Module - 3

Chapter 5

Relations and Functions

In this chapter, the concept of *relation* is introduced and a special class of relations called *functions* are discussed in some depth. The emphasis is on set-theoretic properties of functions and their immediate consequences.

A basic knowledge of *Elementary Set Theory* is a pre-requisite for this and the next chapter.

5.1 Cartesian Product of Sets

Let A and B be two sets. Then the *set of all ordered pairs* (a, b), where $a \in A$ and $b \in B$, is called the ***Cartesian Product***, or *Cross Product* or *Product Set* of A and B (in this order) and is denoted by $A \times B$. Thus,

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}.$$

It is to be noted that the product set $A \times B$ is not the same as the product set $B \times A$; that is, $A \times B \neq B \times A$, in general. Because,

$$B \times A = \{(b, a) \mid b \in B \text{ and } a \in A\}$$

and $(a, b) \neq (b, a)$ in general.*

For example, if $A = \{1, 0, -1\}$ and $B = \{2, 3\}$, then

$$A \times B = \{(1,2), (1,3), (0,2), (0,3), (-1,2), (-1,3)\}$$

$$\text{and} \quad B \times A = \{(2,1), (2,0), (2,-1), (3,1), (3,0), (3,-1)\}.$$

Evidently, $A \times B \neq B \times A$.

It should be noted that $A \times B$ can be defined even when $B = A$. Thus, we can have the product of a set A with itself, and this product is defined by

$$A \times A = \{(a, b) \mid a \in A \text{ and } b \in A\}.$$

The product $A \times A$ is also denoted by A^2.

*If (a, b) and (c, d) are ordered pairs, then $(a, b) = (c, d)$ if and only if $a = c$, $b = d$.

For example, if $A = \{1, 0, -1\}$ we have

$$A^2 = A \times A = \{(1,1),(1,0),(1,-1),(0,1),(0,0),(0,-1),(-1,1),(-1,0),(-1,-1)\}$$

If a set A has m elements and a set B has n elements, then a can be chosen from A in m ways, and with every one of these choices (of a), b can be chosen from B in n ways. Accordingly, (a, b) can be chosen in $m \times n$ ways; this means that $A \times B$ has exactly mn elements. Thus, we have the following result:

*If A and B are finite sets with $|A| = m$ and $|B| = n$, then $A \times B$ is a finite set with $|A \times B| = mn$.**

In other words:

If A and B are finite sets, then

$$|A \times B| = |A|\,|B|.$$

From this result, we note that

$$|B \times A| = |B||A| = |A||B| = |A \times B|, \quad \text{and} \quad |A \times A| = |A|^2.$$

For example, if A has 5 elements and B has 8 elements, then $A \times B$ and $B \times A$ will have $5 \times 8 = 8 \times 5 = 40$ elements each, $A \times A$ will have $5 \times 5 = 25$ elements, and $B \times B$ will have $8 \times 8 = 64$ elements.

The idea of Cartesian product of sets can be extended to any finite number of sets. For any non-empty sets $A_1, A_2, \ldots, A_k$, the ***k-fold product*** $A_1 \times A_2 \times \cdots \times A_k$ is defined as the set of all ordered ***k-tuples*** $(a_1, a_2, \ldots, a_k)$, where $a_i \in A_i$, $i = 1, 2, \ldots, k$. That is,

$$A_1 \times A_2 \times \cdots \times A_k = \{(a_1, a_2, \ldots, a_k) \mid a_i \in A_i,\ i = 1, 2, \ldots, k\}.$$

For example, if $A = \{1, 0\}$, $B = \{2, -2\}$, $C = \{0, -1\}$, then

$$A \times B \times C = \{(1,2,0),(1,2,-1),(1,-2,0),(1,-2,-1),(0,2,0),(0,2,-1),(0,-2,0),(0,-2,-1)\}$$

As with the ordered pairs, if $(a_1, a_2, \ldots, a_k)$, $(b_1, b_2, b_3 \ldots, b_k)$ are *k-tuples*, then $(a_1, a_2, \ldots a_k) = (b_1, b_2, \ldots, b_k)$ if and only if $a_i = b_i$ for $i = 1, 2, \ldots, k$.

A little thinking will indicate that if A_1 has n_1 elements, A_2 has n_2 elements, $\ldots$, A_k has n_k elements, then $A_1 \times A_2 \times \cdots \times A_k$ has $n_1 n_2 n_3 \ldots n_k$ elements. That is,

$$|A_1 \times A_2 \times \cdots \times A_k| = |A_1| \cdot |A_2| \cdots |A_k|.$$

Example 1 *Find x and y in each of the following cases:*

(i) $(2x, x + y) = (6, 1)$ (ii) $(y - 2, 2x + 1) = (x - 1, y + 2)$

▶ (i) We note that $(2x, x + y) = (6, 1)$ if and only if $2x = 6$ and $x + y = 1$. These yield $x = 3$ and $y = 1 - x = -2$.

*For a finite set S the number of elements in S is denoted by $|S|$. This number is called the *order of* S and is also denoted by $O(S)$.

(ii) $(y-2, 2x+1) = (x-1, y+2)$ if and only if $y-2 = x-1$ and $2x+1 = y+2$; that is, $x - y + 1 = 0$ and $2x - y - 1 = 0$. These yield $x = 2$, $y = 3$. ■

Example 2 *Let $A = \{1,3,5\}$, $B = \{2,3\}$, and $C = \{4,6\}$. Write down the following:*

(1) $A \times B$ (2) $B \times A$ (3) $B \times C$ (4) $A \times C$
(5) $(A \cup B) \times C$ (6) $A \cup (B \times C)$ (7) $(A \times B) \cup C$ (8) $A \cap (B \times C)$
(9) $(A \times B) \cup (B \times C)$ (10) $(A \times B) \cap (B \times A)$ (11) $(A \times B) \cap (B \times C)$

▶ By using the definition of the product of sets, we find that

$$A \times B = \{(1,2), (1,3), (3,2), (3,3), (5,2), (5,3)\},$$
$$B \times A = \{(2,1), (2,3), (2,5), (3,1), (3,3), (3,5)\},$$
$$B \times C = \{(2,4), (2,6), (3,4), (3,6)\},$$
$$A \times C = \{(1,4), (1,6), (3,4), (3,6), (5,4), (5,6)\},$$
$$(A \cup B) \times C = \{1,2,3,5,\} \times \{4,6\}$$
$$= \{(1,4), (1,6), (2,4), (2,6), (3,4), (3,6), (5,4), (5,6)\}$$
$$A \cup (B \times C) = \{1,3,5\} \cup \{(2,4), (2,6), (3,4), (3,6)\}$$
$$= \{1,3,5, (2,4), (2,6), (3,4), (3,6)\}$$
$$(A \times B) \cup C = \{(1,2), (1,3), (3,2), (3,3), (5,2), (5,3), 4, 6\}$$
$$A \cap (B \times C) = \Phi.$$
$$(A \times B) \cup (B \times C) = \{(1,2), (1,3), (3,2), (3,3), (5,2), (5,3)$$
$$(2,4), (2,6), (3,4), (3,6)\},$$
$$(A \times B) \cap (B \times A) = \{(3,3)\}$$
$$(A \times B) \cap (B \times C) = \Phi.$$
■

Example 3 *Suppose $A, B, C \subseteq \mathbb{Z} \times \mathbb{Z}$ with $A = \{(x,y) \mid y = 5x - 1\}$, $B = \{(x,y) \mid y = 6x\}$, $C = \{(x,y) \mid 3x - y = -7\}$.*
Find : (i) $A \cap B$, (ii) $B \cap C$, (iii) $\overline{\bar{A} \cup \bar{C}}$, (iv) $\bar{B} \cup \bar{C}$.

▶ (i) We note that

$$(x,y) \in A \cap B \Leftrightarrow (x,y) \in A \text{ and } (x,y) \in B$$
$$\Leftrightarrow y = 5x - 1 \text{ and } y = 6x$$
$$\Leftrightarrow 5x - 1 = y = 6x$$
$$\Leftrightarrow x = -1,\ y = -6.$$

Thus, $A \cap B = \{(-1, -6)\}$.

(ii) We note that

$$\begin{aligned}(x, y) \in B \cap C &\Leftrightarrow (x, y) \in B \text{ and } (x, y) \in C\\ &\Leftrightarrow y = 6x, \text{ and } 3x - y = -7\\ &\Leftrightarrow 6x = 3x + 7\\ &\Leftrightarrow 3x = 7 \quad \text{i.e., } x = 7/3.\end{aligned}$$

which is not possible, because $x \in \mathsf{Z}$.

Thus, $B \cap C = \Phi$.

(iii) We note that $\bar{A} \cup \bar{C} = \overline{A \cap C}$ so that $\overline{\bar{A} \cup \bar{C}} = \overline{\overline{A \cap C}} = A \cap C$. Now,

$$\begin{aligned}(x, y) \in A \cap C &\Leftrightarrow (x, y) \in A \text{ and } (x, y) \in C\\ &\Leftrightarrow y = 5x - 1 \text{ and } 3x - y = -7\\ &\Leftrightarrow 5x - 1 = y = 3x + 7\\ &\Leftrightarrow x = 4,\ y = 19.\end{aligned}$$

Thus, $\overline{\bar{A} \cup \bar{C}} = A \cap C = \{(4, 19)\}$.

(iv) We note that $\bar{B} \cup \bar{C} = \overline{B \cap C}$. It has been seen in (ii) above that $B \cap C = \Phi$. Therefore, $\bar{B} \cup \bar{C} = \overline{B \cap C} = \mathsf{Z} \times \mathsf{Z}$ (universal set). ∎

Example 4 *For any set A, prove that*

$$A \times \Phi = \Phi \times A = \Phi.$$

▶ Suppose $A \times \Phi \neq \Phi$. Then, $A \times \Phi$ has at least one element (a, b) in it such that $a \in A$ and $b \in \Phi$. Now, $b \in \Phi$ means that Φ is not the null set. This is a contradiction. Therefore, $A \times \Phi = \Phi$. Similarly, $\Phi \times A = \Phi$. ∎

Example 5 *Let A, B, C, D be non-empty sets. Prove that $(A \times B) \subseteq (C \times D)$ if and only if $A \subseteq C$ and $B \subseteq D$. What happens to this result if any of the sets is empty?*

▶ First, suppose that $(A \times B) \subseteq (C \times D)$. Take any $a \in A$ and any $b \in B$, then $(a, b) \in (A \times B)$. Since $(A \times B) \subseteq (C \times D)$, it follows that $(a, b) \in (C \times D)$. Therefore, $a \in C$ and $b \in D$. This proves that $A \subseteq C$ and $B \subseteq D$.

Conversely, suppose that $A \subseteq C$ and $B \subseteq D$. Take *any* $(x, y) \in A \times B$. Then $x \in A$ and $y \in B$. Since $A \subseteq C$ and $B \subseteq D$, it follows that $x \in C$ and $y \in D$. Therefore, $(x, y) \in C \times D$. This proves that $(A \times B) \subseteq (C \times D)$. The required result is thus proved.

Next, suppose A is empty, and take $B = \{1, 2\}$, $C = \{2, 4, 5\}$, $D = \{1, 3\}$. We find that $A \times B = \Phi \times B = \Phi$. Also, $\Phi \subseteq C \times D$. Therefore, here, $A \times B \subseteq C \times D$. But $B \nsubseteq D$. Thus, if A is empty, then $(A \times B) \subseteq (C \times D)$ does not imply that $A \subseteq C$ and $B \subseteq D$. ∎

Exercises

1. Find x and y in each of the following cases:

   (1) $(2x-3, 3y+1) = (5,7)$ (2) $(x+2, 4) = (5, 2x+y)$

   (3) $(x,y) = \left(x^2, y^2\right)$ (4) $(x,y) = \left(y^2, x^2\right)$

2. Given $A = \{a,b\}$ and $B = \{1,2,3\}$, find $A\times B$, $B\times A$, $A\times A$ and $B\times B$.

3. Given $A = \{1,2\}$, $B = \{a,b,c\}$ and $C = \{3,4\}$, find $A\times B\times C$ and $B\times C\times A$.

4. Let $A = \{1,2,3,4\}$, $B = \{2,5\}$, $C = \{3,4,7\}$. Write down the following:

$$A\times B,\quad B\times A,\quad A\cup(B\times C),\quad (A\cup B)\times C,\quad (A\times C)\cup(B\times C).$$

5. If $A = \{2,3\}$, $B = \{-1,2\}$ and $C = \{a,b\}$, verify that

   $A\times(B\cup C) = (A\times B)\cup(A\times C)$ and $A\times(B\cap C) = (A\times B)\cap(A\times C)$.

6. If $A = \{1,2,3\}$, $B = \{2,3,4\}$, $C = \{3,4,5\}$, verify that

   $A\times(B\cup C) = (A\times B)\cup(A\times C)$ and $A\times(B\cap C) = (A\times B)\cap(A\times C)$.

7. For which sets A, B, do we have $A\times B = B\times A$?

Answers

1. (1) $x=4, y=2,$ (2) $x=3, y=-2$

   (3) $x=0,1,\ y=0,1,$ (4) $x=y=0$, or $x=y^2$ where $x^3=1$.

2. $A\times B = \{(a,1),(a,2),(a,3),(b,1),(b,2),(b,3)$

   $B\times A = \{(1,4),(1,5),(2,4),(2,5),(3,4),(3,5)\}$

   $A\times A = \{(4,4),(4,5),(5,4),(5,5)\}$

   $B\times B = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\}$

3. $A\times B\times C = \{(1,a,3),(1,a,4),(1,b,3),(1,b,4),(1,c,3),(1,c,4),(2,a,3),(2,a,4),$
   $(2,b,3),(2,b,4),(2,c,3),(2,c,4)\}$

   $B\times C\times A = \{(a,3,1),(a,3,2),(a,4,1),(a,4,2),(b,3,1),(b,3,2),(b,4,1),(b,4,2),$
   $(c,3,1),(c,3,2),(c,4,1),(c,4,2)\}$

4. $A \times B = \{(1,2), (2,2), (3,2), (4,2), (1,5), (2,5), (3,5), (4,5)\}$
$B \times A = \{(2,1), (2,2), (2,3), (2,4), (5,1), (5,2), (5,3), (5,4)\}$
$A \cup (B \times C) = 1, 2, 3, 4, (2,3), (2,4), (2,7), (5,3), (5,4), (5,7)\}$
$(A \cup B) \times C = \{(1,3), (2,3), (3,3), (4,3), (5,3), (1,4), (2,4), (3,4), (4,4), (5,4), (1,7), (2,7), (3,7), (4,7), (5,7)\}$
$= (A \times C) \cup (B \times C)$

7. When A or B is Φ, or when $A = B$.

5.2 Relations

Let A and B be two sets. Then a *subset* of $A \times B$ is called a *binary relation* or just a ***relation from A to B***. Thus, if R is a relation from A to B, then R is a (some) set of ordered pairs (a, b) where $a \in A$ and $b \in B$, and conversely if R is a set of ordered pairs (a, b) where $a \in A$ and $b \in B$, then R is a relation from A to B. If $(a, b) \in R$, we say that "a is related to b by R"; this is denoted by aRb.*

If R is a relation from A to A, that is, if R is a subset of $A \times A$, we say that R is a ***binary relation** on A*.

For example, consider the sets $A = \{0, 1, 2\}$, $B = \{3, 4, 5\}$. Let $R = \{(1,3), (2,4), (2,5)\}$. Evidently, R is a subset of $A \times B$. As such, R is a relation from A to B, and $1R3, 2R4, 2R5$. This relation can be depicted in a diagram as shown below, called the *arrow diagram*:

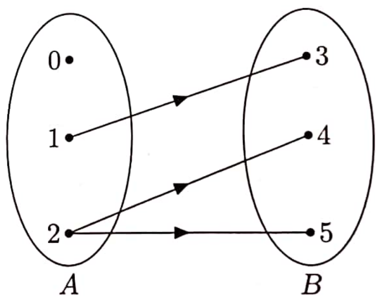


Figure 5.1

As another example, consider the sets $A = \{0, 1, -1\}$ and $B = \{2, -2\}$. Let

$$R_1 = \{(0,2), (1,2), (-1,2)\} \quad \text{and} \quad R_2 = \{(0,-2), (1,-2), (-1,-2)\}.$$

Then R_1 and R_2 are subsets of $A \times B$ and are therefore relations from A to B. We observe that R_1 consists of elements $(a, b) \in A \times B$ for which the relationship $a < b$ holds. Hence here, "aR_1b" is read as "a is less than b", the symbol R_1 standing for the phrase "is less than". Further, R_2 consists of elements $(a, b) \in A \times B$ for which the relationship $a > b$ holds. Hence

*The expressions $(a, b) \in R$ and aRb carry one and the same meaning.

here, "aR_2b" is read as "a is greater than b", the symbol R_2 standing for the phrase "is greater than". The arrow diagrams of R_1 and R_2 are shown in Figures 5.2(a) and 5.2(b) respectively.

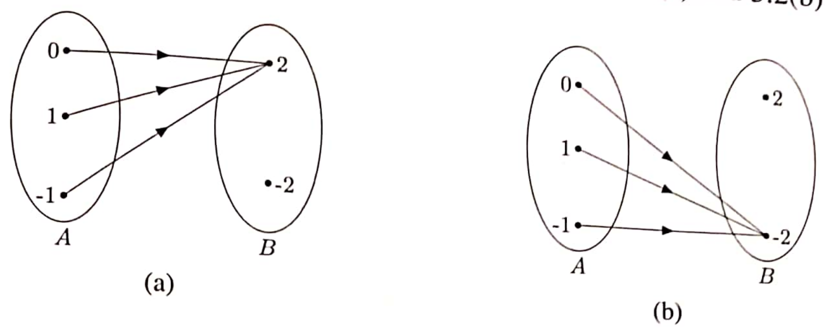


Figure 5.2

As yet another example, consider the set $A = \{1, 2, 3, 4\}$. Let

$$R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$$

Evidently, R is a subset of $A \times A$. Therefore, R is a relation *on* A.

We further note that $(x, y) \in R$ if and only if x divides y. Hence, here xRy stands for the statement "x divides y", and the relation R stands for the word "divides".

Example 1 *If A is a set with m elements and B is a set with n elements, find the number of relations from A to B.*

▶ Since a relation from A to B is a subset of $A \times B$, the set of all relations from A to B is the same as the set of all subsets of $A \times B$. Therefore, the number of relations from A to B is equal to the number of subsets of $A \times B$.

Since $|A| = m$ and $|B| = n$, we have $|A \times B| = mn$. Therefore $A \times B$ has 2^{mn} number of subsets*. This implies that there are 2^{mn} relations from A to B. (For instance, if $|A| = 3$ and $|B| = 2$, then there exist $2^6 = 64$ relations from A to B). ■

Example 2 *Let A and B be finite sets with $|B| = 3$. If there are 4096 relations from A to B, what is $|A|$?*

▶ If $|A| = m$ and $|B| = n$, then there are 2^{mn} relations from A to B. From what is given, we have $n = 3$ and $2^{mn} = 4096$. Thus,

$$2^{3m} = 4096 \quad \text{or} \quad 3m \log_e 2 = \log_e 4096$$

so that

$$m = \frac{\log_e 4096}{3 \log_e 2} = 4.$$

■

Thus, $|A| = 4$.

*Recall that if a set has n elements, its power set has 2^n elements.

Example 3 *Let $A = \{1, 2, 3\}$ and $B = \{2, 4, 5\}$. Determine the following:*

(1) $|A \times B|$.

(2) *Number of relations from A to B.*

(3) *Number of binary relations on A.*

(4) *Number of relations from A to B that contain $(1, 2)$ and $(1, 5)$.*

(5) *Number of relations from A, B that contain exactly five ordered pairs.*

(6) *Number of binary relations on A that contain at least seven ordered pairs.*

▶ We have $|A| = m = 3$, $|B| = n = 3$. Therefore:

(1) $|A \times B| = mn = 9$.

(2) No. of relations from A to B is $2^{mn} = 2^9 = 512$.

(3) No. of binary relations on A is $2^{mm} = 2^{m^2} = 2^9 = 512$.

(4) Let $R_1 = \{(1, 2), (1, 5)\}$. We note that every relation from A to B that contains the elements $(1, 2)$ and $(1, 5)$ is of the form $R_1 \cup R_2$, where R_2 is a subset of $\overline{R_1}$ in $A \times B$. Therefore, the number of such relations is equal to the number of subsets of $\overline{R_1}$. Since $|\overline{R_1}| = |A \times B| - |R_1| = 9 - 2 = 7$, the number of subsets of $\overline{R_1}$ is $2^7 = 128$. Thus, there are $2^7 = 128$ number of relations from A to B that contain the elements $(1, 2)$ and $(1, 5)$.

(5) Since $A \times B$ contains 9 ordered pairs, the number of relations from A to B that contain exactly five ordered pairs is precisely the number of ways of choosing five ordered pairs from nine ordered pairs. This number is ${}^9C_5 = 126$.

(6) Similarly, the number of binary relations on A that contains at least seven elements (ordered pairs) is ${}^9C_7 + {}^9C_8 + {}^9C_9 = 46$. ∎

Example 4 *Let $A = \{1, 2, 3, 4, 6\}$ and R be the relation on A defined by $(a, b) \in R$ if and only if a is a multiple of b. Write down R as a set of ordered pairs.*

▶ From the way R has been defined, we find that

$$\begin{aligned} R &= \{(a, b) \mid a, b \in A \text{ and } a \text{ is a multiple of } b\} \\ &= \{(1, 1), (2, 1), (2, 2), (3, 1), (3, 3), (4, 1), (4, 2), (4, 4), (6, 1), (6, 2), (6, 3), (6, 6)\} \end{aligned}$$ ∎

Example 5 *A relation R on the set of all integers Z is defined recursively by*

(i) $(0, 0) \in R$, *and* (ii) *if $(s, t) \in R$, then $(s + 1,\ t + 7) \in R$.*

Find R as a set of ordered pairs.

▶ We have $(0,0) \in R$, by the first part of the definition. Using this in the second part of the definition, we find that $(0+1, 0+7) \in R$, that is, $(1,7) \in R$. Using this in the second part of the definition, we get $(1+1, 7+7) \in R$; that is $(2,14) \in R$. Proceeding like this, we find (by repeated use of the second part of the definition), $(3,21) \in R$, $(4,28) \in R$, and so on.

Thus,

$$\begin{aligned} R &= \{(0,0),(1,7),(2,14),(3,21),(4,28),\ldots\} \\ &= \{(m,n) \in \mathbb{Z} \times \mathbb{Z} \mid n = 7m\}. \end{aligned}$$

■

Exercises

1. Consider the sets $A = \{0,1,2\}$ and $B = \{8,9\}$. Indicate whether the following sets of ordered pairs are relations from A to B.

$$R_1 = \{(0,8),(1,8),(2,9)\}, \quad R_2 = \{(1,8),(1,9),(2,2)\}, \quad R_3 = \{(1,9),(0,8),(8,0)\}.$$

2. Indicate whether the following sets of ordered pairs are relations on the set $A = \{4,5,7,12\}$:

$$R_1 = \{(4,4),(5,12),(7,7)\}, \quad R_2 = \{(5,4),(4,12),(12,12)\}, \quad R_3 = \{(4,4),(5,5),(12,2)\}.$$

3. Let $A = \{1,2,3,4\}$ and R be the relation on A defined by $(a,b) \in R$ if and only if $a \leq b$. Write down R as a set of ordered pairs.

4. Let A be the set of all positive integers and R be the relation on A defined by aRb if and only if $a = b^k$ for some positive integer k. Find which of the following belong to R?

   (a) $(3,9)$ (b) $(9,3)$ (c) $(2,5)$ (d) $(5,2)$ (e) $(6,6)$ (f) $(32,2)$

5. A binary relation R on $\mathbb{N}$ (the set of all natural numbers) is defined by $R = \{(m,n) \in \mathbb{N} \times \mathbb{N} \mid n = 5m + 2\}$. Give a recursive definition of R. Hence verify that $(7,22) \in R$.

Answers

1. R_1 is a relation from A to B. R_2 and R_3 are not relations from A to B.

2. R_1 and R_2 are relations on A. R_3 is not a relations on A.

3. $R = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\}$

4. $(9,3)$, $(6,6)$, $(32,2)$ belong to R; others do not.

5. R is defined by (i) $(0,2) \in R$, and (ii) if $(m,n) \in R$ then $(m+1, n+5) \in R$.

5.3 Functions

Let A and B be two non-empty sets. Then a ***function*** (or *mapping*) f from A to B is a relation from A to B such that for *each* a in A there is a *unique* b in B such that $(a, b) \in f$. Then we write $b = f(a)$. Here, b is called the ***image*** of a, and a is called a ***preimage*** of b, under f. The element a is also called an *argument* of the function f, and $b = f(a)$ is then called the *value* of the function f for the argument a.

A function f from A to B is denoted by $f : A \to B$. The pictorial representation of f is as shown below.

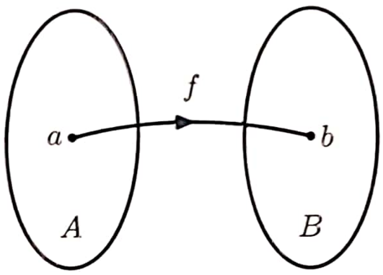


Figure 5.3

It has to be emphasized that *every function is a relation, but a relation need not be a function*. Because, if R is a relation from A to B then an element of A can be related to two different elements of B, under R. This is not the case in respect of a function from A to B; under a function an element of A can be related to only one element of B.*

For the function $f : A \to B$, A is called the ***domain*** of f and B is called the ***codomain*** of f. The subset of B consisting of the images of all elements of A under f is called the ***range*** of f and is denoted by $f(A)$.

The following observations are immediate consequences of the definition of a function $f : A \to B$ and other associated definitions given above:

(1) Every a in A belongs to some pair $(a, b) \in f$, and if $(a, b_1) \in f$ and $(a, b_2) \in f$, then $b_1 = b_2$. This means that every element of A has an image in B (under f) and if an element a of A has two images in B, then the two images cannot be different.

(2) An element $b \in B$ need not have a preimage in A, under f.

(3) If an element $b \in B$ has a preimage $a \in A$ under f, the preimage need not be unique. In other words, two different elements of A can have the same image in B, under f.

*To emphasize this fact, a function is also referred to as a *single-valued function*. On the other hand, a relation which is not a function is referred to as a *multiple-valued function*. The terms "single-valued function" and "multiple-valued function" have become out-dated in mathematical literature. In the present day mathematical terminology, *a function means a single-valued function only*.

(4) The statements $(a, b) \in f$, afb and $b = f(a)$ are equivalent (in the sense that they all carry the same meaning).

(5) If g is a function from A to B (denoted by $g : A \to B$), then $f = g$ if and only if $f(a) = g(a)$ for every $a \in A$.

(6) The range of $f : A \to B$ is given by

$$f(A) = \{ f(x) \mid x \in A \}$$

and $f(A)$ is a subset of B.

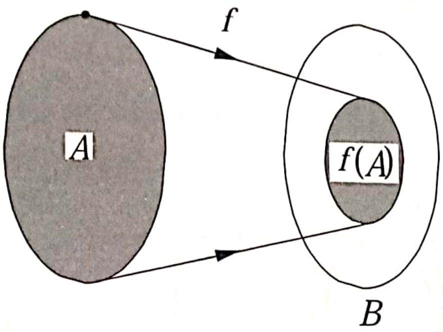


Figure 5.4

(7) For $f : A \to B$, if $A_1 \subseteq A$ and $f(A_1)$ is defined by

$$f(A_1) = \{f(x) \mid x \in A_1\},$$

then $f(A_1) \subseteq f(A)$. (Here $f(A_1)$ is called the *image of* A_1 *under* f).

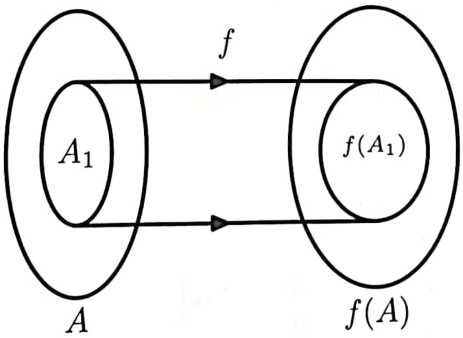


Figure 5.5

(8) For $f : A \to B$, if $b \in B$ and $f^{-1}(b)$ is defined by

$$f^{-1}(b) = \{ x \in A \mid f(x) = b \},$$

then $f^{-1}(b) \subseteq A$. (Here, $f^{-1}(b)$ is called the *preimage set of b* under f.)

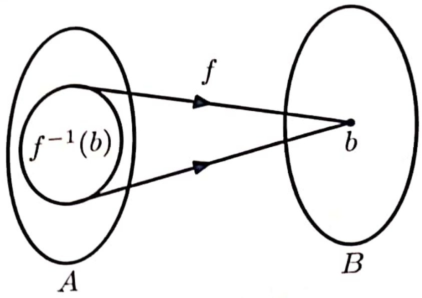


Figure 5.6

(9) For $f : A \to B$, if $B_1 \subseteq B$ and $f^{-1}(B_1)$ is defined by

$$f^{-1}(B_1) = \{\, x \in A \mid f(x) \in B_1 \,\},$$

then $f^{-1}(B_1) \subseteq A$. (Here $f^{-1}(B_1)$ is called the *preimage of* B_1 under f).

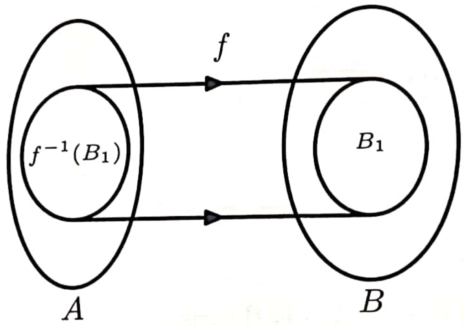


Figure 5.7

Example 1 *Let* $A = \{1, 2, 3\}$ *and* $B = \{-1, 0\}$, *and R be a relation from A to B defined by*

$$R = \{(1, -1),\ (1, 0),\ (2, -1),\ (3, 0)\}.$$

Is R a function from A to B?

▶ The arrow diagram for the given R is shown below:

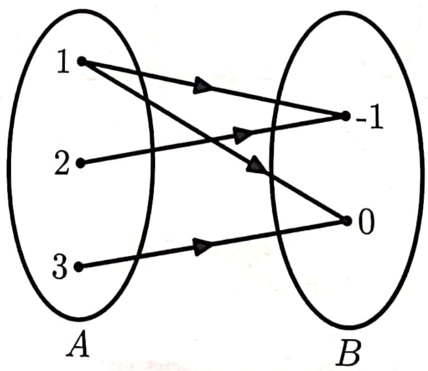


Figure 5.8

We observe that, under R, the element 1 of A is related to two different elements, -1 and 0, of B. Therefore, R is *not* a function. ∎

Example 2 *Let $A = \{1, 2, 3\}$ and $B = \{-1, 0\}$ as in Example 1, and S be a relation from A to B defined by $S = \{(1, -1),\ (2, -1),\ (3, 0)\}$. Is S a function?*

▶ The arrow diagram for the given S is shown below:

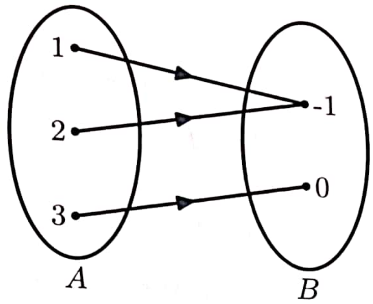


Figure 5.9

We observe that, under S, each element of A is related to a unique element of B: 1 is related only to -1; 2 is related only to -1, and 3 is related only to 0. Therefore, S is a function from A to B. We observe that B is the range of f. ■

Example 3 *Let $A = \{1, 2, 3, 4\}$. Determine whether or not the following relations on A are functions:*

(1) $f = \{(2, 3),\ (1, 4),\ (2, 1),\ (3, 2),\ (4, 4)\}$. (2) $g = \{(3, 1),\ (4, 2),\ (1, 1)\}$.

(3) $h = \{(2, 1),\ (3, 4),\ (1, 4),\ (2, 1),\ (4, 4)\}$.

▶ (1) We note that $(2, 3) \in f$ and $(2, 1) \in f$; this means that the element 2 is related to two different elements, 3 and 1, under f. Therefore, f is *not* a function.

(2) We note that the element 2 is not related to any element in A under g. Therefore, g is *not* a function.

(3) We note that, under h, every element of A is related to a unique element of A. Therefore, h is a function from A to A. Its range is $h(A) = \{1, 4\}$.

In h, the term (2,1) appears twice. This has no special significance; in a set, an element can appear any number of times. ■

Example 4 *Let $A = \{0, \pm 1, \pm 2, 3\}$. Consider the function $f : A \to \mathrm{R}$ (where R is the set of all real numbers) defined by $f(x) = x^3 - 2x^2 + 3x + 1$, for $x \in A$. Find the range of f.*

▶ We find that

$$f(0) = 1,\ f(1) = 3,\ f(-1) = -5,\ f(2) = 7,\ f(-2) = -21,\ f(3) = 19.$$

Hence, the range of f is

$$f(A) = \{1, 3, -5, 7, -21, 19\}.$$

■

Example 5 *Let a function $f : \mathsf{R} \to \mathsf{R}$ be defined by $f(x) = x^2 + 1$. Determine the images of the following subsets of* R:

(i) $A_1 = \{2, 3\}$ (ii) $A_2 = \{-2, 0, 3\}$

(iii) $A_3 = (0, 1)$ (iv) $A_4 = [-6, 3]$

► (i) We have $f(2) = 5$, $f(3) = 10$. Therefore, $f(A_1) = \{5, 10\}$.

(ii) We have $f(-2) = 5$, $f(0) = 1$, $f(3) = 10$. Therefore, $f(A_2) = \{5, 1, 10\}$.

(iii) Here,

$$A_3 = \{\, x \in R \mid 0 < x < 1 \,\}.$$

Therefore, $$f(A_3) = \{\, f(x) \mid 0 < x < 1 \,\} = \left\{\left(x^2 + 1\right) \mid 0 < x < 1\right\}$$

(iv) $f(A_4) = \{\, f(x) \mid -6 \le x \le 3 \,\} = \left\{\left(x^2 + 1\right) \mid -6 \le x \le 3\right\}.$

Example 6 *Let $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{6, 7, 8, 9, 10\}$. If a function $f : A \to B$ is defined by*

$$f = \{(1, 7),\ (2, 7),\ (3, 8),\ (4, 6),\ (5, 9),\ (6, 9)\},$$

determine $f^{-1}(6)$ and $f^{-1}(9)$. If $B_1 = \{7, 8\}$ and $B_2 = \{8, 9, 10\}$, find $f^{-1}(B_1)$ and $f^{-1}(B_2)$.

► We note that

$$f^{-1}(6) = \{\, x \in A \mid f(x) = 6 \,\} = \{4\}$$

and $$f^{-1}(9) = \{x \in A \mid f(x) = 9\,\} = \{5, 6\}$$

For any $B_1 \subseteq B$, we have, by definition,

$$f^{-1}(B_1) = \{\, x \in A \mid f(x) \in B_1 \,\}.$$

For $B_1 = \{7, 8\}$, $f(x) \in B_1$ when $f(x) = 7$ and $f(x) = 8$. From the definition of f, we note that $f(x) = 7$ when $x = 1$ and $x = 2$, and $f(x) = 8$ when $x = 3$. Therefore,

$$f^{-1}(B_1) = \{1, 2, 3\}.$$

Similarly, for $B_2 = \{8, 9, 10\}$, we find that

$$f^{-1}(B_2) = \{\, x \in A \mid f(x) \in B_2 \,\} = \{3, 5, 6\}.$$

Example 7 *Let $f : \mathsf{R} \to \mathsf{R}$ be defined by*

$$f(x) = \begin{cases} 3x - 5 & \textit{for} \quad x > 0 \\ -3x + 1 & \textit{for} \quad x \le 0 \end{cases}$$

(i) *Determine* $f(0)$, $f(-1)$, $f(5/3)$, $f(-5/3)$.

(ii) *Find* $f^{-1}(0)$, $f^{-1}(1)$, $f^{-1}(-1)$, $f^{-1}(3)$, $f^{-1}(-3)$, $f^{-1}(-6)$.

(iii) *What are* $f^{-1}([-5,5])$ *and* $f^{-1}([-6,5])$?

▶(i) By using the definition of the given f, we find that

$$f(0) = (-3\times 0) + 1 = 1, \qquad f(-1) = \{-3\times(-1)\} + 1 = 4,$$
$$f(5/3) = (3\times 5/3) - 5 = 0, \qquad f(-5/3) = \{-3\times(-5/3)\} + 1 = 6.$$

(ii) From the definition of the given f, we find that $f(x) = 0$ only when $x = 5/3$. (Observe that $f(x) \neq 0$ for $x \leq 0$). Therefore,

$$f^{-1}\{0\} = \{5/3\}.$$

Similarly,

$$f^{-1}(1) = \{x \in \mathrm{R} \mid f(x) = 1\} = \{2, 0\}.$$
$$f^{-1}(-1) = \{x \in \mathrm{R} \mid f(x) = -1\} = \{4/3\};\ \text{observe that } f(x) \neq -1 \text{ when } x \leq 0.$$
$$f^{-1}(3) = \{x \in \mathrm{R} \mid f(x) = 3\} = \{8/3, -2/3\}.$$
$$f^{-1}(-3) = \{2/3\}.$$
$$f^{-1}(-6) = \Phi,\ \text{because } f(x) \neq -6 \text{ for } any\ x \in \mathrm{R}.$$

(iii) We note that

$$f^{-1}([-5,5]) = \{x \in \mathrm{R} \mid f(x) \in [-5,5]\}$$
$$= \{x \in \mathrm{R} \mid -5 \leq f(x) \leq 5\}$$

When $x > 0$, we have $f(x) = 3x-5$. Therefore, $-5 \leq f(x) \leq 5$ whenever $-5 \leq (3x-5) \leq 5$, or $0 \leq 3x \leq 10$, or $0 < x \leq 10/3$.

When $x \leq 0$, we have $f(x) = -3x+1$. Therefore, $-5 \leq f(x) \leq 5$ whenever $-5 \leq (-3x+1) \leq 5$, or $-6 \leq -3x \leq 4$, or $2 \geq x \geq -4/3$, or $-4/3 \leq x \leq 2$. Thus,

$$f^{-1}([-5,5]) = \{x \in \mathrm{R} \mid -4/3 \leq x \leq 2 \text{ or } 0 < x \leq 10/3\}$$
$$= \{x \in \mathrm{R} \mid -4/3 \leq x \leq 10/3\}$$
$$= [-4/3, 10/3]$$

Similarly, we find that

$$f^{-1}([-6,5]) = [-4/3, 10/3]$$

■

Example 8 (a) *Let A and B be finite sets with* $|A| = m$ *and* $|B| = n$. *Find how many functions are possible from A to B?*
(b) *If there are 2187 functions from A to B and* $|B| = 3$, *what is* $|A|$?

► (a) Let $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$. Then, a typical function $f : A \to B$ is of the form

$$f = \{(a_1, x),\ (a_2, x), \ldots, (a_m, x)\}$$

where x stands for b_j for some j. Since there are n number of b_j's, there are n choices for x in each of the m ordered pairs belonging to f. Therefore, the total possible number of choices for x is

$$n \times n \times n \times \ldots \times n \quad (m \text{ factors}) = n^m.$$

Thus, there are $n^m = |B|^{|A|}$ = possible functions from A to B.

(b) Here, $n = 3$ and $n^m = 2187$. Thus, we have $3^m = 2187$ so that $m = (\log_e 2187 / \log_e 3) = 7$. Thus, $|A| = 7$. ∎

Exercises

1. Consider the following relations on the set $A = \{1, 2, 3\}$:

   $f = \{(1,3),\ (2,3),\ (3,1)\}$; $g = \{(1,2),\ (3,1)\}$; $h = \{(1,3),\ (2,1),\ (1,2),\ (3,1)\}$

   Which of these are functions?

2. Let $X = \{1, 2, 3, 4\}$ and $Y = \{a, b, c\}$. In each of the following cases, determine whether the given relation R from X to Y is a function. If it is a function, indicate its range.

   (i) $R = \{(1,a),\ (2,b),\ (3,a),\ (4,b)\}$ (ii) $R = \{(1,a),\ (2,b),\ (1,b),\ (3,a),\ (4,b)\}$
   (iii) $R = \{(1,c),\ (2,b),\ (3,a)\}$ (iv) $R = \{(1,a),\ (2,a),\ (3,a),\ (4,a)\}$

3. Determine whether or not the following relations are functions. If a relation is a function, find its range.

   (i) $\{(x,y) \mid x, y \in \mathbb{Z},\ y = 3x + 1\}$ (ii) $\{(x,y) \mid x, y \in \mathbb{Z},\ y = x^2 + 3\}$
   (iii) $\{(x,y) \mid x, y \in \mathbb{R},\ y^2 = x\}$ (iv) $\{(x,y) \mid x, y \in \mathbb{Q},\ x^2 + y^2 = 1\}$

4. If R is a relation from a set A to a set B with $|A| = 5$, $|B| = 6$, $|R| = 6$, prove that R is not a function.

5. If $A = \{0, \pm1, \pm2\}$ and $f : A \to \mathbb{R}$ is defined by $f(x) = x^2 - x + 1$, $x \in A$, find the range of f.

6. If $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^2$, what is the range of f? What is $f(\mathbb{Z})$? What is $f([-2, 1])$?

7. For $A = \{1, 2, 3, 4, 5\}$ and $B = \{w, x, y, z\}$, let a function $f : A \to B$ be given by

   $$f = \{(1,w),\ (2,x),\ (3,x),\ (4,y),\ (5,y)\}.$$

   Find the images of the subsets $A_1 = \{1\}$, $A_2 = \{2, 3\}$, $A_3 = \{1, 2, 3\}$ and $A_4 = \{2, 3, 4, 5\}$ under f.

8. Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{3, 4, 5\}$. A function $f : A \to B$ is defined by

$$f = \{(1, 3),\ (2, 3),\ (3, 4),\ (4, 5),\ (5, 4)\}.$$

Find: $f^{-1}(3)$, $f^{-1}(4)$, $f^{-1}(B_1)$, $f^{-1}(B_2)$, where $B_1 = \{3, 4\}$, $B_2 = \{4, 5\}$.

9. If $f : \mathsf{Z} \to \mathsf{R}$ is defined by $f(x) = x^2 + 5$, find $f^{-1}(\{6\})$, $f^{-1}([6, 7])$, $f^{-1}([6, 10])$, $f^{-1}([-4, 5))$, $f^{-1}([-4, 5])$, and $f^{-1}([5, \infty))$.

10. If $f : \mathsf{R} \to \mathsf{R}$ is defined by $f(x) = x^2 + 5$, find $f^{-1}([6, 7])$, $f^{-1}([6, 10])$, and $f^{-1}([5, \infty))$.

11. If $A = \{1, 2, 3, 4, 5\}$, and $B = \{6, 7, 8, 9, 10, 11, 12\}$, find how many functions $f : A \to B$ are such that $f^{-1}(\{6, 7, 8\}) = \{1, 2\}$?

12. If A is a finite set with $|A| = n$, find how many functions are there from A to A.

Answers

1. f is a function; g and h are not functions.

2. (i) Yes; $\{a, b\}$ (ii) No (iii) No (iv) Yes; $\{a\}$

3. (i) Function; range: set of all integers of the form $3k + 1$
   (ii) Function; range $= \{3, 4, 7, 12, 19, \ldots\}$ (iii) Not a function (iv) Not a function

5. $f(A) = \{1, 3, 7\}$

6. Range $= [0, \infty)$, $f(\mathsf{Z}) = \{0, 1, 4, 9, 16, \ldots\}$, $f([-2, 1]) = [0, 4]$.

7. $f(A_1)=\{w\}$, $f(A_2)=\{x\}$, $f(A_3)=\{w, x\}$, $f(A_4) = \{x, y\}$.

8. $f^{-1}(3) = \{1, 2\}$, $f^{-1}(4) = \{3, 5\}$, $f^{-1}(B_1) = \{1, 2, 3, 5\}$, $f^{-1}(B_2) = \{3, 4, 5\}$.

9. $f^{-1}(\{6\}) = f^{-1}([6, 7]) = \{-1, 1\}$, $f^{-1}([6, 10]) = \{-2, -1, 1, 2\}$,
   $f^{-1}([4, 5)) = \Phi$, $f^{-1}([-4, 5]) = \{0\}$, $f^{-1}([5, \infty)) = \mathsf{Z}$.

10. $f^{-1}([6, 7]) = \left\{x \in \mathsf{R} \mid -\sqrt{2} \le x \le -1 \text{ or } 1 \le x \le \sqrt{2}\right\}$,
    $f^{-1}([6, 10]) = \left\{x \in \mathsf{R} \mid -\sqrt{5} \le x \le -1 \text{ or } 1 \le x \le \sqrt{5}\right\}$,
    $f^{-1}([5, \infty)) = \mathsf{R}$.

11. $3^2 \times 4^3 = 576$ functions.

12. n^n.

5.3.1 Types of functions

Below we consider some special types of functions that are of importance.

Identity function

A function $f : A \to A$ such that $f(a) = a$ for every $a \in A$ is called the *identity function* (or *identity mapping*) on A.

In other words, a function f on a set A is an identity function if the image of every element of A (under f) is itself. In this case, $f(A) = A$.

The identity function defined on a set A is usually denoted by I_A or 1_A.

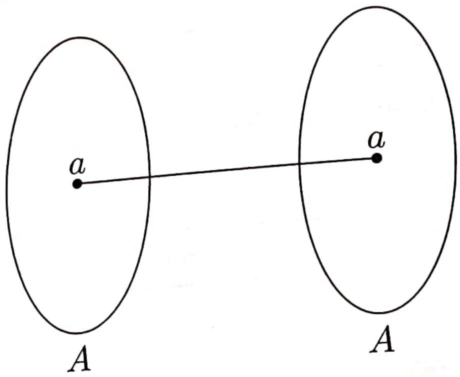


Figure 5.10: Identity Function I_A

Constant function

A function $f : A \to B$ such that $f(a) = c$ for every $a \in A$, where c is a fixed element of B, is called a *constant function*.

In other words, a function f from A to B is a constant function if all elements of A have the same image (say c) in B. In this case $f(A) = \{c\}$.

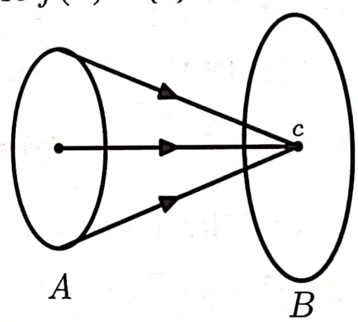


Figure 5.11: Constant function

Onto function

A function $f : A \to B$ is said to be an *onto function* if *for every element p* of B there is an element a of A such that $f(a) = b$.

In other words, f is an onto function from A to B if every element of B has a preimage in A; This amounts to saying that f is an onto function of the range of f is equal to B.

When f is an onto function from A to B, we say that f is a function from A *onto* B.

An onto function is also called a *surjective function*.

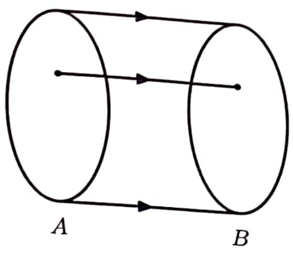


Figure 5.12: Onto function

One-to-one function

A function $f : A \to B$ is said to be a *one-to-one function* (or *one-one function*, written as *1-1* function) if different elements of A have different images in B under f; that is if whenever $a_1, a_2 \in A$ with $a_1 \neq a_2$ then $f(a_1) \neq f(a_2)$; or equivalently that (taking contrapositive) if whenever $f(a_1) = f(a_2)$ for $a_1, a_2 \in A$, then $a_1 = a_2$.

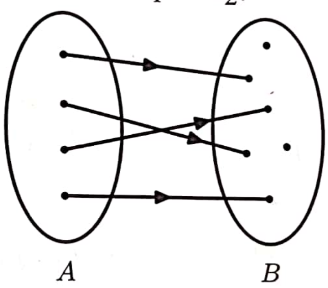


Figure 5.13: One-to-One function

Thus, if $f : A \to B$ is a one-to-one function, then every element of A has a unique image in B and every element of $f(A)$ has a unique preimage in A.

A one-to-one function is also called an *injective function*.

One-to-one correspondence

A function which is both one-to-one and onto is called a *one-to-one correspondence* or a *bijective function* (or a *bijection*). If $f : A \to B$ is such a function, then every element of A has a unique image in B and every element in B has a unique preimage in A.

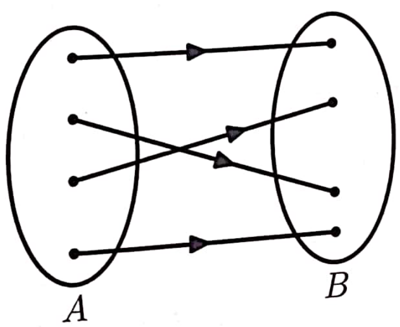


Figure 5.14: One-to-One Correspondence

Example 1 *Find the nature of the following functions defined on* $A = \{1, 2, 3\}$.

(1) $f = \{(1, 1), (2, 2), (3, 3)\}$ (2) $g = \{(1, 2), (2, 2), (3, 2)\}$

(3) $h = \{(1, 2), (2, 2), (3, 1)\}$ (4) $p = \{(1, 2), (2, 3), (3, 1)\}$

► (1) We note that for every $a \in A$, $(a, a) \in f$; that is, $a = f(a)$. Therefore, f is the identity function on A.

(2) We note that every $a \in A$ has 2 as its image; that is, $g(1) = 2$, $g(2) = 2$ and $g(3) = 2$. Therefore, g is a constant function.

(3) We note that h is neither the identity function nor a constant function. The range of h is $\{2, 1\} \subset A$; the element 3 has no preimage under h. Therefore, h is not onto. We further note that both of 1 and 2 have the same image 2 under h. Therefore, h is not one-to-one.

(4) We note that every element of A has a unique image and every element of A has a unique preimage, under p. Therefore, p is both one-to-one and onto; it is a one-to-one correspondence. Furthermore, since p is defined *on* A, it is a permutation. ∎

Example 2 *Let* $A = \{1, 2, 3\}$ *and* $B = \{1, 2, 3, 4, 5\}$. *Find the whether the following functions from A to B are* (a) *one-to-one,* (b) *onto.*

(i) $f = \{(1, 1), (2, 3), (3, 4)\}$ (ii) $g = \{(1, 1), (2, 2), (3, 3)\}$

► (i) We note that, under f, every element of A has a unique image in B and no two elements of A have the same image in B. Therefore, f is one-to-one.

We observe that, under f, the element 5 of B has no preimage in A. Therefore, f is not onto.

(ii) We note that, under g, both of the elements 2 and 3 of A have the same image 3 in B. Therefore, g is not one-to-one.

Since the element 5 of B has no preimage in A under g, it follows that g is not onto. ∎

Example 3 *The functions* $f : \mathrm{R} \to \mathrm{R}$ *and* $g : \mathrm{R} \to \mathrm{R}$ *are defined by* $f(x) = 3x + 7$ *for all* $x \in \mathrm{R}$, *and* $g(x) = x\left(x^3 - 1\right)$ *for all* $x \in \mathrm{R}$. *Verify that f is one-to-one but g is not.*

► For any $x_1, x_2 \in \mathrm{R}$, we have

$$f(x_1) = 3x_1 + 7,\ f(x_2) = 3x_2 + 7.$$

Evidently, if $f(x_1) = f(x_2)$ we have $3x_1 + 7 = 3x_2 + 7$ so that $x_1 = x_2$. Therefore, f is an one-to-one function.

Next, we note that $g(0) = 0$ and $g(1) = 0$. Thus, for $x_1 = 0$ and $x_2 = 1$ we have $g(x_1) = g(x_2)$ but $x_1 \neq x_2$. Therefore, g is not a one-to-one function. ■

Example 4 *The function $f : (\mathbb{Z}\times\mathbb{Z}) \to \mathbb{Z}$ is defined by $f(x, y) = 2x+3y$. Verify that f is onto but not one-to-one.*

If $A = \{(0, n) | n \in \mathbb{Z}^+\}$, find $f(A)$.

▸ Take any $n \in \mathbb{Z}$. We note that

$$n = 4n - 3n = 2(2n) + 3(-n) = f(2n, -n)$$

Thus, every $n \in \mathbb{Z}$ has a preimage $(2n, -n) \in \mathbb{Z}\times\mathbb{Z}$ under f. Therefore, f is an onto function.

Next, we check that $f(0, 2) = 2 \times 0 + 3 \times 2 = 6$ and $f(3, 0) = 2 \times 3 + 3 \times 0 = 6$. Thus, $f(0, 2) = f(3, 0)$, but $(0, 2) \neq (3, 0)$. Therefore, f is not one-to-one.

For any $(0, n) \in A$, we have

$$f(0, n) = 2 \cdot 0 + 3n = 3n$$

Therefore

$$f(A) = \{3n | n \in \mathbb{Z}^+\} = \{3, 6, 9, 12, \cdots\}$$ ■

Example 5 *Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(a) = a+1$ for $a \in \mathbb{Z}$. Find whether f is one-to-one or onto (or both or neither).*

▸ Take any $a_1, a_2 \in \mathbb{Z}$ with $a_1 \neq a_2$. Then $f(a_1) = a_1 + 1$ and $f(a_2) = a_2 + 1$. Since $a_1 \neq a_2$, it is evident that $f(a_1) \neq f(a_2)$. Thus, different elements of $\mathbb{Z}$ have different images under f. Therefore, f is one-to-one.

Take any $b \in \mathbb{Z}$. We check that b has $b - 1$ as its preimage under f; because $f(b - 1) = (b - 1) + 1 = b$. Thus, every element of $\mathbb{Z}$ has a preimage. Therefore, f is onto.

Thus, f is both one-to-one and onto; that is, f is a one-to-one correspondence. ■

Example 6 *Let $A = \mathbb{R}$ and $B = \{x \mid x$ is real and $x \geq 0\}$. Is the function $f : A \to B$ defined by $f(a) = a^2$ an onto function? a one-to-one function?*

▸ Take any $b \in B$. Then b is a non-negative real number. Therefore, its square roots $\pm\sqrt{b}$ exist and are real numbers; that is $\pm\sqrt{b} \in A$. By the definition of f, we note that

$$f\left(\sqrt{b}\right) = \left(\sqrt{b}\right)^2 = b \quad \text{and} \quad f\left(-\sqrt{b}\right) = \left(-\sqrt{b}\right)^2 = b.$$

Thus, $\pm\sqrt{b}$ are preimages of b under f. Since b is an arbitrary element f B, it follows that every element in B has a (at least one) preimage in A. Hence f is an onto function. Since $b \in B$ has two preimages $\pm\sqrt{b} \in A$ under f, it follows that f is not one-to-one. ■

Exercises

1. Let $A = \{a_1, a_2, a_3\}$, $B = \{b_1, b_2, b_3\}$, $C = \{c_1, c_2\}$, and $D = \{d_1, d_2, d_3, d_4\}$. Let $f_1 : A \to B$, $f_2 : A \to D$, $f_3 : B \to C$ and $f_4 : D \to B$ be functions defined as follows:
   (i) $f_1 = \{(a_1, b_2),\ (a_2, b_3),\ (a_3, b_1)\}$
   (ii) $f_2 = \{(a_1, d_2),\ (a_2, d_1),\ (a_3, d_4)\}$
   (iii) $f_3 = \{(b_1, c_2),\ (b_2, c_2),\ (b_3, c_1)\}$
   (iv) $f_4 = \{(d_1, b_1),\ (d_2, b_2),\ (d_3, b_1), (d_4, b_2)\}$

   Find the nature of each of these functions.

2. Prove the following:
   (i) If $A = \{1, 2, 3, 4\}$, $B = \{w, x, y, z\}$, and $f = \{(1, w), (2, x), (3, y), (4, z)\}$, then f is both one-to-one and onto.
   (ii) If $A = \{1, 2, 3, 4, 5\}$, $B = \{w, x, y, z\}$, and $f = \{(1, w), (2, w), (3, x), (4, y), (5, z)\}$, then f is onto but not one-to-one.
   (iii) If $A = \{1, 2, 3, 4\}$, $B = \{v, w, x, y, z\}$, $f = \{(1, v), (2, x), (3, z), (4, y)\}$ and $g = \{(1, v), (2, v), (3, w), (4, x)\}$, then f is one-to-one but not onto, and g is neither one-to-one nor onto.

3. In each of the following cases, sets A and B and a function f from A to B are given. Determine (in each case) whether f is one-to-one or onto (or both or neither).
   (i) $A = B = \{1, 2, 3, 4\}$, $f = \{(1, 1),\ (2, 3),\ (3, 4),\ (4, 2)\}$
   (ii) $A = \{a, b, c\}$; $B = \{1, 2, 3, 4\}$, $f = \{(a, 1),\ (b, 1),\ (c, 3)\}$
   (iii) $A = \{1, 2, 3\}$, $B = \{1, 2, 3, 4, 5\}$, $f = \{(1, 1),\ (2, 3),\ (3, 4)\}$
   (iv) $A = \{1, 2, 3\}$, $B = \{1, 2, 3, 4, 5\}$, $f = \{(1, 1),\ (2, 3),\ (3, 3)\}$
   (v) $A = \{1, 2, 3, 4\}$, $B = \{a, b, c, d\}$, $f = \{(1, a),\ (2, a),\ (3, d),\ (4, c)\}$
   (vi) $A = \mathrm{R}$, $B = \{\, x \mid x \text{ is real and } x \geq 0 \,\}$; $f(a) = |\, a \,|$
   (vii) $A = \mathrm{R} \times \mathrm{R}$, $B = \mathrm{R}$; $f((a, b)) = ab$

4. In each of the following cases, a function $f : \mathrm{R} \to \mathrm{R}$ is given. Determine whether f is one-to-one or onto. If f is not onto, find its range.
   (i) $f(x) = 2x - 3$
   (ii) $f(x) = x^3$
   (iii) $f(x) = x^2$
   (iv) $f(x) = x^2 + x$
   (v) $f(x) = e^{x^2}$
   (vi) $f(x) = \sin x$.

5. A function $f : (\mathrm{Z} \times \mathrm{Z}) \to \mathrm{Z}$ is defined by $f((x, y)) = 4x + 5y$. Prove that f is not one-to-one, but onto. Find $f(A)$ where $A = \{(n, 0) \mid n \in \mathrm{Z}^+\}$.

Answers

1. (i) f_1 is one-to-one and onto, (ii) f_2 is one-to-one, but not onto,
   (iii) f_3 is onto, but not one-to-one,
   (iv) f_4 is neither one-to-one nor onto.

3. (i) Both (ii) Neither
   (iii) One-to-one, but not onto (iv) Neither one-to-one nor onto
   (v) Neither (vi) Onto, but not one-to-one
   (vii) Onto, but not one-to-one (viii) both.

4. (i), (ii): one-to-one and onto.
   (iii) Neither one-to-one nor onto; range $[0, \infty)$.
   (iv) Neither one-to-one nor onto; range $[-\frac{1}{4}, \infty)$.
   (v) Neither one-to-one nor onto, range $[0, \infty)$.
   (vi) Neither one-to-one nor onto; range $[-1, 1]$.

5. $f(A) = \{4n \mid n \in Z^+\}$

5.3.2 Properties of functions

The following theorems contain some basic properties of functions.

Theorem 1. *Let A and B be finite sets and f be a function from A to B. Then the following are true:*

(1) *If f is one-to-one, then $|A| \leq |B|$.*

(2) *If f is onto, then $|B| \leq |A|$.*

(3) *If f is a one-to-one correspondence, then $|A| = |B|$.*

(4) *If $|A| > |B|$, then at least two different elements of A have the same image under f.*

Proof: Here, A and B are finite sets. Let

$$A = \{a_1, a_2, \ldots, a_n\} \quad \text{and} \quad B = \{b_1, b_2, \ldots, b_m\}$$

so that $|A| = n$ and $|B| = m$.

(1) Suppose f is one-to-one. Then the images of the elements of A, namely $f(a_1)$, $f(a_2)$, $\ldots$, $f(a_n)$, are all different and so their number is n. All these images belong to B. Therefore, B must have at least n elements; that is $|B| \geq n$. Since $n = |A|$, this means that $|A| \leq |B|$.

(2) Suppose f is onto. Then with each b in B there is an a in A such that $f(a) = b$. Since f is a function, no two different b's can correspond to the same a. Therefore, the number k of a's which are preimages of b's cannot be less than the number of b's. Thus, we should have $k \geq m$. On the other hand, every a is a preimage of some b (under f). Therefore, $k = n$. Thus, $m \leq k$ and $k = n$. As such; $m \leq n$; that is $|B| \leq |A|$.

(3) Suppose f is a one-to-one correspondence. Then f is both one-to-one and onto. From results proved in the above two paragraphs, it follows that $|A| \leq |B|$ and $|B| \leq |A|$. Therefore, $|A| = B$.

(4) The contrapositive of the result (1) reads: " If $|A| > |B|$, then f is not one-to-one." This means that if $|A| > B$, then at least two different elements of A have the same image under f. This result is true, because the result (1) is true.*

This completes the proof of the theorem.

Remark: The result (4) of the above theorem can be interpreted as follows: If every element of a set A with $|A| = n$ is assigned to a unique element of a set B with $|B| = m$ and if $n > m$, then at least two different elements of A are assigned to the same element of B. In other words, if n objects (pigeons, say) are assigned to m places (pigeonholes), and if $n > m$, then at least two objects (pigeons) are assigned to the same place. This is known as the ***Pigeonhole Principle****.

Theorem 2. *Suppose A and B are finite sets having the same number of elements, and f is a function from A to B. Then f is one-to-one if and only if f is onto.*

Proof: Here $f : A \to B$ where A and B are finite sets with $|A| = |B|$. Let $A = \{a_1, a_2, \ldots, a_n\}$.

First suppose that f is one-to-one. Then, the images of elements of A, namely $f(a_1)$, $f(a_2)$, $\ldots$, $f(a_n)$, are all different. These images constitute the range of f and since they are n in number, we have $|f(A)| = n$. But $f(A) \subseteq B$. Thus, B and its subset $f(A)$ have the same number of elements. Therefore, $f(A) = B$. This means that f is onto.

Conversely, suppose f is onto. Then

$$B = f(A) = \{f(a_1),\ f(a_2), \cdots, f(a_n)\}.$$

*Recall that a conditional and its contrapositive are logically equivalent.

**This principle is discussed separately in Section 5.4.

Since $|B| = n$, it follows that $f(a_1), f(a_2), \cdots, f(a_n)$ are exactly n in number; in other words, $f(a_1), f(a_2), \cdots, f(a_n)$ must all be different. Accordingly, f is one-to-one.

This completes the proof. •

Corollary. *A function f from a finite set A to a finite set B with $|A| = |B|$ is bijective if and only if f is one-to-one or onto.*

Proof: Obvious, by Theorem 2. •

Remark: The contents of Theorem 2 and its corollary can be put together as follows:

If A and B are finite sets with $|A| = |B|$ and $f : A \to B$, then *the following statements are equivalent:*

(1) f is one-to-one.

(2) f is onto.

(3) f is a bijection.

(4) f is a one-to-one correspondence.

Accordingly, if we wish to prove that a certain function $f : A \to B$, where A and B are finite sets with $|A| = |B|$ is bijective (one-to-one correspondence), we need only to show that f is one-to-one *or* onto.

Numbers of one-to-one and onto functions

If A and B are finite sets with $|A| = m$ and $|B| = n$, where $m \geq n$, then :

(1) The number of one-to-one functions possible from A to B is given by the formula

$$P(n, m) = \frac{n!}{(n-m)!}$$

(2) The number of onto functions possible from A to B is given by the formula

$$p(n, m) = \sum_{k=0}^{n} (-1)^k \, ({}^{n}C_{n-k}) \, (n-k)^m$$

The proofs are omitted.

Example 1 *Let A and B be finite sets. If there are 60 one-to-one functions from A to B and $|A| = 3$, what is $|B|$?*

▶ Here, $m = 3$ and $\dfrac{n!}{(n-m)!} = 60$. Thus,

$$\frac{n!}{(n-3)!} = 60. \text{ This yields } n(n-1)(n-2) = 60. \text{ Evidently, } n = 5.$$

Thus, $|B| = 5$. ■

Example 2 *Let $A = \{1,2,3,4,5,6,7\}$ and $B = \{w,x,y,z\}$. Find the number of onto functions from A to B.*

▶ Here $m = |A| = 7$ and $n = |B| = 4$. Therefore, the number of onto functions from A to B is

$$\begin{aligned} p(7,4) &= \sum_{k=0}^{4}(-1)^k\left({}^4C_{4-k}\right)(4-k)^7 \\ &= {}^4C_4 \times 4^7 - {}^4C_3(4-1)^7 + {}^4C_2(4-2)^7 - {}^4C_1(4-3)^7 + 0 \\ &= 4^7 - 4\times 3^7 + 6\times 2^7 - 4 \\ &= 8400. \end{aligned}$$

∎

Example 3 *Let $A = \{1,2,3,4\}$ and $B = \{1,2,3,4,5,6\}$.*

(a) *Find how many functions are there from A to B. How many of these are one-to-one? How many are onto?*

(b) *Find how many functions are there from B to A. How many of these are one-to-one? How many are onto?*

▶ Here, $|A| = m = 4$ and $|B| = n = 6$. Therefore:

(a) The number of functions possible from A to B is $n^m = 6^4 = 1296$.

The number of one-to-one functions possible from A to B is

$$\frac{n!}{(n-m)!} = \frac{6!}{2!} = 360.$$

There is no onto function from A to B.

(b) The number of functions possible from B to A is $m^n = 4^6 = 4096$.

There is no one-to-one function from B to A.

The number of onto functions from B to A is

$$\begin{aligned} p(6,4) &= \sum_{k=0}^{4}(-1)^k({}^4C_{4-k})(4-k)^6 \\ &= 4^6 - 4\times 3^6 + 6\times 2^6 - 4 = 1560. \end{aligned}$$

∎

Exercises

1. If $A = \{2, 6, 9\}$ and $B = \{p, q, r, s, t\}$, find the number of one-to-one functions from A to B.
2. If $A = \{1, 2, 3, 4, 5\}$ and there are 6720 one-to-one functions $f : A \to B$, what is $|B|$?
3. If $A = \{w, x, y, z\}$ and $B = \{1, 2, 3\}$, find how many onto functions are there from A to B.
4. If A and B are finite sets with $|A| = 5$ and $|B| = 3$, find the number of onto functions from A to B.
5. Let $A = \{1, b, c, d, e\}$ and $B = \{1, 2, 3, 4, 5, 6, 7, 8, \}$.

   (i) Determine the number of functions from A to B. How many of these are one-to-one? onto?

   (ii) Determine the number of functions from B to A. How many of these are one-to-one? onto?
6. If a set A has n elements, how many one-to-one correspondences are there from A to A?
7. If $A = \{1, 2, 3, \ldots, n\}$, for some fixed $n \in \mathbb{Z}^+$, find how many bijective functions are there with $f(1) \neq 1$.

Answers

1. 60 **2.** 8 **3.** 36 **4.** 150 **5.** (i) 8^5, 160, 0 (ii) 5^8, 0, 126000. **6.** $n!$ **7.** $n! - (n-1)!$

5.4 The Pigeonhole Principle

As an interpretation of Part (4) of Theorem 1 of Section 5.3.2, we have got the following statement:

If m pigeons occupy n pigeonholes and if m > n, then two or more pigeons occupy the same pigeonhole.

This is often restated as follows:

If m pigeons occupy n pigeonholes, where m > n, then at least one pigeonhole must contain two or more pigeons in it.

As already mentioned, this statement is known as the ***Pigeonhole Principle***.

A simple illustration of the above principle is that if 6 pigeons occupy 4 pigeonholes, then at least one pigeonhole must contain two or more pigeons in it. See Figure 5.15.

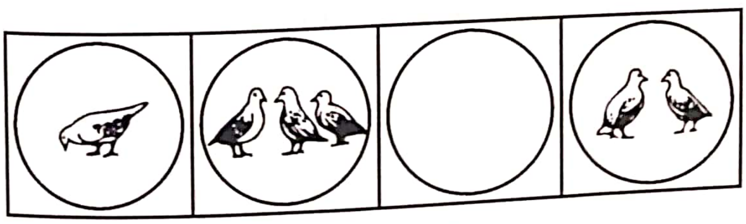


Figure 5.15

As a simple application of the principle, we may note that if 8 children are born in the same week, then two or more children are born on the same day of the week.

Generalization

The following is an extension/generalization of the pigeonhole principle.

If m pigeons occupy n pigeonholes, then at least one pigeonhole must contain $(p + 1)$ *or more pigeons, where* $p = \lfloor (m-1)/n \rfloor$.*

Proof: We prove this principle by the method of contradiction.

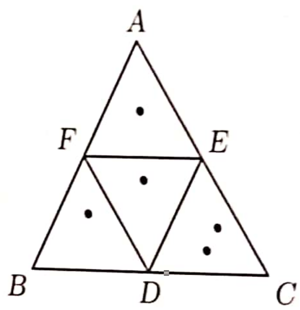
Assume that the conclusion part of the principle is not true. Then, no pigeonhole contains $(p+1)$ or more pigeons. This means that every pigeonhole contains p or less number of pigeons. Then:

$$\text{Total number of pigeons} \le np = n \times \lfloor (m-1)/n \rfloor \le n\left(\frac{m-1}{n}\right) = (m-1).$$

This is a contradiction, because the total number of pigeons is m. Hence our assumption is wrong, and the principle is true. ∎

Example 1 *ABC is an equilateral triangle whose sides are of length 1 cm each. If we select 5 points inside the triangle, prove that at least two of these points are such that the distance between them is less than 1/2 cm.*

▶ Consider the triangle DEF formed by the mid-points of the sides BC, CA and AB of the given triangle ABC; see Figure 5.16. Then the triangle ABC is partitioned into four small equilateral triangles (portions), each of which has sides equal to 1/2 cm. Treating each of these four portions as a pigeonhole and five points chosen inside the triangle as pigeons, we find by using the pigeonhole principle that at least one portion must contain two or more points. Evidently, the distance between such points is less than 1/2 cm.



∎

Figure 5.16

Example 2 *A bag contains 12 pairs of socks (each pair in different color). If a person draws the socks one by one at random, determine at most how many draws are required to get at least one pair of matched socks.*

*For any real number x, the symbol $\lfloor x \rfloor$ represents the greatest integer less than or equal to x. Thus, $\lfloor x \rfloor \le x$. The symbol $\lfloor x \rfloor$ is called the floor of x.

▶ Let n denote the number of the draw. For $n \leq 12$, it is possible that the socks drawn are of different colors, because there are 12 colors. For $n = 13$, all socks cannot have different colors – at least two must have the same color (Treat 13 as the number of pigeons and 12 colors as 12 pigeonholes). Thus, at most 13 draws are required to have at least one pair of socks of the same color. ■

Example 3 *A magnetic tape contains a collection of 5 lakh strings made up of four or fewer number of English letters. Can all the strings in the collection be distinct?*

▶ Each place in an n letter string can be filled in 26 ways. Therefore, the possible number of strings made up of n letters is 26^n. Consequently, the total number of different possible strings made up of four or fewer letters is

$$26^4 + 26^3 + 26^2 + 26 = 4,75,254.$$

Therefore, if there are 5 lakh strings in the tape, then at least one string is repeated. (Treat 4,75,254 strings as pigeonholes and 5 lakh strings as pigeons). Thus, all the strings in the collection cannot be distinct. ■

Example 4 *Prove that if 30 dictionaries in a library contain a total of 61,327 pages, then at least one of the dictionaries must have at least 2045 pages.*

▶ Treating the pages as pigeons and dictionaries as pigeonholes, we find by using the generalized pigeonhole principle that at least one of the dictionaries must contain $p+1$ or more pages, where

$$p = \left\lfloor \frac{61327 - 1}{30} \right\rfloor = \lfloor 2044.2 \rfloor = 2044.$$

This proves the required result. ■

Example 5 *If 5 colours are used to paint 26 doors, prove that at least 6 doors will have the same colour.*

▶ Treating 26 doors as pigeons and 5 colours as pigeonholes, we find by using the generalized pigeonhole principle that at least one of the colours must be assigned to $\left(\frac{26-1}{5}\right) + 1 = 6$ or more doors.

This proves the required result. ■

Example 6 *Prove that in any set of 29 persons at least five persons must have been born on the same day of the week.*

▶ Treating the seven days of a week as 7 pigeonholes and 29 persons as pigeons, we find by using the generalized pigeonhole principle that at least one day of the week is assigned to $\left(\frac{29-1}{7}\right) + 1 = 5$ or more persons. In other words, at least 5 of any 29 persons must have been born on the same day of the week. ■

Example 7 *How many persons must be chosen in order that at least five of them will have birth days in the same calendar month?*

▸ Let n be the required number of persons. Since the number of months over which the birthdays are distributed is 12, the least number of persons who have their birthdays in the same month is, by the generalized pigeonhole principle, equal to $\left\lfloor \frac{(n-1)}{12} \right\rfloor + 1$. This number is 5, if

$$\left\lfloor \frac{(n-1)}{12} \right\rfloor + 1 = 5, \text{ or } n = 49.$$

Thus, the number of persons is 49 (at the least). ∎

Example 8 *Find the least number of ways of choosing three different numbers from 1 to 10 so that all choices have the same sum.*

▸ From the numbers from 1 to 10, we can choose three different numbers in $C(10, 3) = 120$ ways.

The smallest possible sum that we get from a choice is 1+2+3=6 and the largest sum is 8+9+10=27. Thus, the sums vary from 6 to 27 (both inclusive), and these sums are 22 in number.

Accordingly, here, there are 120 choices (pigeons) and 22 sums (pigeonholes). Therefore, the least number of choices assigned to the same sum is, by the generalized pigeonhole principle,

$$\left\lfloor \frac{120-1}{22} \right\rfloor + 1 = \lfloor 6.4 \rfloor \approx 6.$$ ∎

Example 9 *Shirts numbered consecutively from 1 to 20 are worn by 20 students of a class. When any 3 of these students are chosen to be a debating team from the class, the sum of their shirt numbers is used as the code number of the team. Show that if any 8 of the 20 are selected, then from these 8 we may form at least two different teams having the same code number.*

▸ From the 8 of the 20 students selected, the number of teams of 3 students that can be formed is ${}^8C_3 = 56$.

According to the way in which the code number of a team is determined, we note that the smallest possible code number is $1 + 2 + 3 = 6$ and the largest possible code number is $18 + 19 + 20 = 57$. Thus, the code numbers vary from 6 to 57 (inclusive), and these are 52 in number. As such, only 52 code numbers (pigeon holes) are available for the 56 possible teams (pigeons). Consequently, by the pigeonhole principle, at least two different teams will have the same code number. ∎

Example 10 *Prove that every set of 37 positive integers contains at least two integers that leave the same remainder upon division by 36.*

▸ When a positive integer is divided by 36, the possible remainders are $0, 1, 2, \ldots 35$. Let A_r denote the set of all positive integers that leave the remainder r when divided by 36. Thus.

every positive integer belongs to one or the other of the 36 sets: $A_0, A_1, A_2, \ldots, A_{35}$. Hence if we take any 37 positive integers, then at least two of them must belong to one of these A_r's (Treat A_r's as pigeonholes and 37 as the number of pigeons). This proves the required result. ■

Example 11 *Show that if any $n + 1$ numbers from 1 to $2n$ are chosen, then two of them will have their sum equal to $2n + 1$.*

▸ Let us consider the following sets:

$$A_1 = \{1, 2n\}, \quad A_2 = \{2, 2n-1\}, \ldots A_{n-1} = \{n-1, n+2\}, \quad A_n = \{n, n+1\}$$

These are the only sets containing two numbers from 1 to $2n$ whose sum is $2n + 1$.

Since every number from 1 to $2n$ belongs to one of the above sets, each of the $n + 1$ numbers chosen must belong to one of the sets. Since there are only n sets, two of the $n + 1$ chosen numbers have to belong to the same set (according to the pigeonhole principle). These two numbers have their sum equal to $2n + 1$. ■

Example 12 *Show that every set of seven distinct integers includes two integers x and y such that at least one of $x + y$ or $x - y$ is divisible by 10.*

▸ Let $X = \{x_1, x_2, \ldots, x_7\}$ be a set of seven distinct integers and let r_i be the remainder when x_i is divided by 10.

Consider the following subsets of X:

$$A_1 = \{x_i \in X | r_i = 0\}$$
$$A_2 = \{x_i \in X | r_i = 5\}$$
$$A_3 = \{x_i \in X | r_i = 1 \text{ or } 9\}$$
$$A_4 = \{x_i \in X | r_i = 2 \text{ or } 8\}$$
$$A_5 = \{x_i \in X | r_i = 3 \text{ or } 7\}$$
$$A_6 = \{x_i \in X | r_i = 4 \text{ or } 6\}$$

Now, the seven elements of X play the role of pigeons and the six subsets listed above play the role of pigeonholes. As such, at least two elements x, y (say) of X must be in the same subset.

If x and y are in A_1 then x and y are multiples of 10 so that both $x+y$ and $x-y$ are multiples of 10.

If x and y are in A_2, then x and y are of the forms $x = 10k_1 + 5$ and $y = 10k_2 + 5$, where k_1 and k_2 are integers, so that $x + y = 10(k_1 + k_2 + 1)$ and $x - y = 10(k_1 - k_2)$ are both multiples of 10.

If x and y are in any of the other four subsets, then it is easily seen that either $x - y$ or $x + y$ is a multiple of 10, but not both. This proves the required result. ■

Example 13 *Prove that if 101 integers are selected from the set $S = \{1, 2, 3, \ldots, 200\}$, then at least two of these are such that one divides the other.*

▶ Let $X = \{1, 3, 5, \ldots, 199\}$. Then every integer n between 1 and 200 (inclusive) is of the form $n = 2^k x$, where k is an integer ≥ 0 and $x \in X$. Thus, every element of S corresponds to some $x \in X$.

The set X has 100 distinct elements and therefore, if 101 elements of S are selected, then at least two of them say a and b, $a \neq b$ must correspond to the same $x \in X$. Thus, $a = 2^m x$, $b = 2^n x$, for some integers $m, n \geq 0$. Evidently, a divides b if $m \leq n$ and b divides a if $n < m$. This proves the required result. ∎

Example 14 *Suppose that a patient is given a prescription of 45 pills with instructions to take at least one pill a day for 30 days. Prove that there must be a period of consecutive days during which the patient takes a total of exactly 14 pills.*

▶ Let a_i be the number of pills the patient has taken through the end of the i^{th} day. Since the patient takes at least one pill per day and at most 45 pills in 30 days, we have

$$1 \leq a_1 < a_2 < a_3 < \cdots < a_{30} \leq 45$$

These inequalities give

$$a_1 + 14 < a_2 + 14 < a_3 + 14 < \cdots < a_{30} + 14 \leq 45 + 14 \,(= 59)$$

Thus, we have 60 positive integers

$$a_1, a_2, a_3, \ldots, a_{30}, \quad a_1 + 14, a_2 + 14, \ldots a_{30} + 14$$

all of which lie between 1 and 59 (inclusive). (That is, we have 60 pigeons in 59 pigeonholes). So, two of these numbers must be equal. Since $a_1, a_2, \ldots, a_{30}$ are all different and $a_1 + 14$, $a_2 + 14, \ldots, a_{30} + 14$ are all different, it must be that one of $a_1, a_2, \ldots, a_{30}$ is equal to one of $a_1 + 14, a_2 + 14, \ldots, a_{30} + 14$. This means that there are i and j such that $a_i = a_j + 14$.

Thus, between days i and j, the patient takes exactly 14 pills. ∎

Example 15 *Prove the statement: If $m = kn + 1$ pigeons (where $k \geq 1$) occupy n pigeonholes then at least one pigeonhole must contain $k + 1$ or more pigeons.*†

▶ Assume that the conclusion part of the given statement is false. Then every pigeonhole contains k or less number of pigeons. Then, the total number of pigeons would be nk. This is a contradiction. Hence, the assumption made is wrong, and the given statement is true. ∎

Example 16 *A bag contains many red marbles, many white marbles, and many blue marbles. What is the least number of marbles one should take out to be sure of getting at least six marbles of the same color?*

†This is an alternative version of the generalized pigeonhole principle.

▶ Let us treat the marbles as pigeons and colors as pigeonholes. Then, the number of pigeonholes is $n = 3$. Therefore, if $m = 3k + 1$ pigeons, where $k \geq 1$, occupy 3 pigeonholes then at least one pigeonhole must contain $k+1$ or more pigeons. We note that $k+1 = 6$ corresponds to $m = 16$. Therefore, the presence of 16 or more pigeons in 3 pigeonholes will ensure that there are 6 or more pigeons in a hole. Thus, 16 is the least number of marbles to be taken out. ■

Example 17 *Suppose* $m = (p_1 + p_2 + \cdots + p_n - n + 1)$ *pigeons occupy* n *pigeonholes* $H_1, H_2, \ldots, H_n$. *Prove that some pigeonhole* H_j *contains* p_j *or more pigeons.*‡

▶ Assume that the conclusion part of the given statement is false. Then every hole H_j contains $p_j - 1$ or less number of pigeons, $j = 1, 2, \ldots n$. Then the total number of pigeons would be less than or equal to

$$(p_1 - 1) + (p_2 - 1) + \cdots + (p_n - 1) = (p_1 + p_2 + \cdots + p_n - n) = m - 1.$$

This is a contradiction, because the number of pigeons is equal to m. Hence the assumption made is wrong, and the given statement is true. ■

Exercises

1. Prove that in a set of 13 children at least two have birthdays during the same month.
2. If three men and five women form a queue, show that at least two women will be next to each other.
3. If seven cars carry 26 passengers, prove that at least one car must have 4 or more passengers.
4. If six persons have a total of Rs. 2161 with them, show that one or more of them must have at least Rs. 361.
5. Show that if 50 books in a library contain a total of 27551 pages, one of the books must have at least 552 pages.
6. Suppose there are many red socks, many white socks and many blue socks in a box. What is the least number of socks that one should take out from the box to be sure of getting a matching pair? (*Ans*: 4)
7. What should be the minimum number of students so that at least two students have their last names begin with the same English letter. (*Ans*: 27)
8. Show that if five points are selected in a square whose sides have length 1cm each, then there are at least two points whose distance apart is less than $\frac{1}{\sqrt{2}}$ cm.
9. If 10 points are selected from the interior of a triangle whose sides are of length 3 cms (each), show that at least two points are within 1cm apart.

‡This is a further extension of the Pigeonhole Principle.

10. Show that if any 6 numbers from 1 to 10 are chosen, then two of them have their sum equal to 11.

11. Show that if any eight positive integers are chosen, two of them will have the same remainder when divided by 7.

12. Show that there are at least 90 ways to choose six numbers from 1 to 5 so that all the choices have the same sum.

13. A set of children from primary I to IV Standards are assembled in a room. Find the minimum number of children that one needs to choose from the set in order that five of them belong to the same Standard. (*Ans*: 17)

14. Show that if 11 numbers are chosen from the set $\{1, 2, 3, \ldots, 20\}$, one of them is a multiple of another.

15. Prove that if 151 integers are selected from the set $S = \{1, 2, 3, \ldots, 300\}$, then the selection must include two integers m, n such that m divides n or n divides m.

16. Let S be a set of 5 positive integers the maximum of which is at most 9. Prove that the sums of the elements in the nonempty subsets of S cannot all be distinct.

17. If 11 integers are selected from the set $S = \{1, 2, 3, \ldots 100\}$, prove that there are at least two, say x and y, such that $0 < |\sqrt{x} - \sqrt{y}| < 1$.

18. A student plays at least one set of tennis each day, but does not play more than 40 sets in a four-week period. Prove that no matter how he distributes his sets during the four weeks, there is a consecutive span of days during which he will play exactly 15 sets.

19. Show that in any set of $n \geq 2$ persons there will be at least two persons who know the same number of persons in the set.

20. Consider a tournament in which each player plays against every other player and each player wins at least once. Show that there are at least two players having the same number of wins.

5.5 Composition of functions

Consider three non-empty sets A, B, C (which are not necessarily distinct) and the functions $f : A \to B$ and $g : B \to C$. The ***composition*** (or *product*) of these two functions is *defined* as the function $g \circ f : A \to C$ with $(g \circ f)(a) = g\{f(a)\}$, for all $a \in A$.

The pictorial representation of $g \circ f$ is shown in Figure 5.17.

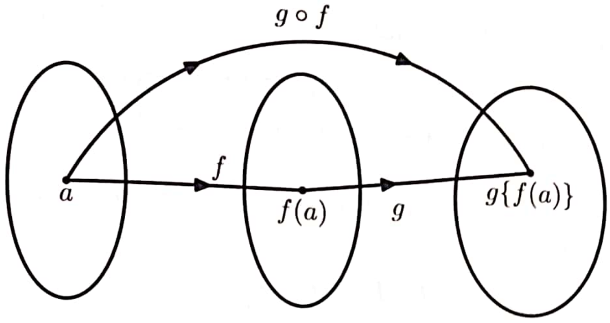


Figure 5.17

For a function $f : A \to A$, $f \circ f$ is denoted by f^2, $f \circ f^2$ is denoted by f^3, and so on. For any integer $n \geq 2$, the function $f^n : A \to A$ is defined *recursively* by

$$f^1 = f, \quad f^n = f \cdot f^{n-1}.$$

Example 1 *Let* $A = \{1, 2, 3, 4\}$, $B = \{a, b, c\}$ *and* $C = \{w, x, y, z\}$ *with* $f : A \to B$ *and* $g : B \to C$ *given by*

$$f = \{(1, a), (2, a), (3, b), (4, c)\}, \text{ and } g = \{(a, x), (b, y), (c, z)\}$$

Find $g \circ f$.

▸ We find, by using the definitions of f and g, that

$$(g \circ f)(1) = g\{f(1)\} = g(a) = x,$$

$$(g \circ f)(2) = g\{f(2)\} = g(a) = x,$$

$$(g \circ f)(3) = g\{f(3)\} = g(b) = y,$$

$$(g \circ f)(4) = g\{f(4)\} = g(c) = z.$$

Thus, $g \circ f = \{(1, x), (2, x), (3, y), (4, z)\}$. ■

Example 2 *Consider the functions* f *and* g *defined by* $f(x) = x^3$ *and* $g(x) = x^2 + 1$, $\forall x \in \mathbb{R}$. *Find* $g \circ f$, $f \circ g$, f^2 *and* g^2.

▸ Here, both f and g are defined on $\mathbb{R}$. Therefore, all of the functions $g \circ f$, $f \circ g$, $f^2 = f \circ f$ and $g^2 = g \circ g$ are defined on $\mathbb{R}$, and we find that

$$(g \circ f)(x) = g\{f(x)\} = g\left(x^3\right) = \left(x^3\right)^2 + 1 = x^6 + 1,$$

$$(f \circ g)(x) = f\{g(x)\} = f\left(x^2 + 1\right) = \left(x^2 + 1\right)^3,$$

$$f^2(x) = (f \circ f)(x) = f\{f(x)\} = f\left(x^3\right) = \left(x^3\right)^3 = x^9,$$

$$g^2(x) = (g \circ g)(x) = g\{g(x)\} = g\left(x^2 + 1\right) = \left(x^2 + 1\right)^2 + 1.$$

The above expressions define the function's $g \circ f$, $f \circ g$, f^2 and g^2 respectively.

Remark: This example illustrates the important fact that the functions $g \circ f$ and $f \circ g$ are *not* one and the same even when they exist. The composition of functions is thus *not commutative*. ∎

Example 3 *Let I_A and I_B denote the identity functions on sets A and B respectively. For any function $f : A \to B$, prove that*

$$f \circ I_A = f = I_B \circ f.$$

▶ For any $a \in A$, we have

$$(f \circ I_A)(a) = f\{I_A(a)\} = f(a)$$

and

$$(I_B \circ f)(a) = I_B\{f(a)\} = f(a)$$

Therefore, $f \circ I_A = f$ and $I_B \circ f = f$. ∎

Example 4 *Let f and g be functions from R to R defined by $f(x) = ax+b$ and $g(x) = 1-x+x^2$. If $(g \circ f)(x) = 9x^2 - 9x + 3$, determine a, b.*

▶ We have

$$\begin{aligned} 9x^2 - 9x + 3 &= (g \circ f)(x) \\ &= g\{f(x)\} = g(ax+b) \\ &= 1 - (ax+b) + (ax+b)^2 \\ &= a^2x^2 + (2ab - a)x + \left(1 - b + b^2\right). \end{aligned}$$

Comparing the corresponding coefficients, we get

$$9 = a^2,\ 9 = a - 2ab,\ 3 = 1 - b + b^2.$$

The first of these gives $a = \pm 3$. For $a = 3$, the second and third are satisfied if $b = -1$. For $a = -3$, we get $b = 2$.

Thus, $a = 3, b = -1$, and $a = -3, b = 2$ are the required values of a, b. ∎

Example 5 *Let f and g be functions from R to R defined by $f(x) = ax+b$ and $g(x) = cx+d$. What relationship must be satisfied by a, b, c, d if $g \circ f = f \circ g$?*

▶ We have, for any $x \in \mathsf{R}$,

$$\begin{aligned} (g \circ f)(x) &= g\{f(x)\} = g(ax+b) \\ &= c(ax+b) + d = cax + (cb+d) \end{aligned}$$

and

$$\begin{aligned} (f \circ g)(x) &= f\{g(x)\} = f(cx+d) \\ &= a(cx+d) + b = cax + (ad+b) \end{aligned}$$

Therefore, if $(g \circ f) = (f \circ g)$, we should have

$$cb + d = ad + b.$$

■

The following theorems contain some important results on composition of functions.

Theorem 1. *Let $f : A \to B$ and $g : B \to C$ be any two functions. Then the following are true:*

(1) *If f and g are one-to-one, so is $g \circ f$.*

(2) *If $g \circ f$ is one-to-one, then f is one-to-one.*

(3) *If f and g are onto, so is $g \circ f$.*

(4) *If $g \circ f$ is onto, then g is onto.*

Proof: First, we note that $g \circ f : A \to C$.

(1) Take any $a_1, a_2 \in A$. We find that

$$\begin{aligned}(g \circ f)(a_1) = (g \circ f)(a_2) &\Rightarrow g(f(a_1)) = g(f(a_2))\\ &\Rightarrow f(a_1) = f(a_2), \text{ because } g \text{ is one-to-one.}\\ &\Rightarrow a_1 = a_2, \text{ because } f \text{ is one-to-one.}\end{aligned}$$

Therefore, $g \circ f$ is one-to-one.

(2) Take any $a_1, a_2 \in A$. Then, $f(a_1), f(a_2) \in B$ and

$$\begin{aligned}f(a_1) = f(a_2) &\Rightarrow g(f(a_1)) = g(f(a_2)), \text{ because } g \text{ is a function from } B.\\ &\Rightarrow (g \circ f)(a_1) = (g \circ f)(a_2)\\ &\Rightarrow a_1 = a_2, \text{ because } (g \circ f) \text{ is one-to-one.}\end{aligned}$$

This shows that f is one-to-one.

(3) Take any $c \in C$. Since g is onto, there is some $b \in B$ such that $g(b) = c$. Since $b \in B$ and f is onto, there is some $a \in A$ such that $f(a) = b$. Consequently,

$$(g \circ f)(a) = g(f(a)) = g(b) = c.$$

Thus, for any $c \in C$, there is some $a \in A$ such that $(g \circ f)(a) = c$. Therefore, $g \circ f$ is onto.

(4) Take any $c \in C$. Since $g \circ f : A \to C$ is onto, there is some $a \in A$ such that $g(f(a)) = c$. Since $f(a) \in B$, this means that, given any $c \in C$, there is an element $f(a)$ in B such that $g(f(a)) = c$. Therefore, g is onto.

■

Theorem 2. *Let $f : A \to B$, $g : B \to C$ and $h : C \to D$ be three functions. Then*

$$(h \circ g) \circ f = h \circ (g \circ f).$$

Proof: We first note that both $(h \circ g) \circ f$ and $h \circ (g \circ f)$ are functions from A to D.

For any $x \in A$, we have

$$[(h \circ g) \circ f](x) = (h \circ g)[f(x)] = (h \circ g)(y), \text{ where } y = f(x)$$

$$= h[g(y)] = h(z), \text{ where } z = g(y) \qquad \text{(i)}$$

and

$$[h \circ (g \circ f)(x)] = h[(g \circ f)(x)]$$

$$= h[g\{f(x)\}] = h[g(y)] = h(z) \qquad \text{(ii)}$$

Results (i) and (ii) show that

$$[(h \circ g) \circ f](x) = [h \circ (g \circ f)](x), \text{ for every } x \in A.$$

Therefore, $(h \circ g) \circ f = h \circ (g \circ f)$; and the theorem is proved.

Remark: The above theorem shows that the *composition of functions is associative*. In view of this result, we write both of $(h \circ g) \circ f$ and $h \circ (g \circ f)$ as $h \circ g \circ f$.

Example 6 *Let f, g, h be functions from $\mathbb{Z}$ to $\mathbb{Z}$ defined by*

$$f(x) = x - 1, \quad g(x) = 3x,$$

$$h(x) = \begin{cases} 0, & \text{if } x \text{ is even} \\ 1, & \text{if } x \text{ is odd.} \end{cases}$$

Determine $(f \circ (g \circ h))(x)$ and $((f \circ g) \circ h)(x)$ and verify that $f \circ (g \circ h) = (f \circ g) \circ h$.

► We have

$$(g \circ h)(x) = g\{h(x)\} = 3h(x)$$

Therefore,

$$f \circ (g \circ h)(x) = f\{(g \circ h)(x)\}$$

$$= f\{3h(x)\} = 3h(x) - 1$$

$$= \begin{cases} -1 & \text{if } x \text{ is even} \\ 2 & \text{if } x \text{ is odd.} \end{cases} \qquad \text{(i)}$$

On the other hand, $(f \circ g)(x) = f\{g(x)\} = g(x) - 1 = 3x - 1$. Therefore,

$$
\begin{aligned}
\{(f \circ g) \circ h\}(x) &= (f \circ g)\{h(x)\} \\
&= 3h(x) - 1 \\
&= \begin{cases} -1 \text{ if } x \text{ is even} \\ 2 \text{ if } x \text{ is odd.} \end{cases} \qquad \text{(ii)}
\end{aligned}
$$

From expression (i) and (ii), it follows that

$$f \circ (g \circ h) = (f \circ g) \circ h.$$

■

Exercises

1. Let $A = B = C = \mathsf{R}$, the set of all real numbers, and $f : A \to B$ and $g : B \to C$ be defined by $f(a) = a - 1$, $a \in A$ and $g(b) = b^2$, $b \in B$. Find

   (i) $(g \circ f)(a)$ (ii) $(f \circ g)(b)$ (iii) $(f \circ f)(a)$ (iv) $(g \circ g)(b)$

2. Let $A = B = \mathsf{Z}$, the set of all integers and C be the set of all even integers. Let $f : A \to B$ and $g : B \to C$ be defined by $f(a) = a + 1$, $a \in A$ and $g(b) = 2b$, $b \in B$. Find $g \circ f$.

3. Let f and g be functions from R to R defined by $f(x) = x^2$ and $g(x) = x+5$. Prove that $g \circ f \neq f \circ g$.

4. Let f, g, h be functions from R to R defined by $f(x) = x + 2$, $g(x) = x - 2$, $h(x) = 3x$, for all $x \in \mathsf{R}$. Find $g \circ f$, $f \circ g$, $f \circ f$, $g \circ g$, $f \circ h$, $h \circ g$, $h \circ f$, $f \circ h \circ g$.

5. Let $f : \mathsf{R}^+ \to \mathsf{R}$ be defined by $f(x) = \log x$, and $g : \mathsf{R} \to \mathsf{R}^+$ be defined by $g(x) = e^x$, where R is the set of all real numbers and R^+ is set of all positive real numbers. Show that $f \circ g$ and $g \circ f$ are identity functions.

6. Let f, g, h be functions from R to R defined by $f(x) = x^2$, $g(x) = x + 5$ and $h(x) = \sqrt{x^2 + 2}$. Verify that $(h \circ g) \circ f = h \circ (g \circ f)$.

7. Let $g : \mathsf{N} \to \mathsf{N}$ be defined by $g(n) = 2n$. If $A = \{1, 2, 3, 4\}$ and $f : A \to \mathsf{N}$ is given by $f = \{(1, 2), (2, 3), (3, 5), (4, 7)\}$, find $g \circ f$.

8. Let $A = \{1, 2, 3\}$, and f, g, h, p be functions on A defined as follows:

$$
\begin{aligned}
f &= \{(1, 2), (2, 3), (3, 1)\}, \quad & g &= \{(1, 2), (2, 1), (3, 3)\}, \\
h &= \{(1, 1), (2, 2), (3, 1)\}, \quad & p &= \{(1, 1), (2, 2), (3, 3)\},
\end{aligned}
$$

   Find $f \circ g$, $g \circ f$, $p \circ g$, $g \circ p$, $f \circ p$, $f \circ h \circ g$.

9. With $A = \{1, 2, 3, 4\}$ and $f : A \to A$ defined by $f = \{(1,2), (2,2), (3,1), (4,3)\}$, find f^2 and f^3.

10. For the functions f, g, h considered in Example 6, determine $g \circ f, h \circ g, f^2, g^2, h^2, f^3, g^3, h^3$.

Answers

1. (i) $(a-1)^2$ (ii) $b^2 - 1$ (iii) $a - 2$ (iv) b^4

2. $(g \circ f)(a) = 2(a+1), a \in A$

4. $g \circ f = f \circ g = \{(x, x)\}$, $f \circ f = \{(x, x+4)\}$, $g \circ g = \{(x, x-4)\}$,
$f \circ h = \{(x, 3x+2)\}$, $h \circ g = \{(x, 3x-6)\}$, $h \circ f = \{(x, 3x+6)\}$, $f \circ h \circ g = \{(x, 3x-4)\}$.
Here, $x \in \mathsf{R}$.

7. $g \circ f = \{(1,4), (2,6), (3,10), (4,14)\}$

8. $f \circ g = \{(1,3), (2,2), (3,1)\}$, $g \circ f = \{(1,1), (2,3), (3,2)\}$,
$p \circ g = \{(1,2), (2,1), (3,3)\} = g = g \circ p$, $f \circ p = \{(1,2), (2,3), (3,1)\} = f$,
$f \circ h \circ g = \{(1,3), (2,2), (3,2)\}$.

9. $f^2 = \{(1,2), (2,2), (3,2), (4,1)\}$, $f^3 = \{(1,2), (2,2), (3,2), (4,2)\}$

10. $(g \circ f)(x) = 3(x-1)$, $(h \circ g)(x) = h(3x) = h(x)$,
$f^2(x) = x - 2$, $f^3(x) = x - 3$, $g^2(x) = 9x$, $g^3(x) = 27x$,
$h^2(x) = h^3(x) = h(x)$.

5.6 Invertible functions

A function $f : A \to B$ is said to be ***invertible*** if there exists a function $g : B \to A$ such that $g \circ f = I_A$ and $f \circ g = I_B$, where I_A is the identity function on A and I_B is the identity function on B.

Then, g is called an *inverse* of f and we write $g = f^{-1}$.

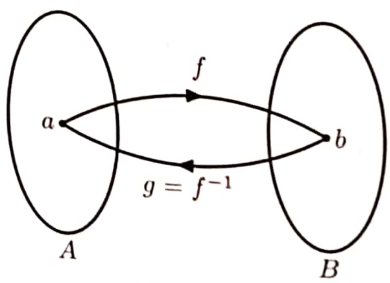


Figure 5.18

Example 1 *Let $A = \{1, 2, 3, 4\}$ and f and g be functions from A to A given by*

$$f = \{(1,4),\ (2,1),\ (3,2),\ (4,3)\} \text{ and } g = \{(1,2),\ (2,3),\ (3,4),\ (4,1)\}.$$

Prove that f and g are inverses of each other.

▸ We check that

$$(g \circ f)(1) = g\{f(1)\} = g(4) = 1 = I_A(1),$$
$$(g \circ f)(2) = g\{f(2)\} = g(1) = 2 = I_A(2),$$
$$(g \circ f)(3) = g\{f(3)\} = g(2) = 3 = I_A(3),$$
$$(g \circ f)(4) = g\{f(4)\} = g(3) = 4 = I_A(4),$$
$$(f \circ g)(1) = f\{g(1)\} = f(2) = 1 = I_A(1),$$
$$(f \circ g)(2) = f\{g(2)\} = f(3) = 2 = I_A(2),$$
$$(f \circ g)(3) = f\{g(3)\} = f(4) = 3 = I_A(3),$$
$$(f \circ g)(4) = f\{g(4)\} = f(1) = 4 = I_A(4),$$

Thus, for all $x \in A$, we have $(g \circ f)(x) = I_A(x)$ and $(f \circ g)(x) = I_A(x)$. Therefore, g is an inverse of f, and f is an inverse of g. ■

Example 2 *Consider the function $f : \mathsf{R} \to \mathsf{R}$ defined by $f(x) = 2x + 5$. Let a function $g : \mathsf{R} \to \mathsf{R}$ be defined by $g(x) = \frac{1}{2}(x - 5)$. Prove that g is an inverse of f.*

▸ Then, we check that, for any $x \in \mathsf{R}$,

$$(g \circ f)(x) = g\{f(x)\} = g(2x + 5)$$
$$= \frac{1}{2}(2x + 5 - 5) = x = I_{\mathsf{R}}(x)$$
$$(f \circ g)(x) = f\{g(x)\} = f\left\{\frac{1}{2}(x - 5)\right\}$$
$$= 2\left\{\frac{1}{2}(x - 5)\right\} + 5 = x = I_{\mathsf{R}}(x).$$

■

These show that g is an inverse of f. It also follows that f is an inverse of g.

Remark: We observe that the functions f considered in the above Examples are one-to-one and onto and are invertible. Theorems 2 to 4 below exhibit the connection between invertibility of a function and its other properties. Theorem 1 establishes that an invertible function cannot have two different inverses.

Theorem 1. *If a function $f : A \to B$ is invertible then it has a unique inverse. Further, if $f(a) = b$, then $f^{-1}(b) = a$.*

Proof: Suppose $f : A \to B$ is invertible and it has g and h as inverses. Then g and h are functions from B to A such that

$$g \circ f = I_A, \quad h \circ f = I_A,$$
$$f \circ g = I_B, \quad h \circ g = I_B,$$

Then, we find that

$h = h \circ I_B = h \circ (f \circ g) = (h \circ f) \circ g = I_A \circ g = g.$

This proves that h and g are not different. Thus, f has a unique inverse (when it is invertible).

Now, suppose that $f(a) = b$. Then, if g is the inverse of f, we have

$$a = I_A(a) = (g \circ f)(a) = g\{f(a)\} = g(b).$$

Since $g = f^{-1}$, this proves that $f^{-1}(b) = a$. •

Remarks

(1) If f is invertible, the statements $f(a) = b$ and $a = f^{-1}(b)$ are equivalent.

(2) If $f = \{(a,b) \mid a \in A, b \in B\}$ is invertible, then
$f^{-1} = \{(b,a) \mid b \in B, a \in A\}$ and conversely.

(3) If f is invertible then f^{-1} is invertible, and $\left(f^{-1}\right)^{-1} = f$.

Theorem 2. *A function $f : A \to B$ is invertible if and only if it is one-to-one and onto.*

Proof: First suppose that f is invertible. Then there exists a unique function $g : B \to A$ such that $g \circ f = I_A$ and $f \circ g = I_B$.

Take any $a_1, a_2 \in A$. Then

$$\begin{aligned} f(a_1) = f(a_2) &\Rightarrow g\{f(a_1)\} = g\{f(a_2)\} \\ &\Rightarrow (g \circ f)(a_1) = (g \circ f)(a_2) \\ &\Rightarrow I_A(a_1) = I_A(a_2) \\ &\Rightarrow a_1 = a_2. \end{aligned}$$

This proves that f is one-to-one.

Next, take any $b \in B$. Then $g(b) \in A$, and $b = I_B(b) = (f \circ g)(b) = f\{g(b)\}$. Thus, b is the image of an element $g(b) \in A$ under f. Therefore, f is onto as well.

Conversely, suppose that f is one-to-one and onto. Then for each $b \in B$ there is a unique $a \in A$ such that $b = f(a)$. Now, consider the function $g : B \to A$ defined by $g(b) = a$. Then

$$(g \circ f)(a) = g\{f(a)\} = g(b) = a = I_A(a), \text{ and}$$
$$(f \circ g)(b) = f\{g(b)\} = f(a) = b = I_B(b).$$

These show that f is invertible with g as the inverse.

This completes the proof of the theorem. •

Theorem 3. *Let A and B be finite sets with $|A| = |B|$ and f be a function from A to B. Then the following statements are equivalent.*

(1) *f is one-to-one* (2) *f is onto* (3) *f is invertible.*

Proof: Suppose $f : A \to B$ is one-to-one. Since A and B are finite sets with $|A| = |B|$, it follows that f is onto[§]. Consequently, f is invertible[¶].

Conversely, suppose f is invertible, then f is one-to-one and onto[**].

Thus, each of the three statements of the theorem implies the other two. The three statements are therefore equivalent. •

Remark: In view of the above theorem, we note that a function $f : A \to B$ where A and B are finite with $|A| = |B|$ is invertible if and only if f is one-to-one *or* onto.

Theorem 4. *If $f : A \to B$ and $g : B \to C$ are invertible functions, then $g \circ f : A \to C$ is an invertible function and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.*

Proof: Since f and g are invertible functions, they are both one-to-one and onto. Consequently, $g \circ f$ is both one-to-one and onto. Therefore, $g \circ f$ is invertible.

Now, the inverse f^{-1} of f is a function from B to A and the inverse g^{-1} of g is a function from C to B. Therefore, if $h = f^{-1} \circ g^{-1}$ then h is a function from C to A.

We find that

$$\begin{aligned}(g \circ f) \circ h &= (g \circ f) \circ \left(f^{-1} \circ g^{-1}\right) = g \circ \left(f \circ f^{-1}\right) \circ g^{-1} = g \circ I_B \circ g^{-1} \\ &= g \circ g^{-1} = I_C\end{aligned}$$

and

$$\begin{aligned}h \circ (g \circ f) &= \left(f^{-1} \circ g^{-1}\right) \circ (g \circ f) \\ &= f^{-1} \circ \left(g^{-1} \circ g\right) \circ f = f^{-1} \circ I_B \circ f \\ &= f^{-1} \circ f = I_A\end{aligned}$$

The above expressions show that h is the inverse of $g \circ f$; that is $h = (g \circ f)^{-1}$. Thus,

$$(g \circ f)^{-1} = h = f^{-1} \circ g^{-1}.$$

•

This completes the proof of the theorem.

[§]See Theorem 2 of Section 5.3.2.
[¶]See Theorem 2 above.

Example 3 *Consider the function $f : \mathsf{R} \to \mathsf{R}$ defined by $f(x) = x^2$, $\forall x \in \mathsf{R}$. Is f invertible?*

▶ For any $a \in \mathsf{R}$, we have

$$f(a) = a^2 \text{ and } f(-a) = (-a)^2 = a^2.$$

Thus, both a and $-a$ have the same image a^2 under f. Therefore, f is not one-to-one. Consequently, f is *not* invertible. ∎

Example 4 *Let $A = \{x \mid x$ is real and $x \geq -1\}$, and $B = \{x \mid x$ is real and $x \geq 0\}$. Consider the function $f : A \to B$ defined by $f(a) = \sqrt{a+1}$, for all $a \in A$. Show that f is invertible and determine f^{-1}.*

▶ Let us first check that f is one-to-one and onto.

Take any $a_1, a_2 \in A$. Then $f(a_1) = \sqrt{a_1 + 1}$ and $f(a_2) = \sqrt{a_2 + 1}$, and $f(a_1) = f(a_2)$ implies

$$\sqrt{a_1 + 1} = \sqrt{a_2 + 1}, \quad \text{or} \quad a_1 + 1 = a_2 + 1, \quad \text{so that } a_1 = a_2.$$

Hence f is one-to-one.

Take any $b \in B$. Then $b = f(a)$ holds if $b = \sqrt{a+1}$ or $b^2 = a + 1$, or $a = b^2 - 1$. Since $b \geq 0$, we note that $b^2 - 1 \geq -1$. Thus, every $b \in B$ has $a = b^2 - 1$ as a preimage in A under f. Hence f is onto as well.

This proves that f is invertible. The inverse of f is given by

$$f^{-1}(b) = b^2 - 1 \quad \text{for all} \quad b \in B.$$ ∎

Example 5 *Find the inverse (if any) of the function $f(x) = e^x$ defined from $\mathsf{R} \to \mathsf{R}^+$.*

▶ For any $x_1, x_2 \in \mathsf{R}$,

$$f(x_1) = f(x_2) \Rightarrow e^{x_1} = e^{x_2}$$
$$\Rightarrow x_1 = x_2.$$

Therefore, f is one-to-one.

Take any $y \in \mathsf{R}^+$, and put $x = \log_e y$. Then $x \in \mathsf{R}$ and $e^x = y$; that is $f(x) = y$. Thus, every $y \in \mathsf{R}^+$ has $x = \log_e y$ as its preimage in R under f. Therefore, f is onto as well.

Accordingly, f is an invertible function. The inverse of f is given by

$$f^{-1}(y) = \log_e y \quad \text{for all} \quad y \in \mathsf{R}^+.$$ ∎

Example 6 *Let $A = B = \mathsf{R}$, the set of all real numbers, and the functions $f : A \to B$ and $g : B \to A$ be defined by*

$$f(x) = 2x^3 - 1, \ \forall x \in A; \quad g(y) = \left\{\frac{1}{2}(y+1)\right\}^{1/3}, \ \forall y \in B.$$

Show that each of f and g is the inverse of the other.

▸ We find that, for any $x \in A$,

$$(g \circ f)(x) = g(f(x)) = g(y) = \left\{\frac{1}{2}(y+1)\right\}^{1/3}, \text{ where } y = f(x)$$

$$= \left\{\frac{1}{2}(2x^3 - 1 + 1)\right\}^{1/3}, \text{ because } y = f(x) = 2x^3 - 1$$

$$= x.$$

Thus, $g \circ f = I_A$.

Next, for any $y \in B$,

$$(f \circ g)(y) = f(g(y)) = f\left(\left\{\frac{1}{2}(y+1)\right\}^{1/3}\right)$$

$$= 2\left[\left\{\frac{1}{2}(y+1)\right\}^{1/3}\right]^3 - 1$$

$$= 2\left[\frac{1}{2}(y+1)\right] - 1 = y$$

Thus, $f \circ g = I_B$.

Accordingly, each of f and g is an invertible function, and further more each is the inverse of the other. ■

Example 7 *Let $A = B = C = \mathbb{R}$, and $f : A \to B$ and $g : B \to C$ be defined by*

$$f(a) = 2a + 1,\ g(b) = \frac{1}{3}b,\ \forall a \in A,\ \forall b \in B.$$

Compute $g \circ f$ and show that $g \circ f$ is invertible. What is $(g \circ f)^{-1}$?

▸ We have

$$(g \circ f)(a) = g(f(a)) = g(2a+1) = \frac{1}{3}(2a+1).$$

Thus, $g \circ f : A \to C$ is defined by $(g \circ f)(a) = \frac{1}{3}(2a+1)$.

We check that f is invertible with $f^{-1}(b) = \frac{1}{2}(b-1)$ and g is invertible with $g^{-1}(c) = 3c$. Therefore, $g \circ f$ is invertible, and its inverse is given by

$$(g \circ f)^{-1}(c) = \left(f^{-1} \circ g^{-1}\right)(c)$$

$$= f^{-1}\left\{g^{-1}(c)\right\} = f^{-1}(3c)$$

$$= \frac{1}{2}(3c-1).$$ ■

Exercises

1. Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c, d\}$. Determine whether the following functions from A to B are invertible or not.

   (1) $f = \{(1, a), (2, a), (3, c), (4, d)\}$ (2) $g = \{(1, a), (2, c), (3, d), (4, d)\}$

2. Let $A = B = \mathrm{R}$. Show that $f : A \to B$ defined by $f(a) = a + 1$ for $a \in A$ is invertible.

3. Find the inverse of the function $f : A \to B$ in each of the following cases:

   (1) $A = B = \{1, 2, 3, 4, 5\}$, $f = \{(1, 3), (2, 2), (3, 4), (4, 5), (5, 1)\}$

   (2) $A = B = \mathrm{R}$, $f(x) = ax + b$, $x \in \mathrm{R}$, where $a \neq 0$ and b are constants.

   (3) $A = \{x \mid x \text{ is real and } x \geq -1\}$, $B = \{y \mid y \text{ is real and } y \geq 0\}$, $f(a) = a^2 - 1$

   (4) $A = \mathrm{R}$, $B = \mathrm{R}^+$, $f(x) = e^{2x+5}$

4. Let $f : \mathrm{Z} \to \mathrm{N}$ be defined by

$$f(x) = \begin{cases} 2x - 1 & \text{if } x > 0 \\ -2x & \text{if } x \leq 0. \end{cases}$$

   Prove that f is a bijection, and find f^{-1}.

5. For the functions f and g from R to R given below, verify that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$:

   (i) $f(x) = 2x$, $g(x) = 3x - 2$, (ii) $f(x) = \frac{1}{2}(x + 1)$, $g(x) = \frac{1}{2}(x - 1)$.

Answers

1. (1) No (2) Yes

3. (1) $f^{-1} = \{(3, 1), (2, 2), (4, 3), (5, 4), (1, 5)\}$

   (2) $f^{-1}(y) = \frac{1}{a}(y - b)$, $y \in \mathrm{R}$

   (3) $f^{-1}(y) = \sqrt{y + 1}$, $y \in B$

   (4) $f^{-1}(y) = (1/2)(\log y - 5)$, $y \in B$.

4. $$f^{-1}(y) = \begin{cases} (1/2)(y + 1), & y = 1, 3, 5, 7, \ldots \\ -y/2, & y = 0, 2, 4, 6, \ldots \end{cases}$$

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6.1 Zero-one Matri

Consider the sets $A = \{a_1, a_2$
Then $A \times B$ consists of *all* or
mn in number. Let R be a rela
Now, let us put $m_{ij} = (a_i,$
Rule:

The $m \times n$ matrix formed by
matrix for R, and is denoted b
$M(R)$ is also called the ***Zero-***
It is to be noted that the
those of B.
When $B = A$, the matrix
with

For example, consider th
defined by

*This matrix is used as a tool fo
structure.