LAPLACE TRANSFORMS

1.1 Definition of Laplace Transform

If $f(t)$ is a real valued function defined for all $t \geq 0$ then the ***Laplace transform*** of $f(t)$ denoted by $L[f(t)]$ is defined by

$$L[f(t)] = \int_{t=0}^{\infty} e^{-st} f(t)\,dt$$

provided the integral exists. On integration of the indefinite integral we will be having a function of s and t. When this is evaluated between the limits $t = 0$ and $t = \infty$ we will be left with a function of s only and we shall denote it by $\bar{f}(s)$ where s is a parameter, real or complex. Thus

$$L[f(t)] = \bar{f}(s)$$

Equivalently we can express this in the form,

$$L^{-1}[\bar{f}(s)] = f(t)$$

and is called the ***inverse Laplace transform.***

Note : *Since linearity property holds good for an integral we can obviously infer that* $L[c_1 f_1(t) \pm c_2 f_2(t)] = c_1 L[f_1(t)] \pm c_2 L[f_2(t)]$ *where* c_1 *and* c_2 *are constants.*

- **Laplace transform of discontinuous functions from the basic definition**

Working procedure for problems

Step-1 Here $f(t)$ is composed of different functions in different subintervals of $(0, \infty)$.

Suppose, $$f(t) = \begin{cases} f_1(t), & 0 < t < a \\ f_2(t), & a < t < b \\ f_3(t), & t > b \end{cases}$$

then we consider the basic definition of $L[f(t)]$. That is

$$L[f(t)] = \int_0^{\infty} e^{-st} f(t)\,dt$$

Step-2 By taking into account the intervals involved in the given $f(t)$ we express the integral in the equivalent form by using a property of definite integrals.

$$\therefore \quad L[f(t)] = \int_0^a e^{-st} f(t)\,dt + \int_a^b e^{-st} f(t)\,dt + \int_b^\infty e^{-st} f(t)\,dt$$

Step-3 We substitute the relevant $f(t)$ as in the data.

$$\text{Now, } L[f(t)] = \int_0^a e^{-st} f_1(t)\,dt + \int_a^b e^{-st} f_2(t)\,dt + \int_b^\infty e^{-st} f_3(t)\,dt$$

Step-4 We evaluate the integrals to obtain $L[f(t)]$ as a function of s.

Note : *If $f(t)$ is a polynomial in t then we have to write $\int e^{-st} f(t)\,dt$ as $\int f(t) e^{-st}\,dt$ and use Bernoulli's generalized rule of integration by parts as follows.*

$$\int u v\,dt = u \int v\,dt - u' \iint v\,dt\,dt + u'' \iiint v\,dt\,dt\,dt - \cdots$$

Illustrative Examples

Ex-1. *Find $L[f(t)]$ where* $f(t) = \begin{cases} t, & 0<t<4 \\ 5, & t>4 \end{cases}$

☞ $$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = \int_0^4 e^{-st} f(t)\,dt + \int_4^\infty e^{-st} f(t)\,dt$$

Using the relevant $f(t)$ in the integrals we have,

$$L[f(t)] = \int_0^4 e^{-st} \cdot t\,dt + \int_4^\infty e^{-st} \cdot 5\,dt$$

$$= \int_0^4 t e^{-st}\,dt + 5\int_4^\infty e^{-st}\,dt$$

Using Bernoulli's rule for the first term in RHS we have,

$$L[f(t)] = \left[t \cdot \frac{e^{-st}}{-s}\right]_{t=0}^4 - \left[1 \cdot \frac{e^{-st}}{s^2}\right]_{t=0}^4 + 5\left[\frac{e^{-st}}{-s}\right]_4^\infty$$

$$= \frac{-1}{s}(4e^{-4s} - 0) - \frac{1}{s^2}(e^{-4s} - 1) - \frac{5}{s}(0 - e^{-4s})$$

Thus, $$\boxed{L[f(t)] = \frac{1}{s}e^{-4s} + \frac{1}{s^2}(1 - e^{-4s})}$$

Ex-2. *Find* $L[f(t)]$ *if* $f(t)=\begin{cases}\sin 2t, & 0<t<\pi\\ 0, & t>\pi\end{cases}$

☞ $$L[f(t)]=\int_0^\infty e^{-st}f(t)\,dt=\int_0^\pi e^{-st}f(t)\,dt+\int_\pi^\infty e^{-st}f(t)\,dt$$

ie., $$=\int_0^\pi e^{-st}\sin 2t\,dt+\int_\pi^\infty e^{-st}\cdot 0\,dt$$

Using, $\int e^{at}\sin bt\,dt=\dfrac{e^{at}}{a^2+b^2}(a\sin bt-b\cos bt)$ we have,

$$L[f(t)]=\left[\frac{e^{-st}}{(-s)^2+4}(-s\sin 2t-2\cos 2t)\right]_{t=0}^{\pi}+0$$

$$=\frac{-1}{s^2+4}\left[e^{-st}(s\sin 2t+2\cos 2t)\right]_{t=0}^{\pi}$$

$$=\frac{-1}{s^2+4}\left[e^{-s\pi}\cdot 2-2\right]$$

$\because$ $\cos 2\pi=1=\cos 0,\ \sin 2\pi=0=\sin 0$

Thus, $$\boxed{L[f(t)]=\frac{2}{s^2+4}(1-e^{-\pi s})}$$

Ex-3. *Find* $L[f(t)]$ *if* $f(t)=\begin{cases}0, & 0<t<1\\ t, & 1<t<2\\ 0, & t>2\end{cases}$

☞ $$L[f(t)]=\int_0^\infty e^{-st}f(t)\,dt$$

$$=\int_0^1 e^{-st}f(t)\,dt+\int_1^2 e^{-st}f(t)\,dt+\int_2^\infty e^{-st}f(t)\,dt$$

On substituting for $f(t)$ we have only one integral in the RHS.

$\therefore$ $$L[f(t)]=\int_1^2 e^{-st}\cdot t\,dt=\int_1^2 t\,e^{-st}\,dt$$

Applying Bernoulli's rule,

1.2 Laplace transform of elementary functions

1. *$L(a)$ where 'a' is a constant.*

$$L(a)=\int_0^\infty e^{-st}\cdot a\,dt = a\left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{-a}{s}(0-1)=\frac{a}{s}\ (e^{-x}\to 0 \text{ as } x-$$

Thus, $L(a)=\frac{a}{s}$ where $s>0$. If $a=1$ then $L(1)=\frac{1}{s}$

2. $L(e^{at})$

$$L(e^{at})=\int_0^\infty e^{-st}\cdot e^{at}\,dt=\int_0^\infty e^{-(s-a)t}\,dt$$

$$L(e^{at})=\left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty = 0-\frac{1}{-(s-a)}=\frac{1}{s-a}$$

Thus, $L(e^{at})=\frac{1}{s-a}$, where $s>a$.

3. $L(\cosh at)$

$$L(\cosh at)=L\left(\frac{e^{at}+e^{-at}}{2}\right)=\frac{1}{2}\left\{L(e^{at})+L(e^{-at})\right\}$$

$$=\frac{1}{2}\left\{\frac{1}{s-a}+\frac{1}{s+a}\right\} \text{ since, } L(e^{at})=\frac{1}{s-a}$$

$$=\frac{1}{2}\left\{\frac{(s+a)+(s-a)}{(s-a)(s+a)}\right\}=\frac{1}{2}\cdot\frac{2s}{s^2-a^2}=\frac{s}{s^2-a^2}$$

Thus, $L(\cosh at)=\frac{s}{s^2-a^2}$ where $s>a$.

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4. $L(\sin h\, at)$

$$L(\sin h\, at) = L\left(\frac{e^{at} - e^{-at}}{2}\right) = \frac{1}{2}\{L(e^{at}) - L(e^{-at})\}$$

$$= \frac{1}{2}\left\{\frac{1}{s-a} - \frac{1}{s+a}\right\} = \frac{1}{2}\left\{\frac{(s+a)-(s-a)}{(s-a)(s+a)}\right\} = \frac{a}{s^2 - a^2}$$

Thus, $L(\sin h\, at) = \dfrac{a}{s^2 - a^2}$ where $s > a$.

5. $L(\cos at)$

$$L(\cos at) = \int_0^\infty e^{-st} \cos at\; dt$$

Using, $\int e^{at} \cos bt\; dt = \dfrac{e^{at}}{a^2 + b^2}(a\cos bt + b\sin bt)$ we have,

$$L(\cos at) = \left[\frac{e^{-st}}{(-s)^2 + a^2}(-s\cos at + a\sin at)\right]_{t=0}^\infty$$

$$= \frac{1}{s^2 + a^2}\left[e^{-st}(-s\cos at + a\sin at)\right]_{t=0}^\infty$$

$$= \frac{1}{s^2 + a^2}\left[0 - e^0(-s\cos 0 + a\sin 0)\right] = \frac{s}{s^2 + a^2}$$

Thus, $L(\cos at) = \dfrac{s}{s^2 + a^2}$ where $s > 0$.

6. $L(\sin at)$

$$L(\sin at) = \int_0^\infty e^{-st} \sin at\; dt$$

Using, $\int e^{at} \sin bt\; dt = \dfrac{e^{at}}{a^2 + b^2}(a\sin bt - b\cos bt)$ we have,

$$L(\sin at) = \left[\frac{e^{-st}}{(-s)^2 + a^2}(-s\sin at - a\cos at)\right]_{t=0}^\infty$$

$$L(\text{Sin}\, at) = \frac{-1}{s^2+a^2}\left[e^{-st}(s\sin at + a\cos at)\right]_0^{\infty} = \frac{-1}{s^2+a^2}(0-a) = \frac{a}{s^2+a^2}$$

Thus, $L(\sin at) = \dfrac{a}{s^2+a^2}$ where $s > 0$.

7. $L(t^n)$

Note : *Definition and properties of gamma function is a pre-requisite.*

$$L(t^n) = \int_0^{\infty} e^{-st}\, t^n\, dt$$

Put $st = x \quad \therefore dt = dx/s$ and x varies from 0 to ∞.

Now, $L(t^n) = \displaystyle\int_{x=0}^{\infty} e^{-x}\left(\frac{x}{s}\right)^n \frac{dx}{s} = \frac{1}{s^{n+1}}\int_0^{\infty} e^{-x} x^n\, dx = \frac{\Gamma(n+1)}{s^{n+1}}$

Thus, $L(t^n) = \dfrac{\Gamma(n+1)}{s^{n+1}}$ where n is a constant.

Remarks :

(i) *We know that $\Gamma(n+1)$ exists if n is a positive real number or n is a negative integer. Hence the expression for $L(t^n)$ is valid for n belonging to t categories of n.*

(ii) *We know that $\Gamma(n+1) = n!$ if n is a positive integer.*

$\therefore \qquad L(t^n) = \dfrac{n!}{s^{n+1}}$ *if n is a positive integer.*

We shall establish this result without the involvement of gamma functions.

8. $L(t^n)$ where n is a positive integer.

$$L(t^n) = \int_0^{\infty} e^{-st} t^n\, dt = \int_0^{\infty} t^n e^{-st}\, dt$$

Integrating by parts we have,

$$L(t^n) = \left[t^n \frac{e^{-st}}{-s}\right]_0^{\infty} - \int_0^{\infty} \frac{e^{-st}}{-s}(n t^{n-1})\, dt = 0 + \frac{n}{s}\int_0^{\infty} e^{-st}\, t^{n-1}\, dt$$

$$\therefore \qquad L(t^n) = \frac{n}{s} L(t^{n-1})$$

Similarly, $L(t^{n-1}) = \frac{n-1}{s} L(t^{n-2}),\ L(t^{n-2}) = \frac{n-2}{s} L(t^{n-3})$ etc.

Also, $L(t^2) = \frac{2}{s} L(t^1),\ L(t^1) = \frac{1}{s} L(t^0)$

Using all these results by back substitution we have,

$$L(t^n) = \frac{n}{s} \cdot \frac{n-1}{s} \cdot \frac{n-2}{s} \cdots \frac{2}{s} \cdot \frac{1}{s} L(t^0)$$

$$= \frac{n!}{s^n} L(1) = \frac{n!}{s^n} \cdot \frac{1}{s} = \frac{n!}{s^{n+1}}$$

Thus, $L(t^n) = \frac{n!}{s^{n+1}}$ where n is a positive integer.

Table of Laplace Transforms

	$f(t)$	$L[f(t)] = \bar{f}(s)$		$f(t)$	$L[f(t)] = \bar{f}(s)$
1.	a	$\frac{a}{s}$	**5.**	$\sinh at$	$\frac{a}{s^2 - a^2}$
2.	e^{at}	$\frac{1}{s-a}$	**6.**	$\sin at$	$\frac{a}{s^2 + a^2}$
3.	$\cosh at$	$\frac{s}{s^2 - a^2}$	**7.**	t^n	$\frac{\Gamma(n+1)}{s^{n+1}}$
4.	$\cos at$	$\frac{s}{s^2 + a^2}$	**8.**	t^n $n = 1, 2, 3, \cdots$	$\frac{n!}{s^{n+1}}$

Observe the following illustrations based on the table of Laplace transforms.

1. $L(4) = \frac{4}{s}$
2. $L(e^{2t}) = \frac{1}{s-2}$
3. $L(e^{-t}) = \frac{1}{s+1}$
4. $L(3e^{4t}) = \frac{3}{s-4}$
5. $L(2\cosh 2t) = \frac{2s}{s^2 - 4}$
6. $L(3\sinh 2t) = \frac{6}{s^2 - 4}$

7. $L(\cos t) = \dfrac{s}{s^2+1}$

8. $L(3\cos 4t) = \dfrac{3s}{s^2+16}$

9. $L(t^5) = \dfrac{5!}{s^6} = \dfrac{120}{s^6}$

10. $L(4t^3) = 4 \cdot \dfrac{3!}{s^4} = \dfrac{24}{s^4}$

11. $L(\sqrt{t}) = L(t^{1/2}) = \dfrac{\Gamma(3/2)}{s^{3/2}}$

12. $L(\sin 2t) = \dfrac{2}{s^2+4}$

ie., $L(\sqrt{t}) = \dfrac{1/2 \cdot \Gamma(1/2)}{s^{3/2}} = \dfrac{\sqrt{\pi}}{2s^{3/2}}$

WORKED PROBLEMS

Find the Laplace transform of the following functions

1. $\cosh^2 3t$
2. $e^{-2t}\sinh 4t$
3. $\sin 5t \cdot \cos 2t$
4. $\cos t \cdot \cos 2t \cdot \cos 3t$ **[June 2018]**
5. $\sin^2(2t+1)$
6. $(3t+4)^3 + 5^t$
7. $3\sqrt{t} + \dfrac{4}{\sqrt{t}}$
8. $\left\{\sqrt{t} - \dfrac{1}{\sqrt{t}}\right\}^3$
9. $t^{-5/2} + t^{5/2}$

Solutions

[1] Let, $f(t) = \cosh^2 3t = \left[\dfrac{e^{3t}+e^{-3t}}{2}\right]^2$

ie., $f(t) = \dfrac{1}{4}(e^{6t}+e^{-6t}+2)$

Thus, $$\boxed{L[f(t)] = \frac{1}{4}\left[\frac{1}{s-6}+\frac{1}{s+6}+\frac{2}{s}\right]}$$

[2] Let, $f(t) = e^{-2t}\cdot\sinh 4t = e^{-2t}\cdot\left(\dfrac{e^{4t}-e^{-4t}}{2}\right)$

ie., $f(t) = \dfrac{1}{2}(e^{2t}-e^{-6t})$

Thus, $$\boxed{L[f(t)] = \frac{1}{2}\left[\frac{1}{s-2}-\frac{1}{s+6}\right] = \frac{4}{(s-2)(s+6)}}$$

[3] Let, $f(t) = \sin 5t \cos 2t = \frac{1}{2}\{\sin(5t+2t)+\sin(5t-2t)\}$

ie., $f(t) = \frac{1}{2}\{\sin 7t + \sin 3t\}$

Thus, $$\boxed{L[f(t)] = \frac{1}{2}\left(\frac{7}{s^2+49}+\frac{3}{s^2+9}\right)}$$

[4] Let, $f(t) = \cos t \cos 2t \cos 3t$

Now, $\cos t \cdot \cos 2t = \frac{1}{2}\{\cos(t+2t)+\cos(t-2t)\} = \frac{1}{2}(\cos 3t + \cos t)$

$\therefore \quad \cos t \cos 2t \cos 3t = \frac{1}{2}\{\cos 3t \cos 3t + \cos 3t \cos t\}$.

ie., $f(t) = \frac{1}{2}\left\{\frac{1}{2}(\cos 6t + \cos 0) + \frac{1}{2}(\cos 4t + \cos 2t)\right\}$

$f(t) = \frac{1}{4}(\cos 6t + 1 + \cos 4t + \cos 2t)$

Thus, $$\boxed{L[f(t)] = \frac{1}{4}\left[\frac{s}{s^2+36}+\frac{1}{s}+\frac{s}{s^2+16}+\frac{s}{s^2+4}\right]}$$

[5] Let, $f(t) = \sin^2(2t+1) = \frac{1}{2}\{1-\cos 2(2t+1)\}$

ie., $f(t) = \frac{1}{2}\{1-\cos(4t+2)\}$

ie., $f(t) = \frac{1}{2}\{1-\cos 4t \cdot \cos 2 + \sin 4t \sin 2\}$

Thus, $$\boxed{L[f(t)] = \frac{1}{2}\left\{\frac{1}{s}-\frac{s\cos 2}{s^2+16}+\frac{4\sin 2}{s^2+16}\right\}}$$

[6] Let, $f(t) = (3t+4)^3 + 5^t$

ie., $f(t) = (27t^3 + 108t^2 + 144t + 64) + e^{\log 5 \cdot t}$

$$\therefore \quad L[f(t)] = 27 \cdot \frac{3!}{s^4} + 108 \cdot \frac{2!}{s^3} + 144 \cdot \frac{1!}{s^2} + \frac{64}{s} + \frac{1}{s - \log 5}$$

Thus, $$\boxed{L[f(t)] = \frac{162}{s^4} + \frac{216}{s^3} + \frac{144}{s^2} + \frac{64}{s} + \frac{1}{s - \log 5}}$$

[7] Let, $f(t) = 3\sqrt{t} + \dfrac{4}{\sqrt{t}} = 3t^{1/2} + 4t^{-1/2}$

$$\therefore \quad L[f(t)] = 3\frac{\Gamma(3/2)}{s^{3/2}} + 4\frac{\Gamma(1/2)}{s^{1/2}} = \frac{3\sqrt{\pi}}{2\,s^{3/2}} + \frac{4\sqrt{\pi}}{\sqrt{s}}$$

Here, $\Gamma(1/2) = \sqrt{\pi}$ and $\Gamma(3/2) = 1/2 \cdot \Gamma(1/2)$

Thus, $$\boxed{L[f(t)] = \sqrt{\frac{\pi}{s}}\left[\frac{3}{2s} + 4\right]}$$

[8] Let, $f(t) = \left(\sqrt{t} - \dfrac{1}{\sqrt{t}}\right)^3 = t^{3/2} - \dfrac{1}{t^{3/2}} - 3\left(\sqrt{t} - \dfrac{1}{\sqrt{t}}\right)$

ie., $f(t) = t^{3/2} - t^{-3/2} - 3t^{1/2} + 3t^{-1/2}$

$$L[f(t)] = \frac{\Gamma(5/2)}{s^{5/2}} - \frac{\Gamma(-1/2)}{s^{-1/2}} - \frac{3\Gamma(3/2)}{s^{3/2}} + \frac{3\Gamma(1/2)}{s^{1/2}} \quad \ldots (1)$$

$$\Gamma(1/2) = \sqrt{\pi},\ \Gamma(3/2) = \frac{\sqrt{\pi}}{2},\ \Gamma(5/2) = \frac{3\sqrt{\pi}}{4},$$

$$\Gamma(-1/2) = \frac{\Gamma(1/2)}{-1/2} = -2\sqrt{\pi}$$

Substituting these values in (1) we get,

$$L[f(t)] = \frac{3\sqrt{\pi}}{4s^{5/2}} + 2\sqrt{\pi}\,\sqrt{s} - \frac{3\sqrt{\pi}}{2s^{3/2}} + \frac{3\sqrt{\pi}}{s^{1/2}}$$

Thus, $$\boxed{L[f(t)] = \sqrt{\pi}\left[\frac{3}{4s^2\sqrt{s}} + 2\sqrt{s} - \frac{3}{2s\sqrt{s}} + \frac{3}{\sqrt{s}}\right]}$$

[9] Let, $f(t) = t^{-5/2} + t^{5/2}$

$$\therefore \quad L[f(t)] = \frac{\Gamma(-3/2)}{s^{-3/2}} + \frac{\Gamma(7/2)}{s^{7/2}} \quad \ldots (1)$$

But $$\Gamma(-3/2) = \frac{\Gamma(-1/2)}{-3/2} = \frac{-2}{3} \cdot \frac{\Gamma(1/2)}{-1/2} = \frac{4\sqrt{\pi}}{3}$$

$$\Gamma(7/2) = \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \sqrt{\pi} = \frac{15\sqrt{\pi}}{8}$$

Substituting these values in (1) we get,

$$L[f(t)] = \frac{4\sqrt{\pi}}{3s^{-3/2}} + \frac{15\sqrt{\pi}}{8s^{7/2}}$$

Thus, $$\boxed{L[f(t)] = \sqrt{\pi}\left[\frac{4}{3}s^{3/2} + \frac{15}{8}\frac{1}{s^{7/2}}\right]}$$

1.3 Properties of Laplace transforms

We discuss three properties. Proofs are also given for the benefit of readers.

Property - 1 If $L[f(t)] = \bar{f}(s)$ then $L[e^{at} f(t)] = \bar{f}(s-a)$

Proof : We have by the definition,

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = \bar{f}(s) \quad \ldots (1)$$

$$\therefore \quad L[e^{at} f(t)] = \int_0^\infty e^{-st}[e^{at} f(t)]\,dt = \int_0^\infty e^{-st+at} f(t)\,dt$$

ie., $$L[e^{at} f(t)] = \int_0^\infty e^{-(s-a)t} f(t)\,dt$$

Comparing this integral with the integral in (1) being denoted by $\bar{f}(s)$ we observe that $(s-a)$ has replaced s.

Thus $L[e^{at} f(t)] = \bar{f}(s-a)$

Note : $L[e^{-at} f(t)] = \bar{f}(s+a)$

Remark : *This property is known as the shifting property.*

Property-2 If $L[f(t)] = \bar{f}(s)$, then

$$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[\bar{f}(s)]$$ where n is a positive integer.

Proof : We establish the result by the principle of mathematical induction.

We have, $\bar{f}(s) = \int_0^\infty e^{-st} f(t)\, dt$

Differentiating w.r.t. 's' on both sides we have,

$$\frac{d}{ds}[\bar{f}(s)] = \int_0^\infty \frac{\partial}{\partial s}[e^{-st}] f(t)\, dt.$$

$$\therefore \quad \frac{d}{ds}\left[\bar{f}(s)\right] = \int_0^\infty e^{-st}(-t) f(t)\, dt$$

or $$(-1)\frac{d}{ds}[\bar{f}(s)] = \int_0^\infty e^{-st}[t f(t)]\, dt = L[t f(t)] \qquad \cdots (1)$$

This verifies the result for $n = 1$.

Let us assume the result to be true for $n = k$.

ie., $$(-1)^k \frac{d^k}{ds^k}[\bar{f}(s)] = L[t^k f(t)] \qquad \cdots (2)$$

or $$(-1)^k \frac{d^k}{ds^k}[\bar{f}(s)] = \int_0^\infty e^{-st}[t^k f(t)]\, dt$$

Differentiating w.r.t. 's' again we get,

$$(-1)^k \frac{d^{k+1}}{ds^{k+1}}[\bar{f}(s)] = \int_0^\infty \frac{\partial}{\partial s}(e^{-st})[t^k f(t)]\, dt$$

ie., $$(-1)^k \frac{d^{k+1}}{ds^{k+1}}[\bar{f}(s)] = \int_0^\infty e^{-st}(-t)[t^k f(t)]\, dt$$

Multiplying by (–1) we get,

$$(-1)^{k+1}\frac{d^{k+1}}{ds^{k+1}}[\bar{f}(s)] = \int_0^\infty e^{-st}[t^{k+1}f(t)]dt$$

ie., $$(-1)^{k+1}\frac{d^{k+1}}{ds^{k+1}}[\bar{f}(s)] = L[t^{k+1}f(t)] \quad \ldots (3)$$

Comparing (2) and (3) we conclude that the result is true for $n = k + 1$. Hence by the principle of mathematical induction the result is true for all positive integral values of n.

Thus $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[\bar{f}(s)] = (-1)^n \bar{f}^{(n)}(s)$

Remark : *This property is called the* ***derivative of the transform*** *property.*

Property - 3 If $L[f(t)] = \bar{f}(s)$ then $L\left[\frac{f(t)}{t}\right] = \int_s^\infty \bar{f}(s)\,ds$

Proof : We have, $\bar{f}(s) = \int_0^\infty e^{-st} f(t)\,dt$

$$\therefore \quad \int_s^\infty \bar{f}(s)\,ds = \int_s^\infty \left[\int_0^\infty e^{-st} f(t)\,dt\right]ds$$

ie., $$= \int_0^\infty \int_s^\infty e^{-st} f(t)\,ds\,dt, \text{ on changing the order of integration.}$$

$$= \int_0^\infty \left[\frac{e^{-st}}{-t}\right]_s^\infty f(t)\,dt = \int_0^\infty \left[0 - \frac{e^{-st}}{-t}\right] f(t)\,dt$$

ie., $$\int_s^\infty \bar{f}(s)\,ds = \int_0^\infty e^{-st}\left[\frac{f(t)}{t}\right]dt = L\left[\frac{f(t)}{t}\right]$$

Thus $L\left[\frac{f(t)}{t}\right] = \int_s^\infty \bar{f}(s)\,ds$

Properties at a glance

If $L[f(t)] = \bar{f}(s)$ then we have,

1. $L[e^{at} f(t)] = \bar{f}(s-a)$

2. $L[t^n f(t)] = (-1)^n \dfrac{d^n}{ds^n}[\bar{f}(s)]$

In particular, $L[t f(t)] = -\dfrac{d}{ds}[\bar{f}(s)]$, $L[t^2 f(t)] = \dfrac{d^2}{ds^2}[\bar{f}(s)]$

3. $L\left[\dfrac{f(t)}{t}\right] = \int_s^\infty \bar{f}(s)\,ds$

WORKED PROBLEMS

Find the Laplace transform of the following functions

10. $e^{-2t}(2\cos 5t - \sin 5t)$ 11. $e^{-t}\cos^2 3t$ 12. $e^{-4t}\, t^{-5/2}$

13. $(1+3t e^{2t})^2$ 14. $\sinh at \sin at$ 15. $\cosh t \sin^3 2t$

Solutions

[10] Let, $f(t) = 2\cos 5t - \sin 5t$

$$\therefore \quad L[f(t)] = 2\cdot\frac{s}{s^2+5^2} - \frac{5}{s^2+5^2} = \frac{2s-5}{s^2+25}$$

Now, $$L[e^{-2t} f(t)] = \left\{\frac{2s-5}{s^2+25}\right\}_{s\to s+2} = \frac{2(s+2)-5}{(s+2)^2+25}$$

Thus, $$\boxed{L[e^{-2t}(2\cos 5t - \sin 5t)] = \frac{2s-1}{s^2+4s+29}}$$

[11] Let, $f(t) = \cos^2 3t = \dfrac{1+\cos 6t}{2}$

$$\therefore \quad L[f(t)] = \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+36}\right]$$

Now, $$L[e^{-t}\cos^2 3t] = \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+36}\right]_{s\to(s+1)}$$

Thus, $$\boxed{L[e^{-t}\cos^2 3t] = \frac{1}{2}\left[\frac{1}{s+1} + \frac{(s+1)}{(s+1)^2+36}\right]}$$

[12] Let, $f(t) = t^{-5/2}$

$$\therefore \quad L[f(t)] = \frac{4\sqrt{\pi}}{3} s^{3/2} \quad \text{(Refer Problem - [9])}$$

Now, $L[e^{-4t} t^{-5/2}] = \frac{4\sqrt{\pi}}{3}\left[s^{3/2}\right]_{s \to s+4}$

Thus, $\boxed{L[e^{-4t} t^{-5/2}] = \frac{4\sqrt{\pi}}{3}(s+4)^{3/2}}$

[13] Let, $f(t) = (1 + 3t\,e^{2t})^2 = 1 + 6t\,e^{2t} + 9t^2 e^{4t}$

$$\therefore \quad L[f(t)] = L(1) + 6L(e^{2t} \cdot t) + 9L(e^{4t} \cdot t^2)$$

$$= \frac{1}{s} + 6\{L(t)\}_{s \to (s-2)} + 9\{L(t^2)\}_{s \to (s-4)}$$

But, $L(t) = \frac{1}{s^2}$ and $L(t^2) = \frac{2}{s^3}$

Thus, $\boxed{L[(1+3t\,e^{2t})^2] = \frac{1}{s} + \frac{6}{(s-2)^2} + \frac{18}{(s-4)^3}}$

[14] Let , $f(t) = \sin h\,at \sin at = \frac{e^{at} - e^{-at}}{2} \cdot \sin at$

ie., $f(t) = \frac{1}{2}(e^{at} \sin at - e^{-at} \sin at)$

$$\therefore \quad L[f(t)] = \frac{1}{2}\left\{L(\sin at)_{s \to s-a} - L(\sin at)_{s \to s+a}\right\}$$

But, $L(\sin at) = \frac{a}{s^2 + a^2}$

Hence, $L(\sin h\,at \sin at) = \frac{1}{2}\left\{\frac{a}{(s-a)^2 + a^2} - \frac{a}{(s+a)^2 + a^2}\right\}$

$$L(\sinh at \sin at) = \frac{a}{2}\left\{\frac{1}{s^2+2a^2-2as} - \frac{1}{s^2+2a^2+2as}\right\}$$

$$= \frac{a}{2}\left\{\frac{4as}{(s^2+2a^2)^2-4a^2s^2}\right\}$$

Thus, $$\boxed{L(\sinh at \sin at) = \frac{2a^2s}{s^4+4a^4}}$$

[15] Let, $f(t) = \sin^3 2t = \frac{1}{4}(3\sin 2t - \sin 6t)$

$$L(\sin^3 2t) = \frac{1}{4}\left(3\cdot\frac{2}{s^2+4} - \frac{6}{s^2+36}\right) = \frac{48}{(s^2+4)(s^2+36)}$$

Now, $L(\cosh t \sin^3 2t) = L\left[\frac{e^t+e^{-t}}{2}\cdot\sin^3 2t\right]$

$$= \frac{1}{2}\{L(e^t\sin^3 2t) + L(e^{-t}\sin^3 2t)\}$$

$$= \frac{1}{2}\left\{L(\sin^3 2t)_{s\to s-1} + L(\sin^3 2t)_{s\to s+1}\right\}$$

$$= \frac{1}{2}\left\{\frac{48}{[(s-1)^2+4][(s-1)^2+36]} + \frac{48}{[(s+1)^2+4][(s+1)^2+36]}\right.$$

$$\boxed{L(\cosh t \sin^3 2t) = 24\left\{\frac{1}{(s^2-2s+5)(s^2-2s+37)} + \frac{1}{(s^2+2s+5)(s^2+2s+37}\right.}$$

Find the Laplace transform of the following functions

16. $t\cos at$ **[Dec 2018]** **17.** $t^2 \sin at$ **18.** $t^3 \sin t$

19. $t^3 \cosh t$ **20.** $t^5 e^{4t} \cosh 3t$ **21.** $t\,e^{-2t} \sin 4t$

☞ **Solutions**

[16] Let, $f(t) = \cos at \quad \therefore \; L[f(t)] = \dfrac{s}{s^2 + a^2}$

Now, $L[t\,f(t)] = \dfrac{-d}{ds}\left[\dfrac{s}{s^2 + a^2}\right] = -\left\{\dfrac{(s^2 + a^2)1 - s \cdot 2s}{(s^2 + a^2)^2}\right\}$

Thus, $$\boxed{L(t\cos at) = \frac{s^2 - a^2}{(s^2 + a^2)^2}}$$

[17] Let, $f(t) = \sin at \quad \therefore \; L[f(t)] = \dfrac{a}{s^2 + a^2}$

Now, $L[t^2 f(t)] = \dfrac{d^2}{ds^2}\left[\dfrac{a}{s^2 + a^2}\right] = \dfrac{d}{ds}\left\{\dfrac{d}{ds}\left(\dfrac{a}{s^2 + a^2}\right)\right\}$

$$L[t^2 \sin at] = \frac{d}{ds}\left\{\frac{-2as}{(s^2 + a^2)^2}\right\}$$

$$L[t^2 \sin at] = \frac{(s^2 + a^2)^2(-2a) + 2as \cdot 2(s^2 + a^2) \cdot 2s}{(s^2 + a^2)^4}$$

$$= \frac{2a(s^2 + a^2)\{-(s^2 + a^2) + 4s^2\}}{(s^2 + a^2)^4}$$

Thus, $$\boxed{L[t^2 \sin at] = \frac{2a(3s^2 - a^2)}{(s^2 + a^2)^3}}$$

[18] Let, $f(t) = \sin t \quad \therefore \; L[f(t)] = \dfrac{1}{s^2 + 1}$

Now, $L[t^3 \sin t] = (-1)^3 \dfrac{d^3}{ds^3}\left(\dfrac{1}{s^2 + 1}\right) = \dfrac{-d}{ds} \cdot \dfrac{d}{ds}\left\{\dfrac{d}{ds}\left(\dfrac{1}{s^2 + 1}\right)\right\}$

$$L[t^3 \sin t] = \frac{-d}{ds}\,\frac{d}{ds}\left\{\frac{-2s}{(s^2 + 1)^2}\right\} = \frac{d}{ds}\left(\frac{(s^2 + 1)^2 \cdot 2 - 2s \cdot 2(s^2 + 1) \cdot 2s}{(s^2 + 1)^4}\right)$$

$$L[t^3 \sin t] = \frac{d}{ds}\left\{\frac{2(s^2 + 1)[s^2 + 1 - 4s^2]}{(s^2 + 1)^4}\right\} = 2\frac{d}{ds}\left\{\frac{1 - 3s^2}{(s^2 + 1)^3}\right\}$$

$$L[t^3 \sin t] = 2\left\{\frac{(s^2+1)^3(-6s)-(1-3s^2)\cdot 3(s^2+1)^2\cdot 2s}{(s^2+1)^6}\right\}$$

$$= -12s(s^2+1)^2\left\{\frac{(s^2+1)+(1-3s^2)}{(s^2+1)^6}\right\}$$

Thus, $$\boxed{L[t^3 \sin t] = \frac{24s(s^2-1)}{(s^2+1)^4}}$$

[19] Let, $f(t) = t^3 \cosh t$

Note : *Here we should not prefer to work the problem similar to the previous* problem as we have cosh t which can be converted to the form $(e^t + e^{-t})/2$ so that *it will be highly convenient to apply the shifting property.*

$$f(t) = t^3\left(\frac{e^t+e^{-t}}{2}\right) = \frac{1}{2}\{e^t t^3 + e^{-t} t^3\}$$

$$L[f(t)] = \frac{1}{2}\{L(t^3)_{s\to s-1} + L(t^3)_{s\to s+1}\}$$

But, $$L(t^3) = \frac{3!}{s^4} = \frac{6}{s^4}$$

Thus, $$\boxed{L[t^3 \cosh t] = \frac{1}{2}\left\{\frac{6}{(s-1)^4}+\frac{6}{(s+1)^4}\right\} = 3\left\{\frac{1}{(s-1)^4}+\frac{1}{(s+1)^4}\right\}}$$

[20] Let, $f(t) = t^5 e^{4t} \cosh 3t = t^5 e^{4t}\cdot\frac{1}{2}(e^{3t}+e^{-3t})$

ie., $$f(t) = \frac{1}{2}(e^{7t} t^5 + e^t t^5)$$

$$L[f(t)] = \frac{1}{2}\{L(t^5)_{s\to s-7} + L(t^5)_{s\to s-1}\}$$

But, $$L(t^5) = \frac{5!}{s^6} = \frac{120}{s^6}$$

Thus, $$\boxed{L(t^5 e^{4t} \cosh 3t) = \frac{1}{2}\left\{\frac{120}{(s-7)^6}+\frac{120}{(s-1)^6}\right\} = 60\left\{\frac{1}{(s-7)^6}+\frac{1}{(s-1)^6}\right\}}$$

[21] Let, $f(t) = t\,e^{-2t}\sin 4t$

$$L(\sin 4t) = \frac{4}{s^2+16} \quad \therefore \; L[e^{-2t}\sin 4t] = \frac{4}{(s+2)^2+16} = \frac{4}{s^2+4s+20}$$

Hence, $L[t\,e^{-2t}\sin 4t] = \dfrac{-d}{ds}\left\{\dfrac{4}{s^2+4s+20}\right\} = \dfrac{4(2s+4)}{(s^2+4s+20)^2}$

Thus, $$\boxed{L[t\,e^{-2t}\sin 4t] = \frac{8(s+2)}{(s^2+4s+20)^2}}$$

[22] *Show that* $\displaystyle\int_0^\infty t^3 e^{-t}\sin t\,dt = 0$

☞ We have, $\displaystyle\int_0^\infty e^{-st}\cdot t^3\sin t\,dt = L(t^3\sin t)$

Referring to Problem-[18] for the RHS we have,

$$\int_0^\infty e^{-st}\cdot t^3\sin t\,dt = \frac{24s(s^2-1)}{(s^2+1)^4}$$

Thus by putting $s = 1$ we get, $\boxed{\displaystyle\int_0^\infty e^{-t}\cdot t^3\sin t\,dt = 0}$

[23] *Show that* $\displaystyle\int_0^\infty t\,e^{-2t}\sin 4t\,dt = \frac{1}{25}$

☞ We have, $\displaystyle\int_0^\infty e^{-st}\,t\sin 4t\,dt = L(t\sin 4t)$ . . . (1)

Now, $L(t\sin 4t) = \dfrac{-d}{ds}L(\sin 4t) = \dfrac{-d}{ds}\left(\dfrac{4}{s^2+16}\right) = \dfrac{8s}{(s^2+16)^2}$

Hence (1) becomes $\displaystyle\int_0^\infty e^{-st}\,t\sin 4t\,dt = \frac{8s}{(s^2+16)^2}$

Thus by putting $s = 2$ we get, $\boxed{\displaystyle\int_0^\infty e^{-2t}\,t\sin 4t\,dt = \frac{16}{400} = \frac{1}{25}}$

[24] *Find the value of* $\int_0^\infty t\, e^{-3t} \cos 2t\, dt$ *using Laplace transforms.*

☞ We have, $\int_0^\infty e^{-st}\, t \cos 2t\, dt = L(t \cos 2t)$ ... (1)

Proceeding as in Problem-[16] we can obtain $L(t\cos 2t) = \dfrac{s^2-4}{(s^2+4)^2}$

Using this result in the RHS of (1) we have,

$$\int_0^\infty e^{-st}\, t \cos 2t\, dt = \frac{s^2-4}{(s^2+4)^2}$$

Thus by putting $s = 3$ we get, $\boxed{\int_0^\infty e^{-3t}\, t\cos 2t\, dt = \dfrac{5}{169}}$

Find the Laplace transform of the following functions.

25. $\dfrac{1-e^{-at}}{t}$ **26.** $\dfrac{\cos at - \cos bt}{t}$ **27.** $\dfrac{\sinh t}{t}$

28. $\dfrac{\sin^2 t}{t}$ **29.** $\dfrac{2\sin t \sin 5t}{t}$ **30.** $\dfrac{\sin at}{t}$ **[Dec. 2**

☞ **Solutions**

[25] Let, $f(t) = 1 - e^{-at}$ $\therefore\ \bar{f}(s) = L[f(t)] = \dfrac{1}{s} - \dfrac{1}{s+a}$

We have, $L\left[\dfrac{f(t)}{t}\right] = \int_s^\infty \bar{f}(s)\, ds$

Hence, $L\left\{\dfrac{1-e^{-at}}{t}\right\} = \int_s^\infty \left(\dfrac{1}{s} - \dfrac{1}{s+a}\right) ds$

$$= [\log s - \log(s+a)]_s^\infty = \left[\log\left(\frac{s}{s+a}\right)\right]_s^\infty$$

ie.,
$$= \left[\lim_{s\to\infty} \log\left(\frac{s}{s+a}\right)\right] - \log\left(\frac{s}{s+a}\right)$$

$$L\left\{\frac{1-e^{-at}}{t}\right\} = \lim_{s\to\infty} \log\left(\frac{s}{s(1+a/s)}\right) - \log($$

Thus, $$\boxed{L\left[\frac{1-e^{-at}}{t}\right] = \log\left(\frac{s+a}{s}\right)}$$

[26] Let , $f(t) = \cos at - \cos bt$ $\therefore$ $\bar{f}(s) = \frac{}{s^2}$

Hence, $$L\left[\frac{f(t)}{t}\right] = \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right) ds$$

$$L\left[\frac{\cos at - \cos bt}{t}\right] = \frac{1}{2}[\log(s^2+a^2) - \log(s^2$$

$$= \frac{1}{2}\left[\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]_s^\infty = \frac{1}{2}\left\{\lim_{s\to\infty} \text{lo}\right.$$

$$= \frac{1}{2}\left\{\log 1 - \log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right\} = \frac{1}{2}\log$$

Thus, $$\boxed{L\left[\frac{\cos at - \cos bt}{t}\right] = \log}$$

[27] Let, $f(t) = \sin ht$ $\therefore$ $\bar{f}(s) = \frac{1}{s^2-1}$

Hence, $$L\left[\frac{\sin ht}{t}\right] = \int_s^\infty \frac{1}{s^2-1}\, ds = \frac{1}{2}\left[\log\left(\frac{s}{s}\right.\right.$$

ie., $$= \lim_{s\to\infty} \frac{1}{2}\log\left[\frac{s(1-1/s)}{s(1+1/s)}\right]$$

$$= \frac{1}{2}\left\{\log 1 - \log\left(\frac{s-1}{s+1}\right)\right.$$

Thus, $$\boxed{L\left[\frac{\sin ht}{t}\right] = \log\sqrt{\frac{s+1}{s-1}}}$$

[28] Let, $f(t) = \sin^2 t = \frac{1}{2}(1-\cos 2t)$

$$\therefore \quad \bar{f}(s) = \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)$$

Hence, $L\left[\frac{f(t)}{t}\right] = \frac{1}{2}\int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+4}\right) ds$

ie., $$L\left[\frac{\sin^2 t}{t}\right] = \frac{1}{2}\left[\log s - \frac{1}{2}\log(s^2+4)\right]_s^\infty = \frac{1}{2}\log\left[\frac{s}{\sqrt{s^2+4}}\right]_s^\infty$$

ie., $$= \lim_{s\to\infty} \frac{1}{2}\log\left[\frac{s}{s\sqrt{1+(4/s^2)}}\right] - \frac{1}{2}\log\left[\frac{s}{\sqrt{s^2+4}}\right]$$

$$= \frac{1}{2}\log 1 - \frac{1}{2}\log\left[\frac{s}{\sqrt{s^2+4}}\right]$$

Thus, $$\boxed{L\left[\frac{\sin^2 t}{t}\right] = \frac{1}{2}\log\left(\frac{\sqrt{s^2+4}}{s}\right)}$$

[29] Let, $f(t) = 2\sin t \sin 5t = 2\cdot\frac{1}{2}[\cos(-4t) - \cos(6t)]$

ie., $f(t) = \cos 4t - \cos 6t \quad \therefore \quad \bar{f}(s) = \frac{s}{s^2+16} - \frac{s}{s^2+36}$

Hence, $L\left[\frac{f(t)}{t}\right] = \int_s^\infty \left(\frac{s}{s^2+16} - \frac{s}{s^2+36}\right) ds = \frac{1}{2}[\log(s^2+16) - \log(s^2 +$

$$L\left[\frac{2\sin t \sin 5t}{t}\right] = \frac{1}{2}\left[\log\left(\frac{s^2+16}{s^2+36}\right)\right]_s^\infty$$

$$= \lim_{s\to\infty}\frac{1}{2}\cdot\log\left[\frac{s^2(1+16/s^2)}{s^2(1+36/s^2)}\right] - \frac{1}{2}\log\left(\frac{s^2+16}{s^2+36}\right.$$

$$L\left[\frac{2\sin t\sin 5t}{t}\right] = \frac{1}{2}\left\{\log 1 - \log\left(\frac{s^2+16}{s^2+36}\right)\right\} = \frac{1}{2}\log\left(\frac{s^2+36}{s^2+16}\right)$$

Thus, $$\boxed{L\left[\frac{2\sin t\sin 5t}{t}\right] = \log\sqrt{\frac{s^2+36}{s^2+16}}}$$

[30] Let, $f(t) = \sin at \quad \therefore \; \bar{f}(s) = \dfrac{a}{s^2+a^2}$

Hence, $$L\left[\frac{f(t)}{t}\right] = \int_s^\infty \frac{a}{s^2+a^2}\,ds$$

ie., $$L\left[\frac{\sin at}{t}\right] = a\cdot\frac{1}{a}\left[\tan^{-1}(s/a)\right]_s^\infty = \tan^{-1}(\infty) - \tan^{-1}(s/a)$$

Thus, $$\boxed{L\left[\frac{\sin at}{t}\right] = \pi/2 - \tan^{-1}(s/a) = \cot^{-1}(s/a)}$$

Evaluate the following integrals using Laplace transforms.

31. $\displaystyle\int_0^\infty \frac{\cos 6t - \cos 4t}{t}\,dt$ **32.** $\displaystyle\int_0^\infty \frac{e^{-at} - e^{-bt}}{t}\,dt$ **33.** $\displaystyle\int_0^\infty \frac{e^{-t}\sin t}{t}\,dt$

☞ **Solutions**

[31] We know that, $\displaystyle\int_0^\infty e^{-st}\left[\frac{\cos 6t - \cos 4t}{t}\right]dt = L\left[\frac{\cos 6t - \cos 4t}{t}\right]$

Proceeding as in Problem-[26] we can obtain

$$L\left[\frac{\cos 6t - \cos 4t}{t}\right] = \log\sqrt{(s^2+16)/(s^2+36)}$$

ie., $$\int_0^\infty e^{-st}\left[\frac{\cos 6t - \cos 4t}{t}\right]dt = \log\sqrt{(s^2+16)/(s^2+36)}$$

Thus by putting $s = 0$ we get,

$$\boxed{\int_0^\infty \frac{\cos 6t - \cos 4t}{t}\,dt = \log\sqrt{16/36} = \log(2/3)}$$

[32] We shall first find $L\left[\frac{e^{-at}-e^{-bt}}{t}\right]$

Now, $L\left[\frac{e^{-at}-e^{-bt}}{t}\right] = \int_s^\infty [L(e^{-at})-L(e^{-bt})]\,dt$

$$= \int_s^\infty \left(\frac{1}{s+a}-\frac{1}{s+b}\right)ds = \left[\log\left(\frac{s+a}{s+b}\right)\right]_s^\infty$$

ie., $= \lim_{s\to\infty} \log\left[\frac{s(1+a/s)}{s(1+b/s)}\right] - \log\left(\frac{s+a}{s+b}\right) = \log 1 - \log\left(\frac{s+a}{s+b}\right.$

$\therefore \quad L\left[\frac{e^{-at}-e^{-bt}}{t}\right] = \log\left(\frac{s+b}{s+a}\right)$

ie., $\int_0^\infty e^{-st}\left[\frac{e^{-at}-e^{-bt}}{t}\right]dt = \log\left(\frac{s+b}{s+a}\right)$

Thus by putting $s = 0$ we get, $\boxed{\int_0^\infty \frac{e^{-at}-e^{-bt}}{t}\,dt = \log(b/a)}$

[33] We shall first find $L\left[\frac{e^{-t}\sin t}{t}\right]$

Now, $L\left[\frac{e^{-t}\sin t}{t}\right] = \int_s^\infty L(e^{-t}\sin t)\,ds = \int_s^\infty \{L(\sin t)_{s\to s+1}\}\,ds$

ie., $= \int_s^\infty \frac{1}{(s+1)^2+1}\,ds = \left[\tan^{-1}(s+1)\right]_s^\infty = \tan^{-1}(\infty) - \tan^{-1}($

$\therefore \quad L\left[\frac{e^{-t}\sin t}{t}\right] = \frac{\pi}{2} - \tan^{-1}(s+1) = \cot^{-1}(s+1)$

ie., $\int_0^\infty e^{-st}\cdot\frac{e^{-t}\sin t}{t}\,dt = \cot^{-1}(s+1)$

Thus by putting $s = 0$ we get, $\boxed{\int_0^\infty \frac{e^{-t}\sin t}{t}\,dt = \cot^{-1}(1) = \frac{\pi}{4}}$

[34] *Find* $L\left[\dfrac{\sin^2 t}{t^2}\right]$

☞ We know that, $L\left[\dfrac{f(t)}{t}\right] = \int_s^\infty \bar{f}(s)\,ds$

Further we have, $L\left[\dfrac{f(t)}{t^2}\right] = \int_s^\infty \left[\int_s^\infty \bar{f}(s)\,ds\right] ds$

$$L\left[\frac{\sin^2 t}{t}\right] = \frac{1}{2}\log\left[\frac{\sqrt{s^2+4}}{s}\right] \qquad \textit{(Refer Problem-[28])}$$

$$\therefore \quad L\left[\frac{\sin^2 t}{t^2}\right] = \int_s^\infty \frac{1}{2}\log\left(\frac{\sqrt{s^2+4}}{s}\right) ds = \int_s^\infty \frac{1}{2}\log\sqrt{\frac{s^2+4}{s^2}}\, ds$$

ie.,
$$= \frac{1}{4}\int_s^\infty \log\left(1+\frac{4}{s^2}\right) ds = \frac{1}{4}\int_s^\infty \log\left(1+\frac{4}{s^2}\right)\cdot 1\, ds$$

Integrating by parts we have,

$$L\left[\frac{\sin^2 t}{t^2}\right] = \frac{1}{4}\left\{\left[\log\left(1+\frac{4}{s^2}\right)\cdot s\right]_s^\infty - \int_s^\infty s\cdot\frac{1}{1+(4/s^2)}\cdot\left(\frac{-8}{s^3}\right) ds\right\}$$

$$= \frac{1}{4}\left\{\left[0-\log\left(1+\frac{4}{s^2}\right)s\right] + \int_s^\infty \frac{8}{s^2+4}\, ds\right\}$$

$$= \frac{-s}{4}\log\left(\frac{s^2+4}{s^2}\right) + \left[\tan^{-1}(s/2)\right]_s^\infty$$

$$= \frac{s}{4}\log\left(\frac{s^2}{s^2+4}\right) + [\pi/2 - \tan^{-1}(s/2)]$$

Thus,
$$\boxed{L\left[\frac{\sin^2 t}{t^2}\right] = \frac{s}{4}\log\left(\frac{s^2}{s^2+4}\right) + \cot^{-1}\left(\frac{s}{2}\right)}$$

MISCELLANEOUS PROBLEMS

[35] *Find the Laplace transform of* $2^t + \dfrac{\cos 2t - \cos 3t}{t} + t\sin t$ [June

☞ The given function be denoted by $f(t)$ and let,

$$f(t) = F(t) + G(t) + H(t)$$

where $F(t) = 2^t$, $G(t) = \dfrac{\cos 2t - \cos 3t}{t}$, $H(t) = t\sin t$

$$\therefore \quad L[f(t)] = L[F(t)] + L[G(t)] + L[H(t)]$$

Now, $L[F(t)] = L[2^t] = L[e^{\log 2 \cdot t}] = \dfrac{1}{s - \log 2}$

Further, $G(t) = \dfrac{\cos 2t - \cos 3t}{t}$

$$\therefore \quad L[G(t)] = \int_s^\infty L(\cos 2t - \cos 3t)\,ds$$

$$= \int_s^\infty \left[\frac{s}{s^2+4} - \frac{s}{s^2+9}\right] ds$$

$$L[G(t)] = \left[\frac{1}{2}\log(s^2+4) - \frac{1}{2}\log(s^2+9)\right]_s^\infty$$

$$= \left[\log\sqrt{s^2+4/s^2+9}\right]_s^\infty$$

$$= \left[\log\sqrt{1+(4/s^2)/1+(9/s^2)}\right]_{s=\infty} - \log\sqrt{s^2+4/s^2+9}$$

$$= \log 1 - \log\sqrt{s^2+4/s^2+9}$$

ie., $L[G(t)] = \log\sqrt{s^2+9/s^2+4}$

Lastly, $H(t) = t\sin t$

$$\therefore \quad L[H(t)] = -\frac{d}{ds}L(\sin t) = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right)$$

Hence $L[H(t)] = \dfrac{2s}{(s^2+1)^2}$

Thus from (1) the required $L[f(t)]$ is given by,

$$\boxed{\frac{1}{s-\log 2} + \log\sqrt{s^2+9/s^2+4} + \frac{2s}{(s^2+1)^2}}$$

[36] *Find the Laplace transforms of* $t^2 e^{-3t} \sin 2t$

☞ We shall first find $L(t^2 \sin 2t)$

We have, $L(t^2 \sin 2t) = (-1)^2 \dfrac{d^2}{ds^2} L(\sin 2t)$

ie., $L(t^2 \sin 2t) = \dfrac{d}{ds} \cdot \dfrac{d}{ds}\left[\dfrac{2}{s^2+4}\right]$

$$= \frac{d}{ds}\left[\frac{-4s}{(s^2+4)^2}\right]$$

$$= \frac{(s^2+4)^2(-4) + 4s \cdot 2(s^2+4)2s}{(s^2+4)^4}$$

$$= \frac{4(s^2+4)[-(s^2+4)+4s^2]}{(s^2+4)^3}$$

ie., $L(t^2 \sin 2t) = \dfrac{4(3s^2-4)}{(s^2+4)^3}$

Thus,

$$\boxed{L(e^{-3t} t^2 \sin 2t) = \frac{4[3(s+3)^2-4]}{[(s+3)^2+4]^3}}$$

[37] *Find the Laplace transforms of* $t e^{2t} - \dfrac{2 \sin 3t}{t}$

☞ Let, $f(t) = t e^{2t} - \dfrac{2\sin 3t}{t} = f_1(t) - f_2(t)$ (*say*)

$\therefore$ $L[f(t)] = L[f_1(t)] - L[f_2(t)]$

Now, $L[f_1(t)] = L(te^{2t}) = \{L(t)\}_{s\to s-2} = \left\{\frac{1}{s^2}\right\}_{s\to s-2}$

$\therefore \quad L[f_1(t)] = \frac{1}{(s-2)^2}$

Next, $L[f_2(t)] = 2L\left[\frac{\sin 3t}{t}\right]$

$$= 2\int_s^\infty L(\sin 3t)\,ds = 2\int_s^\infty \frac{3}{s^2+3^2}\,ds$$

$$L[f_2(t)] = 2[\tan^{-1}(s/3)]_s^\infty = 2\{\pi/2 - \tan^{-1}(s/3)\} = 2\cot^{-1}(s/3)$$

$\therefore \quad L[f_2(t)] = 2\cot^{-1}(s/3)$

Thus the required, $\boxed{L[f(t)] = \frac{1}{(s-2)^2} - 2\cot^{-1}(s/3)}$

[38] *Using Laplace transforms evaluate* $\int_0^\infty e^{-t}\,t\sin^2 3t\,dt$

☞ We shall first find $L(t\sin^2 3t)$

$$\sin^2 3t = \frac{1}{2}(1-\cos 6t)$$

$$L(\sin^2 3t) = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+36}\right]$$

$$L(t\sin^2 3t) = \frac{1}{2}\cdot -\frac{d}{ds}\left[\frac{1}{s} - \frac{s}{s^2+36}\right]$$

$$= \frac{-1}{2}\left[\frac{-1}{s^2} - \frac{(s^2+36)-2s^2}{(s^2+36)^2}\right]$$

$$\therefore \quad L(t\sin^2 3t) = \frac{1}{2}\left[\frac{1}{s^2} + \frac{(36-s^2)}{(s^2+36)^2}\right]$$

Using the basic definition :- -

$$\int_0^\infty e^{-st}\, t \sin^2 3t\, dt = \frac{1}{2}\left[\frac{1}{s^2} + \frac{36 - s^2}{(s^2+36)^2}\right]$$

Thus by putting $s = 1$ we get,

$$\boxed{\int_0^\infty e^{-t}\, t \sin^2 3t\, dt = \frac{1}{2}\left[1 + \frac{35}{(37)^2}\right] = \frac{702}{1369}}$$

[39] *Find* (a) $L[\sin\sqrt{t}\,]$ (b) $L\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right]$

☞ We have the expansion of $\sin x$ given by

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\therefore \quad \sin(\sqrt{t}\,) = \sqrt{t} - \frac{(\sqrt{t}\,)^3}{3!} + \frac{(\sqrt{t}\,)^5}{5!} - \cdots$$

$$L[\sin(\sqrt{t}\,)] = L(t^{1/2}) - \frac{L(t^{3/2})}{6} + \frac{L(t^{5/2})}{120} - \cdots \qquad \ldots (1)$$

$$L(t^{1/2}) = \frac{\Gamma(3/2)}{s^{3/2}} = \frac{1/2\cdot\Gamma(1/2)}{s^{3/2}} = \frac{\sqrt{\pi}}{2s^{3/2}}$$

$$L(t^{3/2}) = \frac{\Gamma(5/2)}{s^{5/2}} = \frac{3/2\cdot 1/2\cdot\sqrt{\pi}}{s^{5/2}} = \frac{3\sqrt{\pi}}{4s^{5/2}}$$

$$L(t^{5/2}) = \frac{\Gamma(7/2)}{s^{7/2}} = \frac{5/2\cdot 3/2\cdot 1/2\cdot\sqrt{\pi}}{s^{7/2}} = \frac{15\sqrt{\pi}}{8s^{7/2}}$$

Substituting these values in (1) we get,

$$L[\sin\sqrt{t}\,] = \frac{\sqrt{\pi}}{2s^{3/2}} - \frac{\sqrt{\pi}}{8s^{5/2}} + \frac{\sqrt{\pi}}{64s^{7/2}} - \cdots$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}\left\{1 - \frac{1}{4s} + \frac{1}{32s^2} - \cdots\right\}$$

$$L[\sin(\sqrt{t})] = \frac{\sqrt{\pi}}{2s^{3/2}}\left\{1 - \frac{1/4s}{1!} + \frac{(1/4s)^2}{2!} - \cdots\right\} = \frac{\sqrt{\pi}}{2s^{3/2}}e^{-1/4s}$$

Thus, $$\boxed{L[\sin(\sqrt{t})] = \frac{1}{2s}\sqrt{\frac{\pi}{s}}\,e^{-1/4s}}$$

(b) On similar lines by using, $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$

we can obtain, $$\boxed{L\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right] = \sqrt{\pi/s}\;e^{-1/4s}}$$

[40] *Prove the following :*

(i) $$L\left[\frac{1}{2a}(\sin at + at\cos at)\right] = \frac{s^2}{(s^2+a^2)^2}$$

(ii) $$L\left[\frac{1}{2a^3}(\sin at - at\cos at)\right] = \frac{1}{(s^2+a^2)^2}$$

☞ We have to first find $L(at\cos at) = aL(t\cos at)$

Referring to Problem-[16], $L(at\cos at) = \frac{a(s^2-a^2)}{(s^2+a^2)^2}$

Also we know that, $L(\sin at) = \frac{a}{s^2+a^2}$

Hence, $$L(\sin at + at\cos at) = \frac{a}{s^2+a^2} + \frac{a(s^2-a^2)}{(s^2+a^2)^2}$$

$$= a\left[\frac{s^2+a^2+s^2-a^2}{(s^2+a^2)^2}\right] = \frac{2as^2}{(s^2+a^2)^2}$$

Thus, $$\boxed{L\left[\frac{1}{2a}(\sin at + at\cos at)\right] = \frac{s^2}{(s^2+a^2)^2}}$$

Also, $$L(\sin at - at\cos at) = a\left[\frac{(s^2+a^2)-(s^2-a^2)}{(s^2+a^2)^2}\right] = \frac{2a^3}{(s^2+a^2)^2}$$

Thus, $$\boxed{L\left[\frac{1}{2a^3}(\sin at - at\cos at)\right] = \frac{1}{(s^2+a^2)^2}}$$

1.4 Laplace transform of periodic function

Definition : A function $f(t)$ is said to be a periodic function of period $T > 0$ if $f(t+nT) = f(t)$ where $n = 1, 2, 3 \ldots$

Example : $\sin t$, $\cos t$ are periodic functions of period 2π because

$$\sin(t+2n\pi) = \sin t,\ \cos(t+2n\pi) = \cos t.$$

Theorem : *If $f(t)$ is a periodic function of period T, then*

$$L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt$$

Proof : We have by the definition,

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = \int_0^\infty e^{-su} f(u)\,du$$

$$= \int_{u=0}^{T} e^{-su} f(u)\,du + \int_{u=T}^{2T} e^{-su} f(u)\,du + \cdots + \int_{u=nT}^{(n+1)T} e^{-su} f(u)\,du + \cdots$$

$$L[f(t)] = \sum_{n=0}^{\infty}\int_{u=nT}^{(n+1)T} e^{-su} f(u)\,du \quad \ldots (1)$$

Now put, $u = t+nT \quad \therefore \quad du = dt$

If $\quad u = nT \quad$ then $\quad t+nT = nT \Rightarrow t = 0$

$\quad u = (n+1)T \quad$ then $\quad t+nT = nT+T \Rightarrow t = T$

Further, $f(u) = f(t+nT) = f(t)$ by the periodic property.

Using these results in the RHS of (1) we obtain,

$$L[f(t)] = \sum_{n=0}^{\infty}\int_{t=0}^{T} e^{-s(t+nT)} f(t)\,dt$$

$$L[f(t)] = \sum_{n=0}^{\infty} e^{-snT}\int_{t=0}^{T} e^{-st} f(t)\,dt \quad \ldots (2)$$

But, $\displaystyle\sum_{n=0}^{\infty} e^{-snT} = \sum_{n=0}^{\infty} (e^{-sT})^n = 1 + (e^{-sT}) + (e^{-sT})^2 + \cdots$

Putting $r = e^{-sT}$ the series involved is a geometric series of the form $1 + r + r^2 + \cdots$ whose sum to infinity is known to be $1/1-r$ where $r < 1$.

Hence, $\sum_{n=0}^{\infty} e^{-snT} = \dfrac{1}{1-e^{-sT}}$ ...

Now using (3) in the RHS of (2) we have,

$$L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt$$

WORKED PROBLEMS

[41] *If $f(t) = t^2, 0<t<2$ and $f(t+2) = f(t)$ for $t > 2$, find $L[f(t)]$.*

$f(t)$ is a periodic function of period 2. $\therefore\ T = 2$

We have, $L[f(t)] = \dfrac{1}{1-e^{-sT}}\displaystyle\int_0^T e^{-st} f(t)\,dt$

Now, $L[f(t)] = \dfrac{1}{1-e^{-2s}}\displaystyle\int_0^2 e^{-st}\cdot t^2\,dt = \dfrac{1}{1-e^{-2s}}\int_0^2 t^2 e^{-st}\,dt$

Applying Bernoulli's rule of integration by parts,

$$L[f(t)] = \frac{1}{1-e^{-2s}}\left\{\left[t^2\cdot\frac{e^{-st}}{-s}\right]_0^2 - \left[2t\cdot\frac{e^{-st}}{s^2}\right]_0^2 + \left[2\cdot\frac{e^{-st}}{-s^3}\right]_0^2\right\}$$

$$= \frac{1}{1-e^{-2s}}\left\{\frac{-1}{s}(4e^{-2s}-0) - \frac{2}{s^2}(2e^{-2s}-0) - \frac{2}{s^3}(e^{-2s}-1)\right\}$$

$$L[f(t)] = \frac{2}{s^3(1-e^{-2s})}\left\{-2s^2 e^{-2s} - 2s\,e^{-2s} - e^{-2s} + 1\right\}$$

Thus, $$\boxed{L[f(t)] = \frac{2}{s^3(1-e^{-2s})}\left\{1-(2s^2+2s+1)e^{-2s}\right\}}$$

[42] *Find the Laplace transform of the periodic function defined by $f(t) = kt/T, 0<t<T$; $f(t+T) = f(t)$*

We have, $L[f(t)] = \dfrac{1}{1-e^{-sT}}\displaystyle\int_0^T e^{-st} f(t)\,dt$

Now, $L[f(t)] = \dfrac{k}{T(1-e^{-sT})}\displaystyle\int_0^T t\,e^{-st}\,dt$

Applying Bernoulli's rule we have,

$$L[f(t)] = \frac{k}{T(1-e^{-sT})}\left[t\cdot\left(\frac{e^{-st}}{-s}\right) - 1\cdot\left(\frac{e^{-st}}{s^2}\right)\right]_0^T$$

$$= \frac{k}{T(1-e^{-sT})}\left[\left(-\frac{1}{s}Te^{-sT}-0\right)-\frac{1}{s^2}(e^{-sT}--1)\right]$$

Thus, $$\boxed{L[f(t)] = \frac{-ke^{-sT}}{s(1-e^{-sT})}+\frac{k}{s^2T}}$$

[43] *Find the Laplace transform of the full wave rectifier* $f(t) = E\sin wt,\ 0<t<\pi/w$ *having period* π/w. **[Dec 2017, 18]**

☞ We have, $L[f(t)] = \dfrac{1}{1-e^{-sT}}\displaystyle\int_0^T e^{-st}f(t)\,dt$

Now, $L[f(t)] = \dfrac{1}{1-e^{-s(\pi/w)}}\displaystyle\int_0^{\pi/w} e^{-st}\,E\sin wt\,dt$

$$= \frac{E}{1-e^{-\pi s/w}}\left[\frac{e^{-st}}{s^2+w^2}(-s\sin wt - w\cos wt)\right]_0^{\pi/w}$$

$$= \frac{E}{(1-e^{-\pi s/w})(s^2+w^2)}[e^{-\pi s/w}\cdot w-(-w)]$$

$\therefore\quad L[f(t)] = \dfrac{Ew(1+e^{-\pi s/w})}{(s^2+w^2)(1-e^{-\pi s/w})}$. We can simplify further.

Multiplying both the numerator and the denominator by $e^{\pi s/2w}$ in RHS, the expression assumes the form,

$$L[f(t)] = \frac{Ew}{s^2+w^2}\cdot\frac{(e^{\pi s/2w}+e^{-\pi s/2w})}{(e^{\pi s/2w}-e^{-\pi s/2w})} = \frac{Ew}{s^2+w^2}\cdot\frac{2\cosh(\pi s/2w)}{2\sinh(\pi s/2w)}$$

Thus, $$\boxed{L[f(t)] = \frac{Ew}{s^2+w^2}\coth(\pi s/2w)}$$

[44] *Given* $f(t) = \begin{cases} E, & 0<t<a/2 \\ -E, & a/2<t<a \end{cases}$ *where* $f(t+a) = f(t)$,

show that $L[f(t)] = E/s \cdot \tanh(as/4)$.

☞ The given function is periodic with period $T = a$.

We have, $L[f(t)] = \dfrac{1}{1-e^{-sT}} \displaystyle\int_0^T e^{-st} f(t)\, dt$

$$= \frac{1}{1-e^{-sa}} \int_0^a e^{-st} f(t)\, dt$$

$$L[f(t)] = \frac{1}{1-e^{-as}} \left\{ \int_0^{a/2} e^{-st} E\, dt + \int_{a/2}^{a} e^{-st}(-E)\, dt \right\}$$

$$= \frac{E}{1-e^{-as}} \left\{ \left[\frac{e^{-st}}{-s} \right]_0^{a/2} + \left[\frac{e^{-st}}{s} \right]_{a/2}^{a} \right\}$$

$$= \frac{E}{s(1-e^{-as})} \left\{ -\left[e^{-st}\right]_0^{a/2} + \left[e^{-st}\right]_{a/2}^{a} \right\}$$

$$= \frac{E}{s(1-e^{-as})} \left\{ -e^{-as/2} + 1 + e^{-as} - e^{-as/2} \right\}$$

$$= \frac{E}{s(1-e^{-as})} (1 - 2e^{-as/2} + e^{-as}) = \frac{E(1-e^{-as/2})^2}{s(1-e^{-as})}$$

$$L[f(t)] = \frac{E(1-e^{-as/2})^2}{s(1-e^{-as/2})(1+e^{-as/2})} = \frac{E(1-e^{-as/2})}{s(1+e^{-as/2})}$$

Multiplying both the numerator and denominator by $e^{as/4}$ we get,

$$L[f(t)] = \frac{E(e^{as/4} - e^{-as/4})}{s(e^{as/4} + e^{-as/4})} = \frac{E \cdot 2\sinh(as/4)}{s \cdot 2\cosh(as/4)}$$

Thus, $$\boxed{L[f(t)] = \frac{E}{s} \tanh\left(\frac{as}{4}\right)}$$

Note : Similar Problem

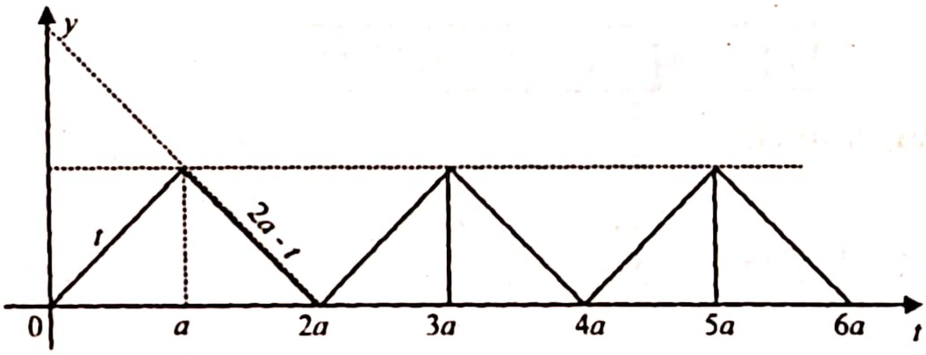
If $f(t) = \begin{cases} E, & 0<t<a \\ -E, & a<t<2a \end{cases}$ *show that* $L[f(t)] = \dfrac{E}{s}\tanh\left(\dfrac{as}{2}\right)$ **[June 2017]**

[45] *If* $f(t) = \begin{cases} t, & 0 \le t \le a \\ 2a - t, & a \le t \le 2a, \end{cases}$ $f(t+2a) = f(t)$

(i) *Sketch the graph of f (t) as a periodic function*

(ii) *Show that* $L[f(t)] = \dfrac{1}{s^2}\tanh(as/2)$ **[June & Dec 2016]**

☞ (i) Let $f(t) = y$ and $y = t$ is a straight line passing through the origin making an angle 45^0 with the t-axis. $y = 2a - t$ or $y + t = 2a$ or $t/2a + y/2a = 1$ is a straight line passing through the points ($2a$, 0) and (0, $2a$).
The graph of $y = f(t)$ is as follows.



The periodic function $f(t)$ is called the *triangular wave function.*

(ii) We have, $T = 2a$ and $L[f(t)] = \dfrac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt$

$$L[f(t)] = \frac{1}{1-e^{-2as}}\int_0^{2a} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2as}}\left\{\int_0^a t\,e^{-st}\,dt + \int_a^{2a}(2a-t)\,e^{-st}\,dt\right\}$$

Applying Bernoulli's rule to each of the integrals we have,

$$L[f(t)] = \frac{1}{1-e^{-2as}}\left\{\left[t\cdot\frac{e^{-st}}{-s} - (1)\frac{e^{-st}}{s^2}\right]_0^a + \left[(2a-t)\frac{e^{-st}}{-s} - (-1)\frac{e^{-st}}{s^2}\right]_a^{2a}\right\}$$

$$L[f(t)] = \frac{1}{1-e^{-2as}}\left\{\frac{-1}{s}(ae^{-as}-0) - \frac{1}{s^2}(e^{-as}-1) - \frac{1}{s}(0-ae^{-as}) + \frac{1}{s^2}(e^{-2as}-e^{-as})\right\}$$

$$L[f(t)] = \frac{1}{s^2(1-e^{-2as})}(-e^{-as}+1+e^{-2as}-e^{-as})$$

$$= \frac{1}{s^2(1-e^{-2as})}(1-2e^{-as}+e^{-2as}) = \frac{(1-e^{-as})^2}{s^2(1-e^{-as})(1+e^{-as})}$$

$$L[f(t)] = \frac{(1-e^{-as})}{s^2(1+e^{-as})} = \frac{e^{as/2}-e^{-as/2}}{s^2(e^{as/2}+e^{-as/2})}$$

where we have multiplied both the numerator and denominator by $e^{as/2}$.

$$\therefore \quad L[f(t)] = \frac{2\sinh(as/2)}{s^2 \cdot 2\cosh(as/2)} = \frac{1}{s^2}\tanh\left(\frac{as}{2}\right)$$

Thus, $\boxed{L[f(t)] = 1/s^2 \cdot \tanh(as/2)}$

Note: Similar Problems

(i) *Find* $L[f(t)]$ *if* $f(t) = \begin{cases} t, & 0 \le t \le 1 \\ 2-t, & 1 \le t \le 2 \end{cases}$ **[June 2015]**

(ii) *Find* $L[f(t)]$ *if* $f(t) = \begin{cases} t, & 0 < t < \pi \\ \pi - t, & \pi < t < 2\pi \end{cases}$ **[June 2018]**

[46] *A periodic function of period $2\pi/\omega$ is defined by*

$$f(t) = \begin{cases} E\sin\omega t, & 0 \le t \le \pi/\omega \\ 0, & \pi/\omega \le t \le 2\pi/\omega \end{cases}$$ *where E and ω are constants.*

Show that $L[f(t)] = \dfrac{E\omega}{(s^2+\omega^2)(1-e^{-\pi s/\omega})}$

☞ We have for a periodic function $f(t)$,

$$L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt.$$ Here, $T = 2\pi/\omega$.

$$L[f(t)] = \frac{1}{1-e^{-2\pi s/\omega}} \int_0^{2\pi/\omega} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2\pi s/\omega}} \left\{ \int_0^{\pi/\omega} e^{-st} E \sin \omega t\,dt + \int_{\pi/\omega}^{2\pi/\omega} e^{-st} \cdot 0\,dt \right\}$$

$$= \frac{E}{1-e^{-2\pi s/\omega}} \left[\frac{e^{-st}}{(-s)^2+\omega^2} (-s \sin \omega t - \omega \cos \omega t) \right]_0^{\pi/\omega}$$

$$= \frac{-E}{(s^2+w^2)(1-e^{-2\pi s/\omega})} \left\{ e^{-s\pi/\omega} (s \sin \pi + \omega \cos \pi) - e^0 (s \sin 0 + \omega \cos 0) \right\}$$

$$= \frac{-E}{(s^2+w^2)(1-e^{-2\pi s/\omega})} \left\{ -\omega e^{-s\pi/\omega} - \omega \right\} = \frac{E\omega(1+e^{-\pi s/\omega})}{(s^2+w^2)(1-e^{-2\pi s/\omega})}$$

$$L[f(t)] = \frac{E\omega(1+e^{-\pi s/\omega})}{(s^2+w^2)(1-e^{-\pi s/\omega})(1+e^{-\pi s/\omega})}$$

Thus, $$\boxed{L[f(t)] = \frac{E\omega}{(s^2+w^2)(1-e^{-\pi s/\omega})}}$$

1.5 Unit step function (Heaviside function)

Definition : The ***unit step function*** $u(t-a)$ or ***Heaviside function*** $H(t-a)$ is defined as follows.

$$u(t-a) = \begin{cases} 0, & t \le a \\ 1, & t > a \end{cases} \quad \text{where } a \text{ is a constant.}$$

Properties associated with the unit step function

(i) $L[u(t-a)] = \dfrac{e^{-as}}{s}$

(ii) $L[f(t-a)u(t-a)] = e^{-as}\,\overline{f}(s)$ where $L[f(t)] = \overline{f}(s)$

Proof : (i) $L[u(t-a)] = \int_0^{\infty} e^{-st} u(t-a)\,dt$

$$L[u(t-a)] = \int_0^a e^{-st} u(t-a)\,dt + \int_a^\infty e^{-st} u(t-a)\,dt$$

$$= \int_0^a e^{-st} \cdot 0\,dt + \int_a^\infty e^{-st} \cdot 1\,dt$$

$$= 0 + \left[\frac{e^{-st}}{-s}\right]_a^\infty = 0 - \frac{e^{-as}}{-s} = \frac{e^{-as}}{s}$$

Thus, $L[u(t-a)] = \dfrac{e^{-as}}{s}$

(ii) $L[f(t-a)u(t-a)] = \int_0^\infty e^{-st} f(t-a)u(t-a)\,dt$

$$= \int_0^a e^{-st} f(t-a) \cdot 0\,dt + \int_a^\infty e^{-st} f(t-a) \cdot 1\,dt$$

ie., $L[f(t-a)u(t-a)] = \int_a^\infty e^{-st} f(t-a)\,dt$

Put $t - a = v \therefore dt = dv$ If $t = a, v = 0$; $t = \infty, v = \infty$

Hence, $L[f(t-a)u(t-a)] = \int_{v=0}^\infty e^{-s(a+v)} f(v)\,dv$

$$= e^{-as} \int_{v=0}^\infty e^{-sv} f(v)\,dv = e^{-as}\,\overline{f}(s)$$

Thus, $L[f(t-a)u(t-a)] = e^{-as}\,\overline{f}(s)$

Remarks

1. *The result (i) follows as a particular case of (ii) when* $f(t-a) = 1$ *as we hav* $f(t)$ *also equal to 1 and hence* $L[f(t)] = 1/s$
Hence (ii) becomes $L[u(t-a)] = e^{-as}/s$

2. *It is possible to express a discontinuous function* $f(t)$ *interms of unit ste function and inturn we can find its Laplace transform by using properties (i) an (ii). The following two results (iii) and (iv) which can be easily verified will be highl useful.*

(iii) *If* $f(t) = \begin{cases} f_1(t), & t \le a \\ f_2(t), & t > a \end{cases}$

Then $f(t) = f_1(t) + [f_2(t) - f_1(t)]u(t-a)$

Proof : RHS $= f_1(t) + [f_2(t) - f_1(t)]\begin{cases} 0, & t \le a \\ 1, & t > a \end{cases}$

$$= f_1(t) + \begin{cases} 0, & t \le a \\ f_2(t) - f_1(t), & t > a \end{cases} = \begin{cases} f_1(t) + 0; & t \le a \\ f_1(t) + f_2(t) - f_1(t), & t > a \end{cases}$$

ie., RHS $= \begin{cases} f_1(t), & t \le a \\ f_2(t), & t > a \end{cases} = f(t)$ = LHS

(iv) *If* $f(t) = \begin{cases} f_1(t), & t \le a \\ f_2(t), & a < t \le b \\ f_3(t), & t > b \end{cases}$

Then $f(t) = f_1(t) + [f_2(t) - f_1(t)]u(t-a) + [f_3(t) - f_2(t)]u(t-b)$

Proof : RHS $= f_1(t) + [f_2(t) - f_1(t)]\begin{cases} 0, & t \le a \\ 1, & t > a \end{cases} + [f_3(t) - f_2(t)]\begin{cases} 0, & t \le b \\ 1, & t > b \end{cases}$

$$\text{RHS} = f_1(t) + \begin{cases} 0, & t \le a \\ f_2(t) - f_1(t), & t > a \end{cases} + \begin{cases} 0, & t \le b \\ f_3(t) - f_2(t), & t > b \end{cases}$$

$$= \begin{cases} f_1(t), & t \le a \\ f_2(t), & t > a \end{cases} + \begin{cases} 0, & t \le b \\ f_3(t) - f_2(t), & t > b \end{cases}$$

$$\text{RHS} = \begin{cases} f_1(t), & t \le a \\ f_2(t), & t > a, t \le b \\ f_3(t); & t > b \end{cases} = \begin{cases} f_1(t), & t \le a \\ f_2(t), & a < t \le b \\ f_3(t), & t > b \end{cases} = f(t) = \text{LHS}$$

Working procedure for problems

Type-1 : *To find $L[F(t)u(t-a)]$ where $F(t)$ is a polynomial in t.*

Step-1 Let $F(t) = f(t-a)$ which implies that $F(t+a) = f(t)$

That is to replace t by $t+a$ to obtain $f(t)$ and find $L[f(t)] = \bar{f}$

Step-2 $L[F(t)u(t-a)] = L[f(t-a)u(t-a)] = e^{-as}\bar{f}(s)$ by property

Type-2 : *Given $f(t)$ as a discontinuous function, to find $L[f(t)]$ by expres $f(t)$ in terms of unit step function*

Step-1 We express $f(t)$ in terms of unit step function by directly making of the result (iii) or (iv) as the case may be.

Step-2 We find $L[f(t)]$ as in Type-1

WORKED PROBLEMS

Find the Laplace transform of the following functions

47. $[e^{t-1} + \sin(t-1)]u(t-1)$
48. $\sin t\, u(t-\pi)$
49. $(3t^2 + 4t + 5)u(t-3)$
50. $(1 - e^{2t})u(t+1)$
51. $(t^3 + t^2 + t + 1)u(t+1)$
52. $(t^2 - 6t + 9)e^{-(t-3)}u(t-$

☞ **Solutions**

[47] Let, $f(t-1) = e^{t-1} + \sin(t-1)$

$\Rightarrow \quad f(t) = e^t + \sin t \quad \therefore \; \bar{f}(s) = \dfrac{1}{s-1} + \dfrac{1}{s^2+1}$

We have, $L[f(t-1)u(t-1)] = e^{-s}\bar{f}(s)$; $(a = 1)$

Thus, $$\boxed{L[e^{t-1} + \sin(t-1)]u(t-1) = e^{-s}\left[\frac{1}{s-1} + \frac{1}{s^2+1}\right]}$$

[48] Let, $f(t-\pi) = \sin t$

$\Rightarrow \quad f(t) = \sin(t+\pi) = -\sin t \quad \therefore \; \bar{f}(s) = \dfrac{-1}{s^2+1}$

We have, $L[f(t-\pi)u(t-\pi)] = e^{-\pi s}\bar{f}(s)$; $(a = \pi)$

Thus, $$\boxed{L[\sin t\, u(t-\pi)] = -e^{-\pi s}/s^2+1}$$

[49] Let, $f(t-3) = 3t^2 + 4t + 5$

$\Rightarrow \quad f(t) = 3(t+3)^2 + 4(t+3) + 5 = 3t^2 + 22t + 44$

$\therefore \quad \bar{f}(s) = 3 \cdot \dfrac{2!}{s^3} + 22 \cdot \dfrac{1!}{s^2} + \dfrac{44}{s} = \dfrac{6}{s^3} + \dfrac{22}{s^2} + \dfrac{44}{s}$

We have, $L[f(t-3)u(t-3)] = e^{-3s}\bar{f}(s); \ (a = 3)$

Thus, $\boxed{L[(3t^2 + 4t + 5)u(t-3)] = e^{-3s}[6/s^3 + 22/s^2 + 44/s]}$

[50] Let, $f(t+1) = 1 - e^{2t}$

$\Rightarrow \quad f(t) = 1 - e^{2(t-1)}$ by replacing t by $(t-1)$.

ie., $\quad f(t) = 1 - e^{-2} \cdot e^{2t} \quad \therefore \bar{f}(s) = \dfrac{1}{s} - e^{-2} \cdot \dfrac{1}{s-2}$

We have, $L[f(t+1)u(t+1)] = e^{s}\bar{f}(s); \ (a = -1)$

Thus, $\boxed{L[(1-e^{2t})u(t+1)] = e^{s}\left[\dfrac{1}{s} - \dfrac{1}{e^2(s-2)}\right] = \dfrac{e^s}{s} - \dfrac{e^{(s-2)}}{s-2}}$

[51] Let, $f(t+1) = t^3 + t^2 + t + 1$

$\Rightarrow \quad f(t) = (t-1)^3 + (t-1)^2 + (t-1) + 1$

$\qquad\qquad = (t^3 - 3t^2 + 3t - 1) + (t^2 - 2t + 1) + (t-1) + 1$

ie., $\quad f(t) = t^3 - 2t^2 + 2t \quad \therefore \ \bar{f}(s) = \dfrac{3!}{s^4} - 2 \cdot \dfrac{2!}{s^3} + 2 \cdot \dfrac{1!}{s^2}$

We have, $L[f(t+1)u(t+1)] = e^{s}\bar{f}(s); \ (a = -1)$

Thus, $\boxed{L[(t^3 + t^2 + t + 1)u(t+1)] = e^{s}\left(\dfrac{6}{s^4} - \dfrac{4}{s^3} + \dfrac{2}{s^2}\right)}$

[52] Let, $f(t-3) = (t^2 - 6t + 9)e^{-(t-3)} = (t-3)^2 e^{-(t-3)}$

$\Rightarrow \quad f(t) = t^2 e^{-t} \quad \therefore \quad \bar{f}(s) = \dfrac{2!}{(s+1)^3} = \dfrac{2}{(s+1)^3}$

We have, $L[f(t-3)u(t-3)] = e^{-3s}\bar{f}(s); \ (a = 3)$

Thus, $\boxed{L[(t^2 - 6t + 9)e^{-(t-3)}u(t-3)] = e^{-3s} \cdot \dfrac{2}{(s+1)^3} = \dfrac{2e^{-3s}}{(s+1)^3}}$

Express the following functions interms of Heaviside unit step function and hence find their Laplace transform.

53. $f(t) = \begin{cases} t, & 0<t<4 \\ 5, & t>4 \end{cases}$

54. $f(t) = \begin{cases} \sin 2t, & 0<t<\pi \\ 0, & t>\pi \end{cases}$

55. $f(t) = \begin{cases} \cos t, & 0<t<\pi \\ \sin t, & t>\pi \end{cases}$

[Dec.2018]

56. $f(t) = \begin{cases} \sin t, & 0<t\le\pi/2 \\ \cos t, & t>\pi/2 \end{cases}$

[June 2017]

57. $f(t) = \begin{cases} 1, & 0<t\le 1 \\ t, & 1<t\le 2 \\ t^2, & t>2 \end{cases}$

[June 2016, 18]

58. $f(t) = \begin{cases} \cos t, & 0<t\le\pi \\ 1, & \pi<t\le 2\pi \\ \sin t, & t>2\pi \end{cases}$

[June 2018]

59. $f(t) = \begin{cases} e^{2t}, & 0<t<1 \\ 2, & t>1 \end{cases}$

60. $f(t) = \begin{cases} \cos t, & 0 < t < \pi \\ \cos 2t, & \pi<t<2\pi \\ \cos 3t, & t>2\pi \end{cases}$

☞ **Solutions**

[53] $f(t) = \begin{cases} t, & 0<t<4 \\ 5, & t>4 \end{cases}$

$f(t) = t+(5-t)u(t-4)$ by a property.

$L[f(t)] = L(t)+L[(5-t)u(t-4)]$ ...(1)

We have, $L(t) = 1/s^2$

Let, $F(t-4) = (5-t)$ $\therefore$ $F(t) = 5-(t+4) = 1-t$

Hence, $\bar{F}(s) = L[F(t)] = \dfrac{1}{s}-\dfrac{1}{s^2}$

But, $L[F(t-4)u(t-4)] = e^{-4s}\bar{F}(s)$

ie., $L[(5-t)u(t-4)] = e^{-4s}\left(1/s-1/s^2\right)$

We shall use these results in (1).

Thus, $\boxed{L[f(t)] = \dfrac{1}{s^2}+e^{-4s}\left(\dfrac{1}{s}-\dfrac{1}{s^2}\right) = \dfrac{1}{s}e^{-4s}+\dfrac{1}{s^2}(1-e^{-4s})}$

[54] $f(t) = \begin{cases} \sin 2t, & 0<t<\pi \\ 0, & t>\pi \end{cases}$

$f(t) = \sin 2t + (0 - \sin 2t)\,u(t-\pi)$ by a property.

$$L[f(t)] = L(\sin 2t) - L[\sin 2t\, u(t-\pi)] \qquad \ldots (1)$$

Let us find, $L[\sin 2t\, u(t-\pi)]$

Taking $F(t-\pi) = \sin 2t, F(t) = \sin 2(t+\pi) = \sin(2\pi + 2t)$

ie., $F(t) = \sin 2t \quad \therefore\ \bar{F}(s) = 2/s^2 + 4$

But $L[F(t-\pi)u(t-\pi)] = e^{-\pi s}\bar{F}(s) = 2e^{-\pi s}/s^2 + 4$

We shall use these results in (1).

Thus, $\boxed{L[f(t)] = \dfrac{2}{s^2+4} - \dfrac{2e^{-\pi s}}{s^2+4} = \dfrac{2(1-e^{-\pi s})}{s^2+4}}$

[55] $f(t) = \begin{cases} \cos t, & 0<t<\pi \\ \sin t, & t>\pi \end{cases}$

$f(t) = \cos t + (\sin t - \cos t)\,u(t-\pi)$ by a property.

$$L[f(t)] = L(\cos t) + L[(\sin t - \cos t)\,u(t-\pi)] \qquad \ldots (1)$$

Now, let $F(t-\pi) = \sin t - \cos t$

$\Rightarrow \quad F(t) = \sin(t+\pi) - \cos(t+\pi) = -\sin t + \cos t$

$$\therefore \quad \bar{F}(s) = \frac{-1}{s^2+1} + \frac{s}{s^2+1} = \frac{s-1}{s^2+1}$$

But $L[F(t-\pi)u(t-\pi)] = e^{-\pi s}\bar{F}(s) = \dfrac{e^{-\pi s}(s-1)}{s^2+1}$

We shall use these results in (1).

Thus, $\boxed{L[f(t)] = \dfrac{s}{s^2+1} + \dfrac{e^{-\pi s}(s-1)}{s^2+1} = \dfrac{s + e^{-\pi s}(s-1)}{s^2+1}}$

[56] $f(t) = \begin{cases} \sin t, & 0 < t \le \pi/2 \\ \cos t, & t > \pi/2 \end{cases}$

$$f(t) = \sin t + (\cos t - \sin t)\, u(t - \pi/2) \text{ by a property.}$$

$$L[f(t)] = L(\sin t) + L[(\cos t - \sin t)\, u(t - \pi/2)] \quad \ldots($$

Now, let $F(t - \pi/2) = \cos t - \sin t$

$\Rightarrow \quad F(t) = \cos(t + \pi/2) - \sin(t + \pi/2) = -\sin t - \cos t$

$\therefore \quad \bar{F}(s) = \dfrac{-1}{s^2+1} - \dfrac{s}{s^2+1} = \dfrac{-(s+1)}{(s^2+1)}$

But, $L[F(t - \pi/2)\, u(t - \pi/2)] = e^{-\pi s/2}\, \bar{F}(s) = \dfrac{-e^{-\pi s/2}(s+1)}{s^2+1}$

We shall use these results in (1).

Thus, $\boxed{L[f(t)] = \dfrac{1}{s^2+1} - \dfrac{e^{-\pi s/2}(s+1)}{s^2+1} = \dfrac{1 - e^{-\pi s/2}(s+1)}{s^2+1}}$

[57] $f(t) = \begin{cases} 1, & 0 < t \le 1 \\ t, & 1 < t \le 2 \\ t^2, & t > 2 \end{cases}$

$$f(t) = 1 + (t-1)\, u(t-1) + (t^2 - t)\, u(t-2) \text{ by a property.}$$

$$L[f(t)] = L(1) + L[(t-1)\, u(t-1)] + L[(t^2 - t)\, u(t-2)] \quad \ldots(1)$$

Let, $F(t-1) = (t-1)\,;\, G(t-2) = t^2 - t$

$\Rightarrow \quad F(t) = t \quad ;\, G(t) = (t+2)^2 - (t+2) = t^2 + 3t + 2$

$\therefore \quad \bar{F}(s) = \dfrac{1}{s^2} \quad ;\, \bar{G}(s) = \dfrac{2}{s^3} + \dfrac{3}{s^2} + \dfrac{2}{s}$

$L[F(t-1)\, u(t-1)] = e^{-s}\, \bar{F}(s)$ and $L[G(t-2)\, u(t-2)] = e^{-2s}\, \bar{G}(s)$

ie., $L[(t-1)\, u(t-1)] = \dfrac{e^{-s}}{s^2}$ and

$$L[(t^2 - t)\, u(t-2)] = e^{-2s}\left(\frac{2}{s^3} + \frac{3}{s^2} + \frac{2}{s}\right)$$

We shall use these results in (1).

Thus, $$\boxed{L[f(t)] = \frac{1}{s} + \frac{e^{-s}}{s^2} + e^{-2s}\left(\frac{2}{s^3} + \frac{3}{s^2} + \frac{2}{s}\right)}$$

[58] $f(t) = \begin{cases} \cos t, & 0 < t \le \pi \\ 1, & \pi < t \le 2\pi \\ \sin t, & t > 2\pi \end{cases}$

$f(t) = \cos t + (1 - \cos t)\,u(t-\pi) + (\sin t - 1)\,u(t - 2\pi)$, by a property.

$L[f(t)] = L(\cos t) + L[(1-\cos t)\,u(t-\pi)] + L[(\sin t - 1)\,u(t-2\pi)]$ . . . (1)

Let, $F(t-\pi) = 1 - \cos t$; $G(t-2\pi) = \sin t - 1$

$\Rightarrow$ $F(t) = 1 - \cos(t+\pi)$; $G(t) = \sin(t+2\pi) - 1$

ie., $F(t) = 1 + \cos t$; $G(t) = \sin t - 1$

$\therefore$ $\bar{F}(s) = \dfrac{1}{s} + \dfrac{s}{s^2+1}$; $\bar{G}(s) = \dfrac{1}{s^2+1} - \dfrac{1}{s}$

$L[F(t-\pi)\,u(t-\pi)] = e^{-\pi s}\,\bar{F}(s)$ and $L[G(t-2\pi)\,u(t-2\pi)] = e^{-2\pi s}\,\bar{G}(s)$

ie., $L[(1-\cos t)\,u(t-\pi)] = e^{-\pi s}\left(\dfrac{1}{s} + \dfrac{s}{s^2+1}\right)$ and

$L[(\sin t - 1)\,u(t-2\pi)] = e^{-2\pi s}\left(\dfrac{1}{s^2+1} - \dfrac{1}{s}\right)$

We shall use these results in (1).

Thus, $$\boxed{L[f(t)] = \frac{s}{s^2+1} + e^{-\pi s}\left(\frac{1}{s} + \frac{s}{s^2+1}\right) + e^{-2\pi s}\left(\frac{1}{s^2+1} - \frac{1}{s}\right)}$$

[59] $f(t) = \begin{cases} e^{2t}, & 0 < t < 1 \\ 2, & t > 1 \end{cases}$

$f(t) = e^{2t} + (2 - e^{2t})\,u(t-1)$, by a property.

$L[f(t)] = L(e^{2t}) + L[(2-e^{2t})\,u(t-1)]$ . . . (1)

Now, let $F(t-1) = 2 - e^{2t} \Rightarrow F(t) = 2 - e^{2(t+1)} = 2 - e^2 \cdot e^{2t}$

$$\therefore \quad \bar{F}(s) = \frac{2}{s} - e^2 \cdot \frac{1}{s-2}$$

But $\quad L[F(t-1)u(t-1)] = e^{-s}\,\bar{F}(s)$

ie., $\quad L[(2-e^{2t})u(t-1)] = e^{-s}\left(\frac{2}{s} - \frac{e^2}{s-2}\right)$

We shall use these results in (1).

Thus $$\boxed{L[f(t)] = \frac{1}{s-2} + e^{-s}\left(\frac{2}{s} - \frac{e^2}{s-2}\right)}$$

[60] $f(t) = \begin{cases} \cos t, & 0<t<\pi \\ \cos 2t, & \pi<t<2\pi \\ \cos 3t, & t>2\pi \end{cases}$

$$f(t) = \cos t + (\cos 2t - \cos t)\,u(t-\pi) + (\cos 3t - \cos 2t)\,u(t-2\pi)$$

$$L[f(t)] = L(\cos t) + L[(\cos 2t - \cos t)\,u(t-\pi)]$$
$$+ L[(\cos 3t - \cos 2t)\,u(t-2\pi)] \quad \ldots$$

Let $\quad F(t-\pi) = \cos 2t - \cos t \quad ; \; G(t-2\pi) = \cos 3t - \cos 2t$

$\Rightarrow \quad F(t) = \cos 2(t+\pi) - \cos(t+\pi)$ and

$$G(t) = \cos 3(t+2\pi) - \cos 2(t+2\pi)$$

ie., $\quad F(t) = \cos 2t + \cos t \quad ; \; G(t) = \cos 3t - \cos 2t$

$$\therefore \quad \bar{F}(s) = \frac{s}{s^2+4} + \frac{s}{s^2+1} \quad ; \; \bar{G}(s) = \frac{s}{s^2+9} - \frac{s}{s^2+4}$$

But $\quad L[F(t-\pi)u(t-\pi)] = e^{-\pi s}\,\bar{F}(s)$ and

$\quad L[G(t-2\pi)u(t-2\pi)] = e^{-2\pi s}\,\bar{G}(s)$

ie., $\quad L[(\cos 2t - \cos t)u(t-\pi)] = e^{-\pi s}\left(\frac{s}{s^2+4} + \frac{s}{s^2+1}\right)$

and $\quad L[(\cos 3t - \cos 2t)u(t-2\pi)] = e^{-2\pi s}\left(\frac{s}{s^2+9} - \frac{s}{s^2+4}\right)$

Hence (1) becomes

$$L[f(t)] = \frac{s}{s^2+1} + e^{-\pi s}\left(\frac{s}{s^2+4} + \frac{s}{s^2+1}\right) + e^{-2\pi s}\left(\frac{s}{s^2+9} - \frac{s}{s^2+4}\right)$$

Thus $\boxed{L[f(t)] = \frac{s}{s^2+1} + s\,e^{-\pi s}\left(\frac{1}{s^2+4} + \frac{1}{s^2+1}\right) - \frac{5s\,e^{-2\pi s}}{(s^2+4)(s^2+9)}}$

[61] *Express the following function in terms of Heaviside unit step function and hence find its Laplace transform where,*

$$f(t) = \begin{cases} t^2, & 0 < t \le 2 \\ 4t, & t > 2 \end{cases}$$

☞ $f(t) = t^2 + (4t - t^2)u(t-2)$, by a property.

$$L[f(t)] = L(t^2) + L[(4t-t^2)u(t-2)] \qquad \ldots (1)$$

We shall find $L[(4t-t^2)u(t-2)]$

Taking $F(t-2) = 4t - t^2$ we have,

$$F(t) = 4(t+2) - (t+2)^2$$

ie., $F(t) = 4 - t^2$

Hence, $\bar{F}(s) = L[F(t)] = \frac{4}{s} - \frac{2}{s^3}$

But $L[F(t-2)u(t-2)] = e^{-2s}\,\bar{F}(s)$, by a property.

$\therefore$ $L[(4t-t^2)u(t-2)] = e^{-2s}\left(\frac{4}{s} - \frac{2}{s^3}\right)$

We shall use this result in (1) along with $L(t^2) = 2/s^3$

Thus, $\boxed{L[f(t)] = \frac{2}{s^3} + e^{-2s}\left(\frac{4}{s} - \frac{2}{s^3}\right)}$

[62] *Express the function* $f(t) = \begin{cases} \pi - t, & 0 < t \le \pi \\ \sin t, & t > \pi \end{cases}$ *interms of unit step function and hence find its Laplace transform.* **[Dec. 2017]**

☞ $f(t) = (\pi - t) + [\sin t - (\pi - t)]u(t-\pi)$ by a standard property.

ie., $f(t) = (\pi - t) + [\sin t - \pi + t]u(t - \pi)$

$$L[f(t)] = L(\pi - t) + L\{[\sin t - \pi + t]u(t - \pi)\}$$

Taking $F(t-\pi) = \sin t - \pi + t$, we have,

$$F(t) = \sin(t+\pi) - \pi + (t+\pi)$$

ie., $F(t) = -\sin t + t$

$$\therefore \quad \bar{F}(s) = L[F(t)] = \frac{-1}{s^2+1} + \frac{1}{s^2}$$

Also $L[F(t-\pi)u(t-\pi)] = e^{-\pi s}\,\bar{F}(s)$

$$\therefore \quad L[(\sin t - \pi + t)u(t-\pi)] = e^{-\pi s}\left(\frac{1}{s^2} - \frac{1}{s^2+1}\right)$$

We shall use this result in (1) along with $L(\pi - t) = \pi/s - 1/s^2$

Thus, $$\boxed{L[f(t)] = \left(\frac{\pi}{s} - \frac{1}{s^2}\right) + \frac{e^{-\pi s}}{s^2(s^2+1)}}$$

[63] *Define Heaviside unit step function. Using unit step function find the Lap transform of*

$$f(t) = \begin{cases} \sin t, & 0 \le t < \pi \\ \sin 2t, & \pi \le t < 2\pi \\ \sin 3t, & t \ge 2\pi \end{cases}$$

[Dec. 2

☞ The given $f(t)$ can be written in the following form by a standard prop

$$f(t) = \sin t + [\sin 2t - \sin t]u(t-\pi) + [\sin 3t - \sin 2t]u(t-2\pi)$$

Now $L[f(t)] = L(\sin t) + L\{[\sin 2t - \sin t]u(t-\pi)\}$

$$+ L\{[\sin 3t - \sin 2t]u(t-2\pi)\}$$

Consider $L\{[\sin 2t - \sin t]u(t-\pi)\}$

Let $F(t-\pi) = \sin 2t - \sin t$

$\Rightarrow \quad F(t) = \sin 2(t+\pi) - \sin(t+\pi)$

ie., $F(t) = \sin(2\pi + 2t) - \sin(\pi + t)$

or $F(t) = \sin 2t + \sin t$

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$$\therefore \quad \bar{F}(s) = L[F(t)] = \frac{2}{s^2+4} + \frac{1}{s^2+1}$$

But $\quad L[F(t-\pi)u(t-\pi)] = e^{-\pi s}\,\bar{F}(s)$

ie., $\quad L\{[\sin 2t - \sin t]u(t-\pi)\} = e^{-\pi s}\left(\frac{2}{s^2+4} + \frac{1}{s^2+1}\right)$

Also let $G(t-2\pi) = \sin 3t - \sin 2t$

$\Rightarrow \quad G(t) = \sin 3(t+2\pi) - \sin 2(t+2\pi)$

ie., $\quad G(t) = \sin 3t - \sin 2t$

$$\therefore \quad \bar{G}(s) = \frac{3}{s^2+9} - \frac{2}{s^2+4}$$

But $\quad L[G(t-2\pi)u(t-2\pi)] = e^{-2\pi s}\,\bar{G}(s)$

ie., $\quad L\{[\sin 3t - \sin 2t]u(t-2\pi)\} = e^{-2\pi s}\left(\frac{3}{s^2+9} - \frac{2}{s^2+4}\right)$

We shall use these results in (1).

Thus, $\boxed{L[f(t)] = \frac{1}{s^2+1} + e^{-\pi s}\left[\frac{2}{s^2+4} + \frac{1}{s^2+1}\right] + e^{-2\pi s}\left[\frac{3}{s^2+9} - \frac{2}{s^2+4}\right]}$

ASSIGNMENT

Find the Laplace transform of the following functions

1. $\sin 3t \sin 2t \sin t$ **2.** $\cos(2t+3) + \cos 7t \cos 3t$

3. $4\sin^2 t \cos t$ **4.** $t\sqrt{t} + 4t^3 + 3^t$

5. $e^{-t}\cos^2 3t$ **6.** $e^{-3t}\sin^3 2t$ **7.** $\cos h\, 2t \cos 2t$

8. $t\sin^2 t$ **9.** $t\sin 3t \cos t$ **10.** $(1 + t\,e^t)^3$

11. $\dfrac{\cos h\, at - \cos bt}{t}$ **12.** $\dfrac{1-\cos at}{t}$ **13.** $\dfrac{2\sin 3t \cos 5t}{t}$

Evaluate the following integrals using Laplace transforms

14. $\int_0^\infty e^{-3t}\, t\sin t\, dt$ **15.** $\int_0^\infty t^2 e^t \cos t\, dt$

16. $\int_0^\infty \frac{e^{-t}-e^{-3t}}{t}dt$ 17. $\int_0^\infty \frac{\sin t}{t}dt$

Find the Laplace transform for the following functions

18. $f(t)=\begin{cases} a, & 0\le t\le a \\ -a, & a<t<2a \end{cases}$ where $f(t+2a)=f(t)$

19. $f(t)=\begin{cases} t, & 0\le t\le \pi \\ 2\pi-t, & \pi\le t\le 2\pi \end{cases}$ where $f(t+2\pi)=f(t)$

20. $f(t)=E\sin\omega t$ in $0<t<\pi/\omega$

Find the Laplace transform of the following functions

21. $(t^2+2t-1)u(t-3)$ 22. $(\sin t+\cos t)u(t-\pi/2)$

23. $e^{-t}u(t-2)$

Express the following functions interms of Heaviside unit step function hence find its Laplace transform

24. $f(t)=\begin{cases} t^2, & 1<t\le 2 \\ 4t, & t>2 \end{cases}$

25. $f(t)=\begin{cases} \sin t, & 0<t<\pi \\ \sin 2t, & \pi<t<2\pi \\ \sin 3t, & t>2\pi \end{cases}$

ANSWERS

1. $\frac{1}{2}\left[\frac{1}{s^2+4}-\frac{3}{s^2+36}+\frac{2}{s^2+16}\right]$

2. $\cos 3\cdot\frac{s}{s^2+4}-\sin 3\cdot\frac{2}{s^2+4}+\frac{s}{2}\left(\frac{1}{s^2+100}+\frac{1}{s^2+16}\right)$

3. $\frac{8s}{(s^2+1)(s^2+9)}$

4. $\dfrac{3\sqrt{\pi}}{4s^{5/2}}+\dfrac{24}{s^4}+\dfrac{1}{s-\log 3}$

5. $\dfrac{s^2+2s+19}{(s+1)(s^2+2s+37)}$

6. $\dfrac{48}{(s^2+6s+13)(s^2+6s+45)}$

7. $\dfrac{s^3}{s^4+64}$

8. $\dfrac{2(3s^2+4)}{s^2(s^2+4)^2}$

9. $\dfrac{6s(s^4+16s^2+96)}{(s^2+16)^2(s^2+4)^2}$

10. $\dfrac{1}{s}+\dfrac{3}{(s-1)^2}+\dfrac{6}{(s-2)^3}+\dfrac{6}{(s-3)^4}$

11. $\log\sqrt{(s^2+b^2)/(s^2-a^2)}$

12. $\log\left(\sqrt{s^2+a^2/s^2}\right)$

13. $\tan^{-1}(s/2)-\tan^{-1}(s/8)$

14. $3/50$ 15. 1 16. $\log 3$ 17. $\pi/2$

18. $a/s\cdot\tan h(as/2)$ 19. $1/s^2\cdot\tan h(\pi s/2)$

20. $\dfrac{E\omega}{s^2+\omega^2}\coth(\pi s/2\omega)$ 21. $e^{-3s}\left(\dfrac{2}{s^3}+\dfrac{8}{s^2}+\dfrac{14}{s}\right)$

22. $\dfrac{(s-1)e^{-\pi s/2}}{s^2+1}$ 23. $\dfrac{e^{-2s}}{e^2(s+1)}$

24. $\dfrac{2}{s^3}+\left(\dfrac{4}{s}-\dfrac{2}{s^3}\right)e^{-2s}$

25. $\dfrac{1}{s^2+1}+e^{-\pi s}\left(\dfrac{2}{s^2+4}+\dfrac{1}{s^2+1}\right)+e^{-2\pi s}\left(\dfrac{3}{s^2+9}-\dfrac{2}{s^2+4}\right)$

of finding the L... We also discuss *convolution theorem* which helps in finding the inverse Laplace transform. Finally we discuss solution of linear differential equations with a given set of initial conditions referred to as *initial value problems.* This method is highly useful in various branches of engineering. We have made a mention of this while defining the Laplace transform of a function $f(t)$.

If $L[f(t)] = \bar{f}(s)$ then $f(t)$ is called the **inverse Laplace transform of** $\bar{f}(s)$ and is denoted by $L^{-1}[\bar{f}(s)]$.

Thus we can say that,

$$L[f(t)] = \bar{f}(s) \Leftrightarrow L^{-1}[\bar{f}(s)] = f(t)$$

Observe the following illustrations.

$$L(1) = \frac{1}{s} \Rightarrow L^{-1}\left(\frac{1}{s}\right) = 1$$

$$L(\cos at) = \frac{s}{s^2+a^2} \Rightarrow L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$$

We revert the table of Laplace transforms of standard functions given earlier to present the basic table of inverse Laplace transforms.

	Function	Inverse Transform		Function	Inverse Transform
1.	$\frac{1}{s}$	1	5.	$\frac{1}{s^2+a^2}$	$\frac{\sin at}{a}$
2.	$\frac{1}{s-a}$	e^{at}	6.	$\frac{1}{s^2-a^2}$	$\frac{\sinh at}{a}$
3.	$\frac{s}{s^2+a^2}$	$\cos at$	7.	$\frac{1}{s^{n+1}}$ $(n>-1)$	$\frac{t^n}{\Gamma(n+1)}$
4.	$\frac{s}{s^2-a^2}$	$\cosh at$	8.	$\frac{1}{s^{n+1}}$ $n = 1, 2, 3 \ldots,$	$\frac{t^n}{n!}$

We present a few illustrative examples based on this table of inverse Laplace transforms.

1. $L^{-1}\left(\frac{1}{s-1}\right)=e^{t}$
2. $L^{-1}\left(\frac{1}{s+1}\right)=e^{-t}$
3. $L^{-1}\left(\frac{s}{s^2+9}\right)=\cos 3t$
4. $L^{-1}\left(\frac{s}{s^2-16}\right)=\cos h\,4t$
5. $L^{-1}\left(\frac{1}{s^2+5}\right)=\frac{1}{\sqrt{5}}\sin(\sqrt{5}\,t)$
6. $L^{-1}\left(\frac{1}{s^2-36}\right)=\frac{1}{6}\sin h\,6t$
7. $L^{-1}\left(\frac{1}{s^4}\right)=\frac{t^3}{3!}$
8. $L^{-1}\left(\frac{1}{s^{3/2}}\right)=\frac{t^{1/2}}{\Gamma(3/2)}=\frac{t^{1/2}}{1/2\cdot\sqrt{\pi}}=2\sqrt{t/\pi}$

Property : $L^{-1}[c_1\bar{f}(s)+c_2\bar{g}(s)]=c_1L^{-1}[\bar{f}(s)]+c_2L^{-1}[\bar{g}(s)]$

WORKED PROBLEMS

Find the inverse Laplace transform of the following functions.

64. $\frac{1}{s+2}+\frac{3}{2s+5}-\frac{4}{3s-2}$

65. $\frac{s+2}{s^2+36}+\frac{4s-1}{s^2+25}$

66. $\frac{2s-5}{4s^2+25}+\frac{8-6s}{16s^2+9}$

67. $\frac{2s-5}{8s^2-50}+\frac{4s}{9-s^2}$

68. $\frac{(s+2)^3}{s^6}$

69. $\frac{3(s^2-1)^2}{2s^5}$

70. $\frac{1}{s\sqrt{s}}+\frac{3}{s^2\sqrt{s}}-\frac{8}{\sqrt{s}}$

71. $\frac{3s+5\sqrt{2}}{s^2+8}$

☞ **Solutions**

[64] $L^{-1}\left[\frac{1}{s+2}\right]+\frac{3}{2}L^{-1}\left[\frac{1}{s+5/2}\right]-\frac{4}{3}L^{-1}\left[\frac{1}{s-2/3}\right]$

Thus the required inverse Laplace transform is

$$\boxed{e^{-2t}+3/2\cdot e^{-5t/2}-4/3\cdot e^{2t/3}}$$

[65] $L^{-1}\left[\frac{s}{s^2+6^2}\right]+2L^{-1}\left[\frac{1}{s^2+6^2}\right]+4L^{-1}\left[\frac{s}{s^2+5^2}\right]-L^{-1}\left[\frac{1}{s^2+5^2}\right]$

Thus the required inverse Laplace transform is

$$\boxed{\cos 6t+1/3\cdot\sin 6t+4\cos 5t-1/5\cdot\sin 5t}$$

[66] $2L^{-1}\left[\frac{s}{4s^2+25}\right]-5L^{-1}\left[\frac{1}{4s^2+25}\right]+8L^{-1}\left[\frac{1}{16s^2+9}\right]-6L^{-1}\left[\frac{s}{16s^2+9}\right]$

$$=\frac{1}{2}L^{-1}\left[\frac{s}{s^2+(5/2)^2}\right]-\frac{5}{4}L^{-1}\left[\frac{1}{s^2+(5/2)^2}\right]$$

$$+\frac{1}{2}L^{-1}\left[\frac{1}{s^2+(3/4)^2}\right]-\frac{3}{8}L^{-1}\left[\frac{s}{s^2+(3/4)^2}\right]$$

$$=\frac{1}{2}\cos(5t/2)-\frac{5}{4}\cdot\frac{2}{5}\sin(5t/2)+\frac{1}{2}\cdot\frac{4}{3}\sin(3t/4)-\frac{3}{8}\cos(3t/4)$$

Thus the required inverse Laplace transform is

$$\boxed{1/2\cdot\cos(5t/2)-1/2\cdot\sin(5t/2)+2/3\cdot\sin(3t/4)-3/8\cdot\cos(3t/4)}$$

[67] $L^{-1}\left[\frac{2s-5}{2(4s^2-25)}\right]-L^{-1}\left[\frac{4s}{s^2-9}\right]$

$$=\frac{1}{2}L^{-1}\left[\frac{1}{2s+5}\right]-4L^{-1}\left[\frac{s}{s^2-3^2}\right]=\frac{1}{2}\cdot\frac{1}{2}L^{-1}\left[\frac{1}{s+(5/2)}\right]-4L^{-1}\left[\frac{s}{s^2-3^2}\right]$$

Thus the required inverse Laplace transform is

$$\boxed{1/4\cdot e^{-5t/2}-4\cosh 3t}$$

[68] $L^{-1}\left[\frac{s^3+6s^2+12s+8}{s^6}\right]$

$$=L^{-1}\left(\frac{1}{s^3}\right)+6L^{-1}\left(\frac{1}{s^4}\right)+12L^{-1}\left(\frac{1}{s^5}\right)+8L^{-1}\left(\frac{1}{s^6}\right)$$

$$=\frac{t^2}{2!}+6\cdot\frac{t^3}{3!}+12\cdot\frac{t^4}{4!}+8\cdot\frac{t^5}{5!}$$

Thus the required inverse Laplace transform is

$$\boxed{\frac{t^2}{2}+t^3+\frac{t^4}{2}+\frac{t^5}{15}}$$

[69] $\dfrac{3}{2}L^{-1}\left[\dfrac{s^4-2s^2+1}{s^5}\right]$

$$=\frac{3}{2}\left[L^{-1}\left(\frac{1}{s}\right)-2L^{-1}\left(\frac{1}{s^3}\right)+L^{-1}\left(\frac{1}{s^5}\right)\right]$$

Thus the required inverse Laplace transform is

$$\boxed{\frac{3}{2}\left[1-2\cdot\frac{t^2}{2!}+\frac{t^4}{4!}\right]=\frac{3}{2}\left[1-t^2+\frac{t^4}{24}\right]}$$

[70] $L^{-1}\left(\dfrac{1}{s^{3/2}}\right)+3L^{-1}\left(\dfrac{1}{s^{5/2}}\right)-8L^{-1}\left(\dfrac{1}{s^{1/2}}\right)$

$$=\frac{t^{1/2}}{\Gamma(3/2)}+3\cdot\frac{t^{3/2}}{\Gamma(5/2)}-8\frac{t^{-1/2}}{\Gamma(1/2)}$$

$$=\frac{\sqrt{t}}{1/2\cdot\Gamma(1/2)}+3\cdot\frac{t\sqrt{t}}{3/2\cdot 1/2\cdot\Gamma(1/2)}-\frac{8}{\sqrt{t}\,\Gamma(1/2)}$$

$$=\frac{2\sqrt{t}}{\sqrt{\pi}}+4\frac{t\sqrt{t}}{\sqrt{\pi}}-\frac{8}{\sqrt{t}\sqrt{\pi}}$$

Thus the required inverse Laplace transform is

$$\boxed{\frac{2}{\sqrt{\pi}}\left[\sqrt{t}+2t\sqrt{t}-\frac{4}{\sqrt{t}}\right]}$$

[71] $3L^{-1}\left[\dfrac{s}{s^2+(\sqrt{8})^2}\right]+5\sqrt{2}\,L^{-1}\left[\dfrac{1}{s^2+(\sqrt{8})^2}\right]$

Thus the required inverse Laplace transform is

$$\boxed{3\cos(2\sqrt{2}\,t)+5/2\cdot\sin(2\sqrt{2}\,t)}$$

$\boxed{1.61}$ Computation of the inverse transform of $e^{-as}\,\overline{f}(s)$

We have proved that $L[f(t-a)u(t-a)]=e^{-as}\,\overline{f}(s)$

$\therefore\qquad L^{-1}[e^{-as}\,\overline{f}(s)]=f(t-a)u(t-a)$

Working procedure for problems

Step-1 In the given function we should observe the presence of e^{-as} first and identify the remaining part of the function to be called as $\bar{f}(s)$.

Step-2 Taking the inverse of $\bar{f}(s)$ we obtain $f(t)$

Step-3 The required inverse of $e^{-as}\,\bar{f}(s)$ is obtained by replacing t by $(t-a)$ in $f(t)$ to be mulitplied by the unit step function $u(t-a)$.

WORKED PROBLEMS

Find the inverse Laplace transforms of the following.

72. $\dfrac{1+e^{-3s}}{s^2}$

73. $\dfrac{3}{s^2}+\dfrac{2e^{-s}}{s^3}-\dfrac{3e^{-2s}}{s}$

74. $\dfrac{\cosh 2s}{e^{3s}s^2}$

75. $\dfrac{(1-e^{-s})(2-e^{-2s})}{s^3}$

76. $\dfrac{e^{-\pi s}}{s^2+1}+\dfrac{s\,e^{-2\pi s}}{s^2+4}$

77. $\dfrac{s\,e^{-s/2}+\pi e^{-s}}{s^2+\pi^2}$

☞ **Solutions**

[72] $L^{-1}\left(\dfrac{1}{s^2}\right)+L^{-1}\left(\dfrac{e^{-3s}}{s^2}\right)$. We have $L^{-1}(1/s^2)=t$

Thus, $$\boxed{L^{-1}\left[\frac{1+e^{-3s}}{s^2}\right]=t+(t-3)\,u(t-3)}$$

[73] $$3L^{-1}\left(\frac{1}{s^2}\right)+2L^{-1}\left(\frac{e^{-s}}{s^3}\right)-3L^{-1}\left(\frac{e^{-2s}}{s}\right) \quad \ldots(1)$$

We have, $L^{-1}\left(\dfrac{1}{s^2}\right)=t,\ L^{-1}\left(\dfrac{1}{s^3}\right)=\dfrac{t^2}{2},\ L^{-1}\left(\dfrac{1}{s}\right)=1$

Hence (1) becomes,

$$3\cdot t+2\frac{(t-1)^2}{2}u(t-1)-3\cdot 1\cdot u(t-2)$$

Thus, $$\boxed{L^{-1}\left[\frac{3}{s^2}+\frac{2e^{-s}}{s^3}-\frac{3e^{-2s}}{s}\right]=3t+(t-1)^2\,u(t-1)-3u(t-2)}$$

[74] $$\frac{\cos h\, 2s}{e^{3s}\cdot s^2} = \frac{e^{-3s}}{s^2}\;\frac{(e^{2s}+e^{-2s})}{2} = \frac{1}{2}\left[\frac{e^{-s}}{s^2}+\frac{e^{-5s}}{s^2}\right]$$

Now, $L^{-1}\left[\dfrac{\cos h\, 2s}{e^{3s}s^2}\right] = \dfrac{1}{2}\left\{L^{-1}\left(\dfrac{e^{-s}}{s^2}\right)+L^{-1}\left(\dfrac{e^{-5s}}{s^2}\right)\right\}$. But $L^{-1}(1/s^2) = t$

Thus, $$\boxed{L^{-1}\left[\frac{\cos h\, 2s}{e^{3s}\cdot s^2}\right] = \frac{1}{2}\{(t-1)\,u(t-1)+(t-5)\,u(t-5)\}}$$

[75] $$\frac{(1-e^{-s})(2-e^{-2s})}{s^3} = \frac{2-2e^{-s}-e^{-2s}+e^{-3s}}{s^3}$$

Now, $L^{-1}\left[\dfrac{(1-e^{-s})(2-e^{-2s})}{s^3}\right]$

$$= 2L^{-1}\left(\frac{1}{s^3}\right)-2L^{-1}\left(\frac{e^{-s}}{s^3}\right)-L^{-1}\left(\frac{e^{-2s}}{s^3}\right)+L^{-1}\left(\frac{e^{-3s}}{s^3}\right)$$

But, $L^{-1}(1/s^3) = t^2/2$

Thus, $L^{-1}\left[\dfrac{(1-e^{-s})(2-e^{-2s})}{s^3}\right]$ is given by

$$\boxed{= t^2-(t-1)^2\,u(t-1)-\frac{(t-2)^2\,u(t-2)}{2}+\frac{(t-3)^2\,u(t-3)}{2}}$$

[76] $$L^{-1}\left(\frac{e^{-\pi s}}{s^2+1}\right)+L^{-1}\left(e^{-2\pi s}\cdot\frac{s}{s^2+4}\right) \qquad \ldots (1)$$

We have, $L^{-1}\left(\dfrac{1}{s^2+1}\right) = \sin t$, $L^{-1}\left(\dfrac{s}{s^2+4}\right) = \cos 2t$

Hence (1) becomes,

$$\sin(t-\pi)\,u(t-\pi)+\cos 2(t-2\pi)\,u(t-2\pi)$$

Thus, $$\boxed{L^{-1}\left[\frac{e^{-\pi s}}{s^2+1}+\frac{s\,e^{-2\pi s}}{s^2+1}\right] = -\sin t\,u(t-\pi)+\cos 2t\,u(t-2\pi)}$$

Hence (1) becomes,

$$\cos\pi(t-1/2)u(t-1/2)+\sin\pi(t-1)u(t-1)$$

$$=\sin\pi t\, u(t-1/2)-\sin\pi t\, u(t-1)$$

Thus, $$\boxed{L^{-1}\left[\frac{se^{-s/2}+\pi e^{-s}}{s^2+\pi^2}\right]=\sin\pi t[u(t-1/2)-u(t-1)]}$$

1.62 Inverse transform by completing the square

We have the property that if $L[f(t)]=\bar{f}(s)$ then $L[e^{at}f(t)]=\bar{f}(s-a)$.

This implies that

$$L^{-1}[\bar{f}(s)]=f(t) \qquad \ldots(1)$$

and $$L^{-1}[\bar{f}(s-a)]=e^{at}f(t) \qquad \ldots(2)$$

Hence, (2) as a consequence of (1) becomes,

$$L^{-1}[\bar{f}(s-a)]=e^{at}L^{-1}[\bar{f}(s)] \qquad \ldots(3)$$

Working procedure for problems

Step-1 Given $\bar{f}(s)=\dfrac{\phi(s)}{(ps^2+qs+r)}$,

we first express (ps^2+qs+r) in the form $(s-a)^2\pm b^2$ and later express $\phi(s)$ in terms of $(s-a)$ so that the given function of s reduces to a function of $(s-a)$.

Step-2 We then use (3) to obtain the result.

Step-3 However if $\bar{f}(s)=\phi(s)/\psi(s-a)$ we only need to express $\phi(s)$ in terms of $(s-a)$ to compute the inverse transform.

WORKED PROBLEMS

Find the inverse Laplace transform of the following functions.

78. $\dfrac{s+5}{s^2-6s+13}$

79. $\dfrac{s^2}{(s+1)^3}$

80. $\dfrac{(s+2)e^{-s}}{(s+1)^4}$

81. $\dfrac{2s-1}{s^2+4s+29}$

82. $\dfrac{s+1}{s^2+6s+9}$

83. $\dfrac{e^{-4s}}{(s-4)^2}$

84. $\dfrac{s+1}{s^2+s+1}$

85. $\dfrac{2s+1}{s^2+3s+1}$

86. $\dfrac{7s+4}{4s^2+4s+9}$

87. $\dfrac{1}{(s+4)^{5/2}}+\dfrac{1}{\sqrt{2s+3}}$

☞ **Solutions**

[78] $L^{-1}\left[\dfrac{s+5}{s^2-6s+13}\right]=L^{-1}\left[\dfrac{s+5}{(s-3)^2+4}\right]$

ie., $\quad = L^{-1}\left[\dfrac{\overline{s-3}+3+5}{(s-3)^2+2^2}\right]=L^{-1}\left[\dfrac{(s-3)+8}{(s-3)^2+2^2}\right]$

Here, $a=3$ and $(s-3)$ changes to s

ie., $\quad = e^{3t}L^{-1}\left[\dfrac{s+8}{s^2+2^2}\right]$

$$= e^{3t}\left\{L^{-1}\left(\frac{s}{s^2+2^2}\right)+8L^{-1}\left(\frac{1}{s^2+2^2}\right)\right\}$$

Thus, $\boxed{L^{-1}\left[\dfrac{s+5}{s^2-6s+13}\right]=e^{3t}(\cos 2t+4\sin 2t)}$

[79] Here we need to express s^2 in terms of $(s+1)$

$$s^2=(s+1)^2-2s-1=(s+1)^2-2(s+1)+2-1$$

ie., $\quad s^2=(s+1)^2-2(s+1)+1$

$$\therefore \quad L^{-1}\left[\frac{s^2}{(s+1)^3}\right] = L^{-1}\left[\frac{(s+1)^2 - 2(s+1)+1}{(s+1)^3}\right]$$

Here, $a = -1$ and $(s+1)$ changes to s. Hence RHS becomes,

$$e^{-t} L^{-1}\left[\frac{s^2 - 2s + 1}{s^3}\right] = e^{-t}\left\{L^{-1}\left(\frac{1}{s}\right) - 2L^{-1}\left(\frac{1}{s^2}\right) + L^{-1}\left(\frac{1}{s^3}\right)\right\}$$

Thus. $$\boxed{L^{-1}\left[\frac{s^2}{(s+1)^3}\right] = e^{-t}\left(1 - 2t + \frac{t^2}{2}\right)}$$

[80] Let, $\bar{f}(s) = \dfrac{s+2}{(s+1)^4}$

We shall first find $L^{-1}[\bar{f}(s)] = f(t)$

$$L^{-1}\left[\frac{s+2}{(s+1)^4}\right] = L^{-1}\left[\frac{(s+1)+1}{(s+1)^4}\right] = e^{-t} L^{-1}\left[\frac{s+1}{s^4}\right]$$

ie., $$L^{-1}\left[\frac{s+2}{(s+1)^4}\right] = e^{-t}\left\{L^{-1}\left(\frac{1}{s^3}\right) + L^{-1}\left(\frac{1}{s^4}\right)\right\}$$

Using $L^{-1}\left(\dfrac{1}{s^{n+1}}\right) = \dfrac{t^n}{n!}$ we get

$$f(t) = e^{-t}\left(\frac{t^2}{2!} + \frac{t^3}{3!}\right) = e^{-t}\left(\frac{t^2}{2} + \frac{t^3}{6}\right)$$

Next we have, $L^{-1}[e^{-s}\bar{f}(s)] = f(t-1)u(t-1)$

Thus, $$\boxed{L^{-1}\left[e^{-s}\frac{s+2}{(s+1)^4}\right] = e^{-(t-1)}\left\{\frac{(t-1)^2}{2} + \frac{(t-1)^3}{6}\right\}u(t-1)}$$

[81] $$L^{-1}\left[\frac{2s-1}{s^2+4s+29}\right] = L^{-1}\left\{\frac{2s-1}{(s+2)^2+25}\right\} = L^{-1}\left\{\frac{2(s+2)-5}{(s+2)^2+25}\right\}$$

ie., $$= e^{-2t} L^{-1}\left\{\frac{2s-5}{s^2+5^2}\right\} = e^{-2t}\left\{2L^{-1}\left(\frac{s}{s^2+5^2}\right) - L^{-1}\left(\frac{5}{s^2+5^2}\right)\right\}$$

Thus, $$\boxed{L^{-1}\left[\frac{2s-1}{s^2+4s+29}\right] = e^{-2t}(2\cos 5t - \sin 5t)}$$

[82] $L^{-1}\left[\dfrac{s+1}{s^2+6s+9}\right] = L^{-1}\left[\dfrac{(s+3)-2}{(s+3)^2}\right] = e^{-3t}\,L^{-1}\left[\dfrac{s-2}{s^2}\right]$

ie., $= e^{-3t}\left\{L^{-1}\left(\dfrac{1}{s}\right) - 2L^{-1}\left(\dfrac{1}{s^2}\right)\right\}$

Thus, $$\boxed{L^{-1}\left[\dfrac{s+1}{s^2+6s+9}\right] = e^{-3t}(1-2t)}$$

[83] Let, $\overline{f}(s) = \dfrac{1}{(s-4)^2}$

$$L^{-1}[\overline{f}(s)] = L^{-1}\left[\dfrac{1}{(s-4)^2}\right] = e^{4t}\,L^{-1}\left(\dfrac{1}{s^2}\right) = e^{4t}\cdot t = f(t)$$

But, $L^{-1}[e^{-4s}\,\overline{f}(s)] = f(t-4)\,u(t-4)$

Thus, $$\boxed{L^{-1}\left[\dfrac{e^{-4s}}{(s-4)^2}\right] = \left\{e^{4(t-4)}(t-4)\right\}u(t-4)}$$

[84] $s^2+s+1 = (s+1/2)^2 - 1/4 + 1 = (s+1/2)^2 + (\sqrt{3}/2)^2$

$$L^{-1}\left[\dfrac{s+1}{s^2+s+1}\right] = L^{-1}\left[\dfrac{(s+1/2)+1/2}{(s+1/2)^2+(\sqrt{3}/2)^2}\right] = e^{-t/2}\,L^{-1}\left[\dfrac{s+1/2}{s^2+(\sqrt{3}/2)^2}\right]$$

ie., $= e^{-t/2}\left\{L^{-1}\left[\dfrac{s}{s^2+(\sqrt{3}/2)^2}\right] + \dfrac{1}{2}L^{-1}\left[\dfrac{1}{s^2+(\sqrt{3}/2)^2}\right]\right\}$

Thus, $$\boxed{L^{-1}\left[\dfrac{s+1}{s^2+s+1}\right] = e^{-t/2}\left\{\cos(\sqrt{3}\,t/2) + 1/\sqrt{3}\cdot\sin(\sqrt{3}\,t/2)\right\}}$$

[85] $s^2+3s+1 = (s+3/2)^2 - 9/4 + 1 = (s+3/2)^2 - (\sqrt{5}/2)^2$

$$L^{-1}\left[\dfrac{2s+1}{s^2+3s+1}\right] = L^{-1}\left[\dfrac{2s+1}{(s+3/2)^2-(\sqrt{5}/2)^2}\right]$$

$$= L^{-1}\left[\dfrac{2(s+3/2)-2}{(s+3/2)^2-(\sqrt{5}/2)^2}\right]$$

$$= e^{-3t/2}\left\{2L^{-1}\left[\frac{s}{s^2-(\sqrt{5}/2)^2}\right]-2L^{-1}\left[\frac{1}{s^2-(\sqrt{5}/2)^2}\right]\right\}$$

Thus, $\boxed{L^{-1}\left[\dfrac{2s+1}{s^2+3s+1}\right] = e^{-3t/2}\left\{2\cosh(\sqrt{5}\,t/2)-4/\sqrt{5}\cdot\sinh(\sqrt{5}\,t/2)\right\}}$

[86] $4s^2+4s+9 = 4(s^2+s+9/4) = 4\{(s+1/2)^2+2\}$

Hence, $7s+4 = 7\,(s+1/2)+1/2$

Now, $L^{-1}\left[\dfrac{7s+4}{4s^2+4s+9}\right] = \dfrac{1}{4}L^{-1}\left[\dfrac{7\,(s+1/2)+1/2}{(s+1/2)^2+2}\right]$

ie., $$= \frac{e^{-t/2}}{4}\left\{7L^{-1}\left[\frac{s}{s^2+(\sqrt{2})^2}\right]+\frac{1}{2}L^{-1}\left[\frac{1}{s^2+(\sqrt{2})^2}\right]\right\}$$

Thus, $\boxed{L^{-1}\left[\dfrac{7s+4}{4s^2+4s+9}\right] = \dfrac{e^{-t/2}}{4}\left\{7\cos(\sqrt{2}\,t)+\dfrac{1}{2\sqrt{2}}\sin(\sqrt{2}\,t)\right\}}$

[87] $L^{-1}\left[\dfrac{1}{(s+4)^{5/2}}\right]+L^{-1}\left[\dfrac{1}{\sqrt{2s+3}}\right]$

$$= e^{-4t}L^{-1}\left[\frac{1}{s^{5/2}}\right]+\frac{1}{\sqrt{2}}L^{-1}\left[\frac{1}{\sqrt{s+3/2}}\right]$$

$$= e^{-4t}\left[\frac{t^{3/2}}{\Gamma(5/2)}\right]+\frac{e^{-3t/2}}{\sqrt{2}}L^{-1}\left(\frac{1}{\sqrt{s}}\right).$$

But $\Gamma(5/2) = \dfrac{3}{2}\cdot\dfrac{1}{2}\cdot\sqrt{\pi} = \dfrac{3\sqrt{\pi}}{4}$

$\therefore$ $L^{-1}\left[\dfrac{1}{(s+4)^{5/2}}\right]+L^{-1}\left[\dfrac{1}{\sqrt{2s+3}}\right]$

$$= \frac{4}{3\sqrt{\pi}}e^{-4t}t^{3/2}+\frac{e^{-3t/2}}{\sqrt{2}}\frac{t^{-1/2}}{\Gamma(1/2)}$$ But $\Gamma(1/2) = \sqrt{\pi}$

Thus the required inverse Laplace transform is given by

$$\boxed{\frac{1}{\sqrt{\pi}}\left\{\frac{4}{3}e^{-4t}\,t^{3/2}+\frac{e^{-3t/2}}{\sqrt{2t}}\right\}}$$

Find the inverse Laplace transform of the following functions.

88. $\dfrac{s}{s^4+4a^4}$ **89.** $\dfrac{s^2}{s^4+4a^4}$ **90.** $\dfrac{s}{s^4+s^2+1}$

Note : *In these problems we factorize the denominator and express the numerator interms of the factors of the denominator by simple adjustment.*

☞ **Solutions**

[88] $s^4+4a^4 = (s^2+2a^2)^2-4a^2s^2$

ie., $s^4+4a^4 = (s^2+2a^2+2as)(s^2+2a^2-2as)$ . . . (1)

Also, $4as = (s^2+2a^2+2as)-(s^2+2a^2-2as)$

Now, $$\frac{s}{s^4+4a^4} = \frac{1}{4a}\left\{\frac{(s^2+2a^2+2as)-(s^2+2a^2-2as)}{(s^2+2a^2+2as)(s^2+2a^2-2as)}\right\}$$

ie., $$\frac{s}{s^4+4a^4} = \frac{1}{4a}\left\{\frac{1}{s^2+2a^2-2as}-\frac{1}{s^2+2a^2+2as}\right\}$$

$$L^{-1}\left[\frac{s}{s^4+4a^4}\right] = \frac{1}{4a}\left\{L^{-1}\left[\frac{1}{(s-a)^2+a^2}\right]-L^{-1}\left[\frac{1}{(s+a)^2+a^2}\right]\right\}$$

$$L^{-1}\left[\frac{s}{s^4+4a^4}\right] = \frac{1}{4a}\left\{e^{at}\,L^{-1}\left[\frac{1}{s^2+a^2}\right]-e^{-at}\,L^{-1}\left[\frac{1}{s^2+a^2}\right]\right\}$$

$$= \frac{1}{4a}\left\{e^{at}\frac{\sin at}{a}-e^{-at}\frac{\sin at}{a}\right\}$$

Thus, $$\boxed{L^{-1}\left[\frac{s}{s^4+4a^4}\right] = \frac{\sin at}{2a^2}\left\{\frac{e^{at}-e^{-at}}{2}\right\} = \frac{\sin at\sinh at}{2a^2}}$$

[89] As in the previous problem, we have,

$$\frac{s}{s^4+4a^4} = \frac{1}{4a}\left\{\frac{1}{(s-a)^2+a^2}-\frac{1}{(s+a)^2+a^2}\right\}.$$

Multiplying by s we have

$$\frac{s^2}{s^4+4a^4} = \frac{1}{4a}\left\{\frac{s}{(s-a)^2+a^2} - \frac{s}{(s+a)^2+a^2}\right\}$$

or
$$\frac{s^2}{s^4+4a^4} = \frac{1}{4a}\left\{\frac{(s-a)+a}{(s-a)^2+a^2} - \frac{(s+a)-a}{(s+a)^2+a^2}\right\}$$

$$\therefore \quad L^{-1}\left[\frac{s^2}{s^4+4a^4}\right] = \frac{1}{4a}\left\{e^{at}L^{-1}\left[\frac{s+a}{s^2+a^2}\right] - e^{-at}L^{-1}\left[\frac{s-a}{s^2+a^2}\right]\right\}$$

ie.,
$$= \frac{1}{4a}\left\{e^{at}(\cos at + \sin at) - e^{-at}(\cos at - \sin at)\right\}$$

$$= \frac{1}{2a}\left\{\frac{e^{at}-e^{-at}}{2}\cos at + \frac{e^{at}+e^{-at}}{2}\sin at\right\}$$

Thus,
$$\boxed{L^{-1}\left[\frac{s^2}{s^4+4a^4}\right] = \frac{1}{2a}(\sinh at \cos at + \cosh at \sin at)}$$

[90] $s^4+s^2+1 = (s^2+1)^2 - s^2 = (s^2+1-s)(s^2+1+s)$

Also, $2s = (s^2+1+s)-(s^2+1-s)$

$$\therefore \quad \frac{s}{s^4+s^2+1} = \frac{1}{2}\left[\frac{(s^2+1+s)-(s^2+1-s)}{(s^2+1+s)(s^2+1-s)}\right]$$

ie.,
$$\frac{s}{s^4+s^2+1} = \frac{1}{2}\left[\frac{1}{s^2-s+1} - \frac{1}{s^2+s+1}\right]$$

Now, $L^{-1}\left[\frac{s}{s^4+s^2+1}\right] = \frac{1}{2}\left\{L^{-1}\left[\frac{1}{s^2-s+1}\right] - L^{-1}\left[\frac{1}{s^2+s+1}\right]\right\}$

ie.,
$$= \frac{1}{2}\left\{L^{-1}\left[\frac{1}{(s-1/2)^2+(\sqrt{3}/2)^2}\right] - L^{-1}\left[\frac{1}{(s+1/2)^2+(\sqrt{3}/2)^2}\right]\right\}$$

$$= \frac{1}{2}\left\{e^{t/2}L^{-1}\left[\frac{1}{s^2+(\sqrt{3}/2)^2}\right] - e^{-t/2}L^{-1}\left[\frac{1}{s^2+(\sqrt{3}/2)^2}\right]\right\}$$

$$L^{-1}\left[\frac{s}{s^4+s^2+1}\right]=\frac{1}{2}\left\{e^{t/2}\cdot\frac{2}{\sqrt{3}}\sin(\sqrt{3}\,t/2)-e^{-t/2}\cdot\frac{2}{\sqrt{3}}\sin(\sqrt{3}\,t/2)\right\}$$

$$=\frac{2}{\sqrt{3}}\sin(\sqrt{3}\,t/2)\left\{\frac{e^{t/2}-e^{-t/2}}{2}\right\}$$

Thus, $$\boxed{L^{-1}\left[\frac{s}{s^4+s^2+1}\right]=\frac{2}{\sqrt{3}}\sin(\sqrt{3}\,t/2)\cdot\sin h(t/2)}$$

1.63 Inverse transform by the method of partial fractions

We know that, the method of partial fractions is a technique of converting an algebraic fraction $\phi(s)/\psi(s)$ [*where degree of* $\phi(s)$ *is less than that of* $\psi(s)$] into a sum. Depending on the nature of terms in $\psi(s)$ we have to split into a sum of various terms with constants *A, B, C, D, . . .* which can be determined. Later the inverse is found term by term.

WORKED PROBLEMS

Find the inverse Laplace transform of the following functions.

91. $\dfrac{1}{s(s+1)(s+2)(s+3)}$

92. $\dfrac{2s^2+5s-4}{s^3+s^2-2s}$

93. $\dfrac{3s+2}{s^2-s-2}$

94. $\dfrac{s^2}{(s^2+1)(s^2+4)}$

95. $\dfrac{4s+5}{(s+1)^2(s+2)}$

96. $\dfrac{s+2}{s^2(s+3)}$

97. $\dfrac{(3s+1)e^{-3s}}{(s-1)(s^2+1)}$

98. $\dfrac{5s+3}{(s-1)(s^2+2s+5)}$

☞ **Solutions**

[91] Let, $$\frac{1}{s(s+1)(s+2)(s+3)}=\frac{A}{s}+\frac{B}{s+1}+\frac{C}{s+2}+\frac{D}{s+3}$$

Multiplying by $s(s+1)(s+2)(s+3)$ we get,

$$1 = A(s+1)(s+2)(s+3) + Bs(s+2)(s+3) + Cs(s+1)(s+3) + Ds(s+1)(s+2)$$

Put $s = 0$: $1 = A(6)$ $\therefore A = 1/6$

Put $s = -1$: $1 = B(-2)$ $\therefore B = -1/2$

Put $s = -2$: $1 = C(2)$ $\therefore C = 1/2$

Put $s = -3$: $1 = D(-6)$ $\therefore D = -1/6$

Now, $L^{-1}\left[\dfrac{1}{s(s+1)(s+2)(s+3)}\right]$

$$= \frac{1}{6}L^{-1}\left[\frac{1}{s}\right] - \frac{1}{2}L^{-1}\left[\frac{1}{s+1}\right] + \frac{1}{2}L^{-1}\left[\frac{1}{s+2}\right] - \frac{1}{6}L^{-1}\left[\frac{1}{s+3}\right]$$

Thus, $$\boxed{L^{-1}\left[\frac{1}{s(s+1)(s+2)(s+3)}\right] = \frac{1}{6} - \frac{1}{2}e^{-t} + \frac{1}{2}e^{-2t} - \frac{1}{6}e^{-3t}}$$

[92] $s^3 + s^2 - 2s = s(s^2 + s - 2) = s(s-1)(s+2)$

Let, $$\frac{2s^2 + 5s - 4}{s(s-1)(s+2)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+2}$$

Multiplying by $s(s-1)(s+2)$ we get,

$$2s^2 + 5s - 4 = A(s-1)(s+2) + Bs(s+2) + Cs(s-1)$$

Put $s = 0$: $-4 = A(-2)$ $\therefore A = 2$

Put $s = 1$: $3 = B(3)$ $\therefore B = 1$

Put $s = -2$: $-6 = C(6)$ $\therefore C = -1$

Now, $L^{-1}\left[\dfrac{2s^2 + 5s - 4}{s(s-1)(s+2)}\right]$

$$= 2L^{-1}\left[\frac{1}{s}\right] + L^{-1}\left[\frac{1}{s-1}\right] - L^{-1}\left[\frac{1}{s+2}\right]$$

Thus, $$\boxed{L^{-1}\left[\frac{2s^2 + 5s - 4}{s^3 + s^2 - 2s}\right] = 2 + e^{t} - e^{-2t}}$$

[93] Note : *The problem can be done by completing the square. Since the quadratic is factorizable, the method of partial fractions is preferred.*

Let, $$\frac{3s+2}{(s-2)(s+1)} = \frac{A}{s-2} + \frac{B}{s+1}$$

or $$3s+2 = A(s+1) + B(s-2)$$

Put $s = 2$: $8 = A(3)$ $\therefore A = 8/3$

Put $s = -1$: $-1 = B(-3)$ $\therefore B = 1/3$

$$L^{-1}\left[\frac{3s+2}{(s-2)(s+1)}\right] = \frac{8}{3}L^{-1}\left[\frac{1}{s-2}\right] + \frac{1}{3}L^{-1}\left[\frac{1}{s+1}\right]$$

Thus, $$\boxed{L^{-1}\left[\frac{3s+2}{s^2-s-2}\right] = \frac{1}{3}(8e^{2t} + e^{-t})}$$

[94] We have, $\dfrac{s^2}{(s^2+1)(s^2+4)}$ and let $s^2 = t$ for convenience.

We now have, $$\frac{t}{(t+1)(t+4)} = \frac{A}{t+1} + \frac{B}{t+4} \quad (say)$$

or $$t = A(t+4) + B(t+1)$$

Put $t = -1$: $-1 = A(3)$ $\therefore A = -1/3$

Put $t = -4$: $-4 = B(-3)$ $\therefore B = 4/3$

Hence $$\frac{t}{(t+1)(t+4)} = \frac{-1}{3}\cdot\frac{1}{t+1} + \frac{4}{3}\cdot\frac{1}{t+4}$$

Substituting $t = s^2$ and taking inverse we have,

$$L^{-1}\left[\frac{s^2}{(s^2+1)(s^2+4)}\right]$$

$$= \frac{-1}{3}L^{-1}\left[\frac{1}{s^2+1}\right] + \frac{4}{3}L^{-1}\left[\frac{1}{s^2+4}\right]$$

Thus, $$\boxed{L^{-1}\left[\frac{s^2}{(s^2+1)(s^2+4)}\right] = \frac{1}{3}(2\sin 2t - \sin t)}$$

[95] Let, $\dfrac{4s+5}{(s+1)^2(s+2)} = \dfrac{A}{s+1} + \dfrac{B}{(s+1)^2} + \dfrac{C}{s+2}$

Multiplying by $(s+1)^2(s+2)$ we get,

$$4s+5 = A(s+1)(s+2) + B(s+2) + C(s+1)^2 \quad \ldots(1)$$

Put $s=-1$: $1 = B(1)$ $\therefore$ $B = 1$

Put $s=-2$: $-3 = C(1)$ $\therefore$ $C = -3$

Equating the coefficient of s^2 on both sides of (1) we get,

$$0 = A + C \quad \therefore \ A = 3$$

Now, $L^{-1}\left[\dfrac{4s+5}{(s+1)^2(s+2)}\right]$

$$= 3L^{-1}\left[\frac{1}{s+1}\right] + L^{-1}\left[\frac{1}{(s+1)^2}\right] - 3L^{-1}\left[\frac{1}{s+2}\right]$$

$$= 3e^{-t} + e^{-t}L^{-1}[1/s^2] - 3e^{-2t}$$

Thus, $$\boxed{L^{-1}\left[\frac{4s+5}{(s+1)^2(s+2)}\right] = 3e^{-t} + e^{-t}\cdot t - 3e^{-2t}}$$

[96] Let, $\dfrac{s+2}{s^2(s+3)} = \dfrac{A}{s} + \dfrac{B}{s^2} + \dfrac{C}{s+3}$

Multiplying by $s^2(s+3)$ we get,

$$s+2 = As(s+3) + B(s+3) + Cs^2 \quad \ldots(1)$$

Put $s=0$: $2 = B(3)$ $\therefore$ $B = 2/3$

Put $s=-3$: $-1 = C(9)$ $\therefore$ $C = -1/9$

Equating the coefficient of s^2 on both sides of (1) we get,

$$0 = A + C \quad \therefore \ A = -C = 1/9$$

Now, $$L^{-1}\left[\frac{s+2}{s^2(s+3)}\right] = \frac{1}{9}L^{-1}\left[\frac{1}{s}\right] + \frac{2}{3}L^{-1}\left[\frac{1}{s^2}\right] - \frac{1}{9}L^{-1}\left[\frac{1}{s+3}\right]$$

Thus, $$\boxed{L^{-1}\left[\frac{s+2}{s^2(s+3)}\right] = \frac{1}{9}(1+6t-e^{-3t})}$$

[97] Let, $\dfrac{3s+1}{(s-1)(s^2+1)} = \dfrac{A}{s-1} + \dfrac{Bs+C}{s^2+1}$

or $$3s+1 = A(s^2+1) + (Bs+C)(s-1) \quad \ldots(1)$$

Put $s = 1$: $4 = A(2)$ $\therefore A = 2$

Put $s = 0$: $1 = A - C$ $\therefore C = 1$

Equating the coefficient of s^2 on both sides of (1) we get,

$$0 = A + B \quad \therefore B = -2.$$

Now, $$L^{-1}\left[\frac{3s+1}{(s-1)(s^2+1)}\right] = 2L^{-1}\left[\frac{1}{s-1}\right] - 2L^{-1}\left[\frac{s}{s^2+1}\right] + L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$= 2e^t - 2\cos t + \sin t$$

Thus, $$\boxed{L^{-1}\left[\frac{(3s+1)e^{-3s}}{(s-1)(s^2+1)}\right] = [2e^{t-3} - 2\cos(t-3) + \sin(t-3)]u(t-3)}$$

[98] Let, $$\frac{5s+3}{(s-1)(s^2+2s+5)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+2s+5}$$

or $$5s + 3 = A(s^2+2s+5) + (Bs+C)(s-1) \qquad \ldots (1)$$

Put $s = 1$: $8 = A(8)$ $\therefore A = 1$

Put $s = 0$: $3 = 5A - C$ $\therefore C = 2$

Equating the coefficient of s^2 on both sides of (1) we get,

$$0 = A + B \quad \therefore B = -1$$

Now we have,

$$\frac{5s+3}{(s-1)(s^2+2s+5)} = \frac{1}{s-1} + \frac{-s+2}{(s^2+2s+5)} = \frac{1}{s-1} + \frac{2-s}{(s+1)^2+4}$$

$$\frac{5s+3}{(s-1)(s^2+2s+5)} = \frac{1}{s-1} + \frac{3-(s+1)}{(s+1)^2+4}$$

$$L^{-1}\left[\frac{5s+3}{(s-1)(s^2+2s+5)}\right] = L^{-1}\left[\frac{1}{s-1}\right] + L^{-1}\left[\frac{3-(s+1)}{(s+1)^2+4}\right]$$

$$= e^t + e^{-t} L^{-1}\left[\frac{3-s}{s^2+4}\right]$$

$$= e^t + e^{-t}\left\{3L^{-1}\left[\frac{1}{s^2+4}\right] - L^{-1}\left[\frac{s}{s^2+4}\right]\right\}$$

Thus, $$\boxed{L^{-1}\left[\frac{5s+3}{(s-1)(s^2+2s+5)}\right] = e^t + e^{-t}\left[\frac{3}{2}\cdot\sin 2t - \cos 2t\right]}$$

1.64 Inverse transform of logarithmic functions and inverse functions

Given $\bar{f}(s)$ we need to find $L^{-1}[\bar{f}(s)] = f(t)$

We have the property : $L[t\,f(t)] = -\bar{f}'(s)$

Equivalently, $L^{-1}[-\bar{f}'(s)] = t\,f(t)$ . . . (1)

Working procedure for problems

Step-1 In the case of logarithmic functions we apply the properties of logarithms and then differentiate w.r.t. s to obtain $\bar{f}'(s)$.

Step-2 We then multiply by -1 and take inverse on both sides.

Step-3 LHS becomes $t\,f(t)$ by (1) and inverses are also found for the terms in RHS with the result we obtain the required $f(t)$.

Step-4 If logarithmic function persists in $\bar{f}'(s)$ we differentiate again w.r.t s to obtain $\bar{f}''(s)$ and use the property that

$L^{-1}[\bar{f}''(s)] = t^2 f(t)$ since $L[t^2 f(t)] = \bar{f}''(s)$

Step-5 In the cases of inverse functions we simply differentiate the given $\bar{f}(s)$ and use the result (1) to obtain $f(t)$.

WORKED PROBLEMS

Find the inverse Laplace transform of the following functions.

99. $\log\left(\dfrac{s+a}{s+b}\right)$ **100.** $\log\left(1-\dfrac{a^2}{s^2}\right)$

101. $\cot^{-1}(s/a)$ **102.** $\log\sqrt{(s^2+1)/(s^2+4)}$

103. $\log\left[\dfrac{s^2+4}{s(s+4)(s-4)}\right]$ **104.** $\tan^{-1}(2/s^2)$

105. $s\log\left(\dfrac{s+4}{s-4}\right)$ **106.** $\cot^{-1}\left(\dfrac{s+a}{b}\right)$

Solutions

[99] Let, $\bar{f}(s) = \log\left(\frac{s+a}{s+b}\right) = \log(s+a) - \log(s+b)$

$$\therefore \quad -\bar{f}'(s) = -\left\{\frac{1}{s+a} - \frac{1}{s+b}\right\} = \frac{1}{s+b} - \frac{1}{s+a}$$

Now $L^{-1}[-\bar{f}'(s)] = L^{-1}\left(\frac{1}{s+b}\right) - L^{-1}\left(\frac{1}{s+a}\right)$

ie., $t\,f(t) = e^{-bt} - e^{-at}$

Thus, $\boxed{f(t) = \frac{e^{-bt} - e^{-at}}{t}}$

[100] Let, $\bar{f}(s) = \log\left(1 - \frac{a^2}{s^2}\right) = \log\left(\frac{s^2 - a^2}{s^2}\right)$

ie., $\bar{f}(s) = \log(s^2 - a^2) - 2\log s$

$$\therefore \quad -\bar{f}'(s) = -\left\{\frac{1}{s^2 - a^2} \cdot 2s - \frac{2}{s}\right\} = 2\left(\frac{1}{s} - \frac{s}{s^2 - a^2}\right)$$

Now $L^{-1}[-\bar{f}'(s)] = 2\left\{L^{-1}\left(\frac{1}{s}\right) - L^{-1}\left(\frac{s}{s^2 - a^2}\right)\right\}$

ie., $t\,f(t) = 2(1 - \cosh at)$

Thus, $\boxed{f(t) = \frac{2(1 - \cosh at)}{t}}$

[101] Let, $\bar{f}(s) = \cot^{-1}(s/a)$

Differentiate w.r.t s and multiply with -1.

$$\therefore \quad \bar{f}'(s) = \frac{-1}{1 + (s/a)^2} \cdot \frac{1}{a} \text{ and } -\bar{f}'(s) = \frac{a}{a^2 + s^2}$$

Taking inverse, $L^{-1}[-\bar{f}'(s)] = L^{-1}\left(\frac{a}{s^2 + a^2}\right)$

$$t\,f(t) = \sin at$$

Thus, $\boxed{f(t) = \frac{\sin at}{t}}$

[102] Let, $\bar{f}(s) = \log\sqrt{s^2+1/s^2+4}$

$$= \frac{1}{2}\{\log(s^2+1) - \log(s^2+4)\}$$

$$\therefore \quad -\bar{f}'(s) = -\frac{1}{2}\left\{\frac{2s}{s^2+1} - \frac{2s}{s^2+4}\right\} = \frac{s}{s^2+4} - \frac{s}{s^2+1}$$

Now, $L^{-1}[-\bar{f}'(s)] = L^{-1}\left(\frac{s}{s^2+2^2}\right) - L^{-1}\left(\frac{s}{s^2+1^2}\right)$

ie., $\quad t\,f(t) = \cos 2t - \cos t$

Thus, $\quad \boxed{f(t) = \frac{\cos 2t - \cos t}{t}}$

[103] Let, $\bar{f}(s) = \log\left[\frac{s^2+4}{s(s+4)(s-4)}\right]$

ie., $\quad \bar{f}(s) = \log(s^2+4) - \log s - \log(s+4) - \log(s-4)$

$$\therefore \quad -\bar{f}'(s) = -\left\{\frac{2s}{s^2+4} - \frac{1}{s} - \frac{1}{s+4} - \frac{1}{s-4}\right\}$$

Now, $L^{-1}[-\bar{f}'(s)] = L^{-1}\left(\frac{1}{s}\right) + L^{-1}\left(\frac{1}{s+4}\right) + L^{-1}\left(\frac{1}{s-4}\right) - 2L^{-1}\left(\frac{s}{s^2+4}\right)$

ie., $\quad t\,f(t) = 1 + e^{-4t} + e^{4t} - 2\cos 2t = 1 + 2\cos h\,4t - 2\cos 2t$

Thus, $\quad \boxed{f(t) = \frac{1 + 2(\cos h\,4t - \cos 2t)}{t}}$

Note : Similar Problem

Find $\quad L^{-1}\left\{\log\left[\frac{s^2+1}{s(s+1)}\right]\right\}$ **[June 2017]**

[104] Let, $\bar{f}(s) = \tan^{-1}(2/s^2)$

$$\therefore \quad \bar{f}'(s) = \frac{1}{1+(4/s^4)} \cdot \frac{-4}{s^3} = \frac{-4s}{s^4+4}$$

Hence, $$L^{-1}[-\bar{f}'(s)] = L^{-1}\left[\frac{4s}{s^4+4}\right]$$

ie., $$t\,f(t) = L^{-1}\left[\frac{4s}{s^4+4}\right] \qquad \ldots (1)$$

Now, $s^4+4 = (s^2+2)^2-4s^2 = (s^2+2+2s)(s^2+2-2s)$

Also, $4s = (s^2+2+2s)-(s^2+2-2s)$

Hence, $$\frac{4s}{s^4+4} = \frac{(s^2+2+2s)-(s^2+2-2s)}{(s^2+2+2s)(s^2+2-2s)}$$

$$= \frac{1}{s^2+2-2s} - \frac{1}{s^2+2+2s}$$

$$\therefore \quad L^{-1}\left[\frac{4s}{s^4+4}\right] = L^{-1}\left[\frac{1}{s^2-2s+2}\right] - L^{-1}\left[\frac{1}{s^2+2s+2}\right]$$

Using (1) in LHS we have,

$$t\,f(t) = L^{-1}\left[\frac{1}{(s-1)^2+1}\right] - L^{-1}\left[\frac{1}{(s+1)^2+1}\right]$$

ie., $$t\,f(t) = e^t\,L^{-1}\left[\frac{1}{s^2+1}\right] - e^{-t}\,L^{-1}\left[\frac{1}{s^2+1}\right]$$

ie., $$t\,f(t) = e^t\sin t - e^{-t}\sin t = \sin t\,(e^t - e^{-t})$$

ie., $$t\,f(t) = \sin t \cdot 2\sinh t$$

Thus, $$\boxed{f(t) = \frac{2\sin t\sinh t}{t}}$$

[105] Let, $\bar{f}(s) = s\log\left(\frac{s+4}{s-4}\right)$

ie., $$\bar{f}(s) = s[\log(s+4) - \log(s-4)]$$

$$\therefore \quad \bar{f}'(s) = \frac{s}{s+4} - \frac{s}{s-4} + [\log(s+4) - \log(s-4)]$$

We need to differentiate w.r.t s again,

ie., $$t^2 f(t) = 4\left\{e^{-4t} L^{-1}\left(\frac{1}{s^2}\right) + e^{4t} L^{-1}\left(\frac{1}{s^2}\right)\right\} + e^{-4t} - e^{4t}$$

$$= 4(e^{-4t} t + e^{4t} \cdot t) - 2\sin h\, 4t$$

$$t^2 f(t) = 8t\cos h\, 4t - 2\sin h\, 4t$$

Thus, $$\boxed{f(t) = \frac{2(4t\cos h\, 4t - \sin h\, 4t)}{t^2}}$$

[106] Let, $\bar{f}(s) = \cot^{-1}\left(\frac{s+a}{b}\right)$

$$\therefore \quad \bar{f}'(s) = \frac{-1}{1 + \frac{(s+a)^2}{b^2}} \cdot \frac{1}{b} = \frac{-b}{b^2 + (s+a)^2}$$

Now, $$L^{-1}[-\bar{f}'(s)] = L^{-1}\left[\frac{b}{(s+a)^2 + b^2}\right] = e^{-at} L^{-1}\left[\frac{b}{s^2 + b^2}\right]$$

ie., $$t f(t) = e^{-at} \sin bt$$

Thus, $$\boxed{f(t) = \frac{e^{-at} \sin bt}{t}}$$

[107] Find (a) $L^{-1}\left[\frac{1}{s}\sin\left(\frac{1}{s}\right)\right]$ (b) $L^{-1}\left[\frac{1}{s}\cos\left(\frac{1}{s}\right)\right]$

☞ We consider the expansion of $\sin x$ and $\cos x$ in the neighbourho
origin (ie., $x = 0$) which are given by

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots ; \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

Replacing x by $1/s$ where we shall assume that s is large in which case $x = 1/s \to 0$.

$$\therefore \quad \sin\left(\frac{1}{s}\right) = \frac{1}{s} - \frac{1}{3!}\frac{1}{s^3} + \frac{1}{5!}\frac{1}{s^5} - \cdots \text{ and } \cos\left(\frac{1}{s}\right) = 1 - \frac{1}{2!}\frac{1}{s^2} + \frac{1}{4!}\frac{1}{s^4} - \cdots$$

Hence, $$\frac{1}{s}\sin\left(\frac{1}{s}\right) = \frac{1}{s^2} - \frac{1}{3!}\frac{1}{s^4} + \frac{1}{5!}\frac{1}{s^6} - \cdots$$

$$\text{and } \frac{1}{s}\cos\left(\frac{1}{s}\right) = \frac{1}{s} - \frac{1}{2!}\frac{1}{s^3} + \frac{1}{4!}\frac{1}{s^5} - \cdots$$

$$\therefore \quad L^{-1}\left[\frac{1}{s}\sin\left(\frac{1}{s}\right)\right] = \frac{t^1}{1!} - \frac{1}{3!}\frac{t^3}{3!} + \frac{1}{5!}\frac{t^5}{5!} - \cdots$$

and $$L^{-1}\left[\frac{1}{s}\cos\left(\frac{1}{s}\right)\right] = 1 - \frac{1}{2!}\frac{t^2}{2!} + \frac{1}{4!}\frac{t^4}{4!} - \cdots$$

Thus, $$\boxed{L^{-1}\left[\frac{1}{s}\sin\left(\frac{1}{s}\right)\right] = t - \frac{t^3}{(3!)^2} + \frac{t^5}{(5!)^2} - \cdots}$$

and $$\boxed{L^{-1}\left[\frac{1}{s}\cos\left(\frac{1}{s}\right)\right] = 1 - \frac{t^2}{(2!)^2} + \frac{t^4}{(4!)^2} - \cdots}$$

ASSIGNMENT

Find the inverse Laplace transform of the following functions.

I

1. $\dfrac{s^2 - 3s + 4}{s^3}$

2. $\dfrac{(s+2)^3}{s^6}$

3. $\dfrac{1}{3s-2} + \dfrac{4}{5s+1} + \dfrac{1}{s\sqrt{s}}$

4. $\dfrac{2s-5}{4s^2+25} + \dfrac{4s-18}{9-s^2}$

5. $\dfrac{(24 - 30\sqrt{s})e^{-s}}{s^4}$

II

6. $\dfrac{s-3}{s^2-6s+13}$

7. $\dfrac{2s-3}{s^2-2s+5}$

8. $\dfrac{s+3}{4s^2+4s+9}$

9. $\dfrac{(3s+1)e^s}{(s+1)^4}$

10. $\dfrac{s\,e^{-2s}}{s^2+8s+16}$

III

11. $\dfrac{2s^2-6s+5}{(s-1)(s-2)(s-3)}$

12. $\dfrac{4s+3}{(s-1)^2(s+2)}$

13. $\dfrac{3s-1}{(s-3)(s^2+4)}$

14. $\dfrac{(s^2+6)e^{2s}}{(s^2+1)(s^2+4)}$

15. $\dfrac{s}{s^4+64}$

16. $\dfrac{s^2 e^{3s}}{s^4+4}$

IV

17. $\dfrac{e^{-3s}}{(s-4)^2}+\dfrac{e^{-5s}}{(s-2)^4}$

18. $\dfrac{s\,e^{-\pi s}}{s^2+16}+\dfrac{e^{-\pi s}}{s^2+9}$

19. $\dfrac{3e^{-3s}}{s}-\dfrac{e^{-s}}{s^2}$

20. $\dfrac{5e^{-3s}}{s}-\dfrac{e^{-s}}{s}$

V

21. $\log\left(1+\dfrac{a^2}{s^2}\right)$

22. $\log\left[\dfrac{s^2+1}{s(s+1)}\right]$

23. $\log\left[\dfrac{s^2+4}{(s-4)^2}\right]$

24. $\dfrac{s+2}{(s^2+4s+5)^2}$

25. $\tan^{-1}(a/s)$

ANSWERS

I

1. $1-3t+2t^2$

2. $\dfrac{1}{30}(15t^2+30t^3+15t^4+2t^5)$

3. $\dfrac{1}{3}e^{2t/3}+\dfrac{4}{5}e^{-t/5}+2\sqrt{t/\pi}$

4. $\frac{1}{2}\{\cos(5t/2)-\sin(5t/2)\}-4\cosh 3t+6\sinh 3t$

5. $\left\{4(t-1)^3-(16/\sqrt{\pi})(t-1)^{5/2}\right\}u(t-1)$

6. $e^{3t}\cos 2t$

7. $e^t/2\cdot(4\cos 2t-\sin 2t)$

8. $\frac{e^{-t/2}}{8\sqrt{2}}\left\{2\sqrt{2}\cos(\sqrt{2}\,t)+5\sin(\sqrt{2}\,t)\right\}$

9. $e^{-(t+1)}\left\{\frac{3(t+1)^2}{2}-\frac{(t+1)^3}{3}\right\}u(t+1)$

10. $e^{-4(t-2)}\{1-4(t-2)\}u(t-2)$

11. $\frac{e^t}{2}-e^{2t}+\frac{5}{2}e^{3t}$

12. $\frac{5}{9}e^t+\frac{7}{3}te^t-\frac{5}{9}e^{-2t}$

13. $\frac{8}{13}e^{3t}-\frac{8}{13}\cos 2t+\frac{15}{26}\sin 2t$

14. $\frac{1}{3}\{5\sin(t+2)-\sin 2(t+2)\}u(t+2)$

15. $\frac{\sin 2t\sinh 2t}{8}$

16. $\frac{1}{2}\{\sinh(t-3)\cos(t-3)+\cosh(t-3)\sin(t-3)\}u(t-3)$

17. $(t-3)e^{4(t-3)}u(t-3)+\frac{(t-5)^3e^{2(t-5)}u(t-5)}{6}$

18. $\left\{\cos 4t-\frac{\sin 3t}{3}\right\}u(t-\pi)$

19. $3\cdot u(t-3)-(t-1)u(t-1)$

20. $5u(t-3)-u(t-1)$

V 21. $\dfrac{2(1-\cos at)}{t}$ 22. $\dfrac{\qquad}{t}$

23. $\dfrac{2(e^{4t}-\cos 2t)}{t}$ 24. $\dfrac{t\,e^{-2t}\sin t}{2}$

25. $\dfrac{\sin at}{t}$

1.7 Convolution

Definition : The convolution of two functions $f(t)$ and $g(t)$ usua denoted by $f(t) * g(t)$ is defined in the form of an integral as follow

$$f(t)*g(t) = \int_{u=0}^{t} f(u)\,g(t-u)\,du$$

Property : $f(t)*g(t) = g(t)*f(t)$

That is to say that the convolution operation $*$ is a commutative.

Proof : We have from the definition of convolution,

$$f(t)*g(t) = \int_{u=0}^{t} f(u)\,g(t-u)\,du \qquad \ldots$$

Put, $t-u=v$ in (1). $\therefore\ -du = dv$ or $du = -dv$

If $u=0, v=t$; If $u=t, v=0$. Also $t-v=u$

Hence, $f(t)*g(t) = \int_{v=t}^{0} f(t-v)\,g(v)\,(-dv)$

ie., $f(t)*g(t) = \int_{v=0}^{t} f(t-v)\,g(v)\,dv$ or $\int_{v=0}^{t} g(v)\,f(t-v)\,dv$

Comparing the RHS with (1) we have,

$$f(t)*g(t) = g(t)*f(t)$$

This proves that the operation $*$ is commutative.

1.71 Convolution theorem

If $L^{-1}[\bar{f}(s)] = f(t)$ *and* $L^{-1}[\bar{g}(s)] = g(t)$ *then*

$$L^{-1}[\bar{f}(s)\cdot\bar{g}(s)] = \int_{u=0}^{t} f(u)g(t-u)\,du$$

Proof : We shall show that

$$L\left[\int_{u=0}^{t} f(u)g(t-u)\,du\right] = \bar{f}(s)\cdot\bar{g}(s)$$

We have LHS by the definition,

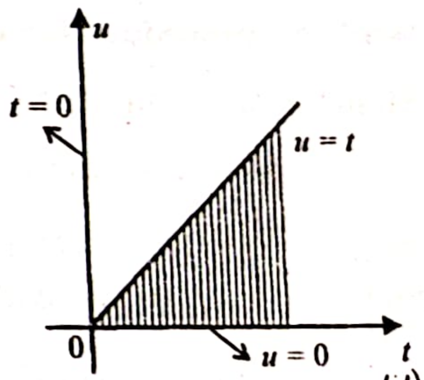
$$\text{LHS} = \int_{t=0}^{\infty} e^{-st}\left[\int_{u=0}^{t} f(u)g(t-u)\,du\right]dt$$

$$\text{LHS} = \int_{t=0}^{\infty}\int_{u=0}^{t} e^{-st} f(u)g(t-u)\,du\,dt \qquad \ldots (1)$$

We shall change the order of integration in respect of this double integral.
Existing region :

$t = 0$ to ∞ (Horizontal strip)

$u = 0$ to t (Vertical strip)



On changing the order :

$u = 0$ to ∞ (Vertical strip)

$t = u$ to ∞ (Horizontal strip)

On changing the order of integration, (1) becomes

$$\text{LHS} = \int_{u=0}^{\infty}\int_{t=u}^{\infty} e^{-st} f(u)g(t-u)\,dt\,du \qquad \ldots (2)$$

Now, let us put $t - u = v$ where u is fixed $\therefore dt = dv$

If $t = u$, $v = 0$; If $t = \infty$, $v = \infty$ and hence (2) becomes

$$\text{LHS} = \int_{u=0}^{\infty}\int_{v=0}^{\infty} e^{-s(u+v)} f(u)g(v)\,dv\,du$$

$$\text{LHS} = \left[\int_{u=0}^{\infty} e^{-su} f(u)\,du\right]\cdot\left[\int_{v=0}^{\infty} e^{-sv} g(v)\,dv\right]$$

$$\text{LHS} = \bar{f}(s)\cdot\bar{g}(s) = \text{RHS}$$

Hence we have proved that

$$L\left[\int_{u=0}^{t} f(u)g(t-u)\,du\right] = \bar{f}(s)\cdot\bar{g}(s)$$

Thus $L^{-1}[\bar{f}(s)\cdot\bar{g}(s)] = \int_{u=0}^{t} f(u)g(t-u)\,du$

This proves the convolution theorem.

Remarks :

1. *The integral in RHS is the convolution of $f(t)$ and $g(t)$ denoted by $f(t) * g(t)$. Thus the convolution theorem can be put in the form.*

$$L^{-1}[\bar{f}(s)\cdot\bar{g}(s)] = f(t)*g(t)$$

2. *Since we have proved that* $f(t)*g(t) = g(t)*f(t)$,

$$L^{-1}[\bar{f}(s)\cdot\bar{g}(s)] = \int_{u=0}^{t} f(u)g(t-u)\,du = \int_{u=0}^{t} f(t-u)g(u)\,du$$

The integral in either of the forms is called as the convolution integral.

1.72 Computation of the inverse transform by using convolution theorem

Working procedure for problems

Step-1 The given function is expressed as the product of two functions say $\bar{f}(s)$ and $\bar{g}(s)$.

Step-2 We find $L^{-1}[\bar{f}(s)] = f(t)$ and $L^{-1}[\bar{g}(s)] = g(t)$.

Step-3 We apply the convolution theorem in one of the form :

$$L^{-1}[\bar{f}(s)\cdot\bar{g}(s)] = \int_{u=0}^{t} f(u)g(t-u)\,du$$

Step-4 We evaluate the convolution integral to obtain the required inverse.

WORKED PROBLEMS

Using convolution theorem obtain the inverse Laplace transform of the following functions

108. $\dfrac{1}{s(s^2+a^2)}$ **[June 2016, 18]**

109. $\dfrac{s}{(s^2+a^2)^2}$ **[Dec 2016]**

110. $\dfrac{s^2}{(s^2+a^2)^2}$

111. $\dfrac{1}{(s^2+a^2)^2}$ **[June 2017]**

112. $\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}$

113. $\dfrac{1}{(s-1)(s^2+1)}$ **[Dec 2017]**

114. $\dfrac{1}{s^2(s+1)^2}$

115. $\dfrac{s+2}{(s^2+4s+5)^2}$

116. $\dfrac{1}{(s^2+4s+13)^2}$

117. $\dfrac{4s+5}{(s-1)^2(s+2)}$

☞ **Solutions**

[108] Let, $\bar{f}(s)=\dfrac{1}{s}\ ;\ \bar{g}(s)=\dfrac{1}{s^2+a^2}$

Taking inverse,

$$f(t)=L^{-1}\left[\frac{1}{s}\right]=1\ ;\ g(t)=L^{-1}\left[\frac{1}{s^2+a^2}\right]=\frac{\sin at}{a}$$

We have convolution theorem,

$$L^{-1}[\bar{f}(s)\cdot\bar{g}(s)]=\int_{u=0}^{t} f(u)g(t-u)\,du$$

$$\therefore \quad L^{-1}\left[\frac{1}{s(s^2+a^2)}\right]=\int_{u=0}^{t} 1\cdot\frac{\sin(at-au)}{a}\,du$$

$$=\left[\frac{\cos(at-au)}{a^2}\right]_{u=0}^{t}=\frac{1}{a^2}(1-\cos at)$$

Thus, $$\boxed{L^{-1}\left[\frac{1}{s(s^2+a^2)}\right]=\frac{1}{a^2}(1-\cos at)}$$

[109] Let, $\bar{f}(s)=\dfrac{1}{s^2+a^2}\ ;\ \bar{g}(s)=\dfrac{s}{s^2+a^2}$

$$\Rightarrow \quad f(t)=L^{-1}[\bar{f}(s)]=\frac{\sin at}{a}\ ;\ g(t)=L^{-1}[\bar{g}(s)]=\cos at$$

We have convolution theorem,

$$L^{-1}[\bar{f}(s)\cdot\bar{g}(s)]=\int_{u=0}^{t} f(u)g(t-u)\,du$$

$$\therefore \quad L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \int_{u=0}^{t} \frac{\sin au}{a} \cdot \cos(at-au)\, du$$

$$= \frac{1}{2a}\int_{u=0}^{t} [\sin(au+at-au)+\sin(au-at+au)]du$$

$$= \frac{1}{2a}\int_{u=0}^{t} [\sin at+\sin(2au-at)]du$$

$$= \frac{1}{2}\left\{\sin at[u]_{u=0}^{t} - \left[\frac{\cos(2au-at)}{2a}\right]_{u=0}^{t}\right\}$$

$$= \frac{1}{2}\left\{\sin at(t-0) - \frac{1}{2a}(\cos at - \cos at)\right\} = \frac{t\sin at}{2a}$$

Thus, $$\boxed{L^{-1} = \left[\frac{s}{(s^2+a^2)^2}\right] = \frac{t\sin at}{2a}}$$

[110] Let, $\bar{f}(s) = \dfrac{s}{s^2+a^2} = \bar{g}(s)$

$\Rightarrow \quad f(t) = L^{-1}[\bar{f}(s)] = \cos at = g(t)$

Now by applying convolution theorem we have,

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right] = \int_{u=0}^{t} \cos au\cos(at-au)\,du$$

$$= \frac{1}{2}\int_{u=0}^{t} [\cos(au+at-au)+\cos(au-at+au)]du$$

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right] = \frac{1}{2}\int_{u=0}^{t} [\cos at+\cos(2au-at)]du$$

$$= \frac{1}{2}\left\{\cos at[u]_0^t + \left[\frac{\sin(2au-at)}{2a}\right]_0^t\right\}$$

$$= \frac{1}{2}\left\{\cos at(t-0) + \frac{1}{2a}(\sin at+\sin at)\right\} = \frac{1}{2}\left\{t\cos at + \frac{\sin at}{a}\right\}$$

Thus, $$\boxed{L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right] = \frac{1}{2a}(at\cos at+\sin at)}$$

11] Let, $\bar{f}(s) = \dfrac{1}{s^2+a^2}$; $\bar{g}(s)$

$\Rightarrow \quad f(t) = L^{-1}[\bar{f}(s)] = \dfrac{\sin at}{a} = g(t)$

low by applying convolution theorem we have,

$$L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = \int_{u=0}^{t} \frac{\sin au}{a} \cdot \frac{\sin(at-au)}{a}\, du$$

$$= \frac{1}{2a^2}\int_{u=0}^{t} [\cos(au-at+au) - \cos(au+at-au)]\, du$$

$$= \frac{1}{2a^2}\int_{u=0}^{t} [\cos(2au-at) - \cos at]\, du$$

$$= \frac{1}{2a^2}\left\{\left[\frac{\sin(2au-at)}{2a}\right]_0^t - \cos at\,[u]_0^t\right\}$$

$$= \frac{1}{2a^2}\left\{\frac{1}{2a}(\sin at + \sin at) - \cos at \cdot t\right\}$$

Thus, $$\boxed{L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = \frac{1}{2a^3}(\sin at - at\cos at)}$$

[112] Let, $\bar{f}(s) = \dfrac{s}{s^2+a^2}$; $\bar{g}(s) = \dfrac{s}{s^2+b^2}$

$\Rightarrow \quad f(t) = L^{-1}[\bar{f}(s)] = \cos at$; $g(t) = L^{-1}[\bar{g}(s)] = \cos bt$

Now by applying convolution theorem we have,

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = \int_{u=0}^{t} \cos au \cdot \cos(bt-bu)\, du$$

Note : *The integral can be evaluated as in problem* [110].

Thus, $$\boxed{L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = \frac{a\sin at - b\sin bt}{a^2-b^2},\ a \neq b}$$

[113] Let, $\bar{f}(s) = \dfrac{1}{s-1}$; $\bar{g}(s) = \dfrac{1}{s^2+1}$

$\Rightarrow$ $f(t) = L^{-1}[\bar{f}(s)] = e^t$; $g(t) = L^{-1}[\bar{g}(s)] = \sin t$

Now by applying convolution theorem we have,

$$L^{-1}\left[\frac{1}{(s-1)(s^2+1)}\right] = \int_{u=0}^{t} e^u \cdot \sin(t-u)\,du$$

But, $\int e^{ax}\sin(bx+c)\,dx = \dfrac{e^{ax}}{a^2+b^2}[a\sin(bx+c) - b\cos(bx+c)]$

$$L^{-1}\left[\frac{1}{(s-1)(s^2+1)}\right] = \left[\frac{e^u}{1+1}\{\sin(t-u)+\cos(t-u)\}\right]_0^t$$

$$= \frac{1}{2}\{e^t(0+1) - 1(\sin t + \cos t)\}$$

Thus, $$\boxed{L^{-1}\left[\frac{1}{(s-1)(s^2+1)}\right] = \frac{1}{2}(e^t - \sin t - \cos t)}$$

Note : Similar problem

Find $L^{-1}\left[\dfrac{1}{(s+1)(s^2+1)}\right]$ *using Convolution theorem.* **[June 2018]**

[114] Let, $\bar{f}(s) = \dfrac{1}{s^2}$; $\bar{g}(s) = \dfrac{1}{(s+1)^2}$

$\Rightarrow$ $f(t) = L^{-1}[\bar{f}(s)] = t$; $g(t) = L^{-1}[\bar{g}(s)] = e^{-t}\,t.$

Now by applying convolution theorem we have,

$$L^{-1}\left[\frac{1}{s^2(s+1)^2}\right] = \int_{u=0}^{t} ue^{-(t-u)}(t-u)\,du = e^{-t}\int_{u=0}^{t}(tu-u^2)e^u\,du$$

We shall apply Bernoulli's rule to evaluate the integral.

ie., $= e^{-t}[(tu-u^2)e^u - (t-2u)e^u + (-2)e^u]_{u=0}^t$

ie., $= e^{-t}[0-\{-te^{t}-t\}-2\{e^{t}-1\}] = t+te^{-t}-2+2e^{-t}$

Thus, $$\boxed{L^{-1}\left[\frac{1}{s^2(s+1)^2}\right] = 2(e^{-t}-1)+t(1+e^{-t})}$$

[115] Let, $\bar{f}(s) = \dfrac{s+2}{s^2+4s+5}$; $\bar{g}(s) = \dfrac{1}{s^2+4s+5}$

$$\Rightarrow \quad f(t) = L^{-1}\left[\frac{s+2}{(s+2)^2+1}\right] \; ; \; g(t) = L^{-1}\left[\frac{1}{(s+2)^2+1}\right]$$

$$f(t) = e^{-2t}L^{-1}\left[\frac{s}{s^2+1}\right] \; ; \; g(t) = e^{-2t}L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$\therefore \quad f(t) = e^{-2t}\cos t \; ; \; g(t) = e^{-2t}\sin t$$

Now by applying convolution theorem we have,

$$L^{-1}\left[\frac{s+2}{(s^2+4s+5)^2}\right] = \int_{u=0}^{t} e^{-2u}\cos u\, e^{-2(t-u)}\sin(t-u)\,du$$

$$= e^{-2t}\int_{u=0}^{t}\sin(t-u)\cos u\,du$$

$$= \frac{e^{-2t}}{2}\int_{u=0}^{t}[\sin(t-u+u)+\sin(t-u-u)]\,du$$

$$= \frac{e^{-2t}}{2}\int_{u=0}^{t}[\sin t+\sin(t-2u)]\,du$$

$$= \frac{e^{-2t}}{2}\left\{\sin t\,[u]_0^t + \frac{\cos(t-2u)}{2}\Big|_{u=0}^{t}\right\}$$

$$= \frac{e^{-2t}}{2}\left\{t\sin t+\frac{1}{2}(\cos t-\cos t)\right\}$$

Thus, $$\boxed{L^{-1}\left[\frac{s+2}{(s^2+4s+5)^2}\right] = \frac{e^{-2t}\,t\sin t}{2}}$$

[116] Let, $\bar{f}(s) = \dfrac{1}{s^2+4s+13} = \bar{g}(s)$

$$\Rightarrow \quad f(t) = L^{-1}\left[\frac{1}{(s+2)^2+3^2}\right] = g(t)$$

ie., $$f(t) = e^{-2t} L^{-1}\left[\frac{1}{s^2+3^2}\right] = \frac{e^{-2t}\sin 3t}{3} = g(t)$$

Now by applying convolution theorem we have,

$$L^{-1}\left[\frac{1}{(s^2+4s+13)^2}\right] = \int_{u=0}^{t} \frac{e^{-2u}\sin 3u}{3}\cdot\frac{e^{-2(t-u)}\sin(3t-3u)}{3}\,du$$

$$= \frac{e^{-2t}}{9}\int_{u=0}^{t} \sin 3u \cdot \sin(3t-3u)\,du$$

$$= \frac{e^{-2t}}{18}\int_{u=0}^{t} [\cos(3u-3t+3u) - \cos(3u+3t-3u)]\,du$$

$$= \frac{e^{-2t}}{18}\int_{u=0}^{t} [\cos(6u-3t) - \cos 3t]\,du$$

$$= \frac{e^{-2t}}{18}\left\{\left[\frac{\sin(6u-3t)}{6}\right]_0^t - \cos 3t\,[u]_0^t\right\}$$

$$= \frac{e^{-2t}}{18}\left\{\frac{\sin 3t + \sin 3t}{6} - \cos 3t \cdot t\right\}$$

Thus, $$\boxed{L^{-1}\left[\frac{1}{(s^2+4s+13)^2}\right] = \frac{e^{-2t}}{54}(\sin 3t - 3t\cos 3t)}$$

[117] Let, $\bar{f}(s) = \dfrac{1}{s+2}$; $\bar{g}(s) = \dfrac{4s+5}{(s-1)^2}$

$$\Rightarrow \quad f(t) = L^{-1}[\bar{f}(s)] = e^{-2t}$$

Also, $$g(t) = L^{-1}[\bar{g}(s)] = L^{-1}\left[\frac{4s+5}{(s-1)^2}\right] = L^{-1}\left[\frac{4(s-1)+9}{(s-1)^2}\right]$$

$$g(t) = e^t L^{-1}\left[\frac{4s+9}{s^2}\right] = e^t(4+9t)$$

Now by applying convolution theorem we have,

$$L^{-1}\left[\frac{1}{s+2}\cdot\frac{4s+5}{(s-1)^2}\right] = \int_{u=0}^{t} e^{-2u}\cdot e^{(t-u)}[4+9(t-u)]\,du$$

$$= e^t \int_{u=0}^{t} e^{-3u}(4+9t-9u)\,du$$

$$= e^t \int_{u=0}^{t} (4+9t-9u)\,e^{-3u}\,du$$

Integrating RHS by parts we get,

$$L^{-1}\left[\frac{4s+5}{(s-1)^2(s+2)}\right] = e^t\left\{(4+9t-9u)\frac{e^{-3u}}{-3} - (-9)\frac{e^{-3u}}{9}\right\}_{u=0}^{t}$$

$$= e^t\left\{4\frac{e^{-3t}}{-3} - \frac{(4+9t)}{-3} + e^{-3t} - 1\right\}$$

$$= e^t\left\{\frac{1}{3} - \frac{1}{3}e^{-3t} + 3t\right\}$$

Thus, $$\boxed{L^{-1}\left[\frac{4s+5}{(s-1)^2(s+2)}\right] = \frac{1}{3}e^t - \frac{1}{3}e^{-2t} + 3t\,e^t}$$

1.8 Solution of linear differential equations using Laplace transforms (Initial value problems)

- **Laplace transform of the derivatives.**

We derive an expression for $L[y'(t)]$ and hence deduce the expressions for $L[y''(t)]$, $L[y'''(t)]$...

$$L[y'(t)] = \int_0^{\infty} e^{-st}\,y'(t)\,dt$$

Integrating by parts we have,

$$L[y'(t)] = \left[e^{-st}y(t)\right]_{t=0}^{\infty} - \int_0^{\infty} y(t)\cdot e^{-st}(-s)\,dt$$

$$L[y'(t)] = [0 - 1 \cdot y(0)] + s\int_0^\infty e^{-st}\, y(t)\,dt$$

$$L[y'(t)] = -y(0) + sL[\ (t)]$$

$$\therefore \quad L[y'(t)] = sL[\ y(t)] - y(0) \qquad \ldots$$

Now, $L[y''(t)] = L[\{y'(t)\}']$ and applying (1) we have,

$$= sL[y'(t)] - y'(0)$$

$$= s\{sL[y(t)] - y(0)\} - y'(0) \text{ by using (1).}$$

$$\therefore \quad L[y''(t)] = s^2 L[y(t)] - sy(0) - y'(0) \qquad \ldots$$

Also, $L[y'''(t)] = s^3 L[y(t)] - s^2 y(0) - sy'(0) - y''(0)$ ...

and so on.

We have already said that a differential equation with a set of initial conditi is called an initial value problem. However if the boundary conditions given the problem is called a boundary value problem.

Laplace transform serves as a useful tool in solving such problems.

Working procedure for problems

Step-1 The given differential equation is expressed in the notation : $y'(t), y''(t), y'''(t) \ldots$ for the derivatives.

Step-2 We take Laplace transform on both sides of the given equation.

Step-3 We use the expressions for $L[y'(t)],\ L[y''(t)] \ldots$

Step-4 We substitute the given initial conditions and simplify to obtain $L[y(t)]$ as a function of s.

Step-5 We find the inverse to obtain $y(t)$.

WORKED PROBLEMS

[118] *Solve by using Laplace transforms :* $\dfrac{d^2y}{dt^2} + k^2 y = 0$ *given that* $y(0) = 2,\ y'(0) = 0$. **[Dec 2018**

☞ The given equation is $y''(t)+k^2y(t)=0$.

Taking Laplace transform on both sides we have,

$$L[y''(t)]+k^2L[y(t)]=L(0)$$

ie., $$\{s^2L[y(t)]-s\,y(0)-y'(0)\}+k^2L[y(t)]=0$$

Using the given initial conditions we obtain,

$$(s^2+k^2)L[y(t)]-2s=0 \text{ or } L[y(t)]=\frac{2s}{s^2+k^2}$$

$$\therefore \quad y(t)=2L^{-1}\left[\frac{s}{s^2+k^2}\right]=2\cos kt.$$

Thus, $$\boxed{y(t)=2\cos kt}$$

[119] *Solve* $y'''+2y''-y'-2y=0$ *given* $y(0)=y'(0)=0$ *and* $y''(0)=6$ *by using Laplace transform method.* **[Dec 2016]**

☞ Taking Laplace transform on both sides of the given equation,

$$L[y'''(t)]+2L[y''(t)]-L[y'(t)]-2L[y(t)]=L(0)$$

ie., $$\{s^3L[y(t)]-s^2y(0)-s\,y'(0)-y''(0)\}+2\{s^2L[y(t)]-s\,y(0)-y'(0)\}$$
$$-\{sL[y(t)]-y(0)\}-2L[y(t)]=0$$

Using the given initial conditions we obtain,

$$L[y(t)]\{s^3+2s^2-s-2\}-6=0$$

ie., $$L[y(t)]\{s^2(s+2)-1(s+2)\}=6$$

or $$L[y(t)]\{(s+2)(s^2-1)\}=6$$

or $$L[y(t)]=\frac{6}{(s+2)(s-1)(s+1)}$$

$$\therefore \quad y(t)=L^{-1}\left\{\frac{6}{(s+2)(s-1)(s+1)}\right\}$$

Let, $$\frac{6}{(s+2)(s-1)(s+1)}=\frac{A}{s+2}+\frac{B}{s-1}+\frac{C}{s+1}$$

or $$6=A(s-1)(s+1)+B(s+2)(s+1)+C(s+2)(s-1)$$

Put $s = -2$: $6 = A(-3)(-1)$ $\therefore A = 2$

Put $s = 1$: $6 = B(3)(2)$ $\therefore B = 1$

Put $s = -1$: $6 = C(1)(-2)$ $\therefore C = -3$

Hence, $$\frac{6}{(s+2)(s-1)(s+1)} = \frac{2}{s+2} + \frac{1}{s-1} + \frac{-3}{s+1}$$

$$\therefore \quad L^{-1}\left\{\frac{6}{(s+2)(s-1)(s+1)}\right\} = 2L^{-1}\left(\frac{1}{s+2}\right) + L^{-1}\left(\frac{1}{s-1}\right) - 3L^{-1}\left(\frac{1}{s+\right.$$

Thus, $$\boxed{y(t) = 2e^{-2t} + e^{t} - 3e^{-t}}$$

[120] *Solve* $\dfrac{d^4y}{dt^4} - k^4 y = 0$ *given* $y(0) = 1$ *and* $y'(0) = 0 = y''(0) = y'''($

☞ The given equation is $y^{(4)}(t) - k^4 y(t) = 0$.

Taking Laplace transform on both sides we have,

$$L[y^{(4)}(t)] - k^4 L[y(t)] = L(0)$$

ie., $\{s^4 L[y(t)] - s^3 y(0) - s^2 y'(0) - sy''(0) - y'''(0)\} - k^4 L[y(t)] =$

Using the given initial conditions we obtain,

$$L[y(t)][s^4 - k^4] - s^3 \cdot 1 = 0 \text{ or } L[y(t)] = \frac{s^3}{s^4 - k^4}$$

$$\therefore \quad y(t) = L^{-1}\left[\frac{s^3}{s^4 - k^4}\right]$$

Now, $s^4 - k^4 = (s^2 - k^2)(s^2 + k^2) = (s-k)(s+k)(s^2+k^2)$

Let, $$\frac{s^3}{(s-k)(s+k)(s^2+k^2)} = \frac{A}{(s-k)} + \frac{B}{(s+k)} + \frac{Cs+D}{s^2+k^2}$$

ie., $s^3 = A(s+k)(s^2+k^2) + B(s-k)(s^2+k^2) + (Cs+D)(s-k)(s+k)$ ...(1

Put $s = k$: $k^3 = A(2k)(2k^2)$ $\therefore A = 1/4$

Put $s = -k$: $-k^3 = B(-2k)(2k^2)$ $\therefore B = 1/4$

Put $s = 0$: $0 = (1/4)(k)(k^2) + (1/4)(-k)(k^2) + D(-k^2$

ie., $0 = (k^3/4) - (k^3/4) + D(-k^2)$ $\therefore D = 0$

Comparing the coefficient of s^3 on both sides of (1) we have,

$$1 = A + B + C \quad ie., \quad 1 = 1/4 + 1/4 + C \qquad \therefore\ C = 1/2$$

Hence, $$\frac{s^3}{(s-k)(s+k)(s^2+k^2)} = \frac{1}{4}\frac{1}{s-k} + \frac{1}{4}\frac{1}{s+k} + \frac{1}{2}\frac{s}{s^2+k^2}$$

$$\therefore \quad L^{-1}\left[\frac{s^3}{s^4 - k^4}\right] = \frac{1}{4}L^{-1}\left[\frac{1}{s-k}\right] + \frac{1}{4}L^{-1}\left[\frac{1}{s+k}\right] + \frac{1}{2}L^{-1}\left[\frac{s}{s^2+k^2}\right]$$

ie., $$y(t) = \frac{1}{4}e^{kt} + \frac{1}{4}e^{-kt} + \frac{1}{2}\cos kt = \frac{1}{4}(e^{kt} + e^{-kt}) + \frac{1}{2}\cos kt$$

ie., $$y(t) = \frac{1}{4}(2\cosh kt) + \frac{1}{2}\cos kt.$$

Thus, $$\boxed{y(t) = \frac{1}{2}(\cosh kt + \cos kt)}$$

[121] *Solve the following initial value problem by using Laplace transforms :*

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = e^{-t} \ ; \ y(0) = 0,\ y'(0) = 0$$ **[June 2016]**

☞ The given equation is $y''(t) + 4y'(t) + 4y(t) = e^{-t}$

Taking Laplace transform on both sides we have,

$$L[y''(t)] + 4L[y'(t)] + 4L[y(t)] = L(e^{-t})$$

ie., $$\{s^2 L[y(t)] - s\,y(0) - y'(0)\} + 4\{s L[y(t)] - y(0)\} + 4L[y(t)] = \frac{1}{s+1}$$

Using the given initial conditions we obtain,

$$L[y(t)]\{s^2 + 4s + 4\} = \frac{1}{s+1} \quad \text{or} \quad L[y(t)] = \frac{1}{(s+1)(s+2)^2}$$

$$\therefore \quad y(t) = L^{-1}\left[\frac{1}{(s+1)(s+2)^2}\right]$$

Let, $$\frac{1}{(s+1)(s+2)^2} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^2}$$

Multiplying with $(s+1)(s+2)^2$ we obtain

$$1 = A(s+2)^2 + B(s+1)(s+2) + C(s+1)$$

Putting $s=-1$ we get $A = 1$
Putting $s=-2$ we get $C = -1$
Putting $s=0$ we have $1 = 1(4) + B(2) - 1(1) \therefore B = -1$

Hence, $\dfrac{1}{(s+1)(s+2)^2} = \dfrac{1}{s+1} + \dfrac{-1}{s+2} + \dfrac{-1}{(s+2)^2}$

$$\therefore \quad L^{-1}\left[\frac{1}{(s+1)(s+2)^2}\right] = L^{-1}\left[\frac{1}{s+1}\right] - L^{-1}\left[\frac{1}{s+2}\right] - L^{-1}\left[\frac{1}{(s+2)^2}\right]$$

ie., $y(t) = e^{-t} - e^{-2t} - e^{-2t} L^{-1}\left(\dfrac{1}{s^2}\right)$

Thus, $\boxed{y(t) = e^{-t} - e^{-2t} - e^{-2t}\, t = e^{-t} - (1+t)e^{-2t}}$

[122] *Employ Laplace transform to solve the equation :* $y'' + 5y' + 6y = 5e^{2x}$, $y(0) = 2,\ y'(0) = 1$.

☞ Taking Laplace transform on both sides of the given equation we have,

$$L[y''(x)] + 5L[y'(x)] + 6L[y(x)] = 5L(e^{2x}) .$$

ie., $\{s^2 L[y(x)] - s\,y(0) - y'(0)\} + 5\{s L[y(x)] - y(0)\} + 6L[y(x)] = \dfrac{5}{s-2}$

Using the given initial conditions we obtain,

$$(s^2+5s+6)L[y(x)] - 2s - 1 - 10 = \frac{5}{s-2}$$

ie., $(s^2+5s+6)L[y(x)] = (2s+11) + \dfrac{5}{s-2}$

$$L[y(x)] = \frac{(2s+11)(s-2)+5}{(s-2)(s^2+5s+6)} = \frac{2s^2+7s-17}{(s-2)(s+2)(s+3)}$$

$$\therefore \quad y(x) = L^{-1}\left[\frac{2s^2+7s-17}{(s-2)(s+2)(s+3)}\right]$$

Let, $$\frac{2s^2+7s-17}{(s-2)(s+2)(s+3)} = \frac{A}{s-2}+\frac{B}{s+2}+\frac{C}{s+3}$$

or $$2s^2+7s-17 = A(s+2)(s+3)+B(s-2)(s+3)+C(s-2)(s+2)$$

Put $s = 2$: $5 = A(4)(5)$ $\therefore A = 1/4$

Put $s = -2$: $-23 = B(-4)(1)$ $\therefore B = 23/4$

Put $s = -3$: $-20 = C(-5)(-1)$ $\therefore C = -4$

Hence, $$L^{-1}\left[\frac{2s^2+7s-17}{(s-2)(s+2)(s+3)}\right]$$

$$= \frac{1}{4}L^{-1}\left[\frac{1}{s-2}\right]+\frac{23}{4}L^{-1}\left[\frac{1}{s+2}\right]-4L^{-1}\left[\frac{1}{s+3}\right]$$

Thus, $$\boxed{y(x) = \frac{1}{4}e^{2x}+\frac{23}{4}e^{-2x}-4e^{-3x}}$$

[123] *Using Laplace transform technique solve* $x''-2x'+x = e^{2t}$ *with* $x(0) = 0,\ x'(0) = -1$ **[June 2017]**

☞ Taking Laplace transform on both sides of the given equation we have,

$$L[x''(t)]-2L[x'(t)]+L[x(t)] = L(e^{2t})$$

ie., $$\{s^2L[x(t)]-s\,x(0)-x'(0)\}-2\{sL[x(t)]-x(0)\}+L[x(t)] = \frac{1}{s-2}$$

Using the given initial conditions we obtain,

$$\{s^2-2s+1\}L[x(t)]+1 = \frac{1}{s-2}$$

ie., $$(s-1)^2L[x(t)] = \frac{1}{s-2}-1 = \frac{3-s}{s-2}$$

or $$L[x(t)] = \frac{3-s}{(s-1)^2(s-2)}$$

$\therefore$ $$x(t) = L^{-1}\left[\frac{3-s}{(s-1)^2(s-2)}\right]$$

Let, $\dfrac{3-s}{(s-1)^2(s-2)} = \dfrac{A}{s-1} + \dfrac{B}{(s-1)^2} + \dfrac{C}{s-2}$

Mulitplying by $(s-1)^2(s-2)$ we get,

$$3-s = A(s-1)(s-2) + B(s-2) + C(s-1)^2 \quad \dots(1)$$

Put $s = 1$: $2 = B(-1)$ $\therefore B = -2$

Put $s = 2$: $1 = C(1)$ $\therefore C = 1$

Equating the coefficient of s^2 on both sides of (1) we get,

$$0 = A + C. \quad \therefore A = -1$$

Hence $L^{-1}\left[\dfrac{3-s}{(s-1)^2(s-2)}\right] = -L^{-1}\left[\dfrac{1}{s-1}\right] - 2L^{-1}\left[\dfrac{1}{(s-1)^2}\right] + L^{-1}\left[\dfrac{1}{s-2}\right]$

Thus, $\boxed{x(t) = -e^t - 2e^t \cdot t + e^{2t} = e^{2t} - (1+2t)e^t}$

[124] *Solve the DE* $y'' + 4y' + 3y = e^{-t}$ *with* $y(0) = 1 = y'(0)$ *using Laplace transforms.*

☞ Taking Laplace transform on both sides of the given equation we have,

$$L[y''(t)] + 4L[y'(t)] + 3L[y(t)] = L(e^{-t})$$

ie., $\{s^2 L[y(t)] - s\,y(0) - y'(0)\} + 4\{s L[y(t)] - y(0)\} + 3L[y(t)] = \dfrac{1}{s+1}$

Using the given initial conditions we obtain,

$$(s^2 + 4s + 3)L[y(t)] - s - 1 - 4 = \frac{1}{s+1}$$

ie., $(s^2 + 4s + 3)L[y(t)] = (s+5) + \dfrac{1}{(s+1)}$

ie., $(s+1)(s+3)L[y(t)] = \dfrac{s^2 + 6s + 6}{s+1}$

or $L[y(t)] = \dfrac{s^2 + 6s + 6}{(s+1)^2(s+3)}$

$\therefore$ $y(t) = L^{-1}\left[\dfrac{s^2 + 6s + 6}{(s+1)^2(s+3)}\right]$

Let $\dfrac{s^2+6s+6}{(s+1)^2(s+3)} = \dfrac{A}{s+1} + \dfrac{B}{(s+1)^2} + \dfrac{C}{s+3}$

Multiplying by $(s+1)^2(s+3)$ we get,

$$s^2+6s+6 = A(s+1)(s+3)+B(s+3)+C(s+1)^2 \qquad \ldots (1)$$

Put $s=-1$: $1 = B(2)$ $\therefore\ B = 1/2$

Put $s=-3$: $-3 = C(4)$ $\therefore\ C = -3/4$

Equating the coefficient of s^2 on both sides of (1) we get,

$$1 = A+C \quad \therefore\ A = 7/4$$

Hence, $L^{-1}\left[\dfrac{s^2+6s+6}{(s+1)^2(s+3)}\right] = \dfrac{7}{4}L^{-1}\left[\dfrac{1}{s+1}\right]+\dfrac{1}{2}L^{-1}\left[\dfrac{1}{(s+1)^2}\right]-\dfrac{3}{4}L^{-1}\left[\dfrac{1}{s+3}\right]$

Thus, $$\boxed{y(t) = \frac{7}{4}e^{-t}+\frac{1}{2}e^{-t}\cdot t-\frac{3}{4}e^{-3t}}$$

[125] *Solve by using Laplace transforms* $\dfrac{d^2x}{dt^2}+2\dfrac{dx}{dt}+x = 3t\,e^{-t}$ *given that* $x=4,\ \dfrac{dx}{dt}=2$ *when* $t=0$.

☞ The given equation is $x''(t)+2x'(t)+x(t) = 3t\,e^{-t}$

Initial conditions are $x(0)=4,\ x'(0)=2$.

Taking Laplace transform on both sides of the equation we have,

$$L[x''(t)]+2L[x'(t)]+L[x(t)] = 3L(e^{-t}\cdot t)$$

ie., $\{s^2L[x(t)]-s\,x(0)-x'(0)\}+2\{sL[x(t)]-x(0)\}+L[x(t)] = \dfrac{3}{(s+1)^2}$

Using the given initial conditions we obtain,

$$(s^2+2s+1)L[x(t)]-4s-2-8 = \frac{3}{(s+1)^2}$$

ie., $$(s+1)^2L[x(t)] = (4s+10)+\frac{3}{(s+1)^2}$$

or $\quad L[x(t)] = \dfrac{4s+10}{(s+1)^2} + \dfrac{3}{(s+1)^4}$

$$\therefore \quad x(t) = L^{-1}\left[\frac{4(s+1)+6}{(s+1)^2}\right] + L^{-1}\left[\frac{3}{(s+1)^4}\right]$$

$$= e^{-t}L^{-1}\left[\frac{4s+6}{s^2}\right] + 3e^{-t}L^{-1}\left[\frac{1}{s^4}\right]$$

$$x(t) = e^{-t}\left\{4L^{-1}\left(\frac{1}{s}\right) + 6L^{-1}\left(\frac{1}{s^2}\right) + 3L^{-1}\left(\frac{1}{s^4}\right)\right\}$$

$$= e^{-t}\left(4 + 6t + 3\cdot t^3/6\right)$$

Thus, $\quad \boxed{x(t) = e^{-t}\left(4 + 6t + t^3/2\right)}$

Note : Similar Problem

Solve using Laplace transform method, $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} + y = t\,e^{-t}$ *with* $y(0) =$ $y'(0) = -2$.

[June 20

[126] *Solve by using Laplace transforms* $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} + 2y = 5\sin t$ *given that* $y(0) = 0 = y'(0)$.

☞ The given equation is $y''(t) + 2y'(t) + 2y(t) = 5\sin t$

Taking Laplace transform on both sides we have,

$$L[y''(t)] + 2L[y'(t)] + 2L[y(t)] = 5L(\sin t).$$

ie., $\{s^2L[y(t)] - s\,y(0) - y'(0)\} + 2\{s\,L[y(t)] - y(0)\} + 2L[y(t)] = \dfrac{5}{s^2+}$

Using the given initial conditions we obtain,

$$L[y(t)](s^2+2s+2) = \frac{5}{s^2+1} \quad \text{or} \quad L[y(t)] = \frac{5}{(s^2+1)(s^2+2s+2}$$

$$\therefore \quad y(t) = L^{-1}\left[\frac{5}{(s^2+1)(s^2+2s+2)}\right]$$

Let, $\dfrac{5}{(s^2+1)(s^2+2s+2)} = \dfrac{As+B}{s^2+1} + \dfrac{Cs+D}{s^2+2s+2}$

ie., $\quad 5 = (As+B)(s^2+2s+2)+(Cs+D)(s^2+1)$

ie., $\quad 5 = (A+C)s^3+(2A+B+D)s^2+(2A+2B+C)s+(2B+D)$

Comparing the coefficients on both sides, we get,

$$A+C=0\ ;\ 2A+B+D=0\ ;\ 2A+2B+C=0\ ;\ 2B+D=5$$

Solving these simultaneously we obtain,

$$A=-2,\ B=1,\ C=2,\ D=3$$

Hence, $\dfrac{5}{(s^2+1)(s^2+2s+2)} = \dfrac{-2s+1}{s^2+1} + \dfrac{2s+3}{s^2+2s+2}$

$$\therefore \quad L^{-1}\left[\frac{5}{(s^2+1)(s^2+2s+2)}\right]$$

$$y(t) = -2L^{-1}\left(\frac{s}{s^2+1}\right)+L^{-1}\left(\frac{1}{s^2+1}\right)+L^{-1}\left(\frac{2s+3}{s^2+2s+2}\right)$$

ie., $\quad y(t) = -2\cos t+\sin t+L^{-1}\left\{\dfrac{2(s+1)+1}{(s+1)^2+1}\right\}$

$$= -2\cos t+\sin t+e^{-t}L^{-1}\left\{\frac{2s+1}{s^2+1}\right\}$$

$$= -2\cos t+\sin t+e^{-t}\left[2L^{-1}\left(\frac{s}{s^2+1}\right)+L^{-1}\left(\frac{1}{s^2+1}\right)\right]$$

Thus, $\quad \boxed{y(t) = -2\cos t+\sin t+e^{-t}(2\cos t+\sin t)}$

[127] *Solve* $y''+6y'+9y = 12t^2e^{-3t}$ *subject to the conditions,*

$y(0)=0=y'(0)$ *by using the Laplace transforms.* **[Dec. 2017]**

☞ The given equation is $y''(t)+6y'(t)+9y(t) = 12t^2e^{-3t}$

Taking Laplace transform on both sides we have,

$$L[y''(t)]+6L[y'(t)]+9L[y(t)] = 12\,L[e^{-3t}\,t^2]$$

ie., $\{s^2L[y(t)]-s\,y(0)-y'(0)\}+6\{s\,L[y(t)]-y(0)\}+9L[y(t)]=\frac{12\cdot 2}{(s+3}$

Using the given initial conditions we obtain,

$$(s^2+6s+9)L[y(t)] = \frac{24}{(s+3)^3} \quad \text{or} \quad L[y(t)] = \frac{24}{(s+3)^5}$$

$$\therefore \quad y(t) = L^{-1}\left[\frac{24}{(s+3)^5}\right]$$

$$y(t) = 24\,e^{-3t}\,L^{-1}\left(\frac{1}{s^5}\right) = 24\,e^{-3t}\,\frac{t^4}{4!}$$

Thus, $\boxed{y(t) = e^{-3t}\,t^4}$

[128] *Solve by using Laplace transform method* $y''(t)+y(t) = H(t-1)$ *gi*
$y(0) = 0$ *and* $y'(0) = 1$.

☞ Taking Laplace transform on both sides of the given equation we ha
$L[y''(t)]+L[y(t)] = L[H(t-1)]$; $H(t-1)$ is the unit step functi

ie., $\{s^2L[y(t)]-s\,y(0)-y'(0)\}+L[y(t)] = \frac{e^{-s}}{s}$

Using the given initial conditions we obtain,

$$(s^2+1)L[y(t)]-1 = \frac{e^{-s}}{s} \quad \text{or} \quad (s^2+1)L[y(t)] = 1+\frac{e^{-s}}{s}.$$

$$\therefore \quad L[y(t)] = \frac{1}{s^2+1}+\frac{e^{-s}}{s(s^2+1)}$$

$$\Rightarrow \quad y(t) = L^{-1}\left[\frac{1}{s^2+1}\right]+L^{-1}\left[\frac{e^{-s}}{s(s^2+1)}\right]$$

ie., $$y(t) = \sin t + L^{-1}\left[\frac{e^{-s}}{s(s^2+1)}\right] \quad \ldots (1)$$

In respect of the second term, let $\bar{f}(s) = \dfrac{1}{s(s^2+1)}$

Also, $\bar{f}(s) = \dfrac{1}{s(s^2+1)} = \dfrac{1}{s} - \dfrac{s}{s^2+1}$ by partial fractions.

Now, $L^{-1}[\bar{f}(s)] = L^{-1}\left[\dfrac{1}{s}\right] - L^{-1}\left[\dfrac{s}{s^2+1}\right]$

ie., $f(t) = 1 - \cos t$

We also have, $L^{-1}[e^{-s}\bar{f}(s)] = f(t-1)H(t-1)$

ie., $$L^{-1}\left[\frac{e^{-s}}{s(s^2+1)}\right] = [1-\cos(t-1)]H(t-1) \quad \ldots (2)$$

We shall use (2) in (1).

Thus, $$\boxed{y(t) = \sin t + [1-\cos(t-1)]H(t-1)}$$

[129] *Solve the following boundary value problem by using Laplace transforms.*

$$y''(t) + y(t) = 0 \; ; \; y(0) = 2, \; y(\pi/2) = 1.$$

☞ Taking Laplace transform on both sides of the given equation we have,

$$L[y''(t)] + L[y(t)] = L(0)$$

ie., $$\{s^2 L[y(t)] - s\,y(0) - y'(0)\} + L[y(t)] = 0 \quad \ldots (1)$$

Let us assume $y'(0) = a$, where a is a constant to be found later and we have $y(0) = 2$ by data.

Hence (1) becomes,

$$(s^2+1)L[y(t)] - 2s - a = 0$$

ie., $$(s^2+1)L[y(t)] = 2s + a \quad \text{or} \quad L[y(t)] = \frac{2s+a}{s^2+1}$$

$$\therefore \quad y(t) = 2L^{-1}\left[\frac{s}{s^2+1}\right] + aL^{-1}\left[\frac{1}{s^2+1}\right]$$

ie., $\quad y(t) = 2\cos t + a\sin t$ ...(2)

Now we shall use the condition $y(\pi/2) = 1$

Hence (2) becomes $y(\pi/2) = 2\cos(\pi/2) + a\sin(\pi/2)$

ie., $\quad 1 = 0 + a \quad \therefore \quad a = 1$

Thus, $\boxed{y(t) = 2\cos t + \sin t}$

[130] *Using Laplace transform method solve,* $\dfrac{d^3y}{dt^3} - 3\dfrac{d^2y}{dt^2} + 3\dfrac{dy}{dt} - y = t^2e^t$

given $y(0) = 1,\ y'(0) = 0,\ y''(0) = -2.$

☞ The given equation is

$$y'''(t) - 3y''(t) + 3y'(t) - y(t) = t^2e^t$$

Taking Laplace transform on both sides we have,

$$L[y'''(t)] - 3L[y''(t)] + 3L[y'(t)] - L[y(t)] = L(e^tt^2)$$

ie., $\{s^3L[y(t)] - s^2y(0) - sy'(0) - y''(0)\} - 3\{s^2L[y(t)] - sy(0) - y'(0)\}$

$$+3\{sL[y(t)] - y(0)\} - L[y(t)] = \frac{2}{(s-1)^3}$$

Using the given initial conditions we have,

ie., $$(s^3 - 3s^2 + 3s - 1)L[y(t)] - s^2 + 2 + 3s - 3 = \frac{2}{(s-1)^3}$$

ie., $$(s-1)^3L[y(t)] = (s^2 - 3s + 1) + \frac{2}{(s-1)^3}$$

or $$L[y(t)] = \frac{s^2 - 3s + 1}{(s-1)^3} + \frac{2}{(s-1)^6}$$

$$\therefore \quad y(t) = L^{-1}\left[\frac{s^2 - 3s + 1}{(s-1)^3}\right] + 2L^{-1}\left[\frac{1}{(s-1)^6}\right] \quad \text{...(i)}$$

Jow, $L^{-1}\left[\dfrac{s^2-3s+1}{(s-1)^3}\right] = L^{-1}\left[\dfrac{\{(s-1)^2+2s-1\}-3s+1}{(s-1)^3}\right]$

$$= L^{-1}\left[\frac{(s-1)^2-s}{(s-1)^3}\right]$$

$$= L^{-1}\left[\frac{(s-1)^2-(s-1)-1}{(s-1)^3}\right]$$

$$= e^t L^{-1}\left[\frac{s^2-s-1}{s^3}\right]$$

$$= e^t\left\{L^{-1}\left[\frac{1}{s}\right]-L^{-1}\left[\frac{1}{s^2}\right]-L^{-1}\left[\frac{1}{s^3}\right]\right\}$$

$$L^{-1}\left[\frac{s^2-3s+1}{(s-1)^3}\right] = e^t\left(1-t-\frac{t^2}{2}\right)$$

lso, $L^{-1}\left[\dfrac{2}{(s-1)^6}\right] = 2e^t L^{-1}\left[\dfrac{1}{s^6}\right] = 2e^t\dfrac{t^5}{5!} = \dfrac{e^t t^5}{60}$

hus by using these results in the RHS of (i) we have,

$$\boxed{y(t) = e^t\left\{1-t-\frac{t^2}{2}+\frac{t^5}{60}\right\}}$$

ASSIGNMENT

pplying convolution theorem find the inverse Laplace transform of the ollowing functions.

1. $\dfrac{s}{(s^2+4)^2}$ 2. $\dfrac{s^2}{(s^2+9)^2}$

3. $\dfrac{1}{(s^2+1)^2}$ 4. $\dfrac{s}{(s^2+a^2)(s^2+b^2)}$

5. $\dfrac{1}{s^2(s^2+a^2)}$

6. $\dfrac{s+1}{(s^2+2s+2)^2}$

Solve the following differential equations by using Laplace transforms.

7. $x''(t)-2x'(t)+x(t)=e^t$; $x(0)=2,\ x'(0)=-1$

8. $x''(t)+4x'(t)+4x(t)=4e^{-2t}$; $x(0)=-1,\ x'(0)=4$

9. $y'''(t)+y'(t)=e^{2t}$; $y(0)=0=y'(0)=y''(0)$

10. $y''(t)+y=6\cos 2t$; $y=3,\ y'=1$ at $t=0$

11. $\dfrac{d^4y}{dx^4}-16y=0$; $y=1\cdot y',\ y'',y'''$ are zero at $x=0$

12. $y''(t)-y(t)=\cosh t,\ y(0)=0=y'(0)$

ANSWERS

1. $\dfrac{t\sin 2t}{4}$

2. $\dfrac{1}{6}(3t\cos 3t+\sin 3t)$

3. $\dfrac{1}{2}(\sin t-t\cos t)$

4. $\dfrac{\cos bt-\cos at}{a^2-b^2}$

5. $\dfrac{1}{a^3}(at-\sin at)$

6. $\dfrac{te^{-t}\sin t}{2}$

7. $x(t)=e^t[2-3t+(t^2/2)]$

8. $x(t)=e^{-2t}(2t^2+2t-1)$

9. $y(t)=\dfrac{-1}{2}+\dfrac{e^{2t}}{10}+\dfrac{2}{5}\cos t-\dfrac{1}{5}\sin t$

10. $y(t)=5\cos t+\sin t-2\cos 2t$

11. $y(x)=\dfrac{1}{2}(\cosh 2x+\cos 2x)$

12. $y(t)=\dfrac{t\sinh t}{2}$

TRANSFORM CALCULUS, FOURIER SERIES & NUMERICAL TECHNIQUES

MODULE - 2

In Differential Calculus we are familiar with the expansion of a differentiable function $f(x)$ in the form of a power series. Taylor's series of $f(x)$ about $x = a$ is an infinite series in ascending powers of $(x - a)$ and Maclaurin's series is an infinite series in ascending powers of x. In many engineering problems it becomes necessary to expand a given function $f(x)$ in a series containing cosine and sine terms which belongs to a class of functions called periodic functions.

*In this module we discuss various aspects of such series referred to as **Fourier series.***

Fourier series is highly useful in the discussion of the Application of Partial Differential Equations (PDE) involving solution of Boundary Value Problems. Fourier series leads to Fourier integrals and further Fourier transforms also.