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Navier–Stokes Equations on Weighted Graphs

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Abstract Navier–Stokes equations arise in the study of incompressible fluid mechanics, star movement inside a galaxy, dynamics of airplane wings, etc. In the case of Newtonian incompressible fluids, we propose an adaptation of such equations to finite connected weighted graphs such that it produces an ordinary differential equation with solutions contained in a linear subspace, this subspace corresponding to the Newtonian conservation law. We discuss the particular case when the graph is the complete graph K_m, with constant weight, and provide a necessary and sufficient condition for it to have solutions.

Keywords Graphs · Partial difference equations · Nonlinear elliptice quations · Laplacian · Navier–Stokes

Mathematics Subject Classification (2000) 05C12 · 39A12 · 35J05

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1 Introduction

Partial differential equations appear naturally when describing some important physical processes. The usual examples in the linear setting being the Laplace, wave and heat equations; while in the non-linear case, *Navier–Stokes equations* arise as the most important paradigm of mathematical physics as they are used in the study of incompressible fluid mechanics, star movement inside a galaxy, dynamics of airplane wings, etc. Navier–Stokes equations are posed on a region $\Omega \subset \mathbb{R}^n$, for $n \geq 2$,

$$\rho\left(\frac{\partial \nu}{\partial t} + \nu \cdot \nabla \nu\right) = -\nabla p + \mu \Delta \nu + f \tag{1}$$

where, if $\nu = (\nu_1, \ldots, \nu_n)$, then

$$\begin{aligned} \nu \cdot \nabla \nu &:= (\nu \cdot \nabla \nu_1, \ldots, \nu \cdot \nabla \nu_n) \\ \Delta \nu &:= (\Delta \nu_1, \ldots, \Delta \nu_n). \end{aligned}$$

In the above, the constant μ represents the dynamic viscosity, $\rho = \rho(x, t)$ is a real function (usually positive) depending on the time variable t and space variable x representing the *density of the fluid*, $\nu = \nu(x, t)$ represents the *incompressible fluid speed*, $p = p(x, t)$ represents the *pressure* and $f(x, t)$ represents an *external force*; the factor $\nu \cdot \nabla \nu$ is called the *convection acceleration* and $\mu \Delta \nu$ the *fluid viscosity*. The fluid is called *Newtonian* if it conserves its mass, that is, it satisfies the *Newton's conservation equation*

$$\nabla \cdot \nu = \sum_{j=1}^{n} \frac{\partial \nu_j}{\partial x_j} = 0 \tag{2}$$

where $x = (x_1, \ldots, x_n)$. One of the greatest problems in mathematics is the existence and smoothness of the solutions of Navier–Stokes equations. A good source for generalities and results on Navier–Stokes equations is [16].

In [12] has been noticed the importance of direct numerical simulations of Navier–Stokes equations as a tool in turbulence research. Numerical methods to obtain solutions of (1) using rectangular and cylindrical geometries have been obtained in [14], and in [7] the finite element method is used to study Navier–Stokes Equations. Another classical method, to provide numerical solutions of partial differential equations, corresponds to partial difference equations. Historically one can trace back the use of partial difference equations to the seminal paper by Courant et al. [6]. In that article, the authors introduce the finite differences method as a convenient way of dealing numerically with partial differential equations. From the introduction of the aforementioned paper we can see the insight of their philosophy: "Problems involving the classical linear partial differential equations of mathematical physics can be reduced to algebraic ones of a very much simpler structure by replacing the differentials by difference quotients on some (say rectilinear) mesh." In recent papers, e.g.

[8,9,13], the study of partial difference equations has appeared as a subject on its own, dealing with problems of existence and qualitative behavior of solutions.

In this paper, based on general ideas of the partial difference equations method, we propose an adaptation of Navier–Stokes equations (the case of Newtonian incompressible fluids) to finite connected weighted graphs. Such an adaptation is an ordinary differential equation with solutions contained in a linear subspace, this subspace corresponding to the Newtonian conservation law. As an example, we discuss the particular case when the graph is either the complete graph K_m or the cyclic graph C_m, with constant weight, and provide a necessary and sufficient condition to have solutions of the adapted Navier–Stokes equation. We hope that the proposed approach may be of use for numerical testing and searching of solutions of the classical Navier–Stokes equations on Riemannian manifolds. At this point, we should note that, in order to use our adaptation to find solutions of Navier–Stokes equations, one has to be careful in the choice of the graphs to be used (see Sect. 5).

This paper is organized as follows. In Sect. 2 we recall some preliminaries on partial derivatives on weighted graphs. In Sect. 3 we describe our adaptation of Navier–Stokes equation to the case of a finite connected weighted graph. In Sect. 4 we discuss the particular case when the graph is either the complete graph K_m or the cyclic graph C_m. In Sect. 5 we discuss some partial ideas to use our adaptation to graphs to discuss classical Navier–Stokes equations in regions $\Omega \subset \mathbb{R}^n$ (we only describe the planar case in order to simplify).

2 Derivation in Graphs

For our purposes, all graphs under consideration will be finite, simple and connected. A graph will be denoted as $\mathcal{G} = (V, E)$, where V denotes its set of vertices and E its set of edges.

2.1 Derivations

To each vertex $v \in V$ there is associated the set $N(v) \subset V$, called the *neighborhood* of v, so that $w \in N(v)$ if and only if $\{v, w\} \in E$; w is called a *neighbor* of v. A good source on graphs is, for instance, the book [2]. Associated to the graph $\mathcal{G}$ is the finite dimensional real vector space $C^0(\mathcal{G})$ consisting of the real functions defined on V. A *derivation* on the graph $\mathcal{G}$ is a linear operator $D : C^0(\mathcal{G}) \to C^0(\mathcal{G})$. The set of derivations on $\mathcal{G}$ is a finite dimensional real algebra $\Xi_{\mathbb{R}}(\mathcal{G})$. A *vector field* on the graph $\mathcal{G}$ is a function $X : V \to V$, so that $X(v) \in N(v) \cup \{v\}$. If $v \in V$ and $w \in N(v)$, then an example of a vector field is the following

$$X_{v,w}(\tau) = \begin{cases} \tau, & \tau \neq v \\ w, & \tau = v. \end{cases} \tag{3}$$

2.2 Weighted Graphs

A *weight* on the graph $\mathcal{G}$ is any positive function $d : E \to (0, +\infty)$. We say that the pair $(\mathcal{G}, d)$ is a *weighted graph*. The connectivity of $\mathcal{G}$ ensure that the weight d

defines a metric space structure on V as follows. If $v, v' \in V$, then, by the connectivity, we may choose a finite collection of vertices $v = w_1, w_2, \ldots, w_n = v' \in V$ so that $\{w_j, w_{j+1}\} \in E$. The weight d permits one to define the length of such a path, say $\sum_{j=1}^{n-1} d(\{w_j, w_{j+1}\})$. The infimum of such lengths is positive for $v \neq v'$, and it provides a metric on V.

If $w \in N(v) \cup \{v\}$, for some $v \in V$ fixed, and $\mu \in C^0(\mathcal{G})$, then the *directional derivative* of μ in v in the direction of w is defined by

$$\partial_w \mu(v) = \begin{cases} \dfrac{\mu(w) - \mu(v)}{d(\{v, w\})}, & v \neq w \\ 0, & v = w. \end{cases} \tag{4}$$

Each vector field X defines a derivation D_X on $\mathcal{G}$ by the rule $D_X(\mu)(v) = \partial_{X(v)} \mu(v)$, for $v \in V$ and $\mu \in C^0(\mathcal{G})$. For instance, $D_{X_{v,w}} \mu(\tau) = 0$ if $\tau \neq v$ and $D_{X_{v,w}} \mu(v) = \partial_w \mu(v)$.

Remark 1 Observe that the derivations arising from vector fields are the natural analogue of differential operators of order one, in the sense that they satisfy a Leibniz-type rule. More explicitly, let $X : V \to V$ be a vector field, let D_X be the induced derivation and consider two arbitrary functions $f, g \in C^0(\mathcal{G})$, then for any vertex $v \in V$ and assuming $X(v) \neq v$

$$\begin{cases} D_X(f \cdot g)(v) = \partial_{X(v)}(f \cdot g)(v) = \dfrac{(f \cdot g)(X(v)) - (f \cdot g)(v)}{d(X(v), v)} \\ \qquad = \dfrac{f(X(v)) \cdot g(X(v)) - f(X(v))g(v) + f(X(v))g(v) - f(v)g(v)}{d(X(v), v)} \\ \qquad = f(X(v)) \cdot \partial_{X(v)} g(v) + g(v) \partial_{X(v)} f(v) \\ \qquad = g(X(v)) \cdot \partial_{X(v)} f(v) + f(v) \partial_{X(v)} g(v). \end{cases} \tag{5}$$

In case that $X(v) = v$, then simply $D_X(f \cdot g)(v) = 0$.

2.3 Discrete Laplace Operator

Let $(\mathcal{G}, d)$ be a finite simple connected weighted graph and set $V = \{v_1, \ldots, v_m\}$. Set $J_k = B_k - A_k$, where $A_k = [a_{ij}(k)]$, $B = [b_{ij}(k)]$, and

$$a_{ij}(k) = \begin{cases} d(\{v_i, v_j\})^{-k}, & \{v_i, v_j\} \in E \\ 0, & \{v_i, v_j\} \notin E \end{cases}$$
$$b_{ij}(k) = \begin{cases} \sum_{r=1}^{m} a_{jr}(k), & i = j \\ 0, & i \neq j. \end{cases}$$

The matrices A_k and B_k are called, respectively, the *adjacency* and *valency* matrix of $(\mathcal{G}, d)$ of degree k. By the definition, J_k is a symmetric positive semi-definite matrix; J_2 is usually called the *Laplacian matrix*. Good sources on the Laplacian matrix and its spectrum are, for instance, [1,3–5,11,15].

In this case, $C^0(\mathcal{G})$ is an m-dimensional real vector space. A canonical basis of $C^0(\mathcal{G})$ is given by $\{\mu_1, \ldots, \mu_m\}$, where $\mu_j(v_j) = 1$ and $\mu_j(v_k) = 0$ for $k \neq j$. By the natural identification of $\mu \in C^0(\mathcal{G})$ with $(\mu(v_1), \ldots, \mu(v_m)) \in \mathbb{R}^m$, a derivation D corresponds to a $m \times m$ real matrix $[D]$. We say that D is *symmetric* if $[D]$ is symmetric and, moreover, that D is *positive semi-definite* if $[D]$ is positive semi-definite. The real algebra $\Xi_{\mathbb{R}}(\mathcal{G})$ of derivations on $\mathcal{G}$ has dimension m^2. Some examples of derivations (on the above weighted graph) are provided by

$$\Delta_k := \sum_{v \in V} \sum_{w \in N(v)} D^k_{X_{v,w}}, \tag{6}$$

which correspond to the discretization of the classical operator

$$\Delta_k f(x) = \sum_{j=1}^{m} \frac{\partial^k f}{\partial x_j^k}(x), \tag{7}$$

where $f : \Omega \subset \mathbb{R}^m \to \mathbb{R}$ has kth order partial derivatives. If we set

$$\partial^k_w \mu(v) = \begin{cases} (-1)^k \dfrac{\mu(v) - \mu(w)}{d(\{v, w\})^k}, & v \neq w \\ 0, & v = w \end{cases} \tag{8}$$

then

$$\Delta_k \mu(v) = \sum_{w \in N(v)} \partial^k_w \mu(v). \tag{9}$$

It follows that Δ_k is represented by the matrix $(-1)^k J_k$, once we have identified each $\mu \in C^0(\mathcal{G})$ with the vector ${}^t[\mu(v_1) \cdots \mu(v_m)]$. The derivation Δ_2 is just the *discrete Laplacian* (some authors use $-\Delta_2$ as the discrete Laplacian; in our definition Δ_2 has non-negative spectrum). Our definition of Δ_2 coincides with that of Neuberger in [13]. One may see that the derivation Δ_k for $(\mathcal{G}, d)$ coincides with Δ_2 for the weighted graph $(\mathcal{G}, d^{2/k})$. Another important type of examples of derivations are provided, for $p \in \{0, 1, 2, 3, \ldots\}$, by

$$\Delta^p = \Delta^{p-1}\Delta_2, \quad \Delta^0 = I. \tag{10}$$

Remark 2 One should notice that in general Δ^p cannot be seen as the discrete Laplace operator by changing the weight d on the graph. In fact, by the definition, the coefficients off the diagonal of the Laplacian matrix are non-positive. In particular, if we consider the complete graph $\mathcal{G} = K_3$, $V = \{v_1, v_2, v_3\}$, and the weight d defined by $d(\{v_1, v_2\}) = 2$, $d(\{v_1, v_3\}) = d(\{v_2, v_3\}) = 1$, then Δ^2 corresponds to the matrix J_2^2 whose coefficients off the diagonal are strictly positive; so it cannot be the discrete Laplace operator for any weight on K_3. The derivations Δ^p and $(-1)^k \Delta_k$ are examples of symmetric and positive semi-definite derivations.

3 Adaptation of Navier–Stokes Equations to Weighted Graphs

Let us consider a finite simple connected weighted graph $(\mathcal{G}, d)$, where $\mathcal{G} = (V, E)$ and $V = \{v_1, \ldots, v_m\}$. Let $0 < n < m$ be so that the degree at each vertex is at least n.

3.1 The n-Gradient

Let us consider n permutations in $\mathfrak{S}_m$, the permutation group on m letters, say

$$\sigma_1, \ldots, \sigma_n : \{1, \ldots, m\} \to \{1, \ldots, m\}$$

so that, for each $j \in \{1, \ldots, m\}$, the following properties hold

(1) $\sigma_r(j) \neq \sigma_s(j)$, if $r \neq s$;
(2) $v_{\sigma_r(j)} \in N(v_j)$.

We define the *n-gradient* of a function $\alpha \in C^0(\mathcal{G})$, associated to the choice of the above permutations, as

$$\nabla_n \alpha(v_j) := \left(\partial_{v_{\sigma_1(j)}} \alpha(v_j), \ldots, \partial_{v_{\sigma_n(j)}} \alpha(v_j) \right) . \tag{11}$$

3.2 The Adaptation

We assume fixed a n-gradient ∇_n defined by a collection of n permutations $\sigma_1, \ldots, \sigma_n$. Let us consider functions

$$\rho, p : V \times \mathbb{R} \to \mathbb{R}$$

both of them differentiable with respect to the real variable, and a function

$$f : V \times \mathbb{R} \to \mathbb{R}^n : (v, t) \mapsto (f_1(v, t), \ldots, f_n(v, t))$$

continuous in the real variable. The corresponding *Discrete Navier–Stokes Equation* is given by

$$\left\{ \begin{array}{c} \rho(v_j, t) \left(\dfrac{\partial \nu}{\partial t}(v_j, t) + \nu(v_j, t) \cdot \nabla_n \nu(v_j, t) \right) \\ || \\ -\nabla_n p(v_j, t) + \mu \Delta \nu(v_j, t) + f(v_j, t), \\ j = 1, \ldots, n. \end{array} \right. \tag{12}$$

A solution of (12) is a function

$$\nu : V \times \mathbb{R} \to \mathbb{R}^n : (v, t) \mapsto (\nu_1(v, t), \ldots, \nu_n(v, t)),$$

differentiable with respect to the real variable, and where

$$\nu \cdot \nabla_n \nu := (\nu \cdot \nabla_n \nu_1, \ldots, \nu \cdot \nabla_n \nu_n)$$
$$\Delta \nu := (-\Delta_2 \nu_1, \ldots, -\Delta_2 \nu_n) .$$

3.3 The Associated Ordinary Equation

The discrete Navier–Stokes equation (12) is equivalent to an ordinary differential equation. In order to state such an equation, we need some notations.

Let J_2 be the Laplacian matrix (representing Δ_2) and let us define, for $j = 1, \ldots, m$ and $k = 1, \ldots, n$, the following:

$$\begin{aligned}
\rho_j(t) &:= \rho(v_j, t), & p_j(t) &:= p(v_j, t)\\
f_j^k(t) &:= f_k(v_j, t), & \nu_j^k(t) &:= \nu_k(v_j, t)\\
\rho &:= {}^t[\rho_1 \cdots \rho_m], & p &:= {}^t[p_1 \cdots p_m]\\
f^k &:= {}^t[f_1^k \cdots f_m^k], & \nu^k &:= {}^t[\nu_1^k \cdots \nu_m^k]
\end{aligned}$$
$$L_k = [l_{ij}^k], \quad k = 1, \ldots, n,$$
$$l_{ij}^k = \begin{cases} \dfrac{-1}{d(\{v_i, v_{\sigma_k(i)}\})}, & i = j \\[2ex] \dfrac{1}{d(\{v_i, v_{\sigma_k(i)}\})}, & \sigma_k(i) = j \\[2ex] 0, & \text{otherwise} \end{cases}$$
$$M = \operatorname{diag}(\rho_1, \rho_2, \ldots, \rho_m)$$
$$S_k = \operatorname{diag}(\nu_1^k, \nu_2^k, \ldots, \nu_m^k)$$

With the previous notations, Eq. (12) may be written as the following system of ordinary differential equations

$$\dot{\nu}^k + \left(\left(\sum_{l=1}^{n} S_l L_l\right) + \mu M^{-1} J_2\right)\nu^k = M^{-1}(f^k - L_k p), \qquad k = 1, \ldots, n. \tag{13}$$

or equivalently, in matrix notation, as

$$\dot{\nu} + \left(SL + \mu M^{-1} J_2\right)\nu = M^{-1}[(f^1 - L_1 p) \cdots (f^n - L_n p)] \tag{14}$$

where $S = [S_1 \cdots S_n]$ and $L = {}^t[{}^tL_1 \cdots {}^tL_n]$.

The Newton conservation equation is given, in this way, by

$$\sum_{k=1}^{n} L_k \nu^k = 0. \tag{15}$$

Remark 3 In the above, we have obtained a system of differential equations with algebraic restrictions

$$\begin{cases} \dot{x} = F(x) \\ E(x) = 0, \end{cases} \tag{16}$$

where $F(x)$ is some vector field and $E(x)$ is a polynomial equation in the coordinates of x.

4 The Complete Graph K_m and the Cyclic Graph C_m

In this section K_m will denote the complete graph with $m \geq 2$ vertices, say $V = \{v_1, \dots, v_m\}$, and similarly C_m will denote the cyclic graph with $m \geq 2$ vertices labeled as before. We assume each edge has constant weight $d > 0$. We set $w = 1/d$.

4.1 The Complete Graph K_2

Let us consider $n = 1$, $\sigma_1(1) = 2$ and $\sigma_1(2) = 1$. In this case, $\nu_1(t) = \nu(v_1, t)$, $\nu_2(t) = \nu(v_2, t)$,

$$L_1 = \begin{bmatrix} -w & w \\ w & -w \end{bmatrix}, \quad J_2 = \begin{bmatrix} w^2 & -w^2 \\ -w^2 & w^2 \end{bmatrix},$$
$$S_1 = \begin{bmatrix} \nu_1 & 0 \\ 0 & \nu_2 \end{bmatrix}, \quad M = \begin{bmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{bmatrix}.$$

The algebraic condition (15) asserts that

$$\nu_1 = \nu_2 \ .$$

In this way, Eq. (13) is given by

$$\begin{bmatrix} \dot{\nu}_1 \\ \dot{\nu}_1 \end{bmatrix} + \begin{bmatrix} \nu_1 & 0 \\ 0 & \nu_1 \end{bmatrix} \underbrace{\begin{bmatrix} -w & w \\ w & -w \end{bmatrix} \begin{bmatrix} \nu_1 \\ \nu_1 \end{bmatrix}}_{0}$$
$$= \begin{bmatrix} -\rho_1^{-1} & 0 \\ 0 & -\rho_2^{-1} \end{bmatrix} \begin{bmatrix} -w & w \\ w & -w \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} - \mu \begin{bmatrix} \rho_1^{-1} & 0 \\ 0 & \rho_2^{-1} \end{bmatrix} \underbrace{\begin{bmatrix} w^2 & -w^2 \\ -w^2 & w^2 \end{bmatrix} \begin{bmatrix} \nu_1 \\ \nu_1 \end{bmatrix}}_{0}$$
$$+ \begin{bmatrix} \rho_1^{-1} & 0 \\ 0 & \rho_2^{-1} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

that is

$$\begin{bmatrix} \dot{\nu}_1 \\ \dot{\nu}_1 \end{bmatrix} = \begin{bmatrix} \rho_1^{-1} w(p_1 - p_2 + f_1/w) \\ \rho_2^{-1} w(-p_1 + p_2 + f_2/w) \end{bmatrix}.$$

Therefore, the following result follows.

Theorem 4 *In the case of K_2 the discrete Navier–Stokes equation* (13) *under the conservation law* (15) *has solutions if and only if*

$$w(\rho_1+\rho_2)(p_1-p_2)=\rho_1 f_2-\rho_2 f_1$$

in which case it has solutions of the form

$$v_1(t)=v_2(t)=w\int_0^t \frac{1}{\rho_1(\tau)}\left(p_1(\tau)-p_2(\tau)+\frac{f_1(\tau)}{w}\right)\,d\tau+a,\quad (a\in\mathbb{R}).$$

4.2 The Complete Graph $K_m, m\geq 3$

Let us consider $n=m-1, \sigma_1(j)=j+1$ (j modulo m), and $\sigma_k=\sigma_1^k$, for $k=2,\ldots,n$. In this case

$$L_1=\cdots=L_n=\begin{bmatrix} -w & w & \cdots & w & w \\ w & -w & w & \cdots & w \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ w & w & \cdots & w & -w \end{bmatrix},\quad J_2=\begin{bmatrix} nw^2 & -w^2 & \cdots & -w^2 & -w^2 \\ -w^2 & nw^2 & -w^2 & \cdots & -w^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -w^2 & -w^2 & \cdots & -w^2 & nw^2 \end{bmatrix},$$

Algebraic condition (15) asserts that

$$\nu^n=-\sum_{j=1}^{n-1}\nu^j$$

and, in particular,

$$S_n=-\sum_{j=1}^{n-1}S_j$$

At this point we note that, from the above,

$$\sum_{l=1}^{n}S_l L_l=L_1\sum_{l=1}^{n}S_l=0,$$

therefore the equation to solve is

$$\dot{\nu}^k+\mu M^{-1}J_2\nu^k=M^{-1}(f^k-L_k p),\qquad k=1,\ldots,n. \tag{17}$$

and a necessary condition to have a solution is

$$\sum_{k=1}^{n}(f^k - L_k p) = 0.$$

Now assuming the above necessary condition, the original system of ordinary differential equations has solutions if and only if the following one has solutions

$$\dot{\nu}^k + \mu M^{-1} J_2 \nu^k = M^{-1}(f^k - L_k p), \qquad k = 1, \ldots, n-1, \tag{18}$$

which can be explicitly solved as it is a linear system of degree one, uniquely determined by the initial conditions for $\nu^k(0)$, $k = 1, \ldots, n-1$.

All the above is summarized in the following.

Theorem 5 *In the case of K_m, with the previous notations, the condition*

$$\sum_{k=1}^{n}(f^k - L_k p) = 0$$

is necessary and sufficient in order for the discrete Navier–Stokes equation (13) *under the conservation law* (15) *to have solution.*

4.3 The Cyclic Graph C_m, $m \geq 3$

If we are interested in the problem of approximation of the solutions to Navier–Stokes equation in the case of a circle, it may be useful to consider the case of a weighted cyclic graph C_m, $m \geq 3$, $n = 1$ and $\sigma(j) = j + 1$ (modulo m). First, observe that incompressibility takes the form

$$\begin{bmatrix} -w & w & 0 & \cdots & 0 & 0 \\ 0 & -w & w & & 0 & 0 \\ 0 & 0 & -w & & 0 & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & & -w & w \\ w & 0 & 0 & \cdots & 0 & -w \end{bmatrix} \begin{bmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \\ \vdots \\ \nu_{m-1} \\ \nu_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

which implies $\nu = \nu_i = \nu_j$ for all $i, j = 1, \ldots, m$. It is easy to see that this restriction forces the discrete Laplace operator to vanish identically. More explicitly

$$\Delta_2 \nu_j = \sum_{w \in N(v_j)} \frac{\nu(w) - \nu(v_j)}{d(w, v_j)^2} = \frac{\nu(v_{j-1}) - 2\nu(v_j) + \nu(v_{j+1})}{d^2} = 0.$$

Thus, Navier–Stokes equation in this case is simply

$$\rho \dot{v} = -w(p_{j+1} - p_j) + f \tag{19}$$

from which we have the following.

Theorem 6 *In the case of C_m, with the previous notations, the discrete Navier–Stokes equation* (13) *under the conservation law* (15) *has solution if and only if $p^+ = p_{j+1} - p_j$ is independent of j. In such case, there is a unique solution with given initial conditions and it is*

$$v(t) = v(0) + \int_0^t \frac{f - wp^+}{\rho} d\tau.$$

5 Possible Application to Planar Regions

In this section, we discuss general ideas of how our adaptation of Navier–Stokes to graphs may be of use to produce numerical solutions of Navier–Stokes equation in a region $\Omega \subset \mathbb{R}^n$. This work is still in progress. In order to simplfy ideas, we only consider the case $n = 2$. Let us assume we have a bounded region $\Omega \subset \mathbb{R}^2$, with boundary Γ, which we assume to be a finite collection of rectifiable Jordan curves.

5.1 First Step: Replacing Ω by a Weighted Graph

We may start with a cellular decomposition Σ of $\mathbb{R}^2$ and define a graph $\mathcal{G}_\Sigma$ as follows. The vertices of this graph are the zero-dimensional components of Σ contained in $\overline{\Omega}$. The edges are the 1-dimensional components of Σ contained inside $\overline{\Omega}$ and extra edges connecting vertices at the boundary Γ (we need to take care in this part to avoid intersections of these extra edges with the previous ones). In Figs. 1 and 2 it is shown such a procedure for two different cellular decompositions of $\mathbb{R}^2$ (the first one corresponds to a cellular decomposition for which every two-dimensional component is a hexagon and in the second one every two-dimensional component is a rectangle). The weight of the graph is defined by the rule that the weight of an edge e is equal to the Euclidean distance between the end vertices.

5.2 Second Step: Choosing Correct Graphs

Unfortunately, not every graph, as constructed above, will work for our purposes. The problem is not the adaptation of Navier–Stokes, it comes from the fact that the corresponding discrete Laplace operator may not approach (a multiple of) the classical Laplace operator. More precise, assume we have a weighted graph, say $\mathcal{G} = (V, E)$ with metric d, as above. If $\mu \in C^0(\mathcal{G})$, $v \in V \cap \Omega$ and $N(v) = \{w_1, \ldots, w_{r_v}\}$, then

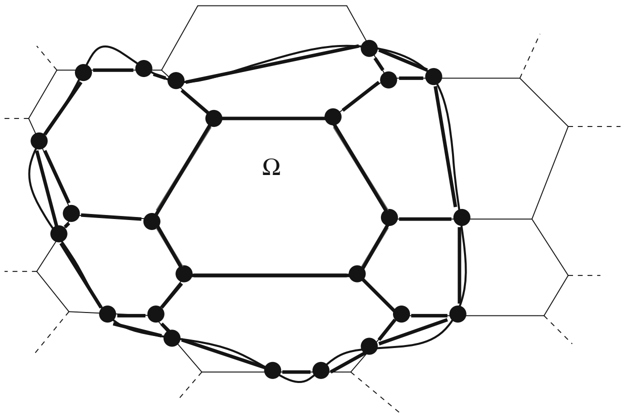


Fig. 1 The graph in thicker lines

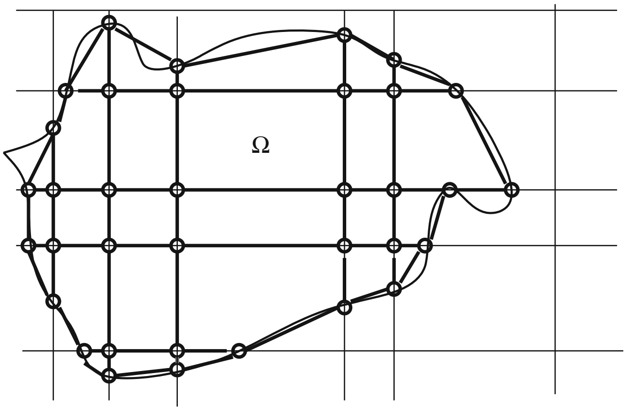


Fig. 2 The graph in thicker lines

$$\Delta_2\mu(v) = \sum_{j=1}^{r_v} \frac{\mu(v) - \mu(w_j)}{d(\{v, w_j\})^2}.$$

Set $w_j = v + (a_{j,v}, b_{j,v})d_{j,v}$, where $a_{j,v}^2 + b_{j,v}^2 = 1$ and $d(\{v, w_j\}) = d_{j,v}$, for each $j = 1, \ldots, r_v$.

If we keep the unit vectors $(a_{j,v}, b_{j,v})$ fixed, then

$$\lim_{(d_{1,v},\ldots,d_{r_v,v})\to(0,\ldots,0)} \sum_{j=1}^{r_v} \frac{\mu(v) - \mu(w_j)}{d_{j,v}^2}$$

$$||$$

$$\frac{1}{2}\left\{\left(\sum_{j=1}^{r_v} a_{j,v}^2\right)\frac{\partial^2\mu(v)}{\partial x^2} + \left(\sum_{j=1}^{r_v} b_{j,v}^2\right)\frac{\partial^2\mu(v)}{\partial y^2} + 2\left(\sum_{j=1}^{r_v} a_{j,v}b_{j,v}\right)\frac{\partial^2\mu(v)}{\partial x\partial y}\right\}.$$

In this way, if we want to use Δ_2 to approach $-\Delta$ (up to a multiple), then we need, for the graph $\mathcal{G}$, the existence of some $D > 0$ with the property that at each vertex $v \in \Omega \cap V$ it holds the following

(P1) $\sum_{j=1}^{r_v} a_{j,v}^2 = D = \sum_{j=1}^{r_v} b_{j,v}^2$,
(P2) $\sum_{j=1}^{r_v} a_{j,v} b_{j,v} = 0$.

Under such properties $\frac{2}{D}\Delta_2$ will approach $-\Delta$. Next, we proceed to describe a nice family of equiangular graphs we may use.

5.3 Equiangular Planar Graphs

From now on, we assume we have a weighted graph, say $\mathcal{G} = (V, E)$ with metric d, as defined in (5.1). As noted in (5.2), we need to assume some extra geometric conditions on such a graph. We assume that at each vertex $v \in V$ the angles between two consecutive edges are the same, that is, if the degree of v is $n_v \geq 3$, then the angle under consideration is $2\pi/n_v$. We say that $\mathcal{G}$ is an *equiangular planar graph*. We first note that (P2) is satisfied for such an equiangular planar graph; at a vertex v it holds that

$$\sum_{j=1}^{r_v} a_{j,v}^2 = \frac{n_v}{2} = \sum_{j=1}^{r_v} b_{j,v}^2$$

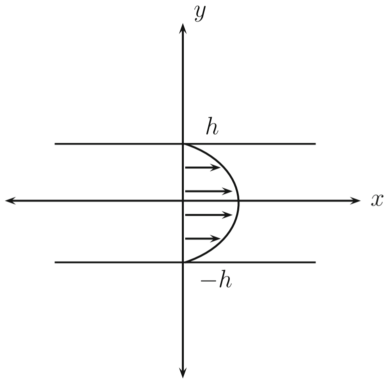
In order, for that equiangular planar graph, to satisfies (P1), we should have $n_v = n$, that is, the degree at each vertex is the same. With this restriction, $D = n/2$. But in this situation, the bounded connected components of the complement of the graph $\mathcal{G}$ are polygons whose interior angles are all equal to $2\pi/n$. Let $L \geq 3$ be the number of sides of one of these polygons. Clearly, $(L-2) = 2L/n$, from which $(n, L) \in \{(3, 6), (4, 4), (6, 3)\}$. In the case $(n, L) = (3, 6)$, so $2/D = 4/3$, we obtain *hexagonal graphs* (see Fig. 1); in the case $(n, L) = (4, 4)$, so $2/D = 1$, we obtain *rectangular graphs* (see Fig. 2); and in the case $(n, L) = (6, 3)$, so $2/D = 2/3$, we obtain *equilateral triangle graphs* (these graphs also define regular hexagonal graphs). Up to translations and rotations, the rectangular and hexagonal graphs depend on infinite number of parameters.

The case $(n, L) = (6, 3)$ seems to be to expensive in computations, in comparison with the other two cases. The case $(n, L) = (4, 4)$ is the classical types of graphs used in the method of finite differences (in the planar setting), but at each vertex we need to take care of 4 computations, one per each neighbor vertex. The case $(n, L) = (3, 6)$ seems to be the less expensive in terms of number of computations of discrete Laplacian.

5.4 The Modified Ordinary Differential Equation

Let us choose an equiangular planar graph (as above), for some fixed (n, L). First, for each vertex $v \in V$, let us enumerate the edges about v counterclockwise starting at one of them. We use the trivial permutation to define the n-gradient in (12). Secondly,

Fig. 3 Channel of the Poiseuille flow



we replace Δ_2 by $\frac{4}{n}\Delta_2$ in (12), which is equivalent to keep Δ_2 but to change the dynamical viscosity μ by $4\mu/n$. With these modifications, (14) is given by

$$\dot{v} + \left(SL + \frac{4\mu}{n} M^{-1} J_2\right) v = M^{-1}[(f^1 - L_1 p) \cdots (f^n - L_n p)] \tag{20}$$

5.5 A Concrete Example

The main problem now is to realize numerically how the solutions of (20), satisfying (15), are related to the solutions of the classical Navier–Stokes equation, satisfying the Newton conservation law. Let us study the case of the planar Poiseuille flow, that is, the stationary state of a Newtonian fluid driven by a constant pressure gradient $\nabla p(x, y) = (-P, 0)$ with constant density ρ and viscosity μ through a channel of width $2h$ (see Fig. 3) with no external force acting. All constants involved are assumed positive.

It is well-known that the solution of the continuous Navier–Stokes equation for this case is given by the classic parabolic profile

$$v(x, y) = \left(\frac{P}{2\mu}(h^2 - y^2), 0\right) \tag{21}$$

as depicted. This solution is obtained by assuming as boundary condition velocity zero on the walls of the channel (Fig. 4).

In order to adapt this to our case, we need to consider a finite domain. To do this, we cut a portion of the channel of size 2ℓ and add the "a posteriori" boundary conditions

$$v(x = \pm\ell, y) = \left(\frac{P}{2\mu}(h^2 - y^2), 0\right).$$

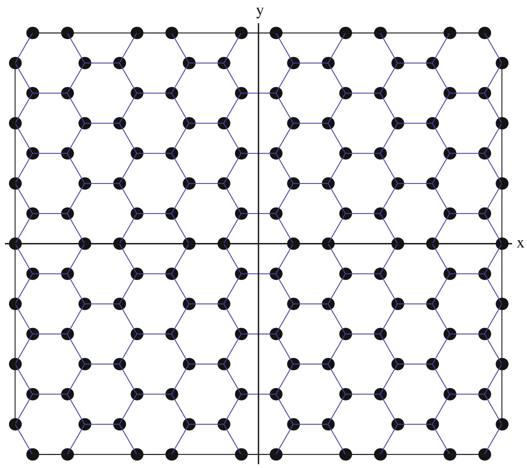


Fig. 4 The graph in the case $p = 3, \sigma = 2$

For each positive integer m, consider the regular hexagonal grid, obtained by translates of the hexagon with vertices

$$(\pm r_m, 0), \ \left(\pm r_m \cos\frac{\pi}{3}, \pm r_m \sin\frac{\pi}{3}\right)$$

where $r_m = \frac{2h}{\sqrt{3}m}$. For simplicity, we assume $m = 2p+1$ odd, $h = 1$, $\ell = (3\sigma+1)r_m$.

For convenience we label the vertices as follows. The vertices located in the rows over the horizontal lines $L_i^{odd} : y = \frac{-2i+2p+3}{m}$, $i = 1, \ldots, 2p+2$ will be referred as α_{ij}, where i indicates the corresponding row and j is the position of the vertex in the respective row (counting from left to right). The vertices in the rows over the horizontal lines $L_i^{even} : y = \frac{-2i+2p+2}{m}$, $i = 1, \ldots, 2p+1$ will be referred as β_{ij}, where i and j are as before. The weights of the edges, associated to the Euclidean distance, are easily seen to be

$$w(\alpha_{ij}, \alpha_{kl}) = \begin{cases} r_m, & \text{if } i = k, |j-l| = 1 \text{ and } \lceil \frac{j}{2} \rceil - \lceil \frac{l}{2} \rceil = 0, \\ 2r_m, & \text{if } i = k, |j-l| = 1 \text{ and } \lceil \frac{j}{2} \rceil - \lceil \frac{l}{2} \rceil \neq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{22}$$

$$w(\alpha_{ij}, \beta_{kl}) = \begin{cases} r_m, & \text{if } i-k = 0 \text{ or } 1 \text{ and } j = l, \\ 0, & \text{otherwise.} \end{cases} \tag{23}$$

$$w(\beta_{ij}, \beta_{kl}) = \begin{cases} \sqrt{3}r_m, & \text{if } |i-k| = 1,\ j = l \text{ and } l \in \{1, 2\sigma+1\}, \\ r_m, & \text{if } i = k, |j-l| = 1 \text{ and } \lceil \frac{j}{2} \rceil - \lceil \frac{l}{2} \rceil \neq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{24}$$

In this case, Eq. (20) takes the form

$$\dot{v} + \left(SL + \frac{4\mu}{3\rho}J_2\right)v = \frac{-1}{\rho}[L_1 p \ L_2 p \ L_3 p] \tag{25}$$

in which J_2 is completely determined by (22)–(24) and the matrices L_k depend on the choice of the permutations defining the discrete gradient.

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