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$$= \int_{-1}^{1} f(x)\, e^{iux}\, dx = \int_{-1}^{1} (1-x^2)\, e^{iux}\, dx$$

$$= \left[1-x^2 \frac{e^{iux}}{iu} - (-2x)\left(\frac{e^{iux}}{i^2u^2}\right) + (-2)\left(\frac{e^{iux}}{i^3u^3}\right) \right]_{-1}^{1}$$

$$= \left[0 + 2(1)\frac{e^{iu}}{i^2u^2} - \frac{2e^{iu}}{i^3u^3} \right] - \left[0 - \frac{2e^{-iu}}{i^2u^2} - \frac{2e^{-iux}}{i^3u^3} \right]$$

$$= \frac{2}{i^2u^2}\left(e^{iu} + e^{-iu}\right) - \frac{2}{i^3u^3}\left(e^{iu} - e^{-iu}\right)$$

$$= \frac{2}{i^2u^2}\left(\cos u + \cancel{i\sin u} + \cos u - \cancel{i\sin u}\right) - \frac{2}{i^3u^3}\left(\cancel{\cos u} + i\sin u - \cancel{\cos u} + i\sin u\right)$$

$$= \frac{2}{i^2u^2}(2\cos u) - \frac{2}{i^{\cancel{3}2}u^3}(2\cancel{i}\sin u)$$

$$= \frac{4\cos u}{-u^2} + \frac{4\sin u}{u^3} \qquad (i^2 = -1).$$

$$\boxed{F(u) = \frac{4(\sin u - u\cos u)}{u^3}}$$

Inverse fourier transform, $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)\, e^{-iux}\, du.$

i) Put $x = 0$, $f(0) = 1 - x^2 = 1 - 0^2 = 1$

$$f(0) = \frac{-4}{2\pi}\int_{-\infty}^{\infty} \frac{u\cos u - \sin u}{u^3} \cdot e^0\, du.$$

$$1 = -\frac{2}{\pi}\int_{-\infty}^{\infty} \frac{u\cos u - \sin u}{u^3}\, du.$$

$$f(u) = \frac{u\cos u - \sin u}{u^3}$$

$$f(-u) = \frac{-u\cos(-u) - \sin(-u)}{-u^3} = \frac{-u\cos u + \sin u}{-u^3}.$$

$$= -\frac{(u\cos u - \sin u)}{-u^3} = \frac{u\cos u - \sin u}{u^3}.$$

$= f(u)$ $\therefore f(u)$ is even function.

Since function is even.

$$\Rightarrow 1 = -\frac{2}{\pi} \, 2\int_0^\infty \frac{u\cos u - \sin u}{u^3}\, du.$$

$$-\frac{\pi}{4} = \int_0^\infty \frac{u\cos u - \sin u}{u^3}\, du.$$

Put $u = x$

$$\int_0^\infty \frac{x\cos x - \sin x}{x^3}\, dx = \frac{-\pi}{4}$$

ii) By inverse fourier transform.

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)\, e^{-iux}\, du.$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4(\sin u - u\cos u)}{u^3}\, e^{-iux}\, du.$$

Put $x = 1/2$. $\qquad$ $1/2$ Lies b/w $-1 \le x \le 1$.

$$f(1/2) = 1 - x^2 = 1 - {1/2}^2 = 1 - 1/4 = 3/4$$

$$f(1/2) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4(\sin u - u\cos u)}{u^3}\, e^{-iu/2}\, du.$$

$$\frac{3}{4} = \frac{-\cancel{4}\,2}{\cancel{2}\pi}\int_{-\infty}^{\infty} \frac{u\cos u - \sin u}{u^3}\, e^{-iu/2}\, du.$$

$$= -\frac{3}{4} \cdot \frac{\pi}{2} = \int_{-\infty}^{\infty} \frac{u\cos u - \sin u}{u^3}\left(\cos u/2 - i\sin u/2\right) du.$$

comparing real terms both sides.

$$\frac{-3\pi}{8} = \int_{-\infty}^{\infty} \frac{u\cos u - \sin u}{u^3} \cdot \cos u/2\, du.$$ (we dont want $i\sin u/2$).

$\therefore f(u)$ is even function.

$$-3\pi/8 = 2\int_0^\infty \left[\frac{u\cos u - \sin u}{u^3}\right] \cos u/2\, du$$

$u = x$

$$= -\frac{3\pi}{16} = \int_0^\infty \frac{x\cos x - \sin x}{x^3} \cos\left(\frac{x}{2}\right) dx$$

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$$= \left[-\frac{1}{(1+iu)^2} + \frac{1}{(1-iu)^2} \right]$$

$$= -\frac{(1-iu)^2 + (1+iu)^2}{(1+iu)^2 (1-iu)^2}$$

$$= -\frac{(1+i^2u^2 - 2ui) + (1+i^2u^2+2ui)}{[(1+iu)(1-iu)]^2}.$$

$$= \frac{4ui}{(i^2 - i^2u^2)^2} = \frac{4ui}{(1+u^2)^2}$$

H.W

5. $f(x) = \begin{cases} x^2, & -\alpha \le x \le \alpha. \\ 0, & |x| > \alpha. \end{cases}$

$$f(u) = \int_{-\infty}^{\infty} f(x) e^{iux} dx.$$

$$= \int_{-\infty}^{-\alpha} a\, e^{iux} dx + \int_{-\alpha}^{\alpha} x^2 e^{iux} dx + \int_{\alpha}^{\infty} a\, e^{iux} dx.$$

$$= \left[a \cdot \frac{e^{iux}}{iu} \right]_{-\infty}^{-\alpha} + \left[x^2 \frac{e^{iux}}{(iu)} - 2x \frac{e^{iux}}{(iu)^2} + 2\frac{e^{iux}}{(iu)^3} \right]_{-\alpha}^{\alpha} + \left[a \cdot \frac{e^{iux}}{iu} \right]_{\alpha}^{\infty}$$

$$= \left[a \cdot \frac{e^{iux}}{iu} \right] + \left[\alpha^2 \cdot \frac{e^{iux}}{iu} - 2\alpha \frac{e^{iux}}{(iu)^2} + 2 \cdot \frac{e^{iu\alpha}}{(iu)^3} \right] -$$

$$\left[\alpha^2 \frac{e^{-iu\alpha}}{(iu)} + 2\alpha \frac{e^{-iu\alpha}}{(iu)^2} + 2\frac{e^{-iu\alpha}}{(iu)^3} \right] + \left[0 - a \cdot \frac{e^{iu\alpha}}{iu} \right].$$

$$= \alpha^2 \cdot \frac{e^{iu\alpha}}{iu} - 2\alpha \frac{e^{iux}}{(iu)^2} + 2 \cdot \frac{e^{iu\alpha}}{(iu)^3} - \alpha^2 \frac{e^{-iu\alpha}}{iu} + 2\alpha \frac{e^{-iu\alpha}}{(iu)^2} + 2\frac{e^{-iu\alpha}}{(iu)^3}.$$

$$= \frac{\alpha^2}{iu} [e^{iu\alpha} - e^{-iu\alpha}] + \frac{2\alpha}{(iu)^2} [e^{-iu\alpha} - e^{iu\alpha}] + \frac{2}{(iu)^3} [e^{iu\alpha} + e^{-iu\alpha}]$$

$$= \frac{\alpha^2}{iu} [\cancel{\cos u\alpha} + i\sin u\alpha - (\cancel{\cos u\alpha} - i\sin u\alpha)] + \frac{2\alpha}{(iu)^2}$$

$$[\cancel{\cos u\alpha} - i\sin u\alpha - (\cancel{\cos u\alpha} + i\sin u\alpha)] + \frac{2\alpha}{(iu)^3} [\cos u\alpha +$$

$$\cancel{i\sin u\alpha} + \cos u\alpha - \cancel{i\sin u\alpha}]$$

$$= \frac{1}{u^3} [2(\alpha^2 u^2 - 2) \sin u + 4a u \cos \alpha u$$

6) $f(x)=e^{-|x|}$

$F(u)=\int_{-\infty}^{\infty} f(x)\, e^{iux}\, dx.$

$=\int_{-\infty}^{\infty} e^{-|x|}\, e^{iux}\, dx$

$=\int_{-\infty}^{0} e^{-(-x)}\, e^{+iux}\, dx+\int_{0}^{\infty} e^{-x}\, e^{iux}\, dx.$

$=\int_{-\infty}^{0} e^{x}\, e^{iux}\, dx+\int_{0}^{\infty} e^{-x}\, e^{iux}\, dx.$

$=\int_{-\infty}^{0} e^{(1+iu)x}\, dx+\int_{0}^{\infty} e^{-(1-iu)x}\, dx.$

$=\left[\frac{e^{(1+iu)x}}{1+iu}\right]_{-\infty}^{0}+\left[\frac{e^{-(1-iu)x}}{-(1-iu)}\right]_{0}^{\infty}.$

$=\left[\frac{1}{1+iu}-0\right]+\left[0+\frac{1}{(1-iu)}\right]^{0}$

$=\left[\frac{1}{1+iu}+\frac{1}{(1-iu)}\right]$

$=\left[\frac{(1-iu)+(1+iu)}{(1+iu)(1-iu)}\right].$

$=\left[\frac{2}{1^2-(iu)^2}\right] \Rightarrow \frac{2}{1^2+u^2}$

Imp!

7. Find the fourier series of $e^{-a^2x^2}$ hence deduce that $e^{-x^2/2}$ is self reciprocal wrt fourier transform.

ans: $F(u)=\int_{-\infty}^{\infty} f(x)\, e^{iux}\, dx.$

$=\int_{-\infty}^{\infty} e^{-a^2x^2}\, e^{iux}\, dx.$

$=\int_{-\infty}^{\infty} e^{-a^2x^2+iux}\, dx.$

$=\int_{-\infty}^{\infty} e^{-a^2\left(x^2-\frac{iux}{a^2}\right)}.\, dx$ $\quad\left|-a^2\left[x^2-\frac{iux}{a^2}\right]\right.$

Taking power,

$\left[-a^2\left[x^2-\frac{iux}{a^2}\right]\right]$

$a^2-2ab+b^2$

$-a^2\left[\left[x^2-2x\left[\frac{iu}{2a^2}\right.\right.\right.$

$=-a^2\left[\left(x-\frac{iu}{2a^2}\right)^2\right.$

$-a^2\left(x-\frac{iu}{2a^2}\right)^2-\frac{u^2}{4a}$

$\int_{-\infty}^{\infty} e^{-a^2\left(x-\frac{iu}{2a^2}\right.}$

$=e^{-u^2/4a^2}\int_{-\infty}^{\infty} e^{-a^2(}$

Now

Put $a\left(x-\frac{iu}{2a^2}\right)=t$

$a\,dx=dt.$

when $x=\infty$ $\quad t=\infty$

$x=-\infty$ $\quad t=-\infty$

$\therefore F(u)=e^{-u^2/4a^2}($

$=\frac{e^{-u^2/4a^2}}{a}\int_{-\infty}^{\infty} e$

$F(e^{-a^2x^2})=\frac{e^{-u}}{a}$

Put $a^2=1/2$.

$F(e^{-x^2/2})=\frac{e^{-u^2/}}{1/\sqrt{}}$

$=\sqrt{2\pi}$

→ Fourier tra

time $e^{-u^2/2}$.

→ $e^{-u^2/2}$ is

fourier tra

8. Imp!

Find the infi

$F_c(u)=\int_0^{\infty} f($

$$-a^2\left\{\left[x^2-2x\left[\frac{iu}{2a^2}\right]+\left(\frac{iu}{2a^2}\right)^2\right]-\frac{i^2u^2}{4a^4}\right\} \quad i^2=-1$$

$$=-a^2\left[\left(x-\frac{iu}{2a^2}\right)^2+\frac{u^2}{4a^4}\right]$$

$$-a^2\left(x-\frac{iu}{2a^2}\right)^2-\frac{u^2}{4a^2}$$

$$\int_{-\infty}^{\infty} e^{-a^2\left(x-\frac{iu}{2a^2}\right)^2}\cdot e^{-u^2/4a^2}\,dx$$

$$=e^{-u^2/4a^2}\int_{-\infty}^{\infty} e^{-a^2\left(x-\frac{iu}{2a^2}\right)^2}dx$$

Put $a\left(x-\overline{\frac{iu}{2a^2}}^{\text{No x term}}\right)=t$

$a\,dx=dt$

when $x=\infty$ $\quad t=\infty$

$x=-\infty$ $\quad t=-\infty$

$$\therefore F(u)=e^{-u^2/4a^2}\int_{-\infty}^{\infty} e^{-t^2}\,dt/a$$

$$=\frac{e^{-u^2/4a^2}}{a}\int_{-\infty}^{\infty} e^{-t^2}dt$$

$\int_{-\infty}^{\infty} e^{-x^2}dx=\sqrt{\pi}$

$$F(e^{-a^2x^2})=\frac{e^{-u^2/4a^2}}{a}\sqrt{\pi}$$

$e^{-a^2x^2} \;\big|\; e^{-x^2/2}$ $\quad a^2=1/2$

Put $a^2=1/2$.

$$F(e^{-x^2/2})=\frac{e^{-u^2/4\times 1/2}\sqrt{\pi}}{1/\sqrt{2}}$$

$$=\underline{\underline{\sqrt{2\pi}\,e^{-u^2/2}}}$$

-> Fourier transform of $e^{-x^2/2}$ is constant time $e^{-u^2/2}$

-> $e^{-u^2/2}$ is a self reciprocal for wrt fourier transform

Q. Find the infinite fourier cosine transform of e^{-x^2}

$F_c(u)=\int_0^{\infty} f(x)\cos ux\,dx$

We cant integrate since e^{-x^2}.

$= \int_0^\infty e^{-x^2} \cos ux \, dx$ — (1)

Libnitz rule of differentiating under integral sign.

$\frac{d}{du} F_c(u) = \int_0^\infty \frac{\partial}{\partial u} (e^{-x^2} \cos ux) dx$.

$F_c'(u) = \int_0^\infty e^{-x^2} (-\sin ux - x) dx$.

$= \int_0^\infty (-x e^{-x^2})(\sin ux) dx$.

$\therefore \int -x e^{-x^2} dx = \sin ux (\frac{1}{2} e^{-x^2}) \Big|_0^\infty - \int_0^\infty (\frac{1}{2} e^{-x^2} \cos ux \cdot u) dx$

$-x^2 = t$.

$-2x\,dx = dt$ $= (0-0) - \frac{1}{2} u \int_0^\infty \underbrace{e^{-x^2} \cos ux \, dx}$.

from (1).

$\int e^t = \frac{dt}{2}$

$= \frac{1}{2} \int e^t dt$

$= \frac{1}{2} e^{-x^2}$

$F_c'(u) = -\frac{1}{2} u F_c(u)$

$\frac{F_c'(u)}{F_c(u)} = -\frac{1}{2} u$.

Integrate wrt u, $\frac{du}{u} = \log u$.

$\log F_c(u) = -\frac{1}{2} u^2/2 + k$.

$= e^{-u^2/4} + k$.

$= e^{-u^2/4} \cdot e^k$.

$F_c(u) = k \cdot e^{-u^2/4}$ — (2).

Put $u=0$ in (1) & (2)

In (1) $F_c(0) = \int_0^\infty e^{-x^2} \cos 0 \, dx$.

$= \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$ → standard value.

In (2) $F_c(0) = k \cdot e^0 = k$.

$k = \sqrt{\pi}/2$.

Sub in (2)

$F_c(u) = \sqrt{\pi}/2 \; e^{-u^2/4}$.

9. Find fourier cosine transform of

$f(x) = \begin{cases} 4x, & 0 < x < 1. \\ 4-x, & 1 < x < 4 \\ 0, & x > 4. \end{cases}$

$F_c(u) = \int_0^1 4x \cos ux \, dx$ ...

$= [4x \cdot (\frac{\sin ux}{u})$ ...

$(0-1)(-\frac{\cos}{u^2}$ ...

$= [(4\frac{\sin u}{u} + \frac{4\cos}{u^2}$ ...

$= [4[\frac{\sin u}{u} + \frac{\cos}{u^2}$ ...

$= \frac{\sin u}{u}$ ...

$= \frac{\sin}{u}$ ...

10. Find the sine ...

evaluate $\int_0^\infty$ ...

By fourier s...

$F_s(u) = \int_0^\infty$ ...

$= \int_0^\infty e^{-|x|} \sin$ ...

$= \int_0^\infty e^{-x}$ ...

(Because x ...

$a = -1, b = u$

$= [\frac{e^{-x}}{(-1)^2 + u^2} (-1$ ...

$= \frac{1}{1^2 + u^2} [0 -$ ...

$= \frac{1}{1+u^2}$

By inverse f...

$f(x) = 2/\pi$ ...

$= 2/\pi$ ...

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$$F_c(u) = \int_0^1 4x\,dx \cos ux + \int_1^4 (4-x)\cos ux\,dx + \int_4^\infty 0\,dx$$

$$= \left[4x\cdot\left(\frac{\sin ux}{u}\right) - 4\left(\frac{-\cos ux}{u^2}\right)\right]_0^1 + \left[(4-x)\left(\frac{\sin ux}{u}\right) - (0-1)\left(-\frac{\cos ux}{u^2}\right)\right]_1^4$$

$$= \left[\left(4\frac{\sin u}{u} + \frac{4\cos u}{u^2}\right) - \left(0 + \frac{4}{u^2}\right)\right] + \left[\left(0 - \frac{\cos 4u}{u^2}\right) - \left(3\frac{\sin u}{u} - \frac{\cos u}{u^2}\right)\right]$$

$$= \left[4\left[\frac{\sin u}{u} + \frac{\cos u}{u^2} - \frac{1}{u^2}\right] - \frac{\cos 4u}{u^2} - 3\frac{\sin u}{u} - \frac{\cos u}{u^2}\right]$$

$$= \frac{\sin u}{u} + \frac{3\cos u}{u^2} - \frac{4}{u^2} - \frac{\cos 4u}{u^2}$$

$$= \frac{\sin u}{u} + \frac{5\cos u - \cos 4u - 4}{u^2}$$

10. Find the sine transform of $f(x) = e^{-|x|}$ & hence evaluate $\int_0^\infty \frac{x \sin mx}{1+x^2}\,dx$, $m > 0$.

By fourier sine transform,

$$F_s(u) = \int_0^\infty f(x) \sin ux\,dx$$

$$= \int_0^\infty e^{-|x|} \sin ux\,dx \quad \rightarrow \text{because } 0-\infty \text{ only positive } x.$$

$$= \int_0^\infty e^{-x} \sin ux\,dx$$

(Because x is linear) $\frac{e^{ax}}{a^2+b^2}(a \sin bx - b\cos bx)$.

$a = -1$, $b = u$.

$$= \left[\frac{e^{-x}}{(-1)^2+u^2}(-1 \sin ux - u\cos ux)\right]_0^\infty$$

$$= \frac{1}{1^2+u^2}\left[0 - e^0(0 - u\cos 0)\right]$$

$$= \frac{1}{1+u^2}(u) = \frac{u}{1+u^2}$$

By invere fourier sine transform.

$$f(x) = 2/\pi \int_0^\infty F_s(u) \sin ux\,du$$

$$= 2/\pi \int_0^\infty \frac{u}{1+u^2} \sin ux\,du$$

Put $x = m$,

$f(m) = e^{-|m|} \quad m > 0.$

$= e^{-m}.$

$f(m) = \frac{2}{\pi} \int_0^\infty \frac{u}{1+u^2} \sin mu \, du.$

$\frac{\pi e^{-m}}{2} = \int_0^\infty \frac{x}{1+x^2} \sin mx \, dx$

11. Find the fourier sine transform of the function $\frac{e^{-ax}}{x}$, $a > 0$.

By fourier sine transform

$Fs(u) = \int_0^\infty f(x) \sin ux \, dx.$

$= \int_0^\infty \frac{e^{-ax}}{x} \sin ux \, dx \quad -(1).$

By Libnitz rule differetiating under integral sign.

$\frac{d}{du}(Fs(u)) = \int_0^\infty \frac{\partial}{\partial u}\left(\frac{e^{-ax}}{x} \cdot \sin ux\right) dx.$

$F's(u) = \int_0^\infty \frac{e^{-ax}}{\cancel{x}} \cos ux \, \cancel{x} \, dx.$

$= \int_0^\infty e^{-ax} \cos ux \, dx.$

$a = -a \qquad b = u.$

$= \frac{e^{-ax}}{(-a)^2 + u^2} (-a \cos ux + u \sin ux) \Big|_0^\infty$

$= \frac{1}{a^2+u^2} \left[0 - e^0 \left[-a + 0\right]\right].$

$= F's(u) = \frac{a}{a^2+u^2}$

integrate on both the sides.

$Fs(u) = a \int \frac{1}{a^2+u^2} du.$

$= \cancel{a} \times 1/\cancel{a} \tan^{-1}[u/a] + k.$

$\tan^{-1}\frac{u}{a} + k$ - (2). $[e^{-x^2} \text{ v Impl}]$

Put $u=0$ in (1) & (2).

From (1) $F_s(u) = 0$.

From (2) $F_s(u) = 0+k$ $\Big\}$ $k=0$.

$\therefore F_s(u) = \tan^{-1} u/a$.

12. S.T $x \cdot e^{-x^2/2}$ is self reciprocal under sine transform.

ans: By fourier sine transform.

$F_s(u) = \int_0^\infty f(x) \sin ux \, dx$.

$= \int_0^\infty x e^{-x^2/2} \sin ux \, dx$.

By integration by parts method,

$= \int_0^\infty x e^{-x^2/2} \sin ux \, dx$.

$= \sin ux \left(-e^{-x^2/2}\right)\Big|_0^\infty - \int_0^\infty \left(-e^{-x^2/2}\right) \cos ux \cdot u \, dx$

$\int uv \, dx = u\int v - \int\int v \cdot du$.

$= (0-0) + u\int_0^\infty e^{-x^2/2} \cos ux \, dx$.

$\int x e^{-x^2/2} \, dx$

$= \frac{x^2}{2} = t$.

$-\frac{2x}{2} dx = dt$

$= -x \, dx = dt$

$= -\int e^t \cdot dt$.

$= -e^t \, dt$.

$= -e^{-x^2/2}$.

$F_s(u) = u\int_0^\infty e^{-x^2/2} \cos ux \, dx$.

consider $F_s(u) = uI$ - $*$.

Use libnitz rule, diff under integral sign.

$\frac{d}{du} F_s(u) = u\int_0^\infty \frac{\partial}{\partial u}\left(e^{-x^2/2} \cos ux\right) dx$

$I = \int_0^\infty e^{-x^2/2} \cos ux \, dx$ - (1).

$\frac{d}{du}\left(F_s(u)\right) = \int_0^\infty e^{-x^2/2} \frac{\partial}{\partial u} (\cos ux) \, dx$.

$$\int_0^\infty e^{-x^2/2}(-\sin ux \cdot x)\,dx$$

$$= -\int_0^\infty \underset{\text{I}}{\sin ux}\,\underset{\text{II}}{(x \cdot e^{-x^2/2})}\,dx$$

By parts.

$$= -\left[\sin ux\,(-e^{-x^2/2})\Big|_0^\infty + \int_0^\infty (-e^{-x^2/2}) \cdot \cos ux \cdot u\,dx\right]$$

$$I' = (0-0) - u\int_0^\infty e^{-x^2/2}\cos ux\,dx$$

$$I' = -uI$$

$$\frac{I'}{I} = -u$$

Integrate wrt u,

$$\log I = \frac{-u^2}{2} + k$$

$$I = e^{-u^2/2 + k}$$

$$I = e^{-u^2/2} \cdot e^k$$

$$I = k\,e^{-u^2/2} \quad -(2)$$

Put $u=0$ in (1) & (2).

$$I = \int_0^\infty e^{-x^2/2}\,dx = \frac{\sqrt{\pi}}{2}$$

$$I = k \cdot e^0 = k$$

$$k = \frac{\sqrt{\pi}}{2}$$

$$\therefore I = \frac{\sqrt{\pi}}{2} e^{-u^2/2}$$

In (*).

$$F_s(u) = u\left(\frac{\sqrt{\pi}}{2} \cdot e^{-u^2/2}\right)$$

$$F_s(u) = \frac{\sqrt{\pi}}{2}\, u \cdot e^{-u^2/2}$$

$$F_s(xe^{-x^2/2}) = \sqrt{\pi}/2\; ue^{-u^2/2}$$

$x \cdot e^{-x^2/2}$ is self reciprocal for unt fourier sine transform

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13. Obtain the fourier sine transform of $f(x)=\begin{cases} x & 0<x<1 \\ 2-x & 1<x<2 \\ 0 & x>2. \end{cases}$

as: $=\int_0^1 x\sin ux\,dx+\int_1^2 (2-x)\sin ux\,dx+\int_2^\infty 0\,dx.$

$=\left[x\left(\frac{-\cos ux}{u}\right)-1\left(\frac{-\sin ux}{u^2}\right)\right]_0^1+\left[(2-x)\left(\frac{-\cos ux}{u}\right)+(0-1)\frac{\sin ux}{u^2}\right]_1^2$

$=-\left[\frac{\cos u}{u}+\frac{\sin}{u^2}-(0+0)\right]+\left[\left(0-\frac{\sin 2u}{u^2}\right)-\left(-\frac{\cos u}{u}-\frac{\sin ux}{u^2}\right)\right].$

$=-\frac{\cancel{\cos u}}{\cancel{u}}+\frac{\sin u}{u^2}-\frac{\sin 2u}{u^2}+\frac{\cancel{\cos u}}{\cancel{u}}+\frac{\sin u}{u^2}$

$=\frac{1}{u^2}\left[2\sin u-\sin 2u\right]$

$=\frac{2\sin u}{u^2}-\frac{\sin 2u}{u^2}.$

$=\frac{2\sin u-\sin 2u}{u^2}.$

extra

page.

Q1

→ Obtain Fourier cosine transform of the functi[on]

$f(x) = \frac{1}{1+x^2}$

ans: $F_c(u) = \int_0^\infty f(x) \cos ux \, dx$

$= \int_0^\infty \frac{1}{1+x^2} \cos ux \, dx \quad — (1)$

By Libentz rule,

$\frac{d}{du} F_c(u) = \int_0^\infty \frac{\partial}{\partial u} \left(\frac{1}{1+x^2} \cos ux \right) dx$

$F_c'(u) = \int_0^\infty \frac{1}{1+x^2} (-\sin ux \cdot x) \, dx$

$= -\int_0^\infty \frac{x^2}{x(1+x^2)} \sin ux \, dx$

$= -\int_0^\infty \frac{(1+x^2)-1}{x(1+x^2)} \sin ux \, dx$

$= -\int_0^\infty \frac{1}{x} \sin ux \, dx + \int_0^\infty \frac{1}{x(1+x^2)} \sin ux \, dx$

$F_c'(u) = -\frac{\pi}{2} + \int_0^\infty \frac{1}{x(1+x^2)} \sin ux \, dx \quad (2) \quad \left[\because \int_0^\infty \frac{\sin ax}{x} dx = \frac{\pi}{2} \right.$

Again apply Libntz rule under integral sign.

$\frac{d}{du} F_c'(u) = 0 + \int_0^\infty \frac{\partial}{\partial u} \left(\frac{1}{x(1+x^2)} \sin ux \right) dx$

$F_c''(u) = \int_0^\infty \frac{1}{x(1+x^2)} \cos ux \cdot x \, dx$

$= \int_0^\infty \frac{1}{1+x^2} \cos ux \, dx$

$F_c''(u) = F_c(u)$

$F_c''(u) - F_c(u) = 0$

A.E: $m^2 - 1 = 0$

$m^2 = 1$

$m = \pm 1$

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$\therefore F(u) = c_1 e^{u} + c_2 e^{-u}$ — (3)

$F_c'(u) = c_1 e^{u} - c_2 e^{-u}$ — (4).

Put $u=0$ in (1) and (3).

In (1) $F_c(0) = \int_0^\infty \frac{1}{1+x^2}\,dx = \tan^{-1}x \Big|_0^\infty$

$= \tan^{-1}\infty - \tan^{-1}0$.

$= \pi/2 - 0 = \pi/2$.

In (3), $F_c(0) = c_1 e^0 + c_2 e^0$.

$= c_1 + c_2$.

$c_1 + c_2 = \pi/2$

Put $u=0$ in (2) and (1).

In (2), $F_c'(u) = -\pi/2 + 0 = -\pi/2$.

In (4), $F_c'(u) = c_1 e^0 - c_2 e^0$.

$= c_1 - c_2 = -\pi/2$.

$c_1 + c_2 = \pi/2$

$c_1 - c_2 = -\pi/2$.

$2c_1 = 0 \Rightarrow c_1 = 0,\ c_2 = \pi/2$.

$\therefore F_c(u) = 0 + \pi/2\, e^{-u}$

$F_c(u) = \pi/2\, e^{-u}$.

HW

→ Obtain sine transform for equation (function) $f(x) = \frac{1}{x(1+x^2)}$.

→ Solve the internal equation $\int_0^\infty f(\theta)\cos\alpha\theta\,d\theta = \begin{cases} 1-\alpha, & 0 \le \alpha \le 1. \\ 0, & \alpha > 1 \end{cases}$ hence evaluate $\int_0^\infty \frac{\sin^2 t}{t}\,dt$.

Ans: $\int_0^\infty f(\theta)\cos\alpha\theta\,d\theta = \begin{cases} 1-\alpha, & 0 \le \alpha \le 1. \\ 0, & \alpha > 1 \end{cases}$ —(*)

(*) is equivalent to

$$F_c(\alpha) = \begin{cases} 1-\alpha, & 0 \le \alpha \le 1 \\ 0, & \alpha > 1 \end{cases}$$

By inverse fourier cosine transform

$$f(x) = \frac{2}{\pi} \int_0^\infty F_c(u) \cos ux \, du$$

$$f(\theta) = \frac{2}{\pi} \int_0^\infty F_c(\alpha) \cos \alpha\theta \, d\alpha$$

$$= \frac{2}{\pi} \left[\int_0^1 (1-\alpha) \cos \alpha\theta \, d\alpha + \int_1^\infty 0 \, d\alpha \right]$$

$$= \frac{2}{\pi} \left[(1-\alpha) \frac{\sin \alpha\theta}{\theta} - (-1) \frac{(-\cos\alpha\theta)}{\theta^2} \right]_0^1 + 0$$

$$= \frac{2}{\pi} \left[\left(0 - \frac{\cos\theta}{\theta^2}\right) - \left(0 - \frac{1}{\theta^2}\right) \right]$$

$$f(\theta) = \frac{2}{\pi\theta^2} (1 - \cos\theta)$$

$$= \frac{2}{\pi\theta^2} \left(2 \sin^2 \frac{\theta}{2} \right)$$

$$= \frac{4 \sin^2 \theta/2}{\pi\theta^2}$$

By fourier cosine transform.

$$F_c(u) = \int_0^\infty f(x) \cos ux \, dx$$

$$F_c(\alpha) = \int_0^\infty f(\theta) \cos \alpha\theta \, d\theta$$

$$= \int_0^\infty \frac{4 \sin^2 \theta/2}{\pi\theta^2} \cos \alpha\theta \, d\theta$$

Put $\alpha = 0$, $F_c(0) = 1 - 0 = 1$.

$$1 = 4/\pi \int_0^\infty \frac{\sin^2 \theta/2}{\theta^2} \cos 0 \, d\theta$$

Put $t = \theta/2$

$dt = d\theta/2$

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when $\theta = 0$, $t = 0$.

$\theta = \infty$, $t = \infty$

$\therefore I = \frac{4}{\pi} \int_0^\infty \frac{\sin^2 t}{(2t)^2} \cdot 2dt.$

$\Rightarrow I = \frac{2}{\pi} \int_0^\infty \frac{\sin^2 t}{t^2} dt.$

$$\pi/2 = \int_0^\infty \frac{\sin^2 t}{t^2} dt$$

Z - Transform:

places a role in discrete analysis. Z transform has many properties similar to those of laplace transform. The main difference is that it operates not on fn/: of continuous arguement, but on sequence of discrete integer valued arguements ie $n = 0, \pm 1, \pm 2 \cdots$
It is used to find solution of difference equation.

Definition: If the functions u_n is defined for discrete values $(0, 1, 2, 3 \cdots)$ and $u_n = 0$ for $n < 0$. Then its Z transform is defined to be

$Z_T(u_n) = U(z)$

$= \sum_{n=0}^{\infty} u_n z^{-n}$ whenever the infinite series converges.

The inverse Z transform is written as Z_T^{-1},

$Z_T^{-1}(U(z)) = u_n$

Some standard ZT:

1) $Z_T(a^n) = \frac{z}{z-a}$