

Model Question Paper-1 with effect from 2019-20 (CBCS Scheme)

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18MAT31

Third Semester B.E.Degree Examination Transform Calculus, Fourier Series and Numerical Techniques (Common to all Programmes)

Time: 3 Hrs

Max.Marks: 100

Note: Answer any FIVE full questions, choosing at least ONE question from each module.

Module-1

1. (a) Find the Laplace transform of (i) $\sqrt{e^{4t+3}} + e^{-2t} \sin 3t$ (ii) $te^{-3t} \sin 4t$ (iii) $(1 - \cos t)/t$ (10 Marks)
- (b) The square wave function $f(t)$ with period "a" is defined by $f(t) = \begin{cases} E, & 0 \leq t < a/2 \\ -E, & a/2 \leq t < a, \end{cases}$
Show that $L\{f(t)\} = (E/s) \tanh(as/4)$. (05 Marks)
- (c) Employ Laplace transform to solve $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 4y = 2e^{-x}$, $y(0) = 1 = y'(0)$. (05 Marks)

OR

2. (a) Find (i) $L^{-1}\left\{\frac{3s+2}{s^2-s-2}\right\}$ (ii) $L^{-1}\left\{(s+5)/(s^2-6s+13)\right\}$ (iii) $L^{-1}[\cot^{-1}\{s/a\}]$ (10 Marks)
- (b) Express $f(t) = \begin{cases} 1, & 0 \leq t \leq 1 \\ t, & t > 1 \end{cases}$ in terms Heaviside's unit step function and hence find its Laplace transform. (05 Marks)
- (c) Find the inverse Laplace transform of $\frac{s^2}{(s^2+a^2)(s^2+b^2)}$, using convolution theorem. (05 Marks)

Module-2

3. (a) Find the Fourier series expansion of $f(x) = \frac{\pi^2}{12} - \frac{x^2}{4}$ in $-\pi \leq x \leq \pi$. Hence deduce that $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \frac{\pi^2}{12}$. (07 Marks)
- (b) Find the half-range cosine series of $f(x) = (x+1)^2$ the interval $0 \leq x \leq 1$. (06 Marks)
- (c) Obtain the Fourier series of $f(x) = \begin{cases} l-x, & \text{for } 0 \leq x \leq l \\ 0, & \text{for } l \leq x \leq 2l \end{cases}$ Hence deduce that $\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}$. (07 Marks)

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OR

4. (a) The displacement y (in cms) of a machine part occurs due to the rotation of x radians is given below:

Rotation x (in radians)	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
Displacement y (in cms)	1.0	1.4	1.9	1.7	1.5	1.2	1.0

Expand y in terms of Fourier series up to second harmonics.

(07 Marks)

- (b) Find the half-range sine series of e^x the interval $0 \leq x \leq 1$.

(06 Marks)

- (c) Find the Fourier series expansion of $f(x) = |x|$ in $-\pi \leq x \leq \pi$. Hence deduce that

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots = \frac{\pi^2}{8}. \quad (07 \text{ Marks})$$

Module-3

5. (a) If $f(x) = \begin{cases} 1-x^2, & \text{for } |x| \leq 1 \\ 0 & \text{for } |x| > 1 \end{cases}$, find the infinite Fourier transform of $f(x)$ and hence evaluate

$$\int_0^\infty \frac{x \cos x - \sin x}{x^3} \cos \frac{x}{2} dx \quad (07 \text{ Marks})$$

- (b) Find the Fourier cosine transform of $f(x) = e^{-2x} + 4e^{-3x}$ (06 Marks)

- (c) Solve: $u_{n+2} - 3u_{n+1} + 2u_n = 2^n$, given $u_0 = 0, u_1 = 1$ by using z-transforms. (07 Marks)

OR

6. (a) Find the Fourier sine transform of $e^{-|x|}$. Hence show that $\int_0^\infty \frac{x \sin mx}{1+x^2} dx = \frac{\pi e^{-m}}{2}, m > 0$. (07 Marks)

- (b) Find the z-transform of $\cos[n\pi/2 + \pi/4]$ (06 Marks)

- (c) Find the inverse z-transform of $\frac{2z^2 + 3z}{(z+2)(z-4)}$ (07 Marks)

Module-4

7. (a) Solve $\frac{dy}{dx} = e^x - y, y(0) = 1$ using Taylor's series method considering up to fourth degree terms and, find the value of $y(0.1)$. (07 Marks)

- (b) Use Runge - Kutta method of fourth order to solve $(x+y)\frac{dy}{dx} = 1, y(0.4) = 1$, to find $y(0.5)$.
(Take $h = 0.1$). (06 Marks)

- (c) Given that $\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}$ and $y(1) = 1, y(1.1) = 0.9960, y(1.2) = 0.9860, \& y(1.3) = 0.9720$
find $y(1.4)$, using Adam-Bashforth predictor-corrector method.

(07 Marks)

OR

8. (a) Solve the differential equation $\frac{dy}{dx} = x\sqrt{y}$ under the initial condition $y(1) = 1$, by using modified Euler's method at the point $x = 1.4$. Perform three iterations at each step, taking $h = 0.2$.
- (b) Use fourth order Runge - Kutta method, to find $y(0.1)$ with $h = 0.1$, given

$$\frac{dy}{dx} + y + xy^2 = 0, y(0) = 1,$$

(06 Marks)

- (c) Apply Milne's predictor-corrector formulae to compute $y(0.3)$ given, $\frac{dy}{dx} = x + y^2$ with

(07 Marks)

x	0.0	0.1	0.2	0.3
y	1.0000	1.1000	1.2310	1.4020

Module-5

9. (a) Solve $\frac{d^2y}{dx^2} - x^2 \frac{dy}{dx} - 2xy = 1$, for $x = 0.1$, correct to four decimal places, using initial conditions $y(0) = 1, y'(0) = 0$, using Runge - Kutta method,
- (b) Find the extremal of the functional $\int_0^1 (y'^2 - y^2 - y) e^{2x} dx$, that passes through the points $(0,0)$ and $(1,1/e)$.
- (c) A heavy cable hangs freely under gravity at two fixed points. Show that the shape of the cable is catenary.

(07 Marks)

(06 Marks)

(07 Marks)

OR

10. (a) Apply Milne's predictor-corrector method to compute $y(0.4)$ given the differential equation $\frac{d^2y}{dx^2} = 1 + \frac{dy}{dx}$ and the following table of initial values:

(07 Marks)

x	0	0.1	0.2	0.3
y	1	1.1103	1.2427	1.3990
y'	1	1.2103	1.4427	1.6990

- (b) Derive Euler's equation in the standard form viz., $\frac{\partial f}{\partial y} - \frac{d}{dx} \left[\frac{\partial f}{\partial y'} \right] = 0$

(06 Marks)

- (c) Find the extremal for the functional $\int_0^{\pi/2} (y^2 - y'^2 - 2y \sin x) dx ; y(0) = 0, y(\pi/2) = 1$.

(07 Marks)

Model Question Paper-2 with effect from 2019-20 (CBCS Scheme)

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18MAT31

Third Semester B.E.Degree Examination Transform Calculus, Fourier Series and Numerical Techniques (Common to all Programmes)

Time: 3 Hrs

Max.Marks: 100

Note: Answer any FIVE full questions, choosing at least ONE question from each module.

Module-1

1. (a) Find the Laplace transform of : (i) $3' + (4t + 5)^3$ (ii) $te^{-4t} \sin 3t$ (iii) $(\cos at - \sin bt)/t$ (10 Marks)
- (b) The triangular wave function $f(t)$ with period "2a" is defined by $f(t) = \begin{cases} t, & 0 \leq t < a \\ 2a-t, & a \leq t < 2a \end{cases}$
Show that $L\{f(t)\} = (1/s^2) \tanh(as/2)$ (05 Marks)
- (c) Using Laplace transform method, solve $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 2y = 5 \sin t$, $y(0) = 0 = y'(0)$. (05 Marks)

OR

2. (a) Find the inverse Laplace transform of (i) $\left\{ \frac{1}{s(s+1)} \right\}$ (ii) $\{(s+1)/(s^2 + 6s + 9)\}$
(iii) $[\log\{(s+a)/(s+b)\}]$ (10 Marks)
- (b) Express $f(t) = \begin{cases} \sin t, & 0 < t \leq \pi/2 \\ \cos t, & t > \pi/2 \end{cases}$ in terms Heaviside's unit step function and hence find its Laplace transform. (05 Marks)
- (c) Find the Laplace transform of $\frac{4}{(s^2 + 2s + 5)^2}$, using convolution theorem. (05 Marks)

Module-2

3. (a) An alternating current $I(x)$ after passing through a rectifier has the form $I(x) = \begin{cases} I_0 \sin x, & \text{for } 0 \leq x < \pi \\ 0, & \text{for } \pi < x \leq 2\pi, \end{cases}$
where I_0 is the maximum current and the period is 2π . Express $I(x)$ as a Fourier series. (07 Marks)
- (b) Find the half-range sine series of $f(x) = \frac{\sinh ax}{\sinh a\pi}$ the interval $(0, \pi)$. (06 Marks)
- (c) Find the Fourier series expansion of $f(x) = x(1-x)(2-x)$ the interval $0 \leq x \leq 2$. Hence deduce the sum of the series that $\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \dots$ (07 Marks)

OR

4. (a) In an electrical research laboratory, scientists have designed a generator which can generate the following currents at different time instant t , in the period T :

Time t (in sec)	0	$T/6$	$T/3$	$T/2$	$2T/3$	$5T/6$	T
$f(x)$	1.98	1.30	1.05	1.30	-0.88	-0.25	1.98

Determine the direct current part and amplitude of the first harmonic from the above data. (07 Marks)

- (b) Find the half-range sine series of $f(x) = \begin{cases} \sin x & \text{for } 0 \leq x < \pi/4 \\ \cos x, & \text{for } \pi/4 < x \leq \pi/2 \end{cases}$ (06 Marks)
- (c) Obtain the Fourier series of $f(x) = x(2\pi - x)$ valid in the interval $(0, 2\pi)$. (07 Marks)

Module-3

5. (a) If $f(x) = \begin{cases} a^2 - x^2, & \text{for } |x| \leq a \\ 0 & \text{for } |x| > a \end{cases}$, find the infinite Fourier transform of $f(x)$ and hence evaluate

$$\int_0^\infty \frac{\sin x - x \cos x}{x^3} dx \quad (07 \text{ Marks})$$

- (b) Find the Fourier sine transform of $f(x) = \begin{cases} x, & \text{if } 0 < x < 1 \\ 2-x, & \text{if } 1 < x < 2 \\ 0, & \text{if } x > 2 \end{cases}$ (06 Marks)

- (c) Solve $u_{n+2} + 6u_{n+1} + 9u_n = 2^n$, $u_0 = 0 = u_1$, by using z-transforms. (07 Marks)

OR

6. (a) If $f(x) = \begin{cases} 1, & \text{for } |x| \leq a \\ 0 & \text{for } |x| > a \end{cases}$, find the infinite Fourier transform of $f(x)$ and hence evaluate

$$\int_0^\infty \frac{\sin x}{x} dx \quad (07 \text{ Marks})$$

- (b) Find the z-transform of $2n + \sin(n\pi/4) + 1$ (06 Marks)

- (c) Find the inverse z-transform of $18z^2 / [(2z-1)(4z+1)]$ (07 Marks)

Module-4

7. (a) Solve $\frac{dy}{dx} = x^3 + y$, $y(1) = 1$ using Taylor's series method considering up to fourth degree terms and, find the $y(1.1)$. (07 Marks)

(b) Use Runge - Kutta method of fourth order to solve $\frac{dy}{dx} = 3x + \frac{y}{2}$, $y(0) = 1$, to find $y(0.2)$.

(Take $h = 0.2$).

(06 Marks)

(c) Given $\frac{dy}{dx} = x^2(1+y)$ and $y(1) = 1$, $y(1.1) = 1.2330$, $y(1.2) = 1.5480$, & $y(1.3) = 1.9790$

find $y(1.4)$, using Adam-Bashforth predictor-corrector method.

(07 Marks)

OR

8. (a) Use modified Euler's method to compute $y(0.2)$, given $\frac{dy}{dx} - xy^2 = 0$ under the initial condition $y(0) = 2$. Perform three iterations at each step, taking $h = 0.1$.

(07 Marks)

(b) Use fourth order Runge - Kutta method, to find $y(0.2)$ with $h = 0.2$, given

$$\frac{dy}{dx} = \sqrt{x+y}, y(0) = 1, \quad (06 \text{ Marks})$$

(c) Apply Milne's predictor-corrector formulae to compute $y(2.0)$ given $\frac{dy}{dx} = \frac{1}{2}(x+y)$ with

(07 Marks)

x	0.0	0.5	1.0	1.5
y	2.0000	2.6360	3.5950	4.9680

Module-5

9. (a) Using Runge - Kutta method , solve $\frac{d^2y}{dx^2} = x\left(\frac{dy}{dx}\right)^2 - y^2$, for $x = 0.2$, correct to four decimal places, using initial conditions $y(0)=1$, $y'(0)=0$.

(07 Marks)

(b) Derive Euler's equation in the standard form viz., $\frac{\partial f}{\partial y} - \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right] = 0$

(06 Marks)

(c) Find the extremal of the functional $\int_0^\pi (y'^2 - y^2 + 4y \cos x) dx$; $y(0) = 0 = y(\pi)$

(07 Marks)

OR

10. (a) Given the differential equation $2\frac{d^2y}{dx^2} - \frac{dy}{dx} - 4x = 0$ and the following table of initial values:

x	1	1.1	1.2	1.3
y	2	2.2156	2.4649	2.7514
y'	2	2.3178	2.6725	2.0657

compute $y(1.4)$ by applying Milne's predictor-corrector method.

(07 Marks)

(b) On what curves can the functional $\int_0^{\pi/2} (y'^2 - y^2 + 2xy) dx$; $y(0) = 0$, $y(\pi/2) = 0$ be extremized?

(06 Marks)

(c) Prove that geodesics of a plane surface are straight lines.

(07 Marks)

||Jai Sri Gurudev||
S. J. C. INSTITUTE OF TECHNOLOGY, CHICKBALLAPUR
Department of Mathematics
UNIVERSITY QUESTION PAPER 1 & 2 SOLUTIONS
TCFS&NT (18MAT31)

Prepared By:

Purushotham.P
Assistant Professor
SJC Institute of Technology
Email id: ppurushothamp48@gmail.com
Phone: 8197481658

Module -1

1.

a) Find the Laplace transform of

$$i) \sqrt{e^{4(t+3)}} + e^{-2t} \sin 3t$$

$$\text{let } F(t) = \sqrt{e^{4(t+3)}} + e^{-2t} \sin 3t$$

$$= [e^{4(t+3)}]^{\frac{1}{2}} + e^{-2t} \sin 3t$$

$$= e^{2(t+3)} + e^{-2t} \sin 3t$$

$$= e^{2t+6} + e^{-2t} \sin 3t$$

$$F(t) = e^{2t} \cdot e^6 + e^{-2t} \sin 3t$$

$$\Rightarrow L[f(t)] = e^6 L[e^{2t}] + L[e^{-2t} \sin 3t]$$

$$= e^6 \frac{1}{s-2} + \left[\frac{3}{s^2+9} \right]_{s \rightarrow s+2}$$

$$= \frac{e^6}{s-2} + \frac{3}{(s+2)^2+9}$$

$$= \frac{e^6}{s-2} + \frac{3}{s^2+2s+4+9}$$

$$L[f(t)] = \frac{e^6}{s-2} + \frac{3}{s^2+2s+13}$$

Q) $t e^{-3t} \sin 4t$

$$L[\sin 4t] = \frac{4}{s^2+16}$$

$$\begin{aligned} L[t \sin 4t] &= (-1)^1 \frac{d}{ds} \left[\frac{4}{s^2+16} \right] \\ &= - \left[\frac{(s^2+16)(0) - 4(2s)}{(s^2+16)^2} \right] \\ &= - \left[\frac{-8s}{(s^2+16)^2} \right] \end{aligned}$$

$$L[t \sin 4t] = \frac{8s}{(s^2+16)^2}$$

$$L[e^{-3t} t \sin 4t] = \left[\frac{8s}{(s^2+16)^2} \right]_{s \rightarrow s+3}$$

$$L[e^{-3t} t \sin 4t] = \frac{8(s+3)}{(s+3)^2+16} = \frac{8(s+3)}{(s^2+9+6s+16)^2}$$

$$L[e^{-3t} t \sin 4t] = \frac{8(s+3)}{(s^2 + cs + 25)^2}$$

ie., $(1 - \cos t) / t$

$$F(t) = 1 - \cos t$$

$$\Rightarrow L[F(t)] = L[1 - \cos t]$$

$$= L[1] - L[\cos t]$$

$$f(s) = \frac{1}{s} - \frac{s}{s^2 + 1}$$

WKT

$$L\left[\frac{F(t)}{t}\right] = \int_s^\infty f(s) ds$$

$$= \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2 + 1} \right) ds$$

$$= \int_s^\infty \frac{1}{s} ds - \int_s^\infty \frac{s}{s^2 + 1} ds$$

$$= [\log s]_s^\infty - \frac{1}{2} \int_s^\infty \frac{2s}{s^2 + 1} ds$$

$$= \left[[\log s] - \frac{1}{2} [\log(s^2 + 1)] \right]_s^\infty$$

$$= \left[\log s - \log(s^2 + 1)^{\frac{1}{2}} \right]_s^\infty$$

$$= \log \left[\frac{s}{(s^2 + 1)^{\frac{1}{2}}} \right]_s^\infty$$

$$= \log \left[\frac{s}{s(s + \frac{1}{s})^{\frac{1}{2}}} \right]_s^\infty$$

$$= \log \left[\frac{1}{(s + \frac{1}{s})^{\frac{1}{2}}} \right]_s^\infty$$

$$= \log 1 - \log \left[\frac{1}{(s + \frac{1}{s})^{\frac{1}{2}}} \right] .$$

$$= \left[0 - \log \frac{s^{\frac{1}{2}}}{(s^2+1)^{\frac{1}{2}}} \right] = -\log \left| \frac{s^{\frac{1}{2}}}{(s^2+1)^{\frac{1}{2}}} \right|$$

$$\mathcal{L} \left[\frac{1-\cos t}{t} \right] = -\log \left| \frac{\sqrt{s}}{\sqrt{s^2+1}} \right|$$

1 b) The Square wave function $f(t)$ with period "a"
is defined by $f(t) = \begin{cases} E, & 0 \leq t < \frac{a}{2} \\ -E, & \frac{a}{2} \leq t < a \end{cases}$ Show that
 $\mathcal{L} \{ f(t) \} = \frac{E}{s} \tanh \left(\frac{as}{4} \right)$

Solⁿ :- $f(t+T) = f(t+a)$
 $\Rightarrow T=a$

WKT

$$\begin{aligned} \mathcal{L} \{ f(t) \} &= \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t) dt \\ &= \frac{1}{1-e^{-sa}} \int_0^a e^{-st} f(t) dt \\ &= \frac{1}{1-e^{-as}} \left[\int_0^{\frac{a}{2}} e^{-st} f(t) dt + \int_{\frac{a}{2}}^a e^{-st} f(t) dt \right] \\ &= \frac{1}{1-e^{-as}} \left[\int_0^{\frac{a}{2}} e^{-st} E dt + \int_{\frac{a}{2}}^a e^{-st} (-E) dt \right] \\ &= \frac{E}{1-e^{-as}} \left[\int_0^{\frac{a}{2}} e^{-st} dt - \int_{\frac{a}{2}}^a e^{-st} dt \right] \\ &= \frac{E}{1-e^{-as}} \left\{ \left[\frac{e^{-st}}{-s} \right]_0^{\frac{a}{2}} - \left[\frac{e^{-st}}{-s} \right]_{\frac{a}{2}}^a \right\} \\ &= \frac{E}{1-e^{-as}} \left\{ -\frac{1}{s} [e^{-as/2} - e^0] + \frac{1}{s} [e^{-as} - e^{-as/2}] \right\} \end{aligned}$$

$$= \frac{E}{S} \left(\frac{1}{1-e^{-as}} \right) \left\{ -e^{-as/2} + 1 + e^{-as} - e^{-as/2} \right\}$$

$$= \frac{E}{S} \left(\frac{1}{1-e^{-as}} \right) \left\{ 1 - 2e^{-as/2} + e^{-as} \right\}$$

$$= \frac{E}{S} \left(\frac{1}{1-e^{-as}} \right) \left\{ 1^2 - 2(1)e^{-as/2} + (e^{-as/2})^2 \right\}$$

$$= \frac{E}{S} \left(\frac{1}{1-e^{-as}} \right) (1 - e^{-as/2})^2$$

$$= \frac{E}{S} \frac{(1 - e^{-as/2})^2}{(1 - e^{-as})}$$

$$= \frac{E}{S} \frac{(1 - e^{-as/2})(1 - e^{-as/2})}{[1^2 - (e^{-as/2})^2]}$$

$$= \frac{E}{S} \frac{(1 - e^{-as/2})(1 - e^{-as/2})}{(1 - e^{-as/2})(1 + e^{-as/2})}$$

$$= \frac{E}{S} \frac{(1 - e^{-as/2})}{(1 + e^{-as/2})} \times 4 e^{as/4} \text{ on } N^r \text{ & } D^r$$

$$= \frac{E}{S} \frac{(1 - e^{-as/2}) e^{as/4}}{(1 + e^{-as/2}) e^{as/4}}$$

$$= \frac{E}{S} \left[\frac{e^{as/4} - e^{-as/4}}{e^{as/4} + e^{-as/4}} \right]$$

$$\tanh \theta = \frac{e^\theta - e^{-\theta}}{e^\theta + e^{-\theta}}$$

$$L[f(t)] = \underline{\underline{\frac{E}{S} \tanh \left(\frac{as}{4} \right)}}$$

1c) Employ Laplace transform to solve

$$\frac{d^2y}{dx^2} - 3 \frac{dy}{dx} - 4y = 2e^{-x}, y(0) = 1 = y'(0)$$

$$\begin{aligned}
 \text{Soln: } & \frac{d^2y}{dx^2} - 3 \frac{dy}{dx} - 4y = 2e^{-x} \\
 & y''(x) - 3y'(x) - 4y(x) = 2e^{-x} \\
 \Rightarrow & L[y''(x)] - 3L[y'(x)] - 4L[y(x)] = L[e^{-x}] \\
 \Rightarrow & [s^2 \bar{y}(s) - sy(0) - y'(0)] - 3[s\bar{y}(s) - y(0)] - 4\bar{y}(s) = 2 \frac{1}{s+1} \\
 \Rightarrow & [s^2 \bar{y}(s) - s - 1] - 3[s\bar{y}(s) - 1] - 4\bar{y}(s) = \frac{2}{s+1} \\
 \Rightarrow & s^2 \bar{y}(s) - 3s\bar{y}(s) - 4\bar{y}(s) - s - 1 + 3 = \frac{2}{s+1} \\
 \Rightarrow & (s^2 - 3s - 4)\bar{y}(s) - (s - 2) = \frac{2}{s+1} \\
 \Rightarrow & (s^2 - 3s - 4)\bar{y}(s) = \frac{2}{s+1} + (s - 2) \\
 \Rightarrow & (s-4)(s+1)\bar{y}(s) = \frac{2 + (s-2)(s+1)}{s+1}
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow (s-4)(s+1)\bar{y}(s) &= \frac{s^2 - s}{s+1} \\
 \Rightarrow \bar{y}(s) &= \frac{s(s-1)}{(s+1)^2(s-4)} \rightarrow ①
 \end{aligned}$$

$$\frac{s(s-1)}{(s+1)^2(s-4)} = \frac{A}{(s+1)} + \frac{B}{(s+1)^2} + \frac{C}{s-4} \rightarrow ②$$

$$s(s-1) = A(s+1)(s-4) + B(s-4) + C(s+1)^2$$

$$\text{when } s=4$$

$$\text{when } s=-1$$

$$4(4-1) = C(4+1)^2 \quad -1(-1-1) = B(-1-4)$$

$$12 = C(25)$$

$$-1(-2) = B(-5)$$

$$C = \frac{12}{25}$$

$$B = -\frac{2}{5}$$

Compare coefficients of s^2

$$1 = A + C \Rightarrow A = 1 - C = 1 - \frac{12}{25} = \frac{25 - 12}{25} = \frac{13}{25}$$

$$\textcircled{2} \Rightarrow \frac{s(s-1)}{(s+1)^2(s-4)} = \frac{13/25}{s+1} - \frac{2/5}{(s+1)^2} + \frac{12/25}{s-4}$$

$$\bar{y}(s) = \frac{13}{25} \frac{1}{s+1} - \frac{2}{5} \frac{1}{(s+1)^2} + \frac{12}{25} \frac{1}{s-4}$$

$$\begin{aligned} L^{-1}[\bar{y}(s)] &= \frac{13}{25} L^{-1}\left[\frac{1}{s+1}\right] - \frac{2}{5} L^{-1}\left[\frac{1}{(s+1)^2}\right] + \frac{12}{25} L^{-1}\left[\frac{1}{s-4}\right] \\ &= \frac{13}{25} e^{-t} - \frac{2}{5} e^{-t} L^{-1}\left[\frac{1}{s^2}\right] + \frac{12}{25} e^{4t} \end{aligned}$$

$$L^{-1}[\bar{y}(s)] = \frac{13}{25} e^{-t} - \frac{2}{5} t e^{-t} + \frac{12}{25} e^{4t}$$

$$y(t) = \underline{\frac{13}{25} e^{-t} - \frac{2}{5} t e^{-t} + \frac{12}{25} e^{4t}}$$

Now find

$$\text{if } L^{-1}\left\{\frac{3s+2}{s^2-s-2}\right\}$$

$$\text{Soln: let } f(s) = \frac{3s+2}{s^2-s-2} = \frac{3s+2}{(s-2)(s+1)}$$

$$\frac{3s+2}{(s-2)(s+1)} = \frac{A}{s-2} + \frac{B}{s+1} \rightarrow \textcircled{1}$$

$$3s+2 = A(s+1) + B(s-2) \rightarrow \textcircled{2}$$

$$\text{when } s=2$$

$$\text{when } s=-1$$

$$\textcircled{2} \Rightarrow 3(2)+2 = A(2+1) \quad \textcircled{2} \Rightarrow 3(-1)+2 = B(-1-2)$$

$$8 = 3A \Rightarrow A = \frac{8}{3}$$

$$-1 = B(-3)$$

$$B = \frac{1}{3}$$

$$\therefore (1) \Rightarrow \frac{3s+2}{(s-2)(s+1)} = \frac{8/3}{s-2} + \frac{1/3}{s+1}$$

$$L^{-1}\left[\frac{3s+2}{(s-2)(s+1)}\right] = \frac{8}{3} L^{-1}\left[\frac{1}{s-2}\right] + \frac{1}{3} L^{-1}\left[\frac{1}{s+1}\right]$$

$$L^{-1}[f(s)] = \frac{8}{3} e^{2t} + \frac{1}{3} e^{-t}$$

$$f(t) = \frac{1}{3}(8e^{2t} + e^{-t})$$

or $L^{-1}\left\{\frac{(s+5)}{(s^2 - 6s + 13)}\right\}$

$$\begin{aligned} f(s) &= \frac{s+5}{s^2 - 6s + 13} \\ &= \frac{s+5}{\underbrace{s^2 - 2(s)(3) + 3^2 - 3^2 + 13}_{(s-3)^2 + 4}} \\ &= \frac{s+5}{(s-3)^2 + 4} \\ &= \frac{(s-3)+3+5}{(s-3)^2 + 4} \end{aligned}$$

$$f(s) = \frac{(s-3)+8}{(s-3)^2 + 4}$$

$$L^{-1}[f(s)] = L^{-1}\left[\frac{(s-3)+8}{(s-3)^2 + 4}\right]$$

$$= e^{3t} L^{-1}\left[\frac{s+8}{s^2+4}\right]$$

$$= e^{3t} L^{-1}\left[\frac{s}{s^2+4} + \frac{8}{s^2+4}\right]$$

$$L^{-1}[f(s)] = e^{3t} \left\{ L^{-1}\left[\frac{s}{s^2+a^2}\right] + 8 L^{-1}\left[\frac{1}{s^2+a^2}\right] \right\}$$

$$= e^{3t} (\cos 2t + 8 \cdot \frac{1}{2} \sin 2t)$$

$$L^{-1}\left\{\frac{s+5}{s^2-6s+13}\right\} = e^{3t} (\cos 2t + 4 \sin 2t)$$

$\Rightarrow L^{-1} [\cot^{-1}(s/a)]$

$$f(s) = \cot^{-1}(s/a)$$

diff w.r.t s

$$f'(s) = \frac{-1}{1+(s/a)^2} \cdot \frac{1}{a}$$

multiply with -1

$$-f'(s) = \frac{a}{a^2+s^2}$$

Taking inverse laplace on both sides

$$L^{-1}[-f'(s)] = L^{-1}\left[\frac{a}{s^2+a^2}\right]$$

$$L[t f(t)] = -f'(s)$$

$$t f(t) = \sin at$$

$$L^{-1}[-f'(s)] = t f(t)$$

$$\therefore f(t) = \boxed{\frac{\sin at}{t}}$$

Qb) Express $f(t) = \begin{cases} 1, & 0 \leq t \leq 1 \\ t, & t > 1 \end{cases}$ in terms heavyside's unit step function and hence find its Laplace transform.

Soln :- $f(t) = 1 + (t-1) u(t-1)$

$$\Rightarrow L[f(t)] = L[1] + L[(t-1) u(t-1)] \rightarrow ①$$

$$\text{let } g(t-1) = t-1 \quad L[1] = \frac{1}{s}$$

$$g(t) = t$$

$$L[g(t)] = L[t]$$

$$\bar{g}(s) = \frac{1}{s^2}$$

$$\therefore L[g(t-1)u(t-1)] = e^{-s} \bar{g}(s)$$

$$L[(t-1)u(t-1)] = e^{-s} \frac{1}{s^2}$$

$$\therefore \textcircled{1} \Rightarrow \bar{f}(s) = \frac{1}{s} + \underline{\underline{\frac{e^{-s}}{s^2}}}$$

Q) Find the inverse laplace transform of $\frac{s^2}{(s^2+a^2)(s^2+b^2)}$
using convolution theorem.

$$\text{Soln} :- \bar{f}(s) \cdot \bar{g}(s) = \frac{s^2}{(s^2+a^2)(s^2+b^2)} = \frac{s}{(s^2+a^2)} \cdot \frac{s}{(s^2+b^2)}$$

$$\bar{f}(s) = \frac{s}{s^2+a^2}, \quad \bar{g}(s) = \frac{s}{s^2+b^2}$$

$$L^{-1}[\bar{f}(s)] = L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at = f(t)$$

$$L^{-1}[\bar{g}(s)] = L^{-1}\left[\frac{s}{s^2+b^2}\right] = \cos bt = g(t)$$

$$\therefore L^{-1}[\bar{f}(s) \cdot \bar{g}(s)] = \int_{u=0}^t f(u) g(t-u) du$$

$$= \int_{u=0}^t \cos au \cdot \cos(bt-bu) du$$

$$= \frac{1}{2} \int_{u=0}^t 2 \cos au \cdot \cos(bt-bu) du$$

$$= \frac{1}{2} \int_{u=0}^t \cos(au+bt-bu) + \cos(au-bt+bu) du$$

$$\begin{aligned}
&= \frac{1}{2} \int_{u=0}^t \left\{ \cos[(a-b)u+bt] + \cos[(a+b)u-bt] \right\} du \\
&= \frac{1}{2} \left\{ \frac{\sin[(a-b)u+bt]}{a-b} + \frac{\sin[(a+b)u-bt]}{a+b} \right\} \Big|_0^t \\
&= \frac{1}{2} \left\{ \left[\frac{\sin at}{a-b} + \frac{\sin at}{a+b} \right] - \left[\frac{\sin bt}{a-b} - \frac{\sin bt}{a+b} \right] \right\} \\
&= \frac{1}{2} \left\{ \left(\frac{1}{a-b} + \frac{1}{a+b} \right) \sin at - \left(\frac{1}{a-b} - \frac{1}{a+b} \right) \sin bt \right\} \\
&= \frac{1}{2} \left[\frac{2a}{a^2-b^2} \sin at - \frac{2b}{a^2-b^2} \sin bt \right] \\
&= \frac{a}{a^2-b^2} \sin at - \frac{b}{a^2-b^2} \sin bt \\
&= \frac{1}{a^2-b^2} [a \sin at - b \sin bt], \quad a \neq b
\end{aligned}$$

Model question paper - 2

- 1 a) Find the Laplace transform of
 if $3^t + (4t+5)^3$

$$f(t) = 3^t + (4t+5)^3$$

$$f(t) = e^{\log(3^t)} + (4t)^3 + 5^3 + 3(4t)^2 5 + 3(4t)(5)^2$$

$$f(t) = e^{(\log 3)t} + 64t^3 + 125 + 240t^2 + 300t$$

$$f(t) = e^{(\log 3)t} + 64t^3 + 240t^2 + 300t + 125$$

$$L[f(t)] = L[e^{(\log 3)t}] + 64 L[t^3] + 240 L[t^2] + 300 L[t] + 125 L[1]$$

$$L[f(t)] = \frac{1}{s-\log 3} + 64 \cdot \frac{6}{s^4} + 240 \cdot \frac{2}{s^3} + 300 \cdot \frac{1}{s^2} + 125 \cdot \frac{1}{s}$$

$$f(s) = \frac{1}{s-1} + \frac{384}{s^4} + \frac{480}{s^3} + \frac{300}{s^2} + \frac{125}{s}$$

iii) $t e^{-4t} \sin 3t$

$$\mathcal{L}[\sin 3t] = \frac{3}{s^2 + 9}$$

$$\mathcal{L}[t \sin 3t] = (-1)^1 \frac{d}{ds} \left[\frac{3}{s^2 + 9} \right]$$

$$= - \left[\frac{(s^2 + 9)(0) - 3(2s)}{(s^2 + 9)^2} \right]$$

$$= - \left[\frac{-6s}{(s^2 + 9)^2} \right]$$

$$\mathcal{L}[t \sin 3t] = \frac{6s}{(s^2 + 9)^2}$$

$$\mathcal{L}[e^{-4t} t \sin 3t] = \left[\frac{6s}{(s^2 + 9)^2} \right]_{s \rightarrow s+4}$$

$$= \frac{6(s+4)}{(s+4)^2 + 9^2}$$

$$\mathcal{L}[e^{-4t} t \sin 3t] = \frac{6(s+4)}{(s^2 + 8s + 25)^2}$$

iii) $(\cos at - \cos bt)/t$

$$F(t) = \cos at - \cos bt$$

$$\mathcal{L}[F(t)] = \mathcal{L}[\cos at] - \mathcal{L}[\cos bt]$$

$$f(s) = \frac{s}{s^2 + a^2} - \frac{s}{s^2 + b^2}$$

$$\text{wkt } \mathcal{L}\left[\frac{F(t)}{t}\right] = \int_s^\infty f(s) ds$$

$$\begin{aligned}
 &= \int_s^\infty \left[\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2} \right] ds \\
 &= \frac{1}{2} \int_s^\infty \left[\frac{2s}{s^2+a^2} - \frac{2s}{s^2+b^2} \right] ds \\
 &= \frac{1}{2} \left[\log(s^2+a^2) - \log(s^2+b^2) \right]_s^\infty \\
 &= \frac{1}{2} \log \left[\frac{s^2+a^2}{s^2+b^2} \right]_s^\infty \\
 &= \frac{1}{2} \log \left[\frac{s^2(1+\frac{a^2}{s^2})}{s^2(1+\frac{b^2}{s^2})} \right]_s^\infty \\
 &= \frac{1}{2} \log \left[\frac{1+a^2/s^2}{1+b^2/s^2} \right]_s^\infty \\
 &= \frac{1}{2} \left[\log \left(\frac{1+0}{1+0} \right) - \log \left(\frac{1+a^2/s^2}{1+b^2/s^2} \right) \right] \\
 &= -\frac{1}{2} \log \left[\frac{s^2+a^2}{s^2+b^2} \right] \\
 &= \log \left(\frac{s^2+a^2}{s^2+b^2} \right)^{-\frac{1}{2}}
 \end{aligned}$$

$$L \left[\frac{\cos at - \cos bt}{t} \right] = \log \sqrt{\frac{s^2+b^2}{s^2+a^2}}$$

1b) The triangular wave function $f(t)$ with period "2a" is defined by $f(t) = \begin{cases} t, & 0 \leq t < a \\ 2a-t, & a \leq t < 2a \end{cases}$ Show that

$$L \{ f(t) \} = \frac{1}{s^2} \tanh \left(\frac{as}{2} \right)$$

$$\text{soln: } f(t+2a) = f(t) \Rightarrow T=2a$$

wkT

$$L \{ f(t) \} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t) dt$$

$$L \{ f(t) \} = \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} f(t) dt$$

$$= \frac{1}{1-e^{-2as}} \left\{ \int_0^a e^{-st} f(t) dt + \int_a^{2a} e^{-st} f(t) dt \right\}$$

$$= \frac{1}{1-e^{-2as}} \left\{ \int_0^a t e^{-st} dt + \int_a^{2a} (2a-t) e^{-st} dt \right\}$$

$$\int_0^a t e^{-st} dt = t \int_0^a e^{-st} dt - \int_0^a [t \int_0^a e^{-st} dt] dt$$

$$= -\frac{1}{s} [t e^{-st}]_0^a + \frac{1}{s} \int_0^a e^{-st} dt$$

$$= -\frac{1}{s} [ae^{-as} - 0] - \frac{1}{s^2} [e^{-as} - e^0]$$

$$= -\frac{1}{s} (ae^{-as}) - \frac{1}{s^2} (e^{-as} - 1)$$

$$\int_0^a t e^{-st} dt = \frac{1}{s^2} - \frac{1}{s^2} e^{-as} - \frac{a}{s} e^{-as}$$

$$\int_a^{2a} (2a-t) e^{-st} dt = (2a-t) \int_a^{2a} e^{-st} dt - \int_a^{2a} [(2a-t) \int_a^{2a} e^{-st} dt] dt$$

$$= -\frac{1}{s} [(2a-t) e^{-st}]_a^{2a} - \frac{1}{s} \int_a^{2a} e^{-st} dt$$

$$= -\frac{1}{s} [0 - ae^{-as}] + \frac{1}{s^2} [e^{-st}]_a^{2a}$$

$$= \frac{a}{s} e^{-as} + \frac{1}{s^2} [e^{-2as} - e^{-as}]$$

$$\int_0^{2a} (2a-t) e^{-st} dt = \frac{1}{s^2} e^{-2as} + \frac{a}{s} e^{-as} - \frac{1}{s^2} e^{-as}$$

$$L[F(t)] = \frac{1}{1-e^{-2as}} \left[\frac{1}{s^2} - \frac{1}{s^2} (e^{-as} - \frac{a}{s} e^{-as} + \frac{1}{s^2} e^{-2as} + \frac{a}{s} e^{-as} - \frac{1}{s^2} e^{-as}) \right]$$

$$= \frac{1}{1-e^{-2as}} \left[\frac{1}{s^2} - \frac{2}{s^2} e^{-as} + \frac{1}{s^2} e^{-2as} \right]$$

$$= \frac{1}{1-e^{-2as}} \frac{1}{s^2} \left[1 - 2e^{-as} + e^{-2as} \right]$$

$$= \frac{1}{s^2} \frac{1}{1-e^{-2as}} \left[1^2 - 2(1) e^{-as} + (e^{-as})^2 \right]$$

$$\begin{aligned}
 &= \frac{1}{s^2} \frac{1}{1-e^{-as}} [1-e^{-as}]^2 \\
 &= \frac{1}{s^2} \frac{(1-e^{-as})^2}{1^2-(e^{-as})^2} \\
 &= \frac{1}{s^2} \frac{(1-e^{-as})^2}{(1-e^{as})(1+e^{-as})} \\
 &= \frac{1}{s^2} \frac{(1-e^{-as})}{(1+e^{-as})} e^{as/2} \text{ on N.R & D.R} \\
 &= \frac{1}{s^2} \frac{(1-e^{-as})}{(1+e^{-as})} e^{as/2} \\
 &= \frac{1}{s^2} \left[\frac{e^{as/2} - e^{-as/2}}{e^{as/2} + e^{-as/2}} \right] \\
 L[f(t)] &= \underline{\frac{1}{s^2} \tanh \left[\frac{as}{2} \right]}
 \end{aligned}$$

ie) Using Laplace transform method, solve

$$\frac{d^2y}{dt^2} + 2 \frac{dy}{dt} + 2y = 5s \sin t, y(0) = 0 = y'(0).$$

Solⁿ

$$\frac{d^2y}{dt^2} + 2 \frac{dy}{dt} + 2y = 5s \sin t$$

$$y''(t) + 2y'(t) + 2y(t) = 5s \sin t$$

$$L[y''(t)] + 2L[y'(t)] + 2L[y(t)] = 5L[s \sin t]$$

$$[s^2 \bar{y}(s) - sy(0) - y'(0)] + 2[s\bar{y}(s) - y(0)] + 2\bar{y}(s) = 5 \frac{1}{s^2 + 1}$$

$$s^2 \bar{y}(s) + 2s \bar{y}(s) + 2\bar{y}(s) = \frac{5}{s^2 + 1}$$

$$(s^2 + 2s + 2) \bar{y}(s) = \frac{5}{s^2 + 1}$$

$$\bar{y}(s) = \frac{5}{(s^2 + 1)(s^2 + 2s + 2)}$$

$$\text{Let } \frac{5}{(s^2+1)(s^2+2s+2)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+2s+2}$$

$$5 = As+B(s^2+2s+2) + (Cs+D)(s^2+1)$$

$$5 = (A+C)s^3 + (2A+B+D)s^2 + (2A+2B+C)s + (2B+D)$$

Comparing the coefficients on both sides, we get

$$A+C=0, 2A+B+D=0, 2A+2B+C=0, 2B+D=5$$

$$A=-2, B=1, C=2, D=3$$

$$\therefore \frac{5}{(s^2+1)(s^2+2s+2)} = -\frac{2s+1}{s^2+1} + \frac{2s+3}{s^2+2s+2}$$

$$Y(t) = -2L^{-1}\left[\frac{s}{s^2+1}\right] + L^{-1}\left[\frac{1}{s^2+1}\right] + L^{-1}\left[\frac{2s+3}{s^2+2s+2}\right]$$

$$= -2\cos st + \sin nt + L^{-1}\left\{\frac{2(s+1)+1}{(s+1)^2+1}\right\}$$

$$= -2\cos st + \sin nt + e^{-t} L^{-1}\left\{\frac{2s+1}{s^2+1}\right\}$$

$$= -2\cos st + \sin nt + e^{-t} \left[2L^{-1}\left(\frac{s}{s^2+1}\right) + L^{-1}\left(\frac{1}{s^2+1}\right) \right]$$

$$Y(t) = -2\cos st + \sin nt + e^{-t} \underline{\underline{[2\cos t + \sin t]}}$$

2a) Find the inverse Laplace transform of

$$\therefore \left\{ \frac{1}{s(s+1)} \right\}$$

$$\text{Sol}^n: \text{let } f(s) = \frac{1}{s(s+1)}$$

$$\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} \rightarrow ①$$

$$1 = A(s+1) + Bs \rightarrow ②$$

$$\text{when } s=0$$

$$\text{when } s=-1$$

$$1 = A(0+1)$$

$$1 = B(-1)$$

$$\boxed{A=1}$$

$$\boxed{B=-1}$$

$$\therefore ① \Rightarrow \frac{1}{s(s+1)} = \frac{1}{s} + \frac{-1}{s+1}$$

$$L^{-1}\left[\frac{1}{s(s+1)}\right] = L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{1}{s+1}\right]$$

$$L^{-1}[f(s)] = 1 - e^{-t}$$

$$f(t) = \underline{\underline{1 - e^{-t}}}$$

$$\therefore \left\{ \frac{(s+1)}{(s^2+6s+9)} \right\}$$

$$\text{Sol}^n: \text{let } f(s) = \frac{s+1}{s^2+6s+9}$$

$$= \frac{s+1}{(s+3)^2}$$

$$= \frac{s+3-3+1}{(s+3)^2}$$

$$f(s) = \frac{(s+3)-2}{(s+3)^2}$$

$$L^{-1} \left[\frac{s+1}{s^2 + 6s + 9} \right] = L^{-1} \left[\frac{(s+3)-2}{(s+3)^2} \right]$$

$$L^{-1}[f(s)] = e^{-3t} L^{-1} \left[\frac{s-2}{s^2} \right]$$

$$= e^{-3t} \left\{ L^{-1} \left[\frac{1}{s} \right] - 2 L^{-1} \left[\frac{1}{s^2} \right] \right\}$$

$$f(t) = e^{-3t} \left[1 - \underline{\underline{2t}} \right]$$

ie } $\log \left[\frac{(s+a)}{(s+b)} \right]$

Solⁿ: let $f(s) = \log \left[\frac{s+a}{s+b} \right]$

$$f(s) = \log(s+a) - \log(s+b)$$

differentiate w.r.t s

$$f'(s) = \frac{1}{s+a} - \frac{1}{s+b}$$

\times^4 -ve on both sides

$$-f'(s) = - \left[\frac{1}{s+a} - \frac{1}{s+b} \right]$$

$$-f'(s) = \frac{1}{s+b} - \frac{1}{s+a}$$

$$L^{-1}[-f'(s)] = L^{-1} \left[\frac{1}{s+b} \right] - L^{-1} \left[\frac{1}{s+a} \right]$$

$$t f(t) = e^{-bt} - e^{-at}$$

$$f(t) = \frac{e^{-bt} - e^{-at}}{t}$$

2b) Express $f(t) = \begin{cases} \sin t, & 0 < t \leq \pi/2 \\ \cos t, & t > \pi/2 \end{cases}$ in terms of heaviside's

unit step function and hence find its Laplace transform.

Solⁿ: $f(t) = \sin t + [\cos t - \sin t] u(t - \pi/2)$

$$\mathcal{L}[f(t)] = \mathcal{L}[\sin t] + \mathcal{L}[\cos t - \sin t] u(t - \pi/2) \rightarrow ①$$

let $g(t - \pi/2) = \cos t - \sin t$

$$\begin{aligned} g(t) &= \cos(t + \pi/2) - \sin(t + \pi/2) \\ &= -\sin t - \cos t \end{aligned}$$

$$\mathcal{L}[g(t)] = -\mathcal{L}[\sin t] - \mathcal{L}[\cos t]$$

$$\bar{g}(s) = -\left[\frac{1}{s^2+1} + \frac{s}{s^2+1} \right]$$

$$\therefore \mathcal{L}[g(t - \pi/2) u(t - \pi/2)] = e^{-\pi/2 s} \cdot \bar{g}(s)$$

$$= e^{-\pi/2 s} - \left[\frac{1}{s^2+1} + \frac{s}{s^2+1} \right]$$

$$\therefore ① \Rightarrow \mathcal{F}(s) = \frac{1}{s^2+1} - e^{-\pi/2 s} \left[\frac{1}{s^2+1} + \frac{s}{s^2+1} \right]$$

c) Find the Laplace transform of $\frac{4}{(s^2+2s+5)^2}$, using convolution theorem.

Solⁿ: given $\mathcal{F}(s) \cdot \bar{g}(s) = \frac{4}{(s^2+2s+5)^2}$

$$\mathcal{F}(s) \cdot \bar{g}(s) = \frac{2}{s^2+2s+5} \cdot \frac{2}{s^2+2s+5}$$

$$\mathcal{F}(s) = \frac{2}{(s+1)^2+4}, \quad \bar{g}(s) = \frac{2}{(s+1)^2+4}$$

$$\Rightarrow \mathcal{L}^{-1}[\mathcal{F}(s)] = \mathcal{L}^{-1}\left[\frac{2}{(s+1)^2+4}\right] = e^{-t} \mathcal{L}^{-1}\left[\frac{2}{s^2+4}\right] = e^{-t} s \operatorname{sn} at$$

$$= f(t)$$

$$L^{-1}[\bar{q}(s)] = L^{-1}\left[\frac{2}{(s+1)^2 + 4}\right] = e^{-t} L^{-1}\left[\frac{2}{s^2 + 4}\right] = e^{-t} \sin 2t = q(t)$$

$$\therefore f(t) = e^{-t} \sin at \quad q(t) = e^{-t} \sin at$$

$$\text{WKT} \quad L^{-1}[f(s) \cdot \bar{g}(s)] = f(t) * g(t)$$

$$L^{-1}\left[\frac{4}{(s^2 + 2s + 5)^2}\right] = \int_{u=0}^t f(u) g(t-u) du$$

$$= \int_{u=0}^t e^{-u} \sin 2u \cdot e^{-(t-u)} \sin(2t-2u) du$$

$$= \int_{u=0}^t e^{tu} \sin 2u \cdot e^{-t} \cdot e^{tu} \sin(2t-2u) du$$

$$= e^{-t} \int_{u=0}^t \sin 2u \cdot \sin(2t-2u) du$$

$$= \frac{e^{-t}}{2} \int_{u=0}^t 2 \sin 2u \sin(2t-2u) du$$

$$= \frac{e^{-t}}{2} \int_{u=0}^t [\cos(2u-2t+2u) - \cos(2u+2t-2u)] du$$

$$= \frac{e^{-t}}{2} \int_{u=0}^t [\cos(4u-2t) - \cos 2t] du$$

$$= \frac{e^{-t}}{2} \left\{ \left[\int_{u=0}^t \cos(4u-2t) du - \cos 2t \int_0^t 1 du \right] \right\}$$

$$= \frac{e^{-t}}{2} \left\{ \left[\frac{\sin(4u-2t)}{4} \right]_{u=0}^t - t \cos 2t \right\}$$

$$= \frac{e^{-t}}{2} \left\{ \frac{\sin 2t}{4} + \frac{\sin 2t}{4} - t \cos 2t \right\}$$

$$= \frac{e^{-t}}{2} \left[\frac{\sin 2t}{2} - t \cos 2t \right]$$

Module -3

5a) If $f(x) = \begin{cases} 1-x^2 & \text{for } |x| \leq 1 \\ 0 & \text{for } |x| > 1 \end{cases}$, find the infinite

Fourier transform of $f(x)$ and hence Evaluate $\int_0^\infty \frac{x \cos x - \sin x}{x^3} \cos \frac{x}{2} dx$

$$\text{Soln: } f(x) = \begin{cases} 1-x^2 & , -1 \leq x \leq 1 \\ 0 & , x > 1 \end{cases}$$

The Fourier transform of $f(x)$ is $F[f(x)] = \int_{-\infty}^{\infty} e^{isx} f(x) dx$

$$= \int_{-\infty}^{-1} e^{-isx} f(x) dx + \int_{-1}^1 e^{isx} f(x) dx + \int_1^{\infty} e^{isx} f(x) dx$$

$$= \int_{-1}^1 e^{isx} f(x) dx$$

$$= \int_{-1}^1 e^{isx} (1-x^2) dx$$

$$= \left\{ \int_{-1}^0 (1-x^2) \int_{-1}^1 e^{isx} dx - \int_{-1}^1 [(-2x) \int_{-1}^1 e^{isx} dx] dx \right\}$$

$$= \frac{1}{is} \left[(1-x^2) e^{isx} \right]_{-1}^1 + \frac{2}{is} \int_{-1}^1 x e^{isx} dx$$

$$= \frac{1}{is} (0-0) + \frac{2}{is} \left\{ x \int_{-1}^1 e^{isx} - \int_{-1}^1 [1 \int_{-1}^1 e^{isx} dx] dx \right\}$$

$$= 0 + \frac{2}{is} \left\{ \frac{1}{is} [x e^{isx}]_{-1}^1 - \frac{1}{i^2 s^2} [e^{isx}]_{-1}^1 \right\}$$

$$= \frac{2}{is} \left\{ \frac{1}{is} [e^{is} + e^{-is}] - \frac{1}{i^2 s^2} [e^{is} - e^{-is}] \right\}$$

$$= \frac{2}{is} \left\{ \frac{1}{is} [\cos s + i \sin s + \cos s - i \sin s] - \frac{1}{i^2 s^2} [\cos s + i \sin s - \cos s - i \sin s] \right\}$$

$$\begin{aligned}
 &= \frac{2}{is} \left\{ \frac{1}{is} (2\cos s) - \frac{1}{i^2 s^2} (2s \sin s) \right\} \\
 &= \frac{4 \cos s}{i^2 s^2} - \frac{4 s \sin s}{i^2 s^3} \\
 &= \frac{-4 \cos s}{s^2} + \frac{4 s \sin s}{s^3} \\
 &= 4 \left[\frac{\sin s}{s^3} - \frac{\cos s}{s^2} \right] \\
 f(s) &= 4 \left[\frac{\sin s - s \cos s}{s^3} \right]
 \end{aligned}$$

WKT The Fourier Inverse transform is

$$\begin{aligned}
 F^{-1}[f(s)] &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(s) e^{-isx} ds = f(x) \\
 \int_{-\infty}^{\infty} f(s) e^{-isx} ds &= 2\pi \begin{cases} 1-x^2 & -1 \leq x \leq 1 \\ 0 & x > 1 \end{cases} \\
 \int_{-\infty}^{\infty} 4 \left[\frac{\sin s - s \cos s}{s^3} \right] [\cos sx - i \sin sx] ds &= 2\pi \begin{cases} 1-x^2, & -1 \leq x \leq 1 \\ 0, & x > 1 \end{cases} \\
 \Rightarrow \int_{-\infty}^{\infty} 4 \left[\frac{\sin s - s \cos s}{s^3} \right] \cos sx - i \int_{-\infty}^{\infty} 4 \left[\frac{\sin s - s \cos s}{s^3} \right] \sin sx ds \\
 &= 2\pi \begin{cases} 1-x^2, & -1 \leq x \leq 1 \\ 0, & x > 1 \end{cases} + i(0) \\
 \int_{-\infty}^{\infty} 4 \left[\frac{\sin s - s \cos s}{s^3} \right] \cos sx ds &= 2\pi \begin{cases} 1-x^2, & -1 \leq x \leq 1 \\ 0, & x > 1 \end{cases} \rightarrow ①
 \end{aligned}$$

When $x = \frac{\pi}{2}$

$$\therefore ① \Rightarrow \int_{-\infty}^{\infty} 4 \left(\frac{\sin s - s \cos s}{s^3} \right) \cos \frac{s}{2} ds = 2\pi \left(1 - \frac{1}{4} \right) = 2\pi \left(\frac{3}{4} \right) = \frac{3\pi}{2}$$

$$\Rightarrow \int_{-\infty}^{\infty} \left(\frac{\sin s - s \cos s}{s^3} \right) \cos \frac{s}{2} ds = \frac{3\pi}{2}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin s - s \cos s}{s^3} \cos \frac{s}{2} ds = \frac{3\pi}{8}$$

$$\Rightarrow 2 \int_0^{\infty} \frac{\sin s - s \cos s}{s^3} \cos \frac{s}{2} ds = \frac{3\pi}{8}$$

$$\Rightarrow \int_0^{\infty} \frac{\sin s - s \cos s}{s^3} \cos \frac{s}{2} ds = \frac{3\pi}{16}$$

$$\Rightarrow \int_0^{\infty} \frac{s \cos s - \sin s}{s^3} \cos \frac{s}{2} ds = -\frac{3\pi}{16}$$

$$\Rightarrow \int_0^{\infty} \frac{x \cos x - \sin x}{x^3} \cos \frac{x}{2} dx = -\frac{3\pi}{16} \text{ when } s=x$$

5b) Find the fourier cosine transform of $f(x) = e^{-2x} + 4e^{-3x}$

Solⁿ: WKT

$$F_c[F(x)] = \int_0^{\infty} f(x) \cos sx dx$$

$$= \int_0^{\infty} [e^{-2x} + 4e^{-3x}] \cos sx dx$$

$$= \int_0^{\infty} e^{-2x} \cos sx dx + 4 \int_0^{\infty} e^{-3x} \cos sx dx$$

$$= \int_0^{\infty} \frac{e^{-2x}}{s^2+4} [-2 \cos sx + s \sin(sx)] dx + 4 \int_0^{\infty} \frac{e^{-3x}}{s^2+9} [-3 \cos(sx) + s \sin(sx)] dx$$

$$= \left[0 - \frac{1}{s^2+4} (-2) \right] + 4 \left[0 - \frac{1}{s^2+9} (-3) \right]$$

$$f_c(s) = \frac{2}{s^2+4} + \frac{12}{s^2+9}$$

5c) Solve: $u_{n+2} - 3u_{n+1} + 2u_n = 2^n$, given $u_0=0$, $u_1=1$ by using z-transforms.

$$\underline{\text{Sol}}^n :- \quad u_{n+2} - 3u_{n+1} + 2u_n = 2^n$$

$$z[u_{n+2}] - 3z[u_{n+1}] + 2z[u_n] = z[2^n]$$

$$\Rightarrow z^2[\bar{u}(z) - u_0 - \frac{u_1}{z}] - 3z[\bar{u}(z) - u_0] + 2\bar{u}(z) = \frac{z}{z-2}$$

$$\Rightarrow z^2[\bar{u}(z) - 0 - \frac{1}{z}] - 3z[\bar{u}(z) - 0] + 2\bar{u}(z) = \frac{z}{z-2}$$

$$\Rightarrow z^2\bar{u}(z) - z^2 \cdot \frac{1}{z} - 3z\bar{u}(z) + 2\bar{u}(z) = \frac{z}{z-2}$$

$$\Rightarrow z^2\bar{u}(z) - 3z\bar{u}(z) + 2\bar{u}(z) - z = \frac{z}{z-2}$$

$$\Rightarrow (z^2 - 3z + 2)\bar{u}(z) = \frac{z}{z-2} + z$$

$$\Rightarrow (z-1)(z-2)\bar{u}(z) = \frac{z^2 - z}{z-2}$$

$$\Rightarrow (z-1)(z-2)\bar{u}(z) = \frac{z(z-1)}{z-2}$$

$$\Rightarrow \bar{u}(z) = \frac{z}{(z-2)^2}$$

$$\bar{u}(z) = \frac{1}{2} \cdot \frac{2z}{(z-2)^2}, \quad z^{-1}[\bar{u}(z)] = \frac{1}{2} z^{-1}\left[\frac{2z}{(z-2)^2}\right]$$

$$u_n = \frac{1}{2} 2^n \cdot n$$

$$u_n = \underline{\underline{2^{n-1} \cdot n}}$$

6a) Find the Fourier Sine transform of $e^{-|x|}$. Hence show that $\int_0^\infty \frac{x \sin mx}{1+x^2} dx = \frac{\pi e^{-m}}{2}$, $m > 0$

Solⁿ: Fourier Sine transform is given by

$$f_s[F(x)] = \int_0^\infty F(x) \sin sx dx$$

$$= \int_0^\infty e^{-|x|} \sin sx dx, \text{ since } |x| = x, x \geq 0$$

$$f_s(s) = \left[\frac{e^{-x}}{(-1)^2 + s^2} [-1 \sin sx - s \cos sx] \right]_0^\infty$$

$$= \left[0 - \frac{1}{1^2 + s^2} [-1(0) - s(1)] \right]$$

$$= -\frac{1}{1^2 + s^2} [-s]$$

$$f_s(s) = \frac{s}{1+s^2}$$

By Inverse Fourier Sine transform we have,

$$\frac{2}{\pi} \int_0^\infty f_s(s) \sin sx ds = f(x)$$

$$\int_0^\infty \frac{s}{1+s^2} \sin sx ds = \frac{\pi}{2} f(x)$$

put $x = m$ while $m > 0$ we have $f(x) = e^{-|m|} = e^{-m}$

$$\int_0^\infty \frac{s \sin ms}{1+s^2} ds = \frac{\pi}{2} e^{-m}$$

$$\int_0^\infty \frac{x \sin mx}{1+x^2} dx = \frac{\pi}{2} e^{-m}$$

when $x = m$

6b) Find the z -transform of $\cos\left[\frac{n\pi}{2} + \frac{\pi}{4}\right]$

Solⁿ: let $f(n) = \cos\left(\frac{n\pi}{2} + \frac{\pi}{4}\right)$

$$f(n) = \cos\left(\frac{n\pi}{2}\right) \cos\frac{\pi}{4} - \sin\left(\frac{n\pi}{2}\right) \sin\frac{\pi}{4}$$

$$Z[f(n)] = \frac{1}{\sqrt{2}} Z\left[\cos\left(\frac{n\pi}{2}\right)\right] - \frac{1}{\sqrt{2}} Z\left[\sin\left(\frac{n\pi}{2}\right)\right] \rightarrow ①$$

WKT $Z[\cos n\theta] = \frac{z^2 - z \cos \theta}{z^2 - 2z \cos \theta + 1}$

$$Z\left[\cos n\frac{\pi}{2}\right] = \frac{z^2 - z \cos \frac{\pi}{2}}{z^2 - 2z \cos \frac{\pi}{2} + 1}$$

$$= \frac{z^2 - 0}{z^2 - 0 + 1}$$

$$Z\left[\cos n\frac{\pi}{2}\right] = \frac{z^2}{z^2 + 1}$$

$$Z[\sin n\theta] = \frac{z \sin \theta}{z^2 - 2z \cos \theta + 1}$$

$$Z\left[\sin n\frac{\pi}{2}\right] = \frac{z \sin \frac{\pi}{2}}{z^2 - 2z \cos \frac{\pi}{2} + 1}$$

$$Z\left[\sin n\frac{\pi}{2}\right] = \frac{z}{z^2 + 1}$$

$$\therefore ① \Rightarrow F(z) = \frac{1}{\sqrt{2}} \left(\frac{z^2}{z^2 + 1} \right) - \frac{1}{\sqrt{2}} \left(\frac{z}{z^2 + 1} \right)$$

$$F(z) = \frac{z^2 - z}{\sqrt{2}(z^2 + 1)}$$

6c) Find the inverse z -transform of $\frac{2z^2 + 3z}{(z+2)(z-4)}$

Solⁿ: let $F(z) = \frac{2z^2 + 3z}{(z+2)(z-4)}$

$$\frac{F(z)}{z} = \frac{2z+3}{(z+2)(z-4)}$$

$$\frac{2z+3}{(z+2)(z-4)} = \frac{A}{z+2} + \frac{B}{z-4} \rightarrow ①$$

$$2z+3 = A(z-4) + B(z+2) \rightarrow ②$$

When $z = -2$

$$① \Rightarrow 2(-2)+3 = A(-2-4)$$

$$-1 = -6A$$

$$A = \frac{1}{6}$$

When $z = 4$

$$② \Rightarrow 2(4)+3 = B(4+2)$$

$$11 = 6B$$

$$B = \frac{11}{6}$$

$$\therefore ① \Rightarrow \frac{F(z)}{z} = \frac{1}{6} \cdot \frac{1}{z+2} + \frac{11}{6} \cdot \frac{1}{z-4}$$

$$F(z) = \frac{1}{6} \frac{z}{z+2} + \frac{11}{6} \frac{z}{z-4}$$

$$z^{-1}[F(z)] = \frac{1}{6} z^{-1}\left[\frac{z}{z+2}\right] + \frac{11}{6} z^{-1}\left[\frac{z}{z-4}\right]$$

$$z^{-1}[F(z)] = \frac{1}{6} z^{-1}\left[\frac{z}{z-(-2)}\right] + \frac{11}{6} z^{-1}\left[\frac{z}{z-4}\right]$$

$$f(n) = \underline{\underline{\frac{1}{6} (-2)^n + \frac{11}{6} (4)^n}}$$

5a) If $f(x) = \begin{cases} a^2 - x^2, & \text{for } |x| \leq a \\ 0, & \text{for } |x| > a \end{cases}$, find the infinite Fourier transform of $f(x)$ & hence evaluate $\int_0^\infty \frac{\sin x - x \cos x}{x^3} dx$

$$\text{Soln: } f(x) = \begin{cases} a^2 - x^2, & |x| \leq a \\ 0, & |x| > a \end{cases}$$

$$f(x) = \begin{cases} a^2 - x^2, & -a \leq x \leq a \\ 0, & x > a \end{cases}$$

WKT The Fourier transform of $F(x)$ is

$$\begin{aligned} F[F(x)] &= \int_{-\infty}^{\infty} e^{isx} F(x) dx \\ &= \int_{-\infty}^{-a} e^{isx} F(x) dx + \int_{-a}^a e^{isx} F(x) dx + \int_a^{\infty} e^{isx} F(x) dx \\ &= \int_{-a}^a e^{isx} (a^2 - x^2) dx \\ &= (a^2 - x^2) \int_{-a}^a e^{isx} dx - \int_{-a}^a [(-2x) \int e^{isx} dx] dx \\ &= \frac{1}{is} \left[(a^2 - x^2) e^{isx} \right]_{-a}^a + \frac{2}{is} \int_{-a}^a x e^{isx} dx \\ &= 0 + \frac{2}{is} \int_{-a}^a x e^{isx} dx \\ &= \frac{2}{is} \left\{ x \int_{-a}^a e^{isx} dx - \int_{-a}^a [1 \int e^{isx} dx] dx \right\} \\ &= \frac{2}{is} \left\{ \frac{1}{is} \left[x e^{isx} \right]_{-a}^a - \frac{1}{i^2 s^2} \left[e^{isx} \right]_{-a}^a \right\} \\ &= \frac{2}{is} \left\{ \frac{1}{is} \left[a e^{ias} - (-a) e^{-ias} \right] + \frac{1}{s^2} [e^{ias} - e^{-ias}] \right\} \end{aligned}$$

$$\begin{aligned}
&= \frac{2}{is} \left\{ \frac{a}{is} [e^{ias} + e^{-ias}] + \frac{1}{s^2} [e^{ias} - e^{-ias}] \right\} \\
&= \frac{2}{is} \left\{ \frac{a}{is} [\cos as + i \sin as + \cos as - i \sin as] + \frac{1}{s^2} [\cos as + i \sin as - \cos as + i \sin as] \right\} \\
&= \frac{2}{is} \left\{ \frac{a}{is} [2 \cos as] + \frac{1}{s^2} (2i \sin as) \right\} \\
&= \frac{2}{is} \left\{ \frac{2a \cos(as)}{is} + \frac{2i \sin as}{s^2} \right\} \\
&= \frac{4a \cos(as)}{i^2 s^2} + \frac{4 \sin as}{s^3} \\
&= \frac{4 \sin as}{s^3} - \frac{4a \cos(as)}{s^2}
\end{aligned}$$

$$f(s) = 4 \left[\frac{\sin as - a \cos(as)}{s^3} \right]$$

WKT Fourier inverse transform is given by

$$\begin{aligned}
F^{-1}[f(s)] &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(s) e^{-isx} ds = F(x) \\
\Rightarrow \int_{-\infty}^{\infty} f(s) e^{-isx} dx &= 2\pi \begin{cases} a^2 - x^2, & -a \leq x \leq a \\ 0, & x > a \end{cases} \\
\Rightarrow \int_{-\infty}^{\infty} 4 \left[\frac{\sin as - a \cos(as)}{s^3} \right] [\cos sx - i \sin sx] ds &= 2\pi \begin{cases} a^2 - x^2, & -a \leq x \leq a \\ 0, & x > a \end{cases} \\
\int_{-\infty}^{\infty} 4 \frac{[\sin as - a \cos(as)]}{s^3} \cos sx ds - i \int_{-\infty}^{\infty} 4 \frac{[\sin as - a \cos(as)]}{s^3} \sin sx ds &= 2\pi \begin{cases} a^2 - x^2, & -a \leq x \leq a \\ 0, & x > a \end{cases} \\
\int_{-\infty}^{\infty} \sin(sx) ds &= 2\pi \begin{cases} a^2 - x^2, & -a \leq x \leq a \\ 0, & x > a \end{cases} + i(0)
\end{aligned}$$

$$\int_{-\infty}^{\infty} 4 \frac{[\sin(as) - a\cos(as)]}{s^3} \cos(sx) dx = 2\pi \begin{cases} a^2 - x^2, & -a \leq x \leq a \\ 0, & x > a \end{cases} \rightarrow ①$$

When $x=0$

$$\begin{aligned} \therefore ① &\Rightarrow \int_{-\infty}^{\infty} 4 \frac{[\sin(as) - a\cos(as)]}{s^3} ds = 2\pi(a^2 - 0) = 2\pi a^2 \\ &\Rightarrow \int_{-\infty}^{\infty} \frac{\sin(as) - a\cos(as)}{s^3} ds = \frac{2\pi a^2}{4} \\ &\Rightarrow 2 \int_0^{\infty} \frac{\sin(as) - a\cos(as)}{s^3} ds = \frac{\pi}{2} a^2 \\ &\Rightarrow \int_0^{\infty} \frac{\sin(as) - a\cos(as)}{s^3} ds = \frac{\pi}{4} a^2 \\ &\Rightarrow \int_0^{\infty} \frac{\sin s - s\cos s}{s^3} ds = \frac{\pi}{4} \text{ for } a=1 \\ &\Rightarrow \int_0^{\infty} \frac{\sin x - x\cos x}{x^3} dx = \frac{\pi}{4} \text{ for } s=x \end{aligned}$$

5b) Find the Fourier Sine transform of $f(x) = \begin{cases} x, & \text{if } 0 < x < 1 \\ 2-x, & \text{if } 1 < x < 2 \\ 0, & \text{if } x > 2 \end{cases}$

Sol:- WKT $F_s[f(x)] = f_s(s) = \int_0^{\infty} f(x) \sin(sx) dx$

$$\Rightarrow f_s(s) = \int_0^1 f(x) \sin(sx) dx + \int_1^2 f(x) \sin(sx) dx + \int_2^{\infty} f(x) \sin(sx) dx$$

$$f_s(s) = \int_0^1 x \sin(sx) dx + \int_1^2 (2-x) \sin(sx) dx \rightarrow ①$$

$$\begin{aligned} \therefore \int_0^1 x \sin(sx) dx &= \left[x \int_0^1 \sin(sx) dx - \int_0^1 \left[\int_0^1 \sin(sx) dx \right] dx \right] \\ &= -\frac{1}{s} \left[x \cos(sx) \right]_0^1 + \frac{1}{s^2} [\sin(sx)]_0^1 \\ &= -\frac{1}{s} [\cos s - 0] + \frac{1}{s^2} [\sin s - 0] \\ &= -\frac{1}{s} \cos s + \frac{1}{s^2} \sin s \end{aligned}$$

$$\begin{aligned}
 \int_1^2 (2-x) \sin(sx) dx &= (2-x) \int_1^2 \sin(sx) dx - \int_1^2 [(-1) \int_1^2 \sin(sx) dx] dx \\
 &= -\frac{1}{s} [(2-x) \cos(sx)]_1^2 - \frac{1}{s^2} [\sin(sx)]_1^2 \\
 &= -\frac{1}{s} [0 - \cos s] - \frac{1}{s^2} [\sin 2s - \sin s] \\
 &= \frac{1}{s} \cos s - \frac{1}{s^2} \sin 2s + \frac{1}{s^2} \sin s
 \end{aligned}$$

$$\therefore \textcircled{1} \Rightarrow f_s(s) = -\frac{1}{s} \cos s + \frac{1}{s^2} \sin s + \frac{1}{s} \cos s - \frac{1}{s^2} \sin 2s + \frac{1}{s^2} \sin s$$

$$f_s(s) = \frac{2}{s^2} \sin s - \frac{1}{s^2} \sin 2s$$

5c) Solve $u_{n+2} + 6u_{n+1} + 9u_n = 2^n$, $u_0 = 0 = u_1$, by using z-transforms.

$$\text{Sol}^n: \text{given } u_{n+2} + 6u_{n+1} + 9u_n = 2^n$$

$$z[u_{n+2}] + 6z[u_{n+1}] + 9z[u_n] = z[2^n]$$

$$z^2[\bar{u}(z) - u_0 - \frac{u_1}{z}] + 6z[\bar{u}(z) - u_0] + 9\bar{u}(z) = \frac{z}{z-2}$$

$$z^2\bar{u}(z) + 6z\bar{u}(z) + 9\bar{u}(z) = \frac{z}{z-2}$$

$$(z^2 + 6z + 9)\bar{u}(z) = \frac{z}{z-2}$$

$$(z+3)^2\bar{u}(z) = \frac{z}{z-2}$$

$$\bar{u}(z) = \frac{z}{(z-2)(z+3)^2}$$

$$\frac{\bar{u}(z)}{z} = \frac{1}{(z-2)(z+3)^2} \rightarrow \textcircled{1}$$

$$\frac{1}{(z-2)(z+3)^2} = \frac{A}{z-2} + \frac{B}{z+3} + \frac{C}{(z+3)^2}$$

$$1 = A(z+3)^2 + B(z-2)(z+3) + C(z-2) \rightarrow ②$$

when $z=2$

$$② \Rightarrow 1 = 25A$$

$$A = \frac{1}{25}$$

when $z=-3$

$$② \Rightarrow 1 = -5C$$

$$C = -\frac{1}{5}$$

Compare coefficients of z^2

$$A+B=0$$

$$B = -A$$

$$B = -\frac{1}{25}$$

$$\therefore ① \Rightarrow \bar{u}(z) = \frac{1}{25} \cdot \frac{1}{z-2} - \frac{1}{25} \cdot \frac{1}{z+3} - \frac{1}{5} \cdot \frac{1}{(z+3)^2}$$

$$\bar{u}(z) = \frac{1}{25} \frac{z}{z-2} - \frac{1}{25} \frac{z}{z+3} - \frac{1}{5} \frac{z}{(z+3)^2}$$

$$\begin{aligned} z^{-1}[\bar{u}(z)] &= \frac{1}{25} z^{-1}\left[\frac{z}{z-2}\right] - \frac{1}{25} z^{-1}\left[\frac{z}{z-(-3)}\right] \\ &\quad + \frac{1}{15} z^{-1}\left[\frac{-3z}{[z-(-3)]^2}\right] \end{aligned}$$

$$u_n = \frac{1}{25} (2)^n - \frac{1}{25} (-3)^n + \frac{1}{15} (-3)^n \cdot n$$

Ques: If $f(x) = \begin{cases} 1, & \text{for } |x| \leq a \\ 0, & \text{for } |x| > a \end{cases}$, find the infinite Fourier transform of $f(x)$ and hence evaluate $\int_0^\infty \frac{\sin x}{x} dx$

Soln: Given $f(x) = \begin{cases} 1, & |x| \leq a \\ 0, & |x| > a \end{cases}$

$$\Rightarrow f(x) = \begin{cases} 1, & -a < x \leq a \\ 0, & x > a \end{cases}$$

WKT

$$\begin{aligned} F[f(x)] &= \int_{-\infty}^{\infty} e^{isx} f(x) dx \\ &= \int_{-\infty}^{-a} e^{isx} f(x) dx + \int_{-a}^a e^{isx} f(x) dx + \int_a^{\infty} e^{isx} f(x) dx \\ &= 0 + \int_{-a}^a e^{isx} 1 dx + 0 \end{aligned}$$

$$\begin{aligned} F[f(x)] &= \int_{-a}^a e^{isx} dx \\ &= \left[\frac{e^{isx}}{is} \right]_{-a}^a \\ &= \frac{1}{is} \left[e^{ais} - e^{-ais} \right] \\ &= \frac{1}{is} [\cos(as) + i \sin(as) - \cos(-as) - i \sin(-as)] \\ &= \frac{1}{is} [2i \sin(as)] \end{aligned}$$

$$f(s) = \frac{2}{s} \sin(as)$$

WKT $F^{-1}[f(s)] = f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-isx} f(s) ds$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} [e^{isx} - \frac{2}{s} \sin as] ds = \begin{cases} 1, & -a < x \leq a \\ 0, & x > a \end{cases}$$

$$\Rightarrow \int_{-\infty}^{\infty} \left[\frac{\cos(sx) \cdot \sin(as)}{s} \right] - i \left[\frac{\sin(as) - \sin(sx)}{s} \right] ds = \pi \begin{cases} 1, & -a < x \leq a \\ 0, & x > a \end{cases}$$

$$\therefore \int_{-\infty}^{\infty} \frac{\cos(sx) \sin(as)}{s} ds = \pi \begin{cases} 1, & -a < x \leq a \\ 0, & x > a \end{cases}$$

When $x=0$

$$① \Rightarrow \int_{-\infty}^{\infty} \frac{\cos 0 \sin as}{s} ds = \pi(1)$$

$$\int_{-\infty}^{\infty} \frac{\sin as}{s} ds = \pi$$

$$② \int_0^{\infty} \frac{\sin as}{s} ds = \pi$$

$$\int_0^{\infty} \frac{\sin as}{s} ds = \frac{\pi}{2}$$

$$\int_0^{\infty} \frac{\sin s}{s} ds = \frac{\pi}{2} \text{ for } a=1$$

$$\Rightarrow \int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2} \text{ for } s=x$$

6b) Find the Z-transform of $\sin n + \sin\left(\frac{n\pi}{4}\right) + 1$

$$\text{soln: let } f(n) = \sin n + \sin\left(\frac{n\pi}{4}\right) + 1$$

$$z[f(n)] = z[z[n]] + z[\sin\left(\frac{n\pi}{4}\right)] + z[1] \rightarrow ①$$

$$z[n] = \frac{z}{(z-1)^2}$$

$$z[\sin n] = \frac{zs^n}{z^2 - 2z \cos \theta + 1}$$

$$z[\sin \frac{n\pi}{4}] = \frac{zs^n \sin \frac{\pi}{4}}{z^2 - 2z \cos \frac{\pi}{4} + 1} = \frac{z\sqrt{2}}{z^2 - 2z(\sqrt{2}) + 1}$$

$$z \left[\sin \frac{n\pi}{4} \right] = \frac{z}{\sqrt{2}z^2 - 2z + \sqrt{2}}$$

$$z[1] = \frac{z}{z-1}$$

$$\therefore ① \Rightarrow F(z) = \frac{2z}{(z-1)^2} + \frac{z}{\sqrt{2}z^2 - 2z + \sqrt{2}} + \frac{z}{z-1}$$

Q) Find the inverse z-transform of $18z^2 / [(2z-1)(4z+1)]$

Soln: Let $F(z) = \frac{18z^2}{(2z-1)(4z+1)}$

$$\frac{F(z)}{z} = \frac{18z}{(2z-1)(4z+1)} \rightarrow ①$$

$$\frac{18z}{(2z-1)(4z+1)} = \frac{A}{2z-1} + \frac{B}{4z+1}$$

$$\Rightarrow 18z = A(4z+1) + B(2z-1) \rightarrow ②$$

when $z = 1/2$

$$② \Rightarrow 9 = 3A$$

$$A = 3$$

when $z = -1/4$

$$② \Rightarrow 18(-1/4) = B[2(-1/4) - 1]$$

$$-9/2 = B[-1/2 - 1]$$

$$-9/2 = B(-3/2)$$

$$B = 3$$

$$\therefore ① \Rightarrow \frac{F(z)}{z} = \frac{3}{2z-1} + \frac{3}{4z+1}$$

$$F(z) = 3 \cdot \frac{z}{2z-1} + 3 \cdot \frac{z}{4z+1}$$

$$F(z) = \frac{3}{2} \frac{z}{(z-1/2)} + \frac{3}{4} \frac{z}{(z+1/4)}$$

$$F(z) = \frac{3}{2} \frac{z}{(z - \gamma_2)} + \frac{3}{4} \frac{z}{[z - (-\gamma_4)]}$$

$$z^{-1}[F(z)] = \frac{3}{2} z^{-1}\left[\frac{z}{z - \gamma_2}\right] + \frac{3}{4} z^{-1}\left[\frac{z}{z - (-\gamma_4)}\right]$$

$$f(n) = \frac{3}{2} (\gamma_2)^n + \frac{3}{4} (-\gamma_4)^n$$

Module - 2

3a) Find the Fourier series expansion of $f(x) = \frac{\pi^2}{12} - \frac{x^2}{4}$

on $-\pi \leq x \leq \pi$. Hence deduce that $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \frac{\pi^2}{12}$

Soln: $f(x) = \frac{\pi^2}{12} - \frac{x^2}{4}$

$$f(-x) = \frac{\pi^2}{12} - \frac{(-x)^2}{4} = \frac{\pi^2}{12} - \frac{x^2}{4} = f(x)$$

$\therefore f(x)$ is an even function $\Rightarrow b_n = 0$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \rightarrow ①$$

$$\begin{aligned} a_0 &= \frac{2}{\pi} \int_0^\pi f(x) dx \\ &= \frac{2}{\pi} \int_0^\pi \frac{\pi^2}{12} - \frac{x^2}{4} dx \\ &= \frac{2}{\pi} \left[\frac{\pi^2}{12} x - \frac{x^3}{12} \right]_0^\pi \\ &= \frac{2}{\pi} \left[\frac{\pi^3}{12} - \frac{\pi^3}{12} - 0 \right] \end{aligned}$$

$$\boxed{a_0 = 0}$$

$$\begin{aligned}
 a_n &= \frac{2}{\pi} \int_0^\pi f(x) \cos nx dx \\
 &= \frac{2}{\pi} \int_0^\pi \left(\frac{\pi^2}{12} - \frac{x^2}{4} \right) \cos nx dx \\
 &= \frac{2}{\pi} \left\{ \left(\frac{\pi^2}{12} - \frac{x^2}{4} \right) \int_0^\pi \cos nx dx - \int_0^\pi \left[\left(-\frac{ex}{4} \right) \int \cos nx dx \right] dx \right\} \\
 &= \frac{2}{\pi} \left\{ \frac{1}{n} \left[\left(\frac{\pi^2}{12} - \frac{x^2}{4} \right) \sin nx \right]_0^\pi + \frac{1}{2n} \int_0^\pi x \sin nx dx \right\} \\
 &= \frac{2}{\pi} \left\{ \frac{1}{n} (0-0) + \frac{1}{2n} \left[x \int_0^\pi \sin nx dx - \int_0^\pi [1 \cdot \int \sin nx] dx \right] \right\} \\
 &= \frac{2}{\pi} \left\{ \frac{1}{2n} \left[-\frac{1}{n} [x \cos nx]_0^\pi + \frac{1}{n^2} [\sin nx]_0^\pi \right] \right\} \\
 &= \frac{2}{\pi} \left\{ \frac{1}{2n} \left[-\frac{1}{n} [\pi \cos n\pi - 0] + \frac{1}{n^2} [\sin n\pi - \sin 0] \right] \right\} \\
 &= \frac{2}{\pi} \left\{ -\frac{1}{2n^2} [\pi (-1)^n] + 0 \right\} \\
 &= \frac{-(-1)^n}{n^2}
 \end{aligned}$$

$$a_n = \frac{(-1)^{n+1}}{n^2}$$

$$\textcircled{1} \Rightarrow f(x) = 0 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos nx$$

$$\frac{\pi^2}{12} - \frac{x^2}{4} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos nx$$

let $x=0$

$$\frac{\pi^2}{12} - 0 = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cdot 1$$

$$\frac{\pi^2}{12} = \frac{(-1)^{1+1}}{1^2} + \frac{(-1)^{2+1}}{2^2} + \frac{(-1)^{3+1}}{3^2} + \frac{(-1)^{4+1}}{4^2} + \dots$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$$

3b) Find the half - range cosine series of $f(x) = (x+1)^2$
in interval $0 \leq x \leq 1$.

Soln:- given $f(x) = (x+1)^2$ $x \in [0, 1]$

WKT the Fourier half range cosine series of $f(x)$ in $[0, 1]$

$$l=1$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \rightarrow ①$$

$$a_0 = \frac{2}{1} \int_0^1 f(x) dx$$

$$= \frac{2}{1} \int_0^1 (x+1)^2 dx$$

$$= 2 \left[\frac{(x+1)^3}{3} \right]_0^1$$

$$= 2 \left[\frac{8}{3} - \frac{1}{3} \right]$$

$$= 2 \left[\frac{7}{3} \right]$$

$$a_0 = \frac{14}{3}$$

$$a_n = \frac{2}{1} \int_0^1 f(x) \cos\left(\frac{n\pi x}{1}\right) dx$$

$$= \frac{2}{1} \int_0^1 (x+1)^2 \cos(n\pi x) dx$$

$$= 2 \left\{ (x+1)^2 \int_0^1 \cos(n\pi x) dx - \int_0^1 [2(x+1)(1) \int_0^1 \cos(n\pi x) dx] dx \right\}$$

$$= 2 \left\{ \frac{1}{n\pi} \left[(x+1)^2 \sin(n\pi x) \right]_0^1 - \frac{2}{n\pi} \int_0^1 (x+1) \sin(n\pi x) dx \right\}$$

$$\begin{aligned}
 &= 2 \left\{ \frac{1}{n\pi} [0 - 0] - \frac{2}{n\pi} \int_0^1 (x+1) \sin n\pi x \, dx \right\} \\
 &= -\frac{4}{n\pi} \int_0^1 (x+1) \sin(n\pi x) \, dx \\
 &= -\frac{4}{n\pi} \left\{ (x+1) \int_0^1 \sin(n\pi x) \, dx - \int_0^1 \left[1 \cdot \int_0^x \sin(n\pi x) \, dx \right] dx \right\} \\
 &= -\frac{4}{n\pi} \left\{ -\frac{1}{n\pi} \left[(x+1) \cos n\pi x \right]_0^1 + \frac{1}{n^2\pi^2} \left[\sin n\pi x \right]_0^1 \right\} \\
 &= -\frac{4}{n\pi} \left\{ -\frac{1}{n\pi} [2 \cos n\pi - 1] + \frac{1}{n^2\pi^2} [\sin n\pi - 0] \right\} \\
 &= \frac{4}{n^2\pi^2} \left\{ [2(-1)^n - 1] + 0 \right\}
 \end{aligned}$$

$$a_n = \frac{4}{n^2\pi^2} [2(-1)^n - 1]$$

$$\begin{aligned}
 \therefore ① \Rightarrow (x+1)^2 &= \frac{14}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2} [2(-1)^n - 1] \cos n\pi x \\
 (x+1)^2 &= \frac{14}{6} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} [2(-1)^n - 1] \cos n\pi x
 \end{aligned}$$

3) To obtain the Fourier Series of $f(x) = \begin{cases} 1-x, & \text{for } 0 \leq x \leq l \\ 0, & \text{for } l \leq x \leq 2l \end{cases}$
Sol: Given $f(x) = \begin{cases} 1-x, & 0 \leq x \leq l \\ 0, & l \leq x \leq 2l \end{cases}$

The given function is neither even nor odd

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \rightarrow ①$$

$$\begin{aligned}
 a_0 &= \frac{1}{l} \int_0^{2l} f(x) \, dx = \frac{1}{l} \int_0^l f(x) \, dx = \frac{1}{l} \int_0^l (1-x) \, dx \\
 &= \frac{1}{l} \left[lx - \frac{x^2}{2} \right]_0^l = \frac{1}{l} \left[l^2 - \frac{l^2}{2} \right] = \frac{1}{l} \left[\frac{l^2}{2} \right] = \frac{l}{2}
 \end{aligned}$$

$$\begin{aligned}
 a_n &= \frac{1}{l} \int_0^l f(x) \cos \frac{n\pi x}{l} dx \\
 &= \frac{1}{l} \int_0^l (1-x) \cos \frac{n\pi x}{l} dx \\
 &= \frac{1}{l} \left\{ \frac{1}{n\pi} \left[(1-x) \sin \frac{n\pi x}{l} \right] \Big|_0^l - \int_0^l \left[(-1) \frac{1}{n\pi} \sin \frac{n\pi x}{l} \right] dx \right\} \\
 &= \frac{1}{l} \left\{ (0-0) + \frac{1}{n\pi} \int_0^l \sin \frac{n\pi x}{l} dx \right\} \\
 &= \frac{1}{l} \left\{ -\frac{l^2}{n^2\pi^2} \left[\cos \frac{n\pi x}{l} \right] \Big|_0^l \right\} \\
 &= \frac{1}{l} \left\{ -\frac{l^2}{n^2\pi^2} [(-1)^n - 1] \right\} \\
 a_n &= \frac{1}{n^2\pi^2} [1 - (-1)^n]
 \end{aligned}$$

$$\begin{aligned}
 b_n &= \frac{1}{l} \int_0^l (1-x) \sin \frac{n\pi x}{l} dx \\
 &= \frac{1}{l} \left\{ -\frac{1}{n\pi} \left[(1-x) \sin \frac{n\pi x}{l} \right] \Big|_0^l - 0 \right\} \\
 &= \frac{1}{l} \left\{ -\frac{1}{n\pi} [0 - l(1)] \right\} \\
 &= \frac{1}{l} \left\{ \frac{l^2}{n\pi} \right\}
 \end{aligned}$$

$$\underline{\underline{b_n = \frac{1}{n\pi}}}$$

$$\therefore \textcircled{1} \Rightarrow f(x) = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{1}{n^2\pi^2} [1 - (-1)^n] \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} \frac{1}{n\pi} \sin \frac{n\pi x}{l} \quad \underline{\underline{=}}$$

(4) a) The displacement (y in cms) of a machine part occurs due to the rotation of x radians is given below.

Rotation x (in radians)	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
Displacement y (in cms)	1.0	1.4	1.9	1.7	1.5	1.2	1.0

Expand y in terms of Fourier Series up to second harmonics.

x	y	$\cos x$	$\cos 2x$	$\sin x$	$\sin 2x$	$y \cos x$	$y \cos 2x$	$y \sin x$	$y \sin 2x$
0	1	1	1	0	0	1	1	0	0
60°	1.4	0.5	-0.5	0.866	0.866	0.7	-0.7	1.2124	1.2124
120°	1.9	-0.5	-0.5	0.866	-0.866	-0.95	-0.95	1.6454	-1.6454
180°	1.7	-1	1	0	0	-1.7	1.7	0	0
240°	1.5	-0.5	-0.5	-0.866	0.866	-0.75	-0.75	-1.299	1.299
300°	1.2	0.5	-0.5	-0.866	-0.866	0.6	-0.6	-1.0392	-1.0392
Total	8.7					-1.1	-0.3	0.5196	-0.1732

$$a_1 = \frac{2}{N} \sum y \cos x = \frac{2}{6} (-1.1) = -0.367 \quad a_0 = \frac{2}{N} \sum y = \frac{2}{6} (8.7) = 2.9$$

$$a_2 = \frac{2}{N} \sum y \cos 2x = \frac{2}{6} (-0.3) = -0.1$$

$$b_1 = \frac{2}{N} \sum y \sin x = \frac{2}{6} (0.5196) = 0.1732$$

$$b_2 = \frac{2}{N} \sum y \sin 2x = \frac{2}{6} (-0.1732) = -0.0577$$

Fourier Series upto second harmonics is given by

$$y = \frac{a_0}{2} + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x)$$

$$y = \frac{0.9}{2} + (-0.367 \cos x + 0.1732 \sin x) + (-0.1 \cos 2x - 0.0577 \sin 2x)$$

$$y = 1.45 + (-0.367 \cos x + 0.1732 \sin x) + (-0.1 \cos 2x - 0.0577 \sin 2x)$$

4c) Find the Fourier Series expansion of $f(x) = |x|$ in $-\pi \leq x \leq \pi$.

Hence deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots = \frac{\pi^2}{8}$

Soln: $f(x) = |x| = \begin{cases} -x, & -\pi \leq x \leq 0 \\ x, & 0 \leq x \leq \pi \end{cases}$

$$\phi(x) = -x, \quad \psi(x) = x$$

$$\phi(-x) = x = \psi(x)$$

$\therefore f(x)$ is an Even function $\Rightarrow b_n = 0$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{--- (1)}$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx = \frac{2}{\pi} \int_0^{\pi} x dx = \frac{2}{\pi} \left[\frac{x^2}{2} \right]_0^{\pi}$$

$$a_0 = \pi$$

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx \\ &= \frac{2}{\pi} \left\{ \int_0^{\pi} x \cos nx dx \right\} \\ &= \frac{2}{\pi} \left\{ x \int_0^{\pi} \cos nx dx - \int_0^{\pi} \left[\int_0^x \cos nx dx \right] dx \right\} \\ &= \frac{2}{\pi} \left\{ \frac{1}{n} \left[x \sin nx \right]_0^{\pi} + \frac{1}{n^2} \left[\cos nx \right]_0^{\pi} \right\} \\ &= \frac{2}{\pi} \left[\frac{1}{n^2} [\cos n\pi - 1] \right] \end{aligned}$$

$$a_n = \frac{2}{n^2\pi} [(-1)^n - 1]$$

$$\textcircled{1} \Rightarrow \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2} [(-1)^n - 1] \cos nx = \begin{cases} -x, & -\pi \leq x \leq 0 \\ x, & 0 \leq x \leq \pi \end{cases} \rightarrow \textcircled{2}$$

when $x = 0$

$$\textcircled{2} \Rightarrow \frac{\pi}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} = 0$$

$$\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} = -\frac{\pi}{2}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} = -\frac{\pi^2}{4}$$

$$\Rightarrow \frac{-2}{1^2} + 0 - \frac{2}{3^2} + 0 - \frac{2}{5^2} + \dots = -\frac{\pi^2}{4}$$

$$-2 \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots \right] = -\frac{\pi^2}{4}$$

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots = \frac{\pi^2}{8}$$

4b) Find the half-range sine series of e^x in the interval $0 \leq x \leq 1$.

$$\text{given } f(x) = e^x$$

The half range sine series is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \rightarrow \textcircled{1}$$

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx$$

$$= \frac{2}{1} \int_0^1 e^x \sin n\pi x dx$$

$$= 2 \left[\frac{e^x}{1+n^2\pi^2} [\sin n\pi x - n\pi \cos n\pi x] \right]_0^1$$

$$= 2 \left[\frac{e}{1+n^2\pi^2} [0 - n\pi(-1)^n] - \frac{1}{1+n^2\pi^2} [0 - n\pi] \right]$$

$$= \frac{2}{1+n^2\pi^2} \left[e^{(-1)^n + 1} n\pi \right] + n\pi$$

$$= \frac{2}{1+n^2\pi^2} \left[2.7183 n\pi (-1)^{n+1} + n\pi \right]$$

$$= \frac{2n\pi}{1+n^2\pi^2} [2.7183 + 1]$$

$$= \frac{2n\pi}{1+n^2\pi^2} (3.7183)$$

$$b_n = \frac{7.436}{1+n^2\pi^2} n\pi$$

$$\therefore ① \Rightarrow e^x = \sum_{n=1}^{\infty} \frac{7.436 n\pi}{1+n^2\pi^2} \sin n\pi x$$

3) If an alternating current $I(x)$ after passing through a rectifier has the form $I(x) = \begin{cases} I_0 \sin x, & \text{for } 0 \leq x < \pi \\ 0, & \text{for } \pi < x \leq 2\pi \end{cases}$,

where I_0 is the maximum current and the period is 2π . Express $I(x)$ as a Fourier Series.

Solⁿ: The Fourier Series of period of 2π is given by

$$I = f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\theta + \sum_{n=1}^{\infty} b_n \sin n\theta \rightarrow ①$$

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(\theta) d\theta$$

$$= \frac{I_0}{\pi} \int_0^{2\pi} \sin \theta d\theta = \frac{I_0}{\pi} \left[\int_0^{\pi} \sin \theta d\theta + \int_{\pi}^{2\pi} \sin \theta d\theta \right]$$

$$= \frac{2I_0}{\pi} \int_0^{\pi} \sin \theta d\theta$$

$$a_0 = \frac{2I_0}{\pi} [-\cos \theta]_0^\pi$$

$$a_0 = \frac{2I_0}{\pi}$$

$$a_n = \frac{1}{\pi} \int_0^{\pi} f(\theta) \cos n\theta d\theta$$

$$= \frac{I_0}{\pi} \left\{ \int_0^{\pi} \sin \theta \cos n\theta d\theta + \int_{-\pi}^{\pi} \sin \theta \cos n\theta d\theta \right\}$$

$$= \frac{I_0}{\pi} \int_0^{\pi} \sin \theta \cos n\theta d\theta$$

$$a_n = \frac{I_0}{\pi} \int_0^{\pi} \sin \theta \cos n\theta d\theta$$

Putting $n=1$ $a_1 = \frac{I_0}{\pi} \int_0^{\pi} \sin \theta \cos \theta d\theta$

$$= \frac{I_0}{\pi} \int_0^{\pi} \frac{\sin 2x}{2} dx$$

$$a_1 = \frac{I_0}{2\pi} \left[-\frac{\cos 2x}{2} \right]_0^\pi$$

$$= \frac{-I_0}{4\pi} (\cos 2\pi - \cos 0)$$

$$= \frac{-I_0}{4\pi} (1-1)$$

$$a_1 = 0$$

and

$$a_n = \frac{I_0}{\pi(n^2-1)} \{1+(-1)^n\} \text{ for } n \neq 1$$

$$b_n = \frac{I_0}{\pi} \int_0^{\pi} \sin \theta \sin n\theta d\theta$$

Put $n=1$

$$b_1 = \frac{I_0}{\pi} \int_0^{\pi} \sin \theta \sin \theta d\theta$$

$$= \frac{I_0}{\pi} \int_0^{\pi} \sin^2 \theta d\theta$$

$$b_1 = \frac{I_0}{\pi} \int_0^\pi \frac{1}{a} (1 - \cos 2\theta) d\theta$$

$$= \frac{I_0}{2\pi} \left[x - \frac{\sin ax}{a} \right]_0^\pi$$

$$= \frac{I_0}{2\pi} (\pi - 0)$$

$$b_1 = \frac{I_0}{a}$$

$$b_n = 0 \text{ for } n \neq 1$$

$$\therefore \textcircled{1} \Rightarrow I = f(\theta) = \frac{I_0}{\pi} + \sum_{n=2}^{\infty} \frac{-I_0}{\pi(n^2-1)} \{ 1 + (-1)^n \} \cos n\theta + \frac{I_0}{2} \sin \theta$$

b) Find the half-angle sine Series of $f(x) = \frac{\sinh ax}{\sinh a\pi}$

in the interval $(0, \pi)$

$$\underline{\text{Soln}}: \text{ given } f(x) = \frac{\sinh ax}{\sinh a\pi}, x \in (0, \pi)$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \rightarrow \textcircled{1}$$

$$\therefore b_n = \frac{2}{\pi} \int_0^\pi f(x) \sin nx dx$$

$$= \frac{2}{\pi} \int_0^\pi \frac{\sinh ax}{\sinh a\pi} \sin nx dx$$

$$= \frac{2}{\pi \sinh a\pi} \int_0^\pi \frac{e^{ax} - e^{-ax}}{2} \sin nx dx$$

$$= \frac{1}{\pi \sinh a\pi} \left\{ \int_0^\pi e^{ax} \sin nx dx - \int_0^\pi e^{-ax} \sin nx dx \right\}$$

$$\therefore \int_0^\pi e^{ax} \sin nx dx = \left[\frac{e^{ax}}{a^2 + n^2} [a \sin nx - n \cos nx] \right]_0^\pi$$

$$= \frac{e^{a\pi}}{a^2+n^2} [n(-1)^n] - \frac{e^0}{a^2+n^2} [0-n] = \frac{e^{a\pi}}{a^2+n^2} n(-1)^{n+1} + \frac{n}{a^2+n^2}$$

$$\begin{aligned} \int_0^\pi e^{-ax} \sin nx dx &= \left[\frac{e^{-ax}}{(-a)^2+n^2} [-a \sin nx - n \cos nx] \right]_0^\pi \\ &= \frac{e^{-a\pi}}{(-a)^2+n^2} [0-n(-1)] - \frac{e^0}{(-a)^2+n^2} [0-n] \\ &= \frac{e^{-a\pi}}{(-a)^2+n^2} [n(-1)^{n+1}] + \frac{n}{a^2+n^2} \end{aligned}$$

$$\therefore b_n = \frac{1}{\pi \sin h a \pi} \left\{ \frac{e^{a\pi}}{a^2+n^2} n(-1)^{n+1} + \frac{n}{a^2+n^2} - \frac{e^{-a\pi}}{a^2+n^2} [n(-1)^{n+1}] + \frac{n}{a^2+n^2} \right\}$$

$$\therefore f(x) = \sum_{n=1}^{\infty} b_n \cdot \sin \frac{nx\pi}{2}$$

c) Find the Fourier Series expansion of $f(x) = x(1-x)(2-x)$ in the interval $0 \leq x \leq 2$. Hence deduce the sum of the series that $\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \dots$

Sol :- $f(x) = x(1-x)(2-x) \quad 2x=2$

$$f(x) = x^3 - 3x^2 + 2x \quad \boxed{d=1}$$

$$f(2-x) = f(2-x) = (2-x)^3 - 3(2-x)^2 + 2(2-x)$$

$$\begin{aligned} x(1-x)(2-x) &= (2-x)(1-2+x)(2-x+x) \\ &= x(2-x)(-1+x) \\ &= -x(2-x)(1-x) \\ &= -f(x) \end{aligned}$$

4 a) In an electrical research laboratory, scientists have designed a generator which can generate the following currents at different time instant t , in the period T :

time t (in sec)	0	$T/6$	$T/3$	$T/2$	$2T/3$	$5T/6$	T
$f(t)$	1.98	1.30	1.05	1.30	-0.88	-0.25	1.98

Determine the direct current part and amplitude of the first harmonic from the above data.

Solⁿ:- Given the time t has been defined in the interval

$$0 \leq t \leq T \Rightarrow \omega t = T$$

$\omega = T/2$ and removing the last term

We have

t	A	$\theta = \frac{\omega \pi t}{T}$	$\cos \theta$	$A \cos \theta$	$\sin \theta$	$A \sin \theta$
0	1.98	0	1	1.98	0	0
$T/6$	1.30	60	0.5	0.65	0.8660	1.1258
$T/3$	1.05	120	-0.5	-0.5250	0.86660	0.9093
$T/2$	1.30	180	-1	-1.30	0	0
$2T/3$	-0.88	240	-0.5	0.44	-0.86660	0.7621
$5T/6$	-0.25	300	0.5	-0.1250	-0.8660	0.2165
ΣA				1.12		3.0137

$$a_0 = \frac{2}{N} \sum A = \frac{2}{6} (4.53) = 1.5100$$

$$\frac{a_0}{2} = \frac{1.5100}{2} = \underline{\underline{0.75A}}$$

$$a_1 = \frac{2}{N} \sum A \cos \theta = \frac{2}{6} (1.12) = 0.3733$$

$$b_1 = \frac{2}{N} \sum A \sin \theta = \frac{2}{6} (3.0137) = 1.0046$$

$$\therefore \text{Amplitude} = \sqrt{a_1^2 + b_1^2} = \sqrt{(0.3733)^2 + (1.0046)^2} = 1.0717$$

4c) obtain the fourier series of $f(x) = x(2\pi - x)$ valid in the interval $(0, 2\pi)$

$$\text{given } f(x) = x(2\pi - x), x \in (0, 2\pi)$$

$$f(2\pi - x) = 2\pi - x(2\pi - (2\pi - x))$$

$$= 2\pi - x [2\pi - 2\pi + x]$$

$$= x(2\pi - x)$$

$$= f(x)$$

$\therefore f(x)$ is an Even function $\Rightarrow b_n = 0$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \rightarrow ①$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x(2\pi - x) dx$$

$$= \frac{2}{\pi} \int_0^{\pi} 2\pi x - x^2 dx$$

$$= \frac{2}{\pi} \left[\pi x^2 - \frac{x^3}{3} \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\pi^3 - \frac{\pi^3}{3} \right]$$

$$a_0 = \frac{4}{3}\pi^2$$

$$\begin{aligned}
 a_n &= \frac{2}{\pi} \int_0^\pi f(x) \cos nx dx \\
 &= \frac{2}{\pi} \int_0^\pi (2\pi x - x^2) \cos nx dx \\
 &= \frac{2}{\pi} \left[(2\pi x - x^2) \int_0^\pi \cos nx dx - \int_0^\pi (2\pi - 2x) \int \cos nx dx \right] \\
 &= \frac{2}{\pi} \left[\frac{1}{n} (2\pi x - x^2) \sin nx \right]_0^\pi + \frac{2}{n} \int_0^\pi (x - \pi) \sin nx dx \\
 &= \frac{4}{n\pi} \int_0^\pi (x - \pi) \sin nx dx \\
 &= \frac{4}{n\pi} \left\{ (x - \pi) \int_0^\pi \sin nx dx - \int_0^\pi \left[\int \sin nx dx \right] dx \right\} \\
 &= \frac{4}{n\pi} \left[-\frac{1}{n} (x - \pi) \cos nx \right]_0^\pi + \frac{1}{n^2} [\sin nx]_0^\pi \\
 &= \frac{4}{n\pi} \left[-\frac{1}{n} (0 + \pi) (1) + \frac{1}{n^2} (0 - 0) \right] \\
 &= \frac{4}{n\pi} \left[-\frac{\pi}{n} \right]
 \end{aligned}$$

$$\boxed{a_n = -\frac{4}{n^2}}$$

$$\therefore \textcircled{1} \Rightarrow x(2\pi - x) = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty} \left(\frac{-4}{n^2} \right) \cos nx$$

$$2\pi x - x^2 = \frac{2\pi^2}{3} - 4 \sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx \rightarrow \textcircled{2}$$

when $x=0$

$$\textcircled{2} \Rightarrow 0 = \frac{2\pi^2}{3} - 4 \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$4 \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{2\pi^2}{3}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}$$

when $x = \pi$

$$② \Rightarrow 2\pi(\pi) - \pi^2 = \frac{2}{3}\pi^2 - 4 \sum_{n=1}^{\infty} \frac{1}{n^2} \cos n\pi$$

$$\pi^2 = \frac{2}{3}\pi^2 - 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$-4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = \pi^2 - \frac{2}{3}\pi^2 = \frac{\pi^2}{3}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$$

$$\Rightarrow -\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \dots = -\frac{\pi^2}{12}$$

$$\Rightarrow \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \underline{\underline{\frac{\pi^2}{12}}}$$

module - 4

7 a) Solve $\frac{dy}{dx} = e^x - y$, $y(0) = 1$ using Taylor's series method considering up to fourth degree terms and find the value of $y(0.1)$.

Solⁿ: $\frac{dy}{dx} = y' = e^x - y \rightarrow ①$

and $y(0) = 1$

$\Rightarrow x_0 = 0, y_0 = 1$

$$y'(x_0) = e^{x_0} - y_0 = e^0 - 1 = 1 - 1 = 0$$

$$y''(x) = e^x - y' \Rightarrow y''(x_0) = e^{x_0} - y'_0 = e^0 - 0 = 1 - 0 = 1$$

$$y'''(x) = e^x - y'' \Rightarrow y'''(x_0) = e^{x_0} - y''_0 = e^0 - (1) = 1 - 1 = 0$$

$$y''''(x) = e^x - y''' \Rightarrow y''''(x_0) = e^{x_0} - y'''_0 = e^0 - 0 = 1 - 0 = 1$$

WKT

$$y(x) = y(x_0) + \frac{(x-x_0)^1}{1!} y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \dots$$

$$y(x) = 1 + \frac{(x-0)}{1!}(0) + \frac{(x-0)^2}{2!}(1) + \frac{(x-0)^3}{3!}(0) + \frac{(x-0)^4}{4!}(1) + \dots$$

$$y(x) = 1 + \frac{x(0)}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

$$y(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots \rightarrow ②$$

At $x = 0.1$

$$② \Rightarrow y(0.1) = 1 + \frac{(0.1)^2}{2} + \frac{(0.1)^4}{24}$$

$$y(0.1) = 1 + 0.0050 + 0$$

$$y(0.1) \approx 1.0050$$

b) Use Runge Kutta method of fourth order to solve

$$(x+y) \frac{dy}{dx} = 1, \quad y(0.4) = 1, \text{ to find } y(0.5)$$

(Take $h = 0.1$).

Solⁿ: $(x+y) \frac{dy}{dx} = 1$

$$\frac{dy}{dx} = \frac{1}{x+y} = f(x, y) \rightarrow ①$$

and $y(0.4) = 1$

$$x_0 = 0.4, y_0 = 1, h = 0.1$$

$$K_1 = hf(x_0, y_0) = 0.1 f(0.4, 1) = \frac{0.1}{0.4+1} = 0.0714$$

$$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right) = 0.1 f\left(0.4 + \frac{0.1}{2}, 1 + \frac{0.0714}{2}\right)$$

$$= 0.1 f(0.45, 1.0357)$$

$$= 0.1 (0.6731)$$

$$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right) = 0.1 f\left(0.4 + \frac{0.1}{2}, 1 + \frac{0.0673}{2}\right)$$

$$= 0.1 f(0.45, 1.0336)$$

$$= 0.1 (0.6740)$$

$$K_3 = 0.06740$$

$$K_4 = hf(x_0 + h, y_0 + K_3) = 0.1 f(0.5, 1.0674)$$

$$= 0.1 (0.6379)$$

$$K_4 = 0.06379$$

WKT $y(x_0 + h) = y(x_1) = y_0 + \frac{1}{6} [K_1 + 2K_2 + 2K_3 + K_4]$

$$y(0.4 + 0.1) = y(0.5) = 1 + \frac{1}{6} [0.0714 + 2(0.0673) + 2(0.0674) + 0.06379]$$

$$y(0.5) \approx \underline{\underline{1.0674}}$$

c) Given that $\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}$ and $y(1) = 1$, $y(1.1) = 0.9960$,
 $y(1.2) = 0.9860$ & $y(1.3) = 0.9720$ find $y(1.4)$, using
 Adam - Bashforth predictor - corrector method.

Soln: Given $\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}$

$$\frac{dy}{dx} = \frac{1}{x^2} - \frac{y}{x}$$

$$\frac{dy}{dx} = \frac{1-x_1 y_1}{x_1^2} = f(x_1, y_1)$$

$$y(1) = 1 \Rightarrow x_0 = 1, y_0 = 1$$

$$x_1 = 1.1 \quad y_1 = 0.9960$$

$$x_2 = 1.2 \quad y_2 = 0.9860$$

$$x_3 = 1.3 \quad y_3 = 0.9720$$

$$x_4 = 1.4 \quad y_4 = ? \quad h = 0.1$$

$$f_0 = f(x_0, y_0) = f(1, 1) = \frac{1 - x_0 y_0}{x_0^2} = \frac{1 - (1)(1)}{1^2} = 0$$

$$f_1 = \frac{1 - x_1 y_1}{x_1^2} = \frac{1 - (1.1)(0.9960)}{(1.1)^2} = -0.0790$$

$$f_2 = \frac{1 - x_2 y_2}{x_2^2} = \frac{1 - (1.2)(0.9860)}{(1.2)^2} = -0.1272$$

$$f_3 = \frac{1 - x_3 y_3}{x_3^2} = \frac{1 - (1.3)(0.9720)}{(1.3)^2} = -0.1560$$

$$y_4^{(P)} = y_3 + \frac{h}{24} [55f_3 - 59f_2 + 37f_1 - 9f_0]$$

$$= 0.9720 + \frac{0.1}{24} [-8.5800 + 7.5048 - 2.9230 - 0]$$

$$y_4^{(P)} = 0.9553$$

$$\therefore f_4^{(p)} = \frac{1 - x_4 y_4^{(p)}}{x_4^2} = \frac{1 - (1.4)(0.9553)}{(1.4)^2} = -0.1721$$

$$y_4^{(c)} = y_3 + \frac{h}{24} [9f_4^{(p)} + 19f_3 - 5f_2 + f_1]$$

$$= 0.9720 + \frac{0.1}{24} [-1.5489 - 2.9640 + 0.6360 + 0.0790]$$

$$y_4^{(c)} = 0.9555$$

$$y(1.4) \cong \underline{0.9555}$$

8 a) Solve the differential equation $\frac{dy}{dx} = x\sqrt{y}$ under the initial condition $y(1)=1$, by using modified Euler's method at the point $x=1.4$. Perform three iterations at each step, taking $h=0.2$.

Solⁿ : Given $\frac{dy}{dx} = x\sqrt{y}$

$$y(1) = 1 \Rightarrow x_0 = 1 \quad y_0 = 1$$

$$h = 0.2 \quad x_1 = x_0 + h = 1 + 0.2 = 1.2$$

To find $y(x_1) = y_1$,

$$\therefore y(x_0 + h) = y_1^{(1)} = y(x_1) = y_0 + h f(x_0, y_0)$$

$$y_1^{(1)} = 1 + 0.2 f(1, 1)$$

$$= 1 + 0.2(1)$$

$$y_1^{(1)} = \underline{1.2000}$$

$$y_1^{(2)} = y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_1, y_1^{(1)})]$$

$$= 1 + \frac{0.2}{2} [f(1, 1) + f(1.2, 1.2)]$$

$$= 1 + 0.1 [1 + 1.3145]$$

$$y_1^{(2)} = 1.23145$$

$$\begin{aligned}
 y_1^{(3)} &= y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_1, y_1^{(2)})] \\
 &= 1 + \frac{0.2}{2} [f(1, 1) + f(1.2, 1.2314)] \\
 &= 1 + 0.1 [1 + 1.23162]
 \end{aligned}$$

$$y_1^{(3)} = 1.23316$$

$$\therefore x_1 \approx 1.2 \quad y_1 = 1.2331$$

To find $y(x_2) = y_2$

$$y(x_1+h) = y(1.2+0.2) = y(1.4)$$

$$\begin{aligned}
 y_2^{(1)} &= y_1 + h f(x_1, y_1) \\
 &= 1.2331 + 0.2 f(1.2, 1.2331) \\
 &= 1.2331 + 0.2 (1.33254)
 \end{aligned}$$

$$y_2^{(1)} = 1.4996$$

$$\begin{aligned}
 y_2^{(2)} &= y_1 + \frac{h}{2} [f(x_0, y_0) + f(x_1, y_1^{(1)})] \\
 &= 1.2331 + \frac{0.2}{2} [f(1, 1) + f(1.2, 1.2331)] \\
 &= 1.2331 + 0.1 [1.3325 + 1.7144]
 \end{aligned}$$

$$y_2^{(2)} = 1.53779$$

$$\begin{aligned}
 y_2^{(3)} &= y_1 + \frac{h}{2} [f(x_1, y_1) + f(x_2, y_2^{(2)})] \\
 &= 1.2331 + \frac{0.2}{2} [f(1.2, 1.2331) + f(1.4, 1.5377)] \\
 &= 1.2331 + 0.1 [1.3325 + 1.7360]
 \end{aligned}$$

$$y_2^{(3)} = 1.5399$$

$$\therefore y(1.4) \approx 1.5399$$

$x = 1.4, y = 1.5399$

by use fourth order Runge - kutta method , to find $y(0.1)$
with $h = 0.1$, given $\frac{dy}{dx} + y + xy^2 = 0$, $y(0) = 1$

given $\frac{dy}{dx} + y + xy^2 = 0$

$$\frac{dy}{dx} = -y - xy^2 = -(y + xy^2) = f(x, y)$$

$$y(0) = 1 \Rightarrow x_0 = 0, y_0 = 1 \quad h = 0.1$$

$$k_1 = h f(x_0, y_0) = 0.1 f(0, 1) = -0.1 (1) = -0.1$$

$$\begin{aligned} k_2 &= h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.1 f\left(\frac{0.1}{2}, 1 + \frac{-0.1}{2}\right) \\ &= 0.1 f(0.0500, 0.9500) \\ &= -0.1 \times 0.9951 \\ k_2 &= -0.0995 \end{aligned}$$

$$\begin{aligned} k_3 &= h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.1 f\left[0.0500, 1 - \frac{0.0995}{2}\right] \\ &= 0.1 f(0.0500, 0.9503) \\ &= 0.1 \times -0.9955 \end{aligned}$$

$$\begin{aligned} k_4 &= h f(x_0 + h, y_0 + k_3) = 0.1 f(0.1, 0.9005) = -0.1 \times 0.9816 = -0.0982 \\ \text{WKT} \quad y(x_0 + h) &= y(x_1) = y_0 + \frac{1}{6} [k_1 + 2k_2 + 2k_3 + k_4] \end{aligned}$$

$$\begin{aligned} y(x_1) &= 1 + \frac{1}{6} [-0.1 + 2(-0.0995) + 2(-0.0995) + (-0.0982)] \\ &= 1 + \frac{1}{6} [-0.5962] \\ &= 1 - 0.0994 \end{aligned}$$

$$y(x_1) \approx \underline{\underline{0.9006}}$$

c) Apply milne's predictor - corrector formulae to compute $y(0.3)$ given, $\frac{dy}{dx} = x + y^2$ with

x	0.0	0.1	0.2	0.3
y	1.0000	1.1000	1.2310	1.4020

Solⁿ: $\frac{dy}{dx} = f(x, y) = x + y^2$

$$x_0 = 0.0 \quad y_0 = 1.0000$$

$$x_1 = 0.1 \quad y_1 = 1.1000$$

$$x_2 = 0.2 \quad y_2 = 1.2310$$

$$x_3 = 0.3 \quad y_3 = 1.4020$$

$$x_4 = 0.4 \quad y_4 = ? \quad h = 0.1$$

$$\therefore f_0 = x_0 + y_0^2 = 0 + 1^2 = 1$$

$$f_1 = x_1 + y_1^2 = 0.1 + 1.1^2 = 1.3100$$

$$f_2 = x_2 + y_2^2 = 0.2 + (1.2310)^2 = 1.7153$$

$$f_3 = x_3 + y_3^2 = 0.3 + (1.4020)^2 = 2.2656$$

$$y_4^{(P)} = y(x_4) = y_0 + \frac{4h}{3} [2f_1 - f_2 + 2f_3]$$

$$= 1 + \frac{4(0.1)}{3} [2(1.31) - (1.7153) + 2(2.2656)]$$

$$y_4^{(P)} = 1.7247$$

$$f_4^{(P)} = x_4 + (y_4^{(P)})^2 = 0.4 + (1.7247)^2 = 3.3745$$

$$y_4^{(C)} = y_2 + \frac{h}{3} [f_2 + 4f_3 + f_4^{(P)}]$$

$$= 1.231 + \frac{0.1}{3} [1.7153 + 4(2.2656) + 3.3745]$$

$$y_4^{(C)} = 1.7027$$

$$y(x_4) = y(0.4) \cong \underline{1.7027}$$

To solve $\frac{dy}{dx} = x^3 + y$, $y(1) = 1$ using Taylor's series method considering up to fourth degree terms and find the $y(1.1)$.

Solⁿ: - $\frac{dy}{dx} = x^3 + y \quad y(1) = 1 \quad x_0 = 1 \quad y_0 = 1$

$$y' = x^3 + y \Rightarrow y'(x_0) = x_0^3 + y_0 = 1^3 + 1 = 2$$

$$y''(x) = 3x^2 + y' \Rightarrow y''(x_0) = 3x_0^2 + y'_0 = 3(1)^2 + 2 = 5$$

$$y'''(x) = 6x + y'' \Rightarrow y'''(x_0) = 6x_0 + y''_0 = 6(1) + 5 = 11$$

WKT $y''''(x) = 6 + y''' \Rightarrow y''''(x_0) = 6 + 11 = 17$

$$\begin{aligned} y(x) &= y(x_0) + \frac{(x-x_0)}{1!} y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + \\ &\quad \frac{(x-x_0)^3}{3!} y'''(x_0) + \frac{(x-x_0)^4}{4!} y''''(x_0) + \dots \\ &= 1 + \frac{(x-1)^2}{1} + \frac{(x-1)^2}{2}(5) + \frac{(x-1)^3}{6}(11) + \frac{(x-1)^4}{24}(17) + \dots \end{aligned}$$

$$y(1.1) = 1 + \frac{(0.1)^2}{1} + \frac{(0.1)^2}{2}(5) + \frac{11}{6}(0.1)^3 + \frac{(0.1)^4}{24}(17) + \dots$$

$$y(1.1) = \underline{\underline{1.22690}}$$

b) Use Runge-Kutta method of fourth order to solve

$$\frac{dy}{dx} = 3x + \frac{y}{2}, \quad y(0) = 1, \text{ to find } y(0.2). \text{ (Take } h = 0.2\text{)}$$

Solⁿ: $\frac{dy}{dx} = 3x + \frac{y}{2} = f(x, y) \rightarrow ①$

$$y(0) = 1$$

$$x_0 = 0 \quad y_0 = 1, \quad h = 0.2$$

$$k_1 = h + (x_0, y_0) = 0.2 + (0, 1) = 0.2(0.5) = 0.1$$

$$k_2 = h + \left(x_0 + \frac{k_1}{2}, y_0 + \frac{k_1}{2} \right) = 0.2 + (0.1, 1.05) = 0.2(0.825)$$

$$k_2 = 0.165$$

$$k_3 = h + \left(x_0 + \frac{3k_2}{2}, y_0 + \frac{3k_2}{2} \right) = 0.2 + (0.1, 1.0825)$$

$$= 0.2 \times 0.84125$$

$$k_3 = 0.1682$$

$$k_4 = h + (x_0 + h, y_0 + k_3) = 0.2 + (0.2, 1.1682)$$

$$= 0.2 \times 1.1841$$

$$k_4 = 0.23682$$

WKT

$$y(x_1) = y_0 + \frac{1}{6} [k_1 + 2k_2 + 2k_3 + k_4]$$

$$= 1 + \frac{1}{6} [0.1 + 2 \times 0.165 + 2 \times 0.1682 + 0.23682]$$

$$= 1 + 0.1672$$

$$y(x_1) \approx \underline{\underline{1.1672}}$$

or Given $\frac{dy}{dx} = x^2(1+y)$ and $y(1) = 1, y(1.1) = 1.2330,$

$y(1.2) = 1.5480$ & $y(1.3) = 1.9790$ find $y(1.4),$

using Adam - Bashforth predictor - corrector method.

Solⁿ: $\frac{dy}{dx} = x^2(1+y)$

$$y(1) = 1 \Rightarrow x_0 = 1, y_0 = 1$$

$$y(1.1) = 1.233 \Rightarrow x_1 = 1.1, y_1 = 1.233$$

$$y(1.2) = 1.548 \Rightarrow x_2 = 1.2, y_2 = 1.548$$

$$y(1.3) = 1.979 \Rightarrow x_3 = 1.3, y_3 = 1.979$$

$$y(1.4) = ? \quad h=0.1$$

$$f_0 = f(x_0, y_0) = f(1, 1) = 2$$

$$f_1 = f(x_1, y_1) = f(1.1, 1.233) = 2.7019$$

$$f_2 = f(x_2, y_2) = f(1.2, 1.548) = 3.6691$$

$$f_3 = f(x_3, y_3) = f(1.3, 1.979) = 5.0345$$

WKT

$$y_4^{(P)} = y_3 + \frac{h}{24} [55f_3 - 59f_2 + 37f_1 - 9f_0]$$

$$= 1.979 + \frac{0.1}{24} [276.8975 - 216.4769 + 99.9703 - 18]$$

$$y_4^{(P)} = 2.5723$$

$$f_4^{(P)} = f(x_4, y_4^{(P)}) = f(1.4, 2.5723)$$

$$= 7.0017$$

$$y_4^{(C)} = y_3 + \frac{h}{24} [9f_4^{(P)} + 19f_3 - 5f_2 + f_1]$$

$$= 1.979 + \frac{0.1}{24} [9(2.5723) + 19(5.0345) - 5(3.6691)]$$

$$+ 2.7019$$

$$= 1.979 + 0.0042 [103.1626]$$

$$y(x_4) = y(1.4) \approx \underline{\underline{2.5123}}$$

8a) Use modified Euler's method to compute $y(0.2)$,

given $\frac{dy}{dx} - xy^2 = 0$ under the initial condition

$y(0) = 2$. perform three iterations at each step,

taking $h = 0.1$.

Solⁿ: given $\frac{dy}{dx} = xy^2 = f(x, y)$

$$y(0) = 2 \Rightarrow x_0 = 0, y_0 = 2$$

$$h = 0.1$$

To find $y(x_1) = y_1$

$$\begin{aligned} y(x_0+h) &= y(x_1) = y_1^{(1)} = y_0 + h f(x_0, y_0) \\ &= 2 + 0.1 f(0, 2) \\ &= 2 + 0.1(0) \end{aligned}$$

$$y_1^{(1)} = 2$$

$$\begin{aligned} y_1^{(2)} &= y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_1, y_1^{(1)})] \\ &= 2 + \frac{0.1}{2} [f(0, 2) + f(0.1, 2)] \\ &= 2 + 0.05 [0 + 0.4] \end{aligned}$$

$$y_1^{(2)} = 2.020$$

$$\begin{aligned} y_1^{(3)} &= y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_1, y_1^{(2)})] \\ &= 2 + \frac{0.1}{2} [f(0, 2) + f(0.1, 2.020)] \\ &= 2 + 0.05 [0 + 0.40804] \end{aligned}$$

$$y_1^{(3)} = 2.0204$$

$$y(0.1) \approx 2.0204$$

$$\Rightarrow x_1 = 0.1 \quad y_1 = 2.0204$$

To find $y(x_2) = y_2$

$$\therefore y(x_1+h) = y(x_2) = y_2^{(1)} = y_1 + h f(x_1, y_1)$$

$$y_2^{(1)} = 2.0204 + \frac{0.1}{2} f(0.1, 2.0204)$$

$$y_2^{(1)} = 2.0612$$

$$\begin{aligned}
 y_2^{(2)} &= y_1 + \frac{h}{2} [f(x_1, y_1) + f(x_2, y_2^{(1)})] \\
 &= 2.0204 + \frac{0.1}{2} [f(0.1, 2.0204) + f(0.2, 2.0833)] \\
 &= 2.0204 + 0.05 [0.4082 + 0.8497]
 \end{aligned}$$

$$y_2^{(2)} = 2.0833$$

$$\begin{aligned}
 y_2^{(3)} &= y_1 + \frac{h}{2} [f(x_1, y_1) + f(x_2, y_2^{(2)})] \\
 &= 2.0204 + \frac{0.1}{2} [f(0.1, 2.0204) + f(0.2, 2.0833)] \\
 &= 2.0204 + 0.05 [0.4082 + 0.8680]
 \end{aligned}$$

$$y_2^{(3)} = 2.0842$$

$$y(0.2) \cong \underline{2.0842}$$

b) Use fourth order Runge - kutta method, to find
 $y(0.2)$ with $h = 0.1$, given $\frac{dy}{dx} = \sqrt{x+y}$,
 $y(0) = 1$.

$$\text{Sol}^n: \quad \frac{dy}{dx} = \sqrt{x+y} = f(x, y)$$

$$\begin{aligned}
 y(0) &= 1 \\
 \Rightarrow x_0 &= 0, y_0 = 1
 \end{aligned}$$

$$h = 0.1$$

$$k_1 = h f(x_0, y_0) = 0.1 f(0, 1) = 0.1 (1) = 0.1$$

$$\begin{aligned}
 k_2 &= h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.1 f(0.1, 1 + \frac{0.1}{2}) \\
 &= 0.1 f(0.1, 1.1) \\
 &= 0.1 \times 1.0954 \\
 &= 0.10954
 \end{aligned}$$

$$k_3 = h + (x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}) = 0.2 + (0.1, 1.1096) \\ = 0.2 \times 1.0998 \\ k_3 = 0.2900$$

$$k_4 = h + (x_0 + h, y_0 + k_3) = 0.2 + (0.2, 1.2200) \\ = 0.2 \times 1.1916 \\ k_4 = 0.2383$$

WKP

$$y(x_0 + h) = y(x_1) = y_0 + \frac{1}{6} [k_1 + 2k_2 + 2k_3 + k_4] \\ y(0+0.2) = 1 + \frac{1}{6} [0.2 + 2(0.2191) + 2(0.2200) + 0.2383]$$

$$y(0.2) = 1 + \frac{1}{6} (1.3165) \\ = 1 + 0.2194$$

$$y(0.2) = y(x_1) \approx \underline{\underline{1.2194}}$$

or Apply milne's predictor - corrector formulae
to compute $y(2.0)$ given $\frac{dy}{dx} = \frac{1}{2}(x+y)$ with

x	0.0	0.5	1.0	1.5
y	2.0000	2.6360	3.5950	4.9680

Soln: given $\frac{dy}{dx} = \frac{1}{2}(x+y) = f(x, y)$

and $x_0 = 0 \quad y_0 = 2$

$$x_1 = 0.5 \quad y_1 = 2.6360$$

$$x_2 = 1.0 \quad y_2 = 3.5950$$

$$x_3 = 1.5 \quad y_3 = 4.9680$$

$$x_4 = 2 \quad y_4 = ? \quad h = 0.5$$

$$f_0 = \frac{x_0 + y_0}{2} = \frac{0 + 2}{2} = 1$$

$$f_1 = \frac{x_1 + y_1}{2} = \frac{0.5 + 2.6360}{2} = 1.5680$$

$$f_2 = \frac{x_2 + y_2}{2} = \frac{1 + 3.5950}{2} = 2.2975$$

$$f_3 = \frac{x_3 + y_3}{2} = \frac{1.5 + 4.9680}{2} = 3.2340$$

$$y_4^{(P)} = y_0 + \frac{4h}{3} [2f_1 - f_2 + 2f_3]$$

$$= 2 + \frac{4(0.5)}{3} [3.1360 - 2.2975 + 6.4680]$$

$$y_4^{(P)} = 6.8710$$

$$f_4^{(P)} = \frac{x_4 + y_4^{(P)}}{2} = \frac{2 + 6.8710}{2} = 4.4355$$

$$y_4^{(C)} = y_2 + \frac{h}{3} [f_2 + 4f_3 + f_4^{(P)}]$$

$$= 3.5950 + \frac{0.5}{3} [2.2975 + 12.9360 + 4.4355]$$

$$y_4^{(C)} = 6.8731$$

$$\Rightarrow y(2.0) \approx \underline{\underline{6.8731}}$$

Module - 05

Q) Solve $\frac{d^2y}{dx^2} - x^2 \frac{dy}{dx} - 2xy = 1$, for $x=0.1$, correct to four decimal places, using initial conditions $y(0)=1$,

$y'(0)=0$, using Runge-Kutta method.

Solⁿ: Given $\frac{d^2y}{dx^2} - x^2 \frac{dy}{dx} - 2xy = 1 \rightarrow (1)$

$$\frac{dy}{dx} = y' = z = f(x, y, z)$$

$$\Rightarrow \frac{dz}{dx} - x^2 z - 2xy = 1$$

$$\frac{dz}{dx} = 1 + 2xy + x^2 z = g(x, y, z)$$

and $y(0)=1$, $y'(0)=0$

$$x_0=0, y_0=1, z_0=0=y'_0$$

but $h=0.1$

$$k_1 = h f(x_0, y_0, z_0) = 0.1 f(0, 1, 0) = 0.1(0) = 0$$

$$l_1 = h g(x_0, y_0, z_0) = 0.1 g(0, 1, 0) = 0.1(1) = 0.1$$

$$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$k_2 = 0.1 f(0.05, 1, 0.05) = (0.1)(0.05) = 0.005$$

$$l_2 = h g\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$l_2 = 0.1 g(0.05, 1, 0.05) = (0.1)(1.1) = 0.11$$

$$k_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$k_3 = 0.1 f(0.05, 1.0025, 0.055)$$

$$k_3 = 0.1(0.055) = 0.0055$$

$$l_3 = 0.1 g(0.05, 1.0025, 0.055) = (0.1)(1.1)$$

$$l_3 = 0.11$$

$$\begin{aligned}
 k_4 &= h + (x_0 + h, y_0 + k_3, z_0 + l_3) \\
 &= 0.1 + (0.1, 1.0055, 0.11) \\
 &= 0.1 (0.11) \\
 &= 0.011
 \end{aligned}$$

WKT $y(x_1) = y_0 + \frac{1}{6} [k_1 + 2k_2 + 2k_3 + k_4]$

$$y(0.1) = 1 + \frac{1}{6} [0 + 0.01 + 0.011 + 0.011]$$

$$y(0.1) \cong \underline{\underline{1.0053}}$$

Q6) Find the extremal of the function $\int (y'^2 - y^2 - ye^{2x}) dx$
that passes through the points $(0,0)$ and $(1, \frac{1}{e})$

Soln:- Let $I = \int_{x_1}^{x_2} f(x, y, y') dx = \int_0^1 (y'^2 - y^2 - ye^{2x}) dx$

$$f(x, y, y') = y'^2 - y^2 - ye^{2x}$$

$$\frac{\partial f}{\partial y} = -2y - e^{2x}$$

$$\frac{\partial f}{\partial y'} = 2y'$$

WKT $\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) = 0$

$$-2y - e^{2x} - \frac{d}{dx} (2y') = 0$$

÷ by ②

$$-y - \frac{1}{2} e^{2x} - y'' = 0$$

$$y'' + y = -\frac{1}{2} e^{2x}$$

$$(D^2 + 1)y = -\frac{1}{2} e^{2x}$$

The A.E is $m^2 + 1 = 0$

$$m = 0 \pm i$$

$$y_c = C_1 \cos x + C_2 \sin x$$

$$y_p = -\frac{1}{2} \frac{e^{2x}}{D^2 + 1}$$

$$= -\frac{1}{2} \frac{e^{2x}}{D^2 + 1}$$

$$= -\frac{1}{2} \frac{e^{2x}}{4+1}$$

$$y_p = -\frac{e^{2x}}{10}$$

$$\therefore y = y_c + y_p$$

$$y = C_1 \cos x + C_2 \sin x - \frac{e^{2x}}{10} \rightarrow ①$$

$$\text{When } x=0 \Rightarrow y=0$$

$$① \Rightarrow 0 = C_1(1) + 0 - \frac{1}{10}$$

$$C_1 = \frac{1}{10} = 0.1$$

$$\text{when } x=1 \Rightarrow y=y_e$$

$$y_e = C_1 \cos 1 + C_2 \sin 1 - \frac{e^2}{10}$$

$$0.3679 = (0.1)(0.5403) + C_2(0.8415) - 0.7389$$

$$C_2 = \frac{1.05277}{0.8415}$$

$$C_2 = 1.2510$$

$$\therefore ① \Rightarrow y = 0.1 \cos x + 1.2510 \sin x - \frac{e^{2x}}{10}$$

c) A heavy cable hangs freely under gravity at two fixed points. Show that the shape of the cable is catenary.

Solⁿ: let P(x₁, y₁) and Q(x₂, y₂) be the two fixed points of the hanging cable.

let us consider an elementary arc length ds of the cable. let ρ be the density (mass/unit length) of the cable so that ρ is the mass of the element.

If g is the acceleration due to gravity then the potential energy of the element ($m \cdot g \cdot h$) is given by $(\rho ds) \cdot g \cdot y$ where x -axis is taken as the line of reference.

\therefore Total potential Energy of the cable is given by

$$T = \int_P^Q (\rho ds) \cdot g y \, dx = \int_{x_1}^{x_2} \rho g y \frac{ds}{dx} \, dx$$

$$\text{But, } \frac{ds}{dx} = \sqrt{1+y'^2}$$

$$\text{Here, } f(x, y, y') = (\rho g) y \sqrt{1+y'^2} = \text{const. } \cdot g \sqrt{1+y'^2}$$

$$\therefore \text{Euler's Equation } f - y' \frac{\partial f}{\partial y'} = \text{constant}$$

$$y \sqrt{1+y'^2} - y' \cdot \frac{y}{2\sqrt{1+y'^2}} \cdot 2y' = c$$

$$\frac{y(1+y'^2) - yy'^2}{\sqrt{1+y'^2}} = c \quad \text{or} \quad c = \frac{y}{\sqrt{1+y'^2}}$$

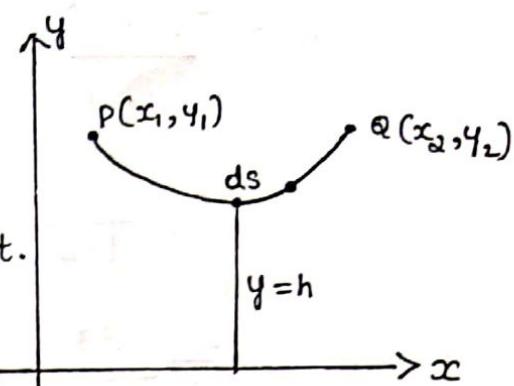
$$y^2 = c^2(1+y'^2) \quad \text{or}$$

$$y'^2 = \frac{y^2 - c^2}{c^2}$$

$$y' = \frac{dy}{dx} = \frac{\sqrt{y^2 - c^2}}{c}$$

$$\text{or} \quad \frac{dy}{\sqrt{y^2 - c^2}} = \frac{1}{c} dx$$

$$\int \frac{dy}{\sqrt{y^2 - c^2}} = \frac{1}{c} \int dx + k$$



$$\cosh^{-1}(\gamma/c) = \frac{x}{c} + k$$

$$\gamma/c = \cosh\left(\frac{x}{c} + k\right)$$

$$\therefore \boxed{y = c \cosh\left(\frac{x+a}{c}\right)}$$

where $a = kc$. This is a catenary and it can be proved that this corresponds to the minimum value of T .

- 10 a) Apply milne's predictor - corrector method to compute $y(0.4)$ given the differential equation $\frac{d^2y}{dx^2} = 1 + \frac{dy}{dx}$ and the following table of initial values:

x	0	0.1	0.2	0.3
y	1	1.1103	1.2427	1.3990
y'	1	1.2103	1.4427	1.6990

Given $\frac{d^2y}{dx^2} = 1 + \frac{dy}{dx} \rightarrow ①$

let $\frac{dy}{dx} = y' = z = f(x, y, z)$

$$① \Rightarrow \frac{dz}{dx} = 1 + z = g(x, y, z)$$

and given

$$x_0 = 0 \quad y_0 = 1 \quad y'_0 = z_0 = 1$$

$$x_1 = 0.1 \quad y_1 = 1.1103 \quad y'_1 = z_1 = 1.2103$$

$$x_2 = 0.2 \quad y_2 = 1.2427 \quad y'_2 = z_2 = 1.4427$$

$$x_3 = 0.3 \quad y_3 = 1.3990 \quad y'_3 = z_3 = 1.6990$$

$$f_1 = f(x_1, y_1, z_1) = z_1 = 1.2103$$

$$f_2 = f(x_2, y_2, z_2) = z_2 = 1.4427$$

$$f_3 = f(x_3, y_3, z_3) = z_3 = 1.6990$$

$$q_1 = q(x_1, y_1, z_1) = 1 + z_1 = 2.2103$$

$$q_2 = q(x_2, y_2, z_2) = 1 + z_2 = 2.4427$$

$$q_3 = q(x_3, y_3, z_3) = 1 + z_3 = 2.6990$$

$$\begin{aligned} y^P(x_4) &= y_0 + \frac{4h}{3} [2f_1 - f_2 + 2f_3] \\ &= 1 + \frac{4(0.1)}{3} [2.4206 - 1.4427 + 3.3980] \end{aligned}$$

$$y_4^{(P)} = 1.5835$$

$$\begin{aligned} z_4^{(P)} &= z_0 + \frac{4h}{3} [2g_1 - g_2 + 2g_3] \\ &= 1 + \frac{4(0.1)}{3} [4.4206 - 2.4427 + 5.3980] \end{aligned}$$

$$z_4^{(P)} = 1.9834$$

$$f_4^{(P)} = f(x_4, y_4^{(P)}, z_4^{(P)}) = z_4^{(P)} = 1.9834$$

$$y_4^{(C)} = y_2 + \frac{h}{3} [f_2 + 4f_3 + f_4^{(P)}]$$

$$y(0.4) = 1.2427 + \frac{0.1}{3} [1.4427 + 6.7900 + 1.9834]$$

$$y(0.4) \cong \underline{1.5834}$$

b) Derive Euler's Equation in the standard form

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \left[\frac{\partial f}{\partial y'} \right] = 0$$

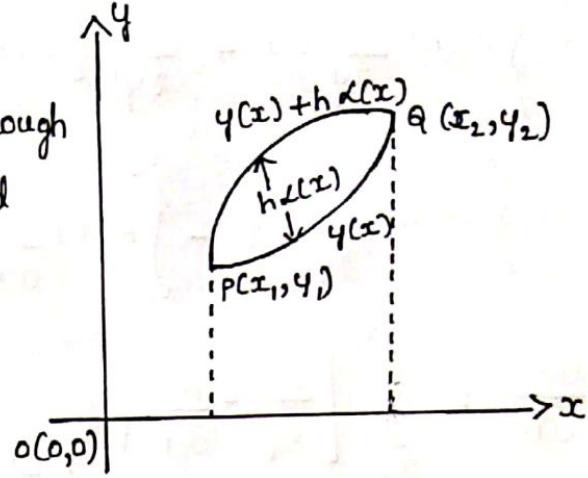
Solⁿ: Euler's Equation : A Necessary condition for the integral

$I = \int_{x_1}^{x_2} f(x, y, y') dx$, where $y(x_1) = y_1$, $y(x_2) = y_2$ to be an

Extremum is $\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) = 0$

Proof :-

Let the curve $y = y(x)$ passing through the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ and make I an extremum. Also let $y = y(x) + h\alpha(x)$ be the neighbouring curve passing through $P(x_1, y_1)$ & $Q(x_2, y_2)$ be an extremum.



The curves both are coincide at P & Q that implies

$$\alpha(x_1) = 0, \alpha(x_2) = 0$$

$$\text{Given } I = \int_{x_1}^{x_2} f(x, y, y') dx \quad \rightarrow ①$$

$$I = \int_{x_1}^{x_2} f\left(x, \frac{y}{y(x) + h\alpha(x)}, \frac{y'}{y'(x) + h\alpha'(x)}\right) dx$$

$$\frac{dI}{dh} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial x} \frac{\partial x}{\partial h} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial h} + \frac{\partial f}{\partial y'} \frac{\partial y'}{\partial h} \right] dx$$

$$= \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial x}(0) + \frac{\partial f}{\partial y} \alpha(x) + \frac{\partial f}{\partial y'} \alpha'(x) \right] dx$$

$$= \int_{x_1}^{x_2} \frac{\partial f}{\partial y} \alpha(x) + \frac{\partial f}{\partial y'} \alpha'(x) dx$$

$$= \int_{x_1}^{x_2} \alpha(x) \frac{\partial f}{\partial y} dx + \int_{x_1}^{x_2} \frac{\partial f}{\partial y'} \alpha'(x) dx$$

$$= \int_{x_1}^{x_2} \alpha(x) \frac{\partial f}{\partial y} dx + \frac{\partial f}{\partial y'} \int_{x_1}^{x_2} \alpha'(x) dx - \int_{x_1}^{x_2} \left[\frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) \int \alpha'(x) dx \right] dx$$

$$= \int_{x_1}^{x_2} \alpha(x) \frac{\partial f}{\partial y} dx + \frac{\partial f}{\partial y} \left[\alpha(x) \right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx} \left[\frac{\partial f}{\partial y'} \right] \alpha(x) dx$$

$$\begin{aligned}
 &= \int_{x_1}^{x_2} L(x) \frac{\partial f}{\partial y} dx + \frac{\partial f}{\partial y'} [L(x_2) - L(x_1)] - \int_{x_1}^{x_2} L(x) \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) dx \\
 &= \int_{x_1}^{x_2} L(x) \frac{\partial f}{\partial y} dx - \int_{x_1}^{x_2} L(x) \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) dx \\
 \frac{dI}{dh} &= \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) \right] L(x) dx
 \end{aligned}$$

For the Extremum of I , then $\frac{dI}{dh} = 0$

$$\Rightarrow \boxed{\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) = 0}$$

c) Find the extremal for the functional $\int_0^{\pi/2} (y^2 - y'^2 - 2ys \sin x) dx$
 $y(0) = 0, \quad y(\pi/2) = 1.$

Soln:- $I = \int_{x_1}^{x_2} f(x, y, y') dx = \int_0^{\pi/2} (y^2 - y'^2 - 2ys \sin x) dx$
 $f(x, y, y') = y^2 - y'^2 - 2ys \sin x$

WKT for the Extremum of I

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) = 0$$

$$(2y - 2s \sin x) - \frac{d}{dx} (-2y') = 0$$

$$(2y - 2s \sin x) - \frac{d}{dx} \left(\frac{dy}{dx} \right) = 0$$

$$y - s \sin x + \frac{d^2 y}{dx^2} = 0$$

$$\frac{d^2 y}{dx^2} + y = s \sin x$$

$$(D^2 + 1)y = \sin x \text{ where } D = \frac{d}{dx}$$

$$\text{A.E } m^2 + 1 = 0$$

$$m^2 = -1$$

$$m = 0 \pm i$$

$$y_c = C_1 \cos x + C_2 \sin x$$

$$y_p = \frac{\sin x}{D^2 + 1}$$

$$\frac{\sin ax}{D^2 + a^2} = \frac{-x}{a^2} \cos ax$$

$$= \frac{-x}{2(1)} \cos x$$

$$y_p = \frac{-x}{2} \cos x$$

$$y = y_c + y_p$$

$$y = C_1 \cos x + C_2 \sin x - \frac{x}{2} \cos x \rightarrow ①$$

$$\text{when } x=0 \Rightarrow y=0$$

$$① \Rightarrow 0 = C_1(1)$$

$$C_1 = 0$$

$$\text{when } x = \frac{\pi}{2} \Rightarrow y=1$$

$$① \Rightarrow 1 = C_1(0) + C_2(1)$$

$$C_2 = 1$$

$$① \Rightarrow y = \sin x - \frac{x}{2} \cos x$$

q a) Using Runge - kutta method , solve $\frac{d^2y}{dx^2} = x \left(\frac{dy}{dx} \right)^2 - y^2$, for $x=0.2$, correct to four decimal places , using initial conditions $y(0)=1$, $y'(0)=0$

$$\text{Sol'n: given } \frac{d^2y}{dx^2} = x \left(\frac{dy}{dx} \right)^2 - y^2$$

$$y'' = xy'^2 - y^2$$

$$\frac{dy}{dx} = y' = z = f(x, y, z)$$

$$\frac{dz}{dx} = xz^2 - y^2 = g(x, y, z)$$

$$x_0 = 0, y_0 = 1, y'(0) = z_0 = 0 \quad h = 0.2$$

$$k_1 = h f(x_0, y_0, z_0) = 0.2 f(0, 1, 0) = 0$$

$$l_1 = h g(x_0, y_0, z_0) = 0.2 g(0, 1, 0) = -0.2$$

$$k_2 = h f(x_1, y_1, z_1) = 0.2 f(0.1, 1, -0.1) = -0.02$$

$$l_2 = h g(x_1, y_1, z_1) = 0.2 g(0.1, 1, -0.1) = -1.00$$

$$k_3 = h f(x_2, y_2, z_2) = 0.2 f(0.2, 0.99, -0.5) = -0.1$$

$$l_3 = h g(x_2, y_2, z_2) = 0.2 g(0.2, 0.99, -0.5) = -0.9551$$

$$k_4 = h f(x_3, y_3, z_3) = 0.2 f(0.2, 0.95, -0.9551) \\ = 0.2 (-0.9551)$$

$$k_4 = -0.1910$$

$$y(x_1) = y_0 + \frac{1}{6} [k_1 + 2k_2 + 2k_3 + k_4]$$

$$y(0.2) = 1 + \frac{1}{6} [0 + 2(-0.02) + 2(-0.1) + (-0.1910)] \\ = 1 - 0.0718$$

$$y(0.2) = \underline{\underline{0.9282}}$$

b) Derive Euler's equation in the standard form viz.,

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) = 0$$

c) Find the extremal of the functional $\int_0^{x_2} (y'^2 - y^2 + 4y \cos x) dx$;

$$y(0) = 0 = y(\pi/2)$$

$$\text{let } I = \int_{x_1}^{x_2} f(x, y, y') dx = \int_0^{\pi/2} (y'^2 - y^2 + 4y \cos x) dx$$

$$\therefore f(x, y, y') = y'^2 - y^2 + 4y \cos x$$

$$\frac{\partial f}{\partial y'} = 0 - 2y + 4 \cos x = -2y + 4 \cos x$$

$$\frac{\partial f}{\partial y'} = 2y'$$

WKT

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) = 0$$

$$-2y + 4 \cos x - \frac{d}{dx}(2y') = 0$$

$$-y + 2 \cos x - y'' = 0$$

$$\Rightarrow y'' + y = 2 \cos x$$

$$(D^2 + 1)y = 2 \cos x$$

$$\text{The A.E is } m^2 + 1 = 0$$

$$m^2 = -1$$

$$m = 0 \pm i$$

$$\therefore y_c = C_1 \cos x + C_2 \sin x$$

$$y_p = \frac{2 \cos x}{D^2 + 1}$$

$$= \frac{2x}{2(1)} \sin x$$

$$y_p = x \sin x$$

$$\therefore y = y_c + y_p$$

$$y = C_1 \cos x + C_2 \sin x + x \sin x \rightarrow ①$$

$$\text{When } x=0 \Rightarrow y=0$$

$$① \Rightarrow 0 = C_1(1) + 0 + 0$$

$$\boxed{C_1 = 0}$$

when $x = \pi/2 \Rightarrow y = 0$

$$\textcircled{1} \Rightarrow 0 = c_1(0) + c_2(1) + \pi/2(1)$$

$$c_2 = -\pi/2$$

$$\therefore y = -\pi/2 \sin x + x \sin x$$

(Q a) Given the differential equation $2 \frac{d^2y}{dx^2} - \frac{dy}{dx} - 4x = 0$ and the following table of initial value:

x	1	1.1	1.2	1.3
y	2	2.2156	2.4649	2.7914
y'	2	2.3178	2.6725	2.0657

Compute $y(1.4)$ by applying milne's predictor-corrector method.

Solⁿ: Given $2 \frac{d^2y}{dx^2} - \frac{dy}{dx} - 4x = 0 \rightarrow \textcircled{1}$

let $\frac{dy}{dx} = z = f(x, y, z)$

$$\textcircled{1} \Rightarrow 2 \frac{dz}{dx} - z - 4x = 0$$

$$\frac{dz}{dx} = \frac{1}{2}(z + 4x) = g(x, y, z)$$

$$x_0 = 1$$

$$y_0 = 2$$

$$y'_0 = z_0 = 2$$

$$x_1 = 1.1$$

$$y_1 = 2.2156$$

$$y'_1 = z_1 = 2.3178$$

$$x_2 = 1.2$$

$$y_2 = 2.4649$$

$$y'_2 = z_2 = 2.6725$$

$$x_3 = 1.3$$

$$y_3 = 2.7914$$

$$y'_3 = z_3 = 2.0657$$

$$f_1 = f(x_1, y_1, z_1) = f(1.1, 2.2156, 2.3178) = 2.3178$$

$$f_2 = f(x_2, y_2, z_2) = f(1.2, 2.4649, 2.6725) = 2.6725$$

$$f_3 = f(x_3, y_3, z_3) = f(1.3, 2.7914, 2.0657) = 2.0657$$

$$q_1 = g(x_1, y_1, z_1) = g(1.1, 2.6356, 2.3178) = 3.3589$$

$$q_2 = g(x_2, y_2, z_2) = g(1.2, 2.4649, 2.6725) = 3.7363$$

$$q_3 = g(x_3, y_3, z_3) = g(1.3, 2.7914, 2.0657) = 3.6329$$

$$\begin{aligned} y^P(x_4) &= y_0 + \frac{4h}{3} [2f_1 - f_2 + 2f_3] \\ &= 2 + \frac{4(0.1)}{3} [4.6356 - 2.6725 + 4.1314] \end{aligned}$$

$$y_4^{(P)} = 2.8126$$

$$\begin{aligned} z_4^{(P)} &= z_0 + \frac{4h}{3} [2g_1 - g_2 + 2g_3] \\ &= 2 + \frac{4(0.1)}{3} [6.7178 - 3.7363 + 7.2658] \end{aligned}$$

$$z_4^{(P)} = 3.3660$$

$$f_4^{(P)} = +(x_4, y_4^{(P)}, z_4^{(P)}) = z_4^{(P)} = 3.3660$$

$$\begin{aligned} y_4^{(C)} &= y_2 + \frac{h}{3} [f_2 + 4f_3 + f_4^{(P)}] \\ &= 2.4649 + \frac{0.1}{3} [2.6725 + 4(2.0657) + 3.3660] \\ &= 2.4649 + 0.0333 (14.3013) \end{aligned}$$

$$y_4^{(C)} = \underline{\underline{2.9416}}$$

b) On what curves can the functional $\int_0^{T_2} (y'^2 - y^2 + 2xy) dx$;
 $y(0) = 0, y(T_2) = 0$ be extremized?

Solⁿ: let $I = \int_0^{T_2} (y'^2 - y^2 + 2xy) dx = \int_{x_1}^{x_2} +(x, y, y') dx$

$$+(x, y, y') = y'^2 - y^2 + 2xy$$

$$\frac{\partial I}{\partial y} = 0 - 2y + 2x = 2x - 2y$$

$$\frac{\partial f}{\partial y'} = \partial y'$$

WKT

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) = 0$$

$$(x-y) - \frac{d}{dx} (\partial y') = 0$$

$$(x-y) - \frac{d}{dx} y' = 0$$

$$x-y-y''=0$$

$$y''+y=x$$

$$(D^2+1)y = x, D = \frac{d}{dx}$$

The A.E is $m^2+1=0$

$$m = 0 \pm i$$

$$\therefore y_c = C_1 \cos x + C_2 \sin x$$

$$y_p = \frac{x}{D^2+1}$$

$$y_p = (1+D^2)^{-1} x$$

$$y_p = (1-D^2+D^4-D^6+\dots)x$$

$$y_p = x - 0$$

$$y_p = x$$

$$\therefore y = y_c + y_p$$

$$y = C_1 \cos x + C_2 \sin x + x \rightarrow ①$$

$$\text{when } x=0 \Rightarrow y=0$$

$$① \Rightarrow 0 = C_1 + 0 + 0$$

$$C_1 = 0$$

$$\text{when } x = \pi/2 \Rightarrow y=0$$

$$① \Rightarrow 0 = C_1(0) + C_2(1) + \pi/2$$

$$C_2 = -\pi/2$$

$$\therefore y = -\frac{1}{2} \sin x + c$$

or prove that geodesics of a plane surface are straight lines.

$$\text{Soln: let } S = \int_{x_1}^{x_2} \frac{\partial S}{\partial x} dx$$

$$= \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$= \int_{x_1}^{x_2} \sqrt{1 + y'^2} dx$$

$$+ (x, y, y') = \sqrt{1 + y'^2}$$

$$\therefore \frac{\partial f}{\partial y} = 0$$

$$\frac{\partial f}{\partial y} = \frac{1}{\sqrt{1+y'^2}} \cdot y'$$

$$\frac{\partial f}{\partial y} = \frac{y'}{\sqrt{1+y'^2}}$$

$$\text{WKT } \frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) = 0$$

$$0 - \frac{d}{dx} \left(\frac{y'}{\sqrt{1+y'^2}} \right) = 0$$

$$\frac{d}{dx} \left[\frac{y'}{\sqrt{1+y'^2}} \right] = 0$$

$$\sqrt{1+y'^2} y'' - y' \frac{1}{x \sqrt{1+y'^2}} \cdot y'y'' = 0$$

$$y'' \sqrt{1+y'^2} - \frac{y'^2 y''}{\sqrt{1+y'^2}} = 0$$

$$y'' (1+y'^2) - y'^2 y'' = 0$$

$$y'' + y'^2 y'' - y'^2 y'' = 0$$

$$\Rightarrow y'' = 0$$

$$D^2 y = 0, D = \frac{d}{dx}$$

$$\text{A.E } \rho, m^2 = 0$$

$$m = 0, 0$$

$$\therefore y = c_1 + \underline{\underline{c_2 x}}$$