$$\frac{2 \cdot d \cdot F_c}{da} = \cancel{a} - aF_c$$

Separating Variable.

$$2 \cdot \frac{dF_c}{F_c} = -a\,F_c\,da$$

integrating

$$\cancel{2\int dF_c = -F_c\int a\,da}$$

$$\cancel{2F_c = -F_c\frac{a^2}{2}+C_1}$$

$$\cancel{2F_c + F_c\frac{a^2}{2} = C_2}$$

$$\cancel{F_c\left[2+\frac{a^2}{2}\right] = C_1}$$

$$\cancel{F_c\left[\frac{4+4a^2}{2}\right] = C_1}$$

$$\cancel{F_c = \frac{2C_1}{4+a^2}}$$

Integrating

$$2\int\frac{dF_c}{F_c} = -\int a\,.da\,.$$

$$2\log F_c = \frac{-a^2}{2}+c$$

$$\log F_c = \frac{-a^2}{4}+\frac{c}{2}$$

$$F_c = e^{-a^2/4+c/2}$$

$$F_c = e^{-a^2/4}\cdot e^{c/2} \qquad C_1 = e^{-c/2}$$

$$F_c = C_1 e^{-a^2/4}$$

(10) Find the fourier sine transform of $\frac{1}{x(1+x^2)}$

$$F_s a = \int_0^\infty f(x)\cdot \sin ax\cdot dx\,.$$

$$F_s(a) = \int_0^\infty \frac{1}{x(1+x^2)}\cdot \sin ax\,.dx\,.$$

We evaluate under the rule of differentiation under integral

diff w.r.t a.

$$\frac{dF_s}{da} = \int_0^\infty \frac{1}{x(1+x^2)} \frac{\partial}{\partial a}(\sin ax) \cdot dx.$$

$$\frac{dF_s}{da} = \int_0^\infty \frac{1}{\cancel{x}(1+x^2)} \cdot \cos ax \cdot \cancel{x} \cdot dx$$

Adding & sub x^2 to numerator of RHS.

$$\frac{dF_s}{da} = \int_0^\infty \frac{(1+x^2)-x^2}{1+x^2} \cdot \cos ax \cdot dx.$$

$$\frac{dF_s}{da} = \int_0^\infty \frac{(1+x^2)}{(1+x^2)} - \frac{x^2}{(1+x^2)} \cdot \cos ax \cdot dx.$$

$$\frac{dF_s}{da} = \int_0^\infty \left(1 - \frac{x^2}{1+x^2}\right) \cdot \cos ax \cdot dx.$$

$$\frac{dF_s}{da} = \int_0^\infty \cos ax \cdot dx - \int_0^\infty \frac{x^2}{1+x^2} \cdot \cos ax \cdot dx.$$

$$\frac{d^2F_s}{da^2} = \int_0^\infty \frac{1}{1+x^2} (-\sin ax)^x \cdot dx = -\int_0^\infty \frac{x}{1+x^2} \sin ax \cdot dx.$$

x to Nr & dr of RHS

+1 & -1 to Nr of RHS

$$\frac{d^2F_s}{da^2} = -\int_0^\infty \frac{x^2+1-1}{x(x^2-1)} \cdot \sin ax \cdot dx.$$

$$= -\int_0^\infty \left[\frac{1}{x} - \frac{1}{x(x^2+1)}\right] \cdot \sin ax \cdot dx$$

$$\frac{d^2F_s}{da^2} = -\int_0^\infty \frac{\sin ax}{x} \cdot dx + \int_0^\infty \frac{1 \sin ax}{x(x^2+1)} \cdot dx = -\frac{\pi}{2} + F_s(a)$$

$$\frac{d^2F_s}{da^2} - F_s = -\pi/2$$

$$(D^2-1)\, F_s = -\pi/2$$

A.E

$$m^2-1 = 0$$

$$m = \pm 1$$

$$y_c = (F_s)_c = c_1 e^{\alpha} + c_2 e^{-\alpha}$$

$$y_p = (F_s)_p = -\frac{\pi/2 \times e^{0x}}{0^2+1} = -\pi/2$$

$$F_s = (F_s)_c + (F_s)_p$$

$$F_s = c_1 e^{\alpha} + c_2 e^{-\alpha} - \pi/2 \quad -(2)$$

$$F_s(0) = c_1 e^0 + c_2 e^0 - \pi/2$$

$$c_1 + c_2 = \pi/2 \quad -(3)$$

Diff (1) w.r.t α

$$F_s'(\alpha) = c_1 e^{\alpha} - c_2 e^{-\alpha}$$

put $\alpha = 0$

$$F_s'(0) = c_1 e^0 - c_2 e^0$$

$$\int_0^\infty \frac{1}{1+x^2}\,dx = c_1 - c_2$$

$$\tan^{-1}x \Big|_0^\infty = c_1 - c_2$$

$$c_1 - c_2 = \pi/2 \quad -(4)$$

(3) + (4)

$$c_1 + c_2 = \pi/2$$
$$c_1 - c_2 = \pi/2$$

$$c_1 = \pi/2$$

Sub c_1 & c_2 in eqn (2)

$$F_s(\alpha) = \pi/2\, e^{\alpha} - \pi/2$$

$$F_s(\alpha) = \pi/2\left[e^{\alpha} - 1\right]$$

→ Type - 3

(1) Solve $\int_0^\infty f(\theta) \cdot \cos\alpha\theta \, d\theta = \begin{cases} 1-\alpha, & 0 \le \alpha \le 1 \\ 0, & \alpha > 1 \end{cases}$ hence evaluate $\int_0^\infty \frac{\sin^2 t}{t^2} \, dt$.

We have to find $f(\theta)$ and we shall consider the inverse fourier cosine transform with

$$F_c(\alpha) = \begin{cases} 1-\alpha, & 0 \le \alpha \le 1 \\ 0, & \alpha > 1 \end{cases}$$

By definition of inverse fourier cosine transform

$$f(\theta) = \frac{2}{\pi} \int_0^\infty F_c(\alpha) \cos\alpha\theta \, d\alpha$$

$$f(\theta) = \frac{2}{\pi} \int_0^1 (1-\alpha) \cos\alpha\theta \, d\alpha$$

$$= \frac{2}{\pi} \left[(1-\alpha) \cdot \frac{\sin\alpha\theta}{\theta} - (-1) \frac{-\cos\alpha\theta}{\theta^2} \right]_0^1$$

$$= \frac{2}{\pi} \left[-\frac{\cos\theta}{\theta^2} - \left[-\frac{1}{\theta^2} \right] \right]$$

$$= \frac{2}{\pi} \left[\frac{-\cos\theta}{\theta^2} + \frac{1}{\theta^2} \right]$$

$$= \frac{2}{\pi} \left[\frac{1-\cos\theta}{\theta^2} \right]$$

$$= \frac{2}{\pi\theta^2} [1-\cos\theta]$$

$$= \frac{2}{\pi\theta^2} [2\sin^2\theta/2]$$

$$= \frac{4\sin^2\theta/2}{\pi\theta^2}$$

Now we evaluate $\int_0^\infty \frac{\sin^2 t}{t^2}\,dt$ using the defn of fourier cosine transform.

$$F_c(\alpha) = \int_0^\infty f(\theta)\cdot \cos\alpha\theta\cdot d\theta$$

$$= \int_0^\infty \frac{4\sin^2\theta/2}{\pi\theta^2}\cdot \cos\alpha\theta\cdot d\theta$$

put $\alpha = 0$

$$F_c(0) = \int_0^\infty \frac{4\sin^2\theta/2}{\pi\theta^2}\cos\alpha\theta\, d\theta$$

$$1 = \frac{1}{\pi}\int_0^\infty \frac{\sin^2\theta}{\theta^2/4}\,d\theta$$

$$\pi = \int_0^\infty \frac{\sin^2\theta/2}{(\theta/2)^2}\cdot d\theta$$

put $\theta/2 = t$, $\theta = 2t$ $\quad d\theta = 2dt$

$$\pi = \int_0^\infty \frac{\sin^2 t}{t}\cdot 2dt$$

$$\pi/2 = \int_0^\infty \sin^2 t/t^2 \cdot dt$$

(2) Solve $F(\alpha) = \begin{cases} a - \alpha/2, & 0 \le \alpha \le 2a \\ 0, & \alpha > 2a \end{cases}$

By defination $F_c(u) = \int_0^\infty f(x)\cdot \cos ux\cdot dx$

replace 'u' by 'α'

$\therefore$ given eqn $\int_0^\infty f(x)\cdot \cos\alpha x\cdot dx = a - \frac{\alpha}{2}, \quad 0 \le \alpha \le 2a$

Now we find $f(x)$ using defin of inverse fourier cosine transform.

ie

$$f(x) = \frac{2}{\pi}\int_0^\infty F_c(u)\cdot \cos ux\cdot dx$$

$$= \frac{2}{\pi}\int_0^\infty F_c(\alpha)\cdot \cos\alpha x\cdot d\alpha$$

$$= \frac{2}{\pi}\int_0^{\infty}\left(a-\frac{\alpha}{2}\right)\cdot \cos\alpha x \cdot d\alpha$$

$$f(x) = \frac{2}{\pi}\left[\left(a-\frac{\alpha}{2}\right)\cdot\frac{\sin\alpha x}{x} - \left(-\frac{1}{2}\right)\left(-\frac{\cos\alpha x}{x^2}\right)\right]_0^{2a}$$

$$= \frac{2}{\pi}\left[0\left(-\frac{\cos 2ax}{2x^2}\right) - \left[-\frac{\cos 0}{2x^2}\right]\right]$$

$$= \frac{2}{\pi}\cdot\frac{1}{2x^2}\left[-\cos 2ax + 1\right]$$

$$= \frac{1}{\pi x^2}\cdot 2\sin^2 ax$$

(3) Show that $x\cdot e^{-\frac{x^2}{2}}$ is self reciprocal under fourier sine transform

(Imp)

By defination $F_s(u) = \int_0^{\infty} f(x)\cdot \sin ux \cdot dx$.

$$= \int_0^{\infty} x\cdot e^{-x^2/2}\cdot \sin ux \cdot dx$$

$$F_s(u) = \int_0^{\infty}(\sin ux)\cdot\left(x\cdot e^{-x^2/2}\right)\cdot dx.$$

by parts

$$= \sin ux\left(\int_0^{\infty} x\cdot e^{-\frac{x^2}{2}}\cdot dx\right) - \int_0^{\infty}\left[\left[\int_0^{\infty} x\cdot e^{-x^2/2}\cdot dx\right]\times\frac{d}{dx}(\sin ux)\right]dx$$

$$= \sin ux\left(-e^{-x^2/2}\right)\Big|_0^{\infty} - \int_0^{\infty}\left(-e^{-x^2/2}\right)\cdot \cos ux\cdot u\,dx.$$

$$F_s(u) = 0 + u\int_0^{\infty} e^{-\frac{x^2}{2}}\cdot \cos ux\cdot dx.$$

$$F_s(u) = u\int_0^{\infty} e^{-\frac{x^2}{2}}\cdot \cos ux\cdot dx.$$

Let $\int_0^\infty e^{-\frac{x^2}{2}} \cdot \cos ux \cdot dx = \phi u$

$$F_s(u) = u\,\phi(u) \quad -①$$

Put $e^{-x^2/2} = t$

$e^{-\frac{x^2}{2}} \cdot (-\frac{1}{2}) \times 2x\,dx = dt$

$-x \cdot e^{-x^2/2} \cdot dx = dt$

$x \cdot e^{-\frac{x^2}{2}} dx = -dt$

$I = \int -dt = -t$

$= -e^{-x^2/2}$

$\int x e^{-x^2/2} dx = -e^{-x^2/2}$

To Compute $\phi(u)$ We apply LDI

diff $\phi(u)$ w.r.t 'u'

$$\phi'(u) = \int_0^\infty e^{-x^2/2} \frac{\partial}{\partial u}(-\sin ux)\,x \cdot dx$$

$$= \int_0^\infty e^{-x^2/2}\,(+\sin ux) \cdot x \cdot dx$$

$$\phi'(u) = -\int_0^\infty (x \cdot e^{-\frac{x^2}{2}}) \cdot \sin ux \cdot dx$$

$$\phi'(u) = -F_s(u)$$

From ①

$$\phi'(u) = -u\,\phi(u)$$

Separating Variable

$$\frac{\phi'(u)}{\phi(u)} = -u.$$

integrating w.r.t 'u'

$$\int \frac{\phi'(u)}{\phi(u)} du = -\int u \cdot du.$$

$$\log \phi(u) = \frac{-u^2}{2} + c$$

$$\phi(u) = e^{-\frac{u^2}{2}+c} = e^{-u^2/2} \cdot e^c$$

$$\phi(u) = c_1 e^{-u^2/2} \quad -②$$ Where $c_1 = e^c$

Substituting in ①

$$F_s(u) = u\left[c_1 e^{-u^2/2}\right]$$

$$F_s(u) = G_u e^{-u^2/2}$$

is the same as form of $f(x)$

→ ## Z transform:

Defn: Consider a function

$$f(n) = u_n \text{ for } n = 0, 1, 2, \ldots \infty$$
$$= 0 \text{ for } n < 0$$

$$Z_T[u_n] = \sum_{n=0}^{\infty} u_n z^{-n}$$

Z transform of U_n is also denoted as $\bar{U}(z)$

→ **Basic properties**

1. $Z_T(k^n) = \dfrac{z}{z-k}$

k is the transform

2. $Z_T(n^k) = -z \dfrac{d}{dz}\left[Z_T(n^{k-1})\right]$

→ **Z_T of standard functions:**

(Imp)

(1) $Z_T(1)$

We have $Z_T(k^n) = \dfrac{z}{z-k}$

put $k = 1$

$$Z_T(1^n) = \frac{z}{z-1}$$

$$\boxed{Z_T(1) = \frac{z}{z-1}}$$

(2) $Z_T(n)$

We have $Z_T(n^k) = -z \cdot \frac{d}{dz}\left[Z_T(n^{k-1})\right]$

put $k=1$

$$Z_T(n) = -z \cdot \frac{d}{dz}\left[Z_T(n^0)\right]$$

$$Z_T(n) = -z \frac{d}{dz}\left[Z_T(1)\right]$$

$$Z_T(n) = -z \frac{d}{dz}\left[\frac{z}{z-1}\right]$$

$$Z_T(n) = -z \left[\frac{(z-1)\cdot 1 - z(1)}{(z-1)^2}\right]$$

$$Z_T(n) = -z \left[\frac{z-1-z}{(z-1)^2}\right]$$

$$\boxed{Z_T(n) = \frac{z}{(z-1)^2}}$$

(3) $Z_T(n^2)$

We have $Z_T(n^k) = -z \frac{d}{dz}\left[Z_T(n^{k-1})\right]$

put $k=2$

$$Z_T(n^2) = -z \cdot \frac{d}{dz}\left[Z_T(n)\right]$$

$$= -z \cdot \frac{d}{dz}\left[\frac{z}{(z-1)^2}\right]$$

$$= -z \cdot \left[\frac{(z-1)^2(1) - z \cdot 2(z-1)}{(z-1)^4}\right]$$

$$= -z \left[\frac{(z-1)\left[z-1-2z\right]}{(z-1)^4}\right] = \frac{-z\left[-z-1\right]}{(z-1)^3}$$

$$\boxed{Z_T(n^2) = \cancel{\frac{z(z+1)}{z+3}} \quad \frac{z(z+1)}{(z-1)^3}}$$

(4) $Z_T(n^3)$

We have $Z_T(n^k) = -z \cdot \frac{d}{dz}\left[Z_T(n^{k-1})\right]$

put $k = 3$:

$$Z_T(n^3) = -z \cdot \frac{d}{dz}\left[Z_T(n^2)\right]$$

$$= -z \cdot \frac{d}{dz}\left[\frac{z(z+1)}{(z-1)^3}\right]$$

$$= -z \cdot \left[\frac{(z-1)^3 \cdot 2z+1 - z(z+1) \cdot 3(z-1)^2}{(z-1)^6}\right]$$

$$= -z\left[\frac{(z-1)^{2}\left[(2z+1)\cdot(z-1) - z(z+1)\cdot 3\right]}{(z-1)^{\cancel{6}\,4}}\right]$$

$$= -z\left[\frac{(2z+1)(z-1) - (z^2+z)\cdot 3}{(z-1)^4}\right]$$

$$= -z\left[\frac{2z^2 - 2z + z - 1 - 3z^2 - 3z}{(z-1)^4}\right]$$

$$= -z\left[\frac{-z^2 - 4z - 1}{(z-1)^4}\right] = z\left[\frac{z^2+4z+1}{(z-1)^4}\right]$$

$$= \cancel{\frac{z(z+2)^2}{(z-1)^4}}$$

$$\boxed{Z_T(n^3) = \frac{z^3+4z^2+z}{(z-1)^4}}$$

z^2+z

$z(z+1)$

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Properties :

(1) **linearity :** If 'U_n' and 'V_n' are any discrete value of function then

$$Z_T[C_1 U_n + C_2 V_n] = C_1 Z_T(U_n) + C_2 Z_T(V_n)$$

(2) **Damping rule :** If $Z_T(U_n) = \bar{U}(z)$. then

$$Z_T(k^n U_n) = \bar{U}\left(\frac{z}{k}\right) = Z_T(U_n)_{z \to \frac{z}{k}}$$

$$Z_T(k^{-n} U_n) = \bar{U}(zk) = Z_T(U_n)_{z \to zk}.$$

(3) **Shifting rule!** If $Z_T(U_n) = \bar{U}(z)$ then

(i) **Right shifting rule:**

$$Z_T(U_n - k) = \sum_{n=0}^{\infty} (U_{n-k}) z^{-n}$$

(ii) **left shifting rule :** We apply this rule to solve difference eq^n

$$Z_T(U_{n+k}) = z^k\left[\bar{U}(z) - U_0 - \frac{U_1}{z} - \frac{U_2}{z^2} \cdots \frac{U_{k-1}}{z^{k-1}}\right]$$

If $k = 1$

$$Z_T(U_{n+1}) = z\left[\bar{U}(z) - U_0\right]; \quad Z_T(U_{n+2}) = z^2\left[\bar{U}(z) - U_0 - \frac{U_1}{z}\right]$$

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List of Z transform of formula:

(1) $\cosh\theta = \frac{e^{\theta}+e^{-\theta}}{2}$

(2) $\sinh\theta = \frac{e^{\theta}-e^{-\theta}}{2}$

(2) $e^{i\theta} = \cos\theta + i\sin\theta$

(3) $(a+ib)(a-ib) = a^2+b^2$

(4) $Z_T(k^n) = \frac{z}{z-k}$

(5) $Z_T(n^k) = -z\frac{d}{dz}\left[Z_T(n^{k-1})\right]$

(6) $Z_T(1) = \frac{z}{z-1}$

(7) $Z_T(n) = \frac{z}{(z-1)^2}$.

(8) $Z_T(n^2) = \frac{z(z+1)}{(z-1)^3} = \frac{z^2+z}{(z-1)^3}$

(9) $Z_T(n^3) = \frac{z^3+4z^2+z}{(z-1)^4}$

(10) $Z_T(kn) = \frac{kz}{(z-k)^2}$

(11) $Z_T(kn^2) = \cancel{\frac{k(z(z+1))}{(z-1)^3}} = \frac{kz^2+k^2z}{(z-k)^3}$

(12) $Z_T(kn^3) = \cancel{\frac{k[z^3+4z^2+z]}{(z-1)^4}} = \frac{kz^3+4k^2z^2+k^3z}{(z-k)^4}$

Type - 1 :

(1) $(n+1)^2$.

$$u_n = (n+1)^2 = n^2+2n+1$$

$$Z_T(u_n) = Z_T[n^2+2n+1]$$

$$= Z_T[n^2] + Z_T[2n] + Z_T[1]$$

$$= \frac{z^2+z}{(z-1)^3} + 2\left[\frac{z}{(z-1)^2}\right] + \frac{z}{(z-1)}$$

(2) $\left(\frac{1}{2}\right)^n + \left(\frac{1}{3}\right)^n$

$$u_n = \left(\frac{1}{2}\right)^n + \left(\frac{1}{3}\right)^n$$

$$Z_T(u_n) = Z_T\left[(1/2)^n + (1/3)^n\right]$$

$$= Z_T\left[\left(\frac{1}{2}\right)^n\right] + Z_T\left[(1/3)^n\right]$$

$$= \frac{z}{z-1/2} + \frac{z}{z-1/3}$$

$$= \frac{z}{\frac{2z-1}{2}} + \frac{z}{\frac{3z-1}{3}}$$

$$= \frac{2z}{2z-1} + \frac{3z}{3z-1}$$

(3) $\sinh n\theta$

$$u_n = \sinh n\theta = \frac{e^{n\theta} - e^{-n\theta}}{2} = \frac{(e^{\theta})^n - (e^{-\theta})^n}{2}$$

$$z_T(U_n) = z_T\left[\frac{(e^{\theta})^n-(e^{-\theta})^n}{2}\right]$$

$$= \frac{1}{2}\, z_T (e^{\theta})^n - z_T(e^{-\theta})^n$$

$$= \frac{1}{2}\left[z_T(e^{\theta})^n - z_T(e^{-\theta})^n\right]$$

$$= \frac{1}{2}\left[\frac{z}{z-e^{\theta}} - \frac{z}{z-e^{-\theta}}\right]$$

(4) $\cosh\left(\frac{n\pi}{2}+\theta\right)$

$$U_n = \cosh\left(\frac{n\pi}{2}+\theta\right)$$

$$= \frac{e^{(n\pi/2+\theta)} + e^{-(n\pi/2+\theta)}}{2}$$

$$z_T(U_n) = \frac{1}{2}\, z_T\left[e^{(n\pi/2+\theta)} + e^{-(n\pi/2+\theta)}\right]$$

$$= \frac{1}{2}\, z_T\left[e^{n\pi/2}\cdot e^{\theta} + e^{-n\pi/2}\cdot e^{-\theta}\right]$$

$$= \frac{1}{2}\, z_T\left[(e^{\pi/2})^n\cdot e^{\theta} + (e^{-\pi/2})^n\cdot e^{-\theta}\right]$$

$$= \frac{1}{2}\, z_T\left[\frac{z}{z-\cancel{\pi/2}\,e^{\pi/2}}\cdot e^{\theta} + \frac{z}{z-e^{-\pi/2}}\cdot e^{-\theta}\right]$$

$$= \frac{1}{2}\left[\frac{z\cdot e^{\theta}}{z-e^{\pi/2}} + \frac{z\cdot e^{-\theta}}{z-e^{-\pi/2}}\right]$$

(5) $\sin(3n+5)$

$U_n = \sin(3n+5) = \sin 3n \cdot \cos 5 + \cos 3n \sin 5$

$Z_T(U_n) = Z_T[\sin 3n \cos 5 + \cos 3n \sin 5]$

$Z_T(U_n) = Z_T(\sin 3n) \cdot \cos 5 + Z_T(\cos 3n) \cdot \sin 5$ — (1).

We have $e^{i3n} = \cos 3n + i\sin 3n$

We find $Z_T[e^{i3n}] = Z_T[(e^{3i})^n] = \dfrac{Z}{Z - e^{3i}}$

$$Z_T = k^n$$

$$e^{i3n} = (e^{3i})^n = \frac{z}{z - [\cos 3 + i\sin 3]} = \frac{Z}{(Z - \cos 3) - i\sin 3}$$

$$= \frac{z[(z - \cos 3) + i\sin 3]}{(z - \cos 3)^2 + (\sin 3)^2}$$

$$Z_T[(e^{i3})^n] = \frac{z^2 - z\cos 3 + iz\sin 3}{z^2 + (\cos 3)^2 - 2z\cos 3 + (\sin 3)^2}$$

$$= \frac{z^2 - z\cos 3 + iz\sin 3}{z^2 + 1 - 2z\cos 3}$$

$$Z_T[\cos 3n + i\sin 3n] = \frac{z^2 - z\cos 3}{z^2 + 1 - 2z\cos 3} + i\frac{z\sin 3}{z^2 + 1 - 2z\cos 3}$$

$$Z_T[\cos 3n] = \frac{z^2 - z\cos 3}{z^2 + 1 - 2z\cos 3} \cancel{+ i\frac{z\sin 3}{z^2+1-2z\cos 3}}$$

$$Z_T[\sin 3n] = \frac{z\sin 3}{z^2 + 1 - 2z\cos 3}$$

Sub (1)

$$Z_T[U_n] = \cos 5\left[\frac{\sin 3}{z^2 + 1 - 2z\cos 3}\right] + \sin 5\left[\frac{z^2 - z\cos 3}{z^2 + 1 - 2z\cos 3}\right]$$

(6) $\cos\left[\frac{n\pi}{2}+\frac{\pi}{4}\right]$

(Imp)

$u_n = \cos\left[\frac{n\pi}{2}+\pi/4\right]$

$Z_T(u_n) = \cos\frac{n\pi}{2}\cdot\cos\frac{\pi}{4} - \sin\frac{n\pi}{2}\cdot\sin\frac{\pi}{4}$

$= \cos\frac{\pi}{4}\cdot Z_T\left[\cos\frac{n\pi}{2}\right] - \sin\frac{\pi}{4}\cdot\left[\sin\frac{n\pi}{4}\right]$ (1)

We have $e^{in\pi/2} = \cos\frac{n\pi}{2} + i\sin\frac{n\pi}{2}$

We find $Z_T\left[e^{i\frac{n\pi}{2}}\right] = Z_T\left[(e^{i\pi/2})^n\right\} = \frac{z}{z-e^{i\pi/2}}$

$= \frac{z}{z-\left[\cos\pi/2 + i\sin\pi/2\right]} = \frac{z}{z-i} + \frac{z+i}{z+i}$

$Z_T[u_n] = \frac{z^2+iz}{z^2+i^2} = \frac{z^2+iz}{z^2+1}$

~~$Z_T(\cos n\pi/2) =$~~

$Z_T[e^{in\pi/2}] = \frac{z^2}{z^2+1} + \frac{iz}{z^2+1}$

$Z_T[\cos n\pi/2] = \frac{z^2}{z^2+1}$

$Z_T\left[\sin\frac{n\pi}{2}\right] = \frac{z}{z^2+1}$

Sub in (1).

$Z_T(u_n) = \frac{1}{\sqrt{2}}\left[\frac{z^2}{z^2+1}\right] - \frac{1}{\sqrt{2}}\left[\frac{z}{z^2+1}\right]$

(7) $2n + \sin\frac{n\pi}{4} + 5a$

$$u_n = 2n + \sin\frac{n\pi}{4} + 5a.$$

$$Z_T[u_n] = Z_T\left[2n + \sin\frac{n\pi}{4} + 5a\right]$$

$$= 2Z_T(n) + Z_T\left[\sin\frac{n\pi}{4}\right] + 5a\, Z_T[1] \quad (1)$$

$$Z_T(n) = \frac{z}{(z-1)^2} \qquad Z_T(1) = \frac{z}{z-1}$$

$$Z_T\left[\sin\frac{n\pi}{4}\right]$$

We have $e^{\frac{in\pi}{4}} = \cos\frac{n\pi}{4} + i\sin\left(\frac{n\pi}{4}\right)$

$$Z_T\left[\left(e^{\frac{i\pi}{4}}\right)^n\right] = Z_T\left[\left(e^{i\pi/4}\right)^n\right] = \frac{z}{z - e^{i\pi/4}}$$

$$= \frac{z}{z - [\cos\pi/4 + i\sin\pi/4]}$$

$$= \frac{z}{z - \left[1/\sqrt{2} + i\frac{1}{\sqrt{2}}\right]} \qquad \times \sqrt{2} \text{ to Nr \& Dr.}$$

$$= \frac{\sqrt{2}z}{\sqrt{2}z - \sqrt{2}\frac{1}{\sqrt{2}} - i\sqrt{2}\cdot\frac{1}{\sqrt{2}}} = \frac{\sqrt{2}z}{(\sqrt{2}z-1) - i} \times \frac{(\sqrt{2}z-1)+i}{(\sqrt{2}z-1)+i}$$

$$= \frac{2z^2 - \sqrt{2}z + i\sqrt{2}z}{(\sqrt{2}z-1)^2 + 1^2} = \frac{2z^2 - \sqrt{2}z + i\sqrt{2}z}{2z^2 + 2 - 2\sqrt{2}z}$$

$$= \frac{2z^2 - \sqrt{2}z}{2z^2 + 2 - 2\sqrt{2}z} + i\frac{z\sqrt{2}}{2z^2 + 2 - 2\sqrt{2}z}$$

$$= Z_T\left[\sin\frac{n\pi}{4}\right] = \frac{\sqrt{2}z}{2z^2 + 2 - 2\sqrt{2}z}$$

in (1)

$$= \frac{z}{(z-1)^2} + \frac{\sqrt{2}z}{2z^2 + 2 - 2\sqrt{2}z} + 5a\frac{z}{z-1}$$

(8) $\frac{1}{n+1}$

$$U_n = \frac{1}{n+1}$$

$$Z_T(U_n) = Z_T\left(\frac{1}{n+1}\right) \quad - (1)$$

To find $Z_T(1)$ We Consider Maclorence Series

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots$$

put $x = \frac{1}{z}$

$$\log(1-1/z) = -\frac{1}{z} - \frac{(1/z)^2}{2} - \frac{(1/z)^3}{3} - \dots$$

$$\log(z - 1/z) = -\frac{1}{z} - \frac{1}{2z^2} - \frac{1}{3z^3} - \dots$$

$x\ -1$

$$-\log\left[\frac{z-1}{z}\right] = \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3z^3} + \dots \quad - (2)$$

from (1)

$$U_n\left[\frac{1}{n+1}\right] = Z_T\left[\frac{1}{n+1}\right]$$

$$= \sum_{n=0}^{\infty} \frac{1}{n+1} z^{-n} = 1 + \frac{1}{2}z^{-1} + \frac{1}{3}z^{-2} + \frac{1}{4}z^{-3}$$

$$= 1 + \frac{1}{2z} + \frac{1}{3z^2} + \frac{1}{4z^3} + \dots \times \frac{1}{z}$$

$$= \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3z^3} + \frac{1}{4z^4} \quad - (3)$$

from (2) & (3)

$$-\log\left[\frac{z-1}{z}\right] = \frac{1}{z} \times Z_T(U_n)$$

$$ZT(U_n) = z\left[-\log\left[\frac{z-1}{z}\right]\right]$$

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Type - 2 :

① Find Z_T transform of $\cos n\theta$, $\sin n\theta$ hence deduce $Z_T(k^n \sin n\theta)$, $Z_T(k^{-n}\cos n\theta)$

$Z_T(e^{-an}\sin n\theta)$, $Z_T(e^{an}\cos n\theta)$

To find: $Z_T(\cos n\theta)$ and $Z_T(\sin n\theta)$ we find

$Z_T(e^{in\theta}) =$

$$Z_T(e^{in\theta}) = Z_T\{(e^{i\theta})^n\} = \frac{z}{z-e^{i\theta}} = \frac{z}{z-[\cos\theta+i\sin\theta]}$$

$$= \frac{z}{(z-\cos\theta)-i\sin\theta} \times \frac{(z-\cos\theta)+i\sin\theta}{(z-\cos\theta)+i\sin\theta}$$

$$= \frac{z^2-z\cos\theta+iz\sin\theta}{(z-\cos\theta)^2+\sin^2\theta} = \frac{z^2-z\cos\theta+iz\sin\theta}{z^2+\cos^2\theta-2z\cos\theta+\sin^2\theta}$$

$$= \frac{z^2-z\cos\theta+iz\sin\theta}{z^2+1-2z\cos\theta} = \frac{z^2-z\cos\theta}{z^2+1-2z\cos\theta} + i\frac{z\sin\theta}{z^2+1-2z\cos\theta}$$

$$Z_T\{\cos n\theta\} = \frac{z^2-z\cos\theta}{z^2+1-2z\cos\theta} \quad -①$$

$$Z_T\{\sin n\theta\} = \frac{z\sin\theta}{z^2+1-2z\cos\theta} \quad -②$$

$$Z_T\{k^n \sin n\theta\} = \left[Z_T\{\sin n\theta\}\right]_{z\to\frac{z}{k}} \quad \text{(by damping rule)}$$

from ②

$$Z_T\{k^n \sin n\theta\} = \frac{z/k \sin\theta}{\left(\frac{z}{k}\right)^2+1-2\frac{z}{k}\cos\theta} \quad \text{. Mul } k^2 \text{ to Nr \& Dr.}$$

$$= \frac{kz\sin\theta}{z^2+k^2-2zk\cos\theta} \;//$$

$$Z_T\{k^{-n}\cos n\theta\} = \left[Z_T\{\cos n\theta\}\right]_{z \to zk}.$$

from ①

$$Z_T\{k^{-n}\cos n\theta\} = \left[\frac{z^2 - z\cos\theta}{z^2+1-2z\cos\theta}\right]_{z \to kz}$$

$$= \frac{k^2z^2 - kz\cos\theta}{k^2z^2+1-2kz\cos\theta}$$

$$Z_T\{e^{-an}\sin n\theta\} = Z_T\{(e^{+a})^{-n}\sin n\theta\}$$

$$= \left[\frac{z\sin n\theta}{z^2+1-2z\cos\theta}\right]_{z \to ze^a} = \frac{ze^a\sin\theta}{z^2(e^a)^2+1-2ze^a\cos\theta}$$

$$Z_T\{e^{an}\cos n\theta\} = Z_T\{(e^a)^{-n}\cos n\theta\}$$

$$= \left[\frac{z^2 - z\cos\theta}{z^2+1-2z\cos\theta}\right]_{z \to \frac{z}{e^a}} = \cancel{\frac{z^2e^{2a}\sin\theta - ze^a\cos\theta}{z^2e^{2a}+1-2ze^a\cos\theta}}$$

$$= \left[\frac{\left(\frac{z}{e^a}\right)^2 - \frac{z}{e^a}\cdot\cos\theta}{\left(\frac{z}{e^a}\right)^2+1-2\left(\frac{z}{e^a}\right)\cdot\cos\theta}\right] \times e^{2a} \text{ to both Nr \& Dr.}$$

$$= \frac{z^2 - ze^a\cos\theta}{z^2+e^{2a}-2ze^a\cos\theta}$$

② $n^2 \cdot e^{n\theta}$

$U_n = n^2 \cdot e^{n\theta} = U_n = (e^{\theta})^n \cdot n^2$ $\quad \Big\backslash\ Z_T(k^n U_n) = Z_T[U_n]_{z \to \frac{z}{k}}$

$Z_T\{U_n\} = \left[Z_T\{n^2\}\right]_{z \to \frac{z}{e^{\theta}}}$

$= \left[\frac{z^2+z}{(z-1)^3}\right]_{z \to \frac{z}{e^{\theta}}}$

$\frac{(z-e^{\theta})^3}{e^{\theta}} = \frac{(z-e^{\theta})^3}{e^{3\theta}}$

$= \frac{\left(\frac{z}{e^{\theta}}\right)^2 + \left(\frac{z}{e^{\theta}}\right)}{\left(\frac{z}{e^{\theta}} - 1\right)^3} = \frac{\frac{z^2}{e^{2\theta}} + \frac{z}{e^{\theta}}}{\left(\frac{z}{e^{\theta}} - 1\right)^3} \times e^{3\theta}$ for both Nr & Dr

$= \frac{z^2 e^{\theta} + z e^{2\theta}}{(z - e^{\theta})^3}$

③ $U_n = 3^n \cos \frac{n\pi}{4}$

$Z_T\{U_n\} = Z_T\left\{\cos \frac{n\pi}{4}\right\}_{z \to \frac{z}{3}}$

We have, $Z_T\{e^{in\pi/4}\} = Z_T\left\{\left(e^{\frac{i\pi}{4}}\right)^n\right\} = \frac{z}{z - e^{i\pi/4}}$

$= \frac{z}{z - \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)} = \frac{z}{z - \left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)}$

$= \frac{z}{\left(z - \frac{1}{\sqrt{2}}\right) - i\frac{1}{\sqrt{2}}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}\, z}{(\sqrt{2}z - 1) - i} \times \frac{(\sqrt{2}z-1)+i}{(\sqrt{2}z-1)+i}$

$= \frac{2z^2 - \sqrt{2}z + \sqrt{2}zi}{(\sqrt{2}z-1)^2 - i^2} = \frac{2z^2 - \sqrt{2}z + i\sqrt{2}z}{2z^2 + 1 - 2\sqrt{2}z + 1}$

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$$= \frac{(2z^2 - \sqrt{2}z) + i\sqrt{2}z}{2z^2 + 2 - 2\sqrt{2}z}$$

$$Z_T\left\{\cos \frac{n\pi}{4}\right\} = \frac{2z^2 - \sqrt{2}z}{2z^2 + 2 - 2\sqrt{2}z}$$

(1) becomes $Z_T(u_n) = \left(\frac{2z^2 - \sqrt{2}z}{2z^2 + 2 - 2\sqrt{2}z}\right)_{z \to \frac{z}{3}}$

$$= \frac{2\left(\frac{z}{3}\right)^2 - \sqrt{2}\left(\frac{z}{3}\right)}{2\left(\frac{z}{3}\right)^2 + 2 - 2\sqrt{2}\left(\frac{z}{3}\right)} \times 9 \text{ for } N_1 \,\&\, D_1$$

$$= \frac{2z^2 - 3\sqrt{2}z}{2z^2 + 18 - 6\sqrt{2}z}$$

(4) $a^{-n} \cdot \sinh n\theta$

$U_n = a^{-n} \cdot \sinh n\theta$

$$= Z_T\{\sinh n\theta\}_{z \to za}$$

$$= Z_T\left[\frac{e^{n\theta} \cancel{+} e^{-n\theta}}{2}\right]_{z \to za}$$

$$= \frac{1}{2}\left[Z_T\left[e^{n\theta} \cancel{+} e^{-n\theta}\right]\right]_{z \to za}$$

$$= \frac{1}{2}\left[(e^{\theta})^n \cancel{+} (e^{-\theta})^n\right]_{z \to za}$$

$$= \frac{1}{2}\left[\frac{z}{z - e^{\theta}} \cancel{+} \frac{z}{z - e^{-\theta}}\right]_{z \to za}$$

$$= \frac{1}{2}\left[\frac{za}{za - e^{\theta}} \cancel{+} \frac{za}{za - e^{-\theta}}\right]$$

(5) $Z_T(n\cos n\theta)$

$u_n = n\cos n\theta$

$Z_T(u_n) = Z_T(n\cos n\theta)$

$$= -z \cdot \frac{d}{dz}\left[Z_T(n^0 \cdot \cos n\theta)\right]$$

$$Z_T(u_n) = -z \cdot \frac{d}{dz}\left[Z_T(\cos n\theta)\right] \quad -(1)$$

$$Z_T\left[e^{in\theta}\right] = Z_T\left[(e^{i\theta})^n\right] = \frac{z}{z-e^{i\theta}}$$

$$= \frac{z}{z-(\cos\theta + i\sin\theta)}$$

$$= \frac{z}{(z-\cos\theta)-i\sin\theta} \times \frac{(z-\cos\theta)+i\sin\theta}{(z-\cos\theta)+i\sin\theta} = \frac{z^2-z\cos\theta+iz\sin\theta}{(z-\cos\theta)^2+(\sin\theta)^2}$$

$$= \frac{z^2-z\cos\theta+iz\sin\theta}{z^2+\cos^2\theta-2z\cos\theta+\sin^2\theta} = \frac{z^2-z\cos\theta+iz\sin\theta}{z^2+1-2z\sin\theta}$$

$$Z_T(\cos n\theta) = R_L\text{ part} = \frac{z^2-z\cos\theta}{z^2+1-2z\cos\theta}$$

Sub in (1)

$$Z_T(u_n) = -z \cdot \frac{d}{dz}\left[\frac{z^2-z\cos\theta}{z^2+1-2z\cos\theta}\right]$$

$$= -z\left[\frac{(z^2+1-2z\cos\theta)(2z-\cos\theta)-(z^2-z\cos\theta)(2z-2\cos\theta)}{(z^2+1-2z\cos\theta)^2}\right]$$

$$= -z\left[\frac{z^3-z^2\cos\theta+2z-\cos\theta-4z^2\cos\theta+2z\cos^2\theta-z^3+2z^2\cos\theta+2z^2\cos\theta-2z\cos^2\theta}{(z^2+1-2z\cos\theta)^2}\right]$$

(6) find $Z_T(e^{an})$ and $Z_T(e^{-an})$ hence deduce ne^{an}, n^2e^{an}, ne^{-an}, n^2e^{-an}

$Z_T(e^{an}) \quad U_n = (e^a)^n$

$$Z_T(U_n) = \frac{z}{z-e^a}$$

$$Z_T((e^{-a})^n) = \frac{z}{z-e^{-a}}$$

$$Z_T[n \cdot e^{an}] = -z \cdot \frac{d}{dz}\left[Z_T(n^0 \cdot e^{an})\right]$$

$$= -z \cdot \frac{d}{dz}\left[Z_T(e^{an})\right]$$

$$= -z \cdot \frac{d}{dz}\left[\frac{z}{z-e^a}\right]$$

$$= -z\left[\frac{z-e^a-z}{(z-e^a)^2}\right] = \frac{z \times e^a}{(z-e^a)^2}$$

$$Z_T(n^2 \cdot e^{an}) = -z \cdot \frac{d}{dz}\left[Z_T(n \cdot e^{an})\right]$$

$$= -z \cdot \frac{d}{dz}\left[\frac{z \cdot e^a}{(z-e^a)^2}\right]$$

$$= -z\left[\frac{(z-e^a)(z-e^a) \cdot e^a - 2ze^a}{(z-e^a)^4}\right]$$

$$= -z\left[\frac{ze^a - e^{2a} - 2ze^a}{(z-e^a)^3}\right] = \frac{ze^a[z+e^a]}{(z-e^a)^3}$$

$$= \frac{z^2e^a + ze^{2a}}{(z-e^a)^3} = \frac{ze^a[z+e^a]}{(z-e^a)^3} = \frac{ze^a}{(z-e^a)^2}$$

$$= -z \cdot \frac{d}{dz}\left[Z_T(n^0 \cdot e^{-an})\right]$$

$$= -z \cdot \frac{d}{dz}\left[Z_T(e^{-a})^n\right] = -z \cdot \frac{d}{dz}\left[\frac{z}{z-e^{-a}}\right]$$

$$= -z \cdot \frac{d}{dz}\left[\frac{(z-e^{-a}) - z(1)}{(z-e^{-a})^2}\right] = -z\left[\frac{z-e^{-a}-z}{(z-e^{-a})^2}\right]$$

$$= \frac{z e^{-a}}{(z-e^{-a})^2}$$

$$Z_T(n^2 e^{-an}) = -z \cdot \frac{d}{dz}\left[Z_T(n \cdot e^{-an})\right]$$

$$= -z \frac{d}{dz}\left[\frac{z - e^{-a}}{(z-e^{-a})^2}\right]$$

$$= -z \cdot \left[\frac{(z-e^{-a})^2 \times e^{-a} - 2[z-e^{-a}] z e^{-a}}{(z-e^{-a})^4}\right]$$

$$= -z \left[\frac{(z-e^{-a})(z-e^{-a}) e^{-a} - 2z e^{-a}}{(z-e^{-a})^4}\right]$$

$$= -z \left[\frac{z e^{-a} - e^{0} - 2z e^{-a}}{(z-e^{-a})^3}\right] = -z\left[\frac{-z e^{-a} - e^{-2a}}{(z-e^{-a})^3}\right]$$

$$= \frac{z^2 e^{-a} + e^{-2a} z}{(z-e^{-a})^3} = \frac{z e^{-a}[z + e^{-a}]}{(z-e^{-a})^3}$$

$$= \frac{z e^{-a}[z + e^{-a}]}{(z-e^{-a})^3}$$

7. S.T $Z_T\left(\frac{1}{n!}\right) = e^{1/z}$ hence find $Z_T\left(\frac{1}{(n+1)!}\right), Z_T\left(\frac{1}{(n+2)!}\right)$

$$Z_T\left(\frac{1}{n!}\right) = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n}$$

By defination $Z_T(u_n) = \sum_{n=0}^{\infty} u_n z^{-n}$

$$Z_T(1/n!) = \frac{1}{0!} z^0 + \frac{1}{1!} z^{-1} + \frac{1}{2!} z^{-2} + \frac{1}{3} z^{-3} + \dots$$

$$Z_T\left(\frac{1}{n!}\right) = 1 + \frac{z^{-1}}{1!} + \frac{z^{-2}}{2!} + \frac{z^{-3}}{3!} + \dots \quad -(1)$$

We have,

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

put $x = \frac{1}{z} = z^{-1}$

$$e^{1/z} = 1 + \frac{(1/z)}{1!} + \frac{(1/z)^2}{2!} + \frac{(1/z)^3}{3!} + \dots$$

$$e^{1/z} = 1 + \frac{z^{-1}}{1!} + \frac{z^{-2}}{2!} + \frac{z^{-3}}{3!} + \dots \quad (2)$$

from (1) & (2) $Z_T\left\{\frac{1}{n!}\right\} = e^{1/z}$

Now we deduce $Z_T\left\{\frac{1}{(n+1)!}\right\}$ & $Z_T\left\{\frac{1}{(n+2)!}\right\}$ using shifting rule.

$(u_{n+1}) = z\left\{\bar{u}(z) - u_0\right\}$ $u_{n+2} = z^2\left\{\bar{u}(z) - u_0 - u_1/z\right\}$

$Z_T\left\{u_{n+1}\right\} = z\left\{\bar{u}(z) - u_0\right\}$ $= z\left\{Z_T(u_n) - u_0\right\} = z\left\{z + \left\{\frac{1}{n!}\right\} - \frac{1}{0!}\right\}$

$Z_T\left\{\frac{1}{(n+1)!}\right\} = z\left\{Z_T\left\{u_n\right\} - u_0\right\} = z$

$= z\left\{e^{1/z} - 1\right\}$ $= z^2\left\{e^{1/z} - 1/0! - \frac{n!}{z}\right\}$

$= z^2\left\{e^{1/z} - 1 - 1/z\right\}$

Initial Value theorm:

Statement: If $Z_T(u_n) = \bar{u}(z)$ then $u_0 = \lim_{z\to\infty} \bar{u}$

Similarly

$$u_1 = \lim_{z\to\infty} z\left[\bar{u}(z) - u_0\right]$$

$$u_2 = \lim_{z\to\infty} z^2\left[\bar{u}(z) - u_0 - \frac{u_1}{z}\right]$$

$$u_3 = \lim_{z\to\infty} z^3\left[\bar{u}(z) - u_0 - \frac{u_1}{z} - \frac{u_2}{z^2}\right]$$

we evaluate it by the rule differentiation under integral sign.

Diff w.r.to u.

$$\frac{dFs}{du} = \int_0^\infty \frac{1}{x(1+x^2)} \frac{\partial}{\partial u}(\sin ux)\,dx$$

$$\frac{dFs}{du} = \int_0^\infty \frac{1}{x(1+x^2)} \cos ux \cdot x\,dx$$

$$\frac{dFs}{du} = \int_0^\infty \frac{1}{1+x^2} \cos ux\,dx.$$

Adding & substracting the term x^2 to the numerator of RHS

$$\frac{dFs}{du} = \int_0^\infty \frac{(1+x^2)-x^2}{1+x^2} \cos ux\,dx$$

ntinue:- $u_n = \cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right) = \cos\frac{n\pi}{2}\cos\frac{\pi}{4} - \sin$

sert.

* Initial value theorem:-

→ If $Z_T(u_n) = \bar{u}(z)$ then $u_0 = \lim_{z\to\infty} \bar{u}(z)$.

Similarly $u_1 = \lim_{z\to\infty} z\left[\bar{u}(z) - u_0\right]$

$$u_2 = \lim_{z\to\infty} z^2\left[\bar{u}(z) - u_0 - \frac{u_1}{z}\right]$$

$$u_3 = \lim_{z\to\infty} z^3\left[\bar{u}(z) - u_0 - \frac{u_1}{z} - \frac{u_2}{z}\right]$$

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$z^4\left(1-\frac{1}{z^2}\right)$

Q. If $\bar{u}(z) = \frac{2z^2+3z+12}{(z-1)^4}$ find value of u_0, u_1, u_2, u_3.

→ we have $u_0 = \lim_{z\to\infty} \bar{u}(z) = \lim_{z\to\infty}\left[\frac{2z^2+3z+12}{(z-1)^4}\right]$ ②

$$= \lim_{z\to\infty} \frac{z^2\left(2+\frac{3}{z}+\frac{12}{z^2}\right)}{z^4\left(1-\frac{1}{z}\right)^4}$$

$$= \lim_{z\to\infty} \frac{1}{z^2} \times \frac{\left(2+3/z+12/z^2\right)}{\left(1-\frac{1}{z}\right)^4}$$

$$= 0 \times \frac{(2+0+0)}{1-0} = 0$$

$$\therefore u_0 = \underline{\underline{0}}.$$

$$u_1 = \lim_{z\to\infty} z\left[\bar{u}(z) - u_0\right].$$

$$= \lim_{z\to\infty} z\left[\frac{2z^2+3z+12}{(z-1)^4} - 0\right]$$

$$= \lim_{z\to\infty} \frac{\cancel{z}.\cancel{z^2}\left[2+3/z+12/z^2\right]}{z^4\left(1-\frac{1}{z}\right)^4}$$

$$= \lim_{z\to\infty} \frac{1}{z} \times \frac{\left(2+3/z+12/z^2\right)}{(1-1/z)^4} = 0 \times \left(\frac{2+0+0}{1-0}\right) = \underline{\underline{0}}.$$

$$u_2 = \lim_{z\to\infty} z^2\left[\bar{u}(z) - u_0 - \frac{u_1}{z}\right].$$

$$= \lim_{z\to\infty} z^2\left[\frac{2z^2+3z+12}{(z-1)^4} - 0\right]$$

$$= \lim_{z\to\infty} \frac{z^2 \cdot z^2 (2 + 3/z + 12/z^2)}{z^4 (1 - 1/z)^4}$$

$$= \frac{2+0+0}{1-0} = \underline{2}$$

$$\rightarrow u_3 = \lim_{z\to\infty} z^3 \left[\bar{u}(z) - u_0 - \frac{u_1}{z} - \frac{u_2}{z^2} \right]$$

$$= \lim_{z\to\infty} z^3 \left[\frac{2z^2 + 3z + 12}{(z-1)^4} - \frac{2}{z^2} \right]$$

$$= \lim_{z\to\infty} \cancel{z^3} \left[\frac{2z^4 + 3z^3 + 12z^2 - 2(z-1)^4}{\cancel{z^2}(z-1)^4} \right]$$

$$(z-1)^4 = [(z-1)^2]^2 = (z^2 - 2z + 1)^2$$

$$= (z^2)^2 + (-2z)^2 + 1^2 + 2 \cdot z^2(-2z) + 2(-2z) \cdot 1 + 2 \cdot z^2 \cdot 1$$

$$= z^4 - 4z^3 - 6z^2 - 4z + 1$$

$$= \lim_{z\to\infty} z \left[\frac{2z^4 + 3z^3 + 12z^2 - 2(z^4 - 4z^3 + 6z^2 - 4z + 1)}{(z-1)^4} \right]$$

$$= \lim_{z\to\infty} z \left[\frac{\cancel{2z^4} + 3z^3 + \cancel{12z^2} - \cancel{2z^4} + 8z^3 - \cancel{12z^2} + 8z - 2}{(z-1)^4} \right]$$

$$= \lim_{z\to\infty} z \left[\frac{11z^3 + 8z - 2}{(z-1)^4} \right]$$

$$= \lim_{z\to\infty} \frac{\cancel{z} \cdot \cancel{z^3} \left(11 + 8/z^2 - \frac{2}{z^3}\right)}{\cancel{z^4} (1 - 1/z)^4} = \frac{11+0}{1} = \underline{11}$$

Prob:-

Ans: $Z_T(u_n) = \frac{2z^2+3z+4}{(z-3)^3}$, $|z|>3$ ②.

S.t $u_3 = 139$

+ Final Value Theorem:-

If $Z_T(u_n) = \bar{u}(z)$ then $\lim_{z\to 1}(z-1)\bar{u}(z) = \lim_{n\to\infty} u_n$.

+ Inverse Z Transform:-

+ We have $\bar{u}(z) = Z_T(u_n)$ the inverse z transform is denoted as Z_T^{-1} & defined as $Z_T^{-1}\{\bar{u}(z)\} = u_n$.

If $Z_T\{k^n\} = \frac{z}{z-k}$ then $Z_T^{-1}\left\{\frac{z}{z-k}\right\} = k^n$.

Standard result of Inverse Z Transform:-

1) $Z_T^{-1}\left\{\frac{z}{z-1}\right\} = 1$

2) $Z_T^{-1}\left\{\frac{z}{z-k}\right\} = k^n$

3) $Z_T^{-1}\left\{\frac{z}{(z-1)^2}\right\} = n$

4) $Z_T^{-1}\left\{\frac{z^2+z}{(z-1)^2}\right\} = n^2$

5) $Z_T^{-1}\left\{\frac{z^3+4z^2+z}{(z-1)^4}\right\} = n^3$

6) $Z_T^{-1}\left\{\frac{kz}{(z-k)^2}\right\} = k^n n$

7) $Z_T^{-1}\left\{\frac{kz^2+k^2z}{(z-k)^3}\right\} = k^n n^2$

8) $Z_T^{-1}\left\{\frac{kz^3+4k^2z^2+k^3z}{(z-k)^4}\right\} = kn^3$

$\frac{3z}{(z-3)^2}$

$3^n n$

Type I:- Finding inverse z transform using resolving into partial fraction method.
This method is applicable to find inverse z-transform of $\frac{f(z)}{g(z)}$ if $g(z)$ contains non-repeated linear factors only.

Prob:- * Obtain the inverse z-transform of.

1) $\frac{2z^2+3z}{(z+2)(z-4)}$

we have $\bar{u}(z) = \frac{2z^2+3z}{(z+2)(z-4)}$

we have non-repeated factors in the denominator

$\therefore$ by z $\frac{\bar{u}(z)}{z} = \frac{2z+3}{(z+2)(z-4)} = \frac{A}{z+2} + \frac{B}{z-4} \rightarrow$ ①

Nr $\rightarrow$ $2z+3 = A(z-4) + B(z+2) \rightarrow$ ②.

To find A & B we put $z-4=0$ i.e $z=4$ in ②.

$2\cdot4 + 3 = A\cdot0 + B(4+2)$

$11 = 6B \quad \therefore B = 11/6$

Put $z+2=0$, $z=-2$ in ②

$2(-2)+3 = A(-2-4) + B\cdot0$

$-1 = -6A \quad \therefore A = 1/6$

$\therefore$ ① becomes.

$$\frac{\bar{u}(z)}{z} = \frac{1/6}{z+2} + \frac{11/6}{z-4}$$

x by z (2).

$$\bar{u}(z) = \frac{1}{6}\cdot\frac{z}{z+2} + \frac{11}{6}\cdot\frac{z}{z-4}$$

taking Z_T^{-1}

$$Z_T^{-1}(\bar{u}(z)) = \frac{1}{6} Z_T^{-1}\left\{\frac{z}{z+2}\right\} + \frac{11}{6} Z_T^{-1}\left\{\frac{z}{z-4}\right\}$$

$$= \frac{1}{6} Z_T^{-1}\left\{\frac{z}{z-(-2)}\right\} + \frac{11}{6} Z_T^{-1}\left\{\frac{z}{z-4}\right\}.$$

$$= \frac{1}{6}\cdot(-2)^n + \frac{11}{6}\cdot 4^n$$

2) $\bar{u}(z) = \dfrac{8z^2}{(2z-1)(4z-1)}$

$\rightarrow$ $\div$ by z, $\dfrac{\bar{u}(z)}{z} = \dfrac{8z}{(2z-1)(4z-1)} = \dfrac{A}{(2z-1)} + \dfrac{B}{(4z-1)} \rightarrow (1).$

Nr $\rightarrow$ $8z = A(4z-1) + B(2z-1) \rightarrow (2)$

To find A & B, Put $4z-1=0$ i.e $z = 1/4$ in (2).

$$8(1/4) = 0 + B(2\cdot 1/4 - 1)$$

$$2 = -1/2\, B. \quad \therefore B = -4.$$

Put $2z-1=0$, $z = 1/2$ in (2).

$$8\cdot 1/2 = A\left(4^2\cdot\frac{1}{2} - 1\right) + B\cdot 0$$

$$4 = A \qquad \therefore A = 4$$

eqn ① becomes,

$$\frac{\bar{u}(z)}{z} = \frac{4}{2z-1} - \frac{4}{4z-1}$$

x by z

$$\bar{u}(z) = 4\frac{z}{2z-1} - \frac{4z}{4z-1}$$

$$= \cancel{4}^{2}\frac{z}{\cancel{2}(z-1/2)} - \cancel{4}\frac{z}{\cancel{4}(z-1/4)}$$

taking Z_T^{-1}

$$Z_T^{-1}\{\bar{u}(z)\} = 2 \cdot Z_T^{-1}\left\{\frac{z}{z-1/2}\right\} - Z_T^{-1}\left\{\frac{z}{z-1/4}\right\}$$

$$= 2\left(\tfrac{1}{2}\right)^n - \left(\tfrac{1}{4}\right)^n$$

Type 2:- Finding Inverse z-transform of $\bar{u}(z) = \frac{f(z)}{g(z)}$ where $g(z)$ contains repeated linear factor.

1) $\frac{z^3 - 20z}{(z-3)^2(z-4)}$

we have repeated factors in the denominator ∴ we use the following standard results to obtain Z inverse transform.

$$Z_T^{-1}\left\{\frac{z}{z-4}\right\} = 4^n ; \quad Z_T^{-1}\left\{\frac{z}{z-3}\right\} = 3^n ; \quad Z_T^{-1}\left\{\frac{3z}{(z-3)^2}\right\} = 3^n n$$

Let $\frac{z^3-20z}{(z-3)^2(z-4)} = A.\frac{z}{z-4} + B.\frac{z}{z-3} + C\frac{3z}{(z-3)^2} \rightarrow ①$

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$\times (z-3)^2(z-4)$ to eqn ①. equate Nr:

$z^3-20z = Az(z-3)^2 + Bz(z-3)(z-4) + 3Cz(z-4) \rightarrow$ ②.

Put $z-3=0$ i.e $z=3$ in ②.

$27-60 = 3C.3(-1)$

$-33 = -9C$.

$C = \frac{-33}{-9} = \frac{11}{3}$

Put $z-4=0$ i.e $z=4$ in ②

$64-80 = A\times 4.1$

$-16 = 4A$

$A = -4$.

Put $z=1$ in ②

$1-20 = A(-2)^2 + B(-2)(-3) + 3C(-3)$

$-19 = 4A + 6B - 9C$

$-19 = 4(-4) + 6B - \cancel{9}^{3} \times 11/\cancel{3}_{1}$

$30 = 6B \quad \therefore B = \frac{30}{6} = 5$

① becomes, $\bar{u}(z) = -4 \cdot \frac{z}{z-4} + 5 \frac{z}{z-3} + \frac{11}{\cancel{3}} \cdot \frac{\cancel{3}z}{(z-3)^2}$

taking Z_T^{-1} $\quad Z_T^{-1}(\bar{u}(z)) = -4 \cdot Z_T^{-1}\left(\frac{z}{z-4}\right) + 5 \cdot Z_T^{-1}\left(\frac{z}{z-3}\right) + \frac{11}{3} Z_T^{-1}\left(\frac{3z}{(z-3)^2}\right)$

$u_n = -4.4^n + 5.3^n + 11/\cancel{3} . \cancel{3}n = -4^{n+1} + 5.3^n + 11n3^n$.

2) $\dfrac{8z-z^3}{(4-z)^3}$

$$\bar{u}(z) = \frac{8z-z^3}{(4-z)^3} = \frac{-[z^3-8z]}{(-1[z-4])^3} = \frac{-(z^3-8z)}{-(z-4)^3}$$

we have,

$$= \frac{z^3-8z}{(z-4)^3}$$

$$Z_T^{-1}\left\{\frac{z}{z-4}\right\} = 4^n ; \quad Z_T^{-1}\left\{\frac{4z}{(z-4)^2}\right\} = 4n ; \quad Z_T^{-1}\left\{\frac{4z^2+4^2\cdot z}{(z-4)^3}\right\} = 4^n$$

Let

$$\bar{u}(z) = \frac{z^3-8z}{(z-4)^3} = A\,\frac{z}{z-4} + B\cdot\frac{4z}{(z-4)^2} + C\cdot\frac{(4z^2+16z)}{(z-4)^3} \to ①$$

$\times (z-4)^3$ to ① and equate Nr.

$$z^3 - 8z = Az(z-4)^2 + 4Bz(z-4) + C(4z^2+16z) \to ②$$

Put $z-4=0$ i.e $z=4$

$64-32 = C[64+64]$

$32 = 128C$

$\therefore C = \dfrac{32}{128} = \dfrac{1}{4}$

Put $z=1$ in ②

$-7 = 9A - 12B + 20C$

$-7 = 9A - 12B + 20 \cdot \dfrac{1}{4}$

$-12 = 9A - 12B$

$\div$ by 3

$-4 = 3A - 4B$

$3A - 4B = -4 \to ③$

Put $z=-1$ in ②

$-1-8(-1) = A(-1)(-5)^2 + 4B(-1)(-5) + C(4(-1)^2+16(-1))$

$7 = -25A + 20B - 12C$

$7 = -25A + 20B - 12\cdot\dfrac{1}{4}$

$10 = -25A + 20B$

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$\div$ by 5, $2 = -5A + 4B$.
$-5A + 4B = 2 \longrightarrow$ (4).

Solving (3) & (4)

$A = 1$, $B = 7/4$

$\therefore$ (1) becomes.

$\bar{u}(z) =$

taking Z_T^{-1} =

3) $\bar{u}(z) = \dfrac{4z^2 - 2z}{z^3 - 5z^2 + 8z - 4}$ $\nearrow$ roots are 1 & 2 & 2 repeats twice.

$$\bar{u}(z) = \frac{4z^2 - 2z}{(z-1)(z-2)^2}$$

In denominator we have repeated linear factor. we have,

$$Z_T^{-1}\left\{\frac{z}{z-1}\right\} = 1, \quad Z_T^{-1}\left\{\frac{z}{z-2}\right\} = 2^n, \quad Z_T^{-1}\left\{\frac{2z}{(z-2)^2}\right\} = 2^n \cdot n$$

Let $\bar{u}(z) = \dfrac{4z^2 - 2z}{(z-1)(z-2)^2} = A \cdot \dfrac{z}{z-1} + B \dfrac{z}{z-2} + C \cdot \dfrac{2z}{(z-2)^2} \rightarrow$ (1).

$\times$ by $(z-1)(z-2)^2$ & equate Nr.

$4z^2 - 2z = Az(z-2)^2 + Bz(z-1)(z-2) + 2Cz(z-1) \rightarrow$ (2).

Put $z - 2 = 0$ i.e $z = +2$ in eqn (2).

$4(2)^2 - 2(2) = 2C2(2-1)$.

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$16 - 4 = 4C.$

$12 = 4C.$

$C = \underline{3}.$

Put $z-1=0$ i.e $z=1$ in eqn (2).

$4-2 = A(1-2)^2 +$

$4-2 = A.$

$\underline{2 = A}.$

Put $z=-1$; $4(-1)^2 - 2(-1) = A(-1)(-1-2)^2 + B(-1)(-1-1)(-1-2)$

$+ 2C(-1)(-1-1)$

$4+2 = -9A - 6B + 4C.$

$6 = -9(2) - 6B + (4\times 3)$

$6 = -18 - 6B + 12.$

$6 = -6 - 6B.$

$12 = -6B. \quad \underline{B = -2}.$

(1) becomes.

$$\bar{u}(z) = 2.\frac{z}{z-1} - 2\frac{z}{z-2} + 3\frac{2z}{(z-2)^2}$$

taking Z_T^{-1}

$$Z_T^{-1}\{\bar{u}(z)\} = 2\, Z_T^{-1}\left\{\frac{z}{z-1}\right\} - 2\, Z_T^{-1}\left\{\frac{z}{z-2}\right\} + 3\, Z^{-1}\left\{\frac{2z}{(z-2)^2}\right\}$$

$u_n = 2.1 - 2.2^n + 3\cdot 2^n \cdot n.$

$= \underline{2 - 2^{n+1} + 3.n.2^n}.$

Type 3:- __Power Series method__ :-

To obtain inverse z transforms using power series method we apply the following infinite series.

1. $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

2. $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$

3. $(1+x)^n = nC_0 \cdot x^0 + nC_1 x^1 + nC_2 x^2 + \dots$ for $n > 0$

OR

$= 1 + nx + \frac{n(n-1)}{2!} x^2 + \dots$

Prob:- 1) Obtain inverse z transform of $\log\left(\frac{z}{z+1}\right)$.

$\rightarrow$ $\bar{u}(z) = \log\left(\frac{z}{z+1}\right) = \log\left[\frac{z}{z(1+1/z)}\right]$

$= \log\left(1 + 1/z\right)^{-1} = -1 \log\left(1 + 1/z\right)$

$\bar{u}(z) = -\log\left(1 + 1/z\right)$

we have,

$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$

Put $x = 1/z$

$\log\left(1 + 1/z\right) = 1/z - \frac{(1/z)^2}{2} + \frac{(1/z)^3}{3} - \frac{(1/z)^4}{4} + \dots$

$\therefore \bar{u}(z) = -\left[z^{-1} - \frac{z^{-2}}{2} + \frac{z^{-3}}{3} - \frac{z^{-4}}{4} + \dots\right]$

$\bar{u}(z) = -z^{-1} + \frac{z^{-2}}{2} - \frac{z^{-3}}{3} + \frac{z^{-4}}{4} \dots$

$$\bar{u}(z) = \sum_{n=1}^{\infty} \frac{z^{-n}}{n} (-1)^n$$

By defn:- $\sum_{n=0}^{\infty} z^{-n} u_n = \sum_{n=1}^{\infty} \frac{z^{-n}}{n} (-1)^n$

$$u_n = \frac{(-1)^n}{n}.$$

2) Obtain Z_T^{-1} of $z\log\left(\frac{z}{z+1}\right)$.

→ $\bar{u}(z) = z\log\left(\frac{z}{z+1}\right) = z\log\left[\frac{\cancel{z}}{\cancel{z}(1+1/z)}\right]$

$$= z\log\left(1+1/z\right)^{-1} = -z\log\left(1+1/z\right)$$

$$\bar{u}(z) = -z\log\left(1+1/z\right).$$

we have,

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

Put $x = \frac{1}{z}$

$$\log(1+1/z) = 1/z - \frac{(1/z)^2}{2} + \frac{(1/z)^3}{3} - \frac{(1/z)^4}{4} + \dots$$

$$\therefore \bar{u}(z) = -z\left[z^{-1} - \frac{z^{-2}}{2} + \frac{z^{-3}}{3} - \frac{z^{-4}}{4} + \dots\right]$$

$$\bar{u}(z) = -1 + \frac{z^{-1}}{2} - \frac{z^{-2}}{3} + \frac{z^{-3}}{4} - \dots$$

$$\sum_{n=0}^{\infty} u_n z^{-n} = \sum_{n=0} \frac{z^{-n}}{n+1} (-1)^{n+1}$$

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$\Rightarrow U_n = \frac{(-1)^{n+1}}{n+1}$.

3) $\frac{z}{(z+1)^2}$

$\rightarrow \bar{u}(z) = \frac{\cancel{z}}{z^{\cancel{2}}\left(1+\frac{1}{z}\right)^2} = \frac{1}{z}\left(1+\frac{1}{z}\right)^{-2} \rightarrow (1)$

we have, $(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$

$(1+x)^{-2} = 1-2x+\frac{(-2)(-3)}{2}x^2 + \frac{(-2)(-3)(-4)}{6}x^3 + \ldots$

$(1+x)^{-2} = 1-2x+3x^2-4x^3+\ldots$

Put $x = \frac{1}{z}$

$(1+1/z)^{-2} = 1-2\cdot 1/z + 3(1/z)^2 - 4(1/z)^3 + \ldots$

using (1)

$\bar{u}(z) = \frac{1}{z}\left(1+\frac{1}{z}\right)^{-2} = \frac{1}{z}\left[1-\frac{2}{z}+3\cdot\frac{1}{z^2}-4\cdot\frac{1}{z^3}+\ldots\right]$

$\bar{u}(z) = 1/z - \frac{2}{z^2} + \frac{3}{z^3} - \frac{4}{z^4} + \ldots$

$\sum_{n=0}^{\infty} z^{-n} U_n = \sum_{n=1}^{\infty} \frac{n}{z^n}(-1)^{n+1}$

$\therefore U_n = n(-1)^{n+1}$.

4 3 3 2 2 1

* Solution of differential Eqn using z-transform:-

Working Procedure:- 1) Express given differential Eqn in standard form i.e in terms of $\bar{u}(z)$ using following results.

$$Z_T(u_n) = \bar{u}(z)$$

$$Z_T(u_{n+1}) = z\left[\bar{u}(z) - u_0\right]$$

$$Z_T(u_{n+2}) = z^2\left[\bar{u}(z) - u_0 - u_1/z\right]$$

$$Z_T(u_{n+3}) = z^3\left[\bar{u}(z) - u_0 - u_1/z - \frac{u_2}{z^2}\right]$$

2) obtain $\bar{u}(z)$ using given initial condition.

3) we have $Z_T(u_n) = \bar{u}(z) \Rightarrow u_n = Z_T^{-1}\left[\bar{u}(z)\right]$ taking inverse z transform both sides & obtain u_n using suitable method.

Prob:- Solve using Z_T.

1) $u_{n+2} - 5u_{n+1} - 6u_n = 2^n$

→ [taking z-transform:]

$$Z_T(u_{n+2}) - 5Z_T(u_{n+1}) - 6Z_T(u_n) = Z_T(2^n)$$

$$z^2\left[\bar{u}(z) - u_0 - u_1/z\right] - 5z\left[\bar{u}(z) - u_0\right] - 6\bar{u}(z) = \frac{z}{z-2}$$

$$z^2\bar{u}(z) - z^2u_0 - z^2 u_1/z - 5z\bar{u}(z) + 5zu_0 - 6\bar{u}(z) = \frac{z}{z-2}$$

Collect co-eff of $\bar{u}(z)$

$$(z^2 - 5z - 6)\bar{u}(z) + (-z^2 + 5z)u_0 - u_1 z = \frac{z}{z-2}$$

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$$(z^2-5z-6)\bar{u}(z) = \frac{z}{z-2} - (-z^2+5z)u_0 + u_1 z$$

no calculation

$$(z-6)(z+1)\bar{u}(z) = \frac{z}{z-2} + (z^2-5z)u_0 + u_1 z$$

$\div (z-6)(z+1)$

$$\bar{u}(z) = \frac{z}{(z-2)(z-6)(z+1)} + \frac{z^2-5z}{(z-6)(z+1)}u_0 + \frac{z}{(z-6)(z+1)}u_1$$

Taking inverse z transform (z_T^{-1})

$$z_T^{-1}\{\bar{u}(z)\} = z_T^{-1}\left\{\frac{z}{(z-2)(z-6)(z+1)} + \frac{z^2-5z}{(z-6)(z+1)}u_0 + \frac{z}{(z-6)(z+1)}u_1\right\}$$

$$U_n = z_T^{-1}\left\{\frac{z}{(z-2)(z-6)(z+1)}\right\} + u_0\, z_T^{-1}\left\{\frac{z^2-5z}{(z-6)(z+1)}\right\} + u_1\, z_T^{-1}\left\{\frac{z}{(z-6)(z+1)}\right\} \to (1)$$

Let $p(z) = \frac{z}{(z-2)(z-6)(z+1)}$; $q(z) = \frac{z^2-5z}{(z-6)(z+1)}$

$$r(z) = \frac{z}{(z-6)(z+1)}$$

we have non-repeated factors in Denominator $\therefore$ we follow type 1.

$$\frac{p(z)}{z} = \frac{1}{(z-2)(z-6)(z+1)} = \frac{A}{z-2} + \frac{B}{z-6} + \frac{C}{z+1} \to (2)$$

$\times (z-2)(z-6)(z+1)$ & equate Nr.

$$1 = A(z-6)(z+1) + B(z-2)(z+1) + C(z-2)(z-6)$$

Put $z-6=0$ i.e $z=6$.

$$1 = B(4)(7)$$

$$\therefore B = 1/28$$

Put $z+1=0$ i.e $z=-1$; Put $z-2=0$ i.e $z=2$

$$1 = C(-3)(-7) \qquad 1 = A(-4)(3)$$

$$C = 1/21 \qquad A = -1/12$$

$\therefore$ (2) becomes :-

$$\frac{p(z)}{z} = \frac{-1/12}{z-2} + \frac{1/28}{z-6} + \frac{1/21}{z+1}$$

x by z $\quad p(z) = -\frac{1}{12}\cdot\frac{z}{z-2} + \frac{1}{28}\frac{z}{z-6} + \frac{1}{21}\frac{z}{z+1}$

$\Rightarrow$ we have, $q(z) = \frac{z^2-5z}{(z-6)(z+1)}$

$\div$ by z $\quad \frac{q(z)}{z} = \frac{z-5}{(z-6)(z+1)} = \frac{D}{z-6} + \frac{E}{z+1} \rightarrow$ (3).

x $(z-6)(z+1)$ & equate Nr.

$$z-5 = D(z+1) + E(z-6)$$

Put $z=-1$; Put $z=6$.

$$-1-5 = C(-1-6) \quad ; \quad 6-5 = D(6+1)$$

$$-6 = 7C \qquad 1 = 7D$$

$$\therefore C = 6/7 \qquad \therefore D = 1/7$$

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(3) becomes, $\frac{q(z)}{z} = \frac{1/7}{z-6} + \frac{6/7}{z+1}$

$\times z$ $q(z) = \frac{1}{7}\frac{z}{z-6} + \frac{6}{7}\frac{z}{z+1}$

→ we have, $r(z) = \frac{z}{(z-6)(z+1)}$

÷ by z

$$\frac{r(z)}{z} = \frac{1}{(z-6)(z+1)} = \frac{F}{(z-6)} + \frac{G}{(z+1)} \rightarrow (4)$$

× by $(z-6)(z+1)$ & equate Nr.

$$1 = F(z+1) + G(z-6).$$

Put $z+1=0$; i.e $z=-1$. ; Put $z-6=0$ i.e $z=6$.

$1 = G(-7)$. ; $1 = F(7)$.

$G = -\frac{1}{7}$; $F = \frac{1}{7}$

(4) becomes

$$r(z)/z = 1/7 \frac{1}{(z-6)} + (-1/7)\frac{1}{(z+1)}$$

$\times z$ $r(z) = \frac{1}{7}\frac{z}{(z-6)} - \frac{1}{7}\frac{z}{(z+1)}$

Substituting $p(z)$, $q(z)$ & $r(z)$ in (1).

$$U_n = Z_T^{-1}\left\{-\frac{1}{12}\frac{z}{z-2} + \frac{1}{28}\frac{z}{z-6} + \frac{1}{21}\frac{z}{z+1}\right\} + u_0 Z_T^{-1}$$

$$\left\{\frac{1}{7}\frac{z}{z-6} + \frac{6}{7}\frac{z}{z+1}\right\} + u_1 Z_T^{-1}\left\{\frac{1}{7}\frac{z}{z-6} - \frac{1}{7}\frac{z}{(z+1)}\right\}$$

$$U_n = -\frac{1}{12} 2^n + \frac{1}{28} 6^n + \frac{1}{21}(-1)^n + u_0\left[\frac{1}{7} 6^n + \frac{6}{7}(-1)^n\right]$$

$$+ u_1\left[\frac{1}{7} 6^n - \frac{1}{7}(-1)^n\right]$$

$$U_n = -\frac{1}{12} \cdot 2^n + \tfrac{1}{28} 6^n + \tfrac{1}{21}(-1)^n + \tfrac{1}{7} 6^n(u_0+u_1) + \tfrac{1}{7}(-1)^n (6u_0 - u_1)$$

$$U_n = -\frac{2^n}{12} + \frac{6^n}{28} + \underline{\frac{(-1)^n}{21}} + c_1\frac{6^n}{7} + \frac{c_2}{7}(-1)^n$$

where $c_1 = u_1 + u_0$, $c_2 = 6u_0 - u_1$.

2) Solve $y_{n+2} - 5y_{n+1} + 6y_n = u_n$ where $y_0 = 1$, $y_1 = 1$ and $u_n = 1$ for $n = 0, 1, 2, 3$.

→ we have $y_{n+2} - 5y_{n+1} + 6y_n = 1$

taking z-transform

$$Z_T\{y_{n+2}\} - 5Z_T\{y_{n+1}\} + 6Z_T\{y_n\} = Z_T(1)$$

$$z^2\left\{\bar{y}(z) - y_0 - \frac{y_1}{z}\right\} - 5z\{\bar{y}(z) - y_0\} + 6\bar{y}(z) = z/z-1$$

Put $y_0 = 0$, $y_1 = 1$

$$z^2\left\{\bar{y}(z) - \frac{1}{z}\right\} - 5z\{\bar{y}(z) - 0\} + 6\bar{y}(z) = z/z-1$$

$$z^2\bar{y}(z) - z - 5z\bar{y}(z) + 6\bar{y}(z) = z/z-1$$

$$(z^2 - 5z + 6)\bar{y}(z) = z/z-1 + z$$

$$(z-2)(z-3)\bar{y}(z) = z/z-1 + z$$

÷ by $(z-2)(z-3)$

$$\bar{y}(z) = \frac{z}{(z-1)(z-2)(z-3)} + \frac{z}{(z-2)(z-3)}$$

take Z_T^{-1} on both sides.

$$Z_T^{-1}\{\bar{y}(z)\} = Z_T^{-1}\left\{\frac{z}{(z-1)(z-2)(z-3)}\right\} + Z_T^{-1}\left\{\frac{z}{(z-2)(z-3)}\right\} \rightarrow ①$$

- Let $p(z) = \dfrac{z}{(z-1)(z-2)(z-3)}$; $q(z) = \dfrac{z}{(z-2)(z-3)}$

÷ by z $\quad \dfrac{p(z)}{z} = \dfrac{1}{(z-1)(z-2)(z-3)} = \dfrac{A}{z-1} + \dfrac{B}{z-2} + \dfrac{C}{z-3}$

$$1 = A(z-2)(z-3) + B(z-1)(z-3) + C(z-1)(z-2)$$

put $z-2=0$ i.e $z=2$

$1 = B(1)(-1)$

$1 = -B$

$\underline{\underline{B = -1}}$

put $z-1=0$ i.e $z=1$.

$1 = A(-1)(-2)$.

$1 = 2A$

$A = \underline{\underline{1/2}}$.

put $z-3=0$ i.e $z=3$.

$1 = C(2)(1)$.

$\underline{\underline{C = 1/2}}$.

$$\frac{p(z)}{z} = \frac{1/2}{z-1} - \frac{1}{z-2} + \frac{1/2}{z-3}.$$

× by z

$$\underline{\underline{p(z) = \frac{1}{2}\frac{z}{z-1} - \frac{z}{z-2} + \frac{1}{2}\frac{z}{z-3}.}}$$

÷ by z

- $\dfrac{q(z)}{z} = \dfrac{1}{(z-2)(z-3)} = \dfrac{A}{(z-2)} + \dfrac{B}{(z-3)}$

$$1 = A(z-3) + B(z-2).$$

put $z-3=0$ i.e $z=3$; put $z-2=0$ i.e $z=2$.

$\underline{1 = B}$. $\underline{A = -1}$.

$$\frac{q(z)}{z} = \frac{1}{z-3} - \frac{1}{z-2}.$$

$$q(z) = \frac{z}{z-3} - \frac{z}{z-2}.$$

Substituting $p(z)$ & $q(z)$ in 1, we get,

$$y_n = Z_T^{-1}\left\{\frac{1}{2}\frac{z}{z-1} - \frac{z}{z-2} + \frac{1}{2}\frac{z}{z-3}\right\} + Z_T^{-1}\left\{\frac{z}{z-3} - \frac{z}{z-2}\right\}$$

$$= \frac{1}{2}\cdot 1^n - 2^n + \frac{1}{2} 3^n + 3^n - 2^n$$

$$= \frac{1}{2} - 2\cdot 2^n + \frac{3}{2}\cdot 3^n$$

$$y_n = \frac{1}{2} - 2^{n+1} + \frac{1}{2} 3^{n+1}$$

3) Solve $U_{n+2} + 2U_{n+1} + U_n = n$ given $u_0 = u_1 = 0$.

→ we have $U_{n+2} + 2U_{n+1} + U_n = n$

taking z transform.

$$Z_T\{U_{n+2}\} + 2Z_T\{U_{n+1}\} + Z_T\{U_n\} = Z_T\{n\}$$

$$z^2[\bar{u}(z) - u_0 - u_1/z] + 2z[\bar{u}(z) - u_0] + \bar{u}(z) = \frac{z}{(z-1)^2}$$

$$z^2\bar{u}(z) - z^2u_0 - zu_1 + 2z\bar{u}(z) - 2zu_0 + \bar{u}(z) = \frac{z}{(z-1)^2}$$

Put $u_0 = u_1 = 0$.

$$z^2\bar{u}(z) + 2z\bar{u}(z) + \bar{u}(z) = \frac{z}{(z-1)^2}.$$

$$\bar{u}(z)(z^2+2z+1) = \frac{z}{(z-1)^2}$$

$$\bar{u}(z)(z+1)(z+1) = \frac{z}{(z-1)^2}$$

$$\bar{u}(z) = \frac{z}{(z-1)^2(z+1)^2}$$

taking Z_T^{-1}

$$Z_T^{-1}\{\bar{u}(z)\} = Z_T^{-1}\left\{\frac{z}{(z-1)^2(z+1)^2}\right\}$$

$Z_T^{-1}\left\{\frac{z}{(z-1)^2}\right\} = n$; $Z_T^{-1}\left\{\frac{z}{z-1}\right\} = 1$; $Z_T^{-1} = \left\{\frac{z}{z+1}\right\} = (-1)^n$;

$Z_T^{-1}\left\{\frac{z}{(z+1)^2}\right\} = (-1)^n \cdot n.$

let $f(z) = \frac{z}{(z-1)^2(z+1)^2} = A\frac{z}{(z-1)^2} + B\frac{z}{z-1} + C\frac{z}{(z+1)^2} + D\frac{z}{z+1}$

x by $(z-1)^2(z+1)^2$,

$$z = Az(z+1)^2 + Bz(z-1)(z+1)^2 + Cz(z-1)^2 + Dz(z+1)(z-1)^2$$

Put $z-1=0$, $z=1$; Put $z+1=0$, $z=-1$

$1 = 4A$; $-1 = C(-1)(4)$

$\underline{\underline{A = 1/4}}$. ; $\underline{\underline{C = 1/4}}$.

Put $z=2$

$2 = 18A + 18B + 2C + 6D$

$2 = 18 \times \frac{1}{4} + 18B + 2 \times \frac{1}{4} + 6D$

$2 = \frac{9}{2} + 18B + \frac{1}{2} + 6D$

$2 = 5 + 18B + 6D$

$\therefore$ by 3 $18B + 6D = -3$

$6B + 2D = -1 \rightarrow (2)$.

Put $z=3$.

$3 = \frac{1}{4}(16 \times 3) + B\,96 + \frac{1}{4} \times 12 + D\,48.$

$3 = 12 + 96B + 3 + 48D.$

$96B + 48D = -12 \rightarrow (3)$.

$B = -1/4 \quad , \quad D = 1/4$

$$F(z) = 1/4 \cdot \frac{z}{(z-1)^2} - 1/4 \frac{z}{z-1} + 1/4 \frac{z}{(z+1)^2} + 1/4 \frac{z}{z+1}$$

$\therefore$ (1) becomes.

$$Z_T^{-1}\{\bar{u}(z)\} = Z_T^{-1}\left\{1/4 \frac{z}{(z-1)^2} - 1/4 \frac{z}{z-1} + 1/4 \frac{z}{(z+1)^2} + 1/4 \frac{z}{z+1}\right\}$$

$$= 1/4 \cdot n - 1/4 \cdot 1 + 1/4\, n(-1)^n + 1/4\, (-1)^n$$

9.189 MODULE - 3 Fourier Transform:-

→ Defn:- Fourier transform of a real value function $f(x)$ denoted as $F(u)$ and defined as ~~$(F(u) = \int_{-\infty}^{\infty} f(x)$~~

$$F(u) = \int_{-\infty}^{\infty} f(x) e^{iux} dx$$

→ Inverse fourier transform of $F(u)$ defined as ~~$f(x) =$~~

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(u) e^{iux} du.$$

→ (~~Inverse~~) Fourier sine-transform defined as ~~$(F_s(u))$~~

$$F_s(u) = \int_{0}^{\infty} f(x) \sin ux \, dx.$$

→ Inverse fourier sine transform :- $f(x) = \frac{2}{\pi} \int_{0}^{\infty} F_s(u) \sin ux \, du.$

→ Fourier cosine-transform :- $F_c(u) = \int_{0}^{\infty} f(x) \cos ux \, dx$

→ Inverse fourier cosine transform :- $f(x) = \frac{2}{\pi} \int_{0}^{\infty} F_c(u) \cos ux \, du$

Note:- Some important result:-

1) $e^{i\theta} = \cos\theta + i\sin\theta$; $e^{-i\theta} = \cos\theta - i\sin\theta$.

2) $e^{i\theta} + e^{-i\theta} = 2\cos\theta$; $e^{i\theta} - e^{-i\theta} = 2i\sin\theta$.

3) $i = \sqrt{-1}$; $i^2 = -1$; $i^3 = -i$.

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4) $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right]$

$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]$

Imp * Application of Fourier transform:-

Type I :- 1) Find fourier transform of $f(x) = \begin{cases} 1-|x| , & |x| \le 1 \\ 0 , & |x| > 1 \end{cases}$ &

p.t $\int_0^{\infty} \frac{\sin^2 t}{t^2}\,dt = \frac{\pi}{2}$

$\rightarrow$ $f(x) = \begin{cases} 1-|x| , & -1 \le x \le 1 \\ 0 , & \text{otherwise} \end{cases}$

given $f(x) = \begin{cases} 1-x , & 0 < x < 1 \\ 1+x , & -1 < x < 0 \\ 0 , & \text{otherwise} \end{cases}$

Fourier transform $F(u) = \int_{-\infty}^{\infty} f(x)e^{iux}\,dx$

$F(u) = \int_{-1}^{0}(1+x)e^{iux}\,dx + \int_0^1 (1-x)e^{iux}\,dx$

$= \left[(1+x)\frac{e^{iux}}{iu} - \frac{e^{iux}}{i^2u^2}\right]_{-1}^{0} + \left[(1-x)\frac{e^{iux}}{iu} - (-1)\frac{e^{iux}}{i^2u^2}\right]_0^1$

$$\left[\left(\frac{1}{iu}-\frac{1}{i^2u^2}\right)-\left(\frac{-e^{-iu}}{i^2u^2}\right)\right]+\left[\left(\frac{e^{iu}}{i^2u^2}\right)-\left(\frac{1}{iu}+\frac{1}{i^2u^2}\right)\right]$$

$$F(u)=\frac{1}{iu}+\frac{1}{u^2}-\frac{e^{-iu}}{u^2}-\frac{e^{iu}}{u^2}-\frac{1}{iu}+\frac{1}{u^2}=\frac{2}{u^2}-\frac{1}{u^2}\left[e^{ui}+e^{-ui}\right]$$

$$=\frac{2}{u^2}-\frac{1}{u^2}\times 2\cos u \quad = 2/u^2\left[1-\cos u\right]$$

$$=\frac{2}{u^2}\times 2\sin^2\frac{u}{2}$$

$$=\frac{4}{u^2}\cdot\sin^2\frac{u}{2}=F(u)$$

Now we P.T.

$$\int_0^\infty \frac{\sin^2 t}{t^2}\,dt=\frac{\pi}{2} \text{ using inverse f.t.}$$

By defn,

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(u)\,e^{iux}\,du$$

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}4/u^2\sin^2\frac{u}{2}\,e^{iux}\,du$$

Put $x=0$

$$f(0)=1/2\pi\int_{-\infty}^{\infty}\frac{\sin^2 u/2}{u^2/4}\cdot e^0\,du$$

$$\frac{f(0+)+f(0^-)}{2}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\sin^2 u/2}{(u/2)^2}\,du$$

$$\frac{1+1}{2} = \frac{1}{2\pi} 2\int_0^\infty \frac{\sin^2 u/2}{(u/2)^2}\, du \quad \left(\because \frac{\sin^2 u/2}{(u/2)^2} \text{ is even}\right)$$

$$\left(\cancel{\tfrac{1}{2}}\cancel{\tfrac{1}{\pi}}\right) 1 = \frac{1}{\pi}\int_0^\infty \frac{\sin^2 u/2}{(u/2)^2}\, du.$$

Put $u/2 = t$, $du = 2\,dt$ when $u=0$, $t=0$,
$u \to \infty, t \to \infty$

$$\pi = \int_0^\infty \frac{\sin^2 t \cdot 2dt}{t^2}$$

$$\pi/2 = \int_0^\infty \frac{\sin^2 t}{t^2}\, dt.$$

2) Find the fourier transform of $f(x) = \begin{cases} 1+x/a, & -a<x<0 \\ 1\cancel{+}x/a, & 0<x<a \\ 0, & \text{otherwise} \end{cases}$

→ By definition F.T of $f(x) =$

$$= Fu = \int_{-\infty}^{\infty} f(x)\, e^{iux}\, dx$$

$$= \int_{-a}^{0} (1+x/a)\, e^{iux}\, dx + \int_0^a (1-x/a)\, e^{iux}\, dx$$

$$= \left[\left(1+x/a\right)\frac{e^{iux}}{iu} - \frac{1}{a}\frac{e^{iux}}{i^2u^2}\right]_{-a}^{0} + \left[\left(1-\frac{x}{a}\right)\frac{e^{iux}}{iu} - \left(-\frac{1}{a}\right)\frac{e^{iux}}{i^2u^2}\right]_0^a$$

$$= \left[\frac{e^0}{iu} - \frac{1}{a}\frac{e^0}{(-1)u^2}\right] - \left[0 - \frac{1}{a}\cdot\frac{e^{-iua}}{(-u^2)}\right] + \left[0 + \frac{e^{iua}}{a(-u^2)}\right] - \left[\frac{e^0}{iu} + \frac{e^0}{a(-u^2)}\right]$$

$$= \frac{1}{iu} + \frac{1}{au^2} - \frac{e^{-iau}}{au^2} - \frac{e^{+iau}}{au^2} - \frac{1}{iu} + \frac{1}{au^2}$$

$i^2u^2 = -1 \cdot u^2 = -u^2$

$e^{i\theta} + e^{-i\theta} = 2\cos\theta$

$$= \frac{2}{au^2} - \frac{1}{au^2}\left[e^{-iau} + e^{iau}\right]$$

$$= \frac{2}{au^2} - \frac{2\cos au}{au^2} = \frac{2}{au^2}\left[1 - \cos au\right]$$ 2).

Imp.

3) Find the complex fourier transform of $f(x) = \begin{cases} 1 & \text{for } |x| < a \\ 0 & \text{for } |x| > a \end{cases}$

hence evaluate $\int_0^\infty \frac{\sin x}{x}\,dx$.

→

$f(x) = \begin{cases} 1, & -a < x < a \\ 0, & \text{otherwise} \end{cases}$

$|x| = x,\ x > 0$
$= -x,\ x < 0$

By def^n, $F(u) = \int_{-\infty}^{\infty} f(x) e^{iux}\,dx$

$$F(u) = \int_{-a}^{a} 1 \cdot e^{iux}\,dx = \frac{e^{iux}}{iu}\Big|_{-a}^{a}$$

$$= \frac{e^{iau}}{iu} - \frac{e^{-iau}}{iu}$$

$$= \frac{1}{iu}\left[e^{iau} - e^{-iau}\right]$$

$$= \frac{1}{\cancel{i}u}\, 2\cancel{i}\sin au$$

$$F(u) = \frac{2\sin au}{u}$$

Now we evaluate $\int_0^\infty \frac{\sin x}{x}\,dx$ using inverse F.T.

By def^n :- $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(u) e^{iux}\,dx$.

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\int_{0}^{\infty} f(x)\,dx \rightarrow \text{even}$$
$$= 0, \; f(x) \text{ is odd.}$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\,\frac{\sin au}{u}\, e^{iux}\, du.$$

Put $x=0$.

$$f(0) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin au}{u}\, du.$$

$$1 = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin au}{u}\, du.$$

Put $a=1$

$$\pi = \int_{-\infty}^{\infty} \frac{\sin u}{u}\, dx.$$

$\frac{\sin u}{u}$ is an even function.

$$g(u) = \frac{\sin u}{u}.$$
$$g(-u) = \frac{\sin(-u)}{-u} = \frac{-\sin u}{-u} = \frac{\sin u}{u} = g(u)$$

$$\pi = 2\int_{0}^{\infty} \frac{\sin u}{u}\, du.$$

$$\pi/2 = \int_{0}^{\infty} \frac{\sin u}{u}\, du.$$

$$\therefore \int_{0}^{\infty} \frac{\sin x}{x}\, dx = \pi/2$$

4) Find the complex ~~function~~ fourier transform of the function $e^{-|x|}$.

By defn:-

$$\rightarrow F(u) = \int_{-\infty}^{\infty} f(x)\, e^{iux}\, dx.$$

$-|x| = -x, \; x>0$
$= x, \; x<0.$

$$F(u) = \int_{-\infty}^{0} e^{x} e^{iux}\, dx + \int_{0}^{\infty} e^{-x} e^{iux}\, dx$$

$e^{-|x|} = e^{-x}, \; x>0.$
$e^{x}, \; x<0.$

e^{x} | e^{-x}

$-\infty \quad x<0 \quad 0 \quad x>0 \quad \infty$

$$= \int_{-\infty}^{0} e^{(1+iu)x}\,dx + \int_{0}^{\infty} e^{x(-1+ui)}\,dx$$

$$= \int_{-\infty}^{0} e^{(1+ui)x}\,dx + \int_{0}^{\infty} e^{-x(1-ui)}\,dx$$

$$= \frac{e^{(1+ui)x}}{1+ui}\Big|_{-\infty}^{0} + \frac{e^{-x(1-ui)}}{-(1-ui)}\Big|_{0}^{\infty}$$

$$= \left[\frac{e^{0}}{1+ui} - \frac{e^{-\infty}}{1+ui}\right] - \left[\frac{e^{-\infty}}{1-ui} - \frac{e^{0}}{1-ui}\right]$$

$$= \frac{1}{1+ui} + \frac{1}{1-ui} \qquad e^{-\infty} \approx \frac{1}{e^{\infty}} \approx \frac{1}{\infty} \approx 0$$

$$= \frac{1-ui+1+ui}{(1+ui)(1-ui)} = \frac{2}{1^2-u^2i^2} = \frac{2}{1+u^2}$$

5) Find the complex fourier transform of $f(x) = xe^{-|x|}$

→ ∴ we have $f(x) \begin{cases} xe^{-x} & x>0 \quad (0,\infty) \\ xe^{x}, & x<0 \quad (-\infty,0) \end{cases}$

$-|x| = -x \quad x \geq 0$

$\quad\quad = x \quad x<0$

$e^{-|x|} = e^{-x} \quad x \geq 0$

$\quad\quad = e^{x} \quad x<0$

By defn F.T

$$F(u) = \int_{-\infty}^{\infty} f(x) e^{iux}\,dx$$

$$= \int_{-\infty}^{0} xe^{x} e^{iux}\,dx + \int_{0}^{\infty} xe^{-x} e^{iux}\,dx$$

$$e^{iux}e^{x} = e^{iux+x} = e^{(iu+1)x}$$

$$e^{iux}e^{-x} = e^{-x+iux} = e^{-x[1-ui]}$$

$e^{0}=1$

$$= \left[x\frac{e^{(1+ui)x}}{1+ui} - 1\frac{e^{(1+ui)x}}{(1+ui)^2}\right]_{-\infty}^{0} + \left[\frac{xe^{-(1-ui)x}}{-(1-ui)} - 1\frac{e^{-(1-ui)x}}{(1-ui)^2}\right]_{0}^{\infty}$$

$$= \left[\left(-\frac{1}{(1+ui)^2}\right) - 0\right] + \left[0 - \left(-\frac{1}{(1-ui)^2}\right)\right]$$

$$= -\frac{1}{(1+ui)^2} + \frac{1}{(1-ui)^2}$$

$$= \frac{-(1-ui)^2 + (1+ui)^2}{(1+ui)^2(1-ui)^2}$$

$$= \frac{-[1+u^2i^2-2ui] + [1^2+u^2 i+2ui]}{[(1+ui)(1-ui)]^2}$$

$$= \frac{-1-u^2i^2+2ui+1+u^2i^2+2ui}{[1^2-u^2i^2]^2} = \frac{4ui}{(1+u^2)^2}$$

6. Find the fourier cosine transform of

$$f(x) = \begin{cases} 4x, & 0<x<1 \\ 4-x, & 1<x<4 \\ 0, & \text{otherwise} \end{cases}$$

→ By defn $F_c(u) = \int_0^\infty f(x)\cos ux\, dx$.

$$= \int_0^1 4x\cos ux\, dx + \int_1^4 (4-x)\cos ux\, dx$$

$$= \left[4x \cdot \frac{\sin ux}{u} - 4\left(\frac{-\cos ux}{u^2}\right)\right]_0^1 + \left[(4-x)\frac{\sin ux}{\cancel{x}u} - (-1)\left(\frac{-\cos ux}{u^2}\right)\right]_1^4$$

$$\frac{e^{ax}}{a^2+b^2}\left[a\,\cancel{\sin}\cos bx + \cancel{b\cos}\sin bx\right]$$

Date :

$$= \left[4\frac{\sin u}{u} + 4\frac{\cos u}{u^2} - \left(0+\frac{4}{u^2}\right)\right] + \left[-\frac{\cos u4}{u^2} - \left(\frac{3\sin u3}{u} - \frac{\cos u}{u^2}\right)\right]$$

$$= \left[\frac{4\sin u}{u} + \frac{4\cos u}{u^2} - \frac{4}{u^2} - \frac{\cos u4}{u^2} - \frac{3\sin u3}{u} + \frac{\cos u}{u^2}\right]$$

$$= \frac{\sin u}{u} + \frac{5\cos u}{u^2} - \frac{4}{u^2} - \frac{\cos 4u}{u^2}$$

7. Find the fourier sine transform of $f(x) = \frac{e^{-ax}}{x}$, $a>0$.

→ By defn :- $F_s(u) = \int_0^\infty f(x)\sin ux\,dx$

$$F_s(u) = \int_0^\infty \frac{e^{-ax}}{x}\sin ux\,dx$$

we evaluate this function by rule of diffrentiation under integral sign.

diff w. r to u

$$\frac{d}{du}[F_s(u)] = \int_0^\infty \frac{e^{-ax}}{x}\frac{d}{du}[\sin ux]\,dx$$

$$\frac{d}{du}F_s(u) = \int_0^\infty \frac{e^{-ax}}{\cancel{x}}(\cos ux \cdot \cancel{x})\,dx$$

$$\frac{d}{du}[F_s(u)] = \int_0^\infty e^{-ax}\cos ux\,dx$$

$$\frac{d}{du}[F_s(u)] = \left[\frac{e^{-ax}}{(-a)^2+u^2}\left[-a\cos ux + u\sin ux\right]\right]_0^\infty$$

$$\frac{d}{du}[F_s(u)] = 0 - \frac{1}{a^2+u^2}\left[-a\times\cos 0 + 0\right]$$

$$\frac{d}{du}F_s(u) = \frac{a}{a^2+u^2}$$

$$\int\frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}.$$

integrate w.r.to 'u'

$$F_s(u) = \int \frac{a}{a^2+u^2}\,du$$

$$F_s(u) = a\int \frac{1}{u^2+a^2}\,du = a \times \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + c$$

$$F_s(u) = \tan^{-1}\left(\frac{u}{a}\right) + c$$

To find 'c'

Put $u = 0$.

$$F_s(0) = \tan^{-1}0 + c$$

$$0 = 0 + c \quad \therefore c = 0.$$

$$\therefore F_s(u) = \tan^{-1}(u/a)$$

v. Imp

7. Find fourier sine transform of $f(x) = e^{-|x|}$, hence evaluate $\int_0^\infty \frac{x \sin mx}{1+x^2}\,dx$, $m > 0$.

→ $f(x) = \begin{cases} e^{-x}, & x > 0 \quad (0,\infty) \\ e^{x}, & x < 0 \quad (-\infty, 0) \end{cases}$

By defn, $F_s(u) = \int_0^\infty f(x)\sin ux\,dx$

$$= \int_0^\infty e^{-x}\sin ux\,dx$$

$$= \frac{e^{-x}}{(-1)^2+u^2}\left[-1.\sin ux - u\cos ux\right]_0^\infty$$

$$= 0 - \left[\frac{1}{1+u^2}(0-u)\right]$$

$F_s(u) = \frac{u}{1+u^2}$ Now we evaluate this by fourier sine inverse transform.

i.e.,

$$f(x) = \frac{2}{\pi}\int_0^\infty F_s(u)\sin ux\, dx$$

$$f(x) = 2/\pi \int_0^\infty \frac{u}{1+u^2}\sin ux\, dx$$

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{u\sin ux}{1+u^2}\, dx$$

Put $x = m$

$$f(m) = 2/\pi \int_0^\infty \frac{u\sin mu}{1+u^2}\, dx$$

Put $u = x$

$$e^{-m} = \frac{2}{\pi}\int_0^\infty \frac{x\sin mx}{1+x^2}\, dx$$

$$\therefore \int_0^\infty \frac{x\sin mx}{1+x^2}\, dx = \frac{\pi e^{-m}}{2}.$$

Imp

Q. Find the fourier sine transform of $\frac{1}{x(1+x^2)}$.

→ $f(x) = \frac{1}{x(1+x^2)}$

$$F_s(u) = \int_0^\infty \frac{1}{x(1+x^2)} \sin ux \, dx$$

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