11
Financial markets and turbulence

One of the objections often leveled at the approach of physicists working with economic systems is that this kind of activity cannot be a branch of physics because the 'equation of motion of the process' is unknown. But if this criterion - requiring that the Hamiltonian of the process be known or obtainable - were to be applied across the board, several fruitful current research fields in physics would be disqualified, e.g., the modeling of friction and many studies in the area of granular matter. Moreover, a number of problems in physics that are described by a well defined equation - such as turbulence [61] - are not analytically solvable, even with sophisticated mathematical and physical tools.

On a qualitative level, turbulence and financial markets are attractively similar. For example, in turbulence, one injects energy at a large scale by, e.g., stirring a bucket of water, and then one observes the manner in which the energy is transferred to successively smaller scales. In financial systems 'information' can be injected into the system on a large scale and the reaction to this information is transferred to smaller scales - down to individual investors. Indeed, the word 'turbulent' has come into common parlance since price fluctuations in finance qualitatively resemble velocity fluctuations in turbulence. Is this qualitative parallel useful on a quantitative level, such that our understanding of turbulence might be relevant to understanding price fluctuations?

In this chapter, we will discuss fully developed turbulence in parallel with the stochastic modeling of stock prices. Our aim is to show that cross-fertilization between the two disciplines might be useful, not that the turbulence analogy is quantitatively correct. We shall find that the formal correspondence between turbulence and financial systems is not supported by quantitative calculations.

11.1 Turbulence

Turbulence is a well defined but unsolved physical problem which is today one of the great challenges in physics. Among the approaches that have been tried are analytical approaches, scaling arguments based on dimensional analysis, statistical modeling, and numerical simulations.

Consider a simple system that exhibits turbulence, a fluid of kinematic viscosity v flowing with velocity V in a pipe of diameter L. The control parameter whose value determines the 'complexity' of this flowing fluid is the Reynolds number,

$$\mathrm{Re} \equiv \frac{LV}{\nu}. \tag{11.1}$$

When Re reaches a particular threshold value, the 'complexities of the fluid explode' as it suddenly becomes turbulent.

The equations describing the time evolution of an incompressible fluid have been known since Navier's work was published in 1823 [128], which led to what are now called the Navier-Stokes equations,

$$\frac{\partial}{\partial t}\mathbf{V}(\mathbf{r},t) + (\mathbf{V}(\mathbf{r},t)\cdot\nabla)\mathbf{V}(\mathbf{r},t) = -\nabla P + \nu\nabla^2\mathbf{V}(\mathbf{r},t), \tag{11.2}$$

and

$$\nabla\cdot\mathbf{V}(\mathbf{r},t) = 0. \tag{11.3}$$

Here V($\mathbf{r}$,t) is the velocity vector at position $\mathbf{r}$ and time t, and P is the pressure. The Navier-Stokes equations characterize completely 'fully developed turbulence', a technical term indicating turbulence at a high Reynolds number. The analytical solution of (11.2) and (11.3) has proved impossible, and even numerical solutions are impossible for very large values of Re.

In 1941, a breakthrough in the description of fully developed turbulence was achieved by Kolmogorov [82-84]. He showed that in the limit of infinite Reynolds numbers, the mean square velocity increment

$$\langle[\Delta V(\ell)]^2\rangle = \langle[V(r+\ell) - V(r)]^2\rangle \tag{11.4}$$

behaves approximately as

$$\langle[\Delta V(\ell)]^2\rangle \sim \ell^{2/3} \tag{11.5}$$

in the inertial range, where the dimensions are smaller than the overall dimension within which the fluid's turbulent behavior occurs and larger than the typical length below which kinetic energy is dissipated into heat.

Kolmogorov's theory describes well the second-order $\langle[\Delta V(\ell)]^2\rangle$ and provides the exact relation for the third-order $\langle[\Delta V(\ell)]^3\rangle$ moments observed in experiments, but fails to describe higher moments.

In fully developed turbulence, velocity fluctuations are characterized by an intermittent behavior, which is reflected in the leptokurtic nature of the pdf of velocity increments. Kolmogorov theory is not able to describe the intermittent behavior of velocity increments. In the experimental studies of fully developed turbulence, experimentalists usually measure the velocity $V(t)$ as a function of time. From this time series, the spatial dependence of the velocity $V(\ell)$ can be obtained by making the Taylor hypothesis [124].

11.2 Parallel analysis of price dynamics and fluid velocity

Turbulence displays both analogies with and differences from the time evolution of prices in a financial market. To see this, we discuss the results of a parallel analysis [112] of two systems, the time evolution of the S&P 500 index and the velocity of a turbulent fluid at high Reynolds number. Both processes display intermittency and non-Gaussian features at short time intervals. Both processes are nonstationary on short time scales, but are asymptotically stationary. A better understanding and modeling of stochastic processes that are only asymptotically stationary is of potential utility to both fields.

Specifically, we consider the statistical properties of (i) the S&P 500 high-frequency time series recorded during the six-year period 1984 to 1989 and (ii) the wind velocity recorded in the atmospheric surface layer about 6 m above a wheat canopy in the Connecticut Agricultural Research Station.† Similarities and differences are already apparent by direct inspection of the time evolutions of the index and the velocity of the fluid, as well as the successive measurements of both time series.

First, we compare the time evolution of the S&P 500 index (Fig. 11.1a) and the time evolution of fluid velocity (Fig. 11.2a). We also display one-hour changes in the S&P 500 index (Fig. 11.1b) and fluid velocity changes at the highest sampling rate (Fig. 11.2b). By analyzing the temporal evolution of successive increments in both signals, we can obtain useful information concerning the statistical properties of the two signals. A quantitative analysis can be performed by considering the volatility for financial data, and the square root of the second moment of velocity fluctuations for turbulence data.

† K. R. Sreenivasan kindly provided the data on fully developed turbulence.

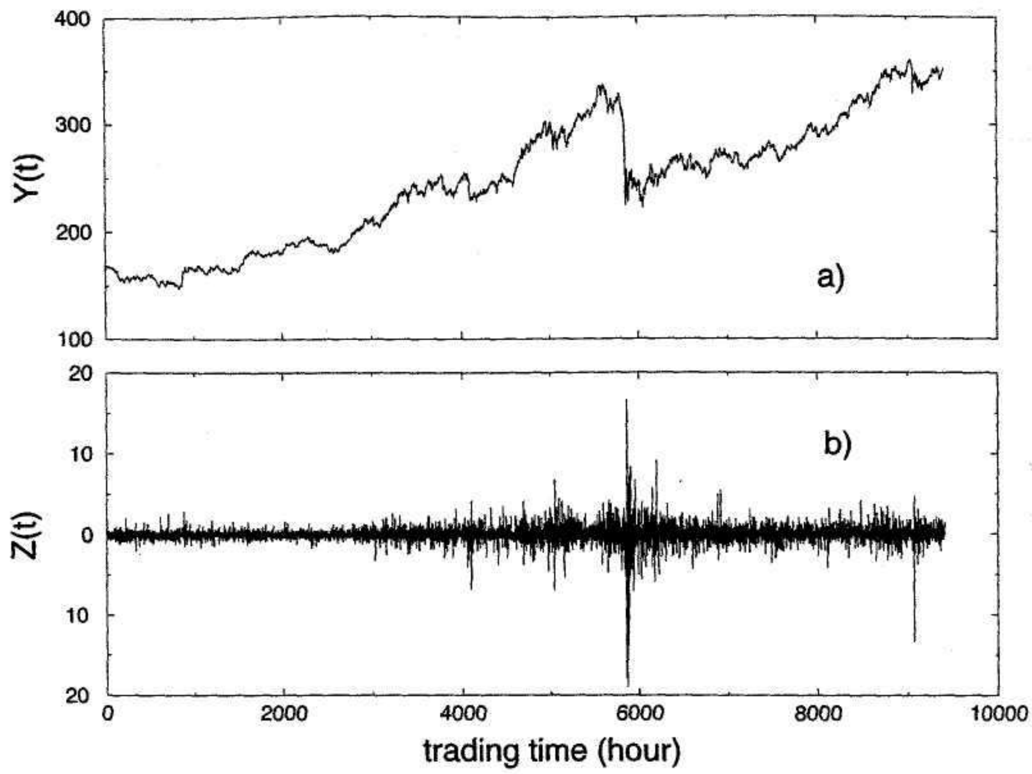

Fig. 11.1. (a) Time evolution of the S&P 500, sampled with a time resolution Δt = 1 h, over the period January 1984 to December 1989. (b) Hourly variations of the S&P 500 index in the 6-year period January 1984 to December 1989.

Both sets of data are seen in Fig. 11.3 to be well described by power laws.

$$\sigma(\Delta t) \sim (\Delta t)^{\nu}, \tag{11.6}$$

but with quite different values of the exponent v. Index changes are essentially uncorrelated (the observed value of ν = 0.53 is extremely close to 1/2, the value expected for uncorrelated changes), while velocity changes are anti-correlated (ν = 0.33 < 1/2). Thus the quantitative difference between the two forms of behavior implies that the nature of the time correlation between two successive changes must be different for the two processes. Indeed, the time evolutions of the index and the velocity in Figs. 11.1a and 11.2a look quite different, since there is a high degree of anticorrelation in the velocity. This difference is also visually apparent, from Fig. 11.2b, which is approximately symmetric about the abscissa, whereas Fig. 11.1b is not.

This difference between the two stochastic processes is also observable in the power spectra of the index and velocity time series (see Fig. 11.4). When both obey Eq. (6.21) over several frequency decades, the exponents η are quite different. For the S&P 500 index, η = 1.98, so the spectral density is essentially identical to the power spectrum of an uncorrelated random

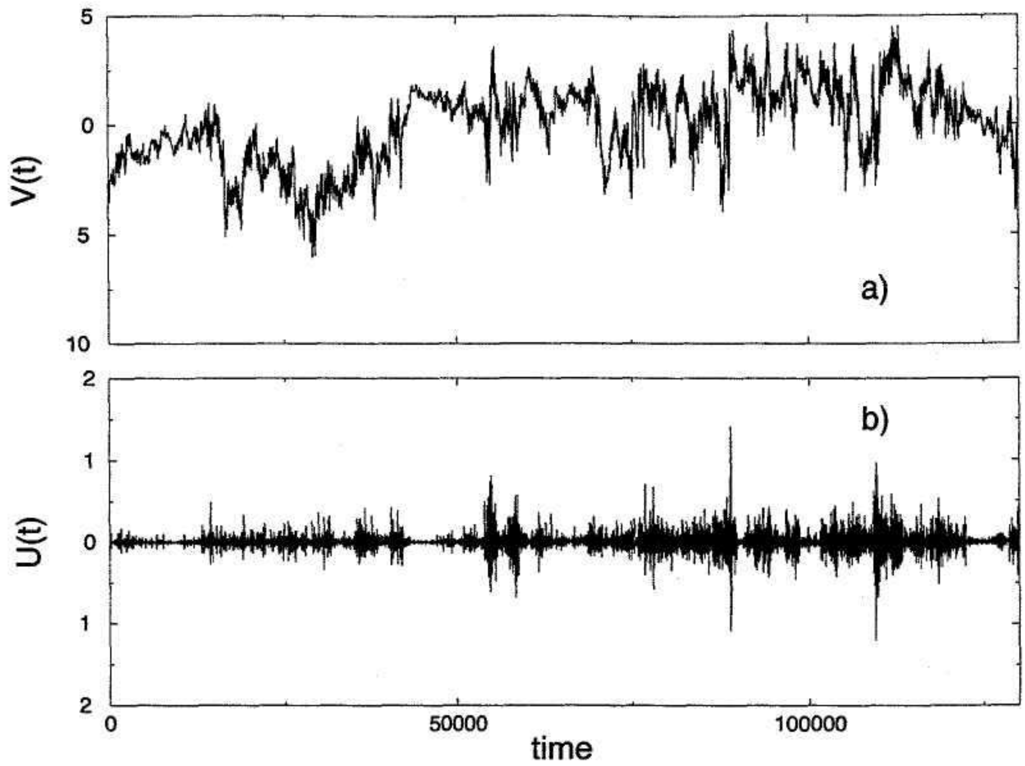

Fig. 11.2. Time evolution of the fluid velocity in fully developed turbulence. (a) Time evolution of the wind velocity recorded in the atmosphere at extremely high Reynolds number; the Taylor microscale Reynolds number is of the order of 1,500. The time units are given in arbitrary units. (b) Velocity differences of the time series given in (a). Adapted from [113].

process ($\eta = 2$). For the velocity time series, $\eta \approx 5/3$ in the inertial range and $\eta \approx 2$ in the dissipative range.

Ghashghaie *et al.* [64] have proposed a formal analogy between the velocity of a turbulent fluid and the currency exchange rate in the foreign exchange market. They supported their conclusion by observing that when measurements are made at different time horizons Δt, the shapes of the pdf of price increments in the foreign exchange market and the pdf of velocity increments in fully developed turbulence both change. Specifically, the shapes of both pdfs display leptokurtic profiles at short time horizons. However, the parallel analysis of the two phenomena [112,113] shows that the time correlation is completely different in the two systems (Fig. 11.4). Moreover, stochastic processes such as the TLF and the GARCH(1,1) processes also describe a temporal evolution of the pdf of the increments which evolves from a leptokurtic to a Gaussian shape, so such behavior is not specific to the velocity fluctuations of a fully turbulent fluid.

To detect the degree of similarity between velocity fluctuations and index

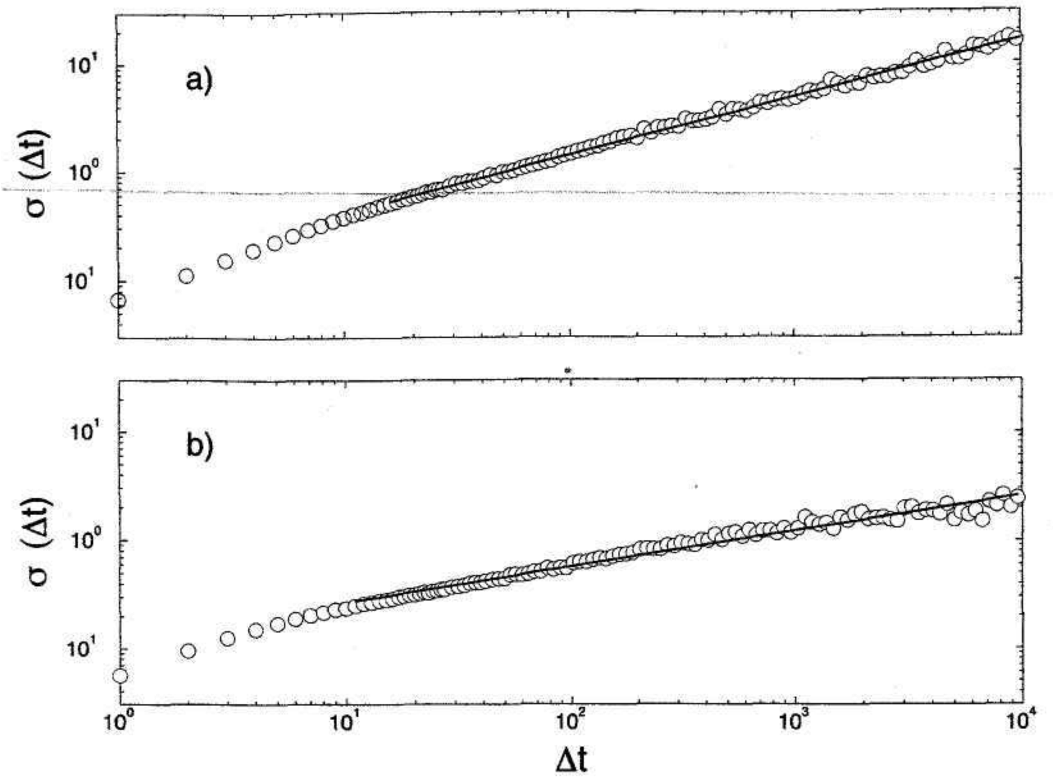

Fig. 11.3. (a) Standard deviation $\sigma(\Delta t)$ of the probability distribution *P(Z)* characterizing the increments $Z_{\Delta t}(t)$ plotted double logarithmically as a function of Δt for the S&P 500 time series. After a time interval of superdiffusive behavior ($0 < \Delta t < 15$ minutes), a diffusive behavior close to the one expected for a random process with uncorrelated increments is observed; the measured diffusion exponent 0.53 (the slope of the solid line) is close to the theoretical value 1/2. (b) Standard deviation $\sigma(\Delta t)$ of the probability distribution *P(U)* characterizing the velocity increments $U_{\Delta t}(t) \equiv V(t + \Delta t) - V(t)$ plotted double logarithmically as a function of Δt for the velocity difference time series in turbulence. After a time interval of superdiffusive behavior ($0 < \Delta t < 10$), a subdiffusive behavior close to the one expected for a fluid in the inertial range is observed. In fact, the measured diffusion exponent 0.33 (the slope of the solid line) is close to the theoretical value 1/3. Adapted from [112].

changes, consider the probability of return to the origin, $P_{\Delta t}(U = 0)$, as a function of Δt for a turbulent fluid, obtained by following the same procedure used to obtain Fig. 9.3. We display in Fig. 11.5 the measured $P_{\Delta t}(U = 0)$. We also show the estimated $P_G(U = 0)$ obtained starting from the measured variance of velocity changes $\sigma(\Delta t)$ and assuming a Gaussian shape for the distribution ($P_G(U = 0) = 1/\sqrt{2\pi}\sigma(\Delta t)$). The difference between each pair of points is a measure of the ratio $P_{\Delta t}/P_G$, and quantifies the degree of non-Gaussian behavior of the velocity differences. We note that the turbulence process becomes increasingly Gaussian as the time interval Δt increases, but we do not observe any scaling regime.

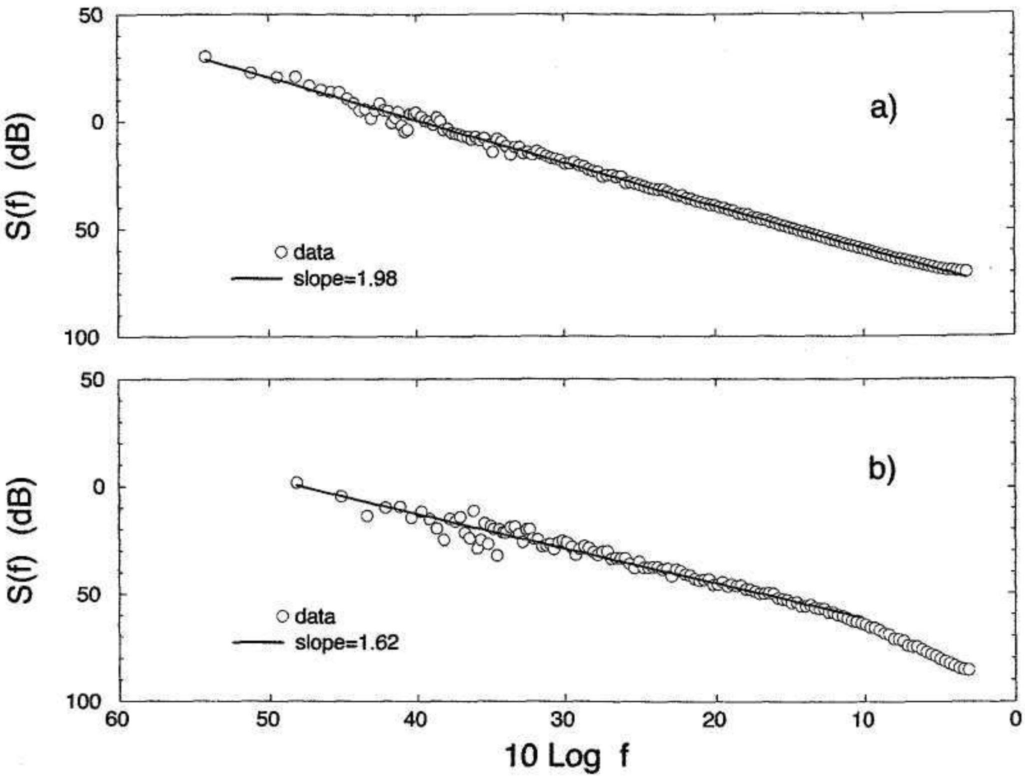

Fig. 11.4. (a) Spectral density of the S&P 500 time series. The $1/f^2$ power-law behavior expected for a random process with increments that are pairwise independent is observed over a frequency interval of more than four orders of magnitude. (b) Spectral density of the velocity time series. The $1/f^{5/3}$ inertial range (low frequency) and the dissipative range (high frequency) are clearly observed. Adapted from [112].

11.3 Scaling in turbulence and in financial markets

The concept of scaling is used in a number of areas in science, emerging when an investigated process does not exhibit a typical scale. A power-law behavior in the variance of velocity measurements in turbulence (see Eq. (11.5)) is an example of a scaling behavior, as is the power-law behavior of volatility at different time horizons in financial markets (see Eq. (11.6)). The reasons underlying the two scaling behaviors are, however, quite different. In the turbulence case, the 2/3 exponent of the distance ℓ is a direct consequence of the fact that, in the inertial range, the statistical properties of velocity fluctuations are uniquely and universally determined by the scale ℓ and by the mean energy dissipation rate per unit mass ϵ.

Next we show that dimensional consistency requires that the mean square velocity increment assumes the form

$$\langle [\Delta V(\ell)]^2 \rangle = C\epsilon^{2/3}\ell^{2/3}, \tag{11.7}$$

where C is a dimensionless constant. This equation is the only one possi-

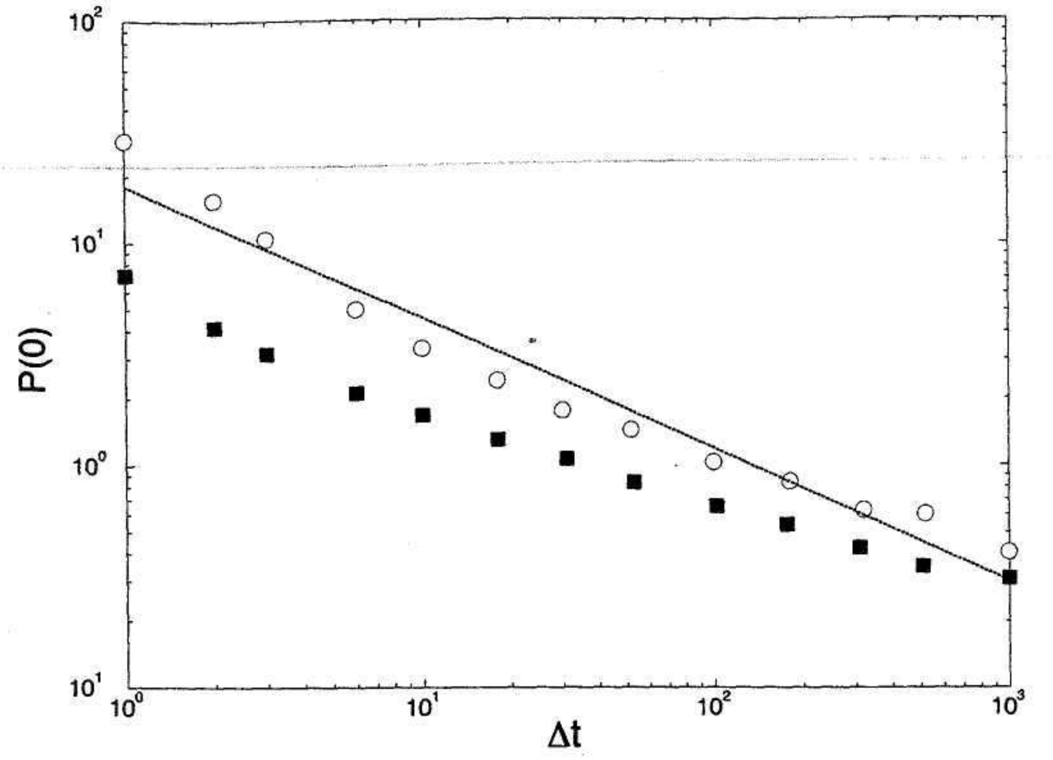

Fig. 11.5. Measured probability of return to the origin of the velocity of a turbulent fluid. Probability of return to the origin P(0) (open circles) and probability of return assuming a Gaussian shape $P_G(0)$ (filled squares) are shown as functions of the time sampling interval Δt. Again, the two measured quantities differ in the full interval, implying that the profile of the PDF must be non-Gaussian. However, in this case, a single scaling power-law behavior does not exist for the entire time interval spanning three orders of magnitude. The slope of the best linear fit (which is of quite poor quality) is —0.59$\pm$0.11, while a Gaussian distribution would have slope -0.5. Adapted from [112].

ble because the energy dissipation rate per unit mass has the dimensions $[L]^2[T]^{-3}$. In fact, if we define a to be the exponent of ϵ, and b to be the exponent of ℓ in Eq. (11.7), then dimensional consistency requires that

$$\frac{[L]^2}{[T]^2} = \frac{[L]^{2a}}{[T]^{3a}}[L]^b, \tag{11.8}$$

where the equality indicates that both sides of the equation have the same dimension. This condition is satisfied by equating powers of L and T,

$$\begin{cases} 2 = 2a + b \\ 2 = 3a. \end{cases} \tag{11.9}$$

Hence a = 2/3 and b = 2/3.

Hence Kolmogorov's law (11.7) is directly related to the observation that the mean energy dissipation rate is the only relevant quantity in the inertial range, and to the requirement of dimensional consistency. In fact, (11.7) loses its validity when other observables become relevant, such as occurs in two-dimensional turbulence, in which there is vorticity conservation.

Note that the scaling properties observed in a stochastic process for the variance at different time horizons and for the probability of return to the origin need not be related. In certain specific (and common) cases they are related, e.g., in Gaussian or fractional Brownian motion stochastic processes. But they are not related

- in turbulent dynamics, where scaling is present in the variance of velocity changes but not present in the probability of return to the origin, and
- in truncated Levy flights, where the scaling exponent of the variance $\sigma^2(t) \sim t$ is always one, but the scaling exponent of the probability of return to the origin is $-1/\alpha$ for time intervals shorter than the crossover time.

For financial markets, the scaling law of the volatility at different time horizons has a different origin, being a direct consequence of two properties. The first is that successive price changes are uncorrelated, while the second is that the variance of price changes is finite. Hence, unlike turbulence, the scaling property of volatility is related to statistical properties of the underlying stochastic process. Thus we have seen that although scaling can be observed in disparate systems, the *causes* of the scaling need not be the same. Indeed, the fundamental reasons that lead to scaling in turbulence differ from those that lead to scaling in financial markets.

11.4 Discussion

The parallel analysis of velocity fluctuations in turbulence and index (or exchange rate) changes in financial markets shows that the same statistical methods can be used to investigate systems with known, but unsolvable, equations of motion, and systems for which a basic mathematical description of the process is still lacking. In the two phenomena we find both

- *similarities:* intermittency, non-Gaussian pdf, and gradual convergence to a Gaussian attractor in probability, and
- *differences:* the pdfs have different shapes in the two systems, and the probability of return to the origin shows different behavior - for turbulence we do not observe a scaling regime whereas for index changes we observe a scaling regime spanning a time interval of more than three orders of magnitude. Moreover, velocity fluctuations are anticorrelated whereas index (or exchange rate) fluctuations are essentially uncorrelated.

A closer inspection of Kolmogorov's theory explains why the observation of this difference is not surprising. The 2/3 law for the evolution of the variance of velocity fluctuations, Eq. (11.5), is valid only for a system in which the dynamical evolution is essentially controlled by the energy dissipation rate per unit mass. We do not see any rational reason supporting the idea that assets in a financial market should have a dynamical evolution controlled by a similar variable. Indeed no analog of the 2/3 law appears to hold for price dynamics.

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