Lecture 13 - Confidence Intervals (II)

Population

X

$\downarrow$

μ

Pop 1 — Pop 2

μ_1 — μ_2

$\mu_1 - \mu_2$?

Sample 1 — Sample 2

n_1 — n_2

lower — upper

$\bar{x}_1 - \bar{x}_2$

$\downarrow$

best guess

$$\mu_1 - \mu_2 = (\bar{x}_1 \pm \bar{x}_2) \pm ?$$

$$\hookrightarrow z_{\frac{\alpha}{2}} \cdot \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

Confidence Intervals

μ_L — μ — μ_U

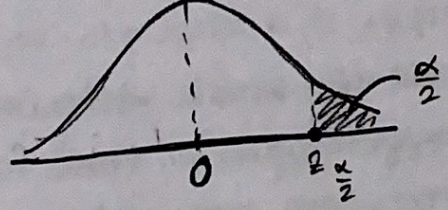
Adım 1: μ'nün en iyi tahmini $\bar{x}$'dır.

($\bar{x}$: sample mean; $\bar{x} = \frac{1}{n} \cdot \sum x_i$)

$\underbrace{\mu_L \quad \bar{x}}_{e} \; \underbrace{\bar{x} \quad \mu_U}_{e}$

$$\mu_L = \bar{x} - e = \bar{x} - z_{\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}}$$

$$\mu_U = \bar{x} + e = \bar{x} + z_{\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}}$$



Paired obs. → 1 sample var
Bu sample için 2 farklı yaklaşıma bakılıyor. Bunlar kıyaslanıyor.

1 Sample

Differences D		Obs. before	Obs. after
-1	V1	5	4
-0.9	V2	5.5	4.6
	⋮	⋮	⋮
-0.2	V10	3.8	3.6

$\bar{d} = \frac{1}{n} \cdot \sum \text{differences}$

$S_d = \frac{1}{n \cdot (n-1)} \left\{ n \cdot \sum d_i^2 - \left(\sum d_i \right)^2 \right\}$

lower (|) upper

? $\bar{d}$?

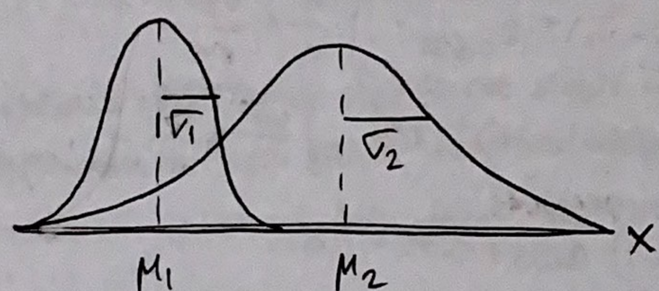
$\mu_d = \bar{d} \pm t_{\frac{\alpha}{2}} \cdot \frac{S_d}{\sqrt{n}}$

Burada tek sample vardır. Fakat hesaplamalar 2 tanedir birbirleriyle bağlıdırlar Fakat 2 ayrı sample için yapılan mean'ler farkı aynı şekilde hesaplanmaz çünkü daha farklıdır

Two Samples: Estimating The Difference Between Two Means

Eger $\bar{x}_1$ ve $\bar{x}_2$, varyansları sırasıyla σ_1^2 ve σ_2^2 ve sizeları n_1 ve n_2 dan 2 bagımsız popülasyona aitse, $\mu_1 - \mu_2$ confidence interval şöyle bulunur:

$$\underset{\text{aralıgı}}{\mu_1 - \mu_2} : (\bar{x}_1 - \bar{x}_2) \pm z_{\frac{\alpha}{2}} \cdot \underbrace{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}_{\text{combined standard error}}$$



Example: The CO_2 level in the atmosphere is measured by 2 instruments. The 1st instrument has a known precision (σ) of 2.66 ppm while the precision of the 2nd is 3.9 ppm. The 1st instrument takes 29 samples and determines the mean concentration to be 401.3 ppm. The 2nd instrument takes 30 samples and determines the mean concentration to be 400.8 ppm. Calculate a 95% confidence interval for the difference between the 2 measurement. Comment the result

$\sigma_1 = 2.66$ $\quad \sigma_2 = 3.9$

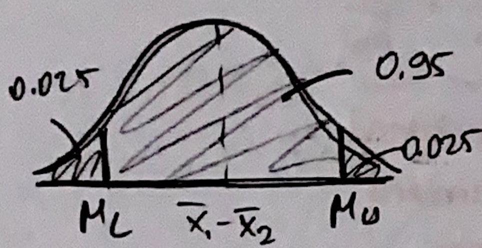
$n_1 = 20$ $\quad n_2 = 30$

$\bar{X}_1 = 401.3$ $\quad \bar{X}_2 = 400.8$

95% confidence interval? $(\mu_1 - \mu_2)$

X_1: Measurements with instr 1

X_2: // // // 2



$1-\alpha = 0.95 \rightarrow \alpha = 0.05 \rightarrow \frac{\alpha}{2} = 0.025$

$$\mu_1 - \mu_2 \,\Big|\, (\bar{x}_1 - \bar{x}_2) \pm z_{0.025} \cdot \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

$$(401.3 - 400.8) \pm 1.96 \cdot \sqrt{\frac{2.66^2}{20} + \frac{3.9^2}{30}} \quad \begin{matrix} \rightarrow 2.318 \ (\mu_U) \\ \rightarrow -1.318 \ (\mu_L) \end{matrix}$$

normcdf

$P(z < z_{0.025}) = 0.025 + 0.95 = 0.975$

$P^{-1}(0.975) = z_{0.025} \rightarrow 1.96$

normcdf^{-1}

$$-1.318 < \mu_1 - \mu_2 < 2.318$$

∴ Popülasyon ortalamaları arasındaki farkın -1.318 ve 2.318 aralığı içerisinde olduğundan %95 eminiz, 0, sınırın içerisinde iyi bir şekilde yer aldığından, sonuç $\mu_1 = \mu_2$ ile tutarlı görünüyor yani 2 araç birbiriyle uyumlu!

instrument

∴ Interval küçük old. ve 0'ı da içerdiği için bu hesaplamaların agree olduklarını söyleyebiliriz.

! Aynı probleme Hypothesis Testing ile de bakabiliriz.
Null hypothesis $H_0: \mu_1 = \mu_2$ yani $\mu_1 - \mu_2 = 0$ alalım.
Alternative // $H_A: \mu_1 - \mu_2 > 0$ olur.

$$P(\bar{X}_1 - \bar{X}_2 > 0.5 \mid \mu_1 - \mu_2 = 0)$$

$$= P\left(\frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}} > \frac{0.5 - 0}{0.9192388}\right)$$

$$P(z > 0.54393) = 1 - \text{normcdf}(0.54393) = 0.293$$

∴ P-value %29.3'tür. Bu significant olmaz ve bu nedenle 2 instrument'in aynı fiilerde olduğunu iddia eden null hypothesis'i reddetmek için yeterli kanıtımız yoktur.

Eğer σ_1 ve σ_2'yi bilmiyorsak? s_1 ve s_2'yi kullanırız:

Unknown Variances

Genellikle σ_1 ve σ_2 standard dev. (popülasyonun) bilmediğimizde, ki genelde bilemeyiz, bunları sample'dan tahmin etmemiz gerekir. σ_1 ve σ_2 yerine s_1 ve s_2 yazılır ve z yerine t yazılır ve t-distribution kullanılır.

$$\mu_1 - \mu_2 : (\bar{X}_1 - \bar{X}_2) \pm t_{\frac{\alpha}{2}} \cdot \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$
aralığı

Tabi burada unutulmaması gereken bir şey t-distribution'ın ν'ye de bağlı olmasıdır.

$$\boxed{\nu = n-1}$$

$$\nu = \frac{\left(s_1^2/n_1 + s_2^2/n_2\right)^2}{(s_1^2/n_1)^2/\nu_1 + (s_2^2/n_2)^2/\nu_2}$$

Önceden bunu kullanıyorduk, çünkü tek sample'ımız vardı. Fakat şu an 2 tane var. Bu yüzden soldaki formülü kullanıyoruz!

! ν mutlaka şu aralıkta olmalıdır:

$$\min(\nu_1, \nu_2) < \nu < \nu_1 + \nu_2$$

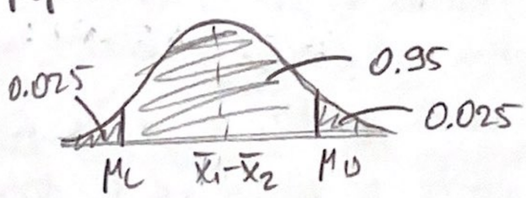
EXAMPLE: A study was conducted by the Dept. of Zoology at the Virginia Tech to estimate the difference in the amounts of the chemical orthophosphorus measured at 2 different stations on the James River. Orthophosphorus was measured in mgrams per liter. Fifteen samples were collected from station 1 and 12 samples were obtained from station 2. The 15 samples from station 1 had an average orthophosphorus content of 3.84 mgrams per liter and a standard dev. of 3.07 mgrams per liter, while the 12 samples from station 2 had an avg. content of 1.49 mgrams per liter and a standard dev. of 0.80 mg per liter. Find a % 95 confidence interval for the difference in the true avg. orthophosphorus contents at these 2 stations, assuming that the observations come from normal populations with different variances.

$n_1 = 15$, $\nu_1 = 14$

$n_2 = 12$, $\nu_2 = 11$

$\bar{x}_1 = 3.84$, $s_1 = 3.07$

$\bar{x}_2 = 1.49$, $s_2 = 0.80$



$$\mu_1-\mu_2=(\bar{x}_1-\bar{x}_2)\pm t_{0.025}\cdot\sqrt{\frac{s_1^2}{n_1}+\frac{s_2^2}{n_2}}$$

$$\mu_1-\mu_2=(3.84-1.49)\pm t_{0.025}\cdot\sqrt{\frac{3.07^2}{15}+\frac{0.8^2}{12}} \quad \sim 0.83$$

$$v=\frac{(s_1^2/n_1+s_2^2/n_2)^2}{(s_1^2/n_1)^2/v_1+(s_2^2/n_2)^2/v_2}=\frac{(3.07^2/15+0.8^2/12)^2}{(3.07^2/15)^2/14+(0.8^2/12)^2/11}\sim 16.32 \approx 16$$

(KONTROL)

$\min(v_1,v_2)<v<v_1+v_2$

$11<16<25$

$$\underbrace{P(T<t_{0.025}}_{tcdf}\mid v=16)=0.975$$

$$tcdf^{-1}(0.975\mid v=16)=t_{0.025}=2.1199$$

$$\mu_1-\mu_2=2.35\pm 2.1199\cdot 0.83 \Rightarrow \begin{array}{l}\mu_U=4.109517\sim 4.10\\ \mu_L=0.59123\sim 0.6\end{array}$$

$$\boxed{0.6<\mu_1-\mu_2<4.10}$$

∴ Bu aralığın, bu iki konum için gerçek ortalama orthophosphor içeriklerinin farkını içerdiğinden %95 eminiz.

! Sample size küçükse normal distr. kullanmak daha iyidir.

Unknown But Equal Variances → Sınavda çıkmayacak

$\sigma_1=\sigma_2$ ise bu yöntemi kullanmak aynı confidence level için daha küçük bir aralık verecektir

$$\begin{array}{ccc} & \sigma_1 & \sigma_2 \\ & \wr & \wr \\ \text{combine} & \{s_1, & s_2\} \rightarrow s_p\end{array}$$

$$sp^2 = \frac{\nu_1 \cdot s_1^2 + \nu_2 \cdot s_2^2}{\nu_1 + \nu_2}$$

Buradan $\mu_1 - \mu_2 = (\bar{x}_1 - \bar{x}_2) \pm t_{\frac{\alpha}{2}} \cdot \sqrt{\frac{sp^2}{n_1} + \frac{sp^2}{n_2}}$

Single Sample : Estimating The Variance

Şimdiye kadar Confidence Interval konusunda sadece μ yü inceledik. σ'yı incelemedik. Şimdi ona bakacağız :)

σ^2 için bir aralık yaklaşımı istatistik sayesinde oluşturulabilir : $\chi^2 = \nu \cdot \frac{s^2}{\sigma^2}$

$\sigma^2 \to$ variance

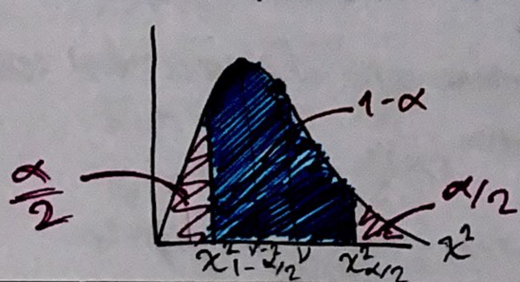
$\sigma \to$ standard dev.

$\sigma_L^2 \quad \sigma^2 \quad \sigma_U^2$

$\to \sigma_L^2 < \sigma^2 < \sigma_U^2$

$\sigma_L < \sigma < \sigma_U$

! Standard dev.'da olasılık hesaplaması yaparken chi-squared distr. kullanılır.



$P(\chi^2 > \chi^2_{1-\frac{\alpha}{2}}) = 1 - \alpha + \frac{\alpha}{2} = 1 - \frac{\alpha}{2}$

$P(\chi^2 > \chi^2_{\frac{\alpha}{2}}) = \frac{\alpha}{2}$

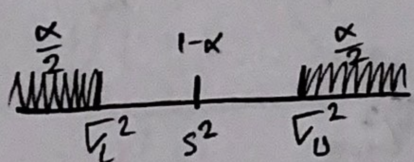
$$P\left(\chi^2_{1-\frac{\alpha}{2}} < \chi^2 < \chi^2_{\frac{\alpha}{2}}\right) = 1-\alpha$$

$$\chi^2 = \nu \cdot \frac{s^2}{\sigma^2} \rightarrow \chi^2 < \chi^2_{\frac{\alpha}{2}} \rightarrow \nu \cdot \frac{s^2}{\sigma^2} < \chi^2_{\frac{\alpha}{2}} \Rightarrow \sigma^2 > \underbrace{\nu \cdot \frac{s^2}{\chi^2_{\frac{\alpha}{2}}}}_{\text{lower bound}}$$

$$\chi^2_{1-\frac{\alpha}{2}} < \chi^2 \rightarrow \chi^2_{1-\frac{\alpha}{2}} < \nu \cdot \frac{s^2}{\sigma^2} \rightarrow \sigma^2 < \underbrace{\nu \cdot \frac{s^2}{\chi^2_{1-\frac{\alpha}{2}}}}_{\text{upper bound}}$$

Yani; $P\left(\chi^2_{1-\frac{\alpha}{2}} < \nu \cdot \frac{s^2}{\sigma^2} < \chi^2_{\frac{\alpha}{2}}\right) = 1-\alpha$

$$\boxed{\nu \cdot \frac{s^2}{\chi^2_{\frac{\alpha}{2}}} < \sigma^2 < \nu \cdot \frac{s^2}{\chi^2_{1-\frac{\alpha}{2}}}}$$



$P(\chi^2 > \chi^2_{1-\frac{\alpha}{2}}) = 1 - \frac{\alpha}{2}$

$\text{chi2cdf}(\chi^2_{1-\frac{\alpha}{2}}, \nu) = \frac{\alpha}{2}$

$P(\chi^2 < \chi^2_{1-\frac{\alpha}{2}})$

EXAMPLE: 100 packages of grass seed are randomly sampled from a store. The sample standard deviation of the weights is found to be 0.739 g. Find the 95% confidence interval for the population standard deviation.

$\sigma^2 \rightarrow$ variance

$\sigma \rightarrow$ standard dev.

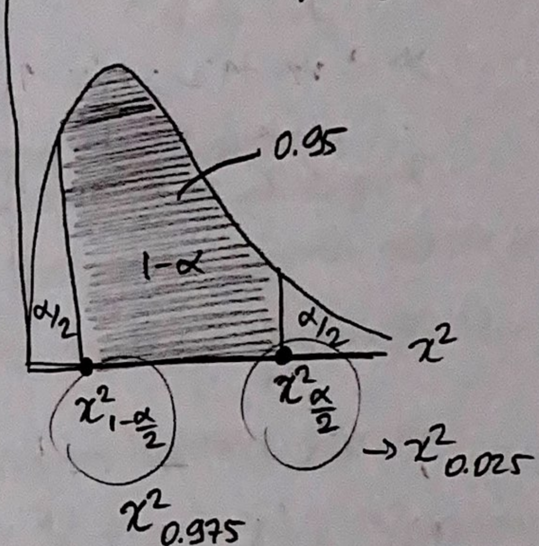
$\nu = n-1 = 99$

$n = 100$

$S = 0.739$

$1-\alpha = 0.95 \rightarrow \alpha = 0.05$

$\frac{\alpha}{2} = 0.025$



$P(\chi^2 < \chi^2_{1-\frac{\alpha}{2}})$

$= \text{chi2cdf}(\chi^2_{1-\frac{\alpha}{2}}, \nu) = \frac{\alpha}{2}$

$= \text{chi2cdf}(\chi^2_{0.975}, 99) = 0.025$

$= \text{chi2cdf}^{-1}(0.025 \mid \nu = 99) = \chi^2_{0.975} \rightarrow 73.3611 \sim 73.36$

$= \text{chi2cdf}(\chi^2_{\frac{\alpha}{2}}, \nu) = 0.975 \rightarrow$

$= \text{chi2cdf}^{-1}(0.975 \mid \nu = 99) = \chi^2_{0.025} \rightarrow 128.4220 \sim 128.42$

$$\nu \cdot \frac{s^2}{\chi^2_{\frac{\alpha}{2}}} < \sigma^2 < \nu \cdot \frac{s^2}{\chi^2_{1-\frac{\alpha}{2}}}$$

$$\frac{99 \cdot (0.739)^2}{(128.42)} < \sigma^2 < 99 \cdot \frac{(0.739)^2}{(73.36)}$$

σ_L^2 σ_U^2

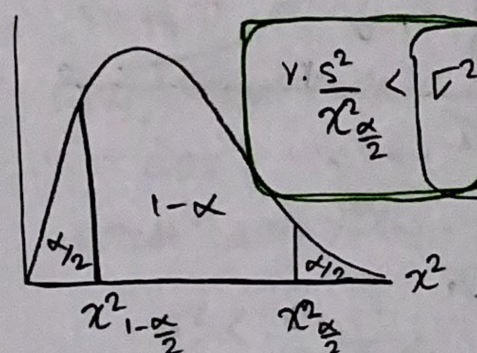
$\sigma_L^2 < \sigma^2 < \sigma_U^2 \rightarrow \quad 0.42 < \sigma^2 < 0.74$

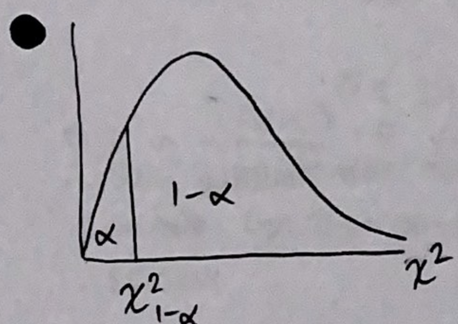
$0.65 < \sigma < 0.86$

∴ Yani standard dev.'in 0.65 ve 0.86 arasında olduğundan %95 eminiz.

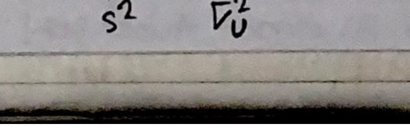
One-Sided Confidence Intervals

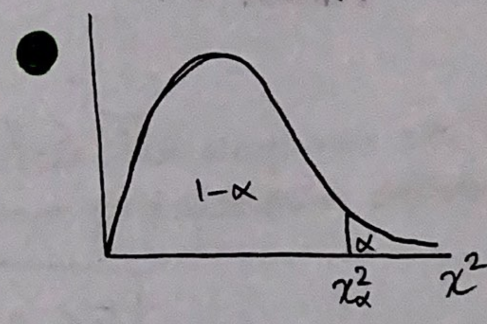
$$\nu \cdot \frac{s^2}{\chi^2_{\frac{\alpha}{2}}} < \sigma^2 < \nu \cdot \frac{s^2}{\chi^2_{1-\frac{\alpha}{2}}}$$



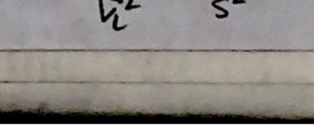


$$\sigma^2 > \nu \cdot \frac{s^2}{\chi^2_{1-\alpha}}$$



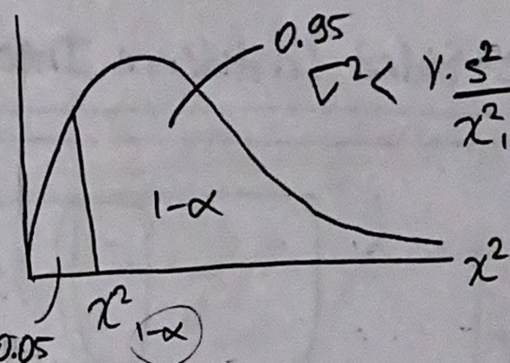


$$\nu \cdot \frac{s^2}{\chi^2_{\alpha}} < \sigma^2$$



EXAMPLE: A client of cheese factory claims that the density of cheese has a standard dev. of 0.3 g/cm³ and so argues that the quality of the product is too varied. We investigated this in lecture 11 and found the standard dev. for a sample of size 8 to be 0.147 g/cm³ and rejected the clients claim. Now form a 95% confidence interval for the upper bound of population standard dev. and comment on the result.

$\sigma = 0.3$
$n = 8$
$s = 0.147$
$\sigma_U = ?$
$v = 7$



$$\sigma^2 < \frac{v \cdot s^2}{\chi^2_{1-\alpha}}$$

$P(\chi^2 < \chi^2_{0.95}) = 0.05$

$\text{chi2cdf}(\chi^2_{0.95} \mid v = 7) = 0.05$

$\text{chi2cdf}^{-1}(0.05 \mid v = 7) = \chi^2_{0.95} \rightarrow 2.1674$

$$\sigma^2 < \frac{v \cdot s^2}{\chi^2_{1-\alpha}} \rightarrow 7 \cdot \frac{(0.147)^2}{2.1674} \sim 0.07$$

$$\boxed{\sigma < 0.26}$$

Yani popülasyon standard dev'ının 0.26 g/cm³'ten az olduğundan %95 eminiz. Popülasyon standard dev.'ının müşterilerin iddia ettiğinden daha az old eminiz çünkü 0.26'dan küçük yani 0.3 olması imkansız!

EXAMPLE: An autonomous vehicle is req. to follow a guideline on the road. Random deviations of the vehicle from the line are expected, but the system is designed to have a population standard dev. that is less than 25 cm. Deviations from the guideline are randomly sampled 21 times during a test drive; the sample standard dev. is measured to be 18 cm. Construct a 95% upper bounded confidence interval for the pop. standard dev. and comment on the result.

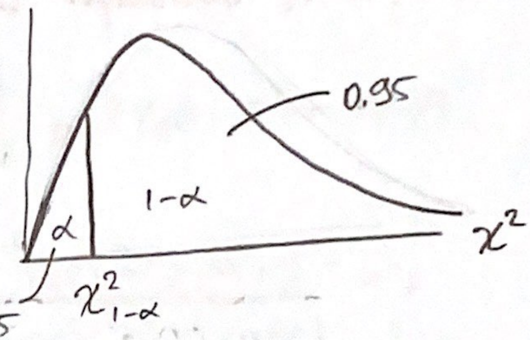
$\sigma < 25$

$n = 21$

$s = 18$

$P(\chi^2 < \chi^2_{0.95} \mid \nu = 20)$

$= 0.05$



$\text{chi2cdf}^{-1}(0.05 \mid \nu = 20) = \chi^2_{0.95} \rightarrow 10.8508$

$$\sigma^2 < \nu \cdot \frac{s^2}{\chi^2_{1-\alpha}} = 20 \cdot \frac{18^2}{10.8508} \sim 597.19$$

$$\sigma < 24.43$$

∴ Yani pop. standard dev. ın 24.43'ten küçük olduğundan %95 eminiz. Bu, 25 cm'den az tasarım gereksinimini de doğrudan karşılar.

SUMMARY

- 2 popülasyonun mean'indeki fark için confidence interval:

1- σ_1 ve σ_2 biliniyor

$$\mu_1-\mu_2=(\bar{x}_1-\bar{x}_2)\pm z_{\frac{\alpha}{2}}\cdot\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}$$

2- σ_1 ve σ_2 bilinmiyor ve $\sigma_1\neq\sigma_2$

$$\mu_1-\mu_2=(\bar{x}_1-\bar{x}_2)\pm t_{\frac{\alpha}{2}}\cdot\sqrt{\frac{s_1^2}{n_1}+\frac{s_2^2}{n_2}}\quad,\ v_1=n_1-1$$

$$v_2=n_2-1$$

$$v=\frac{(s_1^2/n_1+s_2^2/n_2)^2}{(s_1^2/n_1)^2/v_1+(s_2^2/n_2)^2/v_2}\quad,\ \min(v_1,v_2)<v<v_1+v_2$$

- Popülasyonun variance'ının confidence interval'ı:

$$\frac{v\cdot s^2}{\chi^2_{\frac{\alpha}{2}}}<\sigma^2<\frac{v\cdot s^2}{\chi^2_{1-\frac{\alpha}{2}}}\quad,\ v=n-1$$

$\to$ v küsuratlı ise her zaman küçük tam sayıya yuvarlanır!