TRANSFORM CALCULUS, FOURIER SERIES & NUMERICAL TECHNIQUES

MODULE - 5

In this module, we discuss two different topics.

The first one is an extension of the content of the previous module. Here we present two ***Numerical Methods,*** *Runge - Kutta method and Milne's method for solving second order ordinary differential equations.*

The second topic is titled ***Calculus of Variations.***

In calculus of variations, we discuss the method of finding maximum or minimum value of functions represented in the form of integrals and discuss some applications also.

NUMERICAL SOLUTION OF SECOND ORDER ORDINARY DIFFERENTIAL EQAUTIONS

5.1 Introduction and pre-amble

The given second order ODE with two initial conditions will reduce to two first order simultaneous ODEs which can be solved.

We present the method explicitly.

Let $y'' = g(x, y, y')$ with the initial conditions $y(x_0) = y_0$ and $y'(x_0) = y'_0$ be the given second order DE.

Now, let $y' = \dfrac{dy}{dx} = z$. This gives $y'' = \dfrac{d^2y}{dx^2} = \dfrac{dz}{dx}$

The given second order DE assumes the form : $\dfrac{dz}{dx} = g(x, y, z)$ with the conditions $y(x_0) = y_0$ and $z(x_0) = z_0$ where y'_0 is denoted by z_0.

Hence, we now have two first order simultaneous ODEs.

$$\frac{dy}{dx} = z \text{ and } \frac{dz}{dx} = g(x, y, z) \text{ with } y(x_0) = y_0 \text{ and } z(x_0) = z_0$$

Taking $f(x, y, z) = z$, we now have the following system of equations for solving:

$$\frac{dy}{dx} = f(x, y, z),\ \frac{dz}{dx} = g(x, y, z)\ ;\ y(x_0) = y_0 \text{ and } z(x_0) = z_0$$

5.11 Runge - Kutta Method

We have to compute $y(x_0 + h)$ and if required $y'(x_0 + h) = z(x_0 + h)$.

We need to first compute the following :

(Recall the formulae in the case of first order ODE)

$$k_1 = h\, f(x_0, y_0, z_0) \quad ; \quad l_1 = h\, g(x_0, y_0, z_0)$$

$$k_2 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right); \quad l_2 = h\, g\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$k_3 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right); \quad l_3 = h\, g\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$k_4 = h\, f(x_0 + h, y_0 + k_3, z_0 + l_3) \; ; \; l_4 = h\, g(x_0 + h, y_0 + k_3, z_0 + l_3)$$

The required, $y(x_0 + h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

and $\quad y'(x_0 + h) = z(x_0 + h) = z_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$

WORKED PROBLEMS

[1] *Given* $\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} - 2xy = 1, y(0) = 1, y'(0) = 0$. *Evaluate* $y(0.1)$ *using Runge-Kutta method of order 4.*

☞ By data, $\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} - 2xy = 1, y = 1; y' = 0$ at $x = 0$.

Putting, $\frac{dy}{dx} = z$ and differentiating w.r.t x we obtain $\frac{d^2y}{dx^2} = \frac{dz}{dx}$ so that the given equation assumes the form : $\frac{dz}{dx} - x^2z - 2xy = 1$

Hence, we have a system of equations,

$$\frac{dy}{dx} = z \; ; \; \frac{dz}{dx} = 1 + 2xy + x^2z \text{ where } y = 1,\ z = 0,\ x = 0.$$

Let, $\quad f(x, y, z) = z,\ g(x, y, z) = 1 + 2xy + x^2z$

$x_0 = 0,\ y_0 = 1,\ z_0 = 0$ and let us take $h = 0.1$

We shall first compute the following :

$$k_1 = h\,f(x_0, y_0, z_0) = (0.1)\,f(0,1,0) = (0.1)(0) = 0$$

$$l_1 = (0.1)[1+(2)(0)(1)+(0)^2(0)] = 0.1$$

$$k_2 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$k_2 = (0.1)\,f(0.05, 1, 0.05) = (0.1)(0.05) = 0.005$$

$$l_2 = (0.1)[1+(2)(0.05)(1)+(0.05)^2(0.05)] = 0.11$$

$$k_3 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$k_3 = (0.1)\,f(0.05, 1.0025, 0.055) = (0.1)(0.055) = 0.0055$$

$$l_3 = (0.1)[1+(2)(0.05)(1.0025)+(0.05)^2(0.055)] = 0.11004$$

$$k_4 = h\,f(x_0 + h, y_0 + k_3, z_0 + l_3)$$

$$k_4 = (0.1)\,f(0.1, 1.0055, 0.11004) = (0.1)(0.11004) = 0.011$$

$$l_4 = (0.1)[1+(2)(0.1)(1.0055)+(0.1)^2(0.11004)] = 0.12022$$

We have, $y(x_0 + h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$\therefore \quad y(0.1) = 1 + \frac{1}{6}[0 + 2(0.005) + 2(0.0055) + 0.011]$$

Thus, $\boxed{y(0.1) = 1.0053}$

Remark : *The computation of l_1, l_2, l_3, l_4 is just the application of the expression associated with $g(x, y, z)$.*

[2] *By Runge-Kutta method, solve* $\dfrac{d^2y}{dx^2} = x\left(\dfrac{dy}{dx}\right)^2 - y^2$ *for* $x = 0.2$ *correct to four decimal places, using the initial conditions* $y = 1$ *and* $y' = 0$ *when* $x = 0$

[June & Dec 2017].

☞ By data, $\dfrac{d^2y}{dx^2} = x\left(\dfrac{dy}{dx}\right)^2 - y^2$

Putting, $\dfrac{dy}{dx} = z$ and differentiating w.r.t x, we obtain $\dfrac{d^2y}{dx^2} = \dfrac{dz}{dx}$

The given equation becomes,

$$\frac{dz}{dx} = xz^2 - y^2 \text{ with } y = 1,\ z = 0 \text{ at } x = 0.$$

Hence, we have a system of equations $\dfrac{dy}{dx} = z,\ \dfrac{dz}{dx} = xz^2 - y^2$

Let, $f(x,y,z) = z$, $g(x,y,z) = xz^2 - y^2$, $x_0 = 0$, $y_0 = 1$, $z_0 = 0$ and $h = 0.2$

We shall first compute the following.

$$k_1 = h\,f(x_0, y_0, z_0) = (0.2)\,f(0,1,0) = (0.2)(0) = 0$$

$$l_1 = (0.2)[(0)(0)^2 - (1)^2] = -0.2$$

$$k_2 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$k_2 = (0.2)\,f(0.1, 1, -0.1) = (0.2)(-0.1) = -0.02$$

$$l_2 = (0.2)[(0.1)(-0.1)^2 - (1)^2] = -0.1998$$

$$k_3 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$k_3 = (0.2)f(0.1, 0.99, -0.0999) = (0.2)(-0.0999) = -0.01998$$

$$l_3 = (0.2)[(0.1)(-0.0999)^2 - (0.99)^2] = -0.1958$$

$$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3)$$

$$k_4 = (0.2)f(0.2, 0.98002, -0.1958) = (0.2)(-0.1958) = 0.03916$$

$$l_4 = (0.2)[(0.2)(-0.1958)^2 - (0.98002)^2] = -0.19055$$

We have $y(x_0 + h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$\therefore \quad y(0.2) = 1 + \frac{1}{6}[0 + 2(-0.02) + 2(-0.01998) - 0.03916]$$

Thus, $\boxed{y(0.2) = 0.9801}$

[3] *Compute* $y(0.1)$ *given* $\frac{d^2y}{dx^2} = y^3$ *and* $y = 10$, $\frac{dy}{dx} = 5$ *at* $x = 0$ *by Runge-Kutta method of fourth order.*

☞ Putting $\frac{dy}{dx} = z$ and differentiating w.r.t x we obtain $\frac{d^2y}{dx^2} = \frac{dz}{dx}$ so that the given equation assumes the form $\frac{dz}{dx} = y^3$. Hence we have a system of equations:

$$\frac{dy}{dx} = z \; ; \; \frac{dz}{dx} = y^3 \text{ where } y = 10, z = 5, x = 0.$$

Let, $f(x, y, z) = z$, $g(x, y, z) = y^3$, $x_0 = 0$, $y_0 = 10$, $z_0 = 5$ and $h = 0.1$

We shall first compute the following :

$$k_1 = hf(x_0, y_0, z_0) = (0.1)f(0, 10, 5) = (0.1)5 = 0.5$$

$$l_1 = (0.1)[10^3] = 100$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$k_2 = (0.1)f(0.05, 10.25, 55) = (0.1)(55) = 5.5$$

$$l_2 = (0.1)[(10.25)^3] = 107.7$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$k_3 = (0.1)f(0.05, 12.75, 58.85) = (0.1)(58.85) = 5.885$$

$$l_3 = (0.1)(12.75)^3 = 207.27$$

$$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3)$$

$$k_4 = (0.1)f(0.1, 15.885, 212.27) = (0.1)(212.27) = 21.227$$

$$l_4 = (0.1)(15.885)^3 = 400.83$$

We have, $y(x_0 + h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$\therefore$ $y(0.1) = 10 + \frac{1}{6}[0.5 + 2(5.5) + 2(5.885) + 21.227]$

Thus, $\boxed{y(0.1) \doteq 17.4162}$

[4] *Given* $y'' - xy' - y = 0$ *with the initial conditions* $y(0) = 1$, $y'(0) = 0$, *compute* $y(0.2)$ *and* $y'(0.2)$ *using fourth order Runge-Kutta method.*

[June 2018]

☞ Putting $y' = z$, we obtain $y'' = \frac{dz}{dx}$. The given equation becomes

$$\frac{dz}{dx} = xz + y \;;\; y(0) = 1,\; z(0) = 0$$

Hence we have a system of equations,

$$\frac{dy}{dx} = z \; ; \; \frac{dz}{dx} = xz + y \text{ where } y = 1,\ z = 0,\ x = 0$$

Let, $f(x, y, z) = z$, $g(x, y, z) = xz + y$, $x_0 = 0$, $y_0 = 1$, $z_0 = 0$ and $h = 0.2$

We shall first compute the following.

$$k_1 = h\,f(x_0, y_0, z_0) = (0.2)\,f(0,1,0) = (0.2)0 = 0$$

$$l_1 = (0.2)[0 \times 0 + 1] = 0.2$$

$$k_2 = h\,f\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2},\ z_0 + \frac{l_1}{2}\right)$$

$$k_2 = (0.2)\,f(0.1, 1, 0.1) = (0.2)(0.1) = 0.02$$

$$l_2 = (0.2)[0.1 \times 0.1 + 1] = 0.202$$

$$k_3 = h\,f\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2},\ z_0 + \frac{l_2}{2}\right)$$

$$k_3 = (0.2)\,f(0.1,\ 1.01, 0.101) = (0.2)(0.101) = 0.0202$$

$$l_3 = (0.2)[0.1 \times 0.101 + 1.01] = 0.204$$

$$k_4 = h\,f(x_0 + h,\ y_0 + k_3,\ z_0 + l_3)$$

$$k_4 = (0.2)\,f(0.2,\ 1.0202, 0.204) = (0.2)(0.204) = 0.0408$$

$$l_4 = (0.2)[0.2 \times 0.204 + 1.0202] = 0.2122$$

We have, $y(x_0 + h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$z(x_0 + h) = z_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$$

Substituting the appropriate values we obtain $y(0.2) = 1.0202$ and $z(0.2) = 0.204$

Thus,

$$\boxed{y(0.2) = 1.0202 \text{ and } y'(0.2) = 0.204}$$

[5] *Obtain the value of x and* $\dfrac{dx}{dt}$ *when* $t = 0.1$ *given that x satisfies the equation* $\dfrac{d^2x}{dt^2} = t\dfrac{dx}{dt} - 4x$ *and* $x = 3$, $\dfrac{dx}{dt} = 0$ *when* $t = 0$ *initially. Use fourth order Runge-Kutta method.*

☞ Putting, $y = \dfrac{dx}{dt}$ we obtain $\dfrac{dy}{dt} = \dfrac{d^2x}{dt^2}$ The given equation becomes

$$\frac{dy}{dt} = ty - 4x,\ x = 3,\ y = 0 \text{ when } t = 0.$$

Hence we have a system of equations,

$$\frac{dx}{dt} = y,\ \frac{dy}{dt} = ty - 4x,\ x = 3,\ y = 0 \text{ when } t = 0.$$

Let $f(t, x, y) = y$, $g(t, x, y) = ty - 4x$, $t_0 = 0$, $x_0 = 3$, $y_0 = 0$ and $h = 0.1$

We shall first compute the following :

$$k_1 = hf(t_0, x_0, y_0) = (0.1)f(0,3,0) = (0.1)(0) = 0$$

$$l_1 = (0.1)[0 - 12] = -1.2$$

$$k_2 = hf\left(t_0 + \frac{h}{2},\ x_0 + \frac{k_1}{2},\ y_0 + \frac{l_1}{2}\right)$$

$$k_2 = (0.1)f(0.05, 3, -0.6) = (0.1)(-0.6) = -0.06$$

$$l_2 = (0.1)[(0.05)(-0.6) - 12] = -1.203$$

$$k_3 = hf\left(t_0 + \frac{h}{2},\ x_0 + \frac{k_2}{2},\ y_0 + \frac{l_2}{2}\right)$$

$$k_3 = (0.1)f(0.05, 2.97, -0.6015) = (0.1)(-0.6015) = -0.06015$$

$$l_3 = (0.1)[(0.05)(-0.6015) - 4 \times 2.97] = -1.191$$

$$k_4 = hf(t_0 + h,\ x_0 + k_3,\ y_0 + l_3)$$

$$k_4 = (0.1)\,f(0.1, 2.93985, -1.191) = (0.1)(-1.191) = -0.1191$$

$$l_4 = (0.1)[(0.1)(-1.191) - 4 \times 2.93985] = -1.18785$$

We have, $x(t_0 + h) = x_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$y(t_0 + h) = y_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) \text{ where } y = \frac{dx}{dt}$$

Substituting the appropriate values we obtain $x(0.1) = 2.9401$, $y(0.1) = -1.196$

Thus, $\boxed{x = 2.9401 \text{ and } \frac{dx}{dt} = -1.196 \text{ when } t = 0.1}$

5.12 Milne's method

Preamble : We recall [*Module-4, Article-4.21*] Milne's predictor and corrector formulae for solving first order ODE : $y' = f(x, y)$ with $y(x_0) = y_0$, $y(x_1) = y_1$, $y(x_2) = y_2$, $y(x_3) = y_3$. Here x_0, x_1, x_2, x_3 are equidistant values of x distant h.

We have to compute $y(x_4)$ where $x_4 = x_0 + 4h$.

$$y_4^{(P)} = y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3') \qquad \text{[\textit{Predictor formula}]}$$

$$y_4^{(C)} = y_2 + \frac{h}{3}(y_2' + 4y_3' + y_4') \qquad \text{[\textit{Corrector formula}]}$$

Method to solve the ODE $y'' = f(x, y, y')$ given a set of four initial values for y and y' :

- We put $y' = z$ which gives $y'' = \frac{dz}{dx} = z'$.

  The given DE becomes $z' = f(x, y, z)$

- We equip with the following table of values using the given data.

x	x_0	x_1	x_2	x_3
y	y_0	y_1	y_2	y_3
$y' = z$	$y'_0 = z_0$	$y'_1 = z_1$	$y'_2 = z_2$	$y'_3 = z_3$
$y'' = z'$	$y''_0 = z'_0$	$y''_1 = z'_1$	$y''_2 = z'_2$	$y''_3 = z'_3$

- We first apply predictor formula to compute $y_4^{(P)}$ and $z_4^{(P)}$ where,

$$y_4^{(P)} = y_0 + \frac{4h}{3}(2z_1 - z_2 + 2z_3), \text{ since } y' = z.$$

$$z_4^{(P)} = z_0 + \frac{4h}{3}(2z'_1 - z'_2 + 2z'_3)$$

- We compute $z'_4 = f(x_4, y_4, z_4)$ and then apply corrector formula where,

$$y_4^{(C)} = y_2 + \frac{h}{3}(z_2 + 4z_3 + z_4)$$

$$z_4^{(C)} = z_2 + \frac{h}{3}(z'_2 + 4z'_3 + z'_4)$$

- Corrector formula can be applied repeatedly for better accuracy.

WORKED PROBLEMS

[6] *Apply Milne's method to solve* $\dfrac{d^2y}{dx^2} = 1 + \dfrac{dy}{dx}$ *given the following table of initial values. Compute* $y(0.4)$. **[June 2018]**

x	0	0.1	0.2	0.3
y	1	1.1103	1.2427	1.399
y'	1	1.2103	1.4427	1.699

☞ Putting $y' = \dfrac{dy}{dx} = z$, we obtain $y'' = \dfrac{d^2y}{dx^2} = \dfrac{dz}{dx}$

The given equation becomes, $\dfrac{dz}{dx} = 1 + z$ or $z' = 1 + z$.

Further, $z' = 1 + z$ will give us the following values

$$z'(0) = 1 + z(0) = 1 + 1 = 2$$

$$\therefore \quad z'(0.1) = 1 + z(0.1) = 2.2103$$

$$z'(0.2) = 1 + z(0.2) = 2.4427$$

$$z'(0.3) = 1 + z(0.3) = 2.699$$

Now we tabulate these values.

x	$x_0 = 0$	$x_1 = 0.1$	$x_2 = 0.2$	$x_3 = 0.3$
y	$y_0 = 1$	$y_1 = 1.1103$	$y_2 = 1.2427$	$y_3 = 1.399$
$y' = z$	$z_0 = 1$	$z_1 = 1.2103$	$z_2 = 1.4427$	$z_3 = 1.699$
$y'' = z'$	$z'_0 = 2$	$z'_1 = 2.2103$	$z'_2 = 2.4427$	$z'_3 = 2.699$

We first consider Milne's predictor formulae :

$$y_4^{(P)} = y_0 + \frac{4h}{3}(2z_1 - z_2 + 2z_3),$$

$$z_4^{(P)} = z_0 + \frac{4h}{3}(2z'_1 - z'_2 + 2z'_3)$$

Hence, $y_4^{(P)} = 1 + \dfrac{4(0.1)}{3}[2(1.2103) - 1.4427 + 2(1.699)]$

$$z_4^{(P)} = 1 + \frac{4(0.1)}{3}[2(2.2103) - 2.4427 + 2(2.699)]$$

$$\therefore \quad y_4^{(P)} = 1.5835 \text{ and } z_4^{(P)} = 1.9835$$

Next we consider Milne's corrector formulae :

$$y_4^{(C)} = y_2 + \frac{h}{3}(z_2 + 4z_3 + z_4)$$

$$z_4^{(C)} = z_2 + \frac{h}{3}(z_2' + 4z_3' + z_4')$$

We have, $z_4' = 1 + z_4^{(P)} = 1 + 1.9835 = 2.9835$

Hence, $y_4^{(C)} = 1.2427 + \frac{0.1}{3}[1.4427 + 4(1.699) + 1.9835]$

$$z_4^{(C)} = 1.4427 + \frac{0.1}{3}[2.4427 + 4(2.699) + 2.9835]$$

$\therefore \quad y_4^{(C)} = 1.58344$ and $z_4^{(C)} = 1.98344$

Applying the corrector formula again for y_4 we obtain $y_4^{(C)} = 1.583438$

Thus the required, $\boxed{y(0.4) = 1.5834}$

[7] *Apply Milne's method to compute $y(0.8)$ given that $\frac{d^2y}{dx^2} = 1 - 2y\frac{dy}{dx}$ and the following table of initial values.* **[Dec 2017, 18]**

x	0	0.2	0.4	0.6
y	0	0.02	0.0795	0.1762
y'	0	0.1996	0.3937	0.5689

Apply the corrector formula twice in presenting the value of y at $x = 0.8$

☞ Putting $y' = \frac{dy}{dx} = z$ we obtain, $y'' = \frac{d^2y}{dx^2} = z'$

The given equation becomes, $z' = 1 - 2yy' = 1 - 2yz$

Now, $z_0' = 1 - 2(0)(0) = 1$

$$z_1' = 1 - 2(0.02)(0.1996) = 0.992$$

$$z_2' = 1 - 2(0.0795)(0.3937) = 0.9374$$

$$z_3' = 1 - 2(0.1762)(0.5689) = 0.7995$$

We have the following table.

x	$x_0 = 0$	$x_1 = 0.2$	$x_2 = 0.4$	$x_3 = 0.6$
y	$y_0 = 0$	$y_1 = 0.02$	$y_2 = 0.0795$	$y_3 = 0.1762$
$y' = z$	$z_0 = 0$	$z_1 = 0.1996$	$z_2 = 0.3937$	$z_3 = 0.5689$
$y'' = z'$	$z_0' = 1$	$z_1' = 0.992$	$z_2' = 0.9374$	$z_3' = 0.7995$

We first consider Milne's predictor formulae,

$$y_4^{(P)} = y_0 + \frac{4h}{3}(2z_1 - z_2 + 2z_3),$$

$$z_4^{(P)} = z_0 + \frac{4h}{3}(2z_1' - z_2' + 2z_3')$$

On substituting the appropriate values from the table we obtain,

$$y_4^{(P)} = 0.3049 \text{ and } z_4^{(P)} = 0.7055$$

Next we consider Milne's corrector formulae,

$$y_4^{(C)} = y_2 + \frac{h}{3}(z_2 + 4z_3 + z_4)$$

$$z_4^{(C)} = z_2 + \frac{h}{3}(z_2' + 4z_3' + z_4')$$

We have $z_4' = 1 - 2y_4^{(P)} z_4^{(P)} = 1 - 2(0.3049)(0.7055) = 0.5698$

Hence by substituting the appropriate values in the corrector formulae, we obtain,

$$y_4^{(C)} = 0.3045 \text{ and } z_4^{(C)} = 0.7074$$

Applying the corrector formula again for y_4 we have,

$$y_4^{(C)} = 0.0795 + \frac{0.2}{3}[0.3937 + 4(0.5689) + 0.7074] = 0.3046$$

Thus the required, $\boxed{y(0.8) = 0.3046}$

[8] *Obtain the solution of the equation* $2\frac{d^2y}{dx^2} = 4x + \frac{dy}{dx}$ *by computing the value of the dependent variable corresponding to the value 1.4 of the independent variable by applying Milne's method using the following data.* **[June 2017]**

x	1	1.1	1.2	1.3
y	2	2.2156	2.4649	2.7514
y'	2	2.3178	2.6725	3.0657

☞ Dividing the given equation by 2 we have,

$$\frac{d^2y}{dx^2} = 2x + \frac{1}{2}\frac{dy}{dx} \quad \text{or} \quad y'' = 2x + \frac{y'}{2}$$

Putting, $y' = z$ we obtain $y'' = z'$ and the given equation becomes $z' = 2x + \frac{z}{2}$

Now, $z'_0 = 2(1) + \frac{2}{2} = 3$

$$z'_1 = 2(1.1) + \frac{2.3178}{2} = 3.3589$$

$$z'_2 = 2(1.2) + \frac{2.6725}{2} = 3.73625$$

$$z'_3 = 2(1.3) + \frac{3.0657}{2} = 4.13285$$

We have the following table.

x	$x_0 = 1$	$x_1 = 1.1$	$x_2 = 1.2$	$x_3 = 1.3$
y	$y_0 = 2$	$y_1 = 2.2156$	$y_2 = 2.4649$	$y_3 = 2.7514$
$y' = z$	$z_0 = 2$	$z_1 = 2.3178$	$z_2 = 2.6725$	$z_3 = 3.0657$
$y'' = z'$	$z'_0 = 3$	$z'_1 = 3.3589$	$z'_2 = 3.73625$	$z'_3 = 4.13285$

We first consider Milne's predictor formulae,

$$y_4^{(P)} = y_0 + \frac{4h}{3}(2z_1 - z_2 + 2z_3)$$

$$z_4^{(P)} = z_0 + \frac{4h}{3}(2z'_1 - z'_2 + 2z'_3)$$

On substituting the appropriate values from the table we obtain,

$$y_4^{(P)} = 3.0793 \text{ and } z_4^{(P)} = 3.4996$$

Next we consider Milne's corrector formulae,

$$y_4^{(C)} = y_2 + \frac{h}{3}(z_2 + 4z_3 + z_4)$$

$$z_4^{(C)} = z_2 + \frac{h}{3}(z'_2 + 4z'_3 + z'_4)$$

We have, $z'_4 = 2x_4 + \dfrac{z_4^{(P)}}{2} = 2(1.4) + \dfrac{3.4996}{2} = 4.5498$

Hence by substituting the appropriate values in the corrector formulae we obtain

$$y_4^{(C)} = 3.0794 \text{ and } z_4^{(C)} = 3.4997$$

Thus the required, $\boxed{y(1.4) = 3.0794}$

[9] *Given the ODE $y'' + xy' + y = 0$ and the following table of initial values, compute $y(0.4)$ by applying Milne's method.*

x	0	0.1	0.2	0.3
y	1	0.995	0.9801	0.956
y'	0	–0.0995	–0.196	–0.2867

☞ Putting $y' = z$, we get $y'' = z'$.

Also we have, $z' = -(xz + y)$ from the given equation.

Further, $z'(0) = -[0+1] = -1$

$$z'(0.1) = -[(0.1)(-0.0995) + 0.995] = -0.985$$

$$z'(0.2) = -[(0.2)(-0.196) + 0.9801] = -0.941$$

$$z'(0.3) = -[(0.3)(-0.2867) + 0.956] = -0.87$$

We also have the following table.

x	$x_0 = 0$	$x_1 = 0.1$	$x_2 = 0.2$	$x_3 = 0.3$
y	$y_0 = 1$	$y_1 = 0.995$	$y_2 = 0.9801$	$y_3 = 0.956$
$y' = z$	$z_0 = 0$	$z_1 = -0.0995$	$z_2 = -0.196$	$z_3 = -0.2867$
$y'' = z'$	$z'_0 = -1$	$z'_1 = -0.985$	$z'_2 = -0.941$	$z'_3 = -0.87$

We first consider Milne's predictor formulae,

$$y_4^{(P)} = y_0 + \frac{4h}{3}(2z_1 - z_2 + 2z_3)$$

$$z_4^{(P)} = z_0 + \frac{4h}{3}(2z'_1 - z'_2 + 2z'_3)$$

On substituting the appropriate values from the table we obtain

$$y_4^{(P)} = 0.9231 \text{ and } z_4^{(P)} = -0.3692$$

Next we consider Milne's corrector formulae,

$$y_4^{(C)} = y_2 + \frac{h}{3}(z_2 + 4z_3 + z_4)$$

$$z_4^{(C)} = z_2 + \frac{h}{3}(z_2' + 4z_3' + z_4')$$

We have, $z_4' = -(x_4\, z_4^{(P)} + y_4^{(P)}) = -[(0.4)(-0.3692) + 0.9231] = -0.7754$

Hence by substituting the appropriate values in the corrector formulae we obtain

$$y_4^{(C)} = 0.9230 \text{ and } z_4^{(C)} = -0.3692$$

Thus the required, $\boxed{y(0.4) = 0.923}$

[10] *Applying Milne's predictor and corrector formulae compute* $y(0.8)$ *given that y satisfies the equation* $y'' = 2yy'$ *and* y *&* y' *are governed by the following values.*

$$y(0) = 0,\ y(0.2) = 0.2027,\ y(0.4) = 0.4228,\ y(0.6) = 0.6841$$

$$y'(0) = 1,\ y'(0.2) = 1.041,\ y'(0.4) = 1.179,\ y'(0.6) = 1.468$$

Apply corrector formula twice.

☞ Putting $y' = z$ we obtain $y'' = \frac{dz}{dx} = z'$ & the given equation becomes $z' = 2yz$

Now, $z'(0) = 0,\ z'(0.2) = 2(0.2027)(1.041) = 0.422$

$$z'(0.4) = 2(0.4228)(1.179) = 0.997$$

$$z'(0.6) = 2(0.6841)(1.468) = 2.009$$

Now we tabulate all the values.

x	$x_0 = 0$	$x_1 = 0.2$	$x_2 = 0.4$	$x_3 = 0.6$
y	$y_0 = 0$	$y_1 = 0.2027$	$y_2 = 0.4228$	$y_3 = 0.6841$
$y' = z$	$z_0 = 1$	$z_1 = 1.041$	$z_2 = 1.179$	$z_3 = 1.468$
$y'' = z'$	$z_0' = 0$	$z_1' = 0.422$	$z_2' = 0.997$	$z_3' = 2.009$

We first consider Milne's predictor formulae,

$$y_4^{(P)} = y_0 + \frac{4h}{3}(2z_1 - z_2 + 2z_3)$$

$$z_4^{(P)} = z_0 + \frac{4h}{3}(2z'_1 - z'_2 + 2z'_3)$$

On substituting the appropriate values from the table we obtain,

$$y_4^{(P)} = 1.0237 \text{ and } z_4^{(P)} = 2.0307$$

Next we consider Milne's corrector formulae,

$$y_4^{(C)} = y_2 + \frac{h}{3}(z_2 + 4z_3 + z_4)$$

$$z_4^{(C)} = z_2 + \frac{h}{3}(z'_2 + 4z'_3 + z'_4)$$

We have, $z'_4 = 2\,y_4^{(P)}\ z_4^{(P)} = 4.1577$

Hence by substituting the appropriate values in the corrector formulae, we obtain

$$y_4^{(C)} = 1.0282 \text{ and } z_4^{(C)} = 2.0584$$

Applying the corrector formula again we have,

$$y_4^{(C)} = 0.4228 + \frac{0.2}{3}[1.179 + 4(1.468) + 2.0584] = 1.03009$$

Thus the required, $\boxed{y(0.8) = 1.0301}$

ASSIGNMENT

1. Use fourth order Runge- Kutta method to solve the equation $\frac{d^2y}{dx^2} = x\frac{dy}{dx} + y$ given that $y = 1$ and $\frac{dy}{dx} = 0$ when $x = 0$.

   Compute y and $\frac{dy}{dx}$ at $x = 0.2$

2. Solve $y'' + 4y = xy$ given that $y(0) = 3$ and $y'(0) = 0$

Compute $y(0.1)$ using Runge-Kutta method of order 4.

3. Apply Milne's method to compute $y(0.4)$ given the equation $y'' + y' = 2e^x$ and the following table of initial values. Compare the result with theoretical value.

x	0	0.1	0.2	0.3
y	2	2.01	2.04	2.09
y'	0	0.2	0.4	0.6

4. Solve the equation $y'' + y' = 2x$ at $x = 0.4$ by applying Milne's method given that $y = 1$, $y' = -1$ at $x = 0$. Requisite initial values be generated from Taylor's series method.

ANSWERS

1. $y(0.2) = 1.0202$, $y'(0.2) = 0.204$
2. $y(0.1) = 2.94$
3. $y(0.4) = 2.16$
4. $y(0.4) = 0.6897$

5.2 Variation of a function

Let us consider a function of x, y, y'. That is,

$$f(x, y, y') = f(x, y(x), y'(x))$$

Suppose we give small increments to y and y' so that they become respectively, $y + h\alpha(x); y' + h\alpha'(x), h$ is a small parameter independent of x. Now we have

$$f(x, y + h\alpha(x), y' + h\alpha'(x)) = f(x, y, y')$$

$$+\left(h\alpha\frac{\partial}{\partial y} + h\alpha'\frac{\partial}{\partial y'}\right)f + \frac{1}{2!}\left(h\alpha\frac{\partial}{\partial y} + h\alpha'\frac{\partial}{\partial y'}\right)^2 f + \cdots$$

by using Taylor's expansion. [*y and y' are treated as variables since x is fixed*] Neglecting second and higher degree terms since h is a small parameter, we have,

$$f(x, y + h\alpha(x), y' + h\alpha'(x)) - f(x, y, y') = h\alpha\frac{\partial f}{\partial y} + h\alpha'\frac{\partial f}{\partial y'}$$

Denoting the LHS of this equation by δf we have,

$$\delta f = h\alpha\frac{\partial f}{\partial y} + h\alpha'\frac{\partial f}{\partial y'} \qquad \ldots (1)$$

δf is called the **variation of f**.

We have from (1),

$$\delta y = h\alpha\frac{\partial y}{\partial y} + h\alpha'\frac{\partial y}{\partial y'} = h\alpha \cdot 1 + h\alpha' \cdot 0$$

ie.,

$$h\alpha = \delta y \qquad \ldots (2)$$

$$\delta y' = h\alpha\frac{\partial y'}{\partial y} + h\alpha'\frac{\partial y'}{\partial y'} = h\alpha \cdot 0 + h\alpha' \cdot 1$$

ie.,

$$h\alpha' = \delta y' \qquad \ldots (3)$$

Using (2) and (3) in (1) we have,

$$\delta f = \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\delta y' \qquad \dots (4)$$

Remark : *Geometrically* $y(x)$ *and* $y(x)+h\alpha(x)$ *represent two neighbouring curves. Variation in f represents the change in f from curve to curve.*

We now proceed to establish two important properties connected with the variational operator δ, differential operator $\frac{d}{dx}$ and integral $\int$.

Property - 1.

$$\delta\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\delta y)$$

Proof : $$\delta\left(\frac{dy}{dx}\right) = \delta y' = h\alpha', \text{ by using (3).}$$

$$= h\frac{d\alpha}{dx} = \frac{d}{dx}(h\alpha), \quad \text{since } h \text{ is independent of } x.$$

$$= \frac{d}{dx}(\delta y), \text{ by using (2).}$$

Thus, $$\delta\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\delta y)$$

5.21 Functionals

We know that a function is a correspondence (*assignment*) from a set of numbers to a set of numbers. We introduce the concept of a functional as follows.

Let S be a set of functions of a single variable x defined over an interval (x_1, x_2).

Then any function which assigns to each function in S a unique real value is called a **functional.** In other words **functional is a mapping from functions to real numbers.** (*Take a note of the first sentence again and observe the difference between a function and a functional*).

Consider a function of the form $f(x, y, y')$ where y' is the derivative of y w.r.t. x and $x \in (x_1, x_2)$.

The integral $I(y) = \int_{x_1}^{x_2} f(x, y, y')\,dx$ is a functional [*a standard form*]. It can be easily seen that for every $y(x)$, $I(y)$ gives a real value.

Examples of functionals

1. $\int_0^1 x + (y')^2\,dx$ 2. $\int_{x_1}^{x_2} \sqrt{1 + (y')^2}\,dx$

Property - 2.

If $I = \int_{x_1}^{x_2} f(x, y, y')\,dx$ *then* $\delta \int_{x_1}^{x_2} f(x, y, y')\,dx = \int_{x_1}^{x_2} \delta f(x, y, y')\,dx$

That is to say that the variational of a functional associated with $f(x, y, y')$ *is equal to the functional associated with the variation of f.*

Proof : $I = \int_{x_1}^{x_2} f(x, y, y')\,dx$ is the functional.

Since the value of I depends on y and y' we have by using the result connected with the variation [Refer (4)],

$$\delta I = \frac{\partial I}{\partial y}\delta y + \frac{\partial I}{\partial y'}\delta y'$$

$$\therefore \quad \delta I = \left\{ \int_{x_1}^{x_2} \frac{\partial}{\partial y}[f(x, y, y')]\,dx \right\}\delta y + \left\{ \int_{x_1}^{x_2} \frac{\partial}{\partial y'}[f(x, y, y')]\,dx \right\}\delta y'$$

$$\text{ie.,} \quad \delta I = \int_{x_1}^{x_2} \frac{\partial f}{\partial y}\delta y\,dx + \int_{x_1}^{x_2} \frac{\partial f}{\partial y'}\delta y'\,dx$$

$$\text{ie.,} \quad \delta I = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\delta y' \right] dx = \int_{x_1}^{x_2} \delta f\,dx$$

Thus, $\delta \int_{x_1}^{x_2} f(x, y, y')\,dx = \int_{x_1}^{x_2} \delta f(x, y, y')\,dx$

Remark : *From property - 1 and 2 we can say that* δ *and* $\frac{d}{dx}$; δ *and* $\int$ *are commutative with each other.*

In calculus of variations we determine the function $y = y(x)$ [*a curve*] satisfying $y(x_1) = y_1$ and $y(x_2) = y_2$ such that for a given function $f(x, y, y')$, $I(y)$ is an extremum.

In differential calculus we have also discussed applications of maxima and minima. For example,

(1) the maximum rectangle that can be inscribed in a circle is a square.

(2) the rectangular solid of maximum volume that can be inscribed in a sphere is a cube.

The presence of y' in $f(x, y, y')$ in $I(y)$ helps us to discuss a few important applications of calculus of variations analogous to the earlier cited examples in differential calculus.

For example, we can find the curve $y = y(x)$ of minimum length passing through two points (x_1, y_1) and (x_2, y_2). This is equivalent to finding $y(x)$ such that $I(y)$ is a minimum where $f(x, y, y') = \sqrt{1+(y')^2}$ being the formula for the length.

As a matter of comparison, in differential calculus we discuss maxima and minima of functions, whereas in calculus of variations we deal with maxima or minima of functionals.

We now proceed to establish a necessary condition (*referred to as Euler's equation*) for the functional $I(y)$ to be an extremum.

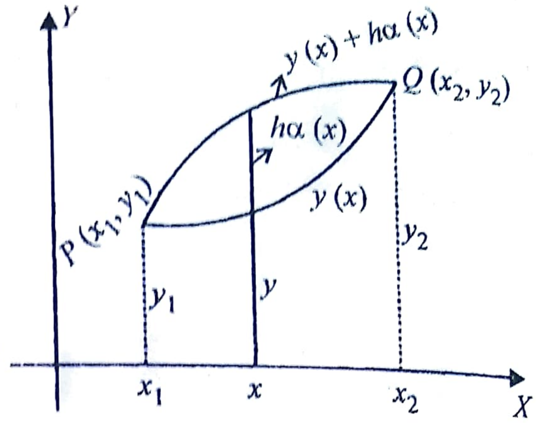
5.22 Euler's Equation

A necessary condition for the integral $I = \int_{x_1}^{x_2} f(x, y, y')\,dx$ *where* $y(x_1) = y_1$ *and* $y(x_2) = y_2$ *to be an extremum is that*

[Dec 2017, 18]

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \quad [\textit{Euler's equation}]$$

Proof :



Let I be an extremum along some curve $y = y(x)$ passing through $P(x_1, y_1)$ and $Q(x_2, y_2)$.

Also, let $y = y(x) + h\alpha(x)$ . . . (1)

be the neighbouring curve (*where h is small*) joining these points so that we must have

$$\alpha(x_1) = 0 \text{ at } P \text{ and } \alpha(x_2) = 0 \text{ at } Q. \quad \ldots (2)$$

When $h = 0$ these two curves coincide thus making I an extremum. That is to say that,

$$I = \int_{x_1}^{x_2} f\left(x, y(x) + h\alpha(x), y'(x) + h\alpha'(x)\right) dx$$

is an extremum when $h = 0$.

This requires $\dfrac{dI}{dh} = 0$ when $h = 0$, treating I to be a function of h.

$$\therefore \quad \frac{dI}{dh} = \int_{x_1}^{x_2} \frac{\partial}{\partial h} f\left(x, y(x) + h\alpha(x), y'(x) + h\alpha'(x)\right) dx$$

Applying chain rule for the partial derivative in RHS, we have,

$$\frac{dI}{dh} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial x}\frac{\partial x}{\partial h} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial h} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial h}\right] dx \quad \ldots (3)$$

But h is independent of x and hence $\dfrac{\partial x}{\partial h} = 0$.

Let us consider (1) and differentiate w.r.t. x.

$$\therefore \quad y' = y'(x) + h\alpha'(x) \quad \ldots (4)$$

Also, we have from (1), $\dfrac{\partial y}{\partial h} = \alpha(x)$ and from (4) $\dfrac{\partial y'}{\partial h} = \alpha'(x)$

Using these results in (3) we have,

$$\frac{dI}{dh}=\int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y}\alpha(x)+\frac{\partial f}{\partial y'}\alpha'(x)\right]dx \qquad \ldots (5)$$

Keeping the first term in the RHS of (5) as it is and integrating the second term by parts we have,

$$\frac{dI}{dh}=\int_{x_1}^{x_2}\frac{\partial f}{\partial y}\alpha(x)\,dx+\left\{\left[\frac{\partial f}{\partial y'}\alpha(x)\right]_{x_1}^{x_2}-\int_{x_1}^{x_2}\alpha(x)\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)dx\right\}$$

$$=\int_{x_1}^{x_2}\frac{\partial f}{\partial y}\alpha(x)\,dx+\left\{\frac{\partial f}{\partial y'}\alpha(x_2)-\frac{\partial f}{\partial y'}\alpha(x_1)\right\}-\int_{x_1}^{x_2}\alpha(x)\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)dx$$

But $\alpha(x_1)=0=\alpha(x_2)$ from (2) and we have by combining the two integrals,

$$\frac{dI}{dh}=\int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\alpha(x)\,dx$$

But we have already stated that $\frac{dI}{dh}$ must be zero when $h=0$ for I to be an extremum. Hence integrand in the RHS must be zero.
Since $\alpha(x)$ is arbitrary we must have

$$\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)=0$$

This is the required **Euler's equation** being the necessary condition for the extremum of the functional $I=\int_{x_1}^{x_2} f(x,y,y')\,dx$.

Theorem : *The necessary condition for the functional* $I=\int_{x_1}^{x_2} f(x,y,y')\,dx$ *to be an extremum is* $\delta I=0$.

Proof :

Note : *We need to simply retrace the steps as in the derivation of Euler's equation upto the stage of arriving at equation* (5).

$$\frac{dI}{dh} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} \alpha(x) + \frac{\partial f}{\partial y'} \alpha'(x) \right] dx \qquad \ldots (5)$$

We have, $\delta I = \delta \int_{x_1}^{x_2} f(x, y, y')\, dx$

Since δ and $\int$ are commutative with each other we have,

$$\delta I = \int_{x_1}^{x_2} \delta f(x, y, y')\, dx$$

Using the expression for the variation of f being δf in the RHS, we have,

$$\delta I = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} \delta y + \frac{\partial f}{\partial y'} \delta y' \right] dx$$

But, $\delta y = h\alpha(x)$ and $\delta y' = h\alpha'(x)$

$$\therefore \quad \delta I = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} h\alpha(x) + \frac{\partial f}{\partial y'} h\alpha'(x) \right] dx$$

ie., $$\delta I = h \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} \alpha(x) + \frac{\partial f}{\partial y'} \alpha'(x) \right] dx$$

ie., $\delta I = h \dfrac{dI}{dh}$, by equation (5).

But $\dfrac{dI}{dh} = 0$ when $h = 0$ is a necessary condition for I to be an extremum.

Thus $\delta I = 0$ also represents the necessary condition for the functional I to be an extremum.

Note : Additional Properties.

If u and v are functions of x, y, y' we have the following properties which can be proved.

(1) $\delta(cu) = c\,\delta u$, c *being a constant* (2) $\delta(c) = 0$

(3) $\delta(c_1 u \pm c_2 v) = c_1 \delta u \pm c_2 \delta v$ (4) $\delta(uv) = u\,\delta v + v\,\delta u$

(5) $\delta\left(\dfrac{u}{v}\right) = \dfrac{v du - u dv}{v^2}$, $(v \neq 0)$ (6) $\delta\left(\dfrac{d^n u}{dx^n}\right) = \dfrac{d^n}{dx^n}(\delta u)$

(7) $\delta(f^n) = n f^{n-1}(\delta f)$

WORKED PROBLEMS

[11] *Show the Euler's equation* $\dfrac{\partial f}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial f}{\partial y'}\right) = 0$ *can be put in the following forms.*

(a) $\dfrac{\partial f}{\partial y} - \dfrac{\partial^2 f}{\partial x\,\partial y'} - \dfrac{\partial^2 f}{\partial y\,\partial y'} y' - \dfrac{\partial^2 f}{\partial y'^2} y'' = 0$

(b) $\dfrac{d}{dx}\left[f - y'\dfrac{\partial f}{\partial y'}\right] = \dfrac{\partial f}{\partial x}$

Further obtain the particular forms of Euler's equation in the following cases.

(c) f does not contain x explicitly.

(d) f does not contain y explicitly.

(e) f does not contain x and y explicitly.

☞ We have Euler's equation : $\dfrac{\partial f}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial f}{\partial y'}\right) = 0$ ... (1)

(a) It is evident that $\dfrac{\partial f}{\partial y'}$ is a function of x, y, y' since f is a function of x, y, y'.

Let $z = \dfrac{\partial f}{\partial y'}$ so that $\dfrac{d}{dx}\left(\dfrac{\partial f}{\partial y'}\right) = \dfrac{dz}{dx}$

Using the expression for the total derivative we have,

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx} + \frac{\partial z}{\partial y'}\frac{dy'}{dx}$$

ie., $$\frac{dz}{dx} = \frac{\partial^2 f}{\partial x\,\partial y'} + \frac{\partial^2 f}{\partial y\,\partial y'} y' + \frac{\partial^2 f}{\partial y'^2} y''$$

We can write Euler's equation in the form :

$$\frac{\partial f}{\partial y} - \frac{dz}{dx} = 0$$

Thus, $$\boxed{\frac{\partial f}{\partial y} - \frac{\partial^2 f}{\partial x\, \partial y'} - \frac{\partial^2 f}{\partial y\, \partial y'} y' - \frac{\partial^2 f}{\partial y'^2} y'' = 0}$$

(b) We again have as a total derivative,

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx}$$

ie., $$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' \quad \ldots (2)$$

Also, $$\frac{d}{dx}\left(y' \frac{\partial f}{\partial y'}\right) = y' \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + y'' \frac{\partial f}{\partial y'} \quad \ldots (3)$$

Now, (2) – (3) will give us,

$$\frac{df}{dx} - \frac{d}{dx}\left(y' \frac{\partial f}{\partial y'}\right) = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' - y' \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)$$

$$= \frac{\partial f}{\partial x} + y' \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]$$

$$= \frac{\partial f}{\partial x} + y' \cdot 0, \text{ by using (1).}$$

$$\therefore \quad \frac{df}{dx} - \frac{d}{dx}\left(y' \frac{\partial f}{\partial y'}\right) = \frac{\partial f}{\partial x}$$

Thus, $$\boxed{\frac{d}{dx}\left[f - y' \frac{\partial f}{\partial y'}\right] = \frac{\partial f}{\partial x}}$$

(c) Suppose f does not contain x explicitly.
This means that f is independent of x.

$\therefore \quad \frac{\partial f}{\partial x} = 0$ and the result (b) becomes,

$$\frac{d}{dx}\left[f - y' \frac{\partial f}{\partial y'} \right] = 0$$

Thus, $\boxed{f - y' \dfrac{\partial f}{\partial y'} = k,}$ where k is a constant.

(d) Suppose f does not contain y explicitly.
This means that f is independent of y.

$\therefore \quad \dfrac{\partial f}{\partial y} = 0$ and hence Euler's equation in the form

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \text{ becomes, } -\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

Thus, $\boxed{\dfrac{\partial f}{\partial y'} = k,}$ where k is a constant.

(e) Suppose f does not contain both x and y explicitly.
This means that f is independent of x and y.

$$\therefore \quad \frac{\partial f}{\partial x} = 0 \text{ and } \frac{\partial f}{\partial y} = 0$$

Differentiating these partially w.r.t y' we obtain,

$$\frac{\partial}{\partial y'}\left(\frac{\partial f}{\partial x}\right) = 0 \text{ and } \frac{\partial}{\partial y'}\left(\frac{\partial f}{\partial y}\right) = 0$$

$$\text{or} \quad \frac{\partial^2 f}{\partial x\, \partial y'} = 0 \text{ and } \frac{\partial^2 f}{\partial y\, \partial y'} = 0$$

Using these along with $\dfrac{\partial f}{\partial y} = 0$ in the equation

$$\frac{\partial f}{\partial y} - \frac{\partial^2 f}{\partial x\, \partial y'} - \frac{\partial^2 f}{\partial y\, \partial y'}\, y' - \frac{\partial^2 f}{\partial y'^2}\, y'' = 0$$

we obtain, $$-\frac{\partial^2 f}{\partial y'^2} y'' = 0$$

Thus, $$\boxed{y'' = 0 \text{ provided } \frac{\partial^2 f}{\partial y'^2} \neq 0}$$

- **Step by step working procedure for solving a variational problem.**

Step-1 Given $I = \int_{x_1}^{x_2} f(x, y, y')\,dx$, solving variational problem is to find the function y such that I is an extremum. The necessary condition is $\delta I = 0$, which is also equivalent to the Euler's equation $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$

Step-2 We adopt Euler's equation for the function $f(x, y, y')$ which will result in an ordinary differential equation (ODE) in y.

Step-3 We solve for y using a suitable method which results in the required y with arbitrary constants (*general solution*).
However if x_1 and x_2 are specifically given with the conditions $y(x_1) = y_1$ and $y(x_2) = y_2$, we use these conditions in the general solution to find the arbitrary constants, there by we get a particular solution.

Important Note : ***Knowledge of solving ODEs of higher order is an essential pre-requiste.*** *(Refer second semester book)*

[12] *Find the extremal of the functional* $\int_{x_1}^{x_2} (y' + x^2 y'^2)\,dx$

or

Solve the Euler's equation for the functional $\int_{z_0}^{x_1} (1 + x^2 y')y'\,dx$ **[June 2017, 18]**

☞ Let, $f(x, y, y') = y' + x^2 y'^2$

Euler's equation $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$ becomes,

$$0-\frac{d}{dx}(1+2x^2\ y')=0 \quad \text{or} \quad \frac{d}{dx}(1+2x^2\ y')=0$$

Integrating w.r.t. x we get,

$1+2x^2\ y'=k_1$, k_1 is a constant.

ie., $$y'=\frac{dy}{dx}=\frac{k_1-1}{2x^2}$$

$\therefore$ $$y=\frac{k_1-1}{2}\int\frac{1}{x^2}dx+c_2$$

ie., $$y=\frac{k_1-1}{2}\left(\frac{-1}{x}\right)+c_2$$ and let $c_1=\dfrac{1-k_1}{2}$

Thus, $$\boxed{y=\frac{c_1}{x}+c_2}$$

[13] *Find the function y which makes the integral* $\displaystyle\int_{x_1}^{x_2}(1+xy'+xy'^2)\,dx$ *an extremum.*

☞ Let, $f(x,y,y')=1+xy'+xy'^2$

Euler's equation, $\dfrac{\partial f}{\partial y}-\dfrac{d}{dx}\left(\dfrac{\partial f}{\partial y'}\right)=0$ becomes,

$$0-\frac{d}{dx}(x+2xy')=0.$$

$\Rightarrow$ $x+2xy'=k_1$, on integration.

$\therefore$ $$y'=\frac{dy}{dx}=\frac{k_1-x}{2x}$$

ie., $$\frac{dy}{dx}=\frac{k_1}{2x}-\frac{1}{2}$$

$\therefore$ $$y=\frac{k_1}{2}\int\frac{1}{x}dx-\int\frac{1}{2}dx+c_2$$

Thus, $$\boxed{y=c_1\log x-\frac{x}{2}+c_2}$$ where $c_1=k_1/2$.

[14] *Find the extremal of the functional* $\int_{x_1}^{x_2} (y'^2 + ky^2)\,dx$

☞ Let, $f(x, y, y') = y'^2 + ky^2$

Euler's equation $\dfrac{\partial f}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial f}{\partial y'}\right) = 0$ becomes,

$$2ky - \frac{d}{dx}(2y') = 0 \quad \text{or} \quad ky - \frac{d^2y}{dx^2} = 0$$

ie., $\dfrac{d^2y}{dx^2} - ky = 0$ or $(D^2 - k)y = 0$ where $D = \dfrac{d}{dx}$

This is a second order ODE and the solution depends on the nature of k.

Case-(i)	:	Let $k = 0$	then	$y = c_1 x + c_2$
Case-(ii)	:	Let $k = +p^2$	then	$y = c_1 e^{px} + c_2 e^{-px}$
Case-(iii)	:	Let $k = -p^2$	then	$y = c_1 \cos px + c_2 \sin px$

[15] *Find the extremal of the functional* $\int_{x_1}^{x_2} (y^2 + y'^2 + 2ye^x)\,dx$

☞ Let, $f(x, y, y') = y^2 + y'^2 + 2ye^x$

Euler's equation, $\dfrac{\partial f}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial f}{\partial y'}\right) = 0$ becomes,

$$(2y + 2e^x) - \frac{d}{dx}(2y') = 0 \quad \text{or} \quad y + e^x - y'' = 0$$

ie., $y'' - y = e^x$ or $(D^2 - 1)y = e^x$ where $D = \dfrac{d}{dx}$

AE is $m^2 - 1 = 0 \quad \therefore\ m = \pm 1$

Hence, $CF = y_c = c_1 e^x + c_2 e^{-x}$

$$PI = y_p = \frac{e^x}{D^2 - 1} = \frac{e^x}{0}, \text{ on replacing } D \text{ by } 1.$$

$$y_p = x\frac{e^x}{2D} = \frac{x\,e^x}{2}$$

We have, $y = y_c + y_p$

Thus, $\boxed{y = c_1 e^x + c_2 e^{-x} + x e^x/2}$

[16] *Find the extremal of the functional* $\int_a^b (x^2 y'^2 + 2y^2 + 2xy)\,dx$

☞ Let, $f(x, y, y') = x^2 y'^2 + 2y^2 + 2xy$

Euler's equation, $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$ becomes,

$$(4y + 2x) - \frac{d}{dx}(2x^2 y') = 0$$

ie., $(4y + 2x) - (2x^2 y'' + 4xy') = 0$

Dividing by -2 we get,

$$x^2 y'' + 2xy' - 2y = x$$

To solve (1), we put, $t = \log x$ or $x = e^t$

Then, $xy' = Dy$; $x^2 y'' = D(D-1)y$ where $D = \frac{d}{dt}$

Now (1) becomes,

$$[D(D-1) + 2D - 2]y = e^t$$

ie., $(D^2 + D - 2)y = e^t$

AE is $m^2 + m - 2 = 0$ or $(m-1)(m+2) = 0$ $\therefore$ $m = 1, -2$

Hence, $y_c = c_1 e^t + c_2 e^{-2t}$ or $y_c = c_1 x + \frac{c_2}{x^2}$

$$y_p = \frac{e^t}{D^2 + D - 2} = \frac{e^t}{0}, \text{ on replacing } D \text{ by } 1.$$

$$= t\frac{e^t}{2D+1} = t\frac{e^t}{2(1)+1} = \frac{t\,e^t}{3} = \frac{x \log x}{3}$$

We have, $y = y_c + y_p$

Thus, $$\boxed{y = c_1 x + \frac{c_2}{x^2} + \frac{x \log x}{3}}$$

[17] *Find the extremal of the functional* $\int_{x_1}^{x_2} (y'^2 - y^2 + 2y \sec x)\, dx$

☞ Let, $f(x, y, y') = y'^2 - y^2 + 2y \sec x$

Euler's equation, $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$ becomes,

$$(-2y + 2\sec x) - \frac{d}{dx}(2y') = 0$$

ie., $-y + \sec x - y'' = 0$ or $y'' + y = \sec x$

We need to apply the method of variation of parameters to solve the ODE.

We have, $(D^2 + 1) y = \sec x$

AE is $m^2 + 1 = 0 \quad \therefore m = \pm i$

Hence, $y_c = c_1 \cos x + c_2 \sin x$

Let $y = A \cos x + B \sin x$ be the solutionof the ODE where A and B are functions of x to be determined. If y_1 and y_2 are the solutions of the homogenous equation $f(D) y = 0$ then we know that

$$A = -\int \frac{y_2\, \phi(x)}{W} dx \; ; \quad B = \int \frac{y_1\, \phi(x)}{W} dx$$

where, $\phi(x) = \sec x$ and $W = y_1 y_2' - y_2 y_1'$

Here we have, $y_1 = \cos x$, $y_2 = \sin x$; $y_1' = -\sin x$; $y_2' = \cos x$

Further, $W = \cos^2 x + \sin^2 x = 1$

Now, $A = -\int \sin x \sec x\, dx$; $B = \int \cos x \sec x\, dx = \int 1\, dx$

ie., $A = -\int \tan x\, dx = -\log(\sec x) + k_1$; $B = x + k_2$

We have, $y = A\cos x + B\sin x$

ie., $y = [-\log(\sec x) + k_1]\cos x + [x + k_2]\sin x$

Thus, $\boxed{y = k_1 \cos x + k_2 \sin x - \cos x \log(\sec x) + x\sin x}$

[18] *Find the curve on which the functional* $\int_0^1 \left[(y')^2 + 12xy\right]dx$ *with* $y(0) = 0$ *and* $y(1) = 1$ *can be determined.*

[Dec 2017]

☞ Let, $I = \int_0^1 \left[(y')^2 + 12xy\right]dx$

Let $f(x, y, y') = (y')^2 + 12xy$

Euler's equation $\dfrac{\partial f}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial f}{\partial y'}\right) = 0$ becomes,

$$12x - \frac{d}{dx}(2y') = 0.$$ That is, $12x - 2y'' = 0$ or $y'' = 6x$

ie., $\dfrac{d^2y}{dx^2} = 6x$ and integrating w.r.t. x we get, $\dfrac{dy}{dx} = 3x^2 + c_1$

Again integrating w.r.t. x we get,

$$y = x^3 + c_1 x + c_2$$

Using the condition $y = 0$ at $x = 0$ and $y = 1$ at $x = 1$,

we obtain $c_1 = 0$ and $c_2 = 0$

Thus, $\boxed{y = x^3}$ is the required curve.

Note : Similar Problem

Solve the variational problem $\delta\int_0^1 (12xy + y'^2)\,dx$ *under the conditions* $y(0) = 3$ *and* $y(1) = 6$.

[Dec 2018]

Here we get $c_1 = 2$ and $c_2 = 3$. $y = x^3 + 2x + 3$ is the curve.

[19] *Solve the variational problem :* $\delta \int_0^1 (x + y + y'^2)\, dx = 0$ *under the conditions* $y(0) = 1$ *and* $y(1) = 2$.

☞ Let, $I = \int_0^1 (x + y + y'^2)\, dx$

$\delta I = 0$ is equivalent to the Euler's equation :

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

Let, $f(x, y, y') = x + y + y'^2$ and Euler's equation becomes,

$$1 - \frac{d}{dx}(2y') = 0. \quad \text{That is,} \quad 2y'' = 1 \quad \text{or} \quad \frac{d^2 y}{dx^2} = \frac{1}{2}$$

Integrating w.r.t. x we get, $\frac{dy}{dx} = \frac{x}{2} + c_1$

and again integrating w.r.t. x we get

$$y = \frac{x^2}{4} + c_1 x + c_2$$

Using the conditions, $y = 1$ at $x = 0$ and $y = 2$ at $x = 1$, we have,

$$1 = c_2 \; : \; 2 = \frac{1}{4} + c_1 + 1 \quad \therefore \; c_1 = \frac{3}{4}$$

Thus, $$\boxed{y = \frac{x^2}{4} + \frac{3}{4}x + 1 \quad \text{or} \quad 4y = x^2 + 3x + 4}$$

Remark : *The equation represents a parabola. This means that the functional I assumes extreme values on the parabola.*

[20] *Solve the variational problem*

$$\delta \int_0^{\pi/2} (y^2 - y'^2)\, dx = 0 \; ; \; y(0) = 0, \; y(\pi/2) = 2$$

☞ Let, $f(x, y, y') = y^2 - y'^2$

Euler's equation, $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$ becomes,

$$2y - \frac{d}{dx}(-2y') = 0 \text{ or } y'' + y = 0$$

$$\therefore \quad y = c_1 \cos x + c_2 \sin x$$

Now, $y = 0$ at $x = 0$ gives $c_1 = 0$, $y = 2$ at $x = \pi/2$ gives $c_2 = 2$.

Thus, $\boxed{y = 2 \sin x}$

[21] *Show that the functional* $\int_{x_1}^{x_2} (y^2 + x^2 y')\,dx$ *assumes extreme values on the striaght line* $y = x$.

☞ Let, $f(x, y, y') = y^2 + x^2 y'$

Euler's equation, $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$ becomes,

$$2y - \frac{d}{dx}(x^2) = 0 \text{ or } 2y - 2x = 0$$

Thus, $\boxed{y = x \text{ is the straight line.}}$

[22] *Show that the extremal of the functional* $\int_0^1 y^2 \{3x(y'^2 - 1) + y\,y'^3\}\,dx$ *subject to the conditions* $y(0) = 0$, $y(1) = 2$ *is the circle* $x^2 + y^2 - 5x = 0$.

☞ Let, $f(x, y, y') = 3xy^2(y'^2 - 1) + y^3 y'^3$

Euler's equation $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$ becomes,

$$[6xy(y'^2 - 1) + 3y^2 y'^3] = \frac{d}{dx}[6xy^2 y' + 3y^3 y'^2] = 0$$

ie.,
$$[6xy(y'^2 - 1) + 3y^2 y'^3] - [6y^2 y' + 12xyy'^2 + 6xy^2 y'' + 6y^3 y' y'' + 9y^2 y'^3] = 0$$

ie., $-6xy\,y'^2 - 6xy - 6y^2y'^3 - 6y^2y' - 6xy^2y'' - 6y^3\,y'y'' = 0$

or $(xy\,y'^2 + xy + xy^2\,y'') + (y^2y'^3 + y^2\,y' + y^3y'\,y'') = 0$

ie., $xy\,(y'^2 + 1 + y\,y'') + y^2\,y'\,(y'^2 + 1 + y\,y'') = 0$

ie., $(xy + y^2\,y')(y'^2 + 1 + y\,y'') = 0$

ie., $y\,(x + y\,y')(y'^2 + 1 + y\,y'') = 0$

Hence we have $y = 0$ and the ODEs

$$x + y\,y' = 0 \qquad \ldots (1)$$

$$y'^2 + 1 + y\,y'' = 0 \qquad \ldots (2)$$

We reject $y = 0$ since the condition $y\,(\,1\,) = 2$ is not satisfied.

Now (1) is $\dfrac{dy}{dx} = -\dfrac{x}{y}$

or $x\,dx + y\,dy = 0$

$\therefore \quad \dfrac{x^2}{2} + \dfrac{y^2}{2} = k$ or $x^2 + y^2 = 2k.$

This gives $k = 0$ for the condition $y\,(\,0\,) = 0$ and $y\,(\,1\,) = 2$ is not satisfied. Hence this soltuion is also rejected.

Now let us consider (2) : $y'^2 + 1 + y\,y'' = 0$ can be written in the form,

$$1 + \frac{d}{dx}\,(y\,y') = 0 \text{ or } \frac{d}{dx}(y\,y') = -1$$

$\therefore \quad y\,y' = -x + c_1$, on integration w.r.t. x.

ie., $y\dfrac{dy}{dx} = -x + c_1$

or $y\,dy = -x\,dx + c_1\,dx$ and by integration we get

$$\frac{y^2}{2} + \frac{x^2}{2} - c_1\,x = c_2 \qquad \ldots (3)$$

Using the condition $y\,(\,0\,) = 0$ we get $c_2 = 0$.

Again $y\,(\,1\,) = 2$ in (3) will give us $c_1 = 5/2$.

Substituting, $c_1 = 5/2$, $c_2 = 0$ in (3) we get,

$$\frac{y^2}{2} + \frac{x^2}{2} - \frac{5x}{2} = 0$$

Thus, $$\boxed{x^2 + y^2 - 5x = 0}$$

This is the circle : $(x - 5/2)^2 + (y - 0)^2 = (5/2)^2$ having centre $(5/2, 0)$ and radius equal to $5/2$.

[23] *Show that the equation of the curve joining the points (1, 0) and (2, 1) for which* $I = \int_1^2 \frac{1}{x}\sqrt{1 + y'^2}\, dx$ *is an extremum is a circle.*

☞ Let, $f(x, y, y') = \frac{1}{x}\sqrt{1 + y'^2}$

Euler's equation, $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$ becomes,

$$-\frac{d}{dx}\left[\frac{1}{x}\frac{1}{2\sqrt{1 + y'^2}} \cdot 2y'\right] = 0 \quad \text{or} \quad \frac{d}{dx}\left[\frac{y'}{x\sqrt{1 + y'^2}}\right] = 0$$

$$\therefore \quad \frac{y'}{x\sqrt{1 + y'^2}} = c_1 \quad \text{or} \quad \frac{y'^2}{x^2(1 + y'^2)} = c_1^2$$

ie., $$y'^2 = c_1^2 x^2 (1 + y'^2)$$

$$y'^2(1 - c_1^2 x^2) = c_1^2 x^2 \quad \text{or} \quad \frac{dy}{dx} = \frac{c_1 x}{\sqrt{1 - c_1^2 x^2}}$$

$$\therefore \quad \int dy = \int \frac{c_1 x}{\sqrt{1 - c_1^2 x^2}}\, dx + c_2$$

or $$y = \int \frac{c_1 x}{\sqrt{1 - c_1^2 x^2}}\, dx + c_2$$

Put $1-c_1^2x^2 = z \quad \therefore \; -2c_1^2 x\,dx = dz$ or $c_1 x\,dx = -dz/2c_1$

Hence, $$y = \frac{-1}{2c_1}\int\frac{dz}{\sqrt{z}} + c_2$$

ie., $$y = \frac{-1}{2c_1}\frac{z^{1/2}}{1/2} + c_2$$

$$\therefore \quad y = \frac{-\sqrt{1-c_1^2x^2}}{c_1} + c_2 \qquad \ldots (1)$$

Since the curve passes through (1, 0) and (2, 1) we have $y = 0$ at $x = 1$ and $y = 1$ at $x = 2$.

Hence, (1) becomes ; $0 = -\dfrac{\sqrt{1-c_1^2}}{c_1} + c_2$, for $x = 1$, $y = 0$

ie., $$\frac{1-c_1^2}{c_1^2} = c_2^2 \text{ or } \frac{1}{c_1^2} - 1 = c_2^2 \qquad \ldots (2)$$

Also for the codnition, $y = 1$ at $x = 2$, (1) becomes

$$1 = \frac{-\sqrt{1-4c_1^2}}{c_1} + c_2 \text{ or } (1-c_2)^2 = \frac{1-4c_1^2}{c_1^2}$$

ie., $(1-c_2)^2 = \dfrac{1}{c_1^2} - 4$. But $\dfrac{1}{c_1^2} = 1 + c_2^2$ from (2).

$$\therefore \quad 1 - 2c_2 + c_2^2 = 1 + c_2^2 - 4 \text{ or } c_2 = 2$$

Hence, $1/c_1^2 = 5 \quad \therefore \; c_1 = 1/\sqrt{5}$

Substituting the values of c_1 and c_2 in (1) we have,

$$y = \left[-\sqrt{1-(x^2/5)}\cdot\sqrt{5}\right] + 2$$

or $$y - 2 = -\sqrt{5-x^2} \text{ or } (y-2)^2 = 5 - x^2$$

Thus $\boxed{x^2 + (y-2)^2 = 5}$ which is a circle with centre (0, 2) and radius $\sqrt{5}$.

[24] *Show that* $\int_{x_1}^{x_2} y^2\, y'^2\, dx$ *has an extremum when* $y(x)$ *is of the form* $c_1 \cdot \sqrt{x + c_2}$

☞ Let, $f(x, y, y') = y^2\, y'^2$

Euler's equation, $\dfrac{\partial f}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial f}{\partial y'}\right) = 0$ becomes,

$$2y\, y'^2 - \frac{d}{dx}(2y^2\, y') = 0$$

or $$y\, y'^2 - \frac{d}{dx}(y^2\, y') = 0$$

ie., $$y\, y'^2 - (y^2\, y'' + 2y\, y'^2) = 0$$

ie., $$y^2\, y'' + y\, y'^2 = 0$$

or $$y\, y'' + y'^2 = 0$$

ie., $$\frac{d}{dx}[y\, y'] = 0 \Rightarrow y\, y' = k_1$$

ie., $$y\frac{dy}{dx} = k_1 \text{ or } y\, dy = k_1\, dx$$

$\therefore$ $$\int y\, dy = \int k_1\, dx + k_2$$

ie., $$\frac{y^2}{2} = k_1 x + k_2$$

or $$y = \sqrt{2(k_1 x + k_2)} = \sqrt{2k_1\,(x + k_2/k_1)}$$

Let us denote, $c_1 = \sqrt{2k_1}$ and $c_2 = k_2/k_1$

Thus, $\boxed{y = c_1 \cdot \sqrt{x + c_2}}$ as required.

[25] *Show that an extremal of* $\int_{x_1}^{x_2} \left(\frac{y'}{y}\right)^2 dx$ *is expressible in the form* $y = a e^{bx}$

☞ Let, $f(x, y, y') = \frac{y'^2}{y^2}$

Euler's equation, $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$ becomes,

$$\frac{-2}{y^3} y'^2 - \frac{d}{dx}\left(\frac{2y'}{y^2}\right) = 0$$

ie., $$\frac{y'^2}{y^3} + \frac{d}{dx}\left(\frac{y'}{y^2}\right) = 0$$

ie., $$\frac{y'^2}{y^3} + \frac{y^2 y'' - 2y\, y'^2}{y^4} = 0$$

ie., $$\frac{y'^2}{y^3} + \frac{y\, y'' - 2\, y'^2}{y^3} = 0$$

ie., $y'^2 + y\, y'' - 2\, y'^2 = 0$ or $y\, y'' - y'^2 = 0$

Now, $y\, y'' - y'^2 = 0$ can be put in the form,

$$\frac{d}{dx}\left(\frac{y'}{y}\right) = 0 \quad \Rightarrow \quad \frac{y'}{y} = c_1$$

Hence, $\int \frac{y'}{y} dx = \int c_1\, dx + c_2$

ie., $\log_e y = c_1 x + c_2$ or $y = e^{c_1 x + c_2} = e^{c_2} \cdot e^{c_1 x}$

Thus, $\boxed{y = a e^{bx}}$ where $a = e^{c_2}$ and $b = c_1$

[26] *Solve the variational equation :*

$\delta \int_1^2 (x^2 y'^2 + 2y^2 + 2xy)\, dx = 0$ *under the conditions* $y(1) = y(2) = 0$.

☞ Refer Problem-[16] where the problem is solved for the integral with general limits a and b. The general solution so obtained is

$$y = c_1 x + \frac{c_2}{x^2} + \frac{x \log x}{3} \qquad \dots (1)$$

We use the given conditions to find the c_1 and c_2 present in (1).

$$y(1) = 0 \Rightarrow c_1 + c_2 + 0 = 0 \text{ or } c_2 = -c_1$$

$$y(2) = 0 \Rightarrow 2c_1 + \frac{c_2}{4} + \frac{2\log 2}{3} = 0$$

ie., $$2c_1 - \frac{c_1}{4} = -\frac{2\log 2}{3}$$

ie., $$\frac{7c_1}{4} = \frac{-\log 4}{3} \text{ or } c_1 = -\frac{4\log 4}{21} \text{ and } c_2 = \frac{4\log 4}{21}$$

Thus, $\boxed{y = \frac{4\log 4}{21}\left(\frac{1}{x^2} - x\right) + \frac{x\log x}{3}}$ is the required solution.

[27] *Solve the Euler's equation obtained for the functional* $I = \int_{x_0}^{x_1} (1 + x^2 y')\, dx$ *to have an extremum.*

☞ Let, $f(x, y, y') = 1 + x^2 y'$

Euler's equation, $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$ becomes,

$$0 - \frac{d}{dx}(x^2) = 0 \text{ or } -2x = 0 \text{ or } x = 0.$$

Thus, $\boxed{x = 0 \text{ being the } y\text{-axis is the solution of the problem.}}$

The given functional assumes extreme values on the y-axis.

[28] *Find the extremal of the functional* $I = \int_0^{\pi/2} (y^2 - y'^2 - 2y\sin x)\, dx$, *under the end conditions* $y(0) = y(\pi/2) = 0$

☞ Let $f(x, y, y') = y^2 - y'^2 - 2y\sin x$

Euler's equation, $\dfrac{\partial f}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial f}{\partial y'}\right) = 0$ becomes,

$$2y - 2\sin x - \frac{d}{dx}(-2y') = 0$$

ie., $\quad y - \sin x + y'' = 0$ or $y'' + y = \sin x$

We have, $(D^2 + 1)y = \sin x$ where $D = \dfrac{d}{dx}$

AE is $m^2 + 1 = 0$ which gives $m = \pm i$.

$$y_c = c_1 \cos x + c_2 \sin x$$

$$y_p = \frac{\sin x}{D^2 + 1}$$

Replacing D^2 by -1, the denominator becomes zero.

$$\therefore \quad y_p = x\frac{\sin x}{2D} = -\frac{x\cos x}{2}$$

General solution, $y = y_c + y_p$

ie., $$y = c_1 \cos x + c_2 \sin x - \frac{x\cos x}{2} \qquad \ldots (1)$$

We use the given conditions to find c_1 and c_2 present in (1).

$$y(0) = 0 \Rightarrow c_1 = 0 \; ; \; y(\pi/2) = 0 \Rightarrow c_2 = 0$$

Thus, $\boxed{y = -x\cos x/2}$ is the required extremal.

[29] *Find the curve on which the functional* $\displaystyle\int_0^{\pi/2} (y'^2 - y^2 + 2xy)\,dx$ *with* $y(0) = y(\pi/2) = 0$ *can be extremised.*

☞ Let, $f(x, y, y') = y'^2 - y^2 + 2xy$

Euler's equation, $\dfrac{\partial f}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial f}{\partial y'}\right) = 0$ becomes,

$$-2y + 2x - \frac{d}{dx}(2y') = 0$$

or $$-y + x - y'' = 0 \quad \text{or} \quad y'' + y = x$$

ie., $$(D^2 + 1)y = x, \text{ where } D = \frac{d}{dx}$$

AE is given by $m^2 + 1 = 0$ which gives $m = \pm i$

$$y_c = c_1 \cos x + c_2 \sin x$$

$$y_p = \frac{x}{D^2 + 1} = \frac{x}{1 + D^2} = x \text{ by division.}$$

$y = y_c + y_p$ is the general solution.

ie., $$y = c_1 \cos x + c_2 \sin x + x$$

Using the conditions, $y(0) = y(\pi/2) = 0$ we obtain,

$$c_1 = 0 \text{ and } c_2 = -\pi/2$$

Thus, $\boxed{y = -(\pi/2)\sin x + x}$ is the required curve.

[30] *Find the extremal of the functional,* $\int_{x_0}^{x_1} (x + y')y'\, dx$

Let, $f(x, y, y') = xy' + y'^2$

Euler's equation $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$ becomes,

$$0 - \frac{d}{dx}(x + 2y') = 0$$

ie., $$1 + 2y'' = 0 \quad \text{or} \quad \frac{d^2y}{dx^2} = -\frac{1}{2}$$

$\Rightarrow$ $$\frac{dy}{dx} = -\frac{x}{2} + c_1, \text{ on integration.}$$

$\Rightarrow \quad y = -\dfrac{x^2}{4} + c_1 x + c_2$, again on integration.

Thus, $\boxed{y = c_1 x + c_2 - (x^2/4)}$ is the required extremal.

5.23 Applications of Calculus of Variations

- **Geodesics**

Given two arbitrary points P and Q on a surface S, there exists infinite number of curves on the surface having P and Q as their extremities. Of these curves that curve whose length is the least is called the ***geodesic*** between the points P and Q on the given surface.

In other words, **a geodesic on a surface is a curve along which the distance between any two points of the surface is a minimum.**

Finding the geodesic on a surface is a variational problem involving the condition for the extremum of the associatied functional.

5.24 Standard variational problems.

[31] *Prove that the shortest distance between two points in a plane is along the straight line joining them* **or** *porve that the geodesics on a plane are straight lines.*

[June 2017, Dec 16, 18]

☞ Let $y = y(x)$ be a curve joining two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ in the XOY plane.

We know that the arc length between P and Q is given by

$$s = \int_{x_1}^{x_2} \frac{ds}{dx}\, dx = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

ie., $\quad s = I = \displaystyle\int_{x_1}^{x_2} \sqrt{1 + y'^2}\, dx$

We need to find the curve $y(x)$ such that I is minimum.

Let, $\quad f(x, y, y') = \sqrt{1 + y'^2}$

Euler's equation, $\dfrac{\partial f}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial f}{\partial y'}\right) = 0$ becomes,

$$0-\frac{d}{dx}\left[\frac{2y'}{2\sqrt{1+y'^2}}\right]=0$$

or $$\frac{d}{dx}\left[\frac{y'}{\sqrt{1+y'^2}}\right]=0$$

ie., $y''\sqrt{1+y'^2}-y'\dfrac{2y'y''}{2\sqrt{1+y'^2}}=0$, by quotient rule and cross multiplying.

ie., $y''(1+y'^2)-y''y'^2=0$ or $y''=0$.

ie., $$\frac{d^2y}{dx^2}=0$$

Let us integrate twice w.r.t x

Thus $\boxed{y=c_1x+c_2}$ which is a straight line.

[32] *Find geodesics on a surface given that the arc length on the surface is*

$$s=\int_{x_1}^{x_2}\sqrt{x(1+y'^2)}\,dx$$ **[June 2018]**

☞ We have, $f=\sqrt{x(1+y'^2)}$ which is independent of y.

$\therefore$ Euler's equation $\dfrac{\partial f}{\partial y}-\dfrac{d}{dx}\left(\dfrac{\partial f}{\partial y'}\right)=0$ reduces to,

$$-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)=0 \text{ or } \frac{\partial f}{\partial y'}=\text{const.}=c$$

ie., $\dfrac{1}{2\sqrt{x(1+y'^2)}}x(2y')=c$ or $\sqrt{x}\cdot\dfrac{y'}{\sqrt{1+y'^2}}=c$

ie., $xy'^2=c^2(1+y'^2)$ or $y'^2(x-c^2)=c^2$

$\therefore$ $$y'=\frac{c}{\sqrt{x-c^2}}$$

$$\Rightarrow \qquad y = c\int \frac{dx}{\sqrt{x-c^2}} + c_1$$

ie., $$y = 2c\sqrt{x-c^2} + c_1$$

Thus, $\boxed{(y-c_1)^2 = 4c^2(x-c^2)}$ is the required geodesic which is a parabola.

[33] *Prove that catenary is the curve which when rotated about a line generates a surface of minimum area.*

☞ We have the expression for the total surface area given by $\int 2\pi y\, ds$ where the curve is rotated about the x-axis.

$$\therefore \qquad I = \int_{x_1}^{x_2} 2\pi y \frac{ds}{dx}\, dx = \int_{x_1}^{x_2} 2\pi y\sqrt{1+y'^2}\, dx$$

Since 2π is a constant we can as well take $f(x, y, y') = y\sqrt{1+y'^2}$ which is independent of x. Therefore it is convenient to take the Euler's equation in the form

$$f - y'\frac{\partial f}{\partial y'} = \text{constant.}$$

ie., $$y\sqrt{1+y'^2} - y' \cdot \frac{y}{2\sqrt{1+y'^2}} \cdot 2y' = c$$

ie., $$\frac{y(1+y'^2) - y\,y'^2}{\sqrt{1+y'^2}} = c \quad \text{or} \quad \frac{y}{\sqrt{1+y'^2}} = c$$

ie., $$y^2 = c^2(1+y'^2) \quad \text{or} \quad y'^2 = \frac{y^2-c^2}{c^2}$$

ie., $$y' = \frac{dy}{dx} = \frac{\sqrt{y^2-c^2}}{c} \quad \text{or} \quad \frac{dy}{\sqrt{y^2-c^2}} = \frac{1}{c}dx$$

$$\therefore \qquad \int \frac{dy}{\sqrt{y^2-c^2}} = \frac{1}{c}\int dx + k$$

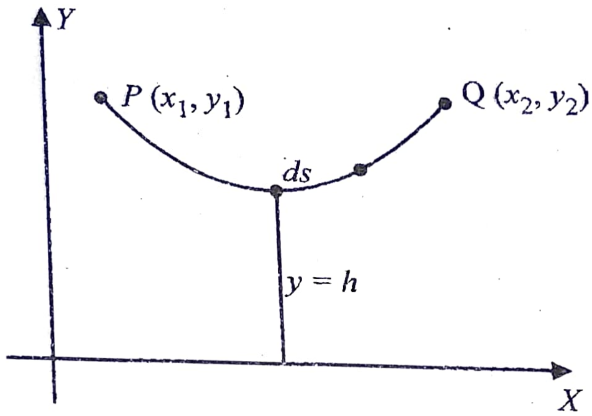
ie., $\cosh^{-1}\left(\frac{y}{c}\right) = \frac{x}{c} + k$ or $\frac{y}{c} = \cosh\left(\frac{x}{c} + k\right)$

Thus, $\boxed{y = c\cosh\left(\frac{x+a}{c}\right)}$ where $a = k\,c$. **This is a catenary.**

Remark : *It can be proved that this curve corresponds to the minimum value of S and the proof is beyond the scope of this book.*

- ***Hanging cable (chain) problem***

[34] *A heavy cable hangs freely under gravity between two fixed points. Show that the shape of the cable is a catenary.* **[Dec 2016, 18]**



Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be the two fixed points of the hanging cable. Let us consider an elementary arc length ds of the cable. Let ρ be the density (*mass/unit length*) of the cable so that $\rho\, ds$ is the mass of the element. If g is the acceleration due to gravity then the potential energy of the element $(m \cdot g \cdot h)$ is given by $(\rho\, ds) \cdot g \cdot y$ where x-axis is taken as the line of reference.

$\therefore$ total potential energy of the cable is given by

$$T = \int_P^Q (\rho\, ds) \cdot g\, y\, dx = \int_{x_1}^{x_2} \rho\, g\, y \frac{ds}{dx}\, dx$$

But, $\frac{ds}{dx} = \sqrt{1 + y'^2}$

Here, $f(x, y, y') = (\rho g)\, y\sqrt{1 + y'^2} = (\text{const.})\, y\sqrt{1 + y'^2}$

This is same as the previous problem.

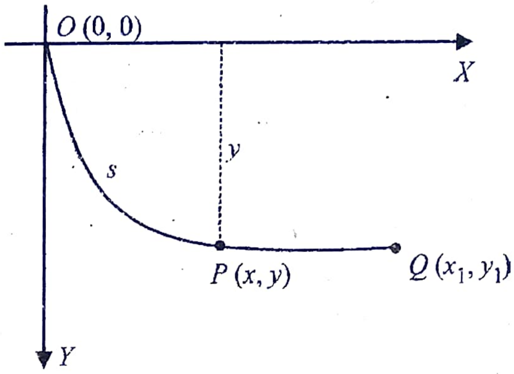
We can obtain, $y = c\cosh\left(\overline{x + a}/c\right)$

Thus, $\boxed{y = c\cosh\left(\dfrac{x+a}{c}\right)}$ **which is a catenary** and it can be proved that this corresponds to the minimum value of T.

- ***Brachistochrone problem***

[35] *Find the path in which a particle, in the absence of friction, will slide from one point to another in the shortest time under the action of gravity.*

☞



Let the particle start from the point O (*initially at rest*) and slide along the curve OQ.

Let the particle be at $P\,(x, y)$ at any time t and let $OP = s$ be the arc length.

We know from the principle of work and energy that the work done in moving the particle form O to P is equal to the difference between the kinetic energy at P and at O.

ie., $\quad mgy = \dfrac{1}{2}mv^2 - 0 \quad$ But, $\quad v = \dfrac{ds}{dt}$

$$\therefore \quad mgy = \frac{1}{2}m\left(\frac{ds}{dt}\right)^2 \quad \text{or} \quad \frac{ds}{dt} = \sqrt{2gy}$$

This is the velocity of the particle at the point $P\,(x, y)$.

Hence the time T required by the particle to move from O to Q is given by

$$T = \int_0^T dt = \int_0^{x_1} \frac{dt}{dx}\,dx = \int_0^{x_1} \frac{dt}{ds}\cdot\frac{ds}{dx}\,dx$$

Using, $\dfrac{dt}{ds} = \dfrac{1}{\sqrt{2gy}}$ and $\dfrac{ds}{dx} = \sqrt{1+y'^2}$ we have,

$$T = \int_0^{x_1} \frac{1}{\sqrt{2gy}}\sqrt{1+y'^2}\,dx$$

ie., $$T = \frac{1}{\sqrt{2g}} \int_0^{x_1} \frac{\sqrt{1+y'^2}}{\sqrt{y}} \, dx$$

We need to find $y(x)$ such that T is minimum.

Let, $f(x, y, y') = \dfrac{\sqrt{1+y'^2}}{\sqrt{y}}$ which is independent of x.

Hence we can take Euler's equation in the form,

$$f - y' \frac{\partial f}{\partial y'} = \text{constant} = c$$

ie., $$\frac{\sqrt{1+y'^2}}{\sqrt{y}} - y' \left\{ \frac{1}{\sqrt{y}} \frac{1}{2\sqrt{1+y'^2}} \cdot 2y' \right\} = c$$

ie., $$\frac{1}{\sqrt{y}\sqrt{1+y'^2}} \left\{ 1 + y'^2 - y'^2 \right\} = c$$

or $$\frac{1}{\sqrt{y}\sqrt{1+y'^2}} = c$$

ie., $$\sqrt{y}\sqrt{1+y'^2} = \frac{1}{c} = \sqrt{a} \ (say)$$

By squaring, $y(1+y'^2) = a$ or $y'^2 = \dfrac{a}{y} - 1$ or $y'^2 = \dfrac{a-y}{y}$

ie., $$\frac{dy}{dx} = \sqrt{\frac{a-y}{y}}$$

or $$dx = \sqrt{\frac{y}{a-y}} \, dy$$

$\therefore$ $$\int dx = \int \sqrt{\frac{y}{a-y}} \, dy + b$$

ie., $$x = \int \sqrt{\frac{y}{a-y}} \, dy + b$$

Put, $y = a\sin^2(\theta/2) \quad \therefore \quad dy = a\cdot 2\sin(\theta/2)\cos(\theta/2)\cdot\frac{1}{2}$

$$\therefore \qquad x = \int \frac{\sqrt{a}\sin(\theta/2)}{\sqrt{a}\cos(\theta/2)}\cdot a\sin(\theta/2)\cos(\theta/2)\,d\theta$$

ie., $$x = a\int \sin^2(\theta/2)\,d\theta + b$$

ie., $$x = a\int \frac{1-\cos\theta}{2}\,d\theta + b$$

ie., $$x = \frac{a}{2}(\theta - \sin\theta) + b$$

Also we have when $\theta = 0,\ y = 0$ and we must have $x = 0$ also since $(0, 0)$ is a point on the curve. This gives $b = 0$.

Hence, $x = \dfrac{a}{2}(\theta - \sin\theta);\ y = a\sin^2\left(\dfrac{\theta}{2}\right) = \dfrac{a}{2}(1-\cos\theta)$

Taking, $a/2 = k$ we have,

$$x = k(\theta - \sin\theta),\ y = k(1-\cos\theta)$$

We have obtained the required equation of the curve in the parametric form which is the equation of a **cycloid.**

Thus we can say that the path of a particle moving in a vertical plane under the action of gravity in least time is a cycloid.

[36] *Find the plane curve of length* l *having end points* (x_1, y_1) *and* (x_2, y_2) *such that the area under the curve between* $x = x_1$ *and* $x = x_2$ *is a maximum.*

☞ We know that,

$$I = \int_{x_1}^{x_2} \sqrt{1+y'^2}\,dx\ ;\ y(x_1) = y_1,\ y(x_2) = y_2 \qquad \ldots (1)$$

Also, area $$A = \int_{x_1}^{x_2} y\,dx \qquad \ldots (2)$$

We need to maximize A subject to the conditions stipulated on l given by (1). We adopt the technique known to us in the Lagrange's method of multipliers.

Let, $$f(x, y, y') = y + \lambda\sqrt{1+y'^2} \qquad \ldots (3)$$

Since 'f' doesnot contain x explicitly, we know that Euler's equation assumes the form $f - y'\dfrac{\partial f}{\partial y'} = k$, where k is a constant.

This equation with reference to (3) yields,

$$\left[y + \lambda\sqrt{1+y'^2} \right] - y' \cdot \lambda \frac{1}{2\sqrt{1+y'^2}} \cdot 2y' = k$$

ie., $$y\sqrt{1+y'^2} + \lambda(1+y'^2) - \lambda y'^2 = k\sqrt{1+y'^2}$$

ie., $$(y-k)\sqrt{1+y'^2} = -\lambda$$

or $$y - k = \frac{-\lambda}{\sqrt{1+y'^2}}$$

Using the substitution $y' = \tan t$ we get,

$$y - k = -\lambda \cos t \quad \ldots (4)$$

Further, $y' = \dfrac{dy}{dx} = \tan t$ be written as,

$dx = dy \cot t$ and we have from (4),

$$dy = \lambda \sin t\, dt$$

Hence, $dx = \lambda \sin t \cot t\, dt$

or $$dx = \lambda \cos t\, dt$$

$\Rightarrow$ $$x = \lambda \sin t + c$$

or $$x - c = \lambda \sin t \quad \ldots (5)$$

Thus from (4) and (5) we obtain,

$$\boxed{(x-c)^2 + (y-k)^2 = \lambda^2} \quad \ldots (6)$$

where c, k and λ can be found using the conditions $y(x_1) = y_1, y(x_2) = y_2$ and it is evident that (6) represents a cirlce.

ASSIGNMENT

Solve the following variational problems [1-4]

1. $\delta \int_0^1 (12xy + y'^2)\, dx = 0 \; ; \; y(0) = 3, y(1) = 6$

2. $\delta \int_0^2 x^3/y'^2 \, dx = 0 \; ; \; y(1) = 1, y(2) = 4$

3. $\delta \int_{x_1}^{x_2} (x + y + y'^2)\, dx = 0$

4. $\delta \int_{x_1}^{x_2} (xy' - y'^2)\, dx = 0; \;\; y(0) = 0, y(4) = 3$

Find the extremals of the following functions [5-8]

5. $\int_1^2 (3x + \sqrt{y'})\, dx \, ; y(1) = 5, y(2) = 7$

6. $\int_0^4 \sqrt{y(1 + y'^2)}\, dx \; ; \; y(0) = 1, y(4) = 5$

7. $\int_0^{\pi/2} (y'^2 - y^2 + 2xy)\; dx \; ; \; y(0) = 0, \; y(\pi/2) = 0$

8. $\int_0^{\pi/2} (y^2 - y'^2 - 2y \sin x)\, dx \; ; \; y(0) = 0 \; ; \; y(\pi/2) = 0$

9. Prove that an extremal of the functional $\int_a^b \frac{1}{y} \sqrt{(1 + y'^2)}\; dx$ is a circle.

10. Prove that an extremal of the function $\int_a^b \sqrt{y(1 + y'^2)}\, dx$ is a parabola.

ANSWERS

1. $y = x^3 + 2x + 3$ **2.** $y = x^2$ **3.** $y = c_1 + c_2 x - x^2/2$

4. $4y = x^2 - x$ **5.** $y = 2x + 3$ **6.** $x^2 = 4(y - 1)$

7. $2y = 2x - \pi \sin x$ **8.** $2y = -x \cos x$