Chapter 6

Relations – II

This chapter is a continuation of Chapter 5. In this chapter, the topic of relations is discussed further. Special types of relations called *equivalence relations* and *partial orders* are dealth-with in some detail. The matrix representation of relations and the pictorial representation of relations (known as *digraphs*) are used to illustrate the concepts and results.

6.1 Zero-one Matrices and Directed graphs

Consider the sets $A = \{a_1, a_2, \dots, a_m\}$ and $B = \{b_1, b_2, \dots, b_n\}$ of orders m and n respectively. Then $A \times B$ consists of *all* ordered pairs of the form (a_i, b_j), $1 \le i \le m$, $1 \le j \le n$, which are mn in number. Let R be a relation from A to B so that R is a subset of $A \times B$.

Now, let us put $m_{ij} = (a_i, b_j)$ and assign the values 1 or 0 to m_{ij} according to the following Rule:

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R. \end{cases}$$

The $m \times n$ matrix formed by these m_{ij}'s is called the ***matrix of the relation R***, or ***the relation matrix*** for R, and is denoted by M_R or $M(R)$.* Since $M(R)$ contains only 0 and 1 as its elements, $M(R)$ is also called the ***Zero-one matrix*** for R.

It is to be noted that the rows of M_R correspond to the elements of A and the columns to those of B.

When $B = A$, the matrix M_R becomes an $n \times n$ matrix whose elements are $m_{ij} = (a_i, a_j)$ with

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, a_j) \in R \\ 0 & \text{if } (a_i, a_j) \notin R. \end{cases}$$

For example, consider the sets $A = \{0, 1, 2\}$ and $B = \{p, q\}$ and the relation R from A to B defined by

$$R = \{(0, p), (1, q), (2, p)\}.$$

*This matrix is used as a tool for the *computer recognition* and analysis of relations and is an example of a *data structure*.

Here, $A = \{a_1, a_2, a_3\} = \{0, 1, 2\}$ and $B = \{b_1, b_2\} = \{p, q\}$. We note that

$$\begin{aligned} m_{11} &= (a_1, b_1) = (0, p) = 1, \text{ because } (0, p) \in R \\ m_{12} &= (a_1, b_2) = (0, q) = 0, \text{ because } (0, q) \notin R \\ m_{21} &= (a_2, b_1) = (1, p) = 0, \\ m_{22} &= (a_2, b_2) = (1, q) = 1, \\ m_{31} &= (a_3, b_1) = (2, p) = 1, \\ m_{32} &= (a_3, b_2) = (2, q) = 0. \end{aligned}$$

Accordingly, the matrix of the relation R is

$$M(R) \equiv M_R = \left[m_{ij}\right] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

As another Example, consider the set $A = \{1, 2, 3, 4\}$ and a relation R defined on A by $R = \{(1, 2), (1, 3), (2, 4), (3, 2)\}$. Thus, here, $A = \{a_1, a_2, a_3, a_4\} = B$ where $a_1 = 1, a_2 = 2, a_3 = 3, a_4 = 4$.

Accordingly, $m_{ij} = \left(a_i, a_j\right) = (i, j), \quad i = 1, 2, 3, 4;\ j = 1, 2, 3, 4$, and we find that

$$\begin{array}{llll} m_{11} = (1,1) = 0, & \text{because} & (1,1) \notin R & \\ m_{12} = (1,2) = 1, & \text{because} & (1,2) \in R & \\ m_{13} = (1,3) = 1, & m_{14} = (1,4) = 0, & & \\ m_{21} = (2,1) = 0, & m_{22} = (2,2) = 0, & m_{23} = (2,3) = 0, & m_{24} = (2,4) = 1, \\ m_{31} = (3,1) = 0, & m_{32} = (3,2) = 1, & m_{33} = (3,3) = 0, & m_{34} = (3,4) = 0, \\ m_{41} = (4,1) = 0, & m_{42} = (4,2) = 0, & m_{43} = (4,3) = 0, & m_{44} = (4,4) = 0. \end{array}$$

Thus, the matrix of R is

$$M_R = \left[m_{ij}\right] = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Remarks

For any relation R from a finite set A to a finite set B, the following results are obvious:

(1) M_R is the zero matrix if and only if $R = \Phi$.

(2) Every element of M_R is 1 if and only if $R = A \times B$.

Digraph of a Relation

Let R be a relation on a finite set A. Then R can be represented pictorially as described below:

Draw a small circle or a bullet for each element of A and label the circle (bullet) with the corresponding element of A. These circles (bullets) are called *vertices* or *nodes*. Draw an arrow, called an *edge*, from a vertex x to a vertex y if and only if $(x, y) \in R$. The resulting pictorial representation of R is called a ***directed graph*** or ***digraph*** of R.

If a relation is pictorially represented by a digraph, a vertex from which an edge leaves is called the *origin* or the *source* for that edge, and a vertex where an edge ends is called the *terminus* for that edge. A vertex which is neither a source nor a terminus of any edge is called an *isolated vertex*. An edge for which the source and terminus are one and the same vertex is called a *loop*. The number of edges (arrows) terminating at a vertex is called the *in-degree* of that vertex and the number of edges (arrows) leaving a vertex is called the *out-degree* of that vertex.

For example, consider the set $A = \{a, b, c, d\}$ and the relation $R = \{(a, b), (b, b), (b, d), (c, b), (c, d), (d, a), (d, c)\}$ defined on A. The digraph of this relation is as shown below:

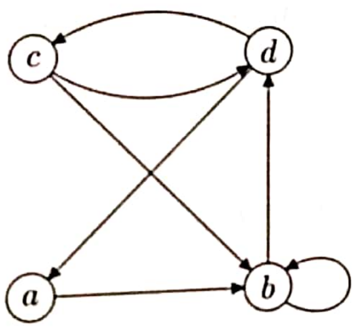


Figure 6.1

Observe that, since $(b, b) \in R$, there is a loop at the vertex b. We also see that the in-degrees of the vertices a, b, c, d are $1, 3, 1, 2$ respectively. Further, the out-degrees of a, b, c, d are $1, 2, 2, 2$ respectively.

As another example, consider the set $A = \{1, 2, 3, 4, 5\}$ and the relation

$$R = \{(1, 1), (1, 2), (1, 4), (3, 2)\}$$

defined on A. The digraph of this relation is as shown below:

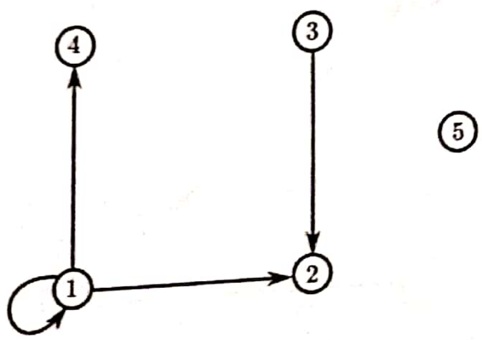


Figure 6.2

We observe that the above digraph has a loop at the vertex 1. Further, the vertex 5 is an isolated vertex; no edge leaves this vertex and no edge terminates at this vertex. We also note that the in-degrees of the vertices 1,2,3,4 are 1,2,0,1 and their out-degrees are 3,0,1,0 respectively.

Example 1 *Let $A = \{1, 2\}$ and $B = \{p, q, r, s\}$ and let the relation R from A to B be defined by*

$$R = \{(1, q), (1, r), (2, p), (2, q), (2, s)\}.$$

Write down the matrix of R.

▶ We first consider all elements of $A \times B$ and make the following observation:

$$(1, p) \notin R, \ (1, q) \in R, \ (1, r) \in R, \ (1, s) \notin R$$

$$(2, p) \in R, \ (2, q) \in R, \ (2, r) \notin R, \ (2, s) \in R$$

In view of these and by using the definition of the matrix of a relation, we find that, for the given R,

$$M_R = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}.$$

∎

Example 2 *Let $A = \{1, 2, 3, 4\}$ and let R be the relation on A defined by xRy if and only if $y = 2x$.*

(a) *Write down R as a set of ordered pairs.*

(b) *Draw the digraph of R.*

(c) *Determine the in-degrees and out-degrees of the vertices in the digraph.*

▶ (a) We observe that for $x, y \in A$, $(x, y) \in R$ if and only if $y = 2x$. Thus,

$$R = \{(1, 2), (2, 4)\}.$$

(b) The digraph of R is as shown below:

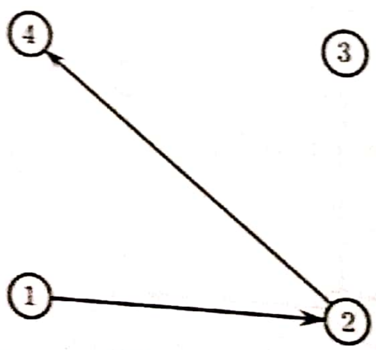


Figure 6.3

(c) From the above digraph, we note that 3 is an isolated vertex and that for the vertices 1, 2, 4, the in-degrees and the out-degrees are as shown in the following Table.

Vertex	1	2	4
In-degree	0	1	1
Out-degree	1	1	0

■

Example 3 *Let $A = \{1, 2, 3, 4\}$ and let R be the relation on A defined by xRy if and only if "x divides y", written $x \mid y$.*

(a) *Write down R as a set of ordered pairs.*

(b) *Draw the digraph of R.*

(c) *Determine the in-degrees and out-degrees of the vertices in the digraph.*

▶ (a) We observe that

$$1 \mid 1,\ 1 \mid 2,\ 1 \mid 3,\ 1 \mid 4,\ 2 \mid 2,\ 2 \mid 4, 3 \mid 3, 4 \mid 4$$

Hence

$$R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}.$$

(b) The digraph of R is as shown below.

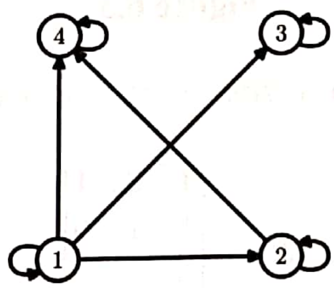


Figure 6.4

(c) By examining the digraph we note that, for the vertices $1, 2, 3, 4$, the in-degrees and the out-degrees are as shown in the following Table.

Vertex	1	2	3	4
In-degree	1	2	2	3
Out-degree	4	2	1	1

■

Example 4 *Let $A = \{1, 2, 3, 4, 6\}$ and R be a relation on A defined by aRb if and only if a is a multiple of b. Represent the relation R as a matrix and draw its digraph.*

▶ From the definition of the given R, we note that

$$R = \{(1,1), (2,1), (2,2), (3,1), (3,3), (4,1), (4,2), (4,4), (6,1), (6,2), (6,3), (6,6)\}$$

By examining the elements of R, we find that the matrix of R is

$$M_R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix}$$

The diagraph of R is as shown below:

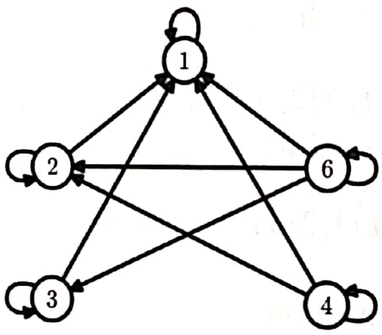


Figure 6.5

Example 5 *Determine the relation R from a set A to a set B as described by the following matrix:*

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

▶ We note that the given M_R is a 4×3 matrix. Therefore, $|A| = 4$ and $|B| = 3$. If $|A| = \{a_1, a_2, a_3, a_4\}$ and $B = \{b_1, b_2, b_3\}$, then by observing the elements of M_R, we find that

$$(a_1, b_1) \in R, (a_1, b_2) \notin R, (a_1, b_3) \in R$$
$$(a_2, b_1) \in R, (a_2, b_2) \in R, (a_2, b_3) \notin R$$
$$(a_3, b_1) \notin R, (a_3, b_2) \notin R, (a_3, b_3) \in R$$
$$(a_4, b_1) \in R, (a_4, b_2) \notin R, (a_4, b_3) \notin R$$

Thus, $R = \{(a_1, b_1), (a_1, b_3), (a_2, b_1), (a_2, b_2), (a_3, b_3), (a_4, b_1)\}$.

Example 6 *Let $A = \{u, v, x, y, z\}$ and R be a relation on A whose matrix is as given below. Determine R and also draw the associated digraph.*

$$M(R) = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

▸ By examining the elements of the given $M(R)$, we find that*

$$R = \{(u, v), (u, x), (v, u), (v, x), (x, u), (x, v), (x, z), (y, u), (y, z), (z, x), (z, y)\}$$

The digraph of this relation is as shown below:

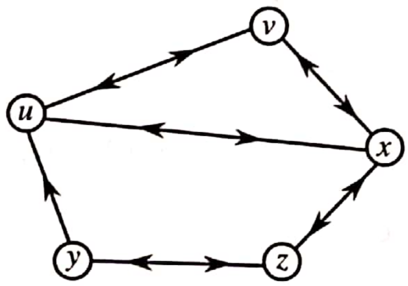


Figure 6.6 ■

Example 7 *Find the relation represented by the digraph given below. Also, write down its matrix.*

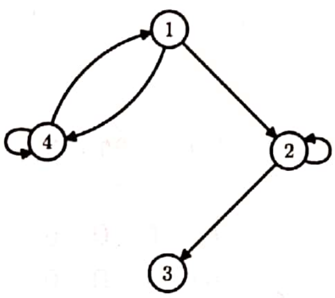


Figure 6.7

▸ By examining the given digraph which has four vertices, we note that the relation R represented by it is defined on the set $A = \{1, 2, 3, 4\}$ and is given by

$$R = \{(1, 2), (1, 4), (2, 2), (2, 3), (4, 1), (4, 4)\}.$$

*The elements of A may be designated as a_1, a_2, a_3, a_4 for the purpose of writing the elements of R.

The matrix of R is

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Example 8 *For $A = \{a, b, c, d, e, f\}$, the digraph in Figure below represents a relation R on A. Determine R as well as its associated relation matrix.*

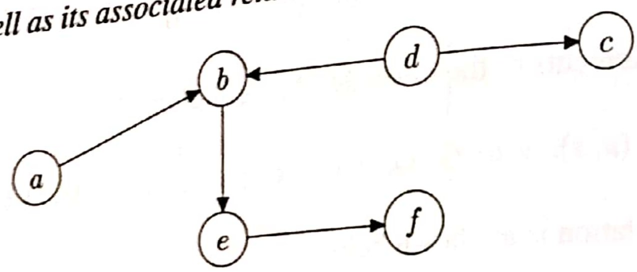


Figure 6.8

▶ By examining the given digraph, we find that

$$R = \{(a, b), (b, e), (d, b), (d, c), (e, f)\}.$$

Also, the matrix of R is given by

$$M(R) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Example 9 *Let $A = \{a, b, c, d\}$ and R be a relation on A that has the matrix*

$$M_R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Construct the digraph of R and list the in-degrees and out-degrees of all vertices.

▶ By examining the entries in the given matrix, we find that the given relation R has the following representation as a set of ordered pairs:

$$R = \{(a, a), (b, b), (c, a), (c, b), (c, c), (d, b), (d, d)\}.$$

The digraph of this relation is as shown below:

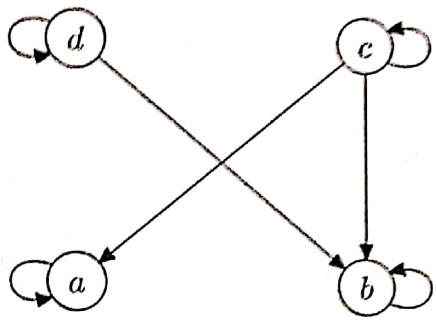


Figure 6.9

The in-degrees and out-degrees of the vertices are shown in the following Table.

Vertex	a	b	c	d
In-degree	2	3	1	1
Out-degree	1	1	3	2

■

Exercises

1. Let $A = \{a, b, c\}$ and $B = \{0, 1\}$, and $R = \{(a, 0), (b, 0), (c, 1)\}$ be a relation from A to B. Write down the matrix of this relation.

2. Let $A = \{1, 2, 3, 4\}$ and R be the relation on A defined by $(a, b) \in R$ if and only if $a \leq b$. Write down R as a set of ordered pairs. Also write down the matrix of this relation.

3. Determine the relation R from a set A to a set B as represented by the following matrix:

$$M_R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

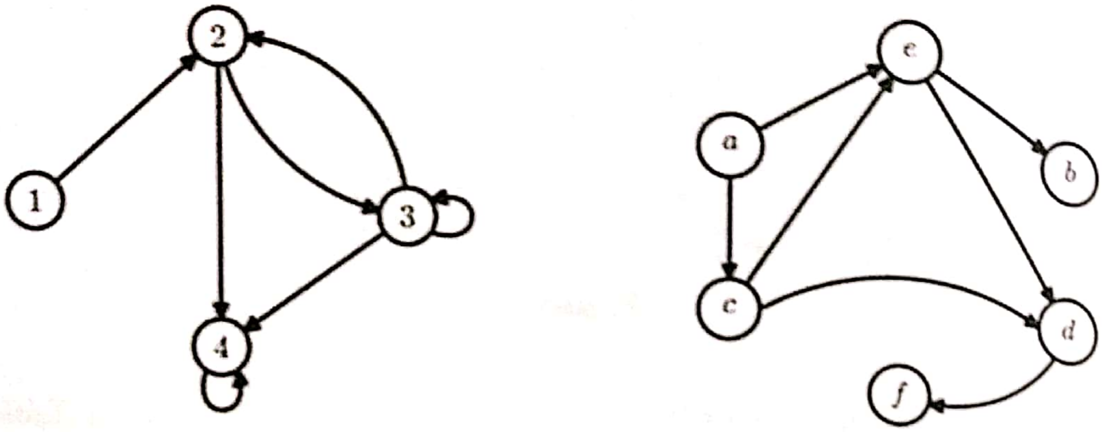
4. Let $A = \{1, 2, 3, 4\}$ and $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1)\}$ be a relation on A. Write down the digraph of R.

5. Let $A = \{a, b, c, d, e, f\}$ and R be the relation on A defined by $R = \{(a, b), (a, d), (b, c), (b, e), (d, b), (d, e), (e, c), (e, f), (f, d)\}$. Draw the digraph of R.

6. If $R = \{(x, y) \mid x > y\}$ is a relation defined on the set $A = \{1, 2, 3, 4\}$, write down the matrix and the digraph of R.

7. Find the relation R determined by each of the digraphs given below. Also, write down the matrix of the relation.

(i) (ii)



(a) (b)

Figure 6.10

8. Let $A = \{2, 4, 5, 7\}$, and let R be the relation on A having the matrix

$$M_R = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Construct the digraph of R.

9. Find the relation R on the set A and write down its digraph, given that $A = \{a, b, c, d, e\}$ and the matrix of R is

$$M_R = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Answers

1. $M_R = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

2. $R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$

$$M_R = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3. If $A = \{a_1, a_2, a_3\}$, $B = \{b_1, b_2, b_3, b_4\}$, then $R = \{(a_1, b_1), (a_2, b_2), (a_3, b_2), (a_3, b_3)\}$

4.

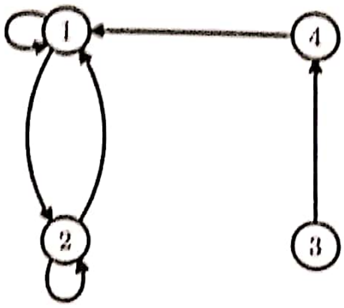


Figure 6.11

5.

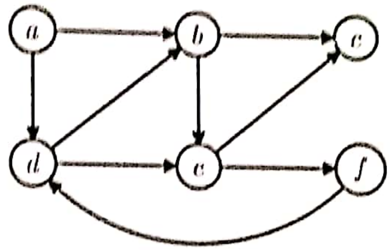


Figure 6.12

6. $M_R = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

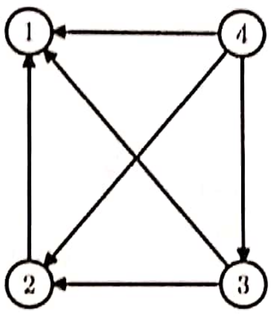


Figure 6.13

7. (i) $R = \{(1,2), (2,3), (2,4), (3,2), (3,3), (3,4), (4,4)\}$

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(ii) $R = \{(a,c), (a,e), (c,e), (c,d), (d,f), (e,b), (e,d)\}$

$$M_R = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

8.

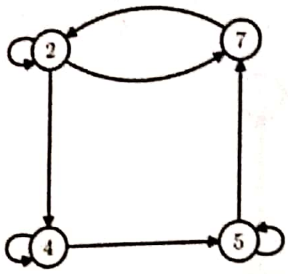


Figure 6.14

9. $R = \{(a,a), (a,b), (b,c), (b,d), (c,d), (c,e), (d,b), (d,c), (e,a)\}$

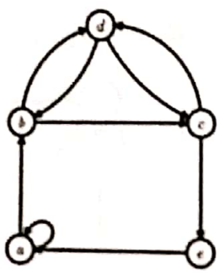


Figure 6.15

6.2 Properties of Relations

In this Section we consider some important properties of relations defined on a set.

Reflexive Relation

A relation R on a set A is said to be ***reflexive*** (or said to have the *reflexive property*) if $(a,a) \in R$, for all $a \in A$.

In other words, a relation R on a set A is reflexive whenever every element a of A is related to itself by R (i.e., aRa, for all $a \in A$).

It follows that R is not *reflexive* if there is some $a \in A$ such that $(a,a) \notin R$.

For example, the relation "is less than or equal to" is a reflexive relation on the set of all real numbers. Because, $a = a$ for every real number a.

It is obvious that the relations "is less than" and "is greater than" are not reflexive on the set of all real numbers.

As another example, we observe that if $A = \{1, 2, 3, 4\}$, then the relation $R = \{(1,1), (2,2), (3,3)\}$ is not reflexive. Because, $4 \in A$ but $(4,4) \notin R$.

The following results are easy to see:

(1) The matrix of a reflexive relation must have 1's on its main diagonal.

(2) At every vertex of the digraph of a reflexive relation there must be a cycle of length 1.

(3) On a set A, the relation Δ_A defined by

$$\Delta_A = \{(a, a) \mid a \in A\}$$

is reflexive.* Furthermore, Δ_A is a *subset of every reflexive relation* on A. The matrix of Δ_A contains 1's on the main diagonal and 0's in all other positions.

Irreflexive Relation

A relation on a set A is said to be *irreflexive* if $(a, a) \notin R$ for any $a \in A$. That is, a relation R is irreflexive if no element of A is related to itself by R.

For example, the relations "is less than" and "is greater than"are irreflexive on the set of all real numbers.

It is to be noted that an irreflexive relation is *not* the same as a non-reflexive relation. A relation can be neither reflexive nor irreflexive. For example, consider the relation $R = \{(1, 1), (1, 2)\}$ defined on the set $A = \{1, 2, 3\}$. This relation is not reflexive because $(2, 2) \notin R$ and $(3, 3) \notin R$. The relation is not irreflexive because $(1, 1) \in R$.

The following results are obvious:

(1) The matrix of an irreflexive relation must have 0's on its main diagonal.

(2) The digraph of an irreflexive relation has no cycle of length 1 at any vertex.

Symmetric Relation

A relation R on a set is said to be *symmetric* (or said to have the *symmetric property*) if $(b, a) \in R$ whenever $(a, b) \in R$ for all $a, b \in A$.

It follows that R is not symmetric if there exist $a, b \in A$ such that $(a, b) \in R$ but $(b, a) \notin R$.

A relation which is not symmetric is called an ***asymmetric relation***.

For example, if $A = \{1, 2, 3\}$ and $R_1 = \{(1, 1), (1, 2), (2, 1)\}$ and $R_2 = \{(1, 2), (2, 1), (1, 3)\}$ are relations on A, then R_1 is symmetric but R_2 is asymmetric; because $(1, 3) \in R_2$ but $(3, 1) \notin R_2$.

It is evident that for the matrix $M_R = [m_{ij}]$ of a symmetric relation the following property holds:

If $m_{ij} = 1$ then $m_{ji} = 1$, and if $m_{ij} = 0$ then $m_{ji} = 0$.

*The relation Δ_A is called the ***equality relation*** on A or the *diagonal line of* $A \times A$.

This means that the matrix M_R of a symmetric relation R is such that the $(i, j)^{th}$ element of M_R is equal to the $(j, i)^{th}$ element of M_R. In other words, the *matrix of a symmetric relation is a symmetric matrix* *.

In the digraph of a symmetric relation, if there is an edge from vertex a to a vertex b, then there is an edge from b to a; this means that if two vertices are connected by an edge, they must always be connected in both directions. Because of this, in a digraph of a symmetric relation, the edges are shown without arrows — the arrows are understood *both ways*. The digraph of a symmetric relation is called the ***graph*** of the relation and an edge connecting two vertices a and b is always a *bidirected edge*; it is denoted by $\{a, b\}$. Two vertices a and b of a graph which are connected by an edge are called *adjacent vertices*.

For example, consider the relation

$$R = \{(1,2), (2,1), (1,3), (3,1)\}$$

on the set $A = \{1,2,3\}$. Evidently, this relation is a symmetric relation, and its graph is as shown below:

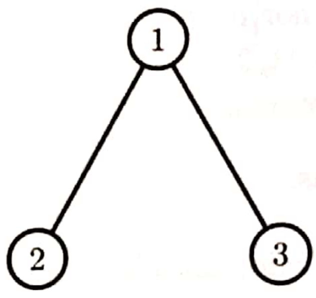


Figure 6.16

In the above graph, 1 and 2 are adjacent vertices, 1 and 3 are adjacent vertices, but 2 and 3 are not adjacent vertices.

Antisymmetric Relation

A relation R on a set A is said to be *antisymmetric* (or said to have the *antisymmetric property*) if whenever $(a, b) \in R$ and $(b, a) \in R$ then $a = b$.

It follows that R is not antisymmetric if there exist $a, b \in A$ such that $(a, b) \in R$ and $(b, a) \in R$ but $a \neq b$.

For example, the relation "is less than or equal to" on the set of all real numbers is an antisymmetric relation (because if $a \leq b$ and $b \leq a$, then $a = b$).

*A square matrix $A = [a_{ij}]$ is said to be *symmetric* if $a_{ij} = a_{ji}$ or equivalently $A = A^T$, the transpose of A.

It should be emphasized that asymmetric (i.e., not symmetric) and antisymmetric relations are *not* one and the same. A relation can be both symmetric and antisymmetric. A relation can be neither symmetric nor antisymmetric.

For example, let $A = \{1,2,3\}$ and $R_1 = \{(1,1),(2,2)\}$ and $R_2 = \{(1,2),(2,1),(2,3)\}$. We check that R_1 is both symmetric and antisymmetric, and R_2 is neither symmetric nor anti-symmetric.

The following results are obvious:

(1) If $M_R = [m_{ij}]$ is the matrix of an antisymmetric relation, then, for $i \neq j$, we have either $m_{ij} = 0$ or $m_{ji} = 0$.

(2) In the digraph of an antisymmetric relation, for two different vertices a and b, there cannot be a bidirectional edge between a and b.

Transitive Relation

A relation R on a set A is said to be *transitive* (or said to have the *transitive property*) if whenever $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$, for all $a,b,c \in A$.

It follows that R is not transitive if there exist $a,b,c \in A$ such that $(a,b) \in R$ and $(b,c) \in R$ but $(a,c) \notin R$.

For example, the relations "is less than or equal to" and "is greater than or equal to" are transitive relations on the set of all real numbers. Because, if $a \leq b$ and $b \leq c$ then $a \leq c$, and if $a \geq b$ and $b \geq c$ then $a \geq c$, for all real numbers a,b,c.

As another example, if we consider the set $A = \{1,2,3\}$ and the relations $R_1 = \{(1,1),(1,2),(2,3),(1,3),(3,1),(3,2)\}$ and $R_2 = \{(1,2),(2,3),(1,3),(3,1)\}$ on A, then R_1 is transitive but R_2 is *not* transitive.

The following result is easy to prove:

A relation R on a set A is transitive if and only if its matrix $M_R = [m_{ij}]$ has the following property:

$$\text{If } m_{ik} = 1 \text{ and } m_{kj} = 1, \text{ then } m_{ij} = 1.$$

Example 1 *Let $A = \{1,2,3\}$. Determine the nature of the following relations on A:*

(i) $R_1 = \{(1,2),(2,1),(1,3),(3,1)\}$ (ii) $R_2 = \{(1,1),(2,2),(3,3),(2,3)\}$

(iii) $R_3 = \{(1,1),(2,2),(3,3)\}$ (iv) $R_4 = \{(1,1),(2,2),(3,3),(2,3),(3,2)\}$

(v) $R_5 = \{(1,1),(2,3),(3,3)\}$ (vi) $R_6 = \{(2,3),(3,4),(2,4)\}$

(vii) $R_7 = \{(1,3),(3,2)\}$.

▶ By examining all ordered pairs present in the relations given, we find that:

R_1 is symmetric and irreflexive, but neither reflexive nor transitive.

R_2 is reflexive and transitive, but not symmetric.

R_3 and R_4 are both reflexive and symmetric.

R_5 is neither reflexive nor symmetric.

R_6 is transitive and irreflexive, but not symmetric.

R_7 is irreflexive, but neither transitive nor symmetric. ∎

Example 2 *Let $A = \{1, 2, 3, 4\}$. Determine the nature of the following relations on A.*

(1) $R_1 = \{(1,1), (1,2), (2,1), (2,2), (3,3), (3,4), (4,3), (4,4)\}$

(2) $R_2 = \{(1,2), (1,3), (3,1), (1,1), (3,3), (3,2), (1,4), (4,2), (3,4)\}$

(3) R_3 *represented by the following digraph:*

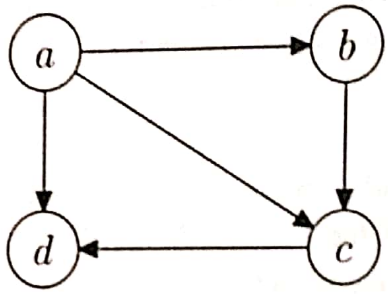


Figure 6.17

► By examining all ordered pairs present in R_1 and R_2, we find that:

(1) R_1 is reflexive, symmetric and transitive, and (2) R_2 is transitive.

By examining the edges in the digraph in Figure 6.17, we find that the relation R_3 is both asymmetric and antisymmetric. ∎

Example 3 *Find the nature of the relations represented by the following matrices:*

(a) $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

► (a) Here, the given matrix is symmetric (that is, $a_{ji} = a_{ij}$ for $i, j = 1, 2, 3$). Therefore, the corresponding relation is symmetric.

(b) Here, the given matrix has 1's on its main diagonal and is symmetric. Therefore, the corresponding relation is reflexive and symmetric.

(c) Here, the given matrix is not symmetric. Therefore, the corresponding relation is not symmetric. Further, the presence of 1 in the $(1,4)^{th}$ and $(4,1)^{th}$ positions of the matrix indicates that the relation is not antisymmetric. ∎

Example 4 *Show that the relation R represented by the matrix*

$$M_R = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

is transitive.

▶ Let $A = \{a, b, c\}$ be the set on which R is defined. Then, by examining the given M_R, we find that

$$R = \{(a,a), (a,b), (a,c), (b,c), (c,c)\}$$

By examining the elements of R, we find that R is transitive. ■

Example 5 *Let $A = \{a, b, c, d, e\}$, and*

$$R = \{(a,d), (d,a), (c,b), (b,c), (c,e), (e,c), (b,e), (e,b), (e,e)\}$$

be a symmetric relation on A. Draw the graph of R.

▶ By examining the elements in R, we find that the graph of R is as shown below:

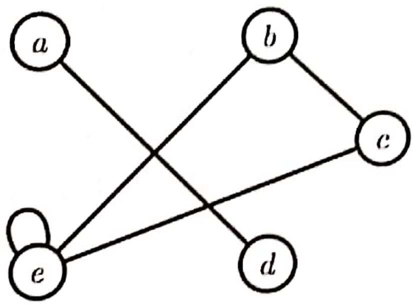


Figure 6.18

■

Example 6 *On the set Z^+, a relation R is defined by aRb if and only if a divides b (exactly)**. *Prove that R is reflexive, transitive and antisymmetric, but not symmetric.*

▶ For any $a \in Z^+$, the statement "a divides a" is true. Thus, aRa for all $a \in Z^+$. Hence R is reflexive.

Next, we note that, for any $a, b \in Z^+$, "a divides b" need not imply that "b divides a" (For instance, 3 divides 6 but 6 does not divide 3). Thus, aRb does not always imply bRa. Hence R is not symmetric.

Further, " a divides b " and " b divides a " imply that $a = b$. Thus aRb and bRa imply $a = b$. Therefore, R is antisymmetric.

Lastly, we note that for any $a, b, c \in Z^+$, "a divides b" and "b divides c" imply that "a divides c". Thus aRb and bRc imply aRc. Hence R is transitive. ■

---

*This relation is called the ***divisibility relation***. When a divides b, we write $a \mid b$. Here, the vertical line | stands for "divides".

Example 7 *Let S be a universal set. On $\mathcal{P}(S)$, define a relation R by $(A, B) \in R$ if and only if $A \subseteq B$ **. Prove that R is reflexive, antisymmetric and transitive, but not symmetric.*

► For any subset A of S (i.e., for any $A \in \mathcal{P}(S)$), we have $A \subseteq A$. This means that $(A, A) \in R$. Hence R is reflexive on $\mathcal{P}(S)$.

Next, we note that for any $A, B \in \mathcal{P}(S)$, $A \subseteq B$ and $B \subseteq A$ imply $A = B$. That is, $(A, B) \in R$ and $(B, A) \in R$ imply $A = B$. Hence R is antisymmetric on $\mathcal{P}(S)$.

Further, for any $A, B, C \in \mathcal{P}(S)$, if $A \subseteq B$ and $B \subseteq C$, we have $A \subseteq C$. That is, if $(A, B) \in R$ and $(B, C) \in R$, then $(A, C) \in R$. Hence R is transitive on $\mathcal{P}(S)$.

Finally, we note that for any $A, B \in \mathcal{P}(S)$, $A \subseteq B$ does not necessarily imply that $B \subseteq A$. That is, $(A, B) \in R$ does not always imply that $(B, A) \in R$. Therefore, R is *not* symmetric on $\mathcal{P}(S)$. ∎

Example 8 *Let R be a non-empty relation on a set A. Prove that if R satisfies any two of the properties: irreflexive, symmetric and transitive, then it cannot satisfy the third.*

► First, suppose that R is irreflexive and symmetric. Assume that R is transitive as well. Take any $(a, b) \in R$. Then, since R is symmetric, $(b, a) \in R$. Thus, $(a, b) \in R$ and $(b, a) \in R$. Since R is assumed to be transitive, this implies $(a, a) \in R$. This is not true, because R is irreflexive. Hence R cannot be transitive, when it is irreflexive and symmetric.

Next, suppose that R is irreflexive and transitive. Assume that R is symmetric as well. Take any $(a, b) \in R$. Then, since R is assumed to be symmetric, $(b, a) \in R$. Since R is transitive, it follows from $(a, b) \in R$ and $(b, a) \in R$ that $(a, a) \in R$. This is not true, because R is irreflexive. Hence, R cannot be symmetric when it is irreflexive and transitive.

Lastly, suppose that R is symmetric and transitive. Take any $(a, b) \in R$. Then, since R is symmetric, $(b, a) \in R$. Consequently, since R is transitive, $(a, a) \in R$. Hence R is not irreflexive.

This completes the proof. ∎

Exercises

1. Consider the following relations on the set $A = \{1, 2, 3\}$:

$$R_1 = \{(1,1), (1,2), (1,3), (3,3)\}, \quad R_2 = \{(1,1), (1,2), (2,1), (2,2), (3,3)\},$$
and $$R_3 = \{(1,1), (1,2), (2,2), (2,3)\}.$$

Which of these are (i) reflexive, (ii) symmetric, (iii) transitive, (iv) antisymmetric?

2. Let $A = \{1, 2, 3, 4\}$ and $R = \{(1,2), (1,3), (4,2)\}$. Is R a transitive relation on A?

3. Let $A = \{1, 2, 3, 4\}$ and $R = \{(1,2), (2,2), (3,4), (4,1)\}$. Verify that the relation R on A is (i) not symmetric, (ii) not asymmetric, (iii) antisymmetric.

This relation is called the ***Subset Relation*.

4. Let $R = \{(1,1),(2,2),(2,3),(3,2),(4,2),(4,4)\}$, and $S = \{(1,3),(1,1),(3,1),(1,2),(3,3),(4,4)\}$ be relations on the set $A = \{1,2,3,4\}$.

   (a) Verify that R is *not* (i) reflexive, (ii) symmetric, (iii) transitive, (iv) antisymmetric.

   (b) Determine whether S is reflexive, irreflexive, symmetric, asymmetric, antisymmetric or transitive.

5. Let $A = \{1,2,3\}$. Verify the following:

   (a) The relation $R_1 = \{(1,1),(2,2)\}$ is both symmetric and antisymmetric on A.

   (b) The relation $R_2 = \{(1,2),(2,1),(2,3)\}$ is neither symmetric nor antisymmetric on A.

   (c) The relation $R_3 = \{(1,1),(1,2),(3,2),(2,3),(3,3)\}$ is not reflexive and not irreflexive on A.

6. Let $A = \{1,2,3\}$ and $R_1 = \{(1,1),(1,2),(2,2),(3,3),(3,1)\}$, $R_2 = \{(1,1),(1,2),(2,1),(2,3),(3,2)\}$ and $R_3 = \{(1,1),(1,2),(2,3),(1,3)\}$ be relations on A. Verify that (i) R_1 is reflexive and antisymmetric, but is neither symmetric nor transitive, (ii) R_2 is symmetric, but is not reflexive, antisymmetric or transitive, and (iii) R_3 is transitive and antisymmetric but not reflexive or symmetric.

7. If $A = \{1,2,3,4\}$, give an example of a relation that is (i) reflexive and symmetric, but not transitive, (ii) reflexive and transitive, but not symmetric, and (iii) symmetric and transitive, but not reflexive.

8. Verify that the relation represented by the following digraph is antisymmetric and transitive.

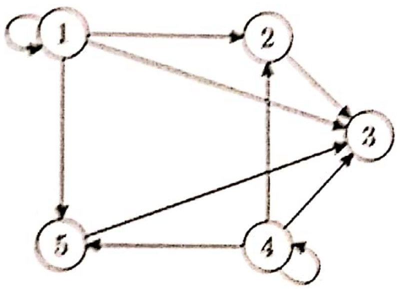


Figure 6.19

9. Verify that the relation R represented by the following matrix is irreflexive and symmetric.

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

10. Let $A = \{1,2,3,4,5\}$, and $R = \{(1,2),(2,1),(1,5),(5,1),(2,2),(3,4),(4,3),(3,5),(5,3)\}$ be a symmetric relation on A. Draw the graph of R.

11. On the set $\mathbb{Z}$, define the relation R by aRb if and only if $ab \geq 0$. Prove that R is reflexive and symmetric, but not transitive.

12. On the set $\mathbb{Z}\times\mathbb{Z}$, define the relation R by $(a,b)R(c,d)$ if and only if $a \leq c$. Prove that R is reflexive and transitive.

13. Prove the following:

    (1) On the set of all integers, the relation R defined by aRb if and only if $a \leq b+1$ is reflexive.

    (2) On the set of all integers, the relation R defined by aRb if and only if $|a-b| = 2$ is irreflexive and symmetric.

    (3) On the set of all positive integers, the relation R defined by aRb if and only if a divides b or b divides a is reflexive and symmetric but not transitive.

    (4) On the set of all positive integers, the relation R defined by aRb if and only if a is equal to some positive integral power of b is reflexive, antisymmetric and transitive.

14. Prove that if a relation on a set is irreflexive and transitive then it is antisymmetric.

Answers

1. R_1 is transitive and antisymmetric; R_2 is reflexive, symmetric and transitive; R_3 is antisymmetric.

2. Yes 4. (b) None

7. (i) $\{(1,1),(2,2),(3,3),(4,4),(1,2),(2,1),(2,3),(3,2)\}$

   (ii) $\{(1,1),(2,2),(3,3),(4,4),(1,2)\}$

   (iii) $\{(1,1),(2,2),(1,2),(2,1)\}$

10.

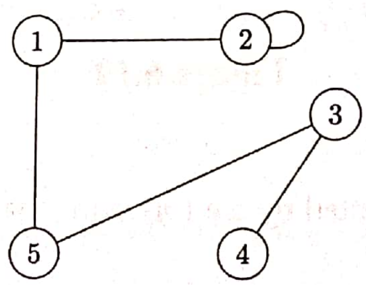


Figure 6.20

6.3 Equivalence Relations

A relation R on a set A is said to be an *equivalence relation* on A if (i) R is *reflexive*, (ii) R is *symmetric*, and (iii) R is *transitive*, on A.

A trivial example of an equivalence relation is the relation "is equal to" on the set of all real numbers, $\mathbb{R}$. An example of a relation which is not an equivalence relation is the relation "is less than " on $\mathbb{R}$.

Example 1 *Let* $A = \{1, 2, 3, 4\}$ *and*

$$R = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,3), (3,3), (4,4)\}$$

be a relation on A. Verify that R is an equivalence relation.

▶ We have to show that R is reflexive, symmetric and transitive.

First, we note that all of $(1,1), (2,2), (3,3), (4,4)$ belong to R. That is, $(a, a) \in R$ for all $a \in A$. Therefore, R is a reflexive relation.

Next, we note the following:

$$(1,2), (2,1) \in R \text{ and } (3,4), (4,3) \in R.$$

That is, if whenever $(a, b) \in R$ then $(b, a) \in R$ for $a, b \in A$. Therefore, R is a symmetric relation.

Lastly, we note that

$$(1,2), (2,1), (1,1) \in R, (2,1), (1,2), (2,2) \in R,$$
$$(4,3), (3,4), (4,4) \in R.$$

That is, if whenever $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$, for $a, b, c \in A$. Therefore, R is a transitive relation. ■

Accordingly, R is an equivalence relation.

Example 2 *Let* $A = \{1, 2, 3, 4\}$, *and*

$$R = \{(1,1), (1,2), (2,1), (2,2), (3,1), (3,3), (1,3), (4,1), (4,4)\}$$

be a relation on A. Is R an equivalence relation?

▶ Here, we have to check whether or not R is reflexive, symmetric and transitive.

By examining the elements of R, we note the following:

(i) $(a, a) \in R$ for every of $a \in R$, Therefore R is reflexive.

(ii) $(4, 1) \in R$, but $(1, 4) \notin R$. Therefore, R is not symmetric.

Since R is not symmetric, R is not an equivalence relation. (We need not check R for transitivity). ■

Example 3 *Let $A = \{1,2,3,4,5,6,7,8,9,10,11,12\}$. On this set define the relation R by $(x,y) \in R$ if and only if $x - y$ is a multiple of 5. Verify that R is an equivalence relation.*

► For any $x \in A$, we have $x - x = 0$ which is a multiple of 5 (because $0 = 5 \cdot 0$). Therefore $(x,x) \in R$ and so R is reflexive.

For any $x, y \in A$, if $(x,y) \in R$ then $x-y = 5k$ for some integer k. Consequently, $y-x = 5(-k)$ so that $(y,x) \in R$. Therefore, R is symmetric.

For any $x, y, z \in A$, if $(x,y) \in R$ and $(y,z) \in R$ then $x - y = 5k_1$ and $y - z = 5k_2$ for some integers k_1 and k_2. Consequently.

$$x - z = (x-y) + (y-z) = 5k_1 + 5k_2 = 5(k_1 + k_2)$$

from which it follows that $(x,z) \in R$. Therefore, R is transitive.

Thus, R is an equivalence relation.

Example 4 *If $A = A_1 \cup A_2 \cup A_3$, where $A_1 = \{1,2\}$, $A_2 = \{2,3,4\}$ and $A_3 = \{5\}$, define the relation R on A by xRy if any only if x and y are in the same set A_i, $i = 1,2,3$. Is R an equivalence relation?*

► We note that xRx for every x in A_i because x and x belong to the same A_i. Therefore, R is reflexive.

Further, if $x, y \in A_i$ then $y, x \in A_i$ for all x, y in A. Therefore, R is symmetric.

Lastly, we observe that $(1,2) \in R$ (because 1 and 2 are in the same set, A_1) and $(2,3) \in R$ (because 2 and 3 are in the same set, A_2) but $(1,3) \notin R$ (because 1 and 3 are not in the same set). Hence R is *not* transitive.

Accordingly, R is *not* an equivalence relation.

Example 5 *A relation R on a set $A = \{a,b,c\}$ is represented by the following matrix:*

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Determine whether R is an equivalence relation.

► By examining the elements of M_R, we find that $R = \{(a,a), (a,c), (b,b), (c,c)\}$. We note that $(a,c) \in R$ but $(c,a) \notin R$. Therefore, R is not symmetric. Accordingly, R is not an equivalence relation.

(That R is not symmetric can also be seen by the fact that the matrix M_R is not symmetric.)

Example 6 *The digraph of a relation R on the set $A = \{1, 2, 3\}$ is as given below. Determine whether R is an equivalence relation.*

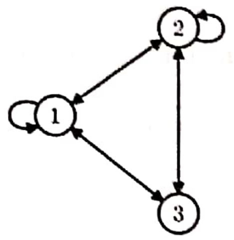


Figure 6.21

▶ By examining the digraph, we note that the given relation is symmetric and transitive but not reflexive; observe that $(3, 3) \notin R$. Therefore, R is not an equivalence relation. ■

Example 7 *Let S be the set of all non-zero integers, and $A = S \times S$. On A, define the relation R by $(a, b)R(c, d)$ if and only if $ad = bc$. Show that R is an equivalence relation.*

▶ First, we note that $(a, a)R(a, a)$, because $aa = aa$ for any $a \in S$. Therefore, R is reflexive on A.

Next, suppose $(a, b)R(c, d)$. Then $ad = bc$ and therefore $cb = da$. Hence $(c, d)R(a, b)$. Accordingly, R is symmetric on A.

Lastly, suppose that $(a, b)R(c, d)$ and $(c, d)R(e, f)$. Then $ad = bc$ and $cf = de$, which yield $af = be$. Hence $(a, b)R(e, f)$. Accordingly, R is transitive on A.

This proves the required result. ■

Example 8 *For a fixed integer $n > 1$, prove that the relation "**congruent modulo n**" is an equivalence relation on the set of all integers, Z.*

▶ For $a, b \in Z$, we say that "*a is congruent to b modulo n*" [written symbolically as $a \equiv b \pmod{n}$] if $a - b$ is a multiple of n, or , equivalently, $a - b = kn$ for some $k \in Z$.

Let us denote this relation by R so that aRb means $a \equiv b \pmod{n}$. We have to prove that R is an equivalence relation.

First, we note that for every $a \in Z$, $a - a = 0$ is a multiple of n; that is, $a \equiv a \pmod{n}$, or aRa. Therefore, R is reflexive.

Next, we note that, for all $a, b \in Z$,

$$\begin{aligned} aRb &\Rightarrow a \equiv b \pmod{n} \\ &\Rightarrow (a - b) \text{ is a multiple of } n \\ &\Rightarrow (b - a) \text{ is a multiple of } n \\ &\Rightarrow b \equiv a \pmod{n} \Rightarrow bRa. \end{aligned}$$

Therefore, R is symmetric.

Lastly, we note that for all $a, b, c \in \mathbb{Z}$,

$$\begin{aligned} aRb \text{ and } bRc &\Rightarrow a \equiv b \pmod{n} \text{ and } b \equiv c \pmod{n} \\ &\Rightarrow a - b \text{ and } b - c \text{ are multiples of } n \\ &\Rightarrow (a-b) + (b-c) = (a-c) \text{ is a multiple of } n \\ &\Rightarrow a \equiv c \pmod{n} \Rightarrow aRc. \end{aligned}$$

Therefore, R is transitive.

This proves that R is an equivalence relation. ∎

Equivalence Classes

Let R be an equivalence relation on a set A and $a \in A$. Then the set of all those elements x of A which are related to a by R is called the *equivalence class* of a with respect to R. This equivalence class is denoted by $R(a)$, or $[a]$, or $\overline{a}$. Thus,

$$\overline{a} = [a] = R(a) = \{x \in A \mid (x, a) \in R\} \tag{*}$$

For example, consider the equivalence relation

$$R = \{(1,1), (1,3), (2,2), (3,1), (3,3)\}$$

defined on the set $A = \{1, 2, 3\}$. We find that the elements x of A for which $(x, 1) \in R$ are $x = 1$, $x = 3$. Therefore, $\{1, 3\}$ is the equivalence class of 1 (with respect to R); that is,

$$[1] = \{1, 3\}.$$

Similarly,

$$[2] = \{2\}, \quad [3] = \{1, 3\}.$$

As another example, consider the relation R on $\mathbb{Z}$ defined by xRy if any only if $x - y$ is even. It is easy to check that R is an equivalence relation. For any $a \in \mathbb{Z}$, the equivalence class of a (w.r.t. R) is

$$\begin{aligned} [a] &= \{x \in \mathbb{Z} \mid xRa\} = \{x \in \mathbb{Z} \mid (x - a) \text{ is even}\} \\ &= \{x \in \mathbb{Z} \mid (x - a) = 2k, k \in \mathbb{Z}\} \\ &= \{x \in \mathbb{Z} \mid x = a + 2k, k = 0, \pm 1, \pm 2, \pm 3, \ldots\} \end{aligned}$$

Thus,

$$
\begin{aligned}
[0] &= \{x \in Z \mid x = 2k, k = 0, \pm 1, \pm 2, \pm 3, \ldots\} \\
&= \{0, \pm 2, \pm 4, \pm 6, \pm 8, \pm \ldots\} \\
[1] &= \{x \in Z \mid x = 1 + 2k, k = 0, \pm 1, \pm 2, \pm 3, \ldots\} \\
&= \{\ldots, -5, -3, -1, 1, 3, 5, 7, \ldots\} \\
[2] &= \{x \in Z \mid x = 2 + 2k, k = 0, \pm 1, \pm 2, \pm 3, \ldots\} \\
&= \{\ldots, -4, -2, 0, 2, 4, 6, \ldots\}
\end{aligned}
$$

and so on.

The following theorems contain some fundamental properties of equivalence classes.

Theorem 1. *Let R be an equivalence relation on a set A and $a \in A$. Then $a \in [a]$.*

Proof: Since R is reflexive, we have aRa. Therefore, $a \in [a]$. •

Theorem 2. *Let R be an equivalence relation on a set A, and let $a, b \in A$. Then aRb if and only if $[a] = [b]$.*

Proof: Suppose aRb. Take any $x \in [a]$. Then xRa. Thus, we have xRa and aRb. Since R is transitive, it follows that xRb so that $x \in [b]$. Thus, $[a] \subseteq [b]$. Similarly, we find that $[b] \subseteq [a]$. Accordingly, $[a] = [b]$.

Conversely, suppose $[a] = [b]$. Since $a \in [a]$ it follows that $a \in [b]$. Thus, aRb.

This completes the proof. •

Theorem 3. *Let R be an equivalence relation on a set A, and let $a, b \in A$. Then the following result is true:*

$$\textit{If} \quad [a] \cap [b] \neq \Phi, \textit{ then } [a] = [b].$$

Proof: Suppose $[a] \cap [b] \neq \Phi$. Then there exists $x \in A$ such that $x \in [a]$ and $x \in [b]$, so that xRa and xRb. These imply aRx and xRb so that aRb, because R is symmetric and transitive. Consequently, $[a] = [b]$, by Theorem 2.

This completes the proof of the theorem. •

Note: Taking the contrapositive of the result proved in the above theorem, we get the following equivalent result:

$$\text{If} \quad [a] \neq [b], \text{ then } [a] \cap [b] = \Phi$$

Remark: The result proved in the above theorem shows that if two equivalence classes are not disjoint then they are equal. In other words, if two equivalence classes have one element in common, then they have all elements in common. Thus,

Two equivalence classes are either disjoint or identical.

Corollary. *For $a, b \in A$, if $b \in [a]$, then $[b] = [a]$.*

Proof: We have $b \in [b]$. Therefore, if $b \in [a]$, we have $[a] \cap [b] \neq \Phi$. This implies $[a] = [b]$.

Partition of a Set

Let A be a nonempty set. Suppose there exist nonempty subsets $A_1, A_2, A_3, \ldots, A_k$ of A such that the following two conditions hold:

(1) A is the union of $A_1, A_2, \ldots, A_k$; that is, $A = A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_k$.

(2) Any two of the subsets $A_1, A_2, \ldots, A_k$ are disjoint; that is, $A_i \cap A_j = \Phi$ for $i \neq j$.

Then the set $P = \{A_1, A_2, A_3, \ldots, A_k\}$ is called a *partition* of A. Also, $A_1, A_2, A_3, \ldots, A_k$ are called the *blocks* or *cells* of the partition.*

A partition of a set A with six blocks (cells) is depicted in Figure 6.22.

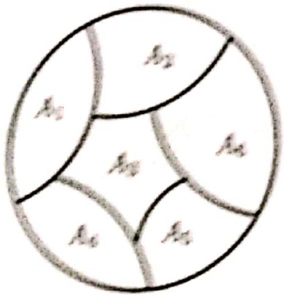


Figure 6.22

For example, consider the set

$$A = \{1, 2, 3, 4, 5, 6, 7, 8\}$$

and its following subsets:

$$A_1 = \{1, 3, 5, 7\},\ A_2 = \{2, 4\},\ A_3 = \{6, 8\}.$$

We observe that A is the union of A_1, A_2, A_3. Also, any two of the subsets A_1, A_2, A_3 are disjoint. Therefore, $P = \{A_1, A_2, A_3\}$ is a partition of A, with A_1, A_2, A_3 as the blocks (cells) of the partition.

In the above example, if $A_4 = \{1, 3, 5\}$ then $P_1 = \{A_2, A_3, A_4\}$ is not a partition of the set A, because although the subsets A_2, A_3 and A_4 are mutually disjoint, A is not the union of these

---

*If $P = \{A_1, A_2, \ldots, A_k\}$ is a partition of a set A, then the representation

$$A = A_1 \cup A_2 \cup \cdots \cup A_k$$

of A is also referred to as a partition of A.

subsets. We find that if $A_5 = \{5, 6, 8\}$, then $P_2 = \{A_1, A_2, A_5\}$ is also not a partition of A; because although A is the union of A_1, A_2, A_5, the sets A_1 and A_5 are not disjoint.

The following theorem, known as the *fundamental theorem on equivalence relations*, brings out the connection between an equivalence relation on a set and a partition of the set.

Theorem 4. *If A is a nonempty set, then:*

(1) *Any equivalence relation R on A induces a partition of A.*

(2) *Any partition of A gives rise to an equivalence relation R on A.*

Proof: (1) Suppose R is an equivalence relation on A and let P be the set of all *distinct* equivalence classes of the elements of A (w.r.t. R); that is

$$P = \{[a] \mid a \in A\}$$

Then, we note that every element a of A belongs to an equivalence class in P, namely $[a]$. Therefore, A is the union of the equivalence classes in P. Also every two (distinct) equivalence classes in P are mutually disjoint.* Therefore, P is a partition of A. Thus, an equivalence relation on A induces a partition of A (whose blocks are the distinct equivalence classes w.r.t. R).

(2) Let $P = \{A_1, A_2, A_3, \ldots, A_k\}$ be a partition of A. Define the relation R on A by aRb if and only if a and b belong to the same block of the partition.

Take any $a \in A$. Then, $a \in A_i$ for some i. As such, aRa, so that R is reflexive.

Suppose that aRb. Then $a, b \in A_i$ for some i. This yields $b, a \in A_i$. Therefore bRa, so that R is symmetric.

Lastly, suppose that aRb and bRc. Then $a, b \in A_i$ for some i, and $b, c \in A_j$ for some j. Thus, $b \in A_i \cap A_j$. Since A_i and A_j are blocks of a partition, this is possible only if $A_i = A_j$. Thus, $a, b, c \in A_i$. Therefore, aRc so that R is transitive.

Thus, R is reflexive, symmetric and transitive. As such, R is an equivalence relation. This equivalence relation is determined by the partition P. •

Example 9 *For the equivalence relation*

$$R = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,3), (3,3)(4,4)\}$$

defined on the set $A = \{1, 2, 3, 4\}$, determine the partition induced.

► By examining the given relation R, we find that the equivalence classes of the elements of A w.r.t. R are

$[1] = \{1, 2\}$, $[2] = \{1, 2\}$, $[3] = \{3, 4\}$, $[4] = \{3, 4\}$.

Of these equivalence classes, only $[1]$ and $[3]$ are distinct. These two distinct equivalence classes constitute the partition

$$P = \{[1], [3]\} = \{\{1, 2\}, \{3, 4\}\}.$$

*See Theorems 1 and 3.

This is the partition of the given A induced by the given R.

We observe that

$$A = [1] \cup [3] = \{1, 2\} \cup \{3, 4\}.$$

■

Example 10 *Consider the set* $A = \{1, 2, 3, 4, 5\}$ *and the equivalence relation*

$$R = \{(1, 1), (2, 2), (2, 3), (3, 2), (3, 3), (4, 4), (4, 5), (5, 4), (5, 5)\}$$

defined on A. Find the partition of A induced by R.

▶ By examining the given R, we find that

$$[1] = \{1\}, \quad [2] = \{2, 3\}, \quad [3] = \{2, 3\}, \quad [4] = \{4, 5\}, \quad [5] = \{4, 5\}.$$

Of these equivalence classes, only [1], [2] and [4] are distinct. These constitute the partition P of A determined by R. Thus, P = {[1], [2], [4]} is the partition induced by R. We observe that

$$A = [1] \cup [2] \cup [4] = \{1\} \cup \{2, 3\} \cup \{4, 5\}$$

■

Example 11 *For the set A and the relation R on A considered in Example 3, find the partition of A induced by R.*

▶ Here, $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$, and R is defined by $(x, y) \in R$ if and only if $x - y$ is a multiple of 5.

By examining the multiples of 5 present in the given set, we find that the equivalence classes of R are

$$[1] = [6] = [11], \quad [2] = [7] = [12],$$
$$[3] = [8], \quad [4] = [9], \quad [5] = [10].$$

All of these classes are distinct.

Therefore, the partition of A induced by R is

$$A = \{1, 6, 11\} \cup \{2, 7, 12\} \cup \{3, 8\} \cup \{4, 9\} \cup \{5, 10\}$$

■

Example 12 *Let* $A = \{a, b, c, d, e\}$. *Consider the partition* P $= \{\{a, b\}, \{c, d\}, \{e\}\}$ *of A. Find the equivalence relation inducing this partition.*

▶ Here, the given partition P consists of three blocks $\{a, b\}$, $\{c, d\}$ and $\{e\}$. Let R be the equivalence relation inducing this partition.

Since a, b belong to the same block, we have aRa, aRb, bRa, bRb.

Since c, d belong to the same block, we have cRc, cRd, dRc, dRd.

Since e belongs to the block $\{e\}$ which contains only e, we have eRe.

Thus, the required equivalence relation R is given by

$$R = \{(a, a), (a, b), (b, a), (b, b), (c, c), (c, d), (d, c), (d, d), (e, e)\}.$$

■

Example 13 *Let $A = \{1,2,3,4,5,6,7\}$ and R be the equivalence relation on A that induces the partition*

$$A = \{1,2\} \cup \{3\} \cup \{4,5,7\} \cup \{6\}.$$

Find R.

▶ Here, the given partition of A has four blocks: $\{1,2\}, \{3\}, \{4,5,7\}, \{6\}$. Let R be the equivalence relation inducing this partition.

Since the elements 1, 2 are in the same block, we have $1R1,\ 1R2,\ 2R1,\ 2R2$.

Since 3 belongs to the block $\{3\}$ which contains only 3, we have $3R3$.

Since 4, 5, 7 belong to the same block, we have

$$4R4,\ 4R5,\ 4R7,\ 5R4,\ 5R5,\ 5R7,\ 7R4,\ 7R5,\ 7R7.$$

Since 6 belongs to $\{6\}$ which contains only 6, we have $6R6$.

Thus, the required relation is

$$R = \{(1,1),(1,2),(2,1),(2,2),(3,3),(4,4),(4,5),(4,7),\\ (5,4),(5,5),(5,7),(7,4),(7,5),(7,7),(6,6)\}.$$ ■

Example 14 *On the set Z of all integers, a relation R is defined by aRb if and only $a^2 = b^2$. Verify that R is an equivalence relation. Determine the partition induced by this relation.*

▶ For any $a \in \mathsf{Z}$, we have $a^2 = a^2$; that is aRa. Hence R is reflexive.

Take any $a,\ b \in \mathsf{Z}$ and suppose that aRb. Then $a^2 = b^2$. This yields $b^2 = a^2$ so that bRa. Therefore, R is symmetric.

Take any $a,\ b,\ c \in \mathsf{Z}$ and suppose that aRb and bRc. Then $a^2 = b^2$ and $b^2 = c^2$. This yields $a^2 = c^2$ so that aRc. Therefore, R is transitive.

Thus, R is reflexive, symmetric and transitive. Therefore, R is an equivalence relation.

Next, for any $a \in \mathsf{Z}$, we have

$$\begin{aligned}[a] &= \{x \in \mathsf{Z} \mid (x,a) \in R\}\\ &= \left\{x \in \mathsf{Z} \mid x^2 = a^2\right\}\\ &= \{x \in \mathsf{Z} \mid x = \pm a\}\end{aligned}$$

This gives

$$[0] = \{0\},$$

and $$[n] = \{n, -n\} = [-n],\ \text{for } n \in \mathsf{Z}^+.$$

The above expressions show that the distinct equivalence classes w.r.t. R in Z are $[0] = \{0\}$ and $[n] = \{n, -n\}$, $n \in \mathsf{Z}^+$. Hence, the partition of Z induced by R is given by

$$\mathsf{P} = \{[0],[n]\}, \quad \text{where} \quad n \in \mathsf{Z}^+$$ ■

Example 15 *Let $A = \{1,2,3,4,5\}$. Define a relation R on $A \times A$ by $(x_1, y_1)\, R\, (x_2, y_2)$ if and only if $x_1 + y_1 = x_2 + y_2$.*

(i) *Verify that R is an equivalence relation on $A \times A$.*

(ii) *Determine the equivalence classes $[(1,3)], [(2,4)]$ and $[(1,1)]$.*

(iii) *Determine the partition of $A \times A$ induced by R.*

► (i) For all $(x, y) \in A \times A$, we have $x+y = x+y$; that is, $(x, y)R(x, y)$. Therefore, R is reflexive.

Next, take any $(x_1, y_1), (x_2, y_2) \in A \times A$ and suppose that $(x_1, y_1)\, R\, (x_2, y_2)$ Then $x_1 + y_1 = x_2 + y_2$. This gives $x_2 + y_2 = x_1 + y_1$ which means that $(x_2, y_2)\, R\, (x_1, y_1)$. Therefore, R is symmetric.

Next, take any $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in A \times A$ and suppose that $(x_1, y_1)\, R\, (x_2, y_2)$ and $(x_2, y_2)\, R\, (x_3, y_3)$. Then $x_1+y_1 = x_2+y_2$ and $x_2+y_2 = x_3+y_3$. This gives $x_1+y_1 = x_3+y_3$; that is, $(x_1, y_1)\, R\, (x_3, y_3)$. Therefore, R is transitive.

Thus, R is reflexive, symmetric and transitive. Therefore, R is an equivalence relation.

(ii) We note that

$$
\begin{aligned}
[(1,3)] &= \{(x, y) \in A \times A \mid (x, y)R(1,3)\} \\
&= \{(x, y) \in A \times A \mid x + y = 1 + 3\} \\
&= \{(1,3), (2,2), (3,1)\}, \quad \text{because } A = \{1,2,3,4,5\}
\end{aligned}
$$

Similarly, $[(2,4)] = \{(1,5), (2,4), (3,3), (4,2), (5,1)\}$

$[(1,1)] = \{(1,1)\}$.

(iii) To determine the partition induced by R, we have to find the equivalence classes of all elements (x, y), of $A \times A$, w.r.t. R. From what has been found above, we note that

$$
\begin{aligned}
&[(1,1)] = \{(1,1)\}, \\
&[(1,3)] = [(2,2)] = [(3,1)], \\
&[(2,4)] = [(1,5)] = [(3,3)] = [(4,2)] = [(5,1)].
\end{aligned}
$$

The other equivalence classes are

$$
\begin{aligned}
&[(1,2)] = \{(1,2), (2,1)\} = [(2,1)] \\
&[(1,4)] = \{(1,4), (2,3), (3,2), (4,1)\} = [(2,3)] = [(3,2)] = [(4,1)] \\
&[(2,5)] = \{(2,5), (3,4), (4,3), (5,2)\} = [(3,4)] = [(4,3)] = [(5,2)] \\
&[(3,5)] = \{(3,5), (4,4), (5,3)\} = [(4,4)] = [(5,3)]
\end{aligned}
$$

$$[(4,5)] = \{(4,5),(5,4)\} = [(5,4)]$$
$$[(5,5)] = \{(5,5)\}$$

Thus, $[(1,1)]$, $[(1,2)]$, $[(1,3)]$, $[(1,4)]$, $[(1,5)]$, $[(2,5)]$, $[(3,5)]$, $[(4,5)]$ and $[(5,5)]$ are the only distinct equivalence classes of $A \times A$ w.r.t. R. Hence the partition of $A \times A$ induced by R is represented by

$$A \times A = [(1,1)] \cup [(1,2)] \cup [(1,3)] \cup [(1,4)] \cup [(1,5)] \cup [(2,5)] \cup [(3,5)] \cup [(4,5)] \cup [(5,5)]. \blacksquare$$

Example 16 *Let R be an arbitrary transitive and reflexive relation on a set A. On the same set A, a relation S is defined by aSb if and only if aRb and bRa. Prove that S is an equivalence relation.*

▶ Since R is reflexive, we have aRa for all $a \in A$. This immediately implies that aSa for all $a \in A$. Therefore, S is reflexive.

Suppose aSb. Then aRb and bRa; that is, bRa and aRb. This implies bSa. Therefore, S is symmetric.

Suppose aSb and bSc. This implies that aRb, bRa, and bRc, cRb; that is, aRb, bRc and cRb and bRa. Since R is transitive, these imply that aRc and cRa. This means that aSc. Therefore, S is transitive.

This proves that S is an equivalence relation. ■

Exercises

1. Let $A = \{1,2,3\}$ and $R = \{(1,1),(2,2),(3,3)\}$. Verify that R is an equivalence relation.

2. Let $A = \{1,2,3,4\}$ and R_1 and R_2 be relations on A as given below:

$$R_1 = \{(1,1),(2,1),(2,2),(3,3),(4,4),(4,3)\}$$
$$R_2 = \{(1,1),(1,2),(2,1),(2,2),(3,1),(3,3),(1,3),(4,1),(4,4)\}$$

Verify that R_1 and R_2 are not equivalence relations.

3. The matrix of a relation R on the set $A = \{1,2,3\}$ is given by

$$M_R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Show that R is an equivalence relation.

4. A relation R on the set $\{a, b, c\}$ is represented by the digraph given below. Show that R is an equivalence relation.

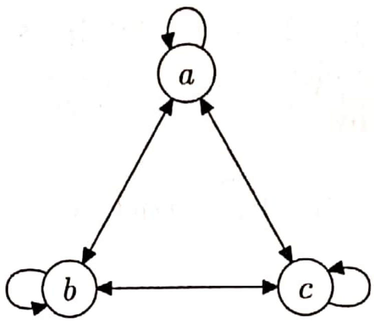


Figure 6.23

5. Let Z be the set of all integers, and let the relation R on Z be defined by aRb if and only if $a \leq b$. Is R an equivalence relation?

6. On the set of all integers, Z, the relation R is defined by $(a, b) \in R$ if and only if $a^2 - b^2$ is an even integer. Show that R is an equivalence relation.

7. On Z, define the relation R by xRy if and only if $4 \mid (x - y)$. Verify that R is an equivalence relation. Find the equivalence class of any integer i w.r.t. R.

8. Let N be the set of all natural numbers. On N × N, the relation R is defined $(a, b)R(c, d)$ if and only if $a + d = b + c$. Show that R is an equivalence relation. Find the equivalence class of the element $(2, 5) \in N \times N$.

9. If $A = \{1, 2, 3, \ldots, 10\}$, prove that, in each of the following cases, the given subsets of A yield a partition of A:

   (a) $A_1 = \{1, 2, 3, 4, 5\}$, $A_2 = \{6, 7, 8, 9, 10\}$,

   (b) $A_1 = \{1, 2, 3\}$, $A_2 = \{4, 6, 7, 9\}$, $A_3 = \{5, 8, 9\}$,

   (c) $A_i = \{i, i + 5\}$, $i = 1, 2, 3, 4, 5$.

10. Let $A = \{a, b, c, d, e, f, g, h\}$, $A_1 = \{d, e\}$, $A_2 = \{a, c, d\}$, $A_3 = \{f, h\}$, $A_4 = \{b, g\}$. Determine whether $\{A_1, A_2, A_3\}$ is a partition of A or not.

11. Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and

    $A_1 = \{1, 2, 3, 4\}$, $A_2 = \{5, 6, 7\}$, $A_3 = \{4, 5, 7, 9\}$, $A_4 = \{4, 8, 10\}$, $A_5 = \{8, 9, 10\}$, $A_6 = \{1, 2, 3, 6, 8, 10\}$.

    Which of the following are partitions of A?

    (a) $\{A_1, A_2, A_3\}$ (b) $\{A_1, A_3, A_5\}$ (c) $\{A_3, A_6\}$ (d) $\{A_2, A_3, A_4\}$

12. Verify that $R = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 1)\}$ is an equivalence relation on the set $A = \{1, 2, 3, 4\}$. Find the corresponding partition of A.

13. If $A = \{1, 2, 3, 4, 5\}$ and R is the equivalence relation on A that induces the partition $A = \{1, 2\} \cup \{3, 4\} \cup \{5\}$, find R.

14. On the set of all integers, Z, define the relation R by aRb if and only if $ab > 0$. Show that R is an equivalence relation and that the corresponding partition contains exactly two infinite sets.

15. Let $A = \{1, 2, 3, \ldots, 19, 20\}$, and R be the equivalence relation on A defined by aRb if and only if $a - b$ is divisible by 5. Find the partition of A induced by R.

16. Given that a relation R on a set A is symmetric and transitive, prove that R is an equivalence relation if and only if for every $a \in A$, there exists $b \in A$ such that $(a, b) \in R$.

17. Let R be a reflexive relation on a set A. Show that R is an equivalence relation if and only if (a, b) and $(a, c) \in R$ imply that $(b, c) \in R$.

18. Find the number of equivalence relations that can be defined on a finite set A with $|A| = 5$.

Answers

5. No 7. $[i] = \{4k + i \mid k \in Z\}$

8. $\{(1, 4), (2, 5), (3, 6), (4, 7), (5, 8), (6, 9)\}$ 10. No

11. (a), (c): Yes; (b), (d): No 12. $\{\{1, 2\}, \{3\}, \{4\}\}$

13. $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (4, 3), (4, 4), (5, 5)\}$.

15. $\{\{1, 6, 11, 16\}, \{2, 7, 12, 17\}, \{3, 8, 13, 18\}, \{4, 9, 14, 19\}, \{5, 10, 15, 20\}\}$.

18. 52.

6.4 Partial Orders

A relation R on set A is said to be a *partial ordering relation* or a ***partial order** on A* if
(i) R is *reflexive*, (ii) R is *antisymmetric*, and (iii) R is *transitive*, on A.

A set A with a partial order R defined on it is called a ***partially ordered set*** or an *ordered set* or a ***poset***, and is denoted by the pair (A, R).

The most familiar partial order is the relation "less than or equal to", denoted by $\leq$, on the set Z of all integers. (Because, this relation is reflexive, antisymmetric and transitive). Thus, $(Z, \leq)$ is a poset.

The relation "is greater than or equal to", denoted by $\geq$, is also a partial order on Z; that is, $(Z, \geq)$ is also a poset.

The "divisibility relation" on the set Z^+ defined by a divides b (denoted by $a|b$) for all $a, b \in Z^+$ is a partial order on Z^+; see Example 6, Section 6.2.

The "subset relation" $\subseteq$ defined on the power set of a set S is a partial order on S; see Example 7, Section 6.2. Thus, for any set S, $(P(S)\subseteq)$ is a poset.

The relations "is less than" and "is greater than" are not partial orders on Z; because, these are not reflexive.

The relation "congruent modulo n" defined on the set of all integers Z is also not a partial order; because this relation is not antisymmetric.*

Total Order

Let R be a partial order on a set A. Then R is called a ***total order*** * on A if *for all* $x, y \in A$, either xRy or yRx. In this case, the poset (A, R) is called a ***totally ordered set.*****

For example, the partial order relation "less than or equal to" is a total order on the set R. Because, for any $x, y \in$ R, we have $x \le y$ or $y \le x$. Thus, $(R, \le)$ is a totally ordered set.

If we consider the divisibility relation on the set $A = \{1, 2, 4, 8\}$, this relation is a total order on A. The same relation is not a total order on the set $A = \{1, 2, 4, 6, 8\}$ although it is a partial order on A. (Observe that neither 4 divides 6 nor 6 divides 4).

The subset relation is also not a total order on the power set of an arbitrary set S although it is a partial order; because for any two subsets S_1 and S_2 of S, neither $S_1 \subseteq S_2$ nor $S_2 \subseteq S_1$ can be true. (For example, if $S = \{1, 2, 3\}$, $S_1 = \{1, 2\}$ and $S_2 = \{1, 3\}$, then $S_1 \subseteq S$ and $S_2 \subseteq S$ but $S_1 \not\subseteq S_2$ and $S_2 \not\subseteq S_1$).

From the definition of a total order and the examples given above it is clear that *every total order is a partial order, but not every partial order is a total order.*

Hasse Diagrams

Since a partial order is a relation on a set, we can think of the digraph of a partial order if the set is finite. Since a partial order is reflexive, at every vertex in the digraph of a partial order there would be a cycle of length 1. In view of this, while drawing the digraph of a partial order, we *need not* exhibit such cycles explicitly; they will be automatically understood (by convention).

If, in the digraph of a partial order, there is an edge from a vertex a to a vertex b and there is an edge from the vertex b to a vertex c, then there should be an edge from a to c (because of transitivity). As such, we ***need not*** exhibit an edge from a to c explicitly; it will be automatically understood (by convention).

To simplify the format of the digraph of a partial order, we represent the vertices by dots (bullets) and draw the digraph in such a way that all edges point upward. With this convention, we ***need not*** put arrows in the edges.

The digraph of a partial order drawn by adopting the conventions indicated in the above paragraphs is called a ***poset digram*** or the ***Hasse diagram*** for the partial order.

---

*Recall the definition of the relation *congruent modulo n* from Example 8, Section 6.3. Note that $a \equiv b \pmod n$ and $b \equiv a \pmod n$ do not always imply that $a = b$. For example, $2 \equiv 8 \pmod 3$ and $8 \equiv 2 \pmod 3$ but $2 \ne 8$.

*Total order is also called a ***linear order***.

A totally ordered set is also called a ***linearly ordered set* or a ***chain.***

Example 1 *Let* $A = \{1, 2, 3, 4\}$, *and*

$$R = \{(1,1), (1,2), (2,2), (2,4), (1,3), (3,3), (3,4), (1,4), (4,4)\}$$

Verify that R *is a partial order on* A. *Also, write down the Hasse diagram for* R.

▸ We observe that the given relation R is reflexive and transitive. Further, R does not contain ordered pairs of the form (a, b) and (b, a) with $b \neq a$. Therefore, R is antisymmetric. As such, R is a partial order on A.

The Hasse diagram for R must exhibit the relationships between the elements of A as defined by R; if $(a, b) \in R$, there must be an upward edge from a to b.

By examining the ordered pairs contained in R, we find that the Hasse diagram of R is as shown below:

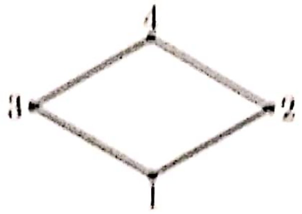


Figure 6.24 ■

Example 2 *Let* R *be a relation on the set* $A = \{1, 2, 3, 4\}$ *defined by* xRy *if and only if* x *divides* y. *Prove that* (A, R) *is a poset. Draw its Hasse diagram.*

▸ From the definition of R, we have

$$\begin{aligned} R &= \{(x, y) \mid x, y \in A \text{ and } x \text{ divides } y\} \\ &= \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\} \end{aligned}$$

We observe that $(a, a) \in R$ for all $a \in A$. Hence R is reflexive on A.

We verify that the elements of R are such that if $(a, b) \in R$ and $a \neq b$, then $(b, a) \notin R$. Therefore, R is antisymmetric on A.

Further, we check that the elements of R are such that if $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$. Therefore, A is transitive on A.

Thus, R is reflexive, antisymmetric and transitive. Hence R is a partial order on A; that is, (A, R) is a poset.

The Hasse diagram for R is as shown below.

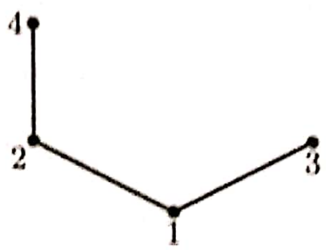


Figure 6.25 ■

Example 3 *Let $A = \{1, 2, 3, 4, 6, 12\}$. On A, define the relation R by aRb if and only if a divides b. Prove that R is a partial order on A. Draw the Hasse digram for this relation.*

► From the definition of R, we note that

$$R = \{(a, b) \mid a, b \in A \text{ and } a \text{ divides } b\}$$
$$= \{(1,1), (1,2), (1,3), (1,4), (1,6), (1,12), (2,2), (2,4), (2,6)$$
$$(2,12), (3,3), (3,6), (3,12), (4,4), (4,12), (6,6), (6,12), (12,12)\}$$

Evidently, $(a, a) \in R$ for all $a \in A$. Therefore, R is reflexive.

We check that the elements of R are such that if $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$. Therefore, R is transitive.

Further, for all $a, b \in A$, if a divides b and b divides a, then $a = b$. Hence R is antisymmetric.

Therefore, R is a partial order on A. The Hasse diagram for R is shown below.

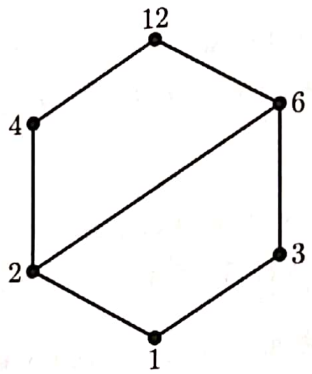


Figure 6.26

Example 4 *Let $A = \{1, 2, 3, 4, 6, 8, 12\}$. On A, define the partial ordering relation R by xRy if and only if $x|y$.* Draw the Hasse diagram for R.*

► By using the definition of R, we note that

$$R = \{(1,1), (1,2), (1,3), (1,4), (1,6), (1,8), (1,12),$$
$$(2,2), (2,4), (2,6), (2,8), (2,12), (3,3), (3,6), (3,12),$$
$$(4,4), (4,8), (4,12), (6,6), (6,12), (8,8), (12,12)\}$$

*The symbol (notation) $x|y$ stands for "x divides y".

The Hasse diagram for this R is as shown below.

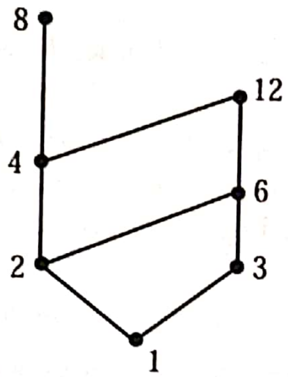


Figure 6.27

■

Example 5 *Draw the Hasse diagram representing the positive divisors of* 36.

▶ The set of all positive divisors of 36 is

$$D_{36} = \{1, 2, 3, 4, 6, 9, 12, 18, 36\} \qquad (*)$$

The relation R of divisibility (that is, aRb if and only if a divides b) is a partial order on this set. The Hasse diagram for this partial order is required here.

We note that, under R,

1 is related to all elements of D_{36},
2 is related to 2, 4, 6, 12, 18, 36;
3 is related to 3, 6, 9, 12, 18, 36;
4 is related to 4, 12, 36;
6 is related to 6, 12, 18, 36;

9 is related to 9, 18, 36;
12 is related to 12 and 36;
18 is related to 18 and 36;
36 is related to 36.

The Hasse diagram for R must exhibit all of the above facts. The diagram is as shown below:

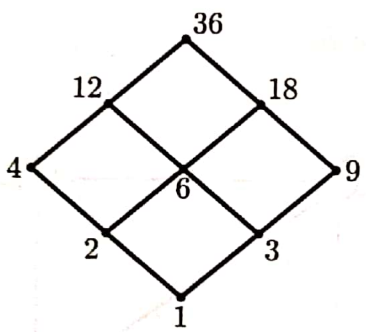


Figure 6.28

■

*For any specified positive integer n, the set of all positive divisors of n is denoted by D_n.

Example 6 *Prove that the set of all positive integers is not totally ordered by the relation of divisibility.*

► For a set A to be totally ordered by a partial order R, we should have aRb or bRa, for every $a, b \in A$.

If R is the divisibility relation on Z^+, aRb or bRa need not hold for every $a, b \in Z^+$. For example, if we take $a = 2$ and $b = 3$, then a does not divide b and b does not divide a.

Therefore, Z^+ is not totally ordered by the relation of divisibility. ∎

Example 7 *In the following cases, consider the partial order of divisibility on the set A. Draw the Hasse diagram for the poset and determine whether the poset is totally ordered or not.*

(i) $A = \{1, 2, 3, 5, 6, 10, 15, 30\}$ (ii) $A = \{2, 4, 8, 16, 32\}$

► The Hasse diagram for the two cases are as shown below:

(i)

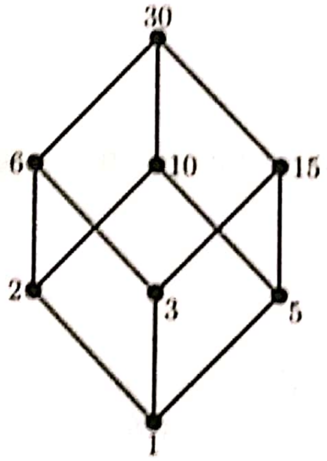


Figure 6.29

(ii)

Figure 6.30

By examining the above Hasse diagrams, we find that the given relation is totally ordered in case (ii), but is not totally ordered in case (i). ∎

Example 8 *A partial order R on the set A = {1, 2, 3, 4} is represented by the following digraph. Draw the Hasse diagram for R.*

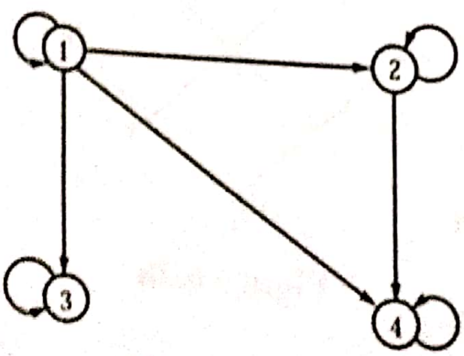


Figure 6.31

Scanned by CamScanner

▸ By observing the given digraph, we note that

$$R = \{(1,1),(2,2),(3,3),(4,4),(1,2),(1,3),(1,4),(2,4)\}.$$

The Hasse diagram for this R is as shown below:

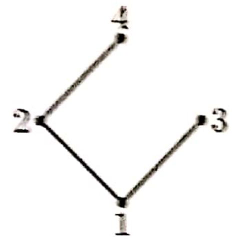


Figure 6.32

■

Example 9 *The digraph for a relation on the set $A = \{1,2,3,6,8\}$ is as shown below:*

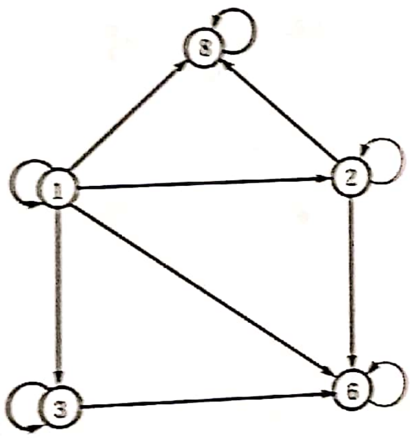


Figure 6.33

Verify that (A, R) is a poset and write down its Hasse diagram.

▸ By examining the given digraph, we find that

$$R = \{(1,1),(1,2),(1,3),(1,6),(1,8),(2,2),(2,6),(2,8),(3,3),(3,6),(6,6),(8,8)\}$$

We check that R is reflexive, transitive and antisymmetric. Therefore, R is a partial order on A. (That is, (A, R) is a poset). The Hasse diagram for R is as shown below:

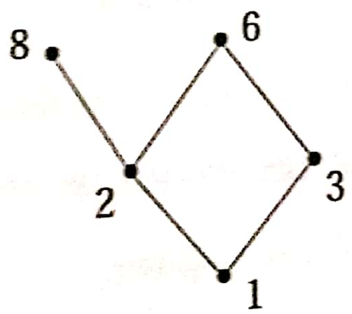


Figure 6.34

■

Example 10 *Draw the Hasse diagram of the relation R on $A = \{1, 2, 3, 4, 5\}$ whose matrix is as given below:*

$$M_R = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

► By examining the given M_R, we note that

$$R = \{(1,1), (1,3), (1,4), (1,5), (2,2), (2,3), (2,4), (2,5), (3,3), (3,4), (3,5), (4,4), (5,5)\}$$

By looking at the elements of R, we can write down the Hasse diagram of R; it is as shown below.

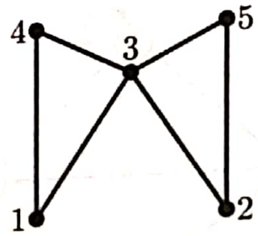


Figure 6.35

Example 11 *The Hasse diagram of a partial order R on the set $A = \{1, 2, 3, 4, 5, 6\}$ is as given below. Write down R as a subset of $A \times A$. Construct its digraph.*

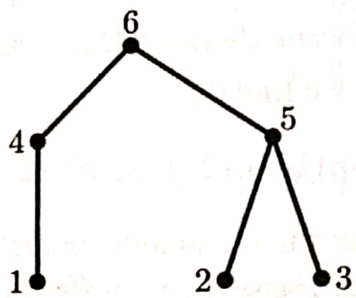


Figure 6.36

► By examining the given Hasse diagram, we note the following:

$$1R4, 1R6, 2R5, 2R6, 3R5, 3R6, 4R6, 5R6.$$

Also, by the convention used in Hasse diagrams,

$$1R1, 2R2, 3R3, 4R4, 5R5, 6R6.$$

Therefore,

$R = \{(1,1), (1,4), (1,6), (2,2), (2,5), (2,6), (3,3), (3,5), (3,6), (4,4), (4,6), (5,5), (5,6), (6,6)\}$

The digraph of this relation is as shown below.

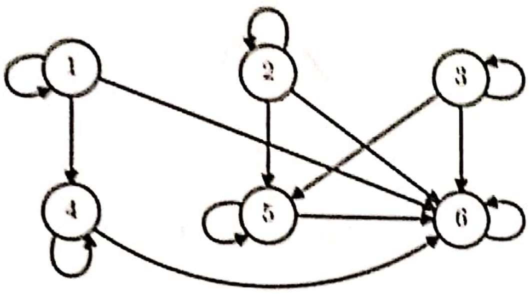


Figure 6.37

■

Example 12 *Determine the matrix of the partial order whose Hasse diagram is given below:*

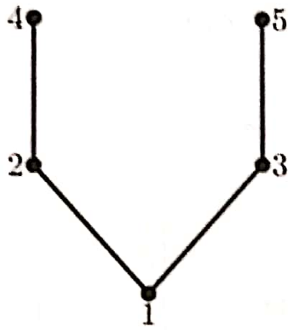


Figure 6.38

▶ By examining the given Hasse diagram, we find that the corresponding partial order R is defined on the set $A = \{1,2,3,4,5\}$ and is given by

$$R = \{(1,1), (1,2), (1,4), (1,3), (1,5), (2,2), (2,4), (3,3), (3,5), (4,4), (5,5)\}.$$

Consequently, the matrix of R is as given below:

$$M_R = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(This matrix can be written down directly by examining the given Hasse diagram. We have written down R explicitly to make things clearer). ■

Example 13 *For* $A = \{a, b, c, d, e\}$, *the Hasse diagram for the poset* (A, R) *is as shown below.*

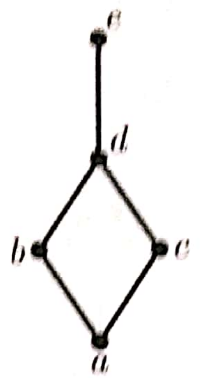


Figure 6.39

(a) *Determine the relation matrix for R.*

(b) *Construct the digraph for R.*

► By examining the given Hasse diagram, we note that

$$R = \{(a,a),(a,b),(a,c),(a,d),(a,e),(b,b),(b,d),(b,e),(c,c),(c,d),(c,e),(d,d),(d,e),(e,e)\}$$

(a) The Matrix of R is as shown below

$$M(R) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) The digraph of R is as shown below.

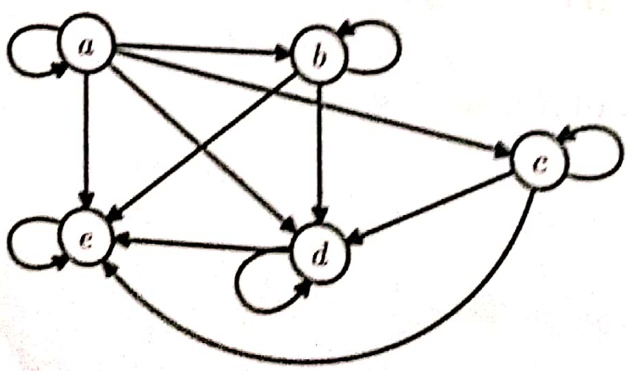


Figure 6.40

Example 14 *The digraph for a relation on set $A = \{1, 2, 3, 4\}$ is as shown below.*

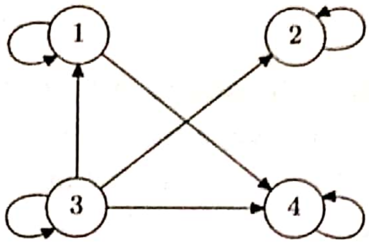


Figure 6.41

(a) *Verify that (A, R) is a poset and draw its Hasse diagram.*

(b) *How many more directed edges are needed in the digraph of R to extend R to a total order ?*

▶ By examining the given digraph, we find that

$$R = \{(1,1), (1,4), (2,2), (3,3), (3,1), (3,2), (3,4), (4,4)\}$$

(a) We check that R is reflexive, transitive and antisymmetric. Therefore, R is a partial order on A. The Hasse diagram of R is as shown below.

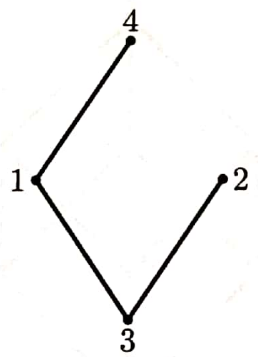


Figure 6.42

(b) By examining the Hasse diagram of R, we find that the vertices that are to be identified as v_1, v_2, v_3, v_4 in the topological sorting process are $v_1 = 4$, $v_2 = 1$ (or 2), $v_3 = 2$ (or 1), $v_4 = 3$. Thus, the total order T on A that contains R as a subset is

$$3 < 2 < 1 < 4 \quad \text{or} \quad 3 < 1 < 2 < 4.$$

(c) We observe that $T = \{(3,3), (3,2), (3,1), (3,4), (2,2), (2,1), (2,4), (1,1), (1,4), (4,4)\}$. By comparing R and T, we find that T contains 2 more directed edge than R, namely $(2,1)$ and $(2,4)$. Thus, two more directed edges are needed in the digraph of R to extend R to a total order T. ■

Example 15 *Let $S = \{1, 2, 3\}$ and $\mathcal{P}(S)$ be the power set of S. On $\mathcal{P}(S)$, define the relation R by XRY if and only if $X \subseteq Y$. Show that this relation is a partial order on $\mathcal{P}(S)$. Draw its Hasse diagram.*

▶ For any set S, the set $\mathcal{P}(S)$ contains all subsets of S. In Example 11 of Section 6.2 we have proved that the subset relation $\subseteq$ is reflexive, antisymmetric and transitive on $\mathcal{P}(S)$ for any set S; this relation is therefore a partial order on S, for any set S. For the given particular set $S = \{1, 2, 3\}$ also, $\subseteq$ is a partial order.*

For the given $S = \{1, 2, 3\}$, the subsets of S are

$$\Phi,\ S_1 = \{1\},\ S_2 = \{2\},\ S_3 = \{3\},\ S_4 = \{1, 2\}$$
$$S_5 = \{2, 3\},\ S_6 = \{3, 1\},\ \text{and } S.$$

We check that

$$\Phi \subseteq S_r,\ r = 1, 2, \dots 6, \text{ and } \Phi \subseteq S,$$
$$S_r \subseteq S, \text{ and } S_r \subseteq S_r,\ r = 1, 2, \dots 6,$$
$$S_1 \subseteq S_4,\ S_1 \subseteq S_6,\ S_2 \subseteq S_4,\ S_2 \subseteq S_5,\ S_3 \subseteq S_5,\ S_3 \subseteq S_6.$$

The Hasse diagram for R must exhibit all these facts. This diagram is shown below.

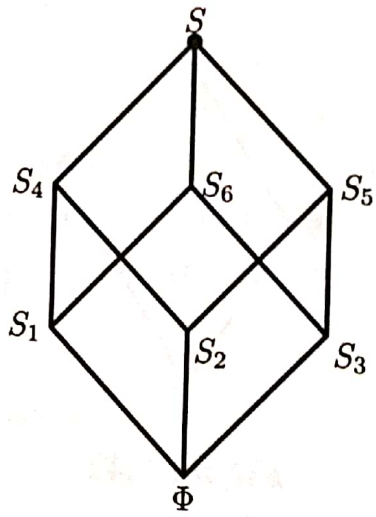


Figure 6.43

▮

Example 16 *Draw the Hasse diagram for the poset $(\mathcal{P}(S), \subseteq)$, where $S = \{1, 2, 3, 4\}$.*

▶ Since $|S| = 4$, we have $|\mathcal{P}(S)| = 2^4 = 16$. Thus, there are 16 subsets of S; these subsets are

$$\Phi,\ S_1 = \{1\},\ S_2 = \{2\},\ S_3 = \{3\},\ S_4 = \{4\},$$
$$S_5 = \{1, 2\},\ S_6 = \{1, 3\},\ S_7 = \{1, 4\},$$
$$S_8 = \{2, 3\},\ S_9 = \{2, 4\},\ S_{10} = \{3, 4\},$$

*The student can verify this fact explicitly b...

$S_{11} = \{1, 2, 3\}$, $S_{12} = \{1, 2, 4\}$, $S_{13} = \{1, 3, 4\}$,

$S_{14} = \{2, 3, 4\}$, and S.

We observe that

$$\Phi \subseteq \Phi, \quad S \subseteq S,$$

$$\Phi \subseteq S_r \subseteq S_r \subseteq S \text{ for } r = 1, 2, \ldots 14,$$

$$S_1 \subseteq S_5,\ S_1 \subseteq S_6,\ S_2 \subseteq S_7,$$

$$S_2 \subseteq S_5,\ S_2 \subseteq S_8,\ S_2 \subseteq S_9,$$

$$S_3 \subseteq S_6,\ S_3 \subseteq S_8,\ S_3 \subseteq S_{10},$$

$$S_4 \subseteq S_7,\ S_4 \subseteq S_9,\ S_4 \subseteq S_{10},$$

$$S_5 \subseteq S_{11},\ S_5 \subseteq S_{12},\ S_6 \subseteq S_{11},\ S_6 \subseteq S_{13},$$

$$S_7 \subseteq S_{12},\ S_7 \subseteq S_{13},\ S_8 \subseteq S_{11},\ S_8 \subseteq S_{14},$$

$$S_9 \subseteq S_{12},\ S_9 \subseteq S_{14},\ S_{10} \subseteq S_{13},\ S_{10} \subseteq S_{14}$$

In view of the above observations, the Hasse diagram for the poset being considered appears as shown below.

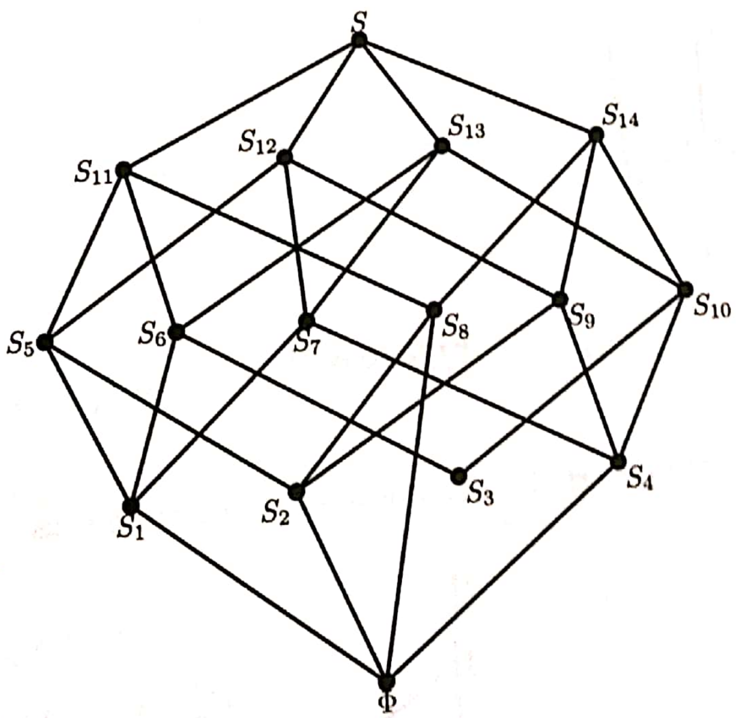


Figure 6.44

■

Exercises

1. In each of the following cases, determine whether the given relation R is a partial order or not on the set of all integers.

   (1) aRb if and only if $a = 3b$.

   (2) aRb if and only if a is multiple of b^2.

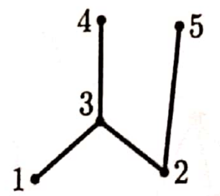
   (3) aRb if and only if a is equal to some positive integral power of b.

2. Let $A = \{1, 2, 3, 4, 5\}$ and R be a relation on A defined by xRy if $x \le y$. Verify that R is a partial order on A.

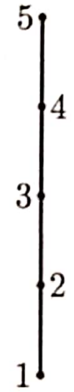
3. Give an example of a relation which is both a partial order and an equivalence relation on a set.

4. Let $A = \{1, 2, 3, 4, 5\}$. Determine the partial orders represented by the following Hasse diagrams.

   (i) (ii)



(a)



(b)

Figure 6.45

5. Determine the matrix of the partial order whose Hasse diagram is as given below:

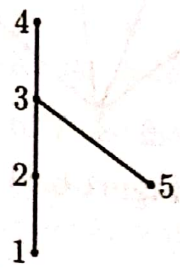


Figure 6.46

6. Write down the matrix of the partial order considered in Worked Example 4.

7. Let $A = \{1, 2, 3, 4, 5\}$, and

$$R = \{(1,1), (2,2), (3,3), (1,3), (3,4), (3,5), (1,4), (4,4), (1,5), (2,3), (2,4)(2,5)(5,5)\}$$

Draw the Hasse diagram for R.

8. Draw the Hasse diagram for the partial order R on the set $A = \{1, 2, 3, 4, 5\}$ whose matrix is

$$M_R = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

9. The digraph of a relation R defined on the set $A = \{1, 2, 3, 4\}$ is as shown below. Verify that (A, R) is a poset and draw the corresponding Hasse diagram.

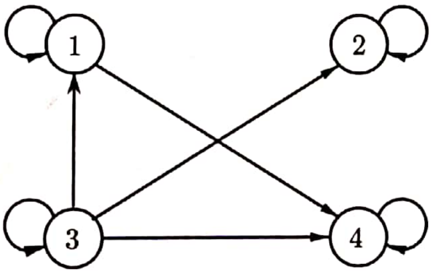


Figure 6.47

10. Write down the Hasse diagram for the positive divisors of 45.

11. Draw the Hasse diagram for the divisibility relation on the set A in each of the following cases:

(a) $A = \{3, 6, 12, 36, 72\}$ (b) $A = \{1, 2, 3, 5, 6, 10, 15, 30\}$

(c) $A = \{2, 3, 6, 12, 24, 36\}$ (d) $A = \{1, 2, 3, 4, 6, 8, 9, 12, 18, 24\}$

(e) $A = \{2, 3, 5, 6, 7, 11, 12, 35, 385\}$

12. Show that the divisibility relation is not a total order on the set $A = \{1, 2, 3, 6\}$ but is a total order on the set $B = \{1, 3, 6, 12, 24\}$.

13. On Z, define the relation R by aRb if $a - b$ is a non-negative even integer. Verify that R is a partial order but not a total order.

Answers

1. (1) No (2) No (3) Yes

3. On any set A, the relation $\Delta_A = \{(a, a) \mid a \in A\}$ is an equivalence relation as well as a partial order.

4. (i) $R = \{(1,1), (1,3), (1,4), (2,2), (2,3), (2,4), (2,5), (3,3), (3,4), (4,4), (5,5)\}$

(ii) $R = \{(1,1), (1,2), (1,3), (1,4), (1,5), (2,2), (2,3), (2,4),$
$(2,5), (3,3), (3,4), (3,5), (4,4), (4,5), (5,5)\}$

5. $M_R = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$

6. $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

7.

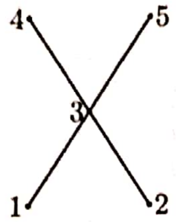


Figure 6.48

8.

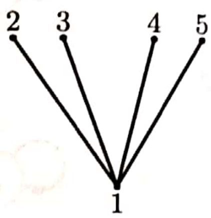


Figure 6.49

9.

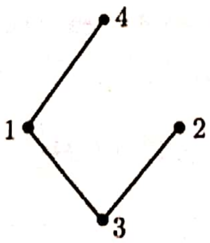


Figure 6.50

10.

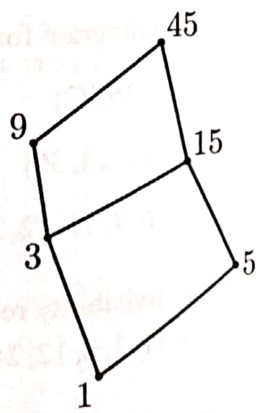


Figure 6.51

11.

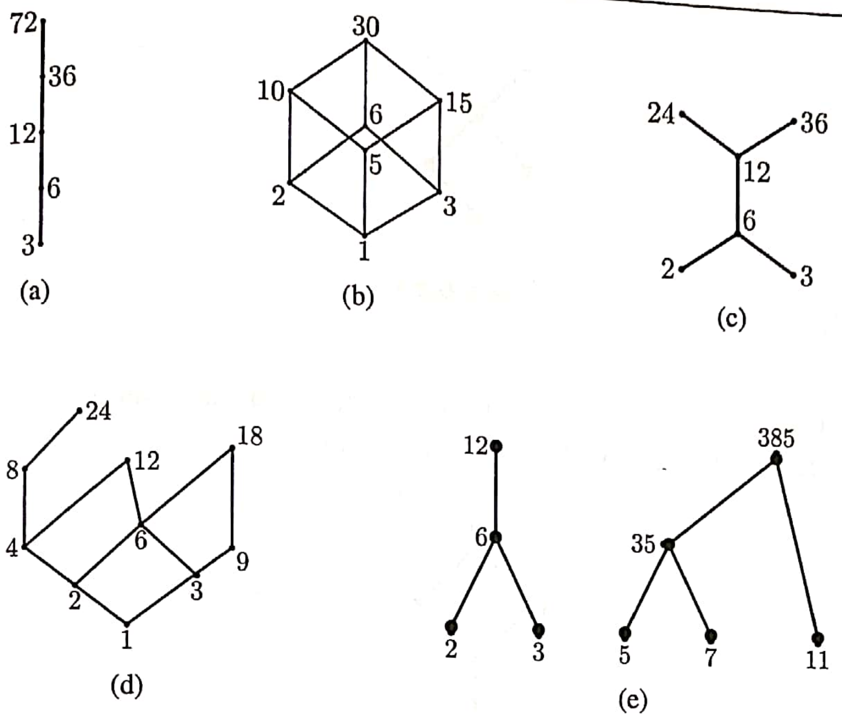


Figure 6.52

6.4.1 Extremal elements in Posets

Consider a poset (A, R). We define below some special elements (called *extremal elements*) that may exist in A.

(1) An element $a \in A$ is called a **maximal element** of A if there exists no element $x \neq a$ in A such that aRx. In other words, $a \in A$ is a maximal element of A if whenever there is $x \in A$ such that aRx then $x = a$.

This means that a is a maximal element of A if (and only if) in the Hasse diagram of R no edge starts at a.

(2) An element $a \in A$ is called a **minimal element** of A if there exists no element $x \neq a$ in A such that xRa. In other words, a is a minimal element of A if whenever there is $x \in A$ such that xRa, then $x = a$.

This means that a is a minimal element of A if (and only if) in the Hasse diagram of R no edge terminates at a.

(3) An element $a \in A$ is called a **greatest element** of A if xRa for all $x \in A$.

(4) An element $a \in A$ is called a **least element** of A if aRx for all $x \in A$.

For example, in the poset represented by the following Hasse diagram, 5 and 6 are maximal elements and 1 is a minimal element. Further, 1 is the least element as well. There is no greatest element.

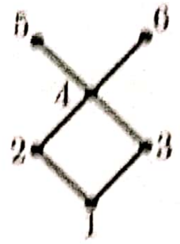


Figure 6.53

In the poset represented by the following diagram, 5 is a maximal as well as a greatest element, and 1 is a minimal as well as a least element.

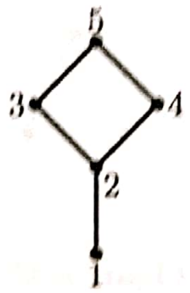


Figure 6.54

(5) An element $a \in A$ is called an **upper bound** ***of a subset*** B ***of*** A if xRa for all $x \in B$.

(6) An element $a \in A$ is called a **lower bound** ***of a subset*** B ***of*** A if aRx for all $x \in B$.

(7) An element $a \in A$ is called the **least upper bound** (LUB) ***of a subset*** B ***of*** A if the following two conditions hold:

(i) a is an upper bound of B. (ii) If a' is an upper bound of B, then aRa'.

Least upper bound is also called ***Supremum***, written as *Sup*.

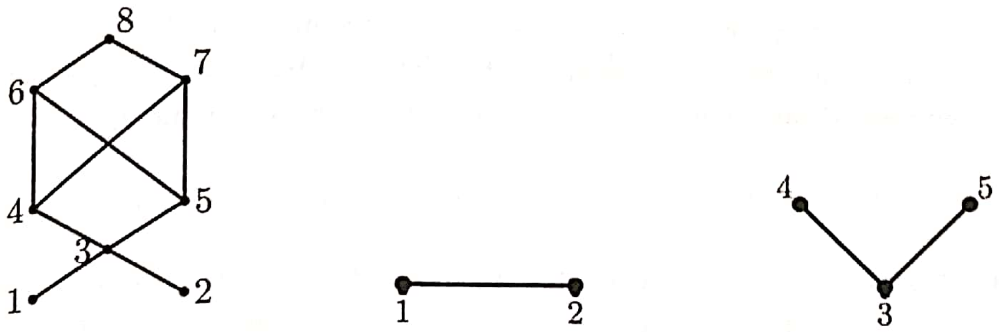
(8) An element $a \in A$ is called the **greatest lower bound** (GLB) ***of a subset*** B ***of*** A if the following two conditions hold:

(i) a is a lower bound of B. (ii) If a' is a lower bound of B, then $a'Ra$.

Greatest lower bound is also called ***Infimum***, written as *Inf*.

Remark: Whereas maximal element, minimal element, greatest element and least element are defined with reference to the set A as a whole, upper bound, lower bound, least upper bound, greatest lower bound are defined with reference to a specified subset of A.

For example, consider the set $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and a partial order on A whose Hasse diagram is as shown in Figure 6.55(a). Consider the subsets $B_1 = \{1, 2\}$ and $B_2 = \{3, 4, 5\}$ of A shown in Figures 6.55(b), (c).



(A, R) B_1 B_2

(a) (b) (c)

Figure 6.55

We make the following observations:

(1) $1R3$, $2R3$. Therefore, 3 is an upper bound of B_1. For a similar reason, 4,5,6,7,8 are also upper bounds of B_1.

(2) The upper bound 3 of B_1 is such that $3Rx$ for all upper bounds x of B_1. Therefore, 3 is a least upper bound (LUB) of B_1; we write this as LUB $(B_1) = 3$

(3) In A, there is no element x such that $xR1$ and $xR2$. Therefore, B_1 has no lower bounds.

(4) Since B_1 has no lower bounds, it has no greatest lower bound.

(5) For each $x \in B_2$, we have $xR6$. Therefore, 6 is an upper bound of B_2. For a similar reason, 7 and 8 are also upper bounds of B_2.

(6) Although 6 is the least of the upper bound of B_2, 6 is not related to the upper bound 7. Therefore, B_2 has no least upper bound.

(7) For each $x \in B_2$, we have $1Rx$. Therefore, 1 is a lower bound for B_2. For a similar reason, 2 and 3 are also lower bounds of B_2.

(8) Among the lower bounds 1,2,3 of B_2, 3 is such that $1R3$, $2R3$ and $3R3$. Therefore, 3 is the greatest lower bound (GLB) of B_2; we write this as GLB $(B_2) = 3$.

The following theorems contain some results on extremal elements.

Theorem 1. *If (A, R) is a poset and A is finite (non-empty), then A has at least one maximal element and at least one minimal element.*

Proof:. Take an element $a \in A$. If a is not maximal, we can find an element $x_1 \in A$ such that $x_1 \neq a$ and aRx_1. If x_1 is not maximal, we can find an element $x_2 \in A$ such that $x_2 \neq x_1$ and x_1Rx_2. Proceeding like this, we end up with a finite chain of the form

$$aRx_1, \quad x_1Rx_2, \quad x_2Rx_3, \ldots$$

which cannot be extended beyond a certain final stage (because A is finite). Hence, we end up with some $x_k \in A$ which is a maximal element of A. Thus, A has at least one maximal element.

A similar argument shows that A has a (at least one) minimal element.

Theorem 2. *Every poset has at most one greatest element and at most one least element.*

Proof:. Assume that a and b are greatest elements of (A, R). Then since b is a greatest element, we have aRb. Similarly, since a is greatest element, we have bRa. Thus, aRb and bRa. Since R is an antisymmetric relation, it follows that $a = b$. Thus, two greatest elements of (A, R), if such exist, cannot be different. In other words, if (A, R) has a greatest element, then that element is unique. Thus, (A, R) has at most one greatest element.

A similar argument shows that (A, R) has at most one least element.

Theorem 3. *If (A, R) is a poset and $B \subseteq A$, then B has at most one* LUB *and at most one* GLB.

Proof:. Assume that a and b are two LUB's of B. Then a and b are upper bounds of B. Since b is an upper bound of B and a is LUB of B, we have aRb. Similarly, bRa. Since R is antisymmetric, it follows that $a = b$. Hence B cannot have two distinct LUB's. This means that B has at most one LUB.

A similar argument shows that B has at most one GLB.

Example 1 *Consider the Hasse diagram of a poset (A, R) given below.*

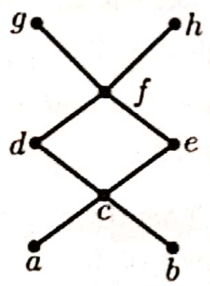


Figure 6.56

If $B = \{c, d, e\}$, find (if they exist)

(i) *all upper bounds of B,*

(ii) *all lower bounds of B,*

(iii) *the least upper bound of B,*

(iv) *the greatest lower bound of B.*

► By examining the given Hasse diagram, we note the following:

(i) All of c, d, e which are in B are related to f, g, h. Therefore, f, g, h are upper bounds of B.

(ii) The elements a, b and c are related to all of c, d, e which are in B. Therefore, a, b and c are lower bounds of B.

(iii) The upper bound f of B is related to the other upper bounds g and h of B. Therefore, f is the LUB of B.

(iv) The lower bounds a and b of B are related to the lower bound c of B. Therefore, c is the GLB of B. ■

Example 2 *Consider the poset whose Hasse diagram is shown below. Find* LUB *and* GLB *of* $B = \{c, d, e\}$.

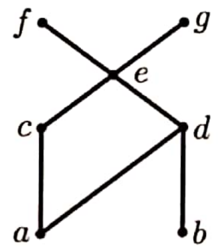


Figure 6.57

▸ By examining all upward paths from c, d, e in the given Hasse diagram, we find that LUB $(B) = e$. By examining all upward paths to c, d, e we find that GLB$(B) = a$. ■

Example 3 *Let* R *be the set of all real numbers with* $\leq$ *as the partial order. Also, let B be the open interval (1, 2). Find* (i) *all upper bounds of B,* (ii) *all lower bounds of B,* (iii) *the* LUB *of B,* (iv) *the* GLB *of B.*

▸ Here, $B = \{x \in \mathrm{R} \mid 1 < x < 2\}$. By examining this B, we note that

(i) Every real number ≥ 2 is an upper bound of B.

(ii) Every real number ≤ 1 is a lower bound of B.

(iii) 2 is the LUB of B.

(iv) 1 is the GLB of B.

Remark: This example illustrates the important fact that LUB and GLB of a set need not belong to the set. (Read the definitions of LUB and GLB carefully). ■

Lattice

Let (A, R) be a poset. This poset is called a ***lattice*** if, for all $x, y \in A$, the elements LUB$\{x, y\}$ and GLB$\{x, y\}$ exist in A.

For example, consider the set N of all natural numbers, and let R be the partial order "less than or equal to". Then for any $x, y \in \mathrm{N}$, we note that LUB$\{x, y\}$ = max $\{x, y\}$ and GLB$\{x, y\}$ = min$\{x, y\}$ and both of these belong to N. Therefore, the poset $(\mathrm{N}, \leq)$, is a lattice.

As another example, consider the poset $(Z^+, |)$, where Z^+ is the set of all positive integers and $|$ is the divisibility relation. We can check that for any $a, b \in Z^+$, the least common multiple of a and b is the LUB$\{a, b\}$ and the greatest common divisor of a and b is the GLB$\{a, b\}$. Since these belong to Z^+, we infer that $(Z^+, |)$ is a lattice.

Lastly, consider the poset whose Hasse diagram is shown below.

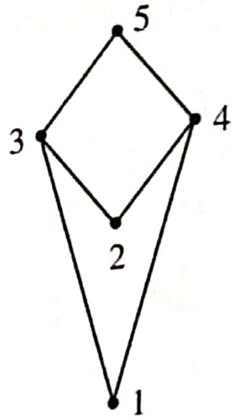


Figure 6.58

By examining the Hasse diagram, we note that GLB$\{3, 4\}$ does not exist. Therefore, the poset is not a lattice. ∎

Exercises

1. For the posets shown in the following Hasse diagrams, find all maximal elements and all minimal elements:

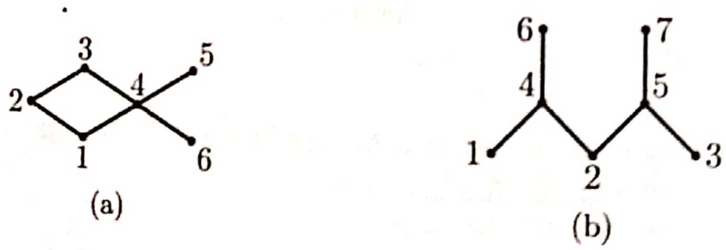
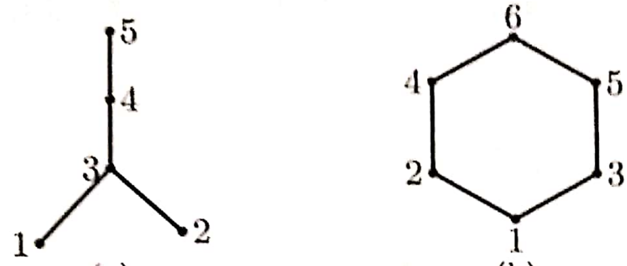


Figure 6.59

2. For the posets shown in the following Hasse diagrams, find the greatest and least elements, if such exist.



(a) (b)

Figure 6.60

3. For the poset $(\mathbb{R}, \leq)$, verify the following:

   (i) If $B = [0, 1]$, then 0 is GLB and 1 is LUB for B.

   (ii) If $B = (0, 1]$, then 0 is GLB and 1 is LUB for B.

   (iii) If $B = \{q \in \mathbb{Q}^* \mid q^2 < 2\}$, then $\sqrt{2}$ is a LUB and $(-\sqrt{2})$ is a GLB for B.

   Identify the maximal and minimal elements.

4. Consider the poset $(\mathcal{P}(S), \subseteq)$, where $S = \{1, 2, 3, 4\}$. For each of the following subsets B(of A), determine the LUB and GLB of B:

   (i) $B = \{\{1\}, \{2\}\}$ (ii) $B = \{\{1\}, \{2\}, \{1, 2\}\}$

   (iii) $B = \{\{1\}, \{2\}, \{3\}, \{1, 2\}\}$ (iv) $B = \{\Phi, \{1\}, \{2\}, \{1, 2\}\}$

   (v) $B = \{\{1\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}\}$

5. For the poset (A, R) defined on the set $A = \{2, 3, 5, 6, 7, 11, 12, 35, 385\}$ as represented by the following Hasse diagram, find (if they exist):

   LUB$\{2, 3\}$, LUB$\{2, 12\}$, LUB$\{3, 6\}$, LUB$\{5, 7\}$, LUB$\{7, 11\}$, LUB$\{11, 35\}$,
   GLB$\{2, 3\}$, GLB$\{3, 6\}$, GLB$\{2, 12\}$, GLB$\{6, 12\}$, GLB$\{35, 385\}$.

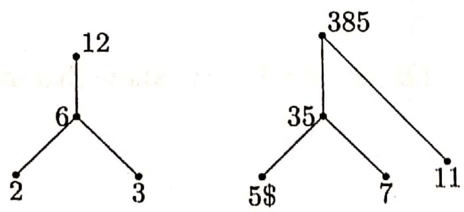


Figure 6.61

*Q denotes the set of all rational numbers

6. For the poset (A, R) represented by the following Hasse diagram, find

(i) GLB $\{b, c\}$ (ii) GLB $\{b, w\}$ (iii) GLB $\{e, x\}$

(iv) LUB $\{c, b\}$ (v) LUB $\{d, x\}$ (vi) LUB $\{c, e\}$ (vii) LUB $\{a, v\}$

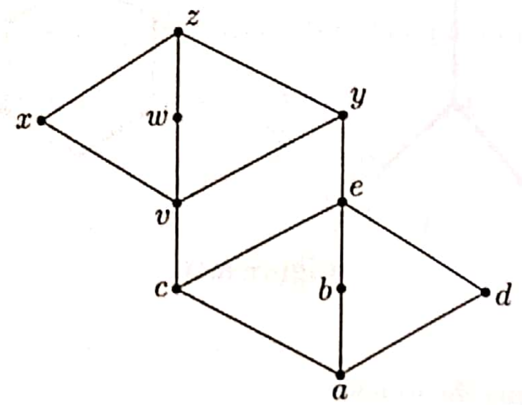


Figure 6.62

7. For the posets shown in the following Hasse diagrams, find (i) all upper bounds, (ii) all lower bounds, and (iii) LUB and GLB of the set B, where $B = \{3, 4, 5\}$ in case (a), $B = \{2, 3, 4\}$ in case (b), $B = \{2, 3, 6\}$ in case (c), and $B = \{6, 7, 10\}$ in case (d).

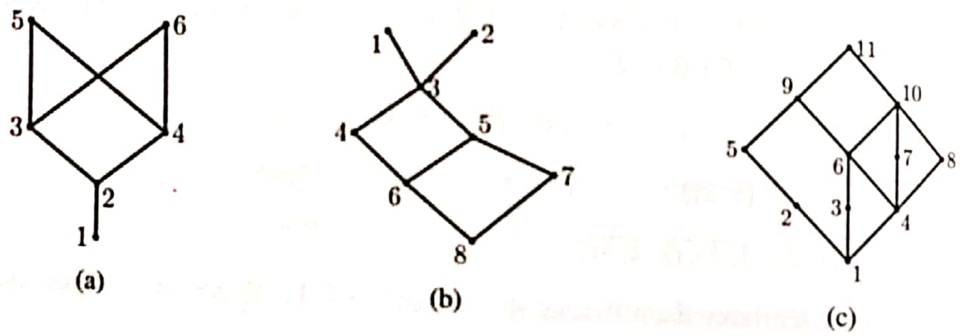
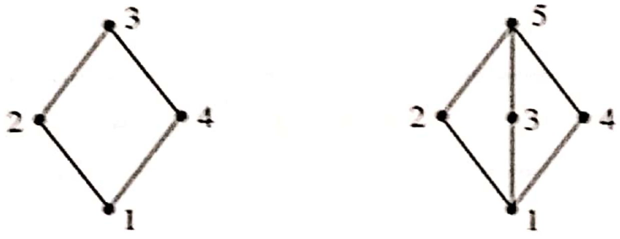


Figure 6.63

8. Let $A = \{0, 1, 2\}$. On $A \times A$ define the relation R by $(a, b)R(c, d)$ if (i) $a < c$ or (ii) $a = c$ and $b \leq d$. Verify that R is a partial order. Also, determine the maximal/minimal and greatest/least elements in A.

9. Let $S = \{1, 2, 3\}$, and let R be the subset relation. Show that the poset $(\mathcal{P}(S), \subseteq)$ is a lattice.

10. Show that the posets represented by the following Hasse diagrams are lattices.



(a) (b)

Figure 6.64

11. Show that the poset D_{36} (whose Hasse diagram is shown in Figure 6.49) is a lattice.

12. Show that the poset considered in Exercise 5 above is not a lattice.

Answers

1. (a) Maximal : 3, 5; Minimal : 1, 6. (b) Maximal : 6, 7; Minimal : 1, 2, 3
2. (a) Greatest : 5; Least : none (b) Greatest : 6; Least :1
4. (i) LUB : {1, 2}, GLB : Φ (ii) LUB : {1, 2}, GLB : Φ
   (iii) LUB : {1, 2, 3}, GLB : Φ (iv) LUB : {1, 2}, GLB : Φ
   (v) LUB : {1, 2, 3}, GLB : {1}
5. LUB {2, 3} = 6, LUB {2, 12} = 12, LUB {3, 6} = 6, LUB {5, 7} = 35,
   LUB {7, 11} = 385, LUB {11, 35} = 385,
   GLB {3, 6} = 3, GLB {2, 12} = 2, GLB {6, 12} = 6, GLB {35, 385} = 35, GLB {2, 3} does not exist.
6. (i) *a* (ii) *a* (iii) *c* (iv) *e* (v) *z* (vi) *e* (vii) *v*
7. (a) (i) 5, 6 (ii) 1, 2 (iii) None, 2 (b) (i) 2 (ii) 6, 8 (iii) 2, 6
   (c) (i) 10, 11 (ii) 1, 4 (iii) 10, 4
8. (0, 0) is the minimal and least element. (2, 2) is the maximal and greatest element.