P}_{(x,y)}\left[T^{(1)}_{\mathsf{C},\mathtt{n}_0} \geq n/2\right] \\
&\leq (1-\varepsilon)^m \mathbf{d}(x,y) + \lambda_1^{n/2}\mathbb{E}_{(x,y)}\left[\lambda_1^{-T^{(m)}_{\mathsf{C},\mathtt{n}_0}+T^{(1)}_{\mathsf{C},\mathtt{n}_0}}\right] + \lambda_1^{n/2}\mathbb{E}_{(x,y)}\left[\lambda_1^{-T^{(1)}_{\mathsf{C},\mathtt{n}_0}}\right] ,
\end{aligned}$$

where we have used the Markov inequality in the last line. Combining this result and Lemma 29, we can conclude that $\mathbb{E}_{(x,y)}[\mathbf{d}(\mathrm{X}_n, \mathrm{Y}_n)] \leq \rho^{n/2}(1+\mathbf{d}(x,y)) + \lambda_1^{n/2}\,\boldsymbol{\Xi}(x,y,\mathtt{n}_0)$ setting

$$m = \lceil n\log(\lambda_1)/\{2\log(1-\varepsilon) - 2\log(M_{\mathsf{C},\mathtt{n}_0})\}\rceil \ .$$

□

Proof of Theorem 27. Let $x, y \in \mathsf{Y}$ and $n \in \mathbb{N}$. Using **H**3(K), we obtain by recursion

$$\mathbb{E}_{(x,y)}[W_2(X_n, Y_n)] \leq \lambda_2^n W_2(x,y) + A_2 \sum_{k=0}^{n-1} \lambda_2^{n-1-k}\ \mathbb{E}_{(x,y)}[\mathbf{d}(X_k, Y_k)] \ . \tag{74}$$

Applying Theorem 26 we obtain

$$\begin{aligned}
&\sum_{k=0}^{n-1} \lambda_2^{n-1-k}\ \mathbb{E}_{(x,y)}[\mathbf{d}(X_k, Y_k)] \\
&\leq \sum_{k=0}^{n-1} \lambda_2^{n-1-k} \min\left[\rho^k(M_{\mathsf{C},\mathtt{n}_0}\boldsymbol{\Xi}(x,y,\mathtt{n}_0) + \mathbf{d}(x,y)), \rho^{k/2}(1+\mathbf{d}(x,y)) + \lambda^{k/2}\boldsymbol{\Xi}(x,y,\mathtt{n}_0)\right] \\
&\leq \min\left[n\tilde{\rho}^{n-1}(\mathbf{d}(x,y) + \boldsymbol{\Xi}(x,y,\mathtt{n}_0)), n\tilde{\rho}^{n/2-1}(1+\mathbf{d}(x,y)) + n\tilde{\lambda}^{n/2-1}\boldsymbol{\Xi}(x,y,\mathtt{n}_0)\right] \ .
\end{aligned}$$

We conclude plugging this result in (74) and using that for any $n \in \mathbb{N}$ and $t \in (0,1)$, $nt^{n/2} \leq 4\log^{-1}(1/t)t^{n/4}$. □

B Minorization conditions for functional autoregressive models

In this section, we extend and complete the results of [25, Section 6] on functional autoregressive models. Let $\mathsf{X} \in \mathcal{B}(\mathbb{R}^d)$ equipped with its trace σ-field $\mathcal{X} = \{\mathsf{A} \cap \mathsf{X} :$

$\mathsf{A} \in \mathcal{B}(\mathbb{R}^d)\}$. In fact, we consider a slightly more general class of models than [25] which is associated with non-homogeneous Markov chains $(X_k^{(\mathrm{a})})_{k\in\mathbb{N}}$ with state space $(\mathsf{X}, \mathcal{X})$ defined for $k \geq 0$ by

$$X_{k+1}^{(\mathrm{a})} = \Pi\left(\mathcal{T}_{k+1}(X_k^{(\mathrm{a})}) + \sigma_{k+1} Z_{k+1}\right) ,$$

where Π is a measurable function from $\mathbb{R}^d$ to X, $(\mathcal{T}_k)_{k\geq 1}$ is a sequence of measurable functions from X to $\mathbb{R}^d$, $(\sigma_k)_{k\geq 1}$ is a sequence of positive real numbers and $(Z_k)_{k\geq 1}$ is a sequence of i.i.d. d dimensional standard Gaussian random variables. We assume that Π satisfies **A**1. We also assume some Lipschitz regularity on the operator $\mathcal{T}_k$ for any $k \in \mathbb{N}^\star$

AR 1 (A). *For all $k \geq 1$ there exists $\varpi_k \in \mathbb{R}$ such that for all $(x, y) \in \mathsf{A}$,*

$$\|\mathcal{T}_k(x) - \mathcal{T}_k(y)\|^2 \leq (1 + \varpi_k) \|x - y\|^2 .$$

The sequence $\{X_k^{(\mathrm{a})},\ k \in \mathbb{N}\}$ is an inhomogeneous Markov chain associated with the family of Markov kernels $(P_k^{(\mathrm{a})})_{k\geq 1}$ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ given for all $x \in \mathbb{R}^d$ and $\mathsf{A} \in \mathbb{R}^d$ by

$$P_k^{(\mathrm{a})}(x, \mathsf{A}) = \frac{1}{(2\pi\sigma_k^2)^{d/2}} \int_{\Pi^{-1}(\mathsf{A})} \exp\left(-\|y - \mathcal{T}_k(x)\|^2 / (2\sigma_k^2)\right) \mathrm{d}y .$$

We denote for all $n \geq 1$ by $Q_n^{(\mathrm{a})}$ the marginal distribution of $X_n^{(\mathrm{a})}$ given by $Q_n^{(\mathrm{a})} = P_1^{(\mathrm{a})} \cdots P_n^{(\mathrm{a})}$. To obtain an upper bound of $\|\delta_x Q_n^{(\mathrm{a})} - \delta_y Q_n^{(\mathrm{a})}\|_{\mathrm{TV}}$ for any $x, y \in \mathbb{R}^d$, $n \in \mathbb{N}^\star$, we introduce a Markov coupling $(X_k^{(\mathrm{a})}, Y_k^{(\mathrm{a})})_{k\in\mathbb{N}}$ such that for any $n \in \mathbb{N}^\star$, the distribution of $X_n^{(\mathrm{a})}$ and $Y_n^{(\mathrm{a})}$ are $\delta_x Q_n^{(\mathrm{a})}$ and $\delta_x Q_n^{(\mathrm{a})}$ respectively, exactly as we have introduced in the homogeneous setting the Markov coupling with kernel K_γ defined by (5) for R_γ defined in (2). For completeness and readability, we recall the construction of $(X_k^{(\mathrm{a})}, Y_k^{(\mathrm{a})})_{k\in\mathbb{N}}$. For all $k \in \mathbb{N}^\star$ and $x, y, z \in \mathbb{R}^d$, define

$$\mathrm{e}_k(x, y) = \begin{cases} \mathrm{E}_k(x, y)/\|\mathrm{E}_k(x, y)\| & \text{if } \mathrm{E}_k(x, y) \neq 0 \\ 0 & \text{otherwise} \end{cases} , \qquad \mathrm{E}_k(x, y) = \mathcal{T}_k(y) - \mathcal{T}_k(x) , \tag{75}$$

$$\begin{aligned} \mathcal{S}_k(x, y, z) &= \mathcal{T}_k(y) + (\mathrm{Id} - 2\mathrm{e}_k(x, y)\mathrm{e}_k(x, y)^\top) z , \\ p_k(x, y, z) &= 1 \wedge \frac{\boldsymbol{\varphi}_{\sigma_{k+1}^2}(\|\mathrm{E}_k(x, y)\| - \langle \mathrm{e}_k(x, y), z\rangle)}{\boldsymbol{\varphi}_{\sigma_{k+1}^2}(\langle \mathrm{e}_k(x, y), z\rangle)} , \end{aligned} \tag{76}$$

where $\boldsymbol{\varphi}_{\sigma_k^2}$ is the one-dimensional zero mean Gaussian distribution function with variance σ_k^2. Let $(U_k)_{k\in\mathbb{N}^\star}$ be a sequence of i.i.d. uniform random variables on $[0, 1]$

and define the Markov chain $(X_k^{(\mathrm{a})}, Y_k^{(\mathrm{a})})_{k\in\mathbb{N}}$ starting from $(X_0^{(\mathrm{a})}, Y_0^{(\mathrm{a})}) \in \mathsf{X}^2$ by the recursion: for any $k \geq 0$

$$\begin{aligned}
&\tilde{X}_{k+1}^{(\mathrm{a})} = \mathcal{T}_{k+1}(X_k^{(\mathrm{a})}) + \sigma_{k+1} Z_{k+1} \;, \\
&\tilde{Y}_{k+1}^{(\mathrm{a})} = \begin{cases} \tilde{X}_{k+1}^{(\mathrm{a})} & \text{if } \mathcal{T}_{k+1}(X_k^{(\mathrm{a})}) = \mathcal{T}_{k+1}(Y_k^{(\mathrm{a})}) \;; \\ W_{k+1}^{(\mathrm{a})} \tilde{X}_{k+1}^{(\mathrm{a})} + (1 - W_{k+1}^{(\mathrm{a})}) \mathcal{S}_{k+1}(X_k^{(\mathrm{a})}, Y_k^{(\mathrm{a})}, \sigma_{k+1} Z_{k+1}) & \text{otherwise} \;, \end{cases}
\end{aligned} \tag{77}$$

where $W_{k+1}^{(\mathrm{a})} = \mathbb{1}_{(-\infty,0]}(U_{k+1} - p_{k+1}(X_k^{(\mathrm{a})}, Y_k^{(\mathrm{a})}, \sigma_{k+1} Z_{k+1}))$ and finally set

$$(X_{k+1}^{(\mathrm{a})}, Y_{k+1}^{(\mathrm{a})}) = (\Pi(\tilde{X}_{k+1}^{(\mathrm{a})}), \Pi(\tilde{Y}_{k+1}^{(\mathrm{a})})) \;. \tag{78}$$

For any $k \in \mathbb{N}^\star$, marginally, the distribution of $X_{k+1}^{(\mathrm{a})}$ given $X_k^{(\mathrm{a})}$ is $P_{k+1}^{(\mathrm{a})}(X_k^{(\mathrm{a})}, \cdot)$, and it is well-know (see e.g. [8, Section 3.3]) that $\tilde{Y}_{k+1}^{(\mathrm{a})}$ and $\mathcal{T}_\gamma(Y_k^{(\mathrm{a})}) + \sigma_{k+1} Z_{k+1}$ have the same distribution given Y_k, and therefore the distribution of Y_{k+1} given Y_k is $P_{k+1}^{(\mathrm{a})}(Y_k, \cdot)$. As a result for any $(x, y) \in \mathsf{X}^2$ and $n \in \mathbb{N}^\star$, $(X_n^{(\mathrm{a})}, Y_n^{(\mathrm{a})})$ with $(X_0^{(\mathrm{a})}, Y_0^{(\mathrm{a})}) = (x, y)$ is coupling between $\delta_x Q_n^{(\mathrm{a})}$ and $\delta_y Q_n^{(\mathrm{a})}$. Therefore, we obtain that $\|\delta_x Q_n^{(\mathrm{a})} - \delta_y Q_n^{(\mathrm{a})}\|_{\mathrm{TV}} \leq \mathbb{P}(X_n^{(\mathrm{a})} \neq Y_n^{(\mathrm{a})})$. Therefore to get an upper bound on $\|\delta_x Q_n^{(\mathrm{a})} - \delta_y Q_n^{(\mathrm{a})}\|_{\mathrm{TV}}$, it is sufficient to obtain a bound on $\mathbb{P}(X_n^{(\mathrm{a})} \neq Y_n^{(\mathrm{a})})$ which is a simple consequence of the following more general result.

Theorem 30. *Let $\mathsf{A} \in \mathcal{B}(\mathbb{R}^{2d})$ and assume* **A***1 and* **AR***1(A). Let $(X_k^{(\mathrm{a})}, Y_k^{(\mathrm{a})})_{k\in\mathbb{N}}$ be defined by (78), with $(X_0^{(\mathrm{a})}, Y_0^{(\mathrm{a})}) = (x, y) \in \mathsf{A}$. Then for any $n \in \mathbb{N}^\star$,*

$$\mathbb{P}\left[X_n^{(\mathrm{a})} \neq Y_n^{(\mathrm{a})} \text{ and for any } k \in \{1, \ldots, n-1\}, (X_k^{(\mathrm{a})}, Y_k^{(\mathrm{a})}) \in \mathsf{A}^2\right] \\ \leq \mathbb{1}_{\Delta_{\mathsf{X}}^{\mathrm{c}}}(x, y) \left\{1 - 2\boldsymbol{\Phi}\left(-\frac{\|x - y\|}{2(\Xi_n^{(\mathrm{a})})^{1/2}}\right)\right\} \;,$$

where $\boldsymbol{\Phi}$ is the cumulative distribution function of the standard normal distribution on $\mathbb{R}$ and $(\Xi_i^{(\mathrm{a})})_{i\geq 1}$ is defined for all $k \geq 1$ by $\Xi_k^{(\mathrm{a})} = \sum_{i=1}^k \{\sigma_i^2 / \prod_{j=1}^i (1 + \varpi_j)\}$.

Proof. Let $(\mathcal{F}_k^{(\mathrm{a})})_{k\in\mathbb{N}}$ be the filtration associated to $(X_k^{(\mathrm{a})}, Y_k^{(\mathrm{a})})_{k\in\mathbb{N}}$. Denote for any $k \in \mathbb{N}$,

$$\mathscr{A}_k = \bigcap_{i=0}^k \{(X_i^{(\mathrm{a})}, Y_i^{(\mathrm{a})}) \in \mathsf{A}\} \;, \qquad \mathscr{A}_{-1} = \mathscr{A}_0 \;,$$

and for all $k_1, k_2 \in \mathbb{N}^*$, $k_1 \leq k_2$, $\Xi_{k_1,k_2}^{(\mathrm{a})} = \sum_{i=k_1}^{k_2} \{\sigma_i^2 / \prod_{j=k_1}^i (1 + \varpi_j)\}$. Let $n \geq 1$ and

$(x, y) \in \mathsf{A}^2$. We show by backward induction that for all $k \in \{0, \cdots, n-1\}$,

$$\mathbb{P}(\{X_n^{(\mathrm{a})} \neq Y_n^{(\mathrm{a})}\} \cap \mathscr{A}_{n-1}) \leq \mathbb{E}\left[\mathbb{1}_{\Delta_{\mathsf{X}}^{\mathrm{c}}}(X_k^{(\mathrm{a})}, Y_k^{(\mathrm{a})})\mathbb{1}_{\mathscr{A}_{k-1}}\left[1 - 2\boldsymbol{\Phi}\left\{-\frac{\left\|X_k^{(\mathrm{a})} - Y_k^{(\mathrm{a})}\right\|}{2\left(\Xi_{k+1,n}^{(\mathrm{a})}\right)^{1/2}}\right\}\right]\right] . \tag{79}$$

Note that the inequality for $k = 0$ will conclude the proof. Using by (77) that $\tilde{X}_n^{(\mathrm{a})} = \tilde{Y}_n^{(\mathrm{a})}$ if $X_{n-1}^{(\mathrm{a})} = Y_{n-1}^{(\mathrm{a})}$ or $W_n = \mathbb{1}_{(-\infty,0]}(U_n - p_n(X_{n-1}^{(\mathrm{a})}, Y_{n-1}^{(\mathrm{a})}, \sigma_n Z_n)) = 1$, where p_n is defined by (76), and (U_n, Z_n) are independent random variables independent of $\mathcal{F}_{n-1}^{(\mathrm{a})}$, we obtain on $\{X_{n-1}^{(\mathrm{a})} \neq Y_{n-1}^{(\mathrm{a})}\}$

$$\begin{aligned}\mathbb{E}\left[\mathbb{1}_{\Delta_{\mathsf{X}}}(\tilde{X}_n^{(\mathrm{a})}, \tilde{Y}_n^{(\mathrm{a})})\middle|\mathcal{F}_{n-1}^{(\mathrm{a})}\right] &= \mathbb{E}\left[p_n(X_{n-1}^{(\mathrm{a})}, Y_{n-1}^{(\mathrm{a})}, \sigma_n Z_n)\middle|\mathcal{F}_{n-1}^{(\mathrm{a})}\right] \\ &= 2\Phi\left\{-\left\|(2\sigma_n)^{-1}\mathrm{E}_n(X_{n-1}^{(\mathrm{a})}, Y_{n-1}^{(\mathrm{a})})\right\|\right\} .\end{aligned}$$

Since $\{X_n^{(\mathrm{a})} \neq Y_n^{(\mathrm{a})}\} \subset \{\tilde{X}_n^{(\mathrm{a})} \neq \tilde{Y}_n^{(\mathrm{a})}\} \subset \{X_{n-1}^{(\mathrm{a})} \neq Y_{n-1}^{(\mathrm{a})}\}$ by (78) and (77), we get

$$\begin{aligned}\mathbb{P}\left[\{X_n^{(\mathrm{a})} \neq Y_n^{(\mathrm{a})}\} \cap \mathscr{A}_{n-1}\right] &\leq \mathbb{E}\left[\mathbb{1}_{\Delta_{\mathsf{X}}^{\mathrm{c}}}(X_{n-1}^{(\mathrm{a})}, Y_{n-1}^{(\mathrm{a})})\mathbb{1}_{\mathscr{A}_{n-1}}\mathbb{E}\left[\mathbb{1}_{\Delta_{\mathsf{X}}^{\mathrm{c}}}(\tilde{X}_n^{(\mathrm{a})}, \tilde{Y}_n^{(\mathrm{a})})\middle|\mathcal{F}_{n-1}^{(\mathrm{a})}\right]\right] \\ &= \mathbb{E}\left[\mathbb{1}_{\Delta_{\mathsf{X}}^{\mathrm{c}}}(X_{n-1}^{(\mathrm{a})}, Y_{n-1}^{(\mathrm{a})})\mathbb{1}_{\mathscr{A}_{n-1}}\left[1 - 2\boldsymbol{\Phi}\left\{-\left\|(2\sigma_n)^{-1}\mathrm{E}_n(X_{n-1}^{(\mathrm{a})}, Y_{n-1}^{(\mathrm{a})})\right\|\right\}\right]\right] ,\end{aligned}$$

Using that $(X_{n-1}^{(\mathrm{a})}, Y_{n-1}^{(\mathrm{a})}) \in \mathsf{A}^2$ on $\mathscr{A}_{n-1}$, **AR**1(A) and (75), we obtain that

$$\|\mathrm{E}_n(X_{n-1}^{(\mathrm{a})}, Y_{n-1}^{(\mathrm{a})})\|^2 \leq (1 + \varpi_n)\|X_{n-1}^{(\mathrm{a})} - Y_{n-1}^{(\mathrm{a})}\|^2 ,$$

showing (79) holds for $k = n-1$ since $\mathscr{A}_{n-2} \subset \mathscr{A}_{n-1}$. Assume that (79) holds for $k \in \{1, \ldots, n-1\}$. On $\{\tilde{X}_k^{(\mathrm{a})} \neq \tilde{Y}_k^{(\mathrm{a})}\}$, we have

$$\left\|\tilde{X}_k^{(\mathrm{a})} - \tilde{Y}_k^{(\mathrm{a})}\right\| = \left|-\left\|\mathrm{E}_k(X_{k-1}^{(\mathrm{a})}, Y_{k-1}^{(\mathrm{a})})\right\| + 2\sigma_k \mathrm{e}_k(X_{k-1}^{(\mathrm{a})}, Y_{k-1}^{(\mathrm{a})})^{\mathrm{T}} Z_k\right| ,$$

which implies by (78) and since Π is non expansive by **A**1

$$\begin{aligned}&\mathbb{1}_{\Delta_{\mathsf{X}}^{\mathrm{c}}}(X_k^{(\mathrm{a})}, Y_k^{(\mathrm{a})})\left[1 - 2\boldsymbol{\Phi}\left\{-\frac{\left\|X_k^{(\mathrm{a})} - Y_k^{(\mathrm{a})}\right\|}{2(\Xi_{k+1,n}^{(\mathrm{a})})^{1/2}}\right\}\right] \\ &\qquad \leq \mathbb{1}_{\Delta_{\mathsf{X}}^{\mathrm{c}}}(X_k^{(\mathrm{a})}, Y_k^{(\mathrm{a})})\left[1 - 2\boldsymbol{\Phi}\left\{-\frac{\left\|\tilde{X}_k^{(\mathrm{a})} - \tilde{Y}_k^{(\mathrm{a})}\right\|}{2(\Xi_{k+1,n}^{(\mathrm{a})})^{1/2}}\right\}\right] \\ &\qquad \leq \mathbb{1}_{\Delta_{\mathsf{X}}^{\mathrm{c}}}(X_k^{(\mathrm{a})}, Y_k^{(\mathrm{a})})\left[1 - 2\boldsymbol{\Phi}\left\{-\frac{\left|2\sigma_k \mathrm{e}_k(X_{k-1}^{(\mathrm{a})}, Y_{k-1}^{(\mathrm{a})})^{\mathrm{T}} Z_k - \left\|\mathrm{E}_k(X_{k-1}^{(\mathrm{a})}, Y_{k-1}^{(\mathrm{a})})\right\|\right|}{2(\Xi_{k+1,n}^{(\mathrm{a})})^{1/2}}\right\}\right] .\end{aligned}$$

Since Z_k is independent of $\mathcal{F}_k^{(\mathrm{a})}$, $\sigma_k \mathrm{e}_k(X_{k-1}^{(\mathrm{a})}, X_{k-1}^{(\mathrm{a})})^{\mathrm{T}} Z_k$ is a real Gaussian random variable with zero mean and variance σ_k^2. Therefore by [25, Lemma 20] and since $\mathscr{A}_{k-1}$ is $\mathcal{F}_{k-1}^{(\mathrm{a})}$-measurable, we get

$$\begin{aligned}
&\mathbb{E}\left[\mathbb{1}_{\Delta_{\mathsf{X}}^{\mathrm{c}}}(X_k^{(\mathrm{a})}, Y_k^{(\mathrm{a})})\mathbb{1}_{\mathscr{A}_{k-1}}\left[1 - 2\boldsymbol{\Phi}\left\{-\frac{\left\|X_k^{(\mathrm{a})} - Y_k^{(\mathrm{a})}\right\|}{2(\Xi_{k+1,n}^{(\mathrm{a})})^{1/2}}\right\}\right]\middle|\mathcal{F}_{k-1}^{(\mathrm{a})}\right] \\
&\qquad\qquad \leq \mathbb{1}_{\mathscr{A}_{k-1}}\mathbb{1}_{\Delta_{\mathsf{X}}^{\mathrm{c}}}(X_{k-1}^{(\mathrm{a})}, Y_{k-1}^{(\mathrm{a})})\left[1 - 2\boldsymbol{\Phi}\left\{-\frac{\left\|\mathrm{E}_k(X_{k-1}^{(\mathrm{a})}, Y_{k-1}^{(\mathrm{a})})\right\|}{2\left(\sigma_k + \Xi_{k+1,n}^{(\mathrm{a})}\right)^{1/2}}\right\}\right] .
\end{aligned}$$

Using by **A**2(A) that $\|\mathrm{E}_k(X_{k-1}^{(\mathrm{a})}, Y_{k-1}^{(\mathrm{a})})\|^2 \leq (1 + \varpi_{k-1})\left\|X_{k-1}^{(\mathrm{a})} - Y_{k-1}^{(\mathrm{a})}\right\|^2$ on $\mathscr{A}_{k-1}$ and $\mathscr{A}_{k-2} \subset \mathscr{A}_{k-1}$ concludes the induction of (79). $\square$

C Quantitative convergence results based on [19, 20]

We start by recalling the following lemma from [19] which is inspired from the results of [20].

Lemma 31 ([19, Lemma 19.4.2])**.** *Let $(\mathsf{Y}, \mathcal{Y})$ be a measurable space and R be a Markov kernel over $(\mathsf{Y}, \mathcal{Y})$. Let Q be a Markov coupling kernel for R. Assume there exist $\mathsf{C} \in \mathcal{Y}^{\otimes 2}$, $M \geq 0$, a measurable function $W : \mathsf{Y} \times \mathsf{Y} \to [1, +\infty)$, $\lambda \in (0,1)$ and $c \geq 0$ such that for any $x, y \in \mathsf{Y}$,*

$$\mathrm{Q}W(x,y) \leq \lambda W(x,y)\mathbb{1}_{\mathsf{C}^{\mathrm{c}}}(x,y) + c\mathbb{1}_{\mathsf{C}}(x,y) .$$

In addition, assume that there exists $\varepsilon > 0$ such that for any $(x,y) \in \mathsf{C}$,

$$\mathrm{Q}((x,y), \Delta_{\mathsf{Y}}^{\mathrm{c}}) \leq 1 - \varepsilon ,$$

where $\Delta_{\mathsf{Y}} = \{(y,y) : y \in \mathsf{Y}\}$. Then there exist $\rho \in (0,1)$ and $C \geq 0$ such that for any $x, y \in \mathcal{Y}$ and $n \in \mathbb{N}^\star$

$$\int_{\mathsf{Y}\times\mathsf{Y}} \mathbb{1}_{\Delta_{\mathsf{Y}}}(\tilde{x}, \tilde{y}) W(\tilde{x}, \tilde{y}) \mathrm{Q}^n((x,y), \mathrm{d}(\tilde{x}, \tilde{y})) \leq C\rho^n W(x,y) ,$$

where

$$\begin{aligned}
C &= 2(1 + c/\{(1-\varepsilon)(1-\lambda)\}) , \\
\log(\rho) &= \{\log(1-\varepsilon)\log(\lambda)\}/\{\log(1-\varepsilon) + \log(\lambda) - \log(c)\} .
\end{aligned}$$

Theorem 32. *Let $(\mathsf{Y}, \mathcal{Y})$ be a measurable space and R be a Markov kernel over $(\mathsf{Y}, \mathcal{Y})$. Let Q be a Markov coupling kernel of R. Assume that there exist $\lambda \in (0,1)$, $A \geq 0$ and a measurable function $W : \mathsf{Y} \times \mathsf{Y} \to [1, +\infty)$, such that Q satisfies $\mathbf{D}_{\mathrm{d}}(W, \lambda, A, \mathsf{Y})$. In addition, assume that there exist $\ell \in \mathbb{N}^\star$, $\varepsilon > 0$ and $M \geq 1$ such that for any $(x,y) \in \mathsf{C}_M = \{(x,y) \in \mathsf{Y} \times \mathsf{Y},\ W(x,y) \leq M\}$,*

$$\mathrm{Q}^\ell((x,y), \Delta_{\mathsf{Y}}^{\mathrm{c}}) \leq 1 - \varepsilon \ , \tag{80}$$

with $\Delta_{\mathsf{Y}} = \{(x,y) \in \mathsf{Y}^2 : x = y\}$ and $M \geq 2A/(1-\lambda)$. Then, there exist $\rho \in (0,1)$ and $C \geq 0$ such that for any $n \in \mathbb{N}$ and $x, y \in \mathsf{Y}$

$$\mathbf{W}_{\mathbf{c}}(\delta_x \mathrm{R}^n, \delta_y \mathrm{R}^n) \leq C \rho^{\lfloor n/\ell \rfloor} W(x,y) \ ,$$

with

$$\begin{aligned} C &= 2(1 + A_\ell)(1 + c_\ell / \{(1-\varepsilon)(1-\lambda_\ell)\}) \ , \\ \lambda_\ell &= (\lambda^\ell + 1)/2 \ , \quad c_\ell = \lambda^\ell M + A_\ell \ , \quad A_\ell = A(1-\lambda^\ell)/(1-\lambda) \ , \\ \log(\rho_\ell) &= \{\log(1-\varepsilon)\log(\lambda_\ell)\} / \{\log(1-\varepsilon) + \log(\lambda_\ell) - \log(c_\ell)\} \ . \end{aligned} \tag{81}$$

Proof. We first show that for any $(x,y) \in \mathsf{C}_M$,

$$\mathrm{Q}^\ell(x,y) \leq \lambda_\ell W(x,y) \mathbb{1}_{\mathsf{C}_M^{\mathrm{c}}}(x,y) + c_\ell \mathbb{1}_{\mathsf{C}_M}(x,y) \ , \tag{82}$$

in order to apply Lemma 31 to R^ℓ with the Markov coupling kernel Q^ℓ. By a straightforward induction, for any $x, y \in \mathsf{Y}$ we have

$$\mathrm{Q}^\ell W(x,y) \leq \lambda^\ell W(x,y) + A(1-\lambda^\ell)/(1-\lambda) \ . \tag{83}$$

We distinguish two cases. If $(x,y) \notin \mathsf{C}_M$, using that $A/M \geq (1-\lambda)/2$ we have that

$$\mathrm{Q}^\ell W(x,y) \leq \lambda^\ell W(x,y) + A(1-\lambda^\ell) W(x,y)/(M(1-\lambda)) \leq 2^{-1}(\lambda^\ell + 1) W(x,y) \ .$$

If $(x,y) \in \mathsf{C}_M$, we have

$$\mathrm{Q}^\ell W(x,y) \leq \lambda^\ell M + A(1-\lambda^\ell)/(1-\lambda) \ .$$

Therefore (82) holds. As a result and since by assumption we have (80), we can apply Lemma 31 to R^ℓ. Then, we obtain that for any $i \in \mathbb{N}$ and $x, y \in \mathsf{Y}$

$$\int_{\mathsf{Y} \times \mathsf{Y}} \mathbb{1}_{\Delta_{\mathsf{Y}}}(\tilde{x}, \tilde{y}) W(\tilde{x}, \tilde{y}) \mathrm{Q}^{\ell i}((x,y), \mathrm{d}(\tilde{x}, \tilde{y})) \leq C_\ell \rho_\ell^{\ell i} W(x,y) \ ,$$

with ρ_ℓ defined by (81) and $\tilde{C}_\ell = 2\left\{1 + c_\ell\left[(1-\lambda_\ell)(1-\varepsilon)\right]^{-1}\right\}$. In addition, using (83), for any $k \in \{0,\ldots,\ell-1\}$ and $x, y \in \mathsf{Y}$, $\mathrm{Q}^k W(x,y) \le (1+A_\ell)W(x,y)$. Therefore, for any $n \in \mathbb{N}$, since $n = i_n\ell + k_n$ with $i_n = \lfloor n/\ell \rfloor$ and $k_n \in \{0,\ldots,\ell-1\}$, we obtain for any $x, y \in \mathsf{Y}$ that

$$\begin{aligned}\mathbf{W}_{\mathbf{c}}(\delta_x\mathrm{R}^n, \delta_y\mathrm{R}^n) \le \tilde{C}_\ell \rho_\ell^{\ell i_n} \int_{\mathsf{Y}\times\mathsf{Y}} \mathbb{1}_{\Delta_{\mathsf{Y}}}(\tilde{x},\tilde{y}) W(\tilde{x},\tilde{y}) \mathrm{Q}^{k_n}((x,y), \mathrm{d}(\tilde{x},\tilde{y}))& \\ \le (1+A_\ell)\tilde{C}_\ell \rho_\ell^{\lfloor n/\ell \rfloor} W(x,y)&\ ,\end{aligned}$$

which concludes the proof. □

D Tamed Euler-Maruyama discretization

In this subsection we consider the following assumption.

T1. $\mathsf{X} = \mathbb{R}^d$ *and* $\Pi = \mathrm{Id}$ *and*

$$\mathcal{T}_\gamma(x) = x + \gamma b(x)/(1+\gamma\|b(x)\|) \text{ for any } \gamma > 0 \text{ and } x \in \mathbb{R}^d \ .$$

Here, we focus on drift b which is no longer assumed to be Lipschitz. Therefore, the ergodicity results obtained in Section 2.2 no longer hold since the minorization condition we derived relied heavily on one-sided Lipschitz condition or Lipschitz regularity for b. We now consider the following assumption on b.

T2. *There exists* $\tilde{\mathtt{L}}, \tilde{\ell} \ge 0$ *such that for any* $x, y \in \mathbb{R}^d$

$$\|b(x) - b(y)\| \le \tilde{\mathtt{L}}(1 + \|x\|^{\tilde{\ell}} + \|y\|^{\tilde{\ell}})\|x-y\| \ .$$

In addition, assume that $b(0) = 0$ *and* $M_{\tilde{\ell}} = \sup_{x\in\mathbb{R}^d}(1+\|x\|^{\tilde{\ell}})(1+\|b(x)\|)^{-1} < +\infty$.

Proposition 33. *Assume* **T**1 *and* **T**2 *then* **A**2$(\mathbb{R}^{2d})$-(iii) *holds with* $\bar{\gamma} > 0$ *and for any* $\gamma \in (0,\bar{\gamma}]$, $\kappa(\gamma) = 2\tilde{\mathtt{L}}_\gamma + \gamma\tilde{\mathtt{L}}_\gamma^2$ *where*

$$\tilde{\mathtt{L}}_\gamma = 2\gamma^{-1}M_{\tilde{\ell}}(1+M_{\tilde{\ell}})\tilde{\mathtt{L}} \ .$$

Proof. Let $x, y \in \mathbb{R}^d$ and assume that $\|x\| \ge \|y\|$. We have the following inequalities

$$\begin{aligned}\left\|\frac{b(x)}{1+\gamma\|b(x)\|} - \frac{b(y)}{1+\gamma\|b(y)\|}\right\| &\le \frac{\|b(x)-b(y)\|}{1+\gamma\|b(x)\|} + \left|\frac{\|b(y)\|}{1+\gamma\|b(x)\|} - \frac{\|b(y)\|}{1+\gamma\|b(y)\|}\right| \\ &\le \gamma^{-1}2M_{\tilde{\ell}}\tilde{\mathtt{L}}\|x-y\| + \gamma\frac{\|b(y)\|\,\|b(x)-b(y)\|}{(1+\gamma\|b(x)\|)(1+\gamma\|b(y)\|)} \\ &\le \gamma^{-1}M_{\tilde{\ell}}(1+M_{\tilde{\ell}})\tilde{\mathtt{L}}\|x-y\| \ .\end{aligned}$$

The same inequality holds with $\|y\| \geq \|x\|$. Therefore, we have

$$\|\mathcal{T}_\gamma(x) - \mathcal{T}_\gamma(y)\|^2 \leq \left(1 + 2\gamma\tilde{\mathtt{L}}_\gamma + \gamma^2\tilde{\mathtt{L}}_\gamma^2\right) \|x - y\|^2 \ ,$$

which concludes the proof. $\square$

Proposition 33 implies that the conclusions of Proposition 3-(c) hold. Note that contrary to the conclusion of Proposition 8, we do not get that $\sup_{\gamma \in (0,\bar{\gamma}]} \kappa(\gamma) < +\infty$. Hence we have for any $\tilde{\ell} \in \mathbb{N}^\star$, $\inf_{\gamma \in (0,\bar{\gamma}]} \alpha_+(\kappa, \gamma, \tilde{\ell}) = 0$.

T 3. *There exist $\tilde{R}$ and $\tilde{\mathtt{m}}^+$ such that for any $x \in \bar{\mathrm{B}}(0, \tilde{R})^{\mathrm{c}}$,*

$$\langle b(x), x\rangle \leq -\tilde{\mathtt{m}}^+ \|x\| \, \|b(x)\| \ .$$

Under **T** 2 and **T** 3 it is shown in [6] that there exists $\bar{\gamma} > 0$, $\lambda \in (0, 1)$ and $A \geq 0$ such that for any $\gamma \in (0, \bar{\gamma}]$, R_γ satisfies $\mathbf{D}_{\mathrm{d}}(V, \lambda^\gamma, A\gamma, \mathsf{X})$ with $V(x) = \exp(a(1 + \|x\|^2)^{1/2})$ for some fixed a.

Theorem 34. *Assume* **T**2 *and* **T**3 *then there exists $\bar{\gamma} > 0$ such that for any $\gamma \in (0, \bar{\gamma}]$ there exist $C_\gamma \geq 0$ and $\rho_\gamma \in (0, 1)$ with for any $\gamma \in (0, \bar{\gamma}]$, $x, y \in \mathbb{R}^d$ and $k \in \mathbb{N}$*

$$\|\delta_x \mathrm{R}_\gamma^k - \delta_y \mathrm{R}_\gamma^k\|_V \leq C_\gamma \rho_\gamma^{k\gamma} \left\{V(x) + V(y)\right\} \ .$$

Proof. The proof is a direct application of Theorem 6-(a). $\square$

It is shown in [6, Theorem 4] that the following result holds: there exists $V : \mathbb{R}^d \to [1, +\infty)$, $\bar{\gamma} > 0$, $C, D \geq 0$ and $\rho \in (0, 1)$ such that for any $k \in \mathbb{N}$, $\gamma \in (0, \bar{\gamma}]$ and $x \in \mathbb{R}^d$

$$\|\delta_x \mathrm{R}_\gamma^k - \pi\|_V \leq C\rho^{k\gamma} V(x) + D\sqrt{\gamma} \ ,$$

where π is the invariant measure for the diffusion with drift b.

E Explicit rates and asymptotics in Theorem 14

We recall that b satisfies

$$\langle b(x), x\rangle \leq -\mathtt{k}_1 \|x\| \mathbb{1}_{\bar{\mathrm{B}}(0,R_3)^{\mathrm{c}}}(x) - \mathtt{k}_2 \|b(x)\|^2 + \mathtt{a}/2 \ ,$$

with $\mathtt{k}_1, \mathtt{k}_2 > 0$ and $R_3, \mathtt{a} \geq 0$. Let $W_3(x, y) = (V(x) + V(y))/2$ with $V(x) = \exp[\mathtt{m}_3^+ \sqrt{1 + \|x\|^2}]$ and $\mathtt{m}_3^+ \in (0, \mathtt{k}_1/2]$. Therefore, by Proposition 13, K_γ satisfies for any $\gamma \in (0, \bar{\gamma}]$, $\mathbf{D}_{\mathrm{d}}(W_3, \lambda^\gamma, A\gamma, \mathsf{X}^2)$ where $\bar{\gamma} \in (0, 2\mathtt{k}_2)$, $R_4 = \max(1, R_3, (d + \mathtt{a})/\mathtt{k}_1)$ and

$$\lambda = e^{-(\mathtt{m}_3^+)^2/2} , \quad A = \exp\left[\bar{\gamma}(\mathtt{m}_3^+(d + \mathtt{a}) + (\mathtt{m}_3^+)^2)/2 + \mathtt{m}_3^+(1 + R_4^2)^{1/2}\right] (\mathtt{m}_3^+(d{+}\mathtt{a})/2{+}(\mathtt{m}_3^+)^2) . \tag{84}$$

We now discuss the rates given by Theorem 14. Let $\tilde{\rho}_{\bar{\gamma},c} \in (0, 1)$ and $\tilde{C}_{\bar{\gamma},c} \geq 0$ the constants given by Theorem 14 such that for any $\gamma \in (0, \bar{\gamma}]$, $k \in \mathbb{N}$ and $x, y \in \mathsf{X}$

$$\mathbf{W}_{\mathbf{c}_3}(\updelta_x \mathrm{R}_\gamma^k, \updelta_y \mathrm{R}_\gamma^k) \leq \tilde{C}_{\bar{\gamma},c} \tilde{\rho}_{\bar{\gamma},c}^{\lfloor k(\ell\lceil 1/\gamma\rceil)^{-1}\rfloor} W(x, y) ,$$

with $\mathbf{c}_3(x, y) = \mathbb{1}_{\Delta_{\mathsf{X}}^{\mathrm{c}}}(x, y)(V(x) + V(y))/2$. Using the fact that $\lfloor k/(\ell \lceil 1/\gamma\rceil)\rfloor \geq k\gamma/(\ell(1 + \bar{\gamma})) - 1$, and letting $C_{\bar{\gamma},c} = \tilde{C}_{\bar{\gamma}} \tilde{\rho}_{\bar{\gamma},c}^{-1}/2$ and $\rho_{\bar{\gamma},c} = \tilde{\rho}_{\bar{\gamma},c}^{1/(\ell(1+\bar{\gamma}))}$, we obtain that for any $\gamma \in (0, \bar{\gamma}]$, $k \in \mathbb{N}$ and $x, y \in \mathsf{X}$

$$\|\updelta_x \mathrm{R}_\gamma^k - \updelta_y \mathrm{R}_\gamma^k\|_V \leq C_{\bar{\gamma},c} \rho_{\bar{\gamma},c}^{k\gamma} \{V(x) + V(y)\} .$$

Using Theorem 6-(b), we obtain that the following limits exist and do not depend on $\mathtt{L}$

$$\rho_c = \lim_{\bar{\gamma}\to 0} \rho_{\bar{\gamma},c} , \qquad C_c = \lim_{\bar{\gamma}\to 0} C_{\bar{\gamma},c} .$$

We now discuss the dependency of ρ_c with respect to the problem constants, depending on the sign of $\mathtt{m}$, based on Theorem 6-(c). Let $\vartheta = \mathtt{m}_3^+ \exp\left[\mathtt{m}_3^+(1 + R_4^2)\right] \{(d + \mathtt{a})/2 + \mathtt{m}_3^+\}\{1 + 2/(\mathtt{m}_3^+)^2\}$ and consider $\mathtt{k}_1 \leq \sqrt{8\log(2)}$ and $A \geq 1$:

(a) If **B**4 holds, let $\ell = \lceil M_{\mathrm{d}}^2 \rceil$ and we have

$$\log(\rho_c) \leq 12\log(2)\log(6\vartheta)(1 + 4\log(4\vartheta)/\mathtt{m}_3^+)/\left[(\mathtt{m}_3^+)^2 \mathbf{\Phi}\{-1/2\}\right] .$$

(b) If **B**3($\mathtt{m}$) with $\mathtt{m} \in \mathbb{R}_-$, we set $\ell = 1$ and we have

$$\log(\rho_c) \leq 12\log(2)\log(6\vartheta)\Big/\left[(\mathtt{m}_3^+)^2 \mathbf{\Phi}\{-(-\mathtt{m})^{1/2}\log(4\vartheta)/(2(\mathtt{m}_3^+)^2(1 - \mathrm{e}^{2\mathtt{m}}))^{1/2}\}\right] .$$

Note that a similar result was already proven in [26, Theorem 10] but the scheme of the proof was different as the authors compared the discretization scheme to the associated diffusion process and used the contraction of the continuous process.

F Proof of Lemma 15

(a) Let $x \in \mathbb{R}^d$ and let $(\mathbf{X}_t)_{t\geq 0}$ a solution of (40) starting from x. Define for any $k \in \mathbb{N}^\star$, $\tau_k = \inf\{t \geq 0 : \|\mathbf{X}_t\| \geq k\}$ and for any $t \geq 0$, $\mathbf{M}_t = \int_0^t \langle \nabla V(\mathbf{X}_s), \mathrm{d}B_s \rangle$. Using the Itô formula we obtain that for every $t \geq 0$ and $k \in \mathbb{N}^\star$

$$\begin{aligned} V(\mathbf{X}_{t\wedge\tau_k})\mathrm{e}^{\zeta(t\wedge\tau_k)} &= \int_0^{t\wedge\tau_k} \left[\mathrm{e}^{\zeta(t\wedge\tau_k)}\mathcal{A}V(\mathbf{X}_u) + \zeta\mathrm{e}^{\zeta u}V(\mathbf{X}_u)\right]\mathrm{d}u + \mathbf{M}_{t\wedge\tau_k} + V(x) \\ &= V(\mathbf{X}_{s\wedge\tau_k})\mathrm{e}^{\zeta(s\wedge\tau_k)} + \mathbf{M}_{t\wedge\tau_k} - \mathbf{M}_{s\wedge\tau_k} + \int_{s\wedge\tau_k}^{t\wedge\tau_k} \left[\mathrm{e}^{\zeta(t\wedge\tau_k)}\mathcal{A}V(\mathbf{X}_u) + \zeta\mathrm{e}^{\zeta u}V(\mathbf{X}_u)\right]\mathrm{d}u \\ &\leq V(\mathbf{X}_{s\wedge\tau_k})\mathrm{e}^{\zeta(s\wedge\tau_k)} + \mathbf{M}_{t\wedge\tau_k} - \mathbf{M}_{s\wedge\tau_k} . \end{aligned}$$

Therefore since for any $k \in \mathbb{N}^\star$, $(\mathbf{M}_{t\wedge\tau_k})_{t\geq 0}$ is a $(\mathcal{F}_t)_{t\geq 0}$-martingale, we get for every $t \geq s \geq 0$ and $k \in \mathbb{N}^\star$

$$\mathbb{E}\left[V(\mathbf{X}_{t\wedge\tau_k})\mathrm{e}^{\zeta(t\wedge\tau_k)}\middle|\mathcal{F}_s\right] \leq V(\mathbf{X}_{s\wedge\tau_k})\mathrm{e}^{\zeta(s\wedge\tau_k)} ,$$

which concludes the proof of (a) taking $k \to +\infty$ and using Fatou's lemma.

(b) Similarly as in (a) we have that $(V(\mathbf{X}_t)\mathrm{e}^{\zeta t} - B(1-\exp(-\zeta t))/\zeta)_{t\geq 0}$ is a $(\mathcal{F}_t)_{t\geq 0}$-supermartingale which concludes the proof of (a) upon taking the expectation of $V(\mathbf{X}_t)\mathrm{e}^{\zeta t} - B(1-\exp(-\zeta t))/\zeta$.