Interbank Runs:

A Network Model of Systemic Liquidity Crunches*

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Abstract

This paper models systemic liquidity crunches on interbank lending networks, and studies how the network structure affects the instability. Interbank runs are modeled as a coordination failure, in which banks run on banks as they mutually reinforce each other to withdraw interbank lending. A mean-field approximation reduces the complex strategic interactions on networks down to a one-dimensional characterization of the system-wide dynamics. It explains how lending-borrowing relationships not only channel the "ripple" effects of local shocks, but also connect the whole network into a self-fulfilling "tsunami". I demonstrate applications in network-based systemic risk measurement and management.

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1 Introduction

A hallmark of the 2007-08 financial crisis is the dramatic and system-wide reduction in bank's funding liquidity.[1] The defining feature is *banks running on banks*—wholesale funding between banks is the prominent form of the liquidity crunch. The experience is reminiscent of but different from classical bank runs, in which depositors mutually reinforce each other's incentive to withdraw from a common borrower. In the modern version, which I call "*interbank runs*", one bank's withdrawal of interbank lending reduces the funding liquidity of its borrowers, which incentivizes further withdrawal. These strategic interactions connected by lending-borrowing relationships fuel a systemic liquidity crunch.

A natural theoretical question arises—why do banks *across the whole system* reduce liquidity at the same time by a discontinuous amount? Notice in interbank runs, one bank's behavior only directly affects those peer banks with a pre-existing lender-borrower link. The links form a decentralized network in which most pairs of banks are not directly connected. Yet the whole system acts in lockstep in crises, as if the run decisions are modulated by a common center. It is understandable that interbank links channel shock propagation, often metaphorically called "ripple effects". In existing models with this feature, a shock indeed dissipates *like ripples* in the sense that the effect attenuates as it spreads further and wider.[2] Such a pattern might describe routine fluctuations in the interbank system. However, it is not obvious how in some situations a local shock can escalate into a system-wide *tsunami*, not to mention how the network structure plays a role in this process. These questions are

[1] Throughout the paper, I use "bank" for financial institutions in a broad sense, including not just depository institutions, but also shadow banks, investment banks, etc.

[2] Some other papers about the systemic risk in financial networks *only* feature the attenuating ripple-like propagation, for example Denbee et al. (2017), Acemoglu et al. (2015), Eisenberg and Noe (2001). See detailed discussions of the "ripple-tsunami" contrast against existing literature in 1.1, in the context of a stylized network in 3.3.3, with general complicated networks in 5.1.

essential to understanding the (in)stability of financial system and have immediate policy implications as central banks are interested, first and foremost, in systemic risks.

To address the questions, I build a model of the banks' interlocking liquidity management problems. It demonstrates precautionary *liquidity hoarding* strategies at the bank level can connect into self-fulfilling runs at the system level, at least with a highly-stylized network. The model becomes hard to analyze when it comes to generic network structures which in reality is complicated and high-dimensional. To show the same economic dynamics carry over, and more importantly to understand how the network structure affects systemic stability, I then apply a network technique called *mean-field approximation*. It extracts the system-wide behavior from the dispersed network by reducing the complicated network game down to a one-dimensional equilibrium condition. The result clearly reveals the dynamics of the whole system as a coordination failure which can be triggered once crossing a tipping point. With these results in place, I apply them to conduct a system-wide stress testing, to construct a stability measure, and to identify the most effective liquidity injection targets for government bail-outs.

The banking model part illustrates interbank lending is beneficial in the sense that it allows the financial system to convert liquid deposits into illiquid real investments. But the desirable arrangement of mutually extending interbank liquidity is subject to a coordination failure, and hence inherently unstable. In detail, with high interbank lending, banks inflate each other's balance sheets although the aggregate deposit flow into the financial system stays constant. An inflated balance sheet alleviates the liquidity constraint imposed by retail depositors, allowing the bank to hoard less cash and allocate more illiquid assets (including real investment and interbank lending).[3] The result is a cooperative equilibrium in which the financial system as a whole functions well in converting deposit funding into illiquid real investment, while satisfying the liquidity demands of depositors.

The instability of this arrangement stems from a bank's tendency to *hoard liquidity* at the individual level. It means as interbank funding decreases, the bank has to not only scale down total assets, but also tilt the asset composition toward cash, in order to prepare liquidity against depositors' run tendencies. Hence, a dollar reduction of interbank funding can lead to an *amplified* reduction of illiquid lending.[4] The banks' individual liquidity hoarding decisions are strategic complements due to the interbank lending connections. When one bank decreases illiquid lending, another bank with pre-existing borrowing connection from it will sustain a reduction in funding liquidity, which will induce withdrawal and further funding liquidity reduction down the borrower chain. The individual "amplifying" best response and the strategic complementarity result in a separate coordination failure equilibrium. In that case, banks self-insure against depositor' demands by holding only cash; interbank lending unravels; and aggregate real investments vanish.

Starting from the cooperative equilibrium, with the right combination of initial trigger and network structure, the equilibrium might be no longer supported. The financial system experiences a discontinuous fall to the coordination failure equilibrium, which characterizes the interbank run.

The mean-field analysis is for extracting the system-wide behavior from the com-

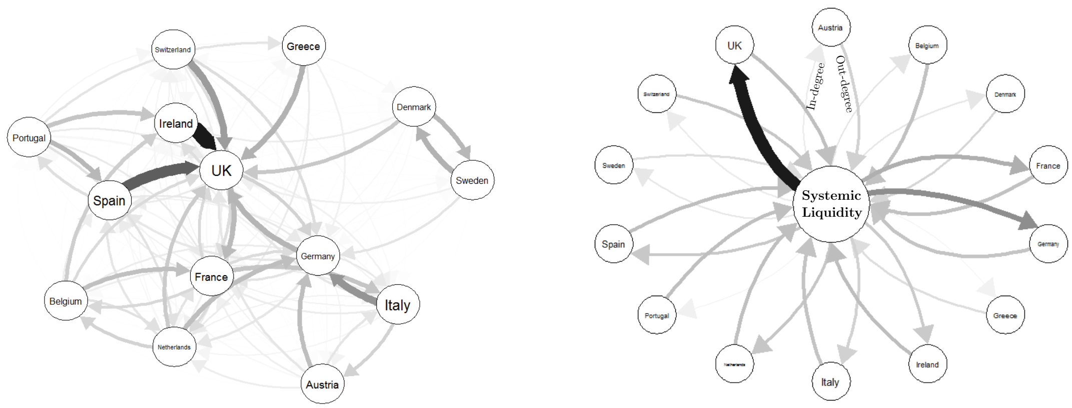
[3]The bank's individual liquidity management problem is similar to that in Ennis and Keister (2006). Banks hold a more liquid portfolio in order to deter retail depositor runs, which limits the fund available for profitable but illiquid investments. A new element here is that acquiring interbank funding can alleviate this constraint.

[4]I assume retail depositors are senior to interbank lenders, while interbank lending decisions are made before the possible retail depositor run. He and Xiong (2012) models the similar staggered maturity structure and the resulting incentive of not rolling over debt *upfront* in order to avoid being attacked by more senior creditors in potential future distresses.

plicated equilibrium condition. The equilibrium condition is complicated for being a high-dimensional non-linear equation set with respect to all banks' strategy profiles. The non-linear best response is necessary for equilibrium multiplicity, which is for characterizing the bank run-like instability and a technical feature that distinguishes this paper from the existing literature. The high dimensionality comes from the network structure which consists heterogeneous connections between each pair of banks (the adjacency matrix). Simulation analysis or numerical solutions (even if feasible) hardly provide any insight into the dynamics of the equilibrium, especially at the system level.

The rationale of the mean-field approximation is that each bank's set of direct lenders is a representative subsample of the population of all banks. Hence, the average lending level extended by the bank's direct lenders is approximately the same as a network-wide (properly weighted) average level called *systemic liquidity*. Therefore, a bank's interbank funding liquidity approximately equals to the product of the amount of its borrowing connections (in-degree) and systemic liquidity. Following this idea, the original equilibrium condition is transformed to a one-dimensional equilibrium condition with respect only systemic liquidity.

The new condition can be understood as a transformed economy (Figure 1). In the original economy (Panel a), the strategic interactions are pairwise, heterogeneous, and dispersed, making system-wide dynamics hard to capture. In the transformed economy (Panel b), all banks are directly interacting via a constructed center node representing systemic liquidity. Each bank accesses interbank funding from systemic liquidity according to its total borrowing connections (in-degree), and then chooses illiquid assets according to the individual best response. A proportion of the illiquid assets is contributed back to the system according to the bank's total lending connections (out-degree). Combined, the aggregate effect of each bank's interaction with the systemic liquidity constitutes the new one-dimensional equilibrium condition.



(a) Original Economy (b) Transformed Economy

Figure 1: Mean-field approximation seen as transformed economy
Note: the width and shade of the arrows are related to the connection intensities. Details about the numerical construction are reported in Subsection 4.4.

I show the approximation is practically accurate in real-world financial networks with a series of numerical exercises. In addition, the approximation method is theoretically justified by showing the approximation errors are asymptotically zero with a random network model.

The mean-field approximation takes a holistic view of the complicated strategic interactions on networks and yields several new insights. First, the one-dimensional equilibrium condition clearly exposes the system-wide dynamics. Some of the classical insights gained with (single) bank run models, including multiple equilibria, tipping points, and self-fulfilling prophecy, indeed carry over to interbank runs. It provide a theoretical explanation of why indirectly connected banks still tend to act in lockstep during a liquidity crisis.

Another important benefit is showing how the network structure affects systemic stability. The approximation method delivers a one-dimensional summary statistic of the network structure, called *effective connectedness*, that determines the systemic behavior. Effective connectedness captures a few interpretable aspects of the network structure. For example, it shows a mismatch between funding connection and lending intensity harms systemic stability, a phenomenon reported with historical interbank networks (Paddrik et al., 2016).

Once the interbank model and the analytical tools are in place, I propose three related applications. First, I demonstrate a system-wide stress testing that simulates local shocks and examine whether an interbank run would be triggered. The stress test considers hypothetical scenarios in which a subset of banks become stressed and are the remaining network suffers from reduced funding sources. As shocks accumulate, the equilibrium systemic liquidity first decreases continuously (like ripple), and then drops to zero after crossing the tipping point (like tsunami). Second, I presents a financial stability measure that summarizes the distance from the given network to the tipping point. The stability measure is built on the abovementioned effective connectedness statistic in order to account for the complicated network structure. Finally, I study the optimal capital injection targets as part of a government bail-out plan. Which banks, depending on their connections in the network, are more effective targets in boosting the stability of the whole system? I find the banks that are constrained in liquidity in the status-quo equilibrium are not necessarily the top-priority targets. Instead, those that would be constrained when the network deteriorates to the tipping point are more effective targets. The result justifies capital injection to the "healthy and viable" banks as opposed to the banks that are already liquidity constrained.[5]

[5] As part of the Troubled Asset Relief Program (TARP), the Treasury invested in "healthy and viable" financial institutions at first—a policy that has been under debate. See more details in Section 5.3.

1.1 Literature

This paper is at the intersection of the liquidity crises literature and the financial networks literature. In the liquidity crises literature, the seminal paper by Diamond and Dybvig (1983) models retail depositors' run on a single bank. The strategic complementarity among lenders' preemptively liquidity demands induces instability. The similar idea is subsequently applied to the settings of investors running on currencies (Morris and Shin, 1998), investors running on a financial market (Bernardo and Welch, 2004), creditors running on non-bank financial institutions (He and Xiong, 2012), financial institutions running on the real-sector (Bebchuk and Goldstein, 2011). This paper's focus is banks running on banks.

Diamond and Rajan (2005), Brunnermeier and Pedersen (2009), Benmelech and Bergman (2012), and Liu (2016), among others, analyze the liquidity crises of a financial system. In these papers, an interbank market serves as the centralized mechanism for the bank-to-bank interactions. This paper breaks the assumptions that atomistic agents directly interact through a monolithic market mechanism. I show a system with discrete banks and dispersed interactions is still subject to discontinuous and system-wide liquidity crunches.

In the literature that studies systemic risks with financial networks, Eisenberg and Noe (2001), Acemoglu et al. (2015), Elliott et al. (2014) among others model the cascading asset loss from an insolvent borrower bank to its lenders. This paper takes the perspective of a liquidity crisis and models liquidity reduction transmitted from the borrower to the lender. The focus on strategic interactions emphasizes that banks do not merely follow domino-style contagions passively. Denbee et al. (2017) study the strategic interactions on liquidity management. In these network models (including the three above) the equilibrium conditions are linear in banks' actions, so that the impulse responses of a exogenous shock dissipate down the transmission chain (like ripple). Without equilibrium multiplicity, such models cannot characterize a system-wide discontinuous changes, which is more relevant in the context of crises and systemic risks.

Allen and Gale (2000) and Nash (2016) study liquidity shortage contagion on stylized network structures. Their models feature ex-post interbank liquidity risk sharing while I model banks' ex-ante coordination in interbank lending. Gai and Kapadia (2010), Gai et al. (2011), and Gofman (2017), resort to simulation methods to study mechanical contagion on the financial network. Relative to simulation methods, mean-field approximation affords succinct and interpretable insights about network structures and system-wide economic dynamics.

Beyond finance applications, Jackson and Yariv (2007) and Galeotti et al. (2009) share similar ideas with the mean-field approximation method. In their models, agents act before knowing the exact identities of their connections. The Bayesian belief of the network population plays a similar role as the systemic liquidity in this paper. The adoption of the mean-field tools is originally inspired by Gao et al. (2016) with an application in ecology networks, which share some similarities with the financial network as suggested by Haldane and May (2011).

2 An Interbank Lending Model

2.1 Big picture configuration and timing

Figure 2 illustrates the big picture configuration of the model. The financial system consists of n banks that lend and borrow with each other. The bilateral lending-borrowing relationships connect the banks into an *interbank network*. Besides internal interactions, the financial system as a whole intermediates between retail depositors and the illiquid real investments as well as the liquid storage asset called cash.

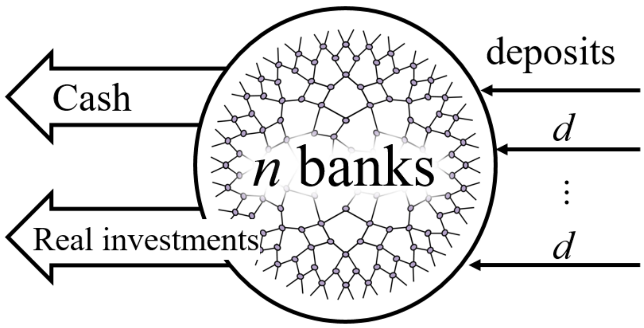


Figure 2: Configuration of the Financial System
Note: The arrows represent time-0 flow of funds to and from the financial system.

The economy has three time periods numbered 0, 1, and 2. Let me introduce them in the reverse order. Time 2 models the "long term" when illiquid assets mature with the high payoff; interbank debts and retail deposits are repaid; and agents consume. The intermediate time 1 represents the an uncertain scenario of liquidity distress. On the asset side, the early liquidation value of illiquid investment is low. Meanwhile, retail depositors might have a panic-driven (single) bank run. A bank must prepare enough short-term liquidity to deter the depositor run to survive through time 1.

Time 0 is the crux of the analysis as it is when interbank runs happen. At this period, all banks make their asset allocation decisions simultaneously, including the amount of interbank lending, real investments, and cash holding. A bank's goal is to maximize its time-2 payoff, provided it has enough liquidity to survive through time 1. Banks liquidity management decisions at time 0 are connected by lending-borrowing relationships which exist before time 0 and are taken as given. The bulk of the analysis is to analyze the equilibrium of the time-0 strategic interactions, which characterizes systemic liquidity crunches as shown further below.[6]

[6]For example, Northern Rock's liquidity crisis was triggered by the non-renewal of wholesale

2.2 Time-0 bank balance sheet and interbank lending

Figure 3 illustrates a generic bank i's balance sheet at time 0. On the funding side, each bank serves a group of retail depositors who supply d dollars of deposits inelastically. In addition, y_i dollars in total are borrowed from some other banks via interbank borrowing. As analyzed later, y_i is an endogenous variable depending on the lenders' decisions. With total funding of $d + y_i$ dollars, bank i chooses an asset allocation of c_i dollars in cash, and x_i dollars in *illiquid lending*.

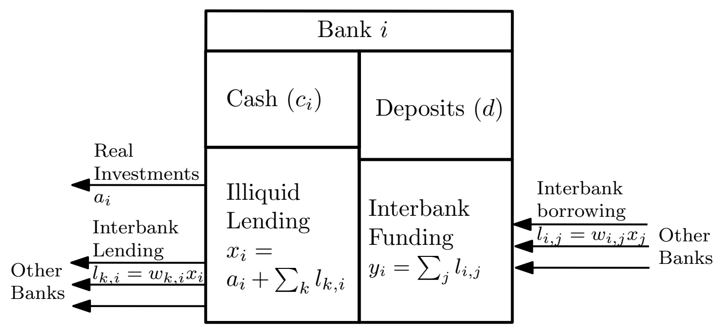


Figure 3: Bank balance sheet at time 0
Note: The items in the graph represent time-0 flow of funds.

Illiquid lending is composed of a homogeneous bundle of real investments (a_i) and interbank lending to other banks ($l_{k,i}$). I assume bank i only chooses the total amount of illiquid lending x_i, but the proportions allocated to the real and interbank borrowers are fixed. This assumption effectively leaves the network formation process unmodeled. Instead, I take the ever-evolving network structure as exogenously given, and focus the analysis on the stability of the status quo configuration, assuming banks withdrawal illiquid lending proportionally in an interbank run. In detail, the proportion of bank i's illiquid lending (x_i) to another bank k is fixed at $w_{k,i}$.[7] Then,

funding from institutional investors (Shin, 2009). "Depositor run ... was an event in the *aftermath* of the liquidity crisis at Northern Rock, rather than the event that triggered its liquidity crisis." The concern of future liquidity need is the reason for the institutional lenders to retract lending preemptively. This is one example that justifies the sequencing of the events in the model.

[7] Notation convention: indexation i always denotes the central subject of the current discussion,

interbank lending from i to k is $l_{k,i} := w_{k,i}x_i$ dollars.

The interbank network constitutes all pairwise lending proportions $\{w_{k,i}\}$, which are arranged in an $n \times n$ *adjacency matrix* $\mathbf{W}$. Notice the network is directed and weighted ($w_{i,j} \neq w_{j,i}$ and each $w_{i,j}$ is continuous rather than binary). Next, I define a few network-related concepts. The *interbank lending intensity* $\delta_i^{\mathrm{O}} := \sum_k w_{k,i}$ of bank i is the proportion of illiquid lending allocated to other banks. Real investments, which flow outside of the financial system take up the remaining proportion: $a_i = (1-\delta_i^{\mathrm{O}})x_i$. In network analysis terminology, δ_i^{O} is also called the *out-degree* of i, calculated as the sum of the i'th column of the adjacency matrix $\mathbf{W}$. Symmetrically, the row sum $\delta_i^{\mathrm{I}} := \sum_j w_{i,j}$ is bank i's *in-degree* or *interbank funding connection*, which measures bank i's access to interbank funding sources. The joint degree distribution of δ_i^{I} and δ_i^{O} will play a central role in the subsequent mean field analysis. The network is typically sparsely connected in the sense that many entries of $\mathbf{W}$ are zero. The banks with a positive lending connection to i are called bank i's (direct) *lenders* (the set $\{j \mid w_{i,j} > 0\}$). And symmetrically, i's (direct) *borrowers* are $\{k \mid w_{k,i} > 0\}$. Overall, the network structure ($\mathbf{W}$) satisfies the following condition.

Condition 1 (Adjacency Matrix)**.** *(1) All entries are nonnegative:* $w_{i,j} \geq 0, \forall i, j$*. (2) diagonal entries are zero:* $w_{i,i} = 0, \forall i$*. (3) interbank lending intensity (out-degrees) are less than 1:* $\delta_i^{\mathrm{O}} = \sum_k w_{k,i} < 1, \forall i$*.*

2.3 Asset returns and liquidation costs

An asset's return structure is given by two numbers, hold-to-maturity payoff and early liquidation value. The two classes of assets, cash and illiquid lending, feature the trade-off between the long and short term payoffs.

while j indexes i's lender and k indexes i's borrower. As a result of this convention, the double subscripts shall always appear like $w_{i,j}$ and $w_{k,i}$.

The returns of cash and real investment are given exogenously by the physical constraints outside the financial system. For one dollar invested at time 0, cash returns one dollar at either time 1 or time 2, whereas real investment returns either $\beta < 1$ at time 1 or $R > 1$ at time 2.

Interbank lending's return structure is assumed to be the same as that of the real investments: β at time 1 or R at time 2. A way to micro-found is to assume that there is a secondary market for the early liquidation at the time of distress (time 1), where both interbank and real debts are priced equally—each share of face value R is at the discounted price of β. The result is that financial assets are as illiquid as real investments. As shown later, the low market liquidity at time 1 in turn causes the funding liquidity problem at time 0, as in one of the two legs of a "liquidity spiral" in Brunnermeier and Pedersen (2009).

In addition to the per-dollar early liquidation cost $(1 - \beta)$, there is a *fixed cost of liquidation* γ to withdraw any positive amount of illiquid lending at time 1. This reflects a bank-level overhead cost during a financial distress to initiate the liquidation process. If the early liquidation value of x_i does not even cover the fixed cost, the bank can (and will) choose not to withdraw any x_i at time 1. As a result, the maximum time-1 total liquidation value from x_i is $\max\{x_i\beta - \gamma, 0\}$.

2.4 Retail deposits and depositor run at time 1

Each bank takes in d dollars of deposits at time 0 which requires the same amount of repayment at time 2. In this sense, deposit funding is cheaper than interbank borrowing which costs $R > 1$ at time 2 per one dollar borrowed at time 0. However, retail deposits imposes a liquidity requirement for the bank's time-0 asset allocation problem, due to depositors' short term liquidity needs and the tendency to run at time 1.

At time 1, retail depositors can demand early withdrawal even though they do not want to consume until time 2. This leaves the possibility of a Diamond Dybvig-style self-fulfilling retail depositor run among the group of depositor who lend to a common bank. I assume in early withdrawal events, retail depositors are more senior to interbank lenders, following depositor preference laws. Therefore, so long as the bank has prepared enough time-1 liquidity that covers the maximum liquidity needs of depositors (d), the panic-driven depositor run will not materialize.

I assume a depositor run is so detrimental to a bank that the bank must deter it by preparing liquidity up front, or no other bank wants to have an interbank interaction with it at time 0. Therefore, the bank's time-0 asset allocation must satisfy the following *depositor-run proof* condition:

$$\max \{x_i \beta - \gamma, 0\} + c_i \geq d.$$

An interbank lender expect that they will be junior to the borrower bank's depositors at time-1, therefore the interbank lender tend to preempt by withdrawing at time 0. The interplay between the retail depositor run and interbank run is similar to the staggered maturities in dynamic runs of He and Xiong (2012).

Alternatively, the depositor-run proof condition can be interpreted as a liquidity coverage regulation on the asset portfolio. The banks are mandated to prepare enough time-1 liquidity to cover the amount of deposits (d). This assumption simplifies the micro-foundation, but it looses some of the abovementioned interpretations. Regardless, depositor-run proof is a key constraint that shapes a bank's time-0 liquidity management strategy, which in turn forms the basis of the systemic instability as shown further below.

3 Model Analysis

3.1 Bank's individual optimal asset allocation

In this subsection, let us first take other banks' lending choices (x_j's) as given, which determine bank i's interbank funding level (y_i), and examine the bank's individual problem of choosing illiquid lending (x_i) given y_i.

Given deposit and interbank funding (d, y_i), each bank chooses an asset portfolio (x_i, c_i) to maximize the time-2 payoff. The bank also faces a liquidity constraint to deter depositor runs. The individual problem is

$$\begin{aligned} \max_{c_i \geq 0, x_i \geq 0} \; & x_i R + c_i - d - y_i R, \\ \text{s.t.} \quad & x_i + c_i \leq d + y_i, \\ & \max\{x_i \beta - \gamma, 0\} + c_i \geq d. \end{aligned}$$

The solution is represented by best response functions $x_i = f(y_i)$ and $c_i = g(y_i)$ with respect to interbank funding y_i. Figure 4 illustrates the piece-wise linear best response functions.[8] Notice the functions are irrelevant of bank identity (no i subscript).

The best response functions feature the key economic mechanism at the individual level—liquidity hoarding. Let us examine how it happens as y_i decreases. The bank would always like to allocate more funding in the high-return illiquid lending (x_i), if there were not the liquidity pressure from depositors. With a high $y_i > y_h$, the depositor-run-proof constraint is not binding. Even if all funding is in illiquid lending ($x_i = d + y_i$ and $c_i = 0$), time-1 liquidity is ample just from x_i's liquidation value ($x_i \beta - \gamma \geq d$). In this region, x_i changes one to one with y_i.

[8] The detailed analytical expressions are in Appendix A.1

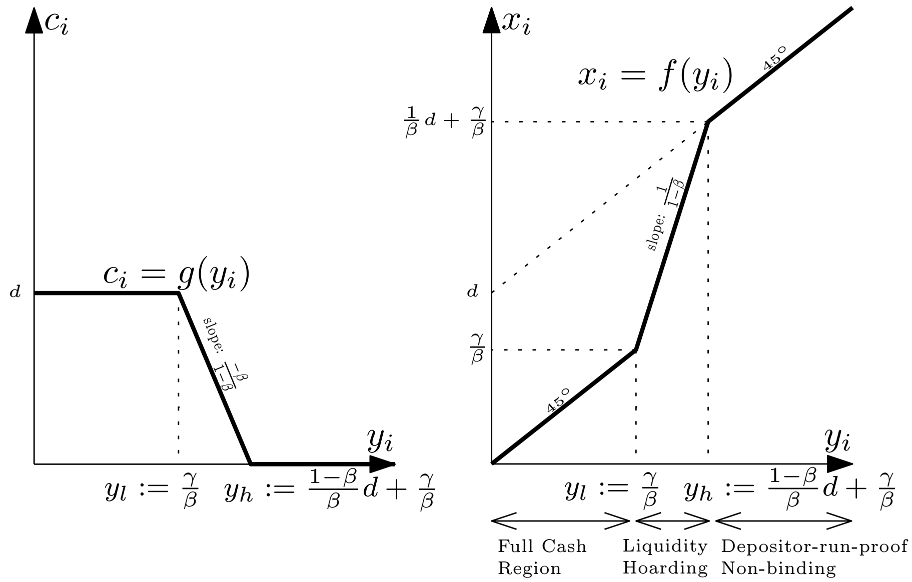


Figure 4: Time-0 asset allocation: $x_i = f(y_i), c_i = g(y_i)$

Then, as interbank liquidity decreases, the bank's balance sheet shrinks. A wholly illiquid asset portfolio would no longer be depositor-run proof. The bank has to switch to a more liquid asset composition to maintain enough ($\geq d$) time-1 asset liquidation value. In detail, for a one-dollar decrease in interbank funding y_i, illiquid lending x_i has to be cut back by more than one dollar ($\frac{1}{1-\beta} > 1$). The excess ($\frac{\beta}{1-\beta}$) is used to accumulate cash, for its greater time-1 liquidation value ($1 > \beta$). In this fashion, a drop in interbank funding y_i causes an *amplified* reduction in illiquid lending x_i. Illustrated in the figure, function f is steeper (slope > 1) in the middle interval $[y_l, y_h]$. This individual amplification effect is the micro-foundation of system-wide instability, as shown later in Proposition 1.

Finally, as y_i further decreases ($y_i < y_l$), x_i cannot even cover its fixed liquidation costs ($x_i\beta < \gamma$). All time-1 liquidity needs (d) have to be prepared in the form of cash ($c_i = d$). Cash locks up funding that cannot be invested in the high return assets. In this full cash region, x_i once again changes one to one with y_i, similar to the $y_i > y_h$ case above.

Interbank borrowing generates profits because it inflates the balance sheet, loosens the depositor-run proof constraint, and thereby allows deposits to be invested in illiquid assets. The optimized time-2 profit equals $(R-1)(d-c_i)$, which is the amount of deposits that are converted into illiquid lending multiplied by the long-term return margin. The reason why some deposits can be freed up from cash and invested in illiquid lending is because of interbank funding. With more y_i comes a larger balance sheet. Although y_i has high funding costs (meaning by itself y_i does not earn profits), it is more junior to deposit claims at time-1. Therefore, should a distress situation occur at time 1, the inflated balance sheet, with a greater asset side, would be able to provide more time-1 liquidity to the depositors. This "sense of safety" of a bigger balance sheet allows some deposit funding to be invested in the high return illiquid assets ex ante.

3.2 Interbank lending equilibrium

Banks' individual decisions are interlinked by the interbank lending network. At time-0, all banks in the network simultaneously solve the same problem of balance sheet management as described above. A bank's lending choice (x_i) depends on its interbank funding (y_i), which is determined by the direct lenders' (j's) lending decisions: $y_i = \sum_j w_{i,j} x_j$. In turn, x_i affects the borrowers' (k's) decisions. The equilibrium condition below formalizes these intertwined strategic interactions.

Define *interbank lending equilibrium* as the time-0 allocations (cash c_i, real investments a_i, and interbank lending $l_{i,j}$) such that each bank i

- utilizes all the interbank borrowing extended by other banks: $y_i = \sum_j l_{i,j}$,

- chooses its optimal asset portfolios following the individual problem: $x_i = f(y_i), c_i = g(y_i)$,

- lends to interbank and real borrowers according to the pre-existing connections $\mathbf{W}$: $l_{k,i} = x_i w_{k,i}, \forall k$ and $a_i = x_i(1 - \delta_i^{\mathrm{O}})$.

In equilibrium, a bank's illiquid lending (x_i) sufficiently describes the bank's other optimal choices (including c_i, a_i, y_i) as well as interbank lending amounts ($l_{i,j}$). Therefore, the interbank lending equilibrium can be characterized by the following n-dimensional fixed-point condition with respect to $\{x_i\}_{i=1}^n$.

Condition 2 (Interbank lending equilibrium condition)**.**

$$x_i = f\left(\sum_j w_{i,j} x_j\right), \qquad \forall i. \tag{1}$$

3.3 A stylized example: loop network

To prepare for analyzing more realist and complicated networks, I examine the equilibrium in a highly-stylized case to illustrate some key mechanisms of interbank runs.

$$\mathbf{W} = \begin{bmatrix} 0 & 0 & \dots & & \color{red}{w} \\ w & 0 & & & 0 \\ 0 & w & & & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & w & 0 \end{bmatrix}_{n \times n}$$

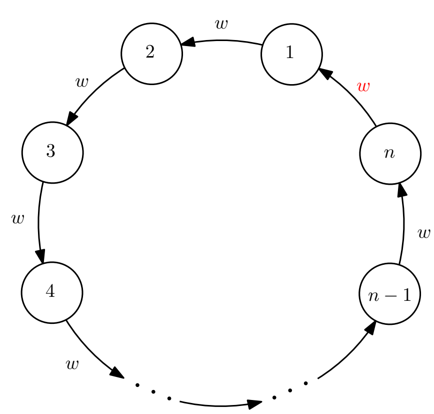
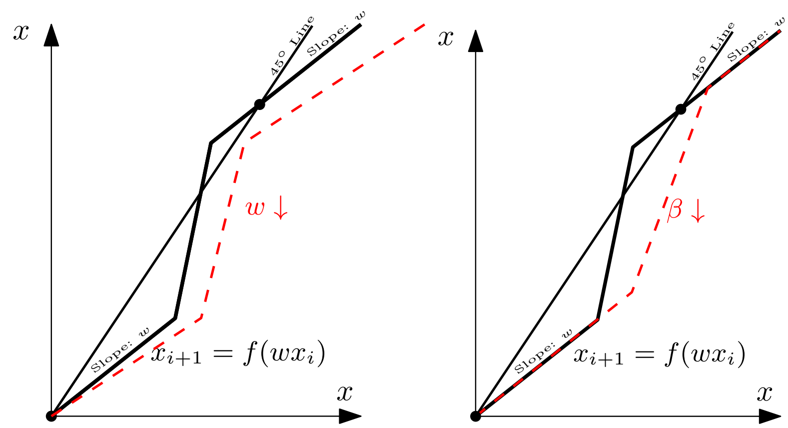


Figure 5: Loop Network
The adjacency matrix and the graphical representation of the loop network. The red "w" marks the target of the comparative static analysis in 3.3.3

A loop network is presented in Figure 5. Each bank i allocates w proportion of illiquid lending (x_i) to the next bank (bank $i+1$, except bank n loops back to 1). The remaining proportion $(1-w)$ is allocated to real investments. Equilibrium condition 2 evaluated at the loop network is $x_{i+1} = f(y_{i+1}) = f(wx_i), \forall i$ (with $n+1$ treated as 1). Because f is increasing, the solution must be symmetric. The condition is reduced to a simple univariate equation, $x = f(wx)$, where x (no subscript) denotes the symmetric equilibrium action. The solutions are shown as the crossing points of the $f(wx)$ curve and the 45° line in Figure 6. The graph of $f(wx)$ is a horizontally stretched version of the S-shaped individual best response function $f(y)$ in Figure 4. The figure shows that there are either one or two (locally stable) equilibria.[9] The origin point ($x=0$) is always an equilibrium. Depending on the parameters, there is (or is not) another equilibrium with positive x's.[10]



(a) w decrease

(b) β decrease

Figure 6: Loop Network Equilibria and Comparative Statics
Note: the crossing points of $f(wx)$ and the 45° line represent the symmetric equilibria. The red dashed lines represent the situations when w or β are reduced.

The multiple equilibria result characterizes the instability of the interbank lending

[9] "Locally stable" equilibrium is characterized by the direction f crosses the 45° line, with the formal definition in Appendix A.2

[10] In detail, it is easy to show that if and only if $w \geq 1 - \frac{\beta}{1+\frac{\gamma}{d}}$, there exist a positive locally stable equilibrium $x_i = \frac{d}{1-w}, \forall i$.

as a coordination failure. At the *cooperative equilibrium* with positive x_i's, all banks are in the depositor-run-proof non-binding region ($y_i > y_h$). The high interbank funding a bank receives allows it to also lend out more to the next bank. Although the aggregate deposit flow into the financial system stays constant (nd), via mutual interbank lending, banks inflate each other's balance sheets. A larger asset side makes every bank become safer from the perspective of its own retail depositors, since depositors know they would be repaid first if a time-1 liquidation happens.[11] The complementary strategies support the coordination on the loop. The financial system in aggregate functions well by converting all deposits (nd) into real investments. The highest possible efficiency and profit level are achieved with no deposit being "wasted" as cash.

At the *coordination failure equilibrium* ($x_i = 0, \forall i$), banks mutually reinforce each other to withdraw interbank lending. All banks hold cash to self-insure against the potential depositor run. The interbank network falls apart with no zero lending flow. Aggregate real investments fall, while liquid assets held in the financial system balloon.

Starting from the cooperative equilibrium, a transition to the coordination failure equilibrium would characterize an interbank run since x_i's across the system would fall in a discontinuous way. What kind of exogenous changes would make the cooperative equilibrium no longer exists and trigger an interbank run? Next, I discuss three such triggers, each of which shows a small change across a critical value leads to the discontinuous and system-wide interbank run.

[11] The mechanism is similar to liquidity sharing in Allen and Gale (2000). Pre-existing interbank lending provides a way for liquidity to be pooled to help a single bank that experiences liquidity needs from depositors.

3.3.1 A network-wide connection reduction

First, I analyze the comparative statics of a network-wide equal reduction in connection intensities (w). Figure 6(a) illustrates that as w decreases, the $f(wx)$ curve stretches to the right. At first, the cooperative equilibrium x decreases continuously. As w drops below a critical value (given in footnote 10), the $f(wx)$ curve no longer crosses the 45° line and the cooperative equilibrium disappears. Coordination failure with zero interbank lending becomes the only equilibrium.

A higher pre-existing w means a greater proportion of illiquid lending is contributed to the borrower bank, which gives the borrower bank a higher incentive to pass on the high lending action to its borrower. Therefore, high interbank connections in general are helpful to sustain the cooperative equilibrium. For an economy that is close the critical level, a small change of w can cause a discontinuously large change around the whole loop.

These (somewhat straight-forward) results are useful preparations for the general complicated networks discussed later. After the mean-field approximation, the complicated network changes will be summarized by a one-dimensional "effective connectedness" statistic whose effect is similar to the simple w here.

3.3.2 A reduction in market liquidity

The second comparative static analysis is with respect to the time-1 early liquidation value of the illiquid assets (β). When β decreases, both the kink points in the f curve move to the upper right (y_l, y_h increase), as shown in Figure 6(b). A decrease in the early liquidation value makes the depositor-run-proof condition binds at a higher interbank funding levels. As β decreases beyond a critical value, the cooperative equilibrium is no longer sustained, and an interbank run is triggered too.

Brunnermeier and Pedersen (2009) discuss a similar effect since β can be interpreted as the market liquidity of the illiquid assets at the time-1 secondary market. Both the models feature a discontinuous liquidity crunch from a small exogenous change of market liquidity. The difference it that their instability stems from the feedback spiral between funding liquidity and market liquidity. I focus on the mutually-reinforcing interactions of funding liquidity on the interbank network, while taking market liquidity (β) as exogenously given.

3.3.3 A local shock

The third trigger is a local shock in the network connections. The analysis displays a clear distinction between the ripple- and tsunami-like propagation patterns allude to at the beginning of the introduction.

The comparative static analysis considers a reduction in $w_{1,n}$ (marked in red in Figure 5), while fixing all other connections in $\mathbf{W}$. It represents a situation where only bank 1 experiences a reduction in funding connection. Notice there is no investment loss. The change is merely that bank n re-allocates some of its illiquid lending composition from bank 1 to real investments.[12]

Since the symmetric property of the loop network is no longer maintained, I rely on numerical solutions for the equilibrium analysis. Figure 7 presents an experiment with seven banks ($n = 7$, which is without loss of generality). At the right extreme is the benchmark starting point where all connections including $w_{1,7}$ equals 0.7. The colored curves plot x_1 to x_7 at the highest equilibrium, as $w_{1,7}$ decreases from 0.7 down to 0.

At higher levels of $w_{1,7}$, the shock propagates like ripples as the impulse responses

[12]The underlying reason for such a change is not important. It might be due to that Bank n is loosing confidence in bank 1. The similar reason of borrowers' insolvency shocks motivates the stress testing in Section 5.1.

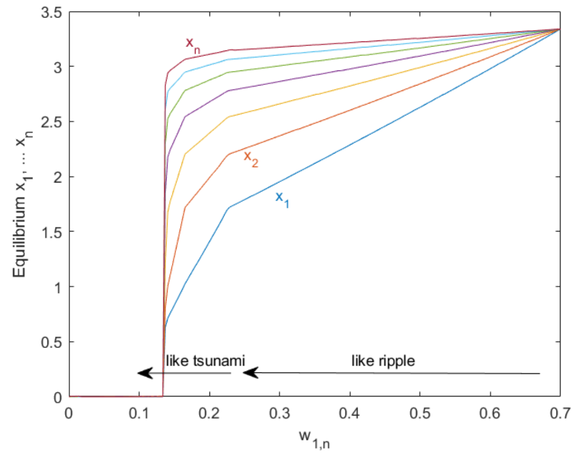


Figure 7: Ripple-like and tsunami-like responses to a local shock
Note: the equilibrium x's are numerically calculated.

dissipate down the borrower chain. By "dissipate" I mean specifically that the responses to the shock are (indeed exponentially) weaker for banks further down the transmission chain. Figure 7 shows that x_1 has the greatest reduction; x_2's reduction is smaller; and so on so forth. At the end of the loop, bank 7 barely "feels" the shock. As a result, in the ripple-like propagation pattern, *local shocks' effects stay local.* Such a pattern might describe routine fluctuations in the interbank system, but it does not match system-wide and discontinuous changes like interbank runs well.

As $w_{1,n}$ further decreases, the tsunami-like system-wide effect kicks in. The banks fall into the liquidity hoarding region one by one, starting with bank 1, and then banks 2, 3...(shown as the progressively steeper slopes of the colored curves). The individual amplification effects snowball on top of each previous bank, dramatically increasing the strength of the propagation. Once bank 7 at the end of the loop also feels the effect, it feeds back to bank 1. The loop network becomes literally a feedback loop. The liquidity hoarding complementarity brings down the whole system in a self-fulfilling way, with just minor further reduction in $w_{1,n}$.

Being able to characterize the tsunami-like propagation is a key innovation of this paper as other models are restricted to the ripple-like pattern. The experiment also reveals that amplifying liquidity hoarding behavior is the source of the tsunami-like systemic effects. This observation will be formalized in Subsection 4.1 in the general case, and will re-emerge in the stress tests against local shocks in 5.1.

4 System-wide Behavior in Complicated Networks

4.1 Systemic instability stems from individual hoarding

I examine what is the underlying reason of the systemic instability, which is captured by the technical feature of equilibrium multiplicity. A necessary condition for more than one equilibria is that the strategic interactions must be in a sense "strong", so that local perturbations are "amplified" down the chain of transmission, and that the whole network can be pushed to a different equilibrium. The idea of "strong" and "amplify" is formalized by *expansive* individual best-response functions:

Definition 1 (Expansive and non-expansive functions)**.**
A function f is called non-expansive if $|f(y_1) - f(y_2)| \leq |y_1 - y_2|, \forall y_1, y_2$. Conversely, if there exist y_1, y_2 such that $|f(y_1) - f(y_2)| > |y_1 - y_2|$, f is expansive.

For a differentiable f, being expansive means the slope is greater than one at least somewhere in the domain.

Proposition 1 (Expansive f is necessary for equilibrium multiplicity)**.**
Given any network $\mathbf{W}$ that satisfies Condition 1, if f is non-expansive, then there is at most one equilibrium as defined by equilibrium Condition 2. Conversely, if there are more than one equilibria, f must be expansive.[13]

[13] The proof is in Appendix A.3. A similar result is reported in Acemoglu et al. (2015).

The specific f considered in this model is expansive. Indeed, it is expansive exactly due to the individual liquidity hoarding behavior, which yields the interval with a slope greater than one (see Figure 4). In this sense, individual bank's precautionary behavior of liquidity hoarding is the source of system-wide instability. Conversely, without the amplification effect, there will be no discontinuous system-wide behavior change no matter how complicated the network is. Local shocks dissipate down the transmission chain and stay local. This difference highlights a major innovation relative to the existing literature.[14]

4.2 Mean-field approximation

Equilibrium multiplicity is a useful feature for characterizing system-wide behavior changes, but it brings immediate difficulties. Condition 2 is a high-dimensional and non-linear fixed-point problem with potentially complicated multiplicity caused by the intricate connections. Even if attainable, numerical solutions would hardly provide any insight into how the network structure affects the number of equilibria and eventually systemic stability.

Mean-field approximation is precisely for overcoming these problems. It provides an easy way to solve the equilibria, albeit not exactly. More importantly, it reduces the n-dimensional condition down to a one-dimension condition, which can be interpreted and visualized as easily as the loop network case and classical bank run models. Moreover, the approximation provides a way to summarize the properties of any network in an interpretable statistic that is closely connected to the systemic behavior,

[14]Besides finance application, this also expands the scope of games on networks literature in general. Different best-response functions represent the various types of strategic interactions of interests. For example, Ballester et al. (2006) study a model with an increasing and linear f, which is applied to skill investment with positive externalities on social networks. Bramoullé et al. (2014) study local public goods provision and free-riding with an f that is decreasing, linear, and bounded from below, representing strategic substitutabilities. See Jackson and Zenou (2014) and Bramoullé and Kranton (2015) for more complete surveys on the topic of games played on fixed network.

allowing for a measure of stability based on the network structure.

Reviewing some network notations: the network is represented by adjacency matrix $\mathbf{W}$. The in- and out-degrees $\delta_i^{\mathrm{I}}, \delta_i^{\mathrm{O}}$ are the row and column sums of $\mathbf{W}$, which represent a bank's interbank funding connection and lending intensity, respectively. Additionally, define μ as the *average degree* (either in or out degrees, which are the same), $\mu := \frac{1}{n}\sum_i \delta_i^{\mathrm{I}} = \frac{1}{n}\sum_i \delta_i^{\mathrm{O}} = \frac{1}{n}\sum_{ij} w_{i,j}$.

Next, I introduce the mean-field approximation of the equilibrium. Decompose bank i's interbank funding (y_i) into the product of two terms: bank i's funding connection and the $w_{i,j}$-weighted average x_j. (Notice $\frac{w_{i,j}}{\delta_i^{\mathrm{I}}}$ forms a set of normalized weights across j's.)

$$y_i = \sum_j w_{i,j} x_j = \delta_i^{\mathrm{I}} \cdot \sum_j \frac{w_{i,j}}{\delta_i^{\mathrm{I}}} x_j, \qquad \forall i. \tag{2}$$

The idea of mean-field approximation is to replace the second term by a system-wide "mean" effect, which is the same across all banks. Loosely speaking, every bank's set of direct lenders is a representative subsample of the population of all banks.[15] With this intuition, the average lending level extended by i's direct lenders is approximated by a network-wide (properly weighted) average level called *systemic liquidity*.

$$\sum_j \frac{w_{i,j}}{\delta_i^{\mathrm{I}}} x_j \approx \sum_j \frac{\delta_j^{\mathrm{O}}}{n\mu} x_j := x^*, \qquad \forall i. \tag{3}$$

Both sides of the approximation are weighted averages of x_j's. The left-hand side is weighted by connection intensities ($w_{i,j}$) toward bank i. The right-hand side is weighted by the lender's out-degree, which is irrelevant the borrower's identity i. An underlying rationale is that the bank with a high lending intensity (δ_j^{O}) is more likely

[15] This is formalized with a random network generating model where all banks' connections are drawn from the same population. See Chapter 4 of the dissertation version of this paper.

to become bank i's direct lender ($w_{i,j} > 0$) and have a high connection intensity $w_{i,j}$ toward i.

Systemic liquidity ($x^* := \sum_j \frac{\delta_j^{\mathrm{O}}}{n\mu} x_j$) represents a system-wide summarization of the supply of interbank liquidity. Substitute x^* back into equation (2), bank i's interbank funding is accordingly approximated as

$$y_i \approx \delta_i^{\mathrm{I}} x^*, \qquad \forall i. \tag{4}$$

Notice x^* is common across i, so that the heterogeneity of banks' funding levels (y_i's) only comes from the aggregate funding connections (δ_i^{I}'s) after the approximation. The detailed composition of the funding connections ($w_{i,j}$ and x_j) are no longer accounted for in calculating y_i.

A somewhat different interpretation of x^* is the network-wide average interbank lending volume per unit of connection. Total interbank lending volume throughout the network is $\sum_{k,i} l_{k,i} = \sum_i \delta_i^{\mathrm{O}} x_i$. This total volume is carried by $\sum_{k,i} w_{k,i} = n\mu$ "units" of total connections. Their ratio, $x^* = \frac{\sum_i \delta_i^{\mathrm{O}} x_i}{n\mu}$, is then the average lending volume per unit of connection. Following this interpretation, (4) says bank i's funding volume approximately equals its funding connection times the network-wide average volume per unit of connection.

4.3 Systemic equilibrium

This subsection defines a new systemic equilibrium that approximates the original equilibrium, which is characterized by a one-dimensional condition.

At the original equilibrium, mean-field approximation holds approximately. So, the exact solution of the approximated condition should approximate the original hard-to-solve equilibrium.[16] In detail, suppose the approximation of interbank funding holds exactly ($y_i = \delta_i^{\mathrm{I}} x^*$), and substitute it into the original equilibrium ($x_i = f(y_i)$).

Definition 2 (Systemic equilibrium)**.**

The systemic liquidity equilibrium is the vector of banks' illiquid lending $\{x_i\}$ *and the associated systemic liquidity* x^**, such that*

$$x_i = f(\delta_i^{\mathrm{I}} x^*), \qquad \forall i. \tag{5}$$

This equilibrium is much easier to characterize since it can be reduced to a one-dimensional condition with respect to *only* x^*. To do that, take the δ^{O}-weighted average on both sides of the above equation set, so that the left hand side combines into x^* too.

Condition 3 (Systemic equilibrium condition)**.**

$$x^* = \sum_i \frac{\delta_i^{\mathrm{O}}}{n\mu} f(\delta_j^{\mathrm{I}} x^*) := f^*(x^*). \tag{6}$$

As a result, the n-dimensional fixed point (Condition 2) is reduced to a univariate fixed-point condition. Systemic liquidity (x^*) becomes the single key object of the following analysis to represent the behavior of the whole network. As long as x^* is pinned down, the individual actions (e.g. x_i) can be recovered via Eq. (5).

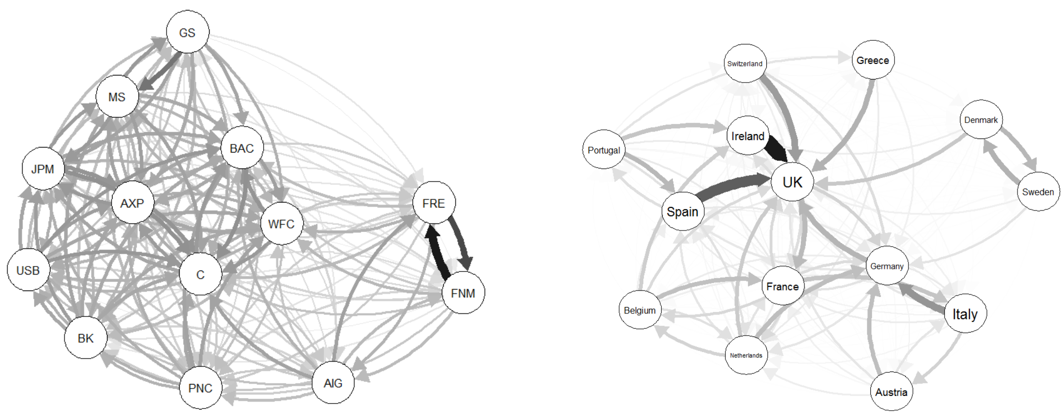
4.4 Numerical verification of the approximation accuracy

This subsection presents numerical experiments to demonstrate that the systemic equilibrium is practically close to the original equilibrium, in order to dismiss doubts

[16] This claim is shown formally in the dissertation version too.

over the application of the mean-field approximation for real-world networks.

The numerical comparisons are conducted with two real-world financial networks shown in Figure 8. The first one, coded `Diebold13`, is the estimated interconnectedness among 13 major U.S. financial institutions by Diebold and Yilmaz (2011). The second, coded `Europe14`, is 14 European country consolidated bank cross-holdings reported by the Bank for International Settlements.[17] The numerical values of the networks are not essential as I am merely using them to illustrate the accuracy of the method.



(a) `Diebold13` (b) `Europe14`

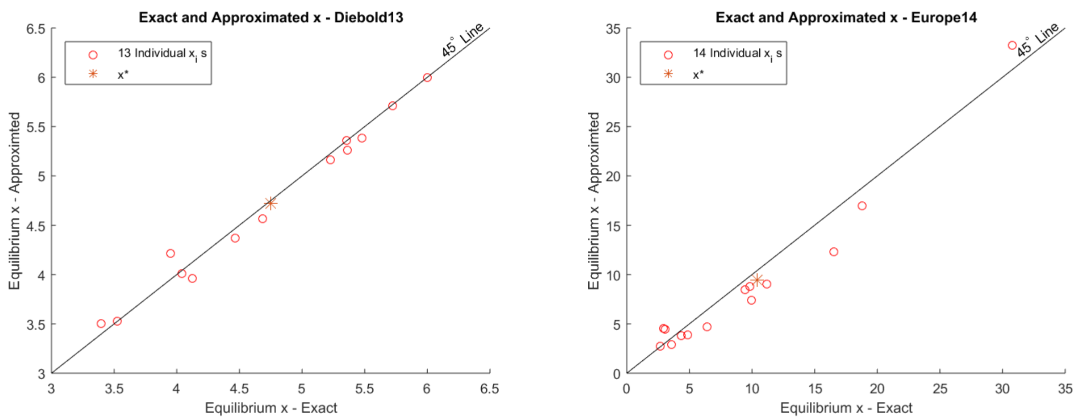
Figure 8: Observed Networks Visualization
The width and shade of the arrows are related to the value of connection intensity ($w_{i,j}$). The layout of the banks is given by an algorithm such that more connected nodes are closer (`R` package `qgraph`).

I conduct the same numerical comparison for each of the two networks. The original equilibrium and the systemic equilibrium are attained by numerically solving Conditions 2 and 3, respectively.[18] Figure 9 plots the solutions of $\{x_i\}$ and x^*, where

[17] The same dataset is used in Elliott et al. (2014) to illustrate their model. They picked 6 European countries, whereas I reconstructed the data for all 14 European countries with observable international lending and borrowing. Using the sub-graph of the 6 countries they pick does not change the results.

[18] The highest solution (cooperative equilibrium) is reported. The coordination failure equilibrium

the horizontal (vertical) coordinates are solved under the original (systemic) equilibrium. The small vertical offsets relative to the 45° line illustrate the accuracy of the approximation using systemic equilibrium.[19] In addition, the numerical accuracy is also supported by a series of other comparisons appeared further below.



(a) `Diebold13` (b) `Europe14`

Figure 9: Original and Systemic Equilibrium Comparison

4.5 Characterizing system-wide equilibrium dynamics

4.5.1 Imagined center, systemic response (f^*), and its instability

Approximating the original equilibrium with the systemic equilibrium can be understood as a transformation of the decentralized network structure to a centralized one. In the original equilibrium (Figure 1a), strategic interactions are pairwise and dispersed on the decentralized lending network. Each bank is directly interacting with a limited number of counterparties. In the transformed structure (Figure 1b), each bank is only directly interacting with the imagined common center that represents systemic liquidity (x^*). Bank i accesses systemic liquidity according to its in-degree

is zeros either way.

[19]Figure 9 is produced with an f function specified in Appendix A.4. The approximation is still accurate when varying the primitive parameters in f, as reported in Appendix A.5.

(δ_i^{I}), obtaining interbank funding of $y_i = \delta_i^{\mathrm{I}} x^*$. According to the individual best response, the bank chooses an illiquid lending of $x_i = f(y_i) = f(\delta_i^{\mathrm{I}} x^*)$. In return, δ_i^{O} portion of x_i is contributed back to the system. Combined, the equilibrium condition is determined by the *aggregate effect* of all banks' direct interaction with the systemic liquidity.

The aggregate effect is captured by the *systemic response function*, defined as $f^*(x) := \sum_i \frac{\delta_i^{\mathrm{O}}}{n\mu} f(\delta_i^{\mathrm{I}} x)$. The systemic equilibrium (Condition 3) is simply the univariate fixed points problem of $x^* = f^*(x^*)$. A typical f^* is illustrated in Figure 12. It displays an S-shape similar to the one in Figure 6 with the loop network, except for being rounded rather than kinked. (If all δ_i^{I} were the same, f^* would fall back to the kinked case.)

The similar shape implies that the intuitions about instability acquired with the loop example indeed carry over to the complex network case. In particular, the higher crossing point with the 45° line marks the cooperative equilibrium. The possibility that the system falls to the coordination failure equilibrium (the origin point in the figure) represents the systemic instability of an interbank run.

The underlying reason why f^* displays the self-fulfilling instability is because it is the aggregation of every bank's direct interaction with x^*, each of which has the amplification effect within the bank's own liquidity hoarding region ($\delta_i^{\mathrm{I}} x^* \in [y_l, y_h]$). These forces in aggregation can lead to a systemic liquidity crunch.

4.5.2 Effective contentedness (W^*) and the pitch of f^*

Next, I discuses how the network structure affect systemic instability. We know f^* contains all the information we are interested in about systemic behavior: the number of equilibria, their position, and the instability of the cooperative equilibrium. Therefore, I analyze how the network structure affects the shape of f^*.

A one-dimensional summary of the network structure, called *effective connectedness*, is a key determinant of f^* as it controls the overall pitch of the S-curve. On top of that, the details of the curvature is determined by the joint distribution of the in and out-degrees.

Define effective connectedness as $W^* := \sum_i \frac{\delta_i^{\mathrm{O}} \delta_i^{\mathrm{I}}}{n\mu}$. To see how it controls the pitch of f^*, I analyze both f^* and its derivative:

$$f^*(x) = W^* x + \sum_i \frac{\delta_i^{\mathrm{O}}}{n\mu} \left(f(\delta_i^{\mathrm{I}} x) - \delta_i^{\mathrm{I}} x \right),$$

$$f^{*\prime}(x) = W^* + W^* \frac{\beta}{1-\beta} \sum_i \frac{\delta_i^{\mathrm{O}} \delta_i^{\mathrm{I}}}{n\mu W^*} \mathbb{1}_{[y_l, y_h)}(\delta_i^{\mathrm{I}} x). \tag{7}$$

Notice $\frac{\delta_i^{\mathrm{O}} \delta_i^{\mathrm{I}}}{n\mu W^*}$ is a set of normalized weights.

A higher W^* makes f^* increases faster with x, making the the S-curve pitch up and condense leftward to the vertical axis (similar to a high w in the loop network example in Figure 6a). As a result, a higher W^* makes the S-curve crosses the 45° line higher, meaning the cooperative equilibrium is higher and further away from the systemic run tipping point. Based on this observation, a systematic stability measure is built in the next section.

The effective connectedness W^* is a summary of the network structure with clear geometric interpretations according to the following decomposition:

$$W^* = \mu \left(1 + \mathrm{corr}(\delta^{\mathrm{O}}, \delta^{\mathrm{I}}) \frac{\sigma(\delta^{\mathrm{I}})}{\mu} \frac{\sigma(\delta^{\mathrm{O}})}{\mu} \right). \tag{8}$$

Firstly, the average degree μ plays a baseline role. All else equal, a more densely connected network yields a higher and more stable cooperative equilibrium. This finding is somewhat expected from the loop example already, where connections are homogeneous, hence $W^* = \mu = w$.

Fixing μ, the correlation between in- and out-degrees also matters for W^*. It means, *asymmetric* networks with mismatches between interbank funding and lending tend to be unstable. In asymmetric networks, the tension between funding liquidity and asset liquidity is accumulated at certain "pinch points" of the network. Some empirical analyses of liquidity crises make similar observations. For example, Paddrik et al. (2016) report that the National Banking Acts led to a concentrated and pyramid-shaped interbank network, in which interbank deposits accumulated from rural banks to regional hubs and eventually to the core banks in New York. But, the core banks' illiquid lending was concentrated in real investments with low contribution back to the interbank system (high δ^{I} but low δ^{O}). When the system reduced lending, the liquidity mismatch was manifested at the core banks, resulting in systemic liquidity crises.

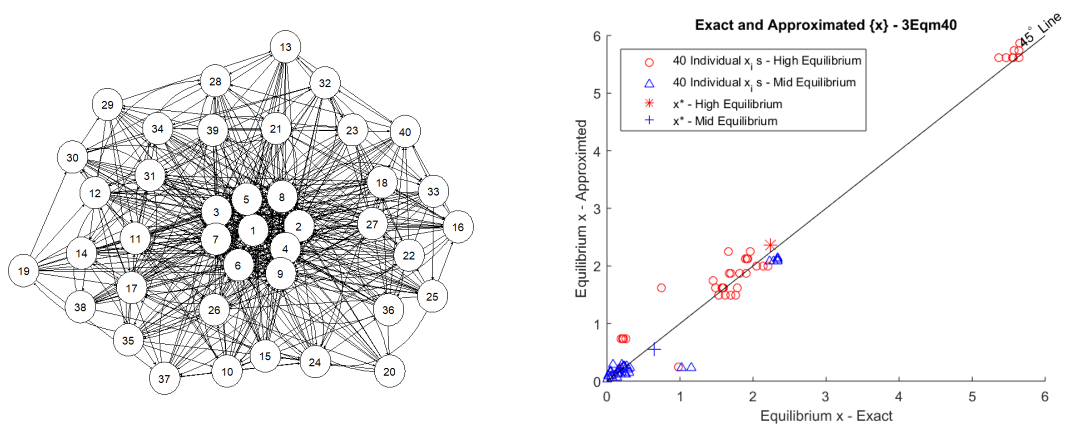
Finally, conditional on a positive correlation between the in- and out-degrees, greater cross-sectional dispersion (measured by the coefficients of variation) increases effective connectedness.

4.5.3 Degree distribution and the curvature of f^*

Fixing W^*, the curvature of f^* can be seen from the second term in Eq. (7). In the summation, $\mathbb{1}_{[y_l,y_h)}(\delta_i^{\mathrm{I}}x)$ is an indicator for bank i being inside the liquidity hoarding region, and $\frac{\delta_i^{\mathrm{O}}\delta_i^{\mathrm{I}}}{n\mu W^*}$ is a set of normalized weights.

For very high or very low (off-equilibrium) levels of systemic liquidity, most of the banks are in the depositor-run-proof non-binding or full-cash regions, respectively. Either way the banks respond to interbank funding changes one-to-one. In aggregate, the system is not experiencing cash-hoarding so that $f^{*\prime}$ is low and close to W^*. In the middle interval however, many banks are in the liquidity hoarding region ($\delta_i^{\mathrm{I}}x^* \in [y_l, y_h]$), where individual response to liquidity reduction is "strong". An initial decrease in systemic liquidity would cause an amplified decrease in lending back to the system, shown as the middle steeper section of the S-curve where $f^{*\prime} > W^*$. This steeper section is the underlying reason why systemic liquidity has multiple equilibria, and why it displays the self-fulfilling instability.

It is even possible to have two "waves" of system-wide liquidity hoarding if there are two distinct groups of banks with different in-degrees (i.e., in-degree distribution is bi-model). I provide a simulated core-peripheral network to demonstrate the situation in Figure 10. The resulting f^* becomes a "double S"-curve which yields three locally stable equilibria in a special knife-edge case.[20] The additional middle equilibrium (plotted in blue) represents a "partial" interbank run in which the peripheral banks collapse while the core ones are still able to support each other due to their dense connections, nonetheless at a lower liquidity level.



(a) Example Network Visualization: `3Eqm40` (b) Exact and Approximation Equilibria: `3Eqm40`

Figure 10: An Example Network with Three Equilibria
Note: the network consists of 40 banks (9 core and 31 peripheral). Panel (b) reports the high (in red) and middle (in blue) locally stable equilibria in the same format as Figure 9.

In summary, these new insights about how the network structure affects the sys-

[20]More details about the core-peripheral network, including the degree distributions and the "double S"-curve are in Appendix A.6

temic response function and eventually the system-wide equilibrium behavior are impossible without the mean-field approximation.

5 Applications

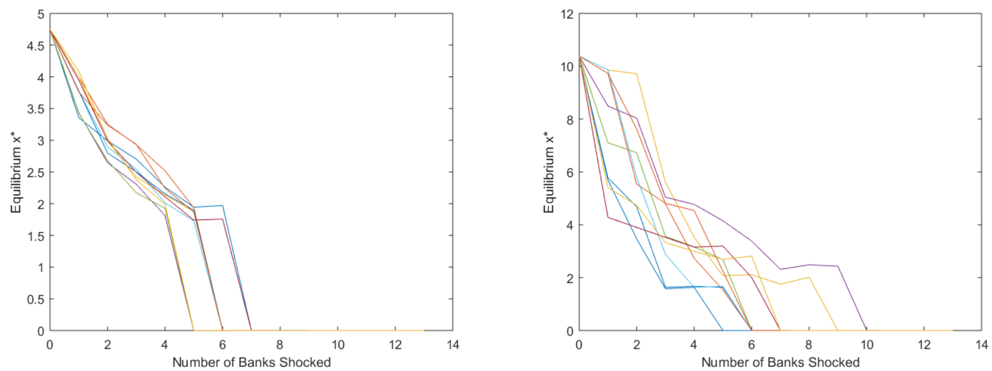
5.1 System-wide stress-testing against local shocks

I demonstrate a system-wide stress testing that examines how much local shocks a given network can sustain before triggering an interbank run. Based on that, the next subsection presents a financial stability measure that summarizes the distance from the given network to the tipping point.

Given a network $\mathbf{W}$, I consider counterfactual scenarios in which a set of shocked banks ($\mathcal{S}$) experience local insolvency shocks, before the start of the time-0 lending game. For example, it is suddenly learned that these banks are exposed to the toxic subprime mortgages. The set of remaining banks ($\mathcal{R}$) respond to the news by cutting lending connections to the shocked banks and redirect the flow of real investments outside of the interbank system. Effectively, $\mathcal{S}$ banks are excluded from the lending game and the stress test considers whether the cooperative interbank equilibrium falls apart in the remaining network denoted $\mathbf{W}_{\mathcal{R}}$.

Notice the stress test considered is not about asset loss contagion from failed borrowers to their lenders as in Eisenberg and Noe (2001) or Acemoglu et al. (2015). No investment is lost at least for the $\mathcal{R}$ banks as they are allowed to redirect lending away from the $\mathcal{S}$ banks ex ante. Instead, the initial trigger is the reduction in funding opportunities coming from the $\mathcal{S}$ banks. The test is about whether the initial funding liquidity reduction multiplies into a systemic liquidity crunch in the remaining network.

Figure 11 demonstrates the stress testing results with the two abovementioned observed networks. In each given network, I randomly generate 10 sequences of $\mathcal{S}$'s by adding banks to the shocked set one by one, in order to capture the expanding extent of local shocks. Each broken line tracks a random sequence and plots the systemic liquidity at the highest equilibrium.



(a) `Diebold13` (b) `Europe14`

Figure 11: Stress testing
The vertical axis is the highest equilibrium x^* at the remaining network $\mathbf{W}_{\mathcal{R}}$, calculated by solving Condition 2 numerically.

As the local shocks expand and the network unravels, the equilibrium systemic liquidity first decreases continuously, and then collapses to zero when the cooperative equilibrium disappears. The drops in the sequences are concrete examples that small and local shocks have triggered a system-wide interbank run.

Depending on the shocked sequence's importance to the system, the run is triggered after different numbers of shocked banks. This difference is more prominent in the case of `Europe14` (Panel b), in which the heterogeneity across banks is greater. The phenomenon implies that it is ineffective to measure how much stress a network can endure by counting the number of shocked banks. As expected, the specific position of a shocked bank in the network matters for how the shock will play out for the system. The next subsection solves this problem by proposing a stability measure rooted in the mean-field analysis that provides a clear outlook of how a systemic liquidity crunch would unfold.

5.2 A systemic stability measure

I present a measure of systemic stability based on how much effective connectedness loss a network can endure before crossing the tipping point and triggering an interbank run.

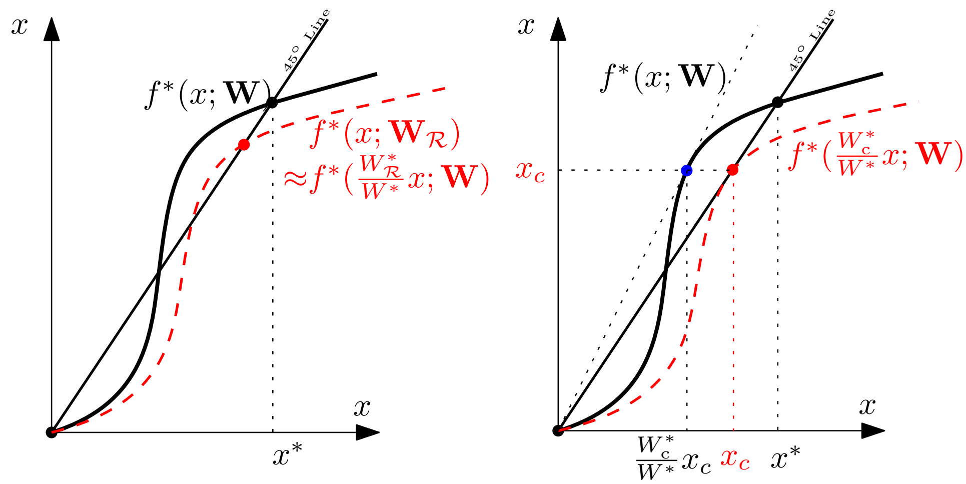
In a stress testing scenario where network $\mathbf{W}$ experiences local shocks and become $\mathbf{W}_{\mathcal{R}}$, the effective connectedness is reduced from W^* to $W^*_{\mathcal{R}}$. Under the approximation in subsection 4.5.2, the effect is a proportional contraction in every bank's funding connection (δ^{I}) to the imagined center x^*. That is, the input to the systemic response function contracts by the ratio of $\frac{W^*_{\mathcal{R}}}{W^*}$,

$$f^*(x; \mathbf{W}_{\mathcal{R}}) \approx f^* \left(\frac{W^*_{\mathcal{R}}}{W^*} x; \mathbf{W} \right). \tag{9}$$

In terms of the graph, the effect is a rightward stretch (pitch down) of the S-curve, shown in Figure 12 Panel (a).

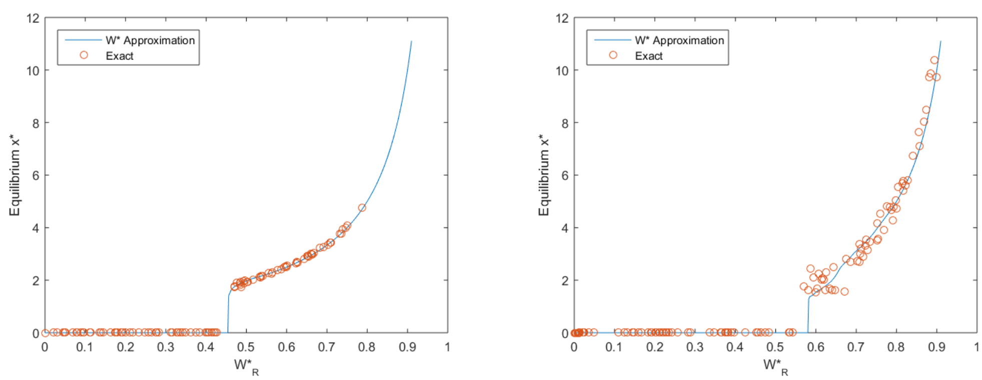
Panel (b) illustrates the tipping point—any further W^* reduction beyond the critical level W^*_{c} will make the cooperative equilibrium disappear and trigger an interbank run. Specifically, the *critical level* is defined as the smallest $W^*_{\mathcal{R}}$ that still supports a cooperative equilibrium, $W^*_{\mathrm{c}} := \min \left\{ W^*_{\mathcal{R}} \text{ s.t. } \exists x > 0, x = f^* \left(\frac{W^*_{\mathcal{R}}}{W^*} x; \mathbf{W} \right) \right\}$. And the *stability measure* of a network is defined as the proportion of tolerable reduction from the status quo relative to the critical level: $\Theta := \frac{W^*}{W^*_{\mathrm{c}}} - 1$. The critical level is normalized to have stability measure of zero.

Figure 13 demonstrates how Θ indeed provides a systemic stability measure. The



(a) small and continuous x^* reduction (b) critical situation

Figure 12: Systemic response functions under stress testing scenarios



(a) `Diebold13` (b) `Europe14`

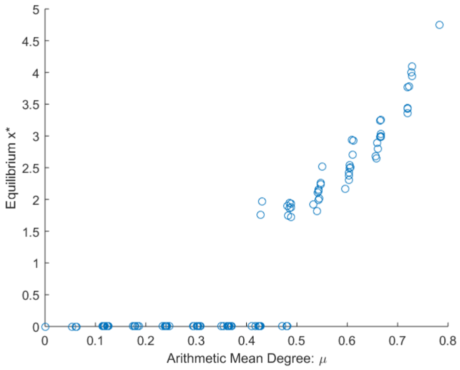
Figure 13: Effective connectedness and equilibrium systemic liquidity
Each circle represents a stress testing scenario ($\mathcal{R}$). The vertical coordinate is equilibrium systemic liquidity, same as Figure 11. The blue curves plot what systemic liquidity would be theoretically according approximation (9). The discontinuous drop is at the critical level W^*_c.

circle markers plot the same simulated stress testing scenarios in Figure 11, except the horizontal coordinates are now rearranged to the remaining effective connectedness ($W^*_{\mathcal{R}}$) instead of the number of banks shocked. The blue curves plot what systemic liquidity would be theoretically according approximation (9) as a function of $W^*_{\mathcal{R}}$. It is solved as the highest fixed point of the approximated systemic response (red dot in Figure 12a). The specific formula of is: $x^*(W^*_{\mathcal{R}}) := \max\left\{x \mid x = f^*\left(\frac{W^*_{\mathcal{R}}}{W^*}x; \mathbf{W}\right)\right\}$.

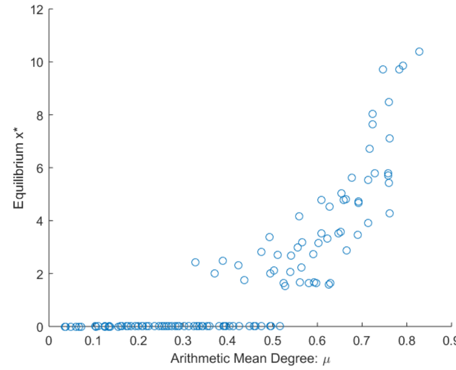
Firstly, figure 13 shows the blue curve traces the circles closely, implying that the approximation is numerically accurate. Importantly, the blue curve is theoretically calculated with the status quo network, without resorting to any simulation experiment. Once we work with the approximated model, a single number $W^*_{\mathcal{R}}$ (or Θ after a simple transformation) summarizes the equilibrium situation across the whole network, regardless of the composition of the shocks ($\mathcal{S}$) or the intricate connections among the remaining banks ($\mathcal{R}$). The blue curve gives a clear picture about how much more shock the status quo network can endure and how a systemic liquidity crunch would unfold. It first decreases continuously as $W^*_{\mathcal{R}}$ decreases, where local shocks propagates like ripples. The discontinuous drop demonstrates that the interbank run would happen when $W^*_{\mathcal{R}}$ decreases to the critical level W^*_{c} (or when Θ decreases to zero).

To further demonstrate Θ's (or $W^*_{\mathcal{R}}$) effectiveness in summarizing of equilibrium situation across the whole network, I contrast it with a less desirable statistic that does not produce such a clear picture. Figure 14 reproduces Figure 13 except with the average degree (μ) of the remaining network ($\mathbf{W}_{\mathcal{R}}$) on the horizontal axis. Different $\mathbf{W}_{\mathcal{R}}$'s with the same μ still behave quite differently in terms of systemic equilibrium, shown as the vertical dispersion of the circle markers. Moreover, there is not a clear cutoff μ between whether the cooperative equilibrium is still sustained or an interbank run is triggered. After all, according to Eq. (8), μ is only part of the decomposition of W^*, and it misses information about the correlation between the in- and out-degrees.

In summary, the orderly arrangement of Figure 13 in contrast to Figures 11 and 14 highlights that W^* effectively summarizes how network structures dictate the system-wide behavior.



(a) `Diebold13`



(b) `Europe14`

Figure 14: Remaining μ and equilibrium systemic liquidity

5.3 Optimal liquidity injection targets

During and after the financial crisis, governments carried out bailout programs by injecting liquidity to financial institutions to boost systemic stability. An important problem is how to identify the optimal liquidity injection targets given complicated interbank networks and limited government resources.

In the U.S., the core of the bailout plan was the Capital Purchase Program (CPP) as part of the Troubled Asset Relief Program (TARP). The policy goal was to prevent systemic failure rather than saving individual banks. As state in an speech by Bernanke in October 2008 "Government assistance should be provided with the greatest reluctance and only when the stability of the financial system, and thus the health of the broader economy, is at risk."[21] The purpose of the Treasury was to "promote financial stability, maintain confidence in the financial system, and permit institutions to continue meeting the credit needs of American consumers and businesses."[22]

One debatable aspect is the injection targets—which banks should receive more

[21]Ben S. Bernanke, "Stabilizing the Financial Markets and the Economy" (Speech, New York, October 15, 2008) https://www.federalreserve.gov/newsevents/speech/bernanke20081015a.htm

[22]Office of the Special Inspector General for the Troubled Asset Relief Program. "Quarterly Report to Congress, Second Quarter 2009," Quarterly Report to Congress (April 21, 2009).

liquidity? Primarily, the Treasury invested in preferred stocks of the "healthy, viable" financial institutions. Nine "systemically important" financial institutions received half of the total CPP investments, in the first round in October 2008. Later, smaller banks were enlisted mostly because it was "not politically feasible" to only favor large financial institutions (Scott, 2016).

This paper's insights about systemic behavior on the interbank network can speak to this problem. Suppose the government lends z_i dollar to each bank i at the same term as interbank lending. Then, bank i's interbank funding is increased to $y_i = \delta_i^{\mathrm{I}} x^* + z_i$, and the systemic response function becomes $f^*(x; \{z_i\}) = \sum_i \frac{\delta_i^{\mathrm{O}}}{n\mu} f\left(\delta_i^{\mathrm{I}} x + z_i\right)$. The marginal increase to systemic response of an additional dollar to bank i is

$$\frac{\partial}{\partial z_i} f^*(x; \{z_i\}) \Big|_{\mathbf{z}=\mathbf{0}} \propto \delta_i^{\mathrm{O}} f'(\delta_i^{\mathrm{I}} x).$$

This marginal benefit expression shows each injection target's different effect on the system and provides a guidance on choosing the optimal targets. First, banks with greater out-degrees (δ_i^{O}) are more effective targets. The liquidity injected to these banks will flow back to the financial system at a greater proportion, as opposed to providing liquidity to the bank's own real investments, if δ_i^{O} is small.

The second term $f'(\delta_i^{\mathrm{I}} x)$ implies that the banks that are in the liquidity hoarding region are more effective targets. A marginal dollar of funding liquidity to such banks will free up hoarded cash and have an amplified effect to illiquid lending ($f' > 1$). In contrast, the banks with too much or too little funding liquidity at most pass through the liquidity one-to-one ($f' = 1$).

There is a caveat about policy interpretation in the point above. Injecting liquidity to banks that are *currently* in the liquidity hoarding region ($\delta_i^{\mathrm{I}} x^* \in [y_l, y_h]$) boosts the status quo equilibrium x^* (the black dot in Figure 12). A more accurate interpretation of the stated policy is maximizing Θ, that is to make the status quo equilibrium more stable in the sense of being further away from the tipping point. Such a goal is achieved by maximizing the section of systemic response function around the critical level (the blue dot in Figure 12b). Therefore, it is justified to take precautions by targeting the banks that *would be* in the liquidity hoarding region in the critical situation. In detail, the critical situation is when x^* drops to x_c and in-degrees contracts by the ratio of $\frac{W_c^*}{W^*}$ (blue dot). That means targeting banks such that $\delta_i^{\mathrm{I}} \frac{W_c^*}{W^*} x_c \in [y_l, y_h]$. These banks are indeed "healthier and more viable" than those that *currently* in the liquidity hoarding region.

6 Conclusion

This paper characterizes interbank runs—banks running on banks as they mutually reinforce each other to withdraw interbank liquidity. The key mechanism is an interplay between the bank-level threats of retail depositor run and the system-level interbank run. The bank-level threat gives each bank an incentive to preemptively hoard cash and withdraw interbank lending. Although it is an individually precautionary behavior to hoard liquidity, the interconnected strategic interactions cause instability at the system level. The discontinuous and system-wide liquidity crunch is modeled as a coordination failure among banks. The model reflects the stylized facts observed during the 2007-2008 financial crises: interbank liquidity drops discontinuously across the financial system; real investments dip; safe and liquid asset holding in the financial system expands.

The model addresses the theoretical question of how small and local shocks can trigger a large and system-wide liquidity crunch, especially in an environment of indirectly connected strategic interactions. It is not to claim that direct and centralized mechanisms, for example public information and centralized markets, do not matter. The paper focuses on a situation in which pair-wise strategic interactions play the primary role, and demonstrate some of the classical dynamics, for example self-fulfilling runs, indeed carry over.

The equilibrium condition is complex due to the high-dimensional network structure and the non-linear best response function, both of which are necessary ingredients to the central message of the paper. The mean-field approximation takes a holistic view of the complicated strategic interactions and produce a one-dimensional characterization of the equilibrium dynamics. It affords succinct and interpretable insights about how the network structure affects the system-wide economic dynamics.

The analysis has immediate policy implications for monitoring and managing systemic risks. I demonstrate a system-wide stress testing against local shocks, provide a stability measure of any given network structure, and discuss the most effective liquidity injection targets for government bail-outs. The method yields a concise description of the complicated strategic interactions on networks and paves the way for further empirical and policy researches that take the financial network as an input.

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Appendix

A Additional Details

A.1 Time-0 asset allocation best response functions

The analytical expressions of the best response functions as illustrated in Figure 4.

$$x_i = f(y_i) = \begin{cases} y_i \\ \frac{y_i-\gamma}{1-\beta} \\ y_i + d \end{cases}, \quad c_i = g(y_i) = \begin{cases} d & y_i \in [0, y_l) \\ \frac{-\beta}{1-\beta} y_i + \frac{1}{1-\beta}\gamma + d & y_i \in [y_l, y_h) \\ 0 & y_i \in [y_h, +\infty) \end{cases}$$

with $y_l = \frac{\gamma}{\beta}$ and $y_h = \frac{1-\beta}{\beta} d + \frac{\gamma}{\beta}$.

A.2 Locally stable equilibrium

Definition 3 (Locally stable equilibrium)**.** *A one-dimensional fixed point x of equation $x = f(x)$ is locally stable if there exist $\epsilon > 0$ such that*

$$\begin{aligned} f(x) &> x, && \forall x \in (x-\epsilon, x) \cap [0, \infty), \textit{ and} \\ f(x) &< x, && \forall x \in (x, x+\epsilon) \cap [0, \infty). \end{aligned}$$

If f is locally differentiable, the condition for locally stable is refined to simply $f'(x) < 1$.

A.3 Proof of Proposition 1

Suppose there are two different vectors $\{x^{[1]}\}, \{x^{[2]}\}$ that both satisfy Condition 2. Then

$$x_i^{[1]} - x_i^{[2]} = f(\sum_j w_{i,j} x_j^{[1]}) - f(\sum_j w_{i,j} x_j^{[2]}), \quad \forall i. \tag{10}$$

$$|x_i^{[1]} - x_i^{[2]}| = |f(\sum_j w_{i,j} x_j^{[1]}) - f(\sum_j w_{i,j} x_j^{[2]})| \tag{11}$$

$$\leq |\sum_j w_{i,j} x_j^{[1]} - \sum_j w_{i,j} x_j^{[2]}| \tag{12}$$

$$= |\sum_j w_{i,j}(x_j^{[1]} - x_j^{[2]})| \tag{13}$$

$$\leq \sum_j w_{i,j} |x_j^{[1]} - x_j^{[2]}|, \quad \forall i. \tag{14}$$

Take sum across i on both sides,

$$\sum_i |x_i^{[1]} - x_i^{[2]}| \leq \sum_j \delta_j^{\mathrm{O}} |x_j^{[1]} - x_j^{[2]}|. \tag{15}$$

Since $0 \leq \delta_j^{\mathrm{O}} < 1, \forall j$, then there is an obvious contradiction. Therefore, there cannot be two distinct $\{x_i\}$ that both satisfy Condition 2.

A.4 Benchmark f

The specific function f for the benchmark numerical exercises in Subsection 4.4 is constructed according to the definition in equation (10), by assigning the primitives as $d = 1, \gamma = 0.05, \beta = 0.9$. Figure 15 contains the exact graph.

A.5 Numerical Comparative Statics with respect to β

The plots are calculated under the same construction as Figure 9, but with changing β at the horizontal axis. The red and blue curves plot the highest systemic liquidity (x^*) calculated for the exact equilibrium and the systemic liquidity equilibrium respectively.

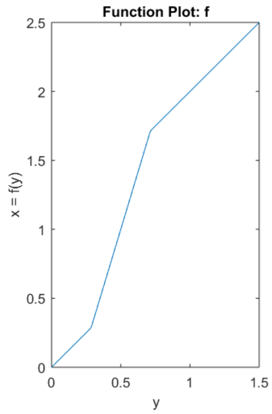
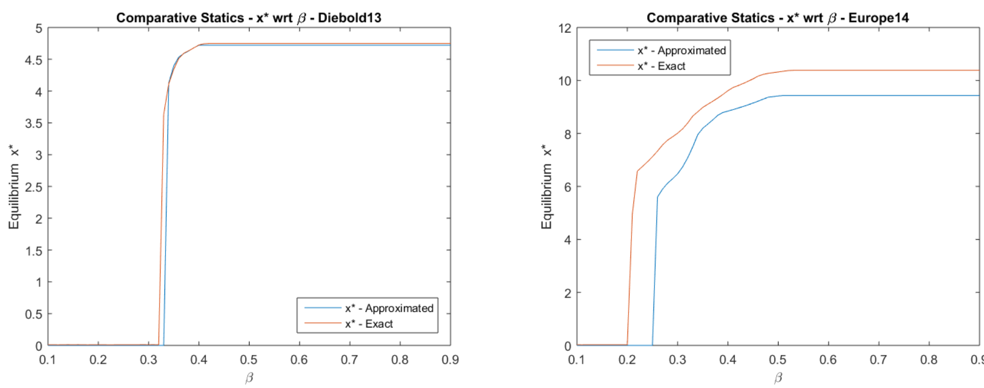


Figure 15: **Benchmark f**



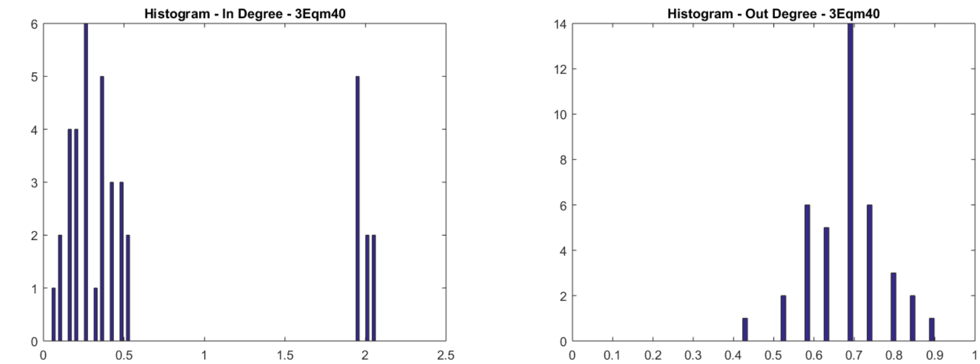
(a) `Diebold13`

(b) `Europe14`

Figure 16: **Numerical Comparative Statics with respect to β**

A.6 Additional details about network `3Eqm40`

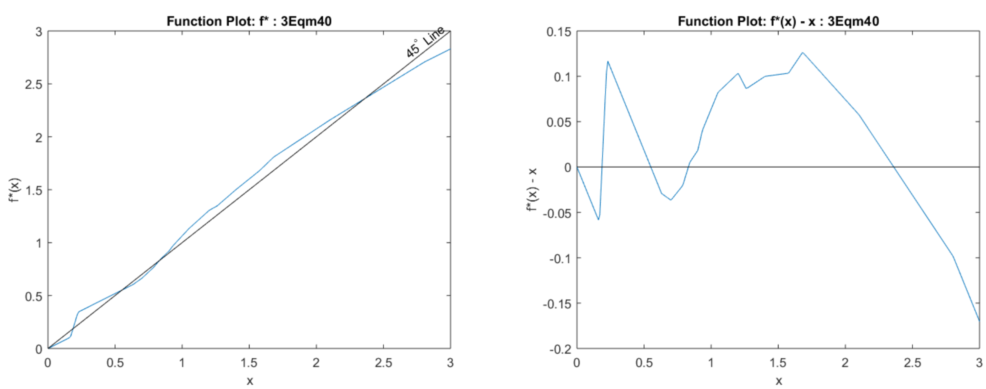
The in-degree distribution is binomial, which is necessary for having more than two (locally stable) equilibria.



(a) In-Degree Distribution

(b) Out-Degree Distribution

Figure 17: Degree Distributions of `3Eqm40` in Figure 10



(a) Function Plot: $f^*(x)$

(b) Function Plot: $f^*(x) - x$

Figure 18: Systemic Response Function f^* of `3Eqm40`as in Figure 10
Panel (a) shows $f^*(x)$ has two consecutive S-shaped regions. Panel (b) takes the difference with x to show how $f^*(x)$ crosses the 40° line several times, constituting three locally stable equilibria.