J Theor Probab (2017) 30:932–960
DOI 10.1007/s10959-016-0666-8

A Discrete-Time Clark–Ocone Formula and its Application to an Error Analysis

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Received: 9 March 2015 / Revised: 12 November 2015 / Published online: 18 January 2016

Abstract In this paper, we will establish a discrete-time version of Clark(–Ocone–Haussmann) formula, which can be seen as an asymptotic expansion in a weak sense. The formula is applied to the estimation of the error caused by the martingale representation. Throughout, we use another distribution theory with respect to Gaussian rather than Lebesgue measure, which can be seen as a discrete Malliavin calculus.

Keywords Discrete Clark–Ocone formula · Discrete Malliavin calculus · Sobolev differentiability-index is the rate of convergence

Mathematics Subject Classification (2010) Primary 60H07; secondary 60F05 · 60G42

1 Introduction

Let $T > 0$, $(W_t)_{0 \le t \le T}$ be a Brownian motion starting from 0, and $(\mathcal{F}_t)_{0 \le t \le T}$ be its natural filtration. Let $X \in L^2(\mathcal{F}_T)$ be differentiable in the sense of Malliavin, for which we may write $X \in \mathbb{D}_{2,1}$ (see e.g., [12]). Then, it holds that

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Jirô Akahori: The First author was supported by JSPS KAKENHI Grant Number 23330109, 24340022, 23654056, 25285102 and the project RARE-318984 (an FP7 Marie Curie IRSES).
Takafumi Amaba: This work was supported by JSPS KAKENHI Grant Number 24·5772.

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$$X = \mathbf{E}[X] + \int_0^T \mathbf{E}\left[D_s X|\mathcal{F}_s\right] \mathrm{d}W_s, \tag{1.1}$$

where D_s means the Malliavin derivative (evaluated at s).

The formula (1.1) is known as Clark–Ocone formula though there are many variants; Clark [3] obtained (1.1) for some Fréchet differentiable functionals and Ocone [16] related it to Malliavin derivatives, while Haussmann [10] extended it to functionals of the solution to a stochastic differential equation. There are yet much more contexts, which we omit here.

In the context of mathematical finance, the formula gives an alternative description of the hedging portfolio in terms of Malliavin derivatives. However, explicit expressions of the Malliavin derivatives of a Wiener functional are not available in general (except for some special cases: see [18]). In the paper, we will introduce a *finite-dimensional approximation* of (1.1) and discuss the "order of the convergence" in a finance-oriented mode. Actually, this kind of finite-dimensional approximation or something similar is commonly used in financial practice. Hence the results presented in this paper might be more insightful and useful for the practitioners in the field.

Let us be more precise. Put $\Delta W_k = W_{k\Delta t} - W_{(k-1)\Delta t}$ for $k \in \mathbb{N}$, where Δt is a fixed constant. Then, for fixed n, the random variable $(\Delta W_1, \cdots, \Delta W_n)$ is distributed as $N(0, \Delta t I)$. Let $\mathcal{G}_k$, $k = 1, \dots N$, be the sigma-algebra generated by $(\Delta W_1, \dots, \Delta W_k)$. Note that $\mathbb{G} := \{\mathcal{G}_k\}_{k=0}^N$ is a filtration, and

$$L^2(\mathcal{G}_N, \mathbf{P}) \simeq L^2\left(\mathbb{R}^N, \mu^N\right),$$

where

$$\mu^N(\mathrm{d}x) = \frac{1}{(2\pi\Delta t)^{\frac{N}{2}}} \mathrm{e}^{-\frac{|x|^2}{2\Delta t}} \mathrm{d}x.$$

With the filtration $\mathbb{G}$, we can discuss "stochastic integral" (which is in fact a Riemannian sum) with respect to the process (random walk) $W^{\Delta t} = \sum \Delta W$. On the other hand, we can naturally define (a precise formulation will be given in Sect. 2.1) a finite-dimensional version of the Malliavin derivative D_s by the weak partial derivatives such as

$$\partial_l X(x_1, \dots, x_N)\,|_{x_k = \Delta W_k, k=1,\dots,N}.$$

Then, one might well guess that a discrete version of the Clark–Ocone formula could be

$$X \stackrel{?}{=} \mathbf{E}[X] + \sum_{l=1}^N \mathbf{E}\left[\partial_l X|\mathcal{G}_{l-1}\right] \Delta W_l$$

but this is not true since the random walk $W^{\Delta t}$ does not have the martingale representation property. In fact, if the martingale representation property holds for a random walk, then we can establish a precise discrete-time Clark–Ocone formula if we define "differentiation" properly. For the binary case, Privault [17] has made a detailed study on the discrete Clark–Ocone formula and related discrete Malliavin calculus.

We should instead ask how much the (martingale representation) error

$$\text{Mart.Err} := X - \mathbf{E}[X] - \sum_{l=1}^{N} \mathbf{E}\left[\partial_l X | \mathcal{G}_{l-1}\right] \Delta W_l,$$

is, though it is dependent on the norm to measure it. Our discrete Clark–Ocone formula states that

$$\text{Mart.Err} = \sum_{m=2}^{\infty} \sum_{l=1}^{N} \frac{(\Delta t)^{m/2}}{\sqrt{m!}} \mathbf{E}\left[\partial_l^m X | \mathcal{G}_{l-1}\right] H_m\left(\frac{\Delta W_l}{\sqrt{\Delta t}}\right),$$

where H_n is the n-th Hermite polynomial (see Theorem 2.1 below). In this paper, we study its asymptotic behavior as $N \to \infty$ with $N\Delta t =$ time horizon T. This is closely related to the problem of so-called *tracking error of the delta hedge* in mathematical finance. If one has a nice finite-dimensional approximation X^N of a Wiener functional X, both defined on the same probability space, then the approximation error (which has the meaning of tracking error in mathematical finance) can be controlled by the (supremum in N of) Mart.Err plus the error caused by the discretization (finite-dimensional approximation) as we see from:

$$\begin{aligned}
\text{Tra.Err} := {} & X - \mathbf{E}[X] - \sum_{l=1}^{N} \mathbf{E}\left[D_{(l\Delta t)} X | \mathcal{F}_{(l\Delta t)}\right] \Delta W_l \\
= {} & \left(X - X^N\right) - \mathbf{E}\left[X - X^N\right] \\
& - \sum_{l=1}^{N} \left(\mathbf{E}\left[D_{(l\Delta t)} X | \mathcal{F}_{(l\Delta t)}\right] - \mathbf{E}\left[\partial_l X^N | \mathcal{G}_{l-1}\right]\right) \Delta W_l + \text{Mart.Err} \\
=: {} & \text{Disc. Err} + \text{Mart.Err}.
\end{aligned}$$

There are considerably many studies on the subject of the approximation error as well. It at least dates back to the paper by Rootzen [19], where the weak convergence of the scaled error was studied. The problem is reformulated as "tracking error of the delta hedge" in Bertsimas et al. [1], where the error was also measured by L^2-norm. Hayashi and Mykland [11] further developed the argument from financial perspectives.

Notable results in this topic are summarized as follows.

(1) The scaled error $N^{-1/2}$Tra.Err converges weakly to B_τ with

$$\tau = \frac{1}{2} \int \left|\mathbf{E}\left[D_s^2 X | \mathcal{F}_s\right]\right|^2 \mathrm{d}s,$$

where B is a Brownian motion independent of τ (Here, actually the differentiability is not required. The expression $\mathbf{E}[D_s X | \mathcal{F}_s]$ should be understood as simply the integrand of the martingale representation of X and the meaning of $\mathbf{E}[D_s^2 X | \mathcal{F}_s]$ will be clarified later).

(2) The error estimated with L^2-norm is in $O(N^{-1/2})$ in the cases of $X = F(S)$ with "ordinary pay-off" F and the solution S of an SDE, while it is in $O(N^{-1/4})$ when

F is "irregular"like Heaviside function (Gobet and Temam [9], Temam [21]). Later the irregularity is associated with differentiability in the *fractional order* $s \in (0, 1)$ by Geiss and Geiss [6]; it is in $O(N^{-s/2})$ for s-differentiable F.

In this paper, we shall establish the corresponding results for the Mart.Err.

This paper is organized as follows. After introducing the Discrete Clark–Ocone formula (Theorem 2.1, Sect. 2.2), we will see that the infinite series form of our Discrete Clark–Ocone formula naturally leads to a multi-level central limit theorem for the error (Theorem 3.1). This corresponds to the result 1 above. Since we will be working on a sequence of Wiener functionals unlike the situations concerning tracking error, we need an additional condition which we call *stationarity*, apart from the fractional regularity. The corresponding result is given in Sect. 3.3. We note that our smoothness is in essence of an infinite-dimensional one in contrast with existing literature.

Section 3.5 is devoted to a study of the asymptotics of the error of the additive functionals. As a case study, we give a detailed estimate of the martingale representation error of the Riemann sum approximation of Brownian occupation time (Theorem 3.8).

The proofs given in this paper are largely based on elementary calculus with a bit of classical Fourier analysis and distribution theory, but nonetheless our methods can be, in spirit, a finite-dimensional reduction in Malliavin-Watanabe's distribution theory. Some detailed discussions on this point of view will be given in Sects. 2.1 and 3.1. We have restricted ourselves to one-dimensional Wiener space case, but this is only for simplicity of notations.

2 A Discrete Version of Clark–Ocone Formula

2.1 Generalized Wiener Functional in Discrete Time

Throughout this section, we fix $N \in \mathbb{N}$ and work on the canonical probability space $(\mathbb{R}^N, \mathfrak{B}(\mathbb{R}^N), \mu^N)$ though we will abuse the notations like ΔW as the coordinate map.

Let $\mathcal{S}_N \equiv \mathcal{S}(\mathbb{R}^N)$ be the Schwartz space; the space of all rapidly decreasing functions and $\mathcal{S}'_N$ be its dual; the space of all tempered distributions (see, e.g., [20]). We (may) call $X \in \mathcal{S}'_N$ a "discrete generalized Wiener functional" and its generalized expectation is defined to be the coupling ${}_{\mathcal{S}'_N}\langle X, p^N\rangle_{\mathcal{S}_N}$, where p^N is the density of μ^N, which is of course in $\mathcal{S}_N$.

The conditional expectation $\mathbf{E}[X|\mathcal{G}_k]$ for $X \in \mathcal{S}'_N$ is then defined as follows. We first note that the inclusion $\mathcal{G}_k \subset \mathcal{G}_N$ induces those of $\mathcal{S}(\mathbb{R}^k) \subset \mathcal{S}(\mathbb{R}^N)$ and as a completion $\mathcal{S}'(\mathbb{R}^k)$ is also embedded to $\mathcal{S}'(\mathbb{R}^N)$. In this sense, we write $\mathcal{S}_k$ and $\mathcal{S}'_k$ for the $\mathcal{G}_k$-measurable Schwartz space and the space of generalized Wiener functionals, respectively, for $k = 1, \ldots, N$. Then, $Y = \mathbf{E}[X|\mathcal{G}_k]$ in $\mathcal{S}'_k$ is defined in terms of the relation

$$\mathbf{E}[XZ] = \mathbf{E}[YZ], \quad \forall Z \in \mathcal{S}_k,$$

which should be understood as

$${}_{\mathcal{S}'_N}\left\langle X, Zp^N\right\rangle_{\mathcal{S}_N} = {}_{\mathcal{S}'_k}\left\langle Y, Zp^k\right\rangle_{\mathcal{S}_k}, \quad \forall Z \in \mathcal{S}_k.$$

In particular, we see that the conditional expectation is well defined by du Bois-Reymond lemma (see e.g., [20]). Note that this generalized conditional expectation reduces to the standard one on $L^1(\mu^N)$, which is included in $\mathcal{S}'_N$ unlike the L^1 space with respect to the Lebesgue measure. Furthermore, differentiations of $X \in \mathcal{S}'_N$ are defined as usual, namely

$$\partial_k X = Y \iff {}_{\mathcal{S}'_N}\langle Y, Z\rangle_{\mathcal{S}_N} = -{}_{\mathcal{S}'_N}\langle X, \partial_k Z\rangle_{\mathcal{S}_N} \quad \forall Z \in \mathcal{S}_N,$$

which imply

$$\mathbf{E}[\partial_k X] = \mathbf{E}\left[X \partial_k \log p^N\right],$$

and so on.

We further note that

$$X\left(p^N\right)^{1/2} \in \mathcal{S} \iff X \in \mathbb{D}^{(N)}_{2,\infty} = \cap_{s>0}\mathbb{D}^{(N)}_{2,s} \tag{2.1}$$

and

$$X\left(p^N\right)^{1/2} \in \mathcal{S}' \iff X \in \mathbb{D}^{(N)}_{2,-\infty} = \cup_{s<0}\mathbb{D}^{(N)}_{2,s}, \tag{2.2}$$

where $\mathbb{D}^{(N)}_{2,s}$ is the completion of $L^2(\mu^N)$ by the norm $\|f\|_{2,s} = \|(1+L)^{s/2} f\|_{L^2(\mu^N)}$ at this stage. Later we will give a remark that they are the right discrete versions of the spaces of the (generalized) Wiener functionals (Watanabe distributions) projected to $\mathcal{G}^N$. Here, L is the Ornstein–Uhlenbeck operator on $\mathbb{R}^N$;

$$L = -\sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^N x_i \frac{\partial}{\partial x_i}.$$

Below we give a brief validation of the equivalences (2.1) and (2.2). Let $\{\phi_n : n \in \mathbb{Z}\}$ be norms defined by

$$\phi_n(f) = \left\|(1+S)^n f\right\|_{L^2(\mathrm{Leb})},$$

where S is the following Schrödinger operator of the harmonic oscillator:

$$S := -\sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} + \frac{1}{4}|x|^2 - \frac{1}{2}.$$

We know that $\mathcal{S}$ is a Fréchet space by the seminorms $\{\phi_n\}$. We also have

$$L(f)\left(p^N\right)^{1/2} = S\left(f\left(p^N\right)^{1/2}\right),$$

which implies $\|f\|_{2,n} = \phi_n(f)$. This proves (2.1). The equivalence (2.2) follows from the following equivalence of the duality:

$${}_{\mathbb{D}^{(N)}_{2,-\infty}}\langle X, Y\rangle_{\mathbb{D}^{(N)}_{2,\infty}} = {}_{\mathcal{S}'}\left\langle X\left(p^N\right)^{1/2}, Y\left(p^N\right)^{1/2}\right\rangle_{\mathcal{S}}.$$

2.2 Clark–Ocone Formula in Discrete Time

We have the following series expansion in Δt:

Theorem 2.1 (A Discrete Version of Clark–Ocone Formula) *For $X \in L^2(\mathcal{G}_N) \simeq L^2(\mu^N)$, we have the following L^2-convergent series expansion:*

$$X - \mathbf{E}[X] = \sum_{m=1}^{\infty} \sum_{l=1}^{N} \frac{(\Delta t)^{m/2}}{\sqrt{m!}} \mathbf{E}\left[\partial_l^m X | \mathcal{G}_{l-1}\right] H_m\left(\frac{\Delta W_l}{\sqrt{\Delta t}}\right) \tag{2.3}$$

where H_m is the m-th Hermite polynomial for $m \in \mathbb{Z}_+$;

$$H_m(x) = \frac{(-1)^m}{\sqrt{m!}} e^{\frac{x^2}{2}} \frac{\mathrm{d}^m}{\mathrm{d}x^m} e^{-\frac{x^2}{2}} \quad (m \in \mathbb{Z}_+)\,. \tag{2.4}$$

Here, the differentiations are understood in the distribution sense, as explained in the previous section.

Moreover, if $X \in \mathbb{D}_{2,s}^{(N)}$, $s \in \mathbb{R}$, the convergence of (2.3) is also attained in $\mathbb{D}_{2,s}^{(N)}$.

Proof Since $\left\{\prod_{i=1}^N H_{k_i}\left(\frac{\Delta W_i}{\sqrt{\Delta t}}\right)\right\}_{k_1,\ldots,k_n \in \mathbb{Z}_+}$ is an orthonormal basis of $L^2(\mathbb{R}^N, \mu^N)$, we have the following orthogonal expansion of $X \in L^2(\mathbb{R}^N, \mu^N)$:

$$X\left(\Delta W_1, \ldots, \Delta W_N\right) = \sum_{k_1,\ldots,k_N} c_{(k_1,\ldots,k_N)} \prod_{i=1}^N H_{k_i}\left(\frac{\Delta W_i}{\sqrt{\Delta t}}\right). \tag{2.5}$$

where we denote

$$c_{(k_1,\ldots,k_N)} := \left\langle X, \prod_{i=1}^N H_{k_i}\left(\frac{\Delta W_i}{\sqrt{\Delta t}}\right)\right\rangle = \mathbf{E}\left[X \prod_{i=1}^N H_{k_i}\left(\frac{\Delta W_i}{\sqrt{\Delta t}}\right)\right].$$

Let us "sort" the series according as the "highest" nonzero k_i;

$$X\left(\Delta W_1, \ldots, \Delta W_N\right) = \mathbf{E}[X] + \sum_{l=1}^N \sum_{k_1,\ldots,k_{l-1}} \sum_{k_l \geq 1} c_{(k_1,\ldots,k_l,0,\ldots,0)} \prod_{i=1}^l H_{k_i}\left(\frac{\Delta W_i}{\sqrt{\Delta t}}\right). \tag{2.6}$$

Here, we claim that

$$\sum_{l=1}^N \sum_{k_1,\ldots,k_{l-1}} c_{(k_1,\ldots,k_l,0,\ldots,0)} \prod_{i=1}^{l-1} H_{k_i}\left(\frac{\Delta W_i}{\sqrt{\Delta t}}\right) = \mathbf{E}\left[X H_{k_l}\left(\frac{\Delta W_i}{\sqrt{\Delta t}}\right) \middle| \mathcal{G}_{l-1}\right]. \tag{2.7}$$

In fact, from the expansion (2.5) we have

$$\mathbf{E}\left[XH_{k_l}\left(\frac{\Delta W_l}{\sqrt{\Delta t}}\right)|\mathcal{G}_{l-1}\right]=\mathbf{E}\left[\sum_{k'_1,\dots,k'_N} c_{(k'_1,\dots,k'_N)}H_{k_l}\left(\frac{\Delta W_l}{\sqrt{\Delta t}}\right)\prod_{i=1}^{N}H_{k'_i}\left(\frac{\Delta W_i}{\sqrt{\Delta t}}\right)|\mathcal{G}_{l-1}\right]$$
$$=\sum_{k'_1,\dots,k'_N;} c_{(k'_1,\dots,k'_N)}\prod_{i=1}^{l-1}H_{k'_i}\left(\frac{\Delta W_i}{\sqrt{\Delta t}}\right)\mathbf{E}\left[H_{k_l}\left(\frac{\Delta W_l}{\sqrt{\Delta t}}\right)\prod_{i=l}^{N}H_{k'_i}\left(\frac{\Delta W_i}{\sqrt{\Delta t}}\right)\right],$$

and we confirm the claim since $\mathbf{E}[H_{k_l}(\frac{\Delta W_l}{\sqrt{\Delta t}})\prod_{i=l}^{n}H_{k'_i}(\frac{\Delta W_i}{\sqrt{\Delta t}})]=0$ unless $k'_l=k_l$ and $k'_i=0$ for $i>l$.

We further claim that

$$\mathbf{E}\left[XH_{k_l}\left(\frac{\Delta W_i}{\sqrt{\Delta t}}\right)|\mathcal{G}_{l-1}\right]=\frac{(\Delta t)^{k/2}}{\sqrt{k!}}\mathbf{E}\left[\partial_l^k X\middle|\mathcal{G}_{l-1}\right],\tag{2.8}$$

which, together with (2.6) and (2.7), will prove the expansion (2.3) in the L^2 case. Here, the conditional expectation should be understood in the generalized sense. Following the definition we have made, it suffices to show that

$$\mathbf{E}\left[XH_{k_l}\left(\frac{\Delta W_l}{\sqrt{\Delta t}}\right)f\left(\Delta W_1,\dots,\Delta W_{l-1}\right)\right]=\frac{(\Delta t)^{k/2}}{\sqrt{k!}}\mathbf{E}\left[\partial_l^k Xf\right]$$

for any $f\in\mathcal{S}_{l-1}$ but this is easy to see if we write down the generalized expectation as the coupling of $\mathcal{S}$ and $\mathcal{S}'$:

$$_{\mathcal{S}'}\left\langle X,H_k\left(x/\sqrt{\Delta t}\right)fp^N\right\rangle_{\mathcal{S}}={}_{\mathcal{S}'}\left\langle X,f(-1)^k\frac{(\Delta t)^{k/2}}{\sqrt{k!}}\partial_l^k p^N\right\rangle_{\mathcal{S}}$$
$$=\frac{(\Delta t)^{k/2}}{\sqrt{k!}}{}_{\mathcal{S}'}\left\langle \partial_l^k X,fp^N\right\rangle_{\mathcal{S}}.$$

The last statement follows from the fact that, by the assumption, the partial sums

$$X_n:=\sum_{k_1+\cdots+k_N\le n}c_{(k_1,\dots,k_N)}\prod_{i=1}^{N}H_{k_i}(x_i),\quad n\in\mathbb{N}$$

form a Cauchy sequence in $\mathbb{D}_{2,s}^{(N)}$. □

3 Asymptotic Analysis of Martingale Representation Errors

In this section, we will consider the asymptotic behavior of the error term when $N\to\infty$ with $N\Delta t=T$. For this purpose, to make explicit the dependence on N we redefine some of the notations. $t_k=:t_k^{(N)}:=\frac{kT}{N}$ for each $k=0,1,\dots,N$. We denote this partition by $\Delta^{(N)}$. We also write $\Delta^{(N)}t := \frac{T}{N}$, $\Delta^{(N)}W_k := (\Delta^{(N)}W)_k := W_{t_k^{(N)}} - W_{t_{k-1}^{(N)}}$ for each k and N, and $\mathcal{G}_k^N := \sigma(\Delta^{(N)}W_l; l = 1, \dots, k)$. We omit the superscript (N) unless it causes any confusion. Further, to facilitate the discussion in the limit, we implement our discrete Malliavin-Watanabe calculus into the classical one in the first subsection.

3.1 Consistency with the Classical Malliavin Calculus

The framework established in Sect. 2 is embedded into Malliavin calculus via the following identity

$$(D_t X)(\omega) = \sum_{l=1}^{N} 1_{\{t_{l-1} < t \le t_l\}} (\partial_l X)(w)$$

for a.a. $(t, \omega) \in [0, T] \times \Omega$ and $X \in \mathbb{D}_{2,1}^{(N)}$. For each $F \in \mathbb{D}_{2,1}$, one can prove that $\mathbf{E}[F|\mathcal{G}_N^N] \in \mathbb{D}_{2,1}^{(N)}$ and $\lim_{N\to\infty} \mathbf{E}[F|\mathcal{G}_N^N] = F$ in $\mathbb{D}_{2,1}$ (consult e.g., [13] Theorem 1.10). Also, one can obtain

$$(D_t F)(\omega) = \lim_{N\to\infty} \sum_{l=1}^{N} 1_{\{t_{l-1} < t \le t_l\}} \partial_l \mathbf{E}\left[F|\mathcal{G}_N^N\right](\omega) \tag{3.1}$$

for a.a. $(t, w) \in [0, T] \times \Omega$. Note that in [13], the derivative D on the Wiener space is defined directly by (3.1) with $N = 2^n$. Following this approach in [13], we define $D^k_{\cdot} X \in L^2[0, T] \otimes L^2(\mathbf{P})$ as the L^2-limit of the sequence $(D^k_{\cdot}\mathbf{E}[X|\mathcal{G}_N^N])_{N=1}^{\infty}$ if it exists (see [13], Theorem 1.10 to consult what condition is sufficient to get this limit).

By the above discussions, we may write

$$D_t^k X := \partial_l^k X \quad \text{if } t_{l-1} < t \le t_l$$

for $X \in \mathbb{D}_{2,n}^{(N)}$, for a.a. $t \in [0, T]$ and $k = 1, 2, \dots, n$. Furthermore, we use the notation $D^k_{t_l^{(N)}} X$ to denote $\partial_l^k X$ for $X \in \mathbb{D}_{2,n}^{(N)}$ though the above equation holds for a.a. t.

3.2 A Central Limit Theorem for the Errors

Suppose that we are given a sequence $(X^N)_{N=1}^{\infty}$ of finite-dimensional Wiener functionals $X^N \in L^2(\mathcal{G}_N^N)$ for each N.

We put, for $n \ge 0$,

$$\mathrm{Err}_N(n) := X^N - \sum_{m=0}^{n} \sum_{l=1}^{N} \frac{(\Delta t)^{m/2}}{\sqrt{m!}} \mathbf{E}\left[D^m_{t_l^{(N)}} X^N \middle| \mathcal{G}_{l-1}^N\right] H_m\left(\frac{\Delta^{(N)} W_l}{\sqrt{\Delta^{(N)} t}}\right).$$

We notice that $\mathrm{Err}_N(1)$ is the same as Mart.Err(X^N) in the introduction.

Theorem 3.1 *Let $n \in \mathbb{N}$. Suppose that $X^N \in \mathbb{D}^{(N)}_{2,n+2}$ for each $N = 1, 2, \ldots$ and for some Wiener functional $X \in \mathbb{D}_{2,n+1}(\mathbb{R})$, we have*

- $X^N \to X$ *in* $L^2(\mathbf{P})$,
- $D^{q+1}_{\cdot} X \in L^2[0,T] \otimes L^2(\mathbf{P})$ *exists and* $\int_0^T \left\| D_t^{q+1} X^N - D_t^{q+1} X \right\|_{L^2}^2 \mathrm{d}t \to 0$

as $N \to \infty$ for each $q = 0, 1, \ldots, n$ and

- $\sup_N \int_0^T \left\| D_t^{n+2} X^N \right\|_{L^2}^2 \mathrm{d}t < \infty.$

Then, we have

$$\begin{pmatrix} Err_N(0) \\ \left(\Delta^{(N)} t\right)^{-1/2} Err_N(1) \\ \vdots \\ \left(\Delta^{(N)} t\right)^{-n/2} Err_N(n) \end{pmatrix} \to \begin{pmatrix} \int_0^T \mathbf{E}\left[D_t X | \mathcal{G}_t\right] \mathrm{d}W_t \\ \frac{1}{\sqrt{2}} \int_0^T \mathbf{E}\left[D_t^2 X | \mathcal{G}_t\right] \mathrm{d}B_t^1 \\ \vdots \\ \frac{1}{\sqrt{(n+1)!}} \int_0^T \mathbf{E}\left[D_t^{n+1} X | \mathcal{G}_t\right] \mathrm{d}B_t^n \end{pmatrix}$$

in probability on an extended probability space as $N \to \infty$, where $(B^1, \ldots, B^n) = (B_t^1, \ldots, B_t^n)_{0 \le t \le T}$ is an n-dimensional Brownian motion independent of $W = (W_t)_{0 \le t \le T}$.

Remark 3.1 (1) Although the Brownian motion $B = (B^1, \ldots, B^n)$ above is not adapted to the filtration $(\mathcal{F}_t)_{0 \le t \le T}$, the above stochastic integrals make sense because it is an $(\mathcal{F}_t \vee \sigma(B_s : 0 \le s \le t))_{0 \le t \le T}$-Brownian motion. (2) The proof uses the technique of so-called *Hermite variation*. Due to the form of Err_N, one may use also the so-called *fourth moment theorem* (see Nualart and Peccati [15]), but our case is too simple to employ it.

Proof In the following, we sometimes omit the superscript (N) from $\Delta^{(N)}$ and write it as Δ. By Theorem 2.1, we have,

$$(\Delta t)^{-q/2} \mathrm{Err}_N(q) = \sum_{m=q+1}^{\infty} \sum_{l=1}^{N} \frac{(\Delta t)^{(m-q)/2}}{\sqrt{m!}} \mathbf{E}\left[D^m_{lT/N} X^N \middle| \mathcal{G}^N_{l-1}\right] H_m\left(\frac{\Delta W_l}{\sqrt{\Delta t}}\right).$$

For $m \ge q + 2$, by using the integration by parts formula (2.8), we see that

$$
\begin{aligned}
&\left\| \sum_{m=q+2}^{\infty} \sum_{l=1}^{N} \frac{(\Delta t)^{(m-q)/2}}{\sqrt{m!}} \mathbf{E}\left[D^{m}_{lT/N} X^N \middle| \mathcal{G}^N_{l-1} \right] H_m \left(\frac{\Delta W_l}{\sqrt{\Delta t}} \right) \right\|^2_{L^2} \\
&\quad = (\Delta t)^2 \sum_{k=0}^{\infty} \sum_{l=1}^{N} \frac{k!}{(k+q+2)!} \left\| \mathbf{E}\left[\left(D^{q+2}_{lT/N} X^N \right) H_k \left(\frac{\Delta W_l}{\sqrt{\Delta t}} \right) \middle| \mathcal{G}^N_{l-1} \right] \right\|^2_{L^2} \\
&\quad \leq (\Delta t) \sum_{k=1}^{\infty} \frac{1}{k^{q+2}} \times \sum_{l=1}^{N} \left\| D^{q+2}_{lT/N} X^N \right\|^2_{L^2} \Delta t \\
&\quad = (\Delta t) \sum_{k=1}^{\infty} \frac{1}{k^{q+2}} \times \int_0^T \left\| D^{q+2}_{t} X^N \right\|^2_{L^2} \mathrm{d}t
\end{aligned} \tag{3.2}
$$

which goes to zero as $N \to \infty$ for each $q = 0, 1, \ldots, n$ by the assumption.

Let us consider the case $m = q + 1$. For each $q = 0, 1, \ldots, n$, we define a right-continuous process $L^{q,N} = \left(L^{q,N}_t \right)_{0 \leq t \leq T}$ with left-hand side limits by

$$
L^{q,N}_t := \sum_{l=1}^{k} H_{q+1} \left(\frac{\Delta^{(N)} W_l}{\sqrt{\Delta^{(N)} t}} \right) \quad \text{if} t_{k-1} \leq t < t_k
$$

for $k = 1, 2, \ldots, N$, and $L^{q,N}_T := L^{q,N}_{t_{N-1}}$.

Since $H_{q+1}\left(\frac{\Delta W_l}{\sqrt{\Delta t}}\right)$, $l = 1, 2, \ldots, N$ are i.i.d. random variables and $H_{q+1}\left(\frac{\Delta W_l}{\sqrt{\Delta t}}\right)$, $q = 0, 1, \ldots, n$ are orthogonal to each other for each $l = 1, 2, \ldots, N$, the central limit theorem of finite-dimensional distributions of $(\Delta t)^{1/2} L^{q,N}$, $N = 1, 2, \ldots$ follows as for each $0 \leq s < t$, with taking $t_{j-1} \leq s < t_j$ and $t_{k-1} \leq t < t_k$,

$$
\begin{aligned}
&\lim_{N \to \infty} \mathbf{E}\left[\mathrm{e}^{i \sum_{q=0}^{n} \xi_q \left\{ (\Delta t)^{1/2} L^{q,N}_t - (\Delta t)^{1/2} L^{q,N}_s \right\}} \middle| \mathcal{F}^{L^{0,N}}_s \vee \mathcal{F}^{L^{1,N}}_s \vee \cdots \mathcal{F}^{L^{n,N}}_s \right] \\
&\quad = \lim_{N \to \infty} \prod_{l=j+1}^{k} \mathbf{E}\left[\mathrm{e}^{i \sum_{q=0}^{n} \left(\xi_q \sqrt{t_k - t_j} \right) \cdot (k-j)^{-1/2} H_{q+1}\left(\frac{\Delta W_l}{\sqrt{\Delta t}} \right)} \right] \\
&\quad = \lim_{N \to \infty} \prod_{l=j+1}^{k} \left\{ 1 - \frac{|\xi|^2}{2(k-j)} \left(t_k - t_j \right) + o\left(\frac{|\xi|^2}{k-j} \right) \right\} = \mathrm{e}^{-\frac{\xi^2}{2}(t-s)}
\end{aligned}
$$

for each $\xi = (\xi_0, \xi_1, \ldots, \xi_n) \in \mathbb{R}^{n+1}$, where $(\mathcal{F}^Z_t)_{0 \leq t \leq T}$ denotes the filtration generated by a stochastic process $Z = (Z_t)_{0 \leq t \leq T}$ and the little-o-notation is with respect to the asymptotics when $N \to \infty$(so that $k - j \to \infty$). This implies that every finite-dimensional distribution of $(n+1)$-dimensional process $\left((\Delta t)^{1/2} L^{q,N} \right)^n_{q=0}$ converges to that of an $(n+1)$-dimensional Brownian motion $(B^0, B^1, \ldots, B^n) = (B^0_t, B^1_t, \ldots, B^n_t)_{0 \leq t \leq T}$.

Besides, using Kolmogorov's inequality, we have for each $q = 0, 1, \ldots, n$,

$$\lim_{K\to\infty} \limsup_{N\to\infty} \mathbf{P}\left(\sup_{0\le t\le T} \left|(\Delta t)^{1/2} L_t^{q,N}\right| \ge K \right) \le \lim_{K\to\infty} \frac{(\Delta t) \left\| L_T^{q,N} \right\|_{L^2}^2}{K^2} = 0$$

and for each $\varepsilon > 0$,

$$\begin{aligned}
&\lim_{\delta\to 0} \limsup_{N\to\infty} \mathbf{P}\left(\inf_{\substack{\{s_j\}_j \subset [0,T]: \\ |s_j - s_{j+1}| > \delta}} \max_j \sup_{t,s\in[s_{j-1},s_j)} (\Delta t)^{1/2} \left| L_t^{q,N} - L_s^{q,N} \right| \ge \varepsilon \right) \\
&\quad \le \limsup_{N\to\infty} \mathbf{P}\left(\max_{j=1,2,\ldots,N} \sup_{t,s\in[s_{j-1},s_j)} (\Delta t)^{1/2} \left| L_t^{q,N} - L_s^{q,N} \right| \ge \varepsilon \right) \\
&\quad = \limsup_{N\to\infty} \mathbf{P}\left(0 \ge \varepsilon\right) = 0.
\end{aligned}$$

They imply the tightness of $\{(\Delta^{(N)} t)^{1/2} L^{q,N}\}_{N=1}^{\infty}$ (see Billingsley [2], Theorem 13.2). Therefore,

$$\left\{ \left(\left(\Delta^{(N)} t\right)^{1/2} L^{0,N}, \left(\Delta^{(N)} t\right)^{1/2} L^{1,N}, \ldots, \left(\Delta^{(N)} t\right)^{1/2} L^{n,N} \right) \right\}_{N=1}^{\infty}$$

also forms a tight family. Hence we have

$$\left(\sqrt{\Delta^{(N)} t}\, L^{0,N}, \sqrt{\Delta^{(N)} t}\, L^{1,N}, \ldots, \sqrt{\Delta^{(N)} t}\, L^{n,N} \right) \to \left(B^0, B^1, \ldots, B^n \right)$$

in law as $N \to \infty$. By the Skorohod representation theorem (see Ikeda and Watanabe [12], Theorem 2.7 and we remark that on the space of all right-continuous functions with left-hand side limits, one can endow the so-called *Skorohod topology* which is metrizable and makes the space a complete separable metric space. For details, see Billingsley [2], Chapter 5.), we may assume that the above convergence is realized as an almost sure convergence on an extended probability space. Note that on the probability space we still have $B^0 = W$ a.s.

Hence we have

$$\begin{aligned}
&\frac{(\Delta t)^{((q+1)-q)/2}}{\sqrt{(q+1)!}} \sum_{l=1}^{N} \mathbf{E}\left[D_{lT/N}^{q+1} X^N \middle| \mathcal{G}_{l-1}^N \right] H_{q+1}\left(\frac{\Delta W_l}{\sqrt{\Delta t}} \right) \\
&\quad = \frac{1}{\sqrt{(q+1)!}} \sum_{l=1}^{N} \mathbf{E}\left[D_{t_l}^{q+1} X^N \middle| \mathcal{G}_{t_{l-1}} \right] \left\{ (\Delta t)^{1/2} L_{t_l}^{q,N} - (\Delta t)^{1/2} L_{t_{l-1}}^{q,N} \right\} \\
&\quad \to \frac{1}{\sqrt{(q+1)!}} \int_0^T \mathbf{E}\left[D_t^{q+1} X \middle| \mathcal{G}_t \right] \mathrm{d}B_t^q \quad \text{in probability as} N \to \infty
\end{aligned}$$

simultaneously for $q = 0, 1, \dots, n$. □

Substituting $q = 0$ into the inequality (3.2) in the proof of Theorem 3.1, we also obtain the following

Corollary 3.2 *If* $\sup_N \int_0^T \|D_t^2 X^N\|_{L^2}^2 \mathrm{d}t < \infty$ *then we have*

$$\left\| X^N - \left\{ \mathbf{E}\left[X^N\right] + \sum_{l=1}^N \mathbf{E}\left[D_{t_l^{(N)}} X^N | \mathcal{G}_{l-1}^N\right] \Delta W_l \right\} \right\|_{L^2} = O\left(N^{-1/2}\right)$$

as $N \to \infty$.

Remark 3.2 (1) The sequence $X^N \equiv (W_T)^2$ satisfies the condition in Corollary 3.2 since we have $\sup_N \int_0^T \|D_t^2 X^N\|_{L^2}^2 \mathrm{d}t = 4T < +\infty$. Then, $\mathrm{Err}_N(1) = \sqrt{2}(\Delta t) \sum_{l=1}^N H_2\left(\frac{\Delta W_l}{\sqrt{\Delta t}}\right)$ and hence $\|\mathrm{Err}_N(1)\|_{L^2} = \sqrt{2} T N^{-1/2}$. Thus, $O(N^{-1/2})$ is tight in Corollary 3.2. (2) Similarly, by substituting $q = n - 1$ for $n \in \mathbb{N}$ into the inequality (3.2), we can deduce the following:

$$\sup_N \int_0^T \left\| D_t^{n+1} X^N \right\|_{L^2}^2 \mathrm{d}t < \infty \Longrightarrow \|\mathrm{Err}_N(n)\|_{L^2} = O\left(N^{-n/2}\right).$$

3.3 The Cases with "Stationary" Sequences

We have seen that the martingale representation error is of an order $1/2$ for a smooth functional. In this section, we will observe that for a non-smooth functional, the order is related to its fractional differentiability if it behaves eventually like a finite-dimensional functional, the property we call *stationarity*. This parallels with the corresponding results in the cases of the tracking error as we have pointed out in Introduction.

Let us start with one-dimensional cases. Let $F \in L^2(\mathbb{R}, \mu_T)$, where μ_T is the Gaussian measure with variance $T > 0$. Then, since

$$\frac{\partial^k}{\partial x_l^k} F\left(x_1 + \cdots + x_N\right) = F^{(k)}\left(x_1 + \cdots + x_N\right),$$

we have, for $k_1 + \cdots + k_l = n$,

$$\begin{aligned}\mathbf{E}\left[D_{t_1^{(N)}}^{k_1} \cdots D_{t_l^{(N)}}^{k_l} F(W_T)\right]^2 &= \mathbf{E}\left[F^{(n)}(W_T)\right]^2 \\ &= \frac{n!}{T^n} \mathbf{E}\left[F(W_T) H_n\left(\frac{W_T}{\sqrt{T}}\right)\right]^2 = \frac{n!}{T^n} \|J_n F(W_T)\|_{L^2}^2,\end{aligned}$$

irrespective of l and N. Here, J_n is the projection to the n-th chaos. With this observation in mind, we understand the following property as a *stationarity* of a sequence; let $\{F^N\}$ be such that each F^N being $\mathcal{G}_N^N$-measurable and that

$$\sup_{k_1+\cdots+k_N=n}\left(\mathbf{E}\left[D^{k_1}_{t_1^{(N)}}\cdots D^{k_N}_{t_N^{(N)}}F^N\right]\right)^2=O\left(\frac{n!\left\|J_nF^N\right\|^2_{L^2}}{T^n}\right)\tag{3.3}$$
$$\text{uniformly in } n=2,3,\dots \text{ as } N\to\infty.$$

Note that a sequence composed of a one-dimensional functional $F(W_T)$ satisfies the above property trivially.

Theorem 3.3 *Suppose that we are given a sequence of* $F^N\in\mathbb{D}^{(N)}_{2,-\infty}$, $N=1,2,\dots$ *satisfying*

$$\sup_N\left\|F^N\right\|^2_{\mathbb{D}_{2,s}}<\infty$$

for some $0\le s\le 1$ *and the "stationary property"* (3.3)*. Then*

$$\left\|\mathit{Mart.Err}\left(F^N\right)\right\|^2_{L^2}=O(N^{-s/2})\quad \text{as } N\to\infty.$$

Proof By observing (2.6), we notice that

$$\left\|\mathrm{Mart.Err}\left(F^N\right)\right\|^2_{L^2}=\sum_{l=1}^N\sum_{\substack{k_1+\cdots+k_l=n\\ k_l\ge 2}}\frac{n!}{k_1!\cdots k_l!}(\Delta t)^n\mathbf{E}\left[\partial_1^{k_1}\cdots\partial_l^{k_l}F^N\right]^2$$

for each $n=2,3,\dots$. By the assumption, there is a constant $C>0$ such that

$$\sup_{k_1+\cdots+k_l=n}\mathbf{E}\left[\partial_1^{k_1}\cdots\partial_l^{k_l}F^N\right]^2\le C\frac{n!\left\|J_nF^N\right\|^2_{L^2}}{T^n}$$

for each $n=2,3,\dots$ and $N=1,2,\dots$ and the multinomial theorem yields that

$$\sum_{\substack{k_1+k_2+\cdots+k_l=n\\ k_l\ge 2}}\frac{n!}{k_1!\cdots k_l!}(\Delta t)^n=\left(\frac{lT}{N}\right)^n-\left(\frac{(l-1)T}{N}\right)^n-n\frac{T}{N}\left(\frac{(l-1)T}{N}\right)^{n-1}.$$

Putting them together, we have

$$\begin{aligned}\left\|\mathrm{Mart.Err}\left(F^N\right)\right\|^2_{L^2}&\le C\sum_{n=2}^\infty\left\{1-n\frac{1}{N}\sum_{l=0}^{N-1}\left(\frac{l}{N}\right)^{n-1}\right\}\left\|J_nF^N\right\|^2_{L^2}\\&=CN^{-s}\sum_{n=2}^\infty\frac{N^s}{n^{s-1}}\left\{\frac{1}{n}-\frac{1}{N}\sum_{l=0}^{N-1}\left(\frac{l}{N}\right)^{n-1}\right\}n^s\left\|J_nF^N\right\|^2_{L^2}\end{aligned}\tag{3.4}$$

for each $s\in\mathbb{R}$.

On the other hand, since we have

$$I_{n,N} := \frac{1}{n} - \frac{1}{N}\sum_{l=0}^{N-1}\left(\frac{l}{N}\right)^{n-1} = \sum_{l=0}^{N-1}\int_{l/N}^{(l+1)/N}\left\{x^{n-1} - \left(\frac{l}{N}\right)^{n-1}\right\}\mathrm{d}x > 0,$$

$I_{n,N} \leq 1/n$, and

$$I_{n,N} \leq \frac{1}{N}\sum_{l=0}^{N-1}\left\{\left(\frac{l+1}{N}\right)^{n-1} - \left(\frac{l}{N}\right)^{n-1}\right\} = \frac{1}{N},$$

we notice that

$$I_{n,N} = \left(I_{n,N}\right)^{s}\left(I_{n,N}\right)^{1-s} \leq \left(\frac{1}{N}\right)^{s}\left(\frac{1}{k}\right)^{1-s} \tag{3.5}$$

for every $0 \leq s \leq 1$.

By (3.4) and (3.5), we finally have

$$\left\|\text{Mart.Err}\left(F^N\right)\right\|_{L^2}^2 \leq CN^{-s}\sum_{n=2}^{\infty} n^s \left\|J_n F^N\right\|_{L^2}^2 \leq CN^{-s}\sup_N \left\|F^N\right\|_{\mathbb{D}_{2,s}^{(N)}}^2.$$

□

Remark 3.3 (1) When we consider a sequence of one-dimensional functionals given identically by $F^N \equiv f(W_T)$, this theorem has been already known in Geiss and Hujo [7]. This can be seen as follows: We know from the proof of Theorem 3.3, that the inequality

$$\left\|f\left(W_T\right) - \mathbf{E}\left[f\left(W_T\right)\right] - \sum_{l=1}^{N}\mathbf{E}\left[f'\left(W_T\right)|\mathcal{G}_{l-1}^N\right]\Delta W_l\right\|_{L^2} \leq \frac{\|f\left(W_T\right)\|_{\mathbb{D}_{2,s}}}{N^{s/2}}$$

holds for every $s \in [0, 1]$, regardless of the finiteness of the right-hand side (Recall that the differentiation and conditional expectations are understood in the generalized sense). At the first glance, one difference between Geiss and Hujo [7] and the current work may be the specification of the class of f to get the rate of convergence. Along any equidistant-time partitions, to obtain $O(N^{-s/2})$-convergence, the former's criterion requires that $f \in (\mathbb{D}_{2,1}(\gamma), L^2(\gamma))_{1-s,W}$, which is an interpolation space between $\mathbb{D}_{2,1}(\gamma) = \mathbb{D}_{2,1}^{(1)}$ and $L^2(\gamma) = \mathbb{D}_{2,0}^{(1)}$ (see Definition 1.2 in [7]. Note that our integrability-index p and differentiability-index s in $\mathbb{D}_{p,s}$ are exchanged in [7]). On the other hand, we require $f(W_T) \in \mathbb{D}_{2,s}$, instead of conditions on f. However, in the case of $T = 1$, the above inequality immediately implies $|f(W_1)|_{1-s,W} \leq \|f(W_1)\|_{\mathbb{D}_{2,s}}$ (see Theorem 1.3 in [7] for the notation $|\cdot|_{\eta,X}$). Hence if $f(W_1) \in \mathbb{D}_{2,s}$ then $f \in (\mathbb{D}_{2,1}(\gamma), L^2(\gamma))_{1-s,W}$, which implies the $O(N^{-s/2})$-convergence by Theorem 1.3 in [7].

(2) Although Theorem 3.3 exhibited a clear relation between fractional regularities and strong convergences in the case of "stationary" sequences, a further question arises

naturally: What is the relation between fractional regularities and weak convergences? A partial answer is given in Geiss and Toivola [8, Corollary4.3] in which they connected the two concepts via L^p-integrabilities of normalized error and fractional derivative (based on the Riemann–Liouville operator) of a given Wiener functional.

(3) In the study of the tracking error, there exist several contexts related to backward stochastic differential equations (BSDE). For instance, see Geiss et al. [5] and Geiss and Steinicke [4] where the fractional regularity like $\|Y_t - \mathbf{E}[Y_t|\mathcal{F}_s]\| = O((t-s)^{\theta/2})$ is considered. Here, (Y, Z) is a solution of a BSDE and $(\mathcal{F}_t)$ stands for the filtration generated by the driving noise of the BSDE. At this moment, we cannot say anything about the relationship between such studies and Theorem 3.3. However, our framework of finite-dimensional approximations might potentially be helpful in investigating the stability of BSDE (in its terminal values) and its rate of convergence in the setting of non-smooth terminal values.

3.4 The Cases with Finite-Dimensional Functionals

For a fixed natural number m, we consider finite-dimensional Wiener functionals $F^N \equiv F = F(\Delta^{(m)} W_1, \cdots, \Delta^{(m)} W_m)$. For each natural number N with $N/m \in \mathbb{N}$, we denote $\ell_k := kN/m$ for $k = 0, 1, \cdots, m$, so that we have $t_k^{(m)} = t_{\ell_k}^{(N)}$. Then, for each $n \geq 2$, $r = 0, 1, \cdots, m-1$ and $l = \ell_r + 1, \ell_r + 2, \cdots, \ell_{r+1}$, we have

$$\begin{aligned} &\sum_{\substack{k_1+\cdots+k_l=n \\ k_l \geq 2}} \frac{(\Delta^{(N)} t)^n}{k_1! \cdots k_l!} \mathbf{E}\left[D^{k_1}_{t_1^{(N)}} \cdots D^{k_l}_{t_l^{(N)}} F\right]^2 \\ &\quad = \sum_{\substack{j_1+\cdots+j_{r+1}=n \\ j_{r+1} \geq 2}} \sum_{\substack{k_{\ell_r+1}+\cdots+k_l=j_{r+1} \\ k_l \geq 2}} \frac{j_{r+1}!}{k_{\ell_r+1}! \cdots k_l!} \left(\frac{m}{N}\right)^{j_{r+1}} \mathbf{E}\left[F_{j_1,\ldots,j_{r+1}}\right]^2 \end{aligned} \tag{3.6}$$

where $F_{j_1,\ldots,j_r}$ is defined by

$$F_{j_1,\ldots,j_r} = \mathbf{E}\left[F \prod_{q=1}^{r} H_{j_q}\left(\frac{\Delta^{(m)} W_q}{\sqrt{\Delta^{(m)} t}}\right)\right] \prod_{q=1}^{r} H_{j_q}\left(\frac{\Delta^{(m)} W_q}{\sqrt{\Delta^{(m)} t}}\right). \tag{3.7}$$

In fact, the relation

$$\mathbf{E}\left[D^{k_1}_{t_1^{(N)}} \cdots D^{k_N}_{t_N^{(N)}} F\right]^2 = \mathbf{E}\left[D^{j_1}_{t_1^{(m)}} \cdots D^{j_m}_{t_m^{(m)}} F\right]^2,$$

where $j_r = k_{\ell_r+1} + \cdots + k_{\ell_{r+1}}$, the integration by parts formula (2.8) along the partition $\Delta^{(m)}$, and the multinomial theorem

$$\sum_{k_{\ell_q+1}+\cdots+k_{\ell_{q+1}}=j_{q+1}} \frac{j_{q+1}!}{k_{\ell_q+1}! \cdots k_{\ell_{q+1}}!} = \left(\ell_{q+1} - \ell_q\right)^{j_{q+1}} = \left(\frac{N}{m}\right)^{j_{q+1}}$$

for $q = 0, 1, \ldots r-1$ lead to the inequality (3.3).

In the following statement, "$m|N \to \infty$" means "$N \to \infty$ in $m\mathbb{N}$," where $m\mathbb{N} := \{mn : n \in \mathbb{N}\}$.

Theorem 3.4 *Suppose that $F \in \mathbb{D}^{(m)}_{2,s}$ for some $0 \le s \le 1$. Then, we have*

$$\left\| Mart.Err\left(F^N\right)\right\|_{L^2} = O\left(N^{-s/2}\right) \quad \text{as } m \mid N \to \infty.$$

In fact, we have

$$\left\| Mart.Err\left(F^N\right)\right\|_{L^2} \le \left(\frac{m}{N}\right)^{s/2} \|F\|_{\mathbb{D}_{2,s}} \tag{3.8}$$

for every $0 \le s \le 1$.

Remark 3.4 It might be worth to notice that $(m/N)^{s/2}$ appears rather than $N^{-s/2}$ in the inequality (3.8). We recall that

$$F^N \equiv F = F\left(\Delta^{(m)}W_1, \ldots, \Delta^{(m)}W_m\right).$$

Hence the inequality (3.8) is alluding that the Clark–Ocone scheme Mart.Err gets worse in the speed of convergence when one takes $m = N \to \infty$, that is, for generic sequences approximating truly infinite-dimensional Wiener functionals. However, there is an example of approximating sequence tending to Brownian occupation time, for which the rate of convergence is $O(N^{-1/2})$ as we will see in Sect. 3.6.

Proof By observing (2.6), we notice that

$$\begin{aligned}\left\| \text{Mart.Err}\left(F^N\right)\right\|^2_{L^2} &= \sum_{l=1}^{N}\sum_{n=2}^{\infty}\sum_{\substack{k_1+\cdots+k_l=n\\ k_l\ge 2}} \frac{\left(\Delta^{(N)}t\right)^n}{k_1!\cdots k_l!}\mathbf{E}\left[\partial_1^{k_1}\cdots\partial_l^{k_l}F\right]^2\\ &= \sum_{r=0}^{m-1}\sum_{l=\ell_r+1}^{\ell_{r+1}}\sum_{n=2}^{\infty}\sum_{\substack{k_1+\cdots+k_l=n\\ k_l\ge 2}} \frac{\left(\Delta^{(N)}t\right)^n}{k_1!\cdots k_l!}\mathbf{E}\left[\partial_1^{k_1}\cdots\partial_l^{k_l}F\right]^2\end{aligned}$$

for each $n = 2, 3, \ldots$. Focusing on the last summation and using the Eq. (3.6),

$$\begin{aligned}&\sum_{\substack{k_1+\cdots+k_l=n\\ k_l\ge 2}} \frac{\left(\Delta^{(N)}t\right)^n}{k_1!\cdots k_l!}\mathbf{E}\left[D^{k_1}_{t_1^{(N)}}\cdots D^{k_N}_{t_N^{(N)}}F\right]^2\\ &\quad = \sum_{\substack{j_1+\cdots+j_{r+1}=n\\ j_{r+1}\ge 2}}\sum_{\substack{k_{\ell_r+1}+\cdots+k_l=j_{r+1}\\ k_l\ge 2}} \frac{j_{r+1}!}{k_{\ell_r+1}!\cdots k_l!}\left(\frac{m}{N}\right)^{j_{r+1}}\mathbf{E}\left[F_{j_1,\ldots,j_{r+1}}\right]^2\end{aligned}$$

for each $N, n, r = 0, 1, \ldots, m-1$ and $l = \ell_r + 1, \ldots, \ell_{r+1}$. Then, the multinomial theorem yields that

$$\sum_{\substack{k_{\ell_r+1}+\cdots+k_l=j_{r+1}\\ k_l\geq 2}} \frac{j_{r+1}!}{k_{\ell_r+1}!\cdots k_l!}\left(\frac{m}{N}\right)^{j_{r+1}}$$
$$= \left(\frac{(l-\ell_r)m}{N}\right)^{j_{r+1}} - \left(\frac{(l-\ell_r-1)m}{N}\right)^{j_{r+1}} - j_{r+1}\frac{m}{N}\left(\frac{(l-\ell_r-1)m}{N}\right)^{j_{r+1}-1}.$$

Putting them together, we have

$$\begin{aligned}
&\left\|\text{Mart.Err}\left(F^N\right)\right\|_{L^2}^2\\
&= \sum_{r=0}^{m-1}\sum_{n=2}^{\infty}\sum_{\substack{j_1+\cdots+j_{r+1}=n\\ j_{r+1}\geq 2}}\left\{1 - j_{r+1}\frac{m}{N}\sum_{l=0}^{(N/m)-1}\left(\frac{lm}{N}\right)^{j_{r+1}-1}\right\}\mathbf{E}\left[F_{j_1,\ldots,j_{r+1}}\right]^2\\
&= N^{-s}\sum_{r=0}^{m-1}\sum_{n=2}^{\infty}\sum_{\substack{j_1+\cdots+j_{r+1}=n\\ j_{r+1}\geq 2}}\frac{N^s}{(j_{r+1})^{s-1}}\left(I_{j_{r+1},N}\right)(j_{r+1})^s\,\mathbf{E}\left[F_{j_1,\ldots,j_{r+1}}\right]^2
\end{aligned} \tag{3.9}$$

for each $s \in \mathbb{R}$, where

$$I_{j_{r+1},N} := \frac{1}{j_{r+1}} - \frac{m}{N}\sum_{l=0}^{(N/m)-1}\left(\frac{lm}{N}\right)^{j_{r+1}-1}.$$

On the other hand, since we have

$$I_{j_{r+1},N} = \sum_{l=0}^{(N/m)-1}\int_{l/(N/m)}^{(l+1)/(N/m)}\left\{x^{j_{r+1}-1} - \left(\frac{lm}{N}\right)^{j_{r+1}-1}\right\}\mathrm{d}x > 0,$$

$I_{j_{r+1},N} \leq 1/j_{r+1}$, and

$$I_{j_{r+1},N} \leq \frac{m}{N}\sum_{l=0}^{(N/m)-1}\left\{\left(\frac{(l+1)m}{N}\right)^{j_{r+1}-1} - \left(\frac{lm}{N}\right)^{j_{r+1}-1}\right\} = \frac{m}{N},$$

we notice that

$$I_{j_{r+1},N} = \left(I_{j_{r+1},N}\right)^s\left(I_{j_{r+1},N}\right)^{1-s} \leq \left(\frac{m}{N}\right)^s\left(\frac{1}{j_{r+1}}\right)^{1-s} \tag{3.10}$$

for every $0 \leq s \leq 1$.

By (3.9) and (3.10), we finally have

$$\begin{aligned}\left\|\text{Mart.Err}\left(F^N\right)\right\|_{L^2}^2 &\le \left(\frac{m}{N}\right)^s \sum_{r=0}^{m-1}\sum_{n=2}^{\infty}\sum_{\substack{j_1+\cdots+j_{r+1}=n\\ j_{r+1}\ge 2}} (j_{r+1})^s \mathbf{E}\left[F_{j_1,\dots,j_{r+1}}\right]^2\\ &\le \left(\frac{m}{N}\right)^s \sum_{n=2}^{\infty} n^s \sum_{j_1+\cdots+j_m=n} \mathbf{E}\left[F_{j_1,\dots,j_{r+1}}\right]^2 \le \left(\frac{m}{N}\right)^s \|F\|_{\mathbb{D}_{2,s}}^2 .\end{aligned}$$

□

3.5 A Study on Additive Functionals

In this subsection, we study sequences of "additive functionals,"

$$\begin{aligned}F^N &:= \sum_{i=1}^{N} f_N\left(t_i^{(N)}, W_{t_i^{(N)}}\right)\left(\Delta^{(N)} t\right)\\ &\text{where } f_N\left(t_i^{(N)},\dots\right), i=1,\dots N \text{ is a sequence in } \mathbb{D}_{2,-\infty}^{(1)}.\end{aligned} \tag{3.11}$$

We are interested in the conditions for the sequence to be "stationary" in the sense of (3.3).

We define an index to control the stationarity. Let

$$A_l := \left(\sum_{i=l}^{N} i^{-n/2}\mathbf{E}\left[f_N\left(t_i^{(N)}, W_{t_i^{(N)}}\right) H_n\left(W_{t_i^{(N)}}/\left(t_i^{(N)}\right)^{1/2}\right)\right]\right)^2$$

and

$$\alpha_{N,n}\left(F^N\right) := \begin{cases} 0 & \text{if } \sum_{l=1}^{N} A_l\{l^n-(l-1)^n\}=0,\\ \frac{N^n \sup A_l}{\sum_{l=1}^{N} A_l\{l^n-(l-1)^n\}} & \text{otherwise.}\end{cases}$$

Then, we have the following criterion.

Proposition 3.5 *The sequence* $\{F_N\}$ *of (3.11) is stationary if and only if*

$$\sup_n \sup_N \alpha_{n,N}\left(F_N\right) < \infty.$$

Proof For arbitrary non-negative integers $k_1, \ldots, k_N$ with $k_1+\cdots+k_N = n$, we have

$$\begin{aligned}\mathbf{E}\left[D_{t_1}^{k_1}\cdots D_{t_N}^{k_N}F^N\right] &= \sum_{i=1}^{N} 1_{\{k_{i+1}=\cdots=k_N=0\}}\mathbf{E}\left[f_N^{(n)}(t_i, W_{t_i})\right]\Delta t\\ &= (n!)^{1/2}(\Delta t)^{(2-n)/2}\sum_{i=1}^{N} 1_{\{k_{i+1}=\cdots=k_N=0\}} i^{-n/2}\mathbf{E}\left[f_N\left(t_i, W_{t_i}\right)H_n\left(\frac{W_{t_i}}{\sqrt{t_i}}\right)\right].\end{aligned}$$

If further $k_l \geq 1$ and $k_{l+1} = \cdots k_N = 0$ for some l, then

$$\begin{aligned}\mathbf{E}\left[D_{t_1}^{k_1}\cdots D_{t_l}^{k_l}F^N\right] &= (n!)^{1/2}(\Delta t)^{(2-n)/2}\sum_{i=l}^{N} i^{-n/2}\mathbf{E}\left[f_N\left(t_i, W_{t_i}\right)H_n\left(\frac{W_{t_i}}{\sqrt{t_i}}\right)\right]\\ &= (n!)^{1/2}(\Delta t)^{(2-n)/2}A_l^{1/2}.\end{aligned}$$

Therefore,

$$\sup_{k_1+\cdots+k_N=n}\left(\mathbf{E}\left[D_{t_1}^{k_1}\cdots D_{t_N}^{k_N}F^N\right]\right)^2 = n!(\Delta t)^{(2-n)}\sup_{l=1,\ldots,N}A_l \tag{3.12}$$

On the other hand, we have

$$\begin{aligned}\left\|J_nF^N\right\|_{L^2}^2 &= \sum_{l=1}^{N}\sum_{\substack{k_1+\cdots+k_l=n\\ k_l\geq 1}}\left(\mathbf{E}\left[F^N H_{k_1}\left(\Delta W_1/\sqrt{\Delta t}\right)\cdots H_{k_l}\left(\Delta W_l/\sqrt{\Delta t}\right)\right]\right)^2\\ &= \sum_{l=1}^{N}\sum_{\substack{k_1+\cdots+k_l=n\\ k_l\geq 1}}\frac{(\Delta t)^n}{k_1!\cdots k_l!}\left(\mathbf{E}\left[D_{t_1}^{k_1}\cdots D_{t_l}^{k_l}F^N\right]\right)^2\\ &= \sum_{l=1}^{N}A_l\sum_{\substack{k_1+\cdots+k_l=n\\ k_l\geq 1}}\frac{(\Delta t)^2 n!}{k_1!\cdots k_l!} = (\Delta t)^2\sum_{l=1}^{N}A_l\left\{l^n-(l-1)^n\right\}.\end{aligned} \tag{3.13}$$

Putting (3.12) and (3.13) together, we have

$$\begin{aligned}\sup_{k_1+\cdots+k_N=n}\frac{\left(\mathbf{E}\left[D_{t_1}^{k_1}\cdots D_{t_N}^{k_N}F^N\right]\right)^2}{\left\|J_nF^N\right\|_{L^2}^2} &= \frac{n!}{T^n}\frac{N^n\sup A_l}{\sum_{l=1}^{N}A_l\left\{l^n-(l-1)^n\right\}}\\ &= \frac{n!}{T^n}\alpha_{N,n}\left(F^N\right).\end{aligned}$$

Note that $\| J_n F^N \|_{L^2}^2 = 0$ implies both $\alpha_{N,n}\left(F^N\right) = 0$ and

$$\sup_{k_1+\cdots+k_N=n} \left(\mathbf{E}\left[D_{t_1}^{k_1}\cdots D_{t_N}^{k_N} F^N\right]\right)^2 = 0.$$

□

Corollary 3.6 *If*

$$\sup_N \frac{\sup_l A_l}{\inf_l A_l} < \infty,$$

then $\left\{F^N\right\}$ *is stationary.*

Proof Since

$$\sum_{l=1}^N A_l \left\{l^n - (l-1)^n\right\} \geq \inf_l A_l \sum_{l=1}^N \left\{l^n - (l-1)^n\right\} = N^n \inf_l A_l,$$

we see that

$$\alpha_{n,N}\left(F^N\right) \leq \frac{\sup_l A_l}{\inf_l A_l}.$$

□

3.6 Asymptotic Analysis of the Martingale Representation Error of a Discretization of Brownian Occupation Time

The sequence of Riemann sum approximations

$$F^N := \sum_{i=1}^N 1_{[0,\infty)}\left(W_{t_i}\right)\Delta t, \quad N \in \mathbb{N} \tag{3.14}$$

of the Brownian occupation time $\int_0^T 1_{[0,\infty)}(W_s)\,ds$ is an interesting example where an explicit calculation is possible. We first claim that the sequence is not stationary in the sense of (3.3). Rather, by a direct calculation the martingale representation error of the sequence is proven to be of order $1/2$.

Proposition 3.7 *The index* $\alpha_{n,N}\left(F^N\right)$ *of the sequence* (3.14) *is not bounded.*

Proof First, we observe that

$$\begin{aligned} A_l &= \left(\sum_{i=l}^N i^{-n/2}\mathbf{E}\left[1_{[0,\infty)}\left(W_{t_i}\right) H_n\left(W_{t_i}/\sqrt{t_i}\right)\right]\right)^2 \\ &= \left(\sum_{i=l}^N i^{-n/2} t_i^{1/2} n^{-1/2}\mathbf{E}\left[\delta_0\left(W_{t_i}\right) H_{n-1}\left(W_{t_i}/\sqrt{t_i}\right)\right]\right)^2 \end{aligned}$$

$$= (2\pi n)^{-1} \left(H_{n-1}(0)\right)^2 \left(\sum_{i=l}^{N} i^{-n/2} \right)^2 .$$

Then, we now see that

$$\alpha_{n,N}\left(F^N\right) = \frac{N^n \left(\sum_{i=1}^N i^{-n/2}\right)^2}{\sum_{l=1}^N \left(\sum_{i=l}^N i^{-n/2}\right)^2 \{l^n - (l-1)^n\}}. \tag{3.15}$$

First, we estimate the numerator of (3.15). We let $n \geq 5$. Then

$$\begin{aligned} N^n \left(\sum_{i=1}^{N} i^{-n/2} \right)^2 &= N^2 \left(\sum_{i=1}^{N} \left(\frac{i}{N}\right)^{-n/2} \frac{1}{N} \right)^2 \\ &\geq N^2 \left(\int_{1/N}^{1} x^{-n/2} dx \right)^2 = N^2 \left\{ \frac{2}{n-2} \left(N^{(n-2)/2} - 1 \right) \right\}^2 . \end{aligned} \tag{3.16}$$

Next, the denominator is estimated as follows:

$$\begin{aligned} \sum_{l=1}^{N} \left(\sum_{i=l}^{N} i^{-n/2} \right)^2 \{l^n - (l-1)^n\} &= N^2 \sum_{l=1}^{N} \left(\sum_{i=l}^{N} \left(\frac{i}{N}\right)^{-n/2} \frac{1}{N} \right)^2 \left\{ \left(\frac{l}{N}\right)^n - \left(\frac{l-1}{N}\right)^n \right\} \\ &\leq N^2 \sum_{l=1}^{N} \left(\int_{l/N}^{1} x^{-n/2} dx + \left(\frac{l}{N}\right)^{-n/2} \frac{1}{N} \right)^2 \\ &\quad \times \left\{ \left(\frac{l}{N}\right)^n - \left(\frac{l-1}{N}\right)^n \right\} \\ &\leq N^2 \sum_{l=1}^{N} \left(\frac{2}{n-2} \left\{ \left(\frac{l}{N}\right)^{(2-n)/2} - 1 \right\} + \left(\frac{l}{N}\right)^{-n/2} \frac{1}{N} \right)^2 \\ &\quad \times \left\{ \left(\frac{l}{N}\right)^n - \left(\frac{l-1}{N}\right)^n \right\} \\ &= N^2 \left\{ J_1^N + J_2^N + J_3^N \right\} \end{aligned}$$

where

$$\begin{aligned} J_1^N &:= (n-2)^{-2} \sum_{l=1}^{N} \left\{ \left(\frac{l}{N}\right)^{(2-n)/2} - 1 \right\}^2 \left\{ \left(\frac{l}{N}\right)^n - \left(\frac{l-1}{N}\right)^n \right\}, \\ J_2^N &:= \frac{2(n-2)^{-1}}{N} \sum_{l=1}^{N} \left\{ \left(\frac{l}{N}\right)^{1-n} - \left(\frac{l}{N}\right)^{-n/2} \right\} \left\{ \left(\frac{l}{N}\right)^n - \left(\frac{l-1}{N}\right)^n \right\} \end{aligned}$$

and

$$J_3^N := \frac{1}{N^2}\sum_{l=1}^{N}\left(\frac{l}{N}\right)^{-n}\left\{\left(\frac{l}{N}\right)^{n}-\left(\frac{l-1}{N}\right)^{n}\right\}.$$

It is easy to see that $\sup_N J_2^N < \infty$ and $\lim_{N\to\infty} J_3^N = 0$. Since J_1^N behaves like

$$(n-2)^{-2}\int_0^1\left\{x^{(2-n)/2}-1\right\}^2 nx^{n-1}\mathrm{d}x < \infty$$

as $N \to \infty$, it is also seen that $\sup_N J_1^N < \infty$. Therefore, there is a constant C_n independent of N but possibly dependent on n such that

$$\sum_{l=1}^{N}\left(\sum_{i=l}^{N} i^{-n/2}\right)^2\{l^n-(l-1)^n\} \le N^2 C_n. \tag{3.17}$$

From (3.16) and (3.17), we see that $\sup_N \alpha_{n,N} = \infty$. □

Our main result in this subsection is the following.

Theorem 3.8 *It holds that*

$$\left\|Mart.Err\left(F^N\right)\right\|_{L^2} = O\left(N^{-1/2}\right).$$

Proof By Theorem 2.1, we have

$$\begin{aligned}\left\|\mathrm{Mart.Err}\left(F^N\right)\right\|_{L^2}^2 &= \sum_{l=1}^{N}\sum_{k=2}^{\infty}\mathbf{E}\left[\mathbf{E}\left[\sum_{i=1}^{N}1_{[0,\infty)}\left(W_{t_i}\right)\Delta t\, H_k\left(\frac{\Delta W_l^N}{\sqrt{\Delta t}}\right)\Big|\mathcal{G}_{l-1}^N\right]^2\right]\\ &= \sum_{l=1}^{N}\sum_{k=2}^{\infty}\frac{(\Delta t)^k}{k!}\mathbf{E}\left[\mathbf{E}\left[\sum_{i=l}^{N}1_{[0,\infty)}^{(k)}\left(W_{t_i}\right)\Delta t\Big|\mathcal{G}_{l-1}^N\right]^2\right].\end{aligned} \tag{3.18}$$

For $l \ge 2$, by the Hermite expansion in $L^2(\mathbf{R}, \mu_{t_{l-1}})$,

$$\mathbf{E}\left[\sum_{i=l}^{N}1_{[0,\infty)}^{(k)}\left(W_{t_i}\right)\Delta t\Big|\mathcal{G}_{l-1}^N\right]=\sum_{n=0}^{\infty}\frac{(t_{l-1})^{n/2}}{\sqrt{n!}}\mathbf{E}\left[\sum_{i=l}^{N}1_{[0,\infty)}^{(n+k)}\left(W_{t_i}\right)\Delta t\right]H_n\left(\frac{W_{t_{l-1}}}{\sqrt{t_{l-1}}}\right),$$

and by Parseval's identity we have

$$\mathbf{E}\left[\mathbf{E}\left[\sum_{i=l}^{N}1_{[0,\infty)}^{(k)}\left(W_{t_i}\right)\Delta t\Big|\mathcal{G}_{l-1}^N\right]^2\right] = \sum_{n=0}^{\infty}\frac{(t_{l-1})^n}{n!}\mathbf{E}\left[\sum_{i=l}^{N}1_{[0,\infty)}^{(n+k)}\left(W_{t_i}\right)\Delta t\right]^2. \tag{3.19}$$

Note that (3.19) is also valid for $l = 1$ with the conventions $t_0 = 0$ and $t_0^0 = 1$. Plugging (3.19) into (3.18), we have

$$\left\|\text{Mart.Err}\left(F^N\right)\right\|_{L^2}^2 = \sum_{l=1}^{N}\sum_{k=2}^{\infty}\sum_{n=0}^{\infty}\frac{(\Delta t)^k}{k!}\frac{(t_{l-1})^n}{n!}\mathbf{E}\left[\sum_{i=l}^{N}1_{[0,\infty)}^{(n+k)}\left(W_{t_i}\right)\Delta t\right]^2.$$

By the renumbering $(n+k, n) \mapsto (k, n)$, we have

$$\left\|\text{Mart.Err}\left(F^N\right)\right\|_{L^2}^2 = \sum_{l=1}^{N}\sum_{k=2}^{\infty}\frac{1}{k!}\mathbf{E}\left[\sum_{i=l}^{N}1_{[0,\infty)}^{(k)}\left(W_{t_i}\right)\Delta t\right]^2$$
$$\sum_{n=0}^{k-2}\frac{k!}{(k-n)!n!}(\Delta t)^k\,(t_{l-1})^n,$$

by keeping the conventions on t_0. With a use of the binomial theorem,

$$\left\|\text{Mart.Err}\left(F^N\right)\right\|_{L^2}^2 = \sum_{l=1}^{N}\sum_{k=2}^{\infty}\frac{1}{k!}\mathbf{E}\left[\sum_{i=l}^{N}1_{[0,\infty)}^{(k)}\left(W_{t_i}\right)\Delta t\right]^2 \times\left\{(t_l)^k-(t_{l-1})^k-k\,(\Delta t)\,(t_{l-1})^{k-1}\right\}.$$

Then, on the one hand, for $l \geq 1$ and $k \geq 2$,

$$\begin{aligned}\mathbf{E}\left[\sum_{i=l}^{N}1_{[0,\infty)}^{(k)}\left(W_{t_{i-1}}\right)\Delta t\right]^2 &= \left\{\sum_{i=l}^{N}\frac{\sqrt{(k-1)!}}{(t_i)^{\frac{k-1}{2}}}\mathbf{E}\left[\delta_0\left(W_{t_i}\right)H_{k-1}\left(\frac{W_{t_i}}{\sqrt{t_i}}\right)\right]\Delta t\right\}^2\\ &= \left\{\sum_{i=l}^{N}\frac{\sqrt{(k-1)!}}{(t_i)^{\frac{k-1}{2}}}H_{k-1}(0)\frac{1}{\sqrt{2\pi t_i}}\Delta t\right\}^2\\ &= k!\cdot\frac{H_{k-1}(0)^2}{2\pi k}\left\{\sum_{i=l}^{N}\frac{\Delta t}{(t_i)^{n/2}}\right\}^2.\end{aligned}$$

By a similar argument, we find

$$\mathbf{E}\left[1_{[0,\infty)}(W_T)H_k\left(\frac{W_T}{\sqrt{T}}\right)\right] = \frac{H_{k-1}(0)}{\sqrt{2\pi k}}$$

and therefore

$$\left\|\text{Mart.Err}\left(F^N\right)\right\|_{L^2}^2 = \sum_{k=2}^{\infty}Z_{N,k}\mathbf{E}\left[1_{[0,\infty)}(W_T)H_k\left(\frac{W_T}{\sqrt{T}}\right)\right]^2$$

where

$$Z_{N,k} := \sum_{l=1}^{N} \left\{ \sum_{i=l}^{N} \frac{\Delta t}{(t_i)^{k/2}} \right\}^2 \left\{ (t_l)^k - (t_{l-1})^k - k(\Delta t)\,(t_{l-1})^{k-1} \right\}. \tag{3.20}$$

On the other hand, by Lemma 3.9 below, we know that there exists a constant $K > 0$ such that

$$Z_{N,k} \leq \frac{K}{N}$$

for each $k = 2, 3, \ldots$ and $N = 3, 4, \ldots$. Hence we have

$$\left\| \text{Mart.Err}\left(F^N\right) \right\|_{L^2}^2 \leq \frac{2K}{N} \left\| 1_{[0,\infty)}(W_T) \right\|_{L^2}^2 .$$

□

Lemma 3.9 *For $k \geq 2$, it holds that*

$$Z_{N,k} \leq \frac{9T^2}{N}. \tag{3.21}$$

where $Z_{N,k}$ is given as above in (3.20).

Proof We may write

$$\begin{aligned} Z_{N,k} = &\sum_{l=1}^{N} \left[\left\{ \sum_{i=l}^{N} \left(\frac{t_l}{t_i} \right)^{k/2} \Delta t \right\}^2 - \left\{ \sum_{i=l}^{N} \left(\frac{t_{l-1}}{t_i} \right)^{k/2} \Delta t \right\}^2 \right] \\ &- k \sum_{l=1}^{N} \left\{ \sum_{i=l}^{N} \frac{(t_{l-1})^{(k-1)/2}}{(t_i)^{k/2}} \Delta t \right\}^2 \Delta t. \end{aligned}$$

For $l \geq 2$, we have

$$\left\{ \sum_{i=l}^{N} \left(\frac{t_{l-1}}{t_i} \right)^{k/2} \Delta t \right\}^2 = \left\{ \sum_{i=l-1}^{N} \left(\frac{t_{l-1}}{t_i} \right)^{k/2} \Delta t \right\}^2 - 2 \sum_{i=l-1}^{N} \left(\frac{t_{l-1}}{t_i} \right)^{k/2} (\Delta t)^2 + (\Delta t)^2,$$

and therefore,

$$\begin{aligned} &\sum_{l=2}^{N} \left[\left\{ \sum_{i=l}^{N} \left(\frac{t_l}{t_i} \right)^{k/2} \Delta t \right\}^2 - \left\{ \sum_{i=l}^{N} \left(\frac{t_{l-1}}{t_i} \right)^{k/2} \Delta t \right\}^2 \right] \\ &= \sum_{l=2}^{N} \left[\left\{ \sum_{i=l}^{N} \left(\frac{t_l}{t_i} \right)^{k/2} \Delta t \right\}^2 - \left\{ \sum_{i=l-1}^{N} \left(\frac{t_{l-1}}{t_i} \right)^{k/2} \Delta t \right\}^2 \right] \end{aligned}$$

$$+2\sum_{l=2}^{N}\sum_{i=l-1}^{N}\left(\frac{t_{l-1}}{t_i}\right)^{k/2}(\Delta t)^2 - N(\Delta t)^2.$$

Using this,

$$\begin{aligned} Z_{N,k} &= (\Delta t)^2 + 2\sum_{l=2}^{N}\sum_{i=l-1}^{N}\left(\frac{t_{l-1}}{t_i}\right)^{n/2}(\Delta t)^2 \\ &\quad - N(\Delta t)^2 - k\sum_{l=2}^{N}\left\{\sum_{i=l}^{N}\frac{(t_{l-1})^{(k-1)/2}}{(t_i)^{k/2}}\Delta t\right\}^2 \Delta t \\ &\le 2\sum_{l=2}^{N}\sum_{i=l-1}^{N}\left(\frac{t_{l-1}}{t_i}\right)^{k/2}(\Delta t)^2 - k\sum_{l=1}^{N}\left\{\sum_{i=l}^{N}\frac{(t_{l-1})^{(k-1)/2}}{(t_i)^{k/2}}\Delta t\right\}^2 \Delta t. \end{aligned} \tag{3.22}$$

We observe that

$$2\sum_{l=2}^{N}\sum_{i=l-1}^{N}\left(\frac{t_{l-1}}{t_i}\right)^{k/2}(\Delta t)^2$$

behaves like

$$2\int_0^T\int_t^T\left(\frac{t}{s}\right)^{k/2}\mathrm{d}s\mathrm{d}t$$

and

$$k\sum_{l=1}^{N}\left\{\sum_{i=l}^{N}\frac{(t_{l-1})^{(k-1)/2}}{(t_i)^{k/2}}\Delta t\right\}^2 \Delta t$$

behaves like

$$k\int_0^T\left\{\int_t^T\frac{t^{(k-1)/2}}{s^{k/2}}\mathrm{d}s\right\}^2\mathrm{d}t$$

as $N \to \infty$, respectively. We note that

$$2\int_0^T\int_t^T\left(\frac{t}{s}\right)^{k/2}\mathrm{d}s\mathrm{d}t = n\int_0^T\left\{\int_t^T\frac{t^{(k-1)/2}}{s^{k/2}}\mathrm{d}s\right\}^2\mathrm{d}t = \begin{cases} \dfrac{T^2}{2} & \text{if } k = 2, \\ \dfrac{2T^2}{k+2} & \text{if } k \ge 2. \end{cases}$$

Based on the observations, we estimate $Z_{N,k}$ by separating it into two terms;

$$Z_{N,k} \le Z^1_{N,k} + Z^2_{N,k}$$

where

$$Z_{N,k}^1 := 2\sum_{l=2}^{N}\sum_{i=l-1}^{N}\left(\frac{t_{l-1}}{t_i}\right)^{k/2}(\Delta t)^2 - 2\int_0^T\int_t^T\left(\frac{t}{s}\right)^{k/2}\mathrm{d}s\mathrm{d}t,$$

$$Z_{N,k}^2 := k\int_0^T\left\{\int_t^T\frac{t^{(k-1)/2}}{s^{k/2}}\mathrm{d}s\right\}^2\mathrm{d}t - k\sum_{l=1}^{N}\left\{\sum_{i=l}^{N}\frac{(t_{l-1})^{(k-1)/2}}{(t_i)^{k/2}}\Delta t\right\}^2\Delta t.$$

We estimate each of them. Firstly, we have

$$\begin{aligned}
Z_{N,n}^1 &\leq 2\sum_{l=2}^{N}\sum_{i=l-1}^{N-1}\int_{t_{l-2}}^{t_{l-1}}\int_{t_i}^{t_{i+1}}\left\{\left(\frac{t_{l-1}}{t_i}\right)^{k/2} - \left(\frac{t}{s}\right)^{k/2}\right\}\mathrm{d}s\mathrm{d}t \\
&\quad + 2\sum_{l=2}^{N}\left(\frac{t_{l-1}}{t_N}\right)^{k/2}(\Delta t)^2 \\
&\leq 2\sum_{l=2}^{N}\sum_{i=l-1}^{N-1}\int_{t_{l-2}}^{t_{l-1}}\int_{t_i}^{t_{i+1}}\left\{\left(\frac{t_{l-1}}{t_i}\right)^{k/2} - \left(\frac{t_{l-2}}{t_{i+1}}\right)^{k/2}\right\}\mathrm{d}s\mathrm{d}t \\
&\quad + 2\sum_{l=2}^{N}\left(\frac{t_{l-1}}{t_N}\right)^{k/2} \qquad (3.23)\\
&= 2(\Delta t)^2\sum_{l=2}^{N}\sum_{i=l-1}^{N-1}\left\{\left(\frac{l-1}{i}\right)^{k/2} - \left(\frac{l-2}{i}\right)^{k/2}\right\} \\
&\quad + 2(\Delta t)^2\sum_{l=2}^{N}\sum_{i=l-1}^{N-1}\left\{\left(\frac{l-2}{i}\right)^{k/2} - \left(\frac{l-2}{i+1}\right)^{k/2}\right\} \\
&\quad + 2\sum_{l=2}^{N}\left(\frac{t_{l-1}}{t_N}\right)^{k/2}(\Delta t)^2.
\end{aligned}$$

By a bit of algebra, the last term in (3.23) is seen to be

$$2(\Delta t)^2\sum_{l=2}^{N}\left\{1 + \left(\frac{l-1}{l}\right)^{k/2}\right\}, \qquad (3.24)$$

which is bounded above by $4T^2/N$.

Next, we estimate $Z_{N,k}^2$. We set

$$I = \sum_{l=1}^{N}\int_{t_{l-1}}^{t_l}t^{k-1}\left\{\int_t^T\frac{ds}{s^{k/2}}\right\}^2\mathrm{d}t - \sum_{l=1}^{N}\int_{t_{l-1}}^{t_l}t^{k-1}\left\{\sum_{i=l}^{N}\frac{\Delta t}{(t_i)^{k/2}}\right\}^2\mathrm{d}t$$

and

$$\mathbb{I} = \sum_{l=1}^{N} \int_{t_{l-1}}^{t_l} t^{k-1} \left\{ \sum_{i=l}^{N} \frac{\Delta t}{(t_i)^{k/2}} \right\}^2 \mathrm{d}t - \sum_{l=1}^{N} \int_{t_{l-1}}^{t_l} (t_{l-1})^{k-1} \left\{ \sum_{i=l}^{N} \frac{\Delta t}{(t_i)^{k/2}} \right\}^2 \mathrm{d}t.$$

Note that $Z_{N,k}^2 = k(I + \mathbb{I})$. For $t_{l-1} \le t \le t_l$, $l = 1, \dots, N$, we have

$$\int_t^T \frac{\mathrm{d}s}{s^{k/2}} - \sum_{i=l}^{N} \frac{\Delta t}{(t_i)^{k/2}} = \sum_{i=l+1}^{N} \int_{t_{i-1}}^{t_i} \left(\frac{1}{s^{k/2}} - \frac{1}{(t_i)^{k/2}} \right) \mathrm{d}s + \int_t^{t_l} \frac{\mathrm{d}s}{s^{k/2}} - \frac{\Delta t}{(t_l)^{k/2}} \ge 0, \tag{3.25}$$

and

$$\begin{aligned} \sum_{i=l+1}^{N} \int_{t_{i-1}}^{t_i} \left(\frac{1}{s^{k/2}} - \frac{1}{(t_i)^{k/2}} \right) \mathrm{d}s &\le \sum_{i=l+1}^{N} \int_{t_{i-1}}^{t_i} \left(\frac{1}{(t_{i-1})^{k/2}} - \frac{1}{(t_i)^{k/2}} \right) \mathrm{d}s \\ &= \Delta t \left(\frac{1}{(t_l)^{k/2}} - \frac{1}{(t_N)^{k/2}} \right). \end{aligned}$$

Combining these two, we have

$$\begin{aligned} &\left\{ \int_t^T \frac{\mathrm{d}s}{s^{k/2}} \right\}^2 - \left\{ \sum_{i=l}^{N} \frac{\Delta t}{(t_i)^{k/2}} \right\}^2 \\ &\quad \le \int_t^{t_l} \frac{\mathrm{d}s}{s^{k/2}} \left(\int_t^T \frac{\mathrm{d}s}{s^{k/2}} + \sum_{i=l}^{N} \frac{\Delta t}{(t_i)^{k/2}} \right) \le 2 \int_t^{t_l} \frac{\mathrm{d}s}{s^{k/2}} \left(\int_t^T \frac{\mathrm{d}s}{s^{k/2}} \right) \\ &\quad = \begin{cases} \dfrac{4}{k-2} (t^{1-\frac{k}{2}} - T^{1-\frac{k}{2}}) \displaystyle\int_t^{t_l} \frac{\mathrm{d}s}{s^{k/2}} \le \frac{4}{k-2} t^{1-\frac{k}{2}} \int_t^{t_l} \frac{\mathrm{d}s}{s^{k/2}} & \text{if } k \ge 3, \\ 2 \displaystyle\int_t^{t_l} \frac{\mathrm{d}s}{s} \log \frac{T}{t} & \text{if } k = 2. \end{cases} \end{aligned}$$

Then for $k \ge 3$,

$$\begin{aligned} I &\le \frac{4}{k-2} \sum_{l=1}^{N} \int_{t_{l-1}}^{t_l} \int_t^{t_l} \left(\frac{t}{s} \right)^{k/2} \mathrm{d}s\mathrm{d}t \le \frac{4}{k-2} \sum_{l=1}^{N} \int_{t_{l-1}}^{t_l} \int_t^{t_l} \mathrm{d}s\mathrm{d}t \\ &= \frac{2}{k-2} \sum_{l=1}^{N} (t_l - t_{l-1})^2 = \frac{2}{k-2} \frac{T^2}{N} \end{aligned} \tag{3.26}$$

and for $k = 2$, we have

$$I \le 2 \sum_{l=1}^{N} \int_{t_{l-1}}^{t_l} \int_t^{t_l} \frac{t}{s} \log \frac{T}{t} \mathrm{d}s\mathrm{d}t \le 2 \sum_{l=1}^{N} \int_{t_{l-1}}^{t_l} \int_t^{t_l} \mathrm{d}s \ \log \frac{T}{t} \ \mathrm{d}t$$

$$\leq 2\Delta t \sum_{l=1}^{N} \left\{ \Delta t \log T - \left[t \log t - t \right]_{t=t_{l-1}+0}^{t=t_l} \right\} = \frac{2T^2}{N}. \tag{3.27}$$

Now we turn to the estimate of $\mathbb{I}$. By (3.25), for $k \geq 3$,

$$\begin{aligned} \mathbb{I} &\leq \sum_{l=1}^{N} \int_{t_{l-1}}^{t_l} \left\{ t^{k-1} - (t_{l-1})^{k-1} \right\} \left(\int_t^T \frac{\mathrm{d}s}{s^{k/2}} \right)^2 \mathrm{d}t \\ &\leq \frac{4}{(k-2)^2} \sum_{l=1}^{N} \int_{t_{l-1}}^{t_l} \left\{ t^{k-1} - (t_{l-1})^{k-1} \right\} t^{2-k} \mathrm{d}t \\ &= \frac{4(k-1)}{(k-2)^2} \sum_{l=1}^{N} \int_{t_{l-1}}^{t_l} \int_{t_{l-1}}^{t} \left(\frac{s}{t} \right)^{k-2} \mathrm{d}s \mathrm{d}t \\ &\leq \frac{4(k-1)}{(k-2)^2} \sum_{l=1}^{N} \int_{t_{l-1}}^{t_l} \int_{t_{l-1}}^{t} \mathrm{d}s \mathrm{d}t = \frac{2(k-1)}{(k-2)^2} \frac{T^2}{N}. \end{aligned} \tag{3.28}$$

For $k = 2$, we have

$$\begin{aligned} \mathbb{I} &\leq \sum_{l=1}^{N} \int_{t_{l-1}}^{t_l} (t - t_{l-1}) \left(\int_t^T \frac{\mathrm{d}s}{s} \right)^2 \mathrm{d}t \\ &\leq \Delta t \sum_{l=1}^{N} \int_{t_{l-1}}^{t_l} \left(\log \frac{T}{t} \right)^2 \mathrm{d}t = \frac{2T^2}{N}. \end{aligned} \tag{3.29}$$

By (3.26), (3.27), (3.28) and (3.29), we have

$$Z_{N,k}^2 \leq \frac{5T^2}{N}. \tag{3.30}$$

Combining (3.24) and (3.30), we obtained (3.21). □

Remark 3.5 A result by Ngo and Ogawa ([14], Theorem 2.2.) tells us that the sequence of processes

$$\left\{ N^{3/4} \left(\frac{1}{N} \sum_{i=0}^{[Nt]} 1_{[0,\infty)} \left(X_{i/N} \right) - \int_0^t 1_{[0,\infty)} (X_s) \mathrm{d}s \right) \right\}_{t \geq 0}$$

is tight for a diffusion $X = (X_t)_{t \geq 0}$ although their results are more general. Moreover, they say that this is optimal in L^2-sense in the case where X is the standard Brownian motion (see [14], Proposition 2.3).

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