Chapter 4

Principles of Counting - I

In this chapter, we state and illustrate two fundamental rules of counting along with their applications to the topic of Permutations and Combinations.

4.1 The Rules of Sum and Product

In many situations of computational work, we employ two basic rules of counting, called the *Sum Rule* and the *Product Rule*. These rules are stated and illustrated in the following paragraphs.

The Sum Rule

Suppose two tasks T_1 and T_2 are to be performed. If the task T_1 can be preformed in m different ways and the task T_2 can be performed in n different ways and if these two tasks cannot be performed simultaneously, then one of the two tasks (T_1 or T_2) can be performed in $m + n$ ways.

More generally, if T_1, T_2, $T_3, \ldots T_k$ are k tasks such that no two of these tasks can be performed at the same time and if the task T_i can be performed in n_i different ways, then one of the k tasks (namely T_1 or T_2 or $T_3 \ldots$, or T_k) can be performed in $n_1 + n_2 + \ldots + n_k$ different ways.

Example 1 . Suppose there are 16 boys and 18 girls in a class and we wish to select one of these students (either a boy or a girl) as the class representative. The number of ways of selecting a boy is 16 and the number of ways of selecting a girl is 18. Therefore, the number of ways of selecting a student (boy or girl) is $16 + 18 = 34$.

Example 2 . Suppose a Hostel library has 12 books on Mathematics, 10 books on Physics, 16 books on Computer Science and 11 books on Electronics. Suppose a student wishes to choose one of these books for study. The number of ways in which he can choose a book is $12 + 10 + 16 + 11 = 49$.

Example 3 . Suppose T_1 is the task of selecting a prime number less than 10 and T_2 is the task of selecting an even number less than 10. Then T_1 can be performed in 4 ways (– by selecting 2 or 3 or 5 or 7), and T_2 can be performed in 4 ways (– by selecting 2 or 4 or 6 or 8). But, since 2 is both a prime and an even number less than 10, the task T_1 or T_2 can be performed in $4 + 4 - 1 = 7$ ways.

The Product Rule

Suppose that two tasks T_1 and T_2 are to be performed one after the other. If T_1 can be performed in N_1 different ways, and for each of these ways T_2 can be performed in n_2 different ways, then both of the tasks can be performed in $n_1 n_2$ different ways.

More generally, suppose that k tasks $T_1, T_2, T_3, \ldots, T_k$ are to be performed in a sequence. If T_1 can be performed in n_1 different ways and for each of these ways T_2 can be performed in n_2 different ways, and for each of $n_1 n_2$ different ways of performing T_1 and T_2 in that order, T_3 can be performed in n_3 different ways, and so on, then the sequence of tasks $T_1, T_2, T_3, \ldots, T_k$ can be performed in $n_1 n_2 n_3 \ldots n_k$ different ways.

Example 4 . Suppose a person has 8 shirts and 5 ties. Then he has $8 \times 5 = 40$ different ways of choosing a shirt and a tie.

Example 5 . Suppose we wish to construct sequences of four symbols in which the first 2 are English letters and the next 2 are single digit numbers. If no letter or digit can be repeated, then the number of different sequences that we can construct is $26 \times 25 \times 10 \times 9 = 58500$.* If repetition of letters and digits is allowed then the number of different sequences that we can construct is $26 \times 26 \times 10 \times 10 = 67600$.

Example 6 . Suppose a restaurant sells 6 South Indian dishes, 4 North Indian dishes, 3 hot beverages and 2 cold beverages. For breakfast, a student wishes to buy 1 South Indian dish and 1 hot beverage, or 1 North Indian dish and 1 cold beverage. Then he can have the first choice in $6 \times 3 = 18$ ways and he can have the second choice in $4 \times 2 = 8$ ways. The total number of ways he can buy his breakfast items is $18 + 8 = 26$**.

The following are some more illustrative Examples.

Example 7 *There are 20 married couple in a party. Find the number of ways of choosing one woman and one man from the party such that the two are not married to each other.*

► From the party, a woman can be chosen in 20 ways. Among the 20 men in the party, one is her husband. Out of the 19 other men, one can be chosen in 19 ways. Therefore, the required number is $20 \times 19 = 380$. ■

Example 8 *A license plate consists of two English letters followed by four digits. If repetitions are allowed, how many of the plates have only vowels (A, E, I, O, U) and even digits?*

► Each of the first two positions in a plate can be filled in 5 ways (with vowels) and each of the remaining four places can be filled in 5 ways (with digits 0, 2, 4, 6, 8). Therefore, the number of possible license plates of the given type is $(5 \times 5) \times (5 \times 5 \times 5 \times 5) = 5^6 = 15,625$. ■

Example 9 *There are four bus routes between the places A and B and three bus routes between the places B and C. Find the number of ways a person can make a round trip from A to A via B if he does not use a route more than once.*

*Note that the number of English letters is 26 and the number of single digits is 10.

**In this example, the product rule as well as the sum rule are used.

▶ The person can travel from A to B in four ways and from B to C in three ways, but only in two ways from C to B and only in three ways from B to A if he does not use a route more than once. Therefore, the number of ways he can make the round trip under the given condition is $4 \times 3 \times 2 \times 3 = 72$. ■

Example 10 *Cars of a particular manufacturer come in 4 models, 12 colours, 3 engize sizes, and 2 transmission types.* (a) *How many distinct cars (of this company) can be manufactured?* (b) *Of these how many have the same colour?*

▶ (a) By the product rule, it follows that the number of distinct cars that can be manufactured is

$$4 \times 12 \times 3 \times 2 = 288.$$

(b) For any chosen colour, the number of distinct cars that can be manufactured is

$$4 \times 3 \times 2 = 24.$$ ■

Example 11 *Let A be a set with n elements. How many different sequences, each of length r*, can be formed using the elements from A if the elements in the sequence may be repeated?*

▶ Since repetition is allowed, each place in the sequence can be filled in n different ways. Thus, in a sequence of length r, there are n^r ways of filling the r places in the sequence. This means that there are n^r possible sequences (of the required type). ■

Example 12 (a) *Find the number of binary sequences of length n.*

(b) *Find the number of binary sequences of length n that contain an even number of* $1's$.

▶ (a) A binary sequence of length n contains n positions and each of these n positions can be filled in two ways (with 0 or 1). Therefore, the number of ways of filling n positions is 2^n. This is precisely the number of binary sequences of length n.

(b) If a binary sequence of length $n - 1$ has an even number of 1's, we append the digit 0 to it to obtain a binary sequence of length n which contains an even number of 1's. If a binary sequence of length $n - 1$ contains an odd number of 1's, we append the digit 1 to it to obtain a sequence of length n which contains an even number of 1's. As such, the number of binary sequences of length n with an even number of 1's is equal to the number of binary sequences of length $n - 1$, which is 2^{n-1}. ■

Example 13 *A bit is either* 0 *or* 1. *A byte is a sequence of* 8 *bits. Find* (i) *the number of bytes,* (ii) *the number of bytes that begin with* 11 *and end with* 11, (iii) *the number of bytes that begin with* 11 *and do not end with* 11, *and* (iv) *the number of bytes that begin with* 11 *or end with* 11.

▶ (i) Since each byte contains 8 bits and each bit is a 0 or 1 (two choices), the number of bytes is $2^8 = 256$.

*A sequence of length r means a sequence containing r terms (places).

(ii) In a byte beginning and ending with 11, there occur 4 open positions. These can be filled in $2^4 = 16$ ways. Therefore, there are 16 bytes which begin and end with 11.

(iii) There occur six open positions in a byte beginning with 11. These positions can be filled in $2^6 = 64$ ways. Thus, there are 64 bytes that begin with 11. Since there are 16 bytes that begin and end with 11, the number of bytes that begin with 11 but do not end with 11 is $64 - 16 = 48$.

(iv) As in (iii), the number of bytes that end with 11 is 64. Also the number of bytes that begin and end with 11 is 16. Therefore, the number of bytes that begin or end with 11 is $64 + 64 - 16 = 112$.*

■

Example 14 *A telegraph can transmit two different signals : a dot and a dash. What length of these symbols is needed to encode 26 letters of the English alphabet and the ten digits $0, 1, 2, \ldots, 9$?*

► Since there are two choices for each signal, the number of different sequences of length k of these signals is 2^k. Therefore, the number of nontrivial sequences of length n or less is

$$2 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 2.$$

To encode 26 letters and 10 digits, we require at least $26 + 10 = 36$ sequences of the above type; that is,

$$2^{n+1} - 2 \geq 36.$$

The least value of n (positive integer) for which this inequality holds is $n = 5$.

Hence, the length of the symbols needed to encode 26 alphabets and 10 digits is at least 5. ■

Example 15 *Find the number of 3-digit even numbers with no repeated digits.*

► Here we consider numbers of the form xyz, where each of x, y, z represents a digit under the given restrictions. Since xyz has to be even, z has to be 0, 2, 4, 6, or 8. If z is 0, then x has 9 choices and if z is 2, 4, 6 or 8 (4 choices) then x has 8 choices. (Note that x cannot be zero). Therefore, z and x can be chosen in $(1 \times 9) + (4 \times 8) = 41$ ways. For each of these ways, y can be chosen in 8 ways. Hence, the desired number is $41 \times 8 = 328$.

■

Example 16 *How many among the first 100,000 positive integers contain exactly one 3, one 4 and one 5 in their decimal representations?*

► The number 100,000 does not contain 3 or 4 or 5. Therefore, we have to consider all possible positive integers with 5 places that meet the given conditions. In a 5-place integer the digit 3 can be in any one of the 5 places. Subsequently, the digit 4 can be in any one of the 4 remaining places. Then the digit 5 can be in any one of the 3 remaining places. There are 2 places left and either of these may be filled by 7 digits. (-digits from 0 to 9 other than 3, 4, 5). Thus, there are $5 \times 4 \times 3 \times 7 \times 7 = 2940$ integers of the required type.

■

*Recall that, for two finite sets A and B, $|A \cup B| = |A| + |B| - |A \cap B|$. Here, $|A|$ denotes the number of elements in A. Similar meanings are there for $|B|$ and $|A \cup B|$.

Example 17 *Find the total number of positive integers that can be formed from the digits* 1, 2, 3, 4 *if no digit is repeated in any one integer.*

▶ We first note that no integer of the required type can contain more than 4 digits. Let s_1, s_2, s_3, s_4 denote the number of integers of the required type containing one, two, three, four digits respectively.

Since there are four digits, there are four integers containing exactly one digit (i.e. $s_1 = 4$), there are $4 \times 3 = 12$ integers containing exactly two digits (i.e. $s_2 = 12$), there are $4 \times 3 \times 2 = 24$ integers containing exactly three digits (i.e. $s_3 = 24$) and there are $4 \times 3 \times 2 \times 1 = 24$ integers containing exactly four digits (i.e. $s_4 = 24$). Therefore, the required number is

$$s_1 + s_2 + s_3 + s_4 = 64.$$

■

Example 18 *Find the number of proper divisors* of* 441000.

▶ We note that $441000 = 2^3 \times 3^2 \times 5^3 \times 7^2$. Therefore, every divisor of $n = 441000$ must be of the form $d = 2^p \times 3^q \times 5^r \times 7^s$ where $0 \leq p \leq 3$, $0 \leq q \leq 2$, $0 \leq r \leq 3$, $0 \leq s \leq 2$. Thus, for a divisor d, p can be chosen in 4 ways, q in 3 ways, r in 4 ways and s in 3 ways. Accordingly, the number of possible d's is $4 \times 3 \times 4 \times 3 = 144$. Of these, two divisors (namely 1 and n) are not proper divisors. Therefore, the number of proper divisors of the given number is $144 - 2 = 142$. ■

Exercises

1. In how many ways can three different coins be placed in two different purses?

2. A coin is tossed four times and the result of each toss is recorded. How many different sequences of heads and tails are possible?

3. A six-faced die is tossed four times and the numbers shown are recorded in a sequence. How many different sequences are there?

4. In how many ways could a student answer a 10-question true-false test if it is possible to leave a question unanswered in order to avoid an extra penalty for a wrong answer?

5. A sports committee of 3 in a college is to be formed consisting of one representative each from boy students, girl students and teachers. If there are 3 possible representatives from boy students, 2 from girl students and 4 from teachers, determine how many different committees can be formed.

6. A label identifier for a computer programme consists of one letter of the English alphabet followed by two digits. If repetitions are allowed, how many distinct label identifiers are possible?

*For a positive integer n, the divisors of n other than 1 and n are called *proper divisors*.

7. Suppose the license plates require three English letters followed by four digits. How many different plates can be made if repetitions of letters and digits are allowed?

8. Find the number of license plates that can be made where each plate contains two distinct English letters followed by three different digits with the first digit different from 0.

9. Suppose that a valid computer password consists of seven characters, the first of which is one of the letters A, B, C, D, E, F, G and the remaining six characters are letters chosen from the English alphabet or a digit. How many different passwords are possible?

10. Find the number of

(i) 2-digit even numbers,

(ii) 2-digit odd numbers,

(iii) 2-digit odd numbers with distinct digits, and

(iv) 2-digit even numbers with distinct digits.

11. How many 3-digit numbers can be formed by using the 6 digits 2, 3, 4, 5, 6, 8 if

(i) repetitions of digits are allowed?

(ii) repetitions are not allowed?

(iii) the number is to be odd and repetitions are not allowed?

(iv) the number is to be even and repetitions are not allowed?

(v) the number is to be a multiple of 5 and repetitions are not allowed?

(vi) the number must contain the digit 5 and repetitions are not allowed?

(vii) the number must contain the digit 5 and repetitions are allowed?

12. Find the number of 5-digit integers that contain the digit 6 exactly once.

Answers

1. 8 **2.** 16 **3.** 6^4 **4.** 3^{10} **5.** $3 \times 2 \times 4$

6. 2600 **7.** $26^3 \times 10^4$ **8.** 421200 **9.** $7 \times (26 + 10)^6$

10. (i) 9×5 (ii) 9×5 (iii) 5×8 (iv) $(1 \times 9) + (4 \times 8)$

11. (i) 6^3 (ii) $6 \times 5 \times 4$ (iii) $2 \times 5 \times 4$ (iv) $(6 \times 5 \times 4) - (2 \times 5 \times 4)$

(v) $1 \times 5 \times 4$ (vi) $3 \times 5 \times 4$ (vii) $6^3 - 5^3$

12. $(1 \times 9^4) + (8 \times 4 \times 9^3)$

4.2 Permutations

Suppose that we are given n distinct objects and wish to arrange r of these objects in a line. Since there are n ways of choosing the first object, and, after this is done, $n-1$ ways of choosing the second object,..., and finally $n-r+1$ ways of choosing the r^{th} object, it follows by the Product Rule of counting (stated in the preceding section) that the number of different arrangements, or ***permutations*** (as they are commonly called) is $n(n-1)(n-2)\cdots(n-r+1)$. We denote this number by $P(n,r)$* and is referred to as the *number of permutations of size r of n objects*. Thus (by definition),

$$P(n,r) = n(n-1)(n-2)\cdots(n-r+1).$$

Using the *factorial notation* defined by

$$k! = k(k-1)(k-2)\cdots 2\cdot 1,$$

for any positive integer k, and $0! = 1$, we find that

$$\begin{aligned} P(n,r) &= n(n-1)(n-2)\cdots(n-r+1) \\ &= \frac{n(n-1)(n-2)\cdots(n-r+1)(n-r)(n-r-1)\cdots 2\cdot 1}{(n-r)(n-r-1)\cdots 2\cdot 1} \\ &= \frac{n!}{(n-r)!} \end{aligned}$$

As a particular case of this, we get

$$P(n,n) = n!.$$

That is, the number of different arrangements (permutations) of n distinct objects, taken *all* at a time, is $n!$. This is simply called the *number of permutations of n distinct objects*.

In the above analysis, we have considered the situation where all the objects that are to be arranged are *distinct*.

A generalization

Suppose it is required to find the number of permutatuions that can be formed from a collection of n objects of which n_1 are of one type, n_2 are of a second type,..., n_k are of k^{th} type, with $n_1 + n_2 + \cdots + n_k = n$. Then, the number of permutations of the n objects (taken *all* of them at a time) is

$$\frac{n!}{n_1!\, n_2! \cdots n_k!}$$

Proof: There are $n!$ permutations when all the n objects are different. We must therefore divide $n!$ by $n_1!$ to acc<...>nt for the fact that the n_1 objects which are alike will identify $n_1!$

*Some other notations . (n, r) are: ^nP_r, $_nP_r$ and $P_{n,r}$.

of these permutations (for any given set of positions of the n_1 objects in the permutation). Similarly, we must divide $n!$ by $n_2!, n_3!, \ldots, n_k!$, which are the numbers of permutations of the corresponding alike objects. Thus, $n!$ divided by all of $n_1!$, $n_2!, \ldots,$ $n_k!$ gives the required number of permutations. •

Example 1 *In how many ways can n persons be seated at a round table if arrangements are considered the same when one can be obtained from the other by rotation?*

▶ Let one of them be seated anywhere. Then the remaining $n - 1$ persons can be seated in $(n - 1)!$ ways. This is the total number of ways of arranging the n persons in a circle. ■

Example 2 *It is required to seat 5 men and 4 women in a row so that the women occupy the even places. How many such arrangements are possible?*

▶ The 5 men may be seated in odd places in 5! ways and the 4 women may be seated in even places in 4! ways, and corresponding to each arrangement of the men there is an arrangement of the women. Therefore, the total number of arrangements of the desired type is

$$5! \times 4! = 120 \times 24 = 2880.$$ ■

Example 3 *In how many ways can 6 men and 6 women be seated in a row*

(i) *if any person may sit next to any other?*

(ii) *if men and women must occupy alternate seats?*

▶ (i) If any person may sit next to any other, no distinction need be made between men and women in their seating. Accordingly, since there are 12 persons in all, the number of ways they can be seated is

$$12! = 479{,}001{,}600.$$

(ii) When men and women are to occupy alternate seats, the six men can be seated in 6! ways in odd places and the six women can be seated in 6! ways in even places, and corresponding to each arrangement of the men there is an arrangement of the women. Therefore, the number of ways in which the men occupy the odd places and the women the even places is

$$6! \times 6! = 720 \times 720 = 518400.$$

Similarly, the number of ways in which the women occupy the odd places and the men the even places is 518400. Accordingly, the total number of ways is

$$518400 + 518400 = 1{,}036{,}800.$$ ■

Example 4 *Four different mathematics books, five different computer science books and two different control theory books are to be arranged in a shelf. How many different arrangements* *are possible if* (a) *the books in each particular subject must all be together?* (b) *only the mathematics books must be together?*

▶ (a) The mathematics books can be arranged among themselves in 4! different ways, the computer science books in 5! ways, the control theory books in 2! ways, and the three groups in 3! ways. Therefore the number of possible arrangements is

$$4! \times 5! \times 2! \times 3! = 24 \times 120 \times 2 \times 6 = 34{,}560.$$

(b) Consider the four mathematics books as one single book. Then we have 8 books which can be arranged in 8! ways. In all of these ways the mathematics books are together. But the mathematics books can be arranged among themselves in 4! ways. Hence, the number of arrangements is

$$8! \times 4! = 40320 \times 24 = 967{,}680. \qquad ■$$

Example 5 *How many 8-digit numbers have one or more repeated digits?*

▶ The number of 8-digit numbers in which repetitions of integers is allowed is 10^8. Of these, $P(10,8)$ numbers do not contain repetitions. Therefore, the required number is $10^8 - P(10,8)$.■

Example 6 *How many different strings (sequences) of length* 4 *can be formed using the letters of the word* FLOWER*?*

▶ The given word has 6 letters all of which are distinct. The required number of strings is the same as the number of permutations of these six letters chosen 4 at a time. This number is

$$P(6,4) = \frac{6!}{(6-4)!} = \frac{6!}{2!} = \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 1} = 360. \qquad ■$$

Example 7 *Find the number of permutations of (all) the letters of the word* SUCCESS.

▶ The given word has 7 letters, of which 3 are S; 2 are C and 1 each are U and E. Therefore, the required number of permutations is

$$\frac{7!}{3! \cdot 2! \cdot 1! \cdot 1!} = \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{(3 \times 2) \times (2 \times 1) \times 1 \times 1} = 420. \qquad ■$$

Example 8 *How many 9 letter "words"* can be formed by using the letters of the word* DIFFICULT*?*

▶ The given word contains 9 letters of which there are 2 F's, 2 I's, and 1 each of D, C, U, L, T. The number of permutations of these letters is the required number of "words". This number is

$$\frac{9!}{2!\,2!\,1!\,1!\,1!\,1!} = 90720. \qquad ■$$

*Here "word" means an arrangement of letters which may or may not carry any meaning.

Example 9 *Find the number of permutations of the letters of the word* MASSASAUGA. *In how many of these, all four A's are together? How many of them begin with S ?*

▶ The given word has 10 letters of which 4 are A, 3 are S and 1 each are M, U and G. Therefore, the required number of permutations is

$$\frac{10!}{4!\ 3!\ 1!\ 1!\ 1!} = 25,200.$$

If, in a permutation, all A's are to be together, we treat all of A's as one single letter. Then the letters to be permuted read (AAAA), S, S, S, M, U, G (which are 7 in number), and the number of permutations is

$$\frac{7!}{1!\ 3!\ 1!\ 1!\ 1!} = 840.$$

For permutations beginning with S, there occur nine open positions to fill, where two are S, four are A, and one each are M, U, G. The number of such permutations is

$$\frac{9!}{2!\ 4!\ 1!\ 1!\ 1!} = 7560.$$ ■

Example 10 (a) *How many arrangements are there for all letters in the word* SOCIOLOGICAL?

(b) *In how many of these arrangements* (i) A *and* G *are adjacent?* (ii) *all the vowels are adjacent?*

▶ (a) The given word has 12 letters of which 3 are O, 2 each are C, I, L and 1 each are S, A, G. Therefore, the number of arrangements of these letters is

$$\frac{12!}{3!\ 2!\ 2!\ 2!\ 1!\ 1!\ 1!} = 99,79,200.$$

(b) (i) If, in an arrangement, A and G are to be adjacent, we treat A and G together as a single letter, say X so that we have 3 number of O's, 2 each of C, I, L and one each of S and X, totalling to 11 letters. These can be arranged in

$$\frac{11!}{3!\ 2!\ 2!2!\ 1!} \text{ ways.}$$

Further, the letters A and G can be arranged among themselves in two ways.

Therefore, the total number of arrangements in this case is

$$\frac{11!}{3!\ 2!\ 2!\ 2!\ 1!} \times 2 = 16,63,200.$$

(ii) If, in an arrangement, all the vowels are to be adjacent, we treat all the vowels present in the given word (namely A, O, I) as a single letter, say Y, so that we have 2 each of C and L and 1 each of S, G and Y, totalling to 7 letters. These can be arranged in

$$\frac{7!}{2!\,2!\,1!\,1!\,1!} \text{ ways.}$$

Further, since the given word contains three O's, two I's and one A, the letters A, O, I (clubbed as Y) can be arranged among themselves in

$$\frac{6!}{3!\,2!\,1!} \text{ ways.}$$

Therefore, the total number of arrangements in this case is

$$\frac{7!}{2!\,2!\,1!\,1!\,1!} \times \frac{6!}{3!\,2!\,1!} = 75,600.$$ ■

Example 11 *How many positive integers n can we form using the digits* 3, 4, 4, 5, 5, 6, 7 *if we want n to exceed* 5, 000, 000*?*

▶ Here n must be of the form

$$n = x_1x_2x_3x_4x_5x_6x_7$$

where $x_1, x_2, \ldots, x_7$ are the given digits with $x_1 = 5, 6$ or 7. Suppose we take $x_1 = 5$. Then $x_2x_3x_4x_5x_6x_7$ is an arrangement of the remaining 6 digits which contains two 4's and one each of 3, 5, 6, 7. The number of such arrangements is

$$\frac{6!}{2!\,1!\,1!\,1!\,1!} = 360.$$

Next, suppose we take $x_1 = 6$. Then, $x_2x_3x_4x_5x_6x_7$ is an arrangement of 6 digits which contains two each of 4 and 5 and one each of 3 and 7. The number of such arrangements is

$$\frac{6!}{1!\,2!\,2!\,1!} = 180.$$

Similarly, if we take $x_1 = 7$, the number of arrangements is

$$\frac{6!}{1!\,2!\,2!\,1!} = 180.$$

Accordingly, by the Sum Rule, the number of n's of the desired type is

$$360 + 180 + 180 = 720.$$ ■

Example 12 *If k is a positive integer, and $n = 2k$, prove that $\frac{n!}{2^k}$ is a positive integer.*

▶ Consider the symbols $x_1, x_1, x_2, x_2, x_3, x_3, \ldots, x_k$ in which each of $x_1, x_2, \ldots, x_k$ is 2 in number. Evidently, the number of these symbols is $2k$. Therefore, the number of permutations of these $2k$ symbols is

$$\frac{(2k)!}{\underbrace{2!\,2!\ldots2!}_{k-factors}} = \frac{n!}{2^k}.$$

This number has to be a positive integer. ■

Example 13 *Prove that $(n!)!$ is divisible by $(n!)^{(n-1)!}$.*

▶ Let us set $n! = N$ so that $(n-1)! = \frac{n!}{n} = \frac{N}{n} = M$ (say).

Consider a collection of N objects of M types with n objects in each type. The number of permutations of these N objects is

$$\frac{N!}{\underbrace{(n!)(n!)\ldots(n!)}_{M\ factors}} = \frac{N!}{(n!)^M}.$$

This has to be a positive integer. This means that $N!$ is divisible by $(n!)^M$. This proves the required result. ■

Example 14 *Find the value of n so that $2P(n,2) + 50 = P(2n,2)$.*

▶ From the given condition, we have

$$2 \times \frac{n!}{(n-2)!} + 50 = \frac{(2n)!}{(2n-2)!}$$

i.e., $$2n(n-1) + 50 = 2n(2n-1), \quad \text{or} \quad n^2 = 25.$$

This gives $n = 5$ or -5. Since n cannot be negative, the value of n should be 5. ■

Example 15 *For non-negative integers n and r, if $n + 1 > r$, prove that*

$$P(n+1,r) = \left(\frac{n+1}{n+1-r}\right) P(n,r).$$

▶ We have

$$P(n,r) = \frac{n!}{(n-r)!} \quad \text{and} \quad P(n+1,r) = \frac{(n+1)!}{(n+1-r)!}$$

Therefore,

$$\frac{P(n+1,r)}{P(n,r)} = \frac{(n+1)!}{(n+1-r)!} \cdot \frac{(n-r)!}{n!}$$
$$= \frac{(n+1)!}{n!} \cdot \frac{(n-r)!}{(n+1-r)!} = \frac{n+1}{n+1-r}$$

so that
$$P(n+1,r) = \left(\frac{n+1}{n+1-r}\right) P(n,r).$$ ■

Exercises

1. Find the value of n in each of the following cases:

   (i) $P(n,2) = 90$, (ii) $P(n,3) = 3P(n,2)$, (iii) $P(n,4) = 42P(n,2)$.

2. How many six-digit numbers can one make using the digits 1, 3, 3, 7, 7, 8?

3. Find the number of distinguishable permutations of all the letters in the following words:

   (1) BASIC (2) PASCAL (3) BANANA
   (4) PEPPER (5) CALCULUS (6) DISCRETE
   (7) STRUCTURES (8) ENGINEERING (9) MATHEMATICS

4. How many different arrangements of all letters in the word BOUGHT can be formed if the vowels must be kept next to each other?

5. Find the number of permutations of all letters of the word BASEBALL if the words are to begin and end with a vowel.

6. Find the number of permutations of the letters of the word MISSISSIPPI. How many of these begin with an I? How many of these begin and end with an S?

7. How many different signals, each consisting of six flags hung in a vertical line, can be formed from four identical red flags and two identical blue flags?

8. In how many ways can the symbols $a, b, c, d, e, e, e, e, e$ be arranged so that no e is adjacent to another e?

9. Consider the permutations of the letters a, c, f, g, i, t, w, x. How many of these start with t? How many start with t and end with c?

10. Five red pens, two black pens and three blue pens are arranged in a row. If the pens of the same colour are not distinguishable, how many different arrangements are possible?

11. In how many ways can seven books be arranged on a shelf if (a) any arrangement is allowed? (b) three particular books must always be together? (c) two particular books must occupy the ends?

12. A student has three books on C++ and four books on Java. In how many ways can he arrange three books on a shelf (i) if there are no restrictions ? (ii) if the languages should alternate? (iii) if all the C++ books must be next to each other? (iv) if all the C++ books must be next to each other and all the Java books must be next to each other?

13. How many four-digit numbers can be formed with the 10 digits 0, 1, 2, ..., 9 if (a) repetitions are allowed? (b) repetitions are not allowed? (c) the last digit must be zero and repetitions are not allowed?

14. How many different three-digit numbers can be formed with 3 fours, 4 two's and 2 three's?

15. In how many ways can 7 women and 3 men be arranged in a row if the 3 men must always stand next to each other?

16. In how many ways can seven people be seated at a round table if two particular persons must sit next to each other?

17. In how many ways can three men and three women be seated at a round table if (a) no restriction is imposed? (b) two particular women must not sit together? (c) each women is to be between two men?

18. Find the number of ways of seating m women and n men at a round table so that (i) between every two women there is a man when $m = n$, and (ii) no two women sit side by side when $m < n$.

19. In how many ways can the letters of the English alphabet be arranged so that there are exactly 5 letters between the letters a and b?

20. If k is a positive integer and $n = 3k$, prove that $\frac{n!}{6^k}$ is a positive integer.

21. Show that if m and n are positive integers, then $(mn)!$ is divisible by $(m!)^m$.

22. Prove the following identities:

(i) $P(n+1, r) = (n+1) \cdot P(n, r-1)$

(ii) $P(n, r) = (n-r+1) \cdot P(n, r-1)$

Answers

1. (i) 10 (ii) 5 (iii) 9. **2.** 180

3. (1) 120 (2) 360 (3) 10 (4) 60 (5) 5040
(6) 20160 (7) 226,800 (8) 277,200 (9) 4,989,000

4. 240 **5.** 540 **6.** 34650, 12600, 7560 **7.** 15 **8.** 24 **9.** 7!, 6! **10.** 2520

11. (a) 5040 (b) 720 (c) 240 **12.** (i) 7! (ii) $4! \times 3!$ (iii) $5! \times 3!$ (iv) 288

13. (a) 9000 (b) 4536 (c) 504 **14.** 26 **15.** 241,920 **16.** 240

17. (a) 120 (b) 72 (c) 12 **18.** (i) $m!(m-1)!$ (ii) $(n-1)!P(n,m)$ **19.** $P(24,5) \times 20! \times 2$

4.3 Combinations

Suppose we are interested in selecting (choosing) a *set* of r objects from a set of $n \geq r$ objects without regard to order. The set of r objects being selected is traditionally called a *combination of r objects* (or briefly *r-combination*).

The total number of combinations of r different objects that can be selected from n different objects can be obtained by proceeding in the following way. Suppose this number is equal to C, say; that is, suppose there is a total of C number of combinations of r different objects chosen from n different objects. Take any one of these combinations. The r objects in this combination can be arranged in $r!$ different ways. Since there are C combinations, the total number of permutations is $C \cdot r!$. But this is equal to $P(n, r)$. Thus,

$$C \cdot r! = P(n,r), \quad \text{or} \quad C = \frac{P(n,r)}{r!}.$$

Thus, the total number of combinations of r different objects that can be selected from n different objects is equal to $P(n,r)/r!$. This number is denoted by $C(n,r)$, or $\binom{n}{r}$*. Thus,

$$C(n,r) \equiv \binom{n}{r} \equiv \frac{P(n,r)}{r!} = \frac{n!}{(n-r)!r!} \quad \text{for} \quad 0 \leq r \leq n.$$

Replacing r by $n-r$ in this expression, we get

$$C(n,n-r) = \binom{n}{n-r} = \frac{n!}{r!(n-r)!} = C(n,r) = \binom{n}{r} \quad \text{for} \quad 0 \leq r \leq n.$$

*Some other notations for this number are ${}_nC_r$, ${}^nC_r$ and $C_{n,r}$.

Consequently, we have

$$C(n,n) = \binom{n}{n} = C(n,0) = \binom{n}{0} = 1 \quad \text{and} \quad C(n,1) = \binom{n}{1} = C(n,n-1) = \binom{n}{n-1} = n.$$

Note: For $r > n$, $C(n,r) \equiv \binom{n}{r}$ is *defined* to be equal to zero.

Example 1 *How many committees of 5 with a given chairperson can be selected from 12 persons?*

▶ The chairperson can be chosen in 12 ways, and, following this, the other four on the committee can be chosen in $C(11,4)$ ways. Therefore, the possible number of such committees is

$$12 \times C(11,4) = 12 \times \frac{11!}{4!\,7!} = 12 \times 330 = 3960.$$ ■

Example 2 *Find the number of committees of 5 that can be selected from 7 men and 5 women if the committee is to consist of at least 1 man and at least 1 woman.*

▶ From the given 12 persons the number of committees of 5 that can be formed is $C(12,5)$. Among these possible committees, there are $C(7,5)$ committees consisting of 5 men and $1 = C(5,5)$ committee consisting of 5 women. Accordingly, the number of committees containing at least one man and one woman is

$$\begin{aligned} C(12,5) - C(7,5) - 1 &= \frac{12!}{7!\,5!} - \frac{7!}{5!\,2!} - 1 \\ &= 792 - 21 - 1 = 770. \end{aligned}$$ ■

Example 3 *At a certain college, the housing office has decided to appoint, for each floor, one male and one female residential advisor. How many different pairs of advisors can be selected for a seven-floor building from 12 male and 15 female candidates?*

▶ From 12 male candidates, 7 candidates can be selected in $C(12,7)$ ways. From 15 female candidates, 7 candidates can be selected in $C(15,7)$ ways. Therefore, the total number of possible pairs of advisors of the required type is

$$C(12,7) \times C(15,7) = \frac{12!}{7!\,5!} \times \frac{15!}{7!\,8!} = 792 \times 6435 = 5{,}096{,}520.$$ ■

Example 4 *A certain question paper contains two parts A and B each containing 4 questions. How many different ways a student can answer 5 questions by selecting at least 2 questions from each part?*

▶ The different ways a student can select his 5 questions are

(I) 3 questions from Part A and 2 questions from Part B. This can be done in $C(4,3) \times C(4,2) = 4 \times 6 = 24$ ways.

(II) 2 questions from Part A and 3 questions from Part B. This can be done in

$C(4,2) \times C(4,3) = 24$ ways.

Therefore, the total number of ways a student can answer 5 questions under the given restrictions is $24 + 24 = 48$. ■

Example 5 *A certain question paper contains three parts* A, B, C *with four questions in part* A*, five questions in part* B *and six questions in part* C*. It is required to answer seven questions selecting at least two questions from each part. In how many different ways can a student select his seven questions for answering?*

▶ The different possible ways in which a student can make a selection are

(I) 2 questions from Part A, 2 from Part B and 3 from Part C.

(II) 2 questions from Part A, 3 from Part B and 2 from Part C.

(III) 3 questions from Part A, 2 from Part B and 2 from Part C.

Now, selection (I) can be made in

$$C(4,2) \times C(5,2) \times C(6,3) = 6 \times 10 \times 20 = 1200 \quad \text{ways},$$

the selection (II) can be made in

$$C(4,2) \times C(5,3) \times C(6,2) = 6 \times 10 \times 15 = 900 \quad \text{ways},$$

and the selection (III) can be made in

$$C(4,3) \times C(5,2) \times C(6,2) = 4 \times 10 \times 15 = 600 \quad \text{ways}.$$

Consequently, the total number of possible selections is

$$1200 + 900 + 600 = 2700.$$ ■

Example 6 *A woman has 11 close relatives and she wishes to invite 5 of them to dinner. In how many ways can she invite them in the following situations:*

(i) *There is no restriction on the choice.*

(ii) *Two particular persons will not attend separately.* → (2)

(iii) *Two particular persons will not attend together.*

► (i) Since there is no restriction on the choice of invitees, five out of 11 can be invited in

$$C(11,5) = \frac{11!}{6!\,5!} = 462 \quad \text{ways.}$$

(ii) Since two particular persons will not attend separately, they should both be invited or not invited. If both of them are invited, then three more invitees are to be selected from the remaining 9 relatives. This can be done in

$$C(9,3) = \frac{9!}{6!\,3!} = 84 \quad \text{ways.}$$

If both of them are not invited, then five invitees are to be selected from 9 relatives. This can be done in

$$C(9,5) = \frac{9!}{5!\,4!} = 126 \quad \text{ways.}$$

Therefore, the total number of ways in which the invitees can be selected in this case is

$$84 + 126 = 210.$$

(iii) Since two particular persons (say P_1 and P_2) will not attend together, only one of them can be invited or none of them can be invited. The number of ways of choosing the invitees with P_1 invited is

$$C(9,4) = \frac{9!}{5!\,4!} = 126.$$

Similarly the number of ways of choosing the invitees with P_2 invited is 126.

If both P_1 and P_2 are not invited, the number of ways of choosing the invitees is

$$C(9,5) = 126.$$

Thus, the total number of ways in which the invitees can be selected in this case is

$$126 + 126 + 126 = 378.$$

■

Example 7 *From seven consonants and five vowels, how many sets consisting of four different consonants and three different vowels can be formed?*

► The four different consonants can be selected in $C(7,4)$ different ways and three different vowels can be selected in $C(5,3)$ ways, and the resulting seven different letters (four consonants and three vowels) can then be arranged among themselves in 7! ways. Therefore, the number of possible sets is

$$C(7,4) \times C(5,3) \times 7! = \frac{7!}{4!\,3!} \times \frac{5!}{3!\,2!} \times 7! = 35 \times 10 \times 5040 = 1,764,000.$$

■

Example 8 *Find the number of ways of seating r persons out of n persons around a circular table, and the others around another circular table.*

▶ First, choose a set of r persons for the first table – this can be done in $C(n, r)$ ways. These r persons can be seated around the first table in $(r-1)!$ ways. The remaining $(n-r)$ persons can be seated around the second table in $(n-r-1)!$ ways. So, the required number is

$$C(n, r) \times (r-1)! \times (n-r-1)!. \qquad \blacksquare$$

Example 9 *A party is attended by n persons. If each person in the party shakes hands with all the others in the party, find the number of handshakes.*

▶ Each handshake is determined by exactly two persons. Therefore, if each person shakes hands with all the other persons, the total number of handshakes is equal to the number of combinations of two persons that can be selected from the n persons. This number is

$$C(n, 2) = \frac{n!}{(n-2)!2!} = \frac{1}{2}n(n-1) \qquad \blacksquare$$

Example 10 *There are n married couples attending a party. Each person shakes hands with every person other than his or her spouse. Find the total number of handshakes.*

▶ The number of persons at the party is $2n$. These $2n$ persons fall into $C(2n, 2)$ pairs out of which n pairs are married couples. Thus, the number of pairs who are not married couples is

$$C(2n, 2) - n = \frac{(2n!)}{(2n-2)!\,2!} - n = \frac{1}{2} \cdot 2n(2n-1) - n = 2n(n-1).$$

This number is identical with the number of handshakes. ■

Example 11 (a) *How many diagonals are there in a regular polygon with n sides?*
(b) *Which regular polygon has the same number of diagonals as sides?*

▶ (a) A regular polygon of n sides has n vertices. Any two vertices determine either a side or a diagonal. Thus, the number of sides plus the number of diagonals is $C(n, 2)$. Consequently, the number of diagonals is

$$C(n, 2) - n = \frac{n!}{(n-2)!\,2!} - n = \frac{1}{2}n(n-1) - n = \frac{1}{2}n(n-3).$$

(b) If the number of diagonals is the same as the number of sides, we should have

$$\frac{1}{2}n(n-3) = n, \quad \text{or} \quad n^2 - 5n = 0, \quad \text{or} \quad n(n-5) = 0.$$

Since $n > 0$, we should have $n = 5$. Thus, the regular polygon which has the same number of diagonals as sides must have 5 sides; that is, it must be a pentagon. ■

Example 12 *A string of length n is a sequence of the form $x_1x_2x_3\dots x_n$, where each x_i is a digit. The sum $x_1+x_2+x_3\cdots+x_n$ is called the* weight *of the string. If each x_i can be one of* 0, 1, *or* 2, *find the number of strings of length $n = 10$. Of these, find the number of strings whose weight is an even number.*

► There are 10 positions in a string of length 10, and each of these positions can be filled in 3 ways (with 0, 1, or 2). Therefore, the number of ways of filling the 10 positions of a string of length 10 is 3^{10}. This means that there are 3^{10} number of strings of length 10 (with 0, 1 or 2 as its digits).

Since each digit in the strings being considered here is 0, 1, or 2, the weight of a string is even only when it contains an even number of 1's. Thus, strings of even weight have zero, two, four, six, eight or ten number of 1's.

If a string has no 1's, then all its places are filled by 0's and 2's. The number of such strings is 2^{10}.

If a string has two 1's, it can have two 1's in $C(10,2)$ number of locations. For each of these locations, the remaining eight locations are filled by 0's and 2's. Therefore, the number of strings having two 1's is $C(10,2)\times 2^8$.

Similarly, the numbers of strings having four 1's, six 1's and eight 1's are $C(10,4)\times 2^6$, $C(10,6)\times 2^4$ and $C(10,8)\times 2^2$ respectively.

Lastly, the number of strings having ten 1's is evidently only one.

Accordingly, the number of strings that have even weight is

$$2^{10} + C(10,2)\times 2^8 + C(10,4)\times 2^6 + C(10,6)\times 2^4 + C(10,8)\times 2^2 + 1.$$

■

Example 13 *Find how many distinct four-digit integers one can make from the digits* $1,3,3,7,7,8$.

► The four-digit integers that can be formed using the given digits fall into the following three categories.

(1) Two of the four digits are identical and another two are identical.

There occurs *one* combination under this category (consisting of two 3's and two 7's). The number of permutations in this combination is

$$\frac{4!}{2!\,2!} = 6$$

(2) Two of the four digits are identical and another two are distinct.

There occur *six* combinations* under this category. In each of these combinations, the number of permutations is

$$\frac{4!}{2!\,1!\,1!} = 12$$

*Identify these combinations!

The total number of permutations in this category is $6 \times 12 = 72$.

(3) All the four digits are distinct.

There occurs *one* combination under this category. The number of permutations in this combination is

$$\frac{4!}{1!\,1!\,1!\,1!} = 24$$

The total number of integers (of the required type) is precisely the total number of permutations in all of the categories mentioned above. This number is

$$6 + 72 + 24 = 102. \qquad \blacksquare$$

Example 14 *Find the number of arrangements of all the letters in* TALLAHASSEE. *How many of these arrangements have no adjacent* A*'s?*

▶ The number of letters in the given word is 11 of which 3 are A's, 2 each are L's, S's, E's, and 1 each are T and H. Therefore, the number of arrangements (permutations) of the letters in the given word is

$$\frac{11!}{3!\,2!\,2!\,2!\,1!\,1!} = 8,31,600.$$

If we disregard the A's, the remaining 8 letters can be arranged in

$$\frac{8!}{2!\,2!\,2!\,1!\,1!} = 5040 \quad \text{ways.}$$

In each of these arrangements, there are 9 possible locations for the three A's*. These locations can be chosen in $C(9,3)$ ways. Therefore, the number of arrangements having no adjacent A's is

${}^9C_3$

$$5040 \times C(9,3) = 5040 \times \frac{9!}{3!\,6!} = 5040 \times 84 = 4,23,360. \qquad \blacksquare$$

Example 15 *Prove the following identities:*

(i) $C(n, r-1) + C(n, r) = C(n+1, r)$

(ii) $C(m, 2) + C(n, 2) = C(m+n, 2) - mn$

*The nine possible locations for A's in one of the arrangements of the 8 remaining letters are (shown by arrows):

$$\uparrow T \uparrow L \uparrow L \uparrow H \uparrow S \uparrow S \uparrow E \uparrow E \uparrow .$$

► We have

(i) $$C(n,r-1)+C(n,r)=\frac{n!}{(r-1)!\,(n-r+1)!}+\frac{n!}{r!\,(n-r)!}$$
$$=\frac{n!}{(r-1)!(n-r)!}\left\{\frac{1}{n-r+1}+\frac{1}{r}\right\}$$
$$=\frac{n!}{(r-1)!(n-r)!}\cdot\frac{n+1}{r(n-r+1)}=\frac{(n+1)!}{r!\,(n-r+1)!}$$
$$=C(n+1,r).$$

(ii) $$C(m,2)+C(n,2)=\frac{m!}{(m-2)!\cdot 2}+\frac{n!}{(n-2)!\cdot 2}$$
$$=\tfrac{1}{2}\{m(m-1)+n(n-1)\}=\tfrac{1}{2}(m^2+n^2-m-n)$$
$$=\tfrac{1}{2}(m+n)(m+n-1)-mn=\frac{(m+n)!}{2(m+n-2)!}-mn$$
$$=C(m+n,2)-mn$$ ■

Example 16 *Prove the identity*

$$C(n,r)\cdot C(r,k)=C(n,k)\cdot C(n-k,r-k),\quad \textit{for}\quad n\geq r\geq k.$$

► We have

$$C(n,r)\cdot C(r,k)=\frac{n!}{r!\,(n-r)!}\cdot\frac{r!}{k!\,(r-k)!}$$
$$=\frac{n!}{(n-k)!\,k!}\cdot\frac{(n-k)!}{(n-r)!(r-k)!}$$
$$=\frac{n!}{(n-k)!\,k!}\cdot\frac{(n-k)!}{\{(n-k)-(r-k)\}!\,(r-k)!}$$
$$=C(n,k)\cdot C(n-k,r-k).$$ ■

Exercises

1. In how many different ways can a committee of 5 teachers and 4 students be selected from 9 teachers and 15 students?

2. How many different seven-person committees can be formed each containing 3 women and 4 men from an available set of 20 women and 30 men?

3. A bag contains 5 red marbles and 6 white marbles. Find the number of ways that 4 marbles can be drawn from the bag if the 4 marbles must be of the same colour.

4. How many 10-digit binary numbers are there with exactly six 1's?

5. There are 21 consonants and 5 vowels in the English alphabet. Consider only 8-letter words with 3 different vowels and 5 different consonants. (a) How many of such words can be formed ? (b) How many begin with a and end with b? (c) How many contain the letters a, b, and c?

6. Find the number of 5-letter words which contain three different consonants and two different vowels of the English alphabet.

7. Find the number of ways of selecting 4 persons out of 12 persons to a party if two of them will not attend the party together.

8. A man has 15 close friends of whom 6 are women. In how many ways can he invite 3 or more of these friends to a party if he wants the same number of men (including himself) as women?

9. A multiple choice test has 15 questions and 4 choices for each answer. How many ways can the 15 questions be answered so that 3 answers are correct?

10. There are 12 students in a class. In how many ways can these students take four different tests if three students are to take each test?

11. In a class social there were 40 students and 5 teachers present. Each student had to shake hands with every other student and also with each of the teachers. How many handshakes were there?

12. How many arrangements of the letters in MISSISSIPPI have no consecutive S's?

13. Find the number of arrangements of letters in the word TALLAHASSEE that begin with T and end with E.

14. A question paper contains 10 questions of which 7 are to be answered. In how many ways a student can select the 7 questions (i) if he can choose any seven? (ii) if he should select three questions from the first five and four questions from the last five? (iii) if he should select at least three from the first five?

15. How many bytes contain (i) exactly two 1's? (ii) exactly four 1's? (iii) exactly six 1's? (iv) at least six 1's?

16. Out of five mathematicians and seven engineers, a committee consisting of two mathematicians and three engineers is to be formed. In how many ways can this be done if (a) any mathematician and any engineer can be included? (b) one particular engineer must be on the committee? (c) two particular mathematicians cannot be on the committee?

17. A box contains 15 IC chips of which 7 are defective and 8 are non-defective. In how many ways 5 chips car [illegible] hosen so that (a) all are non-defective? (b) all are defective? (c) 2 are non-defective? [illegible] are non-defective?

18. Find how many distinct five-digit integers one can form from the set of digits consisting of five 1's, three 2's, two 3's, three 4's, two 5's, one 6 and four 7's.

19. Prove the following identities:

(i) $C(2n, 2) = 2C(n, 2) + n^2$

(ii) $C(3n, 3) = 3C(n, 3) + 6nC(n, 2) + n^3$

(iii) $C(n + r + 1, r) = C(n + 1, 1) + C(n + 2, 2) + \cdots + C(n + r, r)$

(iv) $C(n + 1, r + 1) = C(r, r) + C(r + 1, 2) + \cdots + C(n, r)$

(v) $n\,C(m + n, m) = (m + 1)\,C\,(m + n, m + 1)$

(vi) $C(m + n, n) = \sum_{r=0}^{n} C(m, r)C(n, r)$

(vii) $C(n + r + 1, r) = \sum_{k=0}^{r} C(n + k, k)$

Answers

1. $C(9, 5) \times C(15, 4)$ **2.** $C(20, 3) \times C(30, 4)$ **3.** $C(5, 4) + C(6, 4)$ **4.** $C(10, 6)$

5. (a) $C(5, 3) \times C(21, 5) \times 8!$ (b) $C(4, 2) \times C(20, 4) \times 6!$ (c) $C(4, 2) \times C(19, 3) \times 8!$

6. $C(21, 3) \times C(5, 2) \times 5!$ **7.** 450 **8.** $\sum_{r=1}^{5} \binom{9}{r}\binom{6}{r+1}$

9. $C(15, 3) \times 3^{12}$ **10.** 39600 **11.** 980

12. 7350 **13.** 15,120 **14.** (i) 120 (ii) 50 (iii) 110

15. (i) 28 (ii) 70 (iii) 28 (iv) 37 **16.** (a) 350 (b) 150 (c) 105

17. (a) 350 (b) 21 (c) 980 (d) 1176 **18.** 13, 431.

4.4 Binomial and Multinomial Theorems

One of the basic properties of $C(n, r) \equiv \binom{n}{r}$ is that it is the coefficient of $x^r y^{n-r}$ and $x^{n-r} y^r$ in the expansion of the expression $(x + y)^n$, where x and y are any real numbers. In other words,

$$(x + y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r} = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r$$

This result is known as the ***Binomial Theorem for a positive integral index.***

The numbers $\binom{n}{r}$ for $r = 0, 1, 2, \ldots n$, namely $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}$, in the above result are called the *binomial coefficients.*

The student is already familiar with the proof by mathematical induction of the above-mentioned binomial theorem.

The following is a generalization of the binomial theorem, known as the ***Multinomial Theorem***.

Theorem* : *For positive integers n and k, the coefficient of $x_1^{n_1} x_2^{n_2} x_3^{n_3} \ldots x_k^{n_k}$ in the expansion of $(x_1 + x_2 + \cdots + x_k)^n$ is*

$$\frac{n!}{n_1!\, n_2!\, n_3! \ldots n_k!}$$

where each n_i is a nonnegative integer $\leq n$, and $n_1 + n_2 + n_3 + \cdots + n_k = n$.

Proof: We note that in the expansion of $(x_1 + x_2 + \cdots + x_k)^n$ the coefficient of $x_1^{n_1} x_2^{n_2} \ldots x_k^{n_k}$ is the number of ways we can select x_1 from n_1 of the n factors, x_2 from n_2 of the $n - n_1$ remaining factors, x_3 from n_3 of the $n - n_1 - n_2$ remaining factors, and so on. Therefore, this coefficient is, by the product rule,

$$\begin{aligned} &C(n, n_1) \cdot C(n - n_1, n_2) \cdot C(n - n_1 - n_2, n_3) \cdots\cdots C(n - n_1 - n_2 - \cdots - n_{k-1}, n_k) \\ &= \frac{n!}{n_1!\,(n-n_1)!} \cdot \frac{(n-n_1)!}{n_2!\,(n-n_1-n_2)!} \cdot \frac{(n-n_1-n_2)!}{n_3!(n-n_1-n_2-n_3)!} \\ &\qquad \cdots \frac{(n-n_1-n_2-\cdots-n_{k-1})!}{n_k!\,(n-n_1-n_2-\cdots-n_{k-1}-n_k)!} \\ &= \frac{n!}{n_1!\, n_2!\, n_3! \cdots n_k!} \end{aligned}$$

This proves the required result. •

Note: (1) Another way of stating the Multinomial Theorem is:

When n is a positive integer, the general term in the expansion of

$$(x_1 + x_2 + x_3 + \cdots + x_k)^n \quad \textit{is} \quad \frac{n!}{n_1!\, n_2! \cdots n_k!} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k},$$

where $n_1, n_2, \ldots n_k$ are nonnegative integers not exceeding n and $n_1 + n_2 + n_3 + \cdots + n_k = n$.

The expression $\dfrac{n!}{n_1!\, n_2! \cdots n_k!}$ is also written as

$$\binom{n}{n_1, n_2, n_3, \ldots, n_k}$$

and is called a ***multinomial coefficient***.

*The case $k = 2$ of this Theorem corresponds to the Binomial theorem for a positive integral index.

(2) The Multinomial Theorem can also be stated in the following form:

$$(x_1 + x_2 + x_3 + \cdots + x_k)^n = \sum_{n_i} \binom{n}{n_1, n_2 \ldots, n_k} x_1^{n_1} x_2^{n_2} \ldots x_k^{n_k}$$

Here $\sum_{n_i}$ denotes summation over all of n_i's.

Example 1 *Prove the following identities for a positive integer n:*

(i) $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$

(ii) $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} + \cdots + (-1)^n \binom{n}{n} = 0$

► The Binomial theorem for a positive integral index n reads

$$(x + y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}.$$

Taking $x = 1$ and $y = 1$ in this we get

$$2^n = \sum_{r=0}^{n} \binom{n}{r} 1^r \cdot 1^{n-r} = \sum_{r=0}^{n} \binom{n}{r} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n},$$

Next, by taking $x = -1$ and $y = 1$, we get

$$0 = \sum_{r=0}^{n} \binom{n}{r} (-1)^r = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} + \cdots + (-1)^n \binom{n}{n}.$$ ■

Example 2 *Evaluate :* $\binom{12}{5, 3, 2, 2}$.

► We have

$$\binom{12}{5, 3, 2, 2} = \frac{12!}{5!\ 3!\ 2!\ 2!} = 166320.$$ ■

Example 3 *Find the sums of all the coefficients in the expansions of*

(a) $(x + y)^n$ **(b)** $(x_1 + x_2 + \cdots + x_k)^n$

Hence obtain the sum of all the coefficients in the expansions of each of the following

(i) $(x + y)^{10}$ (ii) $(x + y + z + w)^5$ (iii) $(x + 2y - 3z + 2u + 3v)^{12}$

▶ **(a)** We have

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}$$

Taking $x = 1$, $y = 1$ in this we get (as in Example 1 above)

$$2^n = \sum_{r=0}^{n} \binom{n}{r} = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}.$$

Thus, the sum of all the coefficients in the expansion of $(x+y)^n$ is 2^n.

(b) We have, by the Multinomial Theorem,

$$\sum_{n_i} \binom{n}{n_1, n_2 \ldots, n_k} x_1^{n_1} x_2^{n_2} \ldots x_k^{n_k} = (x_1 + x_2 + \cdots + x_k)^n$$

Taking $x_1 = 1$, $x_2 = 1, \ldots, x_k = 1$ in this we get

$$\sum_{n_i} \binom{n}{n_1, n_2 \ldots, n_k} = (1 + 1 + \cdots + 1)^n = k^n.$$

The sum in the left hand side of the above expression is the sum of the coefficients in the expansion of $(x_1 + x_2 + \cdots + x_k)^n$. This sum is equal to k^n as shown.

Next :

(i) It follows from the result in **(a)** that the sum of the coefficients in the expansion of $(x+y)^{10}$ is 2^{10}.

(ii) It follows from the result in **(b)** that the sum of the coefficients in the expansion of $(x+y+z+w)^5$ is 4^5.

(iii) We note (by virtue of the multinomial theorem) that

$$\begin{aligned}(x + 2y - 3z + 2u + 3v)^{12} &= \sum_{n_i} \binom{12}{n_1, n_2 \ldots, n_5} x^{n_1}(2y)^{n_2}(-3z)^{n_3}(2u)^{n_4}(3v)^{n_5} \\ &= \sum_{n_i} \binom{12}{n_1, n_2 \ldots, n_5} (2^{n_2})(-3)^{n_3}(2^{n_4})(3^{n_5}) x^{n_1} y^{n_2} z^{n_3} u^{n_4} v^{n_5}.\end{aligned}$$

Taking $x = 1$, $y = 1$, $z = 1$, $u = 1$, $v = 1$ in the above expression we get

$$(1 + 2 - 3 + 2 + 3)^{12} = \sum_{n_i} \binom{12}{n_1, n_2 \ldots, n_5} 2^{n_2}(-3)^{n_3} 2^{n_4} 3^{n_5}$$

The sum in the right hand side of the above expression is the sum of the coefficients in the expansion of $(x + 2y - 3z + 2u + 3v)^{12}$. This sum is $(1 + 2 - 3 + 2 + 3)^{12} = 5^{12}$ as shown. ∎

Example 4 *Find the coefficient of*

(i) x^9y^3 *in the expansion of* $(2x-3y)^{12}$.

(ii) x^0 *in the expansion of* $\left(3x^2-\frac{2}{x}\right)^{15}$.

(iii) x^{12} *in the expansion of* $x^3(1-2x)^{10}$.

(iv) x^k *in the expansion of* $(1+x+x^2)(1+x)^n$, *where n is a positive integer and* $0 \le k \le n+2$.

▶ (i) We have, by the binomial theorem

$$(2x-3y)^{12} = \sum_{r=0}^{12}\binom{12}{r}\cdot(2x)^r(-3y)^{12-r}$$
$$= \sum_{r=0}^{12}\binom{12}{r}2^r(-3)^{12-r}\cdot x^r y^{12-r}$$

In this expansion, the coefficient of x^9y^3 (which corresponds to $r = 9$) is

$$\binom{12}{9}2^9(-3)^3 = -(2^9\times 3^3)\times\frac{12!}{9!\,3!} = -2^9\times 3^3\times\frac{12\times 11\times 10}{6}$$
$$= -2^{10}\times 3^3\times 11\times 10$$

(ii) By Binomial theorem, we have

$$\left(3x^2-\frac{2}{x}\right)^{15} = \sum_{r=0}^{15}\binom{15}{r}(3x^2)^r\left(-\frac{2}{x}\right)^{(15-r)}$$
$$= \sum_{r=0}^{15}\binom{15}{r}3^r(-2)^{(15-r)}x^{3r-15}$$

The coefficient of x^0 (namely the constant term) which corresponds to $r = 5$ in this is

$$\binom{15}{5}\times 3^5\times(-2)^{10} = \frac{15!}{10!\,5!}\times 3^5\times 2^{10}.$$

(iii) By Binomial theorem, we have

$$(1-2x)^{10} = \sum_{r=0}^{10}\binom{10}{r}1^{10-r}(-2x)^r$$

Therefore,

$$x^3(1-2x)^{10} = \sum_{r=0}^{10} \binom{10}{r}(-2)^r x^{r+3}$$

The coefficient of x^{12} in this expansion (which corresponds to $r = 9$) is

$$\binom{10}{9}(-2)^9 = -(10 \times 2^9) = -5120.$$

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(iv) Using Binomial theorem, we find that

$$\begin{aligned}(1+x+x^2)(1+x)^n &= (1+x+x^2) \times \sum_{r=0}^{n} \binom{n}{r} 1^{n-r} x^r \\ &= \sum_{r=0}^{n} \binom{n}{r} x^r + \sum_{r=0}^{n} \binom{n}{r} x^{r+1} + \sum_{r=0}^{n} \binom{n}{r} x^{r+2}\end{aligned}$$

The coefficient of x^k in this is

$$\binom{n}{k} + \binom{n}{k-1} + \binom{n}{k-2}$$

Note: For example, the coefficients of x^7 and x^{18} are

$$\binom{n}{7} + \binom{n}{6} + \binom{n}{5} \quad \text{and} \quad \binom{n}{18} + \binom{n}{17} + \binom{n}{16}$$

respectively. ■

Example 5 *Determine the coefficient of*

(i) xyz^2 *in the expansion of* $(2x - y - z)^4$.

(ii) $x^2y^2z^3$ *in the expansion of* $(3x - 2y - 4z)^7$.

(iii) $x^{11}y^4$ *in the expansion of* $(2x^3 - 3xy^2 + z^2)^6$

(iv) $a^2b^3c^2d^5$ *in the expansion of* $(a + 2b - 3c + 2d + 5)^{16}$.

► (i) By the Multinomial theorem, we note that the general term in the expansion of $(2x - y - z)^4$ is

$$\binom{4}{n_1,\ n_2,\ n_3}(2x)^{n_1}(-y)^{n_2}(-z)^{n_3}.$$

For $n_1 = 1, n_2 = 1$ and $n_3 = 2$, this becomes

$$\binom{4}{1,\ 1,\ 2}(2x)(-y)(-z)^2 = \binom{4}{1,\ 1,\ 2} \times 2 \times (-1) \times (-1)^2 xyz^2$$

This shows that the required coefficient is

$$\binom{4}{1, 1, 2} \times 2 \times (-1) \times (-1)^2 = \frac{4}{1!\ 1!\ 2!} \times (-2)$$

$$= -\cancel{12}.\ -24$$

(ii) The general term in the expansion of $(3x - 2y - 4z)^7$ is

$$\binom{7}{n_1,\ n_2,\ n_3}(3x)^{n_1}(-2y)^{n_2}(-4z)^{n_3}$$

For $n_1 = 2, n_2 = 2, n_3 = 3$, this becomes

$$\binom{7}{2,\ 2,\ 3}(3x)^2(-2y)^2(-4z)^3$$

$$= \binom{7}{2,\ 2,\ 3}\left\{3^2 \times (-2)^2 \times (-4)^3\right\} x^2y^2z^3$$

This shows that the required coefficient is

$$\left\{3^2 \times (-2)^2 \times (-4)^3\right\} \times \binom{7}{2,\ 2,\ 3} = -(9 \times 4 \times 64) \times \frac{7!}{2!\ 2!\ 3!}$$

$$= -2304 \times \frac{7 \times 6 \times 5 \times 4}{4}$$

$$= -4,83,840.$$

(iii) The general term in the expansion of $(2x^3 - 3xy^2 + z^2)^6$ is

$$\binom{6}{n_1,\ n_2,\ n_3}(2x^3)^{n_1}(-3xy^2)^{n_2}(z^2)^{n_3}$$

For $n_3 = 0, n_2 = 2, n_1 = 3$ this becomes

$$\binom{6}{3,\ 2,\ 0}(2^3x^9)(3^2x^2y^4) = 72 \times \frac{6!}{3!\ 2!\ 0!}x^{11}y^4.$$

(i

hus, the required coefficient is

$$72 \times \frac{6 \times 5 \times 4}{2} = 4320.$$

(iv) The general term in the expansion of $(a + 2b - 3c + 2d + 5)^{16}$ is

$$\binom{16}{n_1,\ n_2,\ n_3,\ n_4,\ n_5}(a)^{n_1}(2b)^{n_2}(-3c)^{n_3}(2d)^{n_4}(5)^{n_5}.$$

For $n_1 = 2, n_2 = 3, n_3 = 2, n_4 = 5$ and $n_5 = 16 - (2 + 3 + 2 + 5) = 4$, this becomes

$$\binom{16}{2, 3, 2, 5, 4}a^2(2b)^3(-3c)^2(2d)^5 5^4 = \binom{16}{2, 3, 2, 5, 4} \times 2^3 \times (-3)^2 \times 2^5 \times 5^4 \times a^2b^3c^2d^5$$

Therefore, the coefficient of $a^2b^3c^2d^5$ is

$$2^8 \times 3^2 \times 5^4 \times \frac{16!}{2!\ 3!\ 2!\ 5!\ 4!} = 3 \times 2^5 \times 5^3 \times \frac{16!}{(4!)^2}$$ ■

Exercises

1. Find the coefficient of

(i) x^9y^3 in the expansion of $(x + 2y)^{12}$ (ii) x^5y^2 in the expansion of $(2x - 3y)^7$.

2. Compute the following:

(i) $\binom{7}{2, 3, 2}$ (ii) $\binom{8}{4, 2, 2, 0}$ (iii) $\binom{10}{5, 3, 2, 2}$.

3. Find the coefficient of:

(i) xyz^5 in the expansion of $(x + y + z)^7$.

(ii) $x^2y^2z^3$ in the expansion of $(x + y + z)^7$.

(iii) x^3z^4 in the expansion of $(x + y + z)^7$.

(iv) xyz^{-2} in the expansion of $(x - 2y + 3z^{-1})^4$.

(v) $x^3y^3z^2$ in the expansion of $(2x - 3y + 5z)^8$.

(vi) $w^3x^2yz^2$ in the expansion of $(2w - x + 3y - 2z)^8$.

(vii) $x_1^2x_3x_4^3x_5^4$ in the expansion of $(x_1 + x_2 + x_3 + x_4 + x_5)^{10}$.

4. Find sum of all the coefficients in the expansion of

(i) $(x + y + z)^{12}$ (ii) $(2s - 3t + 5u + 6v - 11w + 3x + 2y)^{10}$

5. Prove that, if n is a nonnegative integer,

$$\frac{1}{2}\{(1+x)^n + (1-x)^n\} = \binom{n}{0} + \binom{n}{2}x^2 + \cdots + \binom{n}{k}x^k,$$

where $k = \begin{cases} n \text{ if } n \text{ is even} \\ n-1 \text{ if } n \text{ is odd.} \end{cases}$

Deduce that (with k defined as above)

$$\binom{n}{0} + \binom{n}{2} + \cdots + \binom{n}{k} = \begin{cases} 2^{n-1} \text{ for } n > 0 \\ 1 \text{ if } n = 0. \end{cases}$$

6. Prove the following:

(i) $\binom{m}{m} + \binom{m+1}{m} + \cdots + \binom{m+n}{m} = \binom{m+n+1}{m+1}$ for $m \geq 0, n \geq 1$

(ii) $\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$ for $n \geq 0$

(iii) $\binom{n}{0}^2 - \binom{n}{1}^2 + \binom{n}{2}^2 - \cdots + (-1)^n\binom{n}{n}^2 = \begin{cases} (-1)^{n/2}\binom{n}{n/2} & \text{for even } n \\ 0 & \text{for odd } n \end{cases}$

Answers

1. (i) 1760 (ii) 6048

2. (i) 210 (ii) 420 (iii) meaningless (why ?)

3. (i) 42 (ii) 210 (iii) 35 (iv) −210 (v) −3024000 (vi) 161280 (vii) 12600

4. (i) 3^{12} (ii) 4^{10}

4.5 Combinations with Repetitions

Suppose we wish to select, with *repetition*, a combination of r objects from a set of n distinct objects. The number of such selections is given by*

$$C(n+r-1, r) \equiv \frac{(n+r-1)!}{r!(n-1)!} \equiv C(r+n-1, n-1)$$

In other words, $C(n+r-1, r) \equiv C(r+n-1, n-1)$ represents the number of combinations of n distinct objects, taken r at a time, with repetitions allowed.

The following are other interpretations of this number:

(1) $C(n+r-1, r) \equiv C(r+n-1, n-1)$ represents the number of ways in which r identical objects can be distributed among n distinct containers.

(2) $C(n+r-1, r) \equiv C(r+n-1, n-1)$ represents the number of nonnegative integer solutions† of the equation

$$x_1 + x_2 + \cdots + x_n = r.$$

Example 1 *A bag contains coins of seven different denominations, with at least one dozen coins in each denomination. In how many ways can we select a dozen coins from the bag?*

► The selection consists in choosing *with repetitions*, $r = 12$ coins of $n = 7$ distinct denominations. The number of ways of making this selection is

$$C(7+12-1, 12) = C(18, 12) = \frac{18!}{12!\,6!} = 18,564. \qquad \blacksquare$$

Example 2 *In how many ways can we distribute* 10 *identical marbles among* 6 *distinct containers?*

► The selection consists in choosing with *repititions*, $r = 10$ marbles for $n = 6$ distinct containers. The required number is

$$C(6+10-1, 10) = C(15, 10) = \frac{15!}{10!\,5!} = 3003. \qquad \blacksquare$$

Example 3 *A cake shop sells* 20 *kinds of cakes. If there are at least a dozen cakes of each kind, in how many ways a dozen cakes can be chosen?*

► The required number is

$$C(20+12-1, 12) = C(31, 12) = 141,120,525 \qquad \blacksquare$$

*The proof of this statement is omitted.

†A nonnegative integer solution of the equation $x_1 + x_2 + \cdots + x_n = r$ is an n-tuple $(x_1, x_2, x_3, \ldots x_n)$, where $x_1, x_2, \ldots x_n$ are nonnegative integers whose sum is r.

Example 4 *Find the number of nonnegative integer solutions of the equation*

$$x_1 + x_2 + x_3 + x_4 + x_5 = 8.$$

n r

► The required number is

$$C(5 + 8 - 1, 8) = C(12, 8) = 495.$$ ■

Example 5 *Find the number of distinct terms in the expansion of*

$$(x_1 + x_2 + x_3 + x_4 + x_5)^{16}.$$

► Every term in the expansion is of the form (by multinomial theorem)

$$\binom{16}{n_1,\ n_2,\ n_3,\ n_4,\ n_5} x_1^{n_1} x_2^{n_2} x_3^{n_3} x_4^{n_4} x_5^{n_5}$$

where each n_i is a nonnegative integer, and these n_is sum to 16. Therefore, the number of distinct terms in the expansion is precisely equal to the number of nonnegative integer solutions of the equation

$$n_1 + n_2 + n_3 + n_4 + n_5 = 16.$$

This number is

$$C(5 + 16 - 1, 16) = C(20, 16) = 4845.$$ ■

Example 6 *Find the number of nonnegative integer solutions of the inequality*

$$x_1 + x_2 + x_3 + \cdots + x_6 < 10. \quad (9 \cdot x_7$$

► We have to find the number of nonnegative integer solutions of the equation

$$x_1 + x_2 + x_3 + \cdots + x_6 = 9 - x_7$$

where $9 - x_7 \leq 9$ so that x_7 is a nonnegative integer. Thus, the required number is the number of nonnegative solutions of the equation

$$x_1 + x_2 + x_3 + \cdots + x_7 = 9.$$

$n = 7,\ r = 9$

This number is

$$C(7 + 9 - 1, 9) = C(15, 9) = \frac{15!}{9!\,6!} = 5005.$$ ■

Example 7 *Find the number of positive integer solutions of the equation $x_1 + x_2 + x_3 = 17$.*

▶ Here, we require $x_1 \geq 1$, $x_2 \geq 1$, $x_3 \geq 1$. Let us set $y_1 = x_1 - 1$, $y_2 = x_2 - 1$, $y_3 = x_3 - 1$. Then, y_1, y_2, y_3 are all nonnegative integers.

When written in terms of y's, the given equation reads

$$(y_1 + 1) + (y_2 + 1) + (y_3 + 1) = 17, \quad \text{or} \quad y_1 + y_2 + y_3 = 14.$$

$n = 3$ $\lambda = 14$

The number of nonnegative integer solutions of this equation is the required number. This number is

$$C(3 + 14 - 1, 14) = C(16, 14) = \frac{16!}{14!\,2!} = \frac{16 \times 15}{2} = 120. \qquad \blacksquare$$

Example 8 *Find the number of integer solutions of*

$$x_1 + x_2 + x_3 + x_4 + x_5 = 30$$

where $x_1 \geq 2$, $x_2 \geq 3$, $x_3 \geq 4$, $x_4 \geq 2$, $x_5 \geq 0$.

▶ Let us set $y_1 = x_1 - 2$, $y_2 = x_2 - 3$, $y_3 = x_3 - 4$, $y_4 = x_4 - 2$, $y_5 = x_5$. Then $y_1, y_2, \ldots y_5$ are all nonnegative integers. When written in terms of y's, the given equation reads

$$(y_1 + 2) + (y_2 + 3) + (y_3 + 4) + (y_4 + 2) + y_5 = 30, \quad \text{or} \quad y_1 + y_2 + y_3 + y_4 + y_5 = 19.$$

The number of nonnegative integer solutions of this equation is the required number, and the number is

$$C(5 + 19 - 1, 19) = C(23, 19) = \frac{23!}{19!\,4!} = 8855. \qquad \blacksquare$$

Example 9 *In how many ways can 10 identical pencils be distributed among 5 children in the following cases:*

(i) *There are no restrictions.*

(ii) *Each child gets at least one pencil.*

(iii) *The youngest child gets atleast two pencils.*

▶ (i) The required number is (like in Example 2)

$$C(5 + 10 - 1, 10) = C(14, 10) = \frac{14!}{10!\,4!} = 1001.$$

(ii) First we distribute one pencil to each child. Then there remian 5 pencils to be distributed. The number of ways of distributing these 5 pencils to 5 children is the required number. This number is

$$C(5 + 5 - 1, 5) = C(9, 5) = \frac{9!}{5!\,4!} = 126$$

(iii) First we give two pencils to the youngest child. Then there remain 8 pencils to be distributed. The number of ways of distributing these 8 pencils to 5 children is the required number. This number is

$$C(5+8-1,8) = C(12,8) = \frac{12!}{8!\,4!} = 495.$$

■

Example 10 *In how many ways can one distribute eight identical balls into four distinct containers so that* (i) *no container is left empty?* (ii) *the fourth container gets an odd number of balls?*

► (i) First we distribute one ball into each container. Then we distribute the remaining 4 balls into 4 containers. The number of ways of doing this is the required number. This number is

$$C(4+4-1,4) = C(7,4) = \frac{7!}{4!\,3!} = 35.$$

(ii) If the fourth container has to get an odd number of balls, we have to put one or three or five or seven balls into it.

Suppose we put one ball into it (the fourth container). Then the remaining 7 balls can be distributed into the remaining three containers in

$$C(3+7-1,7) = C(9,7) \quad \text{ways.}$$

Similarly, putting 3 balls into the fourth container and the remaining 5 into the remaining 3 containers can be done in

$$C(3+5-1,5) = C(7,5) \quad \text{ways}$$

Next, putting 5 balls into the fourth container and the remaining 3 into the remaining 3 containers can be done in

$$C(3+3-1,3) = C(5,3) \quad \text{ways}$$

Lastly, putting 7 balls into the fourth container and the remaining 1 into the remaining 3 container can be done in

$$C(3+1-1,1) = C(3,1) = 3 \quad \text{ways.}$$

Thus, the total number ways of distributing the given balls so that the fourth container gets an odd number of balls is

$$C(9,7)+C(7,5)+C(5,3)+3 = \frac{9!}{7!\,2!}+\frac{7!}{5!\,2!}+\frac{5!}{3!\,2!}+3$$
$$= 36+21+10+3 = 70$$

■

Example 11 *A total amount of ₹1500 is to be distributed to 3 poor students A, B, C of a class. In how many ways the distribution can be made in multiples of ₹100*

(i) *if everyone of these must get at least ₹300?*

(ii) *if* A *must get at least ₹500, and* B *and* C *must get at least ₹400 each?*

► Taking ₹100 as a unit, there are 15 units for distribution.

In case (i), each of the three students must get at least 3 units. Let us first distribute 3 units to each of the 3 students. Then there remain 6 units for distribution. The number of ways of distributing these 6 units to A, B, C is the required number (in this case). This number is

$$C(3+6-1,6) = C(8,6) = 28.$$

In case (ii), A must get at least 5 units, B and C must get at least 4 units each. Let us distribute 5 units to A and 4 units to each of B and C. Then there remain 2 units for distribution. Accordingly, the number of ways of making the distribution in this case is

$$C(3+2-1,2) = C(4,2) = 6.$$

■

Example 12 *In how many ways can we distribute 7 apples and 6 oranges among 4 children so that each child gets at least 1 apple?*

► Suppose we first give 1 apple to each child. This exhausts 4 apples. The remaining 3 apples can be distributed among the 4 children in $C(4+3-1,3) = C(6,3)$ ways. Also, 6 oranges can be distributed among the 4 children in $C(4+6-1,6) = C(9,6)$ ways. Therefore, by the product rule, the number of ways of distributing the given fruits under the given condition is

$$C(6,3)\times C(9,6) = \frac{6!}{3!\,3!}\times\frac{9!}{6!\,3!} = 20\times 84 = 1680.$$

■

Example 13 *Find the number of ways of giving 10 identical gift boxes to 6 persons* A, B, C, D, E, F *in such a way that the total number of boxes given to* A *and* B *together does not exceed* 4.

► Of the 10 boxes, suppose r boxes are given to A and B together. Then $0 \le r \le 4$. The number of ways of giving r boxes to A and B is

$$C(2+r-1,r) = C(r+1,r) = r+1. \tag{i}$$

The number of ways in which the remaining $(10-r)$ boxes can be given to C, D, E, F is

$$C(4+(10-r)+1,(10-r)) = C(13-r,10-r) = C(13-r,3). \tag{ii}$$

Consequently, the number of ways in which r boxes may be given to A and B and $10-r$ boxes to C, D, E, F is, by the product rule,

$$(r+1)\times C(13-r,3). \tag{iii}$$

Since $0 \le r \le 4$, the total number of ways in which the boxes may be given is, by the sum rule,

$$\sum_{r=0}^{4}(r+1)\times C(13-r,3).$$ ■

Example 14 *A message is made up of 12 different symbols and is to be transmitted through a communication channel. In addition to the 12 symbols, the transmitter will also send a total of 45 blank spaces between the symbols, with at least three spaces between each pair of consecutive symbols. In how many ways can the transmitter send such a message?*

► The 12 symbols can be arranged in 12! ways. For each of these arrangements, there are 11 positions between the 12 symbols. Since there must be at least three spaces between successive symbols, 33 of the 45 spaces will be used up. The remaining 12 spaces are to be accommodated in 11 positions. This can be done in $C(11+12-1,12) = C(22,12)$ ways. Consequently, by the product rule, the required number is

$$12!\times C(22,12) = \frac{22!}{10!} = 3.097445\times 10^{14}.$$ ■

Example 15 *Show that $C(n-1+r,r)$ represents the number of binary numbers that contains $(n-1)$ number of 1's and r number of 0's.*

► A binary number that contains $(n-1)$ number of 1's and r number of 0's, has $n-1+r$ positions and is determined by r positions of 0's. The number of such binary numbers is therefore $C(n-1+r,r)$. ■

Example 16 *Given positive integers m, n with $m \ge n$, show that the number of ways to distribute m identical objects into n distinct containers such that each container gets at least r objects, where $r \le (m/n)$, is*

$$C(m-1+(1-r)n, n-1).$$

► Suppose we place r of the m identical objects into each of the n distinct containers. Then, there remain $(m-nr)$ identical objects to be distributed into n distinct containers. The number of ways of doing this is the required number. This number is

$$\begin{aligned} C(n+(m-nr)-1, m-nr) &= C(n+(m-nr)-1, n-1)^* \\ &= C(m-1+(1-r)n, n-1) \end{aligned}$$ ■

*Recall that $C(n,r) = C(n,n-r)$.

Exercises

1. In how many ways can 20 similar books be placed on 5 different shelves?

2. Find the number of ways of placing 8 identical balls in 5 numbered boxes.

3. Determine the number of nonnegative integer solutions of the equation $x_1 + x_2 + x_3 + x_4 = 7$.

4. Find the number of distinct terms in the expansion of $(w + x + y + z)^{10}$.

5. How many integer solutions are there to the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 20$ where each $x_i \geq 2$?

6. How many integer solutions are there to the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 20$, where $x_1 \geq 3$, $x_2 \geq 2$, $x_3 \geq 4$, $x_4 \geq 6$, $x_5 \geq 0$?

7. Determine the number of integer solutions of the equation $x_1+x_2+x_3+x_4 = 32$, where $x_1, x_2 \geq 5$, $x_3, x_4 \geq 7$.

8. Find the number of solutions of the equation $x_1 + x_2 + x_3 + x_4 = 15$ in (a) non-negative integers, (b) positive integers, (c) integers satisfying $x_1 > 2$, $x_2 > -2$, $x_3 > 0$, $x_4 > -3$.

9. Find the number of nonnegative integer solutions of each of the following inequalities:

   (i) $x_1 + x_2 + x_3 + x_4 \leq 6$ **(ii)** $x_1 + x_2 + x_3 + x_4 + x_5 \leq 19$.

10. Find the number of ways of placing 20 identical balls into 5 boxes with at least one ball put into each box.

11. A total of ₹10000 is to be distributed to four persons A, B, C, D in multiples of ₹1000. In how many ways can the distribution be done (i) if there is no restriction? (ii) if every one of these persons should receive at least ₹1000? (iii) if every one should receive at least ₹1000 and A in particular should receive at least ₹5000?

12. How many ways are there to place 12 marbles of the same size in five different jars (a) if the marbles are all of the same colour? (b) if the marbles are all of different colours?

13. Find the number of ways of distributing 7 identical pens and 7 identical pencils to 5 children so that each gets at least 1 pen and at least 1 pencil.

14. How many different outcomes are possible by tossing 10 similar coins?

15. Six distinct symbols are transmitted through a communication channel. A total of 12 blanks are to be inserted between the symbols with at least 2 blanks between every pair of symbols. In how many ways can the symbols and the blanks be arranged?

Answers

1. $C(24, 20)$	**2.** 495	**3.** 120
4. 286	**5.** $C(14, 10)$	**6.** $C(19, 5)$
7. $C(11, 8)$	**8.** $C(18, 3)$, $C(14, 3)$, $C(17, 3)$	**9.** (i) $C(10, 4)$ (ii) $C(24, 19)$
10. $C(19, 15)$	**11.** (i) 286 (ii) 84 (iii) 10	**12.** (a) $C(16, 12)$ (b) 5^{12}
13. $C(6, 4) \times C(6, 4)$	**14.** $C(2-1+10, 10)$	**15.** $C(5-1+2, 2) \times 6!$