Table of Contents

Module - 1

Module - 2

Scanned by CamScanner

Module - 3

Module - 4

Module - 5

Scanned by CamScanner

Scanned by CamScanner

Sahyadri College of Engineering & Management
CENTRAL LIBRARY
15 OCT 2019
Acc. No.
N.H. 48, Adyar, Mangalore - 575 007

Module - 1

Chapter 1
Mathematical Logic-I

Logic is the science dealing with the methods of reasoning. Reasoning plays a very important role in every area of knowledge, particularly mathematics. A symbolic language has been developed over the past two centuries to express the principles of logic in precise and unambiguous terms. Logic expressed in such a language has come to be called "Symbolic logic" or "Mathematical logic". Some basic notions of this subject are introduced in this chapter.

1.1 Propositions

Consider the following sentences:

(1) Bengaluru is in Karnataka. (2) Three is a prime number.

(3) Seven is divisible by 3. (4) Every rectangle is a square.

Each of the above sentences is a statement (declaration) which can be decisively said to be either true or false. In fact, sentences 1 and 2 are true statements whereas sentences 3 and 4 are false statements. Sentences of this type are called *propositions*.

A proposition is formally defined as follows:

A proposition is a statement (declaration) which, in a given context, can be said to be either true or false, but not both.

It should be noted that not all sentences are propositions. For example, consider the following sentences:

1. Take a triangle ABC. 2. $xy = yx$.

We note that the first of these sentences is not at all a statement; as such, it is not a proposition. The second sentence is a statement; but we cannot decisively say whether it is true or not unless we know what x and y are. As such, it is also not a proposition. However, if the context tells that x and y are real numbers, then it becomes a true proposition.

Propositions are usually represented by small letters such as $p, q, r, s \ldots$.* The truth or the falsity of a proposition is called its ***truth value***. If a proposition is *true*, we will indicate its truth value by the symbol 1 and if it is *false* by the symbol 0 **.

*Some writers use capital letters also.

**Some authors denote the truth value of a true proposition by T and the truth value of a false proposition by F.

Scanned by CamScanner

For example, if we represent the proposition "Three is a prime number" by p, then the truth value of p is 1. Similarly, if we represent the proposition "Every rectangle is a square" by q, then the truth value of q is 0.

1.1.1 Logical Connectives and Truth Tables

New propositions are obtained by starting with given propositions with the aid of words or phrases like *'not'*, *'and'*, *'if... then'*, and *'if and only if'*. Such words or pharses are called ***logical connectives***. The new propositions obtained by the use of connectives are called *compound propositions*. The original propositions from which a compound proposition is obtained are called the *components* or the *primitives* of the compound proposition. Propositions which do not contain any logical connective are called *simple propositions*.

Negation

A proposition obtained by inserting the word *'not'* at an appropriate place in a given proposition is called the *negation* of the given proposition.

The negation of a proposition p is denoted by $\neg p$ (read *'not* p'), the symbol $\neg$ denoting the word *not*. *

For example, let the proposition "3 is a prime number" be denoted by p; that is ,

$$p : 3 \text{ is a prime number.}$$

Then the negation of p is "3 is not a prime number"; that is,

$$\neg p : 3 \text{ is not a prime number.}$$

It is obvious that, for any proposition p, *if p is true, then $\neg p$ is false,* and *if p is false, then $\neg p$ is true*. In other words, if the truth value of a proposition p is 1 then the truth value of $\neg p$ is 0 and if the truth value of p is 0 then the truth value of $\neg p$ is 1.

This observation can be recorded in the form of a **Truth Table** as given below:

Table 1.1: Truth Table for Negation

p	$\neg p$
0	1
1	0

The first column of the above table gives the possible truth values of p and the second column gives the corresponding truth values of $\neg p$.

Conjunction

A compound proposition obtained by combining two given propositions by inserting the word *'and'* in between them is called the *conjunction* of the given propositions.

The conjunction of two propositions p and q is denoted by $p \wedge q$ (read "p and q"), the symbol $\wedge$ denoting the word *and*.

*Some authors write $\sim p$ for the negation of p.

The following *Rule* is adopted in deciding the truth value of a conjunction:
The conjunction $p \wedge q$ is true only when p is true and q is true; in all other cases it is false.

In other words, the truth value of the conjunction $p \wedge q$ is 1 only when the truth value of p is 1 and the truth value of q is 1; in all other cases the truth value of $p \wedge q$ is 0.

This rule can be expressed more explicitly in the form of a truth table as given below:

Table 1.2: Truth Table for Conjunction

p	q	$p \wedge q$
0	0	0
0	1	0
1	0	0
1	1	1

In the above table, the last column gives the truth values of $p \wedge q$ for all possible combinations of the truth values of p and q given in the first two columns*.

As an illustration, let us consider the following propositions.

p : $\sqrt{2}$ is an irrational number. q : 9 is a prime number.

r : All triangles are equilateral. s : $2 + 5 = 7$.

Here, p and s are true propositions and q and r are false propositions.

Then

$p \wedge q$: $\sqrt{2}$ is an irrational number and 9 is a prime number.

$q \wedge r$: 9 is a prime number and all triangles are equilateral.

$r \wedge s$: All triangles are equilateral and $2 + 5 = 7$

$s \wedge p$: $2 + 5 = 7$ and $\sqrt{2}$ is an irrational number.

Since p is true and q is false, $p \wedge q$ is false. Since both q and r are false, $q \wedge r$ is false. Since r is false and s is true, $r \wedge s$ is false. Since both s and p are true, $s \wedge p$ is true.

Disjunction

A compound proposition obtained by combining two given propositions by inserting the word '*or*' in between them is called the *disjunction* of the given propositions.

The disjunction of two propositions p and q is denoted by $p \vee q$ (read "p or q"), the symbol $\vee$ denoting the word *or*.

The following *Rule* is adopted in deciding the truth value of a disjunction:
The disjunction $p \vee q$ is false only when p is false and q is false; in all other cases it is true.

In other words, the truth value of the disjunction $p \vee q$ is 0 only when the truth value of p is 0 and the truth value of q is 0; in all other cases the truth value of $p \vee q$ is 1.

*The four possible truth values for p and q can be listed in any order. We have chosen a particular, standard order.

Scanned by CamScanner

This rule can be expressed more explicitly in the form of a truth table given below:

Table 1.3: Truth Table for Disjunction

p	q	$p \vee q$
0	0	0
0	1	1
1	0	1
1	1	1

In the above table, the last column gives the truth values of $p \vee q$ for all possible combinations of the truth values of p and q given in the first two columns.

As an illustration, let us consider the following propositions:

p : $\sqrt{2}$ is an irrational number. q : 9 is a prime number.

r : All triangles are equilateral. s : $2 + 5 = 7$.

Then

$p \vee q$: $\sqrt{2}$ is an irrational number or 9 is a prime number.

$q \vee r$: 9 is a prime number or all triangles are equilateral.

$r \vee s$: All triangles are equilateral or $2 + 5 = 7$.

$s \vee p$: $2 + 5 = 7$ or $\sqrt{2}$ is an irrational number.

We note that p and s are true, and q and r are false. By using the truth table 1.3, we infer that $p \vee q$ is true, $q \vee r$ is false, $r \vee s$ is true, and $s \vee p$ is true.

Exclusive Disjunction

In the conjunction $p \vee q$ of two propositions p and q, the symbol $\vee$ is used in the *inclusive sense*. That is, $p \vee q$ is taken to be true when p or q or both p and q are true. Some times we require the use of the word *or* in the *exclusive sense*. That is, we require that the compound proposition "p or q" to be true only when either p is true or q is true *but not both*. The exclusive *or* is denoted by the symbol $\underline{\vee}$.

The compound proposition $p \underline{\vee} q$ (read as either p or q but not both) is called the *exclusive disjunction* of the propositions p and q*. Its truth table is as given below.

Table 1.4: Truth Table for Exclusive Disjunction

p	q	$p \underline{\vee} q$
0	0	0
0	1	1
1	0	1
1	1	0

*Some authors use the symbol $\overline{\vee}$ for exclusive disjunction.

Scanned by CamScanner

Observe that this truth table differs from the truth table for disjunction (Table 1.3) in the last line (row).

For example, let

p : $\sqrt{2}$ is an irrational number $\quad$ $q : 2 + 3 = 5$

r : 9 is a prime number. $\quad$ s : All triangles are iscoseles.

Then

$p \underline{\vee} q$: Either $\sqrt{2}$ is an irrational number or $2 + 3 = 5$, but not both.

$q \underline{\vee} r$: Either $2 + 3 = 5$ or 9 is a prime number, but not both.

$r \underline{\vee} s$: Either 9 is a prime number or all triangles are isosceles, but not both.

$s \underline{\vee} p$: Either all triangles are isoceles or $\sqrt{2}$ is an irrational number, but not both.

Since p and q are true and r and s are false, we note that $p \underline{\vee} q$ is false, $q \underline{\vee} r$ is true, $r \underline{\vee} s$ is false, and $s \underline{\vee} p$ is true.

Conditional

A compound proposition obtained by combining two given propositions by using the words *'if'* and *'then'* at appropriate places is called a *conditional*.

Given two propositions p and q, we can form the conditionals "If p, then q" and "If q, then p". The conditional *"If p, then q"* is denoted by $p \to q$ and the conditional *"If q then p"* is denoted by $q \to p$.

It is important to note that the conditional $q \to p$ is not the same as the conditional $p \to q$.

The following *Rule* is adopted in deciding the truth value of a conditional.

The conditional $p \to q$ is false only when p is true and q is false; in all other cases it is true.

In other words, the truth value of the conditional $p \to q$ is 0 only when the truth value of p is 1 and the truth value of q is 0; in all other cases, the truth value of $p \to q$ is 1.

This rule can be expressed more explicitly in the form of a truth table as given below:

Table 1.5: Truth Table for Conditional

p	q	$p \to q$
0	0	1
0	1	1
1	0	0
1	1	1

Note that $p \to q$ is taken to be true even when (i) p is false and q is true, and (ii) p and q are both false.

As an illustration, let us consider the following propositions.

p : 2 is a prime number. $\quad$ q : 3 is a prime number.

r : 6 is a perfect square. $\quad$ s : 9 is a multiple of 6.

Scanned by CamScanner

Here, p and q are true propositions, and r and s are false propositions.
Now,

$p \to q$: If 2 is a prime number, then 3 is a prime number.
$p \to r$: If 2 is a prime number then 6 is a perfect square.
$r \to p$: If 6 is a perfect square, then 2 is a prime number.
$r \to s$: If 6 is a perfect square, then 9 is a multiple of 6.

According to the truth table 1.5, we note that $p \to q$ is true, $p \to r$ is false, $r \to p$ is true and $r \to s$ is true.

Biconditional

Let p and q be two propositions. Then the conjunction of the conditionals $p \to q$ and $q \to p$ is called the *biconditional* of p and q; it is denoted by $p \leftrightarrow q$. Thus, $p \leftrightarrow q$ is the same as $(p \to q) \wedge (q \to p)$. As such, $p \leftrightarrow q$ is read as *'If p then q and if q then p'**.

The following is the truth table for $p \leftrightarrow q$:

Table 1.6: Truth Table for Biconditional

p	q	$p \to q$	$q \to p$	$p \leftrightarrow q$
0	0	1	1	1
0	1	1	0	0
1	0	0	1	0
1	1	1	1	1

From the above table, it is clear that $p \leftrightarrow q$ is true only when both p and q have the same truth values.

For example, let

p: 2 is a prime number. q: 3 is a prime number.
r: 6 is a perfect square. s: 9 is a multiple of 6.

Then the biconditionals $p \leftrightarrow q$ and $r \leftrightarrow s$ are *true* and the biconditionals $p \leftrightarrow r$, $p \leftrightarrow s$, $q \leftrightarrow r$ and $q \leftrightarrow s$ are *false*.

The Truth Tables 1.1 to 1.6 can be put together in the following combined form for ready reference:

Table 1.7: Combined Truth Table

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \veebar q$	$p \to q$	$p \leftrightarrow q$
0	0	1	0	0	0	1	1
0	1	1	0	1	1	1	0
1	0	0	0	1	1	0	0
1	1	0	1	1	0	1	1

*Another notation for $p \leftrightarrow q$ is $p \rightleftarrows q$.

Scanned by CamScanner

Compound propositions containing more than one connective are also often encountered. The truth values of such compound propositions are determined by using the combined Truth Table 1.7.

Example 1 *Let*

p: *A circle is a conic.* q: $\sqrt{5}$ *is a real number.*

r: *Exponential series is convergent.*

Express the following compound propositions in words:

(i) $p \wedge (\neg q)$ (ii) $(\neg p) \vee q$ (iii) $p \veebar (\neg q)$

(iv) $q \rightarrow (\neg p)$ (v) $p \rightarrow (q \veebar r)$ (vi) $\neg p \leftrightarrow q$

► (i) $p \wedge (\neg q)$: A circle is a conic and $\sqrt{5}$ is not a real number.

(ii) $(\neg p) \vee q$: A circle is not a conic or $\sqrt{5}$ is a real number.

(iii) $p \veebar (\neg q)$: Either a circle is a conic or $\sqrt{5}$ is not a real number (but not both).

(iv) $q \rightarrow (\neg p)$: If $\sqrt{5}$ is a real number, then a circle is not a conic.

(v) $p \rightarrow (q \veebar r)$: If a circle is a conic then either $\sqrt{5}$ is a real number or the exponential series is convergent (but not both).

(vi) $\neg p \leftrightarrow q$: If a circle is not a conic then $\sqrt{5}$ is a real number and if $\sqrt{5}$ is a real number then a circle is not a conic. ■

Example 2 *Construct the truth tables for the following compound propositions:*

(i) $p \wedge (\neg q)$ (ii) $(\neg p) \vee q$ (iii) $p \rightarrow (\neg q)$ (iv) $(\neg p) \veebar (\neg q)$

► The desired truth tables are obtained by considering all possible combinations of the truth values of p and q. The combined form of the required truth tables is shown below:

p	q	$\neg p$	$\neg q$	$p \wedge (\neg q)$	$(\neg p) \vee q$	$p \rightarrow (\neg q)$	$(\neg p) \veebar (\neg q)$
0	0	1	1	0	1	1	0
0	1	1	0	0	1	1	1
1	0	0	1	1	0	1	1
1	1	0	0	0	1	0	0

■

Example 3 *Let p and q be primitive statements for which the conditional $p \rightarrow q$ is false. Determine the truth values of the following compound propositions:*

(i) $p \wedge q$ (ii) $(\neg p) \vee q$ (iii) $q \rightarrow p$ (iv) $(\neg q) \rightarrow (\neg p)$

► Since $p \rightarrow q$ is given to be false, p has to be true and q has to be false. Consequently, $\neg p$ has to be false and $\neg q$ has to be true. Therefore:

(i) The truth value of $p \wedge q$ is 0. (ii) The truth value of $(\neg p) \vee q$ is 0.

(iii) The truth value of $q \rightarrow p$ is 1. (iv) The truth value of $(\neg q) \rightarrow (\neg p)$ is 0. ■

Scanned by CamScanner

Example 4 *Let p, q and r be propositions having truth values 0, 0 and 1 respectively. Find the truth values of the following compound propositions:*

(1) $(p \vee q) \vee r$ (2) $(p \wedge q) \wedge r$ (3) $(p \wedge q) \to r$
(4) $p \to (q \wedge r)$ (5) $p \wedge (r \to q)$ (6) $p \to (q \to (\neg r))$

► (1) Since both p and q are false, $p \vee q$ is false. Since r true, it follows that $(p \vee q) \vee r$ is true. Thus, the truth value of $(p \vee q) \vee r$ is 1.

(2) Since both p and q are false, $p \wedge q$ is false. Since $p \wedge q$ is false and r is true, $(p \wedge q) \wedge r$ is false. Thus, the truth value of $(p \wedge q) \wedge r$ is 0.

(3) Since $p \wedge q$ is false and r is true, $(p \wedge q) \to r$ is true. Thus, the truth value of $(p \wedge q) \to r$ is 1.

(4) Since q is false and r is true, $q \wedge r$ is false. Also, p is false. Therefore, $p \to (q \wedge r)$ is true. Thus, the truth value of $p \to (q \wedge r)$ is 1.

(5) Since r is true and q is false, $r \to q$ is false. Also, p is false. Hence, $p \wedge (r \to q)$ is false. Thus, the truth value of $p \wedge (r \to q)$ is 0.

(6) Since r is true, $\neg r$ is false, Since q is false, $q \to (\neg r)$ is true. Also, p is false. Therefore, $p \to (q \to (\neg r))$ is true. Thus, the truth value of $p \to (q \to (\neg r))$ is 1. ■

Example 5 *Find the possible truth values of p,q and r in the following cases:*

(i) $p \to (q \vee r)$ *is false* (ii) $p \wedge (q \to r)$ *is true.*

► (i) $p \to (q \vee r)$ can be false only when p is true and $q \vee r$ is false. Also, $q \vee r$ is false only when both q and r are false. Hence, the truth values of p, q, r are 1, 0, 0 respectively.

(ii) $p \wedge (q \to r)$ can be true only when p is true and $q \to r$ is true. Also, $q \to r$ can be true when (a) r is true and q is true or false, and (b) r is false and q is false. Hence the possible truth values of p, q, r are as shown in the following table:

p	q	r
1	1	1
1	0	1
1	0	0

■

Example 6 *Construct the truth tables for the following compound propositions:*

(i) $(p \vee q) \wedge r$ (ii) $p \vee (q \wedge r)$

► Each of the two given combined propositions contains three primitive propositions p, q, r. Each of these three primitive propositions has two possible truth values. Therefore, there exist $2^3 = 8$ sets of possible truth values of p, q, r, and each such set gives a truth value of the compound proposition containing p, q, r.

Scanned by CamScanner

The required truth tables are shown below in a combined form.

p	q	r	$p \vee q$	$(p \vee q) \wedge r$	$q \wedge r$	$p \vee (q \wedge r)$
0	0	0	0	0	0	0
0	0	1	0	0	0	0
0	1	0	1	0	0	0
0	1	1	1	1	1	1
1	0	0	1	0	0	1
1	0	1	1	1	0	1
1	1	0	1	0	0	1
1	1	1	1	1	1	1

Observe that the truth values of $(p \vee q) \wedge r$ and $p \vee (q \wedge r)$ are *not* identical. ■

Example 7 *Construct the truth tables for the following compound propositions:*

(i) $(p \wedge q) \rightarrow (\neg r)$ (ii) $q \wedge ((\neg r) \rightarrow p)$

▶ The required truth tables are shown below in a combined form.

p	q	r	$\neg r$	$(p \wedge q)$	$(p \wedge q) \rightarrow (\neg r)$	$(\neg r) \rightarrow p$	$q \wedge ((\neg r) \rightarrow p)$
0	0	0	1	0	1	0	0
0	0	1	0	0	1	1	0
0	1	0	1	0	1	0	0
0	1	1	0	0	1	1	1
1	0	0	1	0	1	1	0
1	0	1	0	0	1	1	0
1	1	0	1	1	1	1	1
1	1	1	0	1	0	1	1

■

Example 8 *If a proposition q has the truth value 1, determine all truth value assignments for the primitive propositions p, r and s for which the truth value of the following compound proposition is 1.*

$$[q \rightarrow \{(\neg p \vee r) \wedge \neg s\}] \wedge \{\neg s \rightarrow (\neg r \wedge q)\}$$

▶ The given compound proposition is of the form $u \wedge v$, where

$$u \equiv q \rightarrow \{(\neg p \vee r) \wedge (\neg s)\} \quad \text{and} \quad v \equiv \neg s \rightarrow (\neg r \wedge q).$$

Since the truth value of this compound proposition is 1, the truth value of each of u and v is 1.

Since q has the truth value 1, and u has the truth value 1 it follows that the truth value of $(\neg p \vee r) \wedge \neg s$ must also be 1. Consequently, $\neg p \vee r$ has the truth value 1 and $\neg s$ has the truth value 1. Consequently, *s has the truth value* 0.

Scanned by CamScanner

Since $\neg s$ has the truth value 1 and v has the truth value 1, the truth value of $(\neg r \wedge q)$ must be 1. Since q has the truth value 1, it follows that $\neg r$ must also have the truth value 1; that is, *r must have the truth value* 0.

Since $(\neg p \vee r)$ has the truth value 1 and r has the truth value 0, it follows that $\neg p$ must have truth value 1; that is, *p must have the truth value* 0.

Thus, all of p, r, s must have the truth value 0 *. ■

Example 9 *Indicate how many rows are needed in the truth table for the compound proposition*

$$(p \vee \neg q) \leftrightarrow \{(\neg r \wedge s) \rightarrow t\}.$$

Find the truth value of this proposition if p and r are true and q, s, t are false.

► The given compound proposition contains five primitives (components) p, q, r, s, t. Therefore, the number of possible combinations of the truth values of these components which we have to consider is $2^5 = 32$. Hence, 32 rows are needed in the truth table for the given compound proposition.

Next, suppose that p and r are true and q, s, t are false. Then $\neg q$ is true and $\neg r$ is false.

Since p is true and $\neg q$ is true, $p \vee (\neg q)$ is true. On the other hand, since $\neg r$ is false and s is false, $\neg r \wedge s$ is false. Also, t is false. Hence $(\neg r \wedge s) \rightarrow t$ is true.

Since $p \vee (\neg q)$ is true and $(\neg r \wedge s) \rightarrow t$ is true, it follows that the truth value of the given proposition $(p \vee (\neg q)) \leftrightarrow \{(\neg r \wedge s) \rightarrow t\}$ is 1. ■

Exercises

1. State which of the following sentences are propositions.

(1) A triangle contains three sides. (2) $x + 2$ is a positive integer.
(3) 5 divides x. (4) Is $\sqrt{2}$ a rational number?
(5) 21 is an even number. (6) India had a woman prime minister.

2. Indicate the negation of each of the following propositions.

(1) $2 + 3 = 5$ (2) 5 divides 27
(3) Computer Science is a hard subject. (4) Bangalore has pleasant weather.

3. Give the conjunction and disjunction of p and q in the following cases; in each case indicate the truth value.

(1) p : 4 is a perfect square. (2) p : 5 is divisible by 2.
q : 27 is a prime number. q : 7 is a multiple of 5.

*This result can also be got by constructing the truth tables for u and v and using the given data.

Scanned by CamScanner

4. Indicate the truth value of each of the following propositions:

(1) If $3+4=7$, then $5+3=8$. (2) If $3+4=5$, then $4+6=10$.

(3) If $4+2=7$, then $3+2=4$. (4) If $4+5=9$, then $3+1=6$.

5. Consider the following propositions concerned with a certain triangle ABC.

$p : ABC$ is isosceles. $q : ABC$ is equilateral. $r : ABC$ is equiangular.

Write down the following propositions in words.

(1) $p \wedge (\neg q)$ (2) $(\neg p) \vee q$ (3) $p \to q$

(4) $q \to p$ (5) $(\neg r) \to (\neg q)$ (6) $p \leftrightarrow (\neg q)$.

6. From the information given in each of the following, determine the truth value required.

(1) $p \wedge q$ is false and p is true; find the truth value of q.

(2) $p \vee q$ is false and q is false; find the truth value of p.

(3) $p \to q$ is true and q is false; find the truth value of p.

(4) $p \leftrightarrow q$ is true and p is false; find the truth value of q.

7. Given that p is true and q is false, find the truth values of the following:

(1) $(\neg p) \wedge q$ (2) $\neg (p \to (\neg q))$

(3) $(p \wedge q) \to (p \vee q)$ (4) $\neg (p \wedge q) \vee \{\neg (q \leftrightarrow p)\}$

(5) $(p \to q) \vee \{\neg (p \leftrightarrow \neg q)\}$ (6) $\{(p \to (\neg q)\} \vee \{q \to (\neg p)\}$

8. Construct the truth tables for the following:

(1) $p \vee (\neg q)$ (2) $p \wedge (q \wedge p)$ (3) $p \wedge (q \vee p)$

(4) $(p \vee q) \wedge (\neg p)$ (5) $\neg (p \vee \neg q)$ (6) $q \leftrightarrow (\neg p \vee \neg q)$

(7) $p \to (q \to r)$ (8) $(p \to q) \to r$ (9) $[(p \wedge q) \vee (\neg r)] \leftrightarrow p$

Answers

1. (1) True proposition. (2) Not a proposition. (3) Not a proposition.

(4) Not a proposition. (5) False proposition. (6) True proposition.

2. (1) $2+3 \neq 5$. (2) 5 does not divide 27.

(3) Computer Science is not a hard subject. (4) Bangalore does not have pleasant weather.

3. (1) $p \wedge q$: 4 is a perfect square and 27 is a prime number; *truth value* 0.

$p \vee q$: 4 is a perfect square or 27 is a prime number; *truth value* 1.

Scanned by CamScanner

(2) $p \wedge q$: 5 is divisible by 2 and 7 is a multiple of 5; *truth value* 0.

$p \vee q$: 5 is divisible by 2 or 7 is a multiple of 5; *truth value*: 0.

4. (1) 1 (2) 1 (3) 1 (4) 0

5. (1) *ABC* is isosceles and is not equilateral.

(2) *ABC* is not isosceles or *ABC* is equilateral.

(3) If *ABC* is isosceles, then it is equilateral.

(4) If *ABC* is equilateral, then it is isoceles.

(5) If *ABC* is not equiangular then it is not equilateral.

(6) If *ABC* is isoceles then it is not equilateral and if *ABC* is not equilateral then it is isoceles.

6. (1) 0 (2) 0 (3) 0 (4) 0

7. (1) 0 (2) 0 (3) 1 (4) 1 (5) 0 (6) 1

8. (1), (2), (3):

p	q	$p \vee (\neg q)$	$p \wedge (q \wedge p)$	$p \wedge (q \vee p)$
0	0	1	0	0
0	1	0	0	0
1	0	1	0	1
1	1	1	1	1

(4), (5), (6):

p	q	$(p \vee q) \wedge \neg p$	$\neg (p \vee \neg q)$	$q \leftrightarrow (\neg p \vee \neg q)$
0	0	0	0	0
0	1	1	1	1
1	0	0	0	0
1	1	0	0	0

(7), (8), (9):

p	q	r	$p \to (q \to r)$	$(p \to q) \to r$	$[(p \wedge q) \vee \neg r] \leftrightarrow p$
0	0	0	1	0	0
0	0	1	1	1	1
0	1	0	1	0	0
0	1	1	1	1	1
1	0	0	1	1	1
1	0	1	1	1	0
1	1	0	0	0	1
1	1	1	1	1	1

Scanned by CamScanner

1.1.2 Tautology; Contradiction

A compound proposition which is *true* for all possible truth values of its components is called a ***tautology*** (or a *logical truth* or a *universally valid statement*).

A compound proposition which is *false* for all possible truth values of its components is called a ***contradiction*** or an ***absurdity***.

A compound proposition that can be true or false (depending upon the truth values of its components) is called a **contingency**. In other words, a contingency is a compound proposition which is neither a tautology nor a contradiction.

Example 1 *Prove that, for any proposition p, the compound proposition* $p \vee \neg p$ *is a tautology and the compound proposition* $p \wedge \neg p$ *is a contradiction.*

► Let us construct the truth table giving the truth values of $p \vee \neg p$ and $p \wedge \neg p$ for all possible truth values of p and $\neg p$. The truth table is as shown below.

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
0	1	1	0
1	0	1	0

We note that $p \vee \neg p$ is always true; hence it is tautology. On the other hand, $p \wedge \neg p$ is always false; hence it is a contradiction. ■

Example 2 *Show that, for any two propositions p and q,*

(i) $(p \veebar q) \vee (p \leftrightarrow q)$ *is a tautology*

(ii) $(p \veebar q) \wedge (p \leftrightarrow q)$ *is a contradiction*

(iii) $(p \veebar q) \wedge (p \rightarrow q)$ *is a contingency.*

► From Table 1.7, we find that, for all possible truth values of p and q, the compound propositions $p \veebar q$ and $p \leftrightarrow q$ have opposite truth values. Therefore, the truth value of $(p \veebar q) \vee (p \leftrightarrow q)$ is always 1 and the truth value of $(p \veebar q) \wedge (p \leftrightarrow q)$ is always 0. This means that $(p \veebar q) \vee (p \leftrightarrow q)$ is a tautology and $(p \veebar q) \wedge (p \leftrightarrow q)$ is a contradiction.

From Table 1.7, we also find that $(p \veebar q) \wedge (p \rightarrow q)$ can have the truth value 1 or 0.

Thus, $(p \veebar q) \wedge (p \rightarrow q)$ is neither a tautology nor a contradiction; it is a contingency. ■

Example 3 *Show that, for any propositions p and q, the compound proposition* $p \rightarrow (p \vee q)$ *is a tautology and the compound proposition* $p \wedge (\neg p \wedge q)$ *is a contradiction.*

► Let us first prepare the truth tables for $p \rightarrow (p \vee q)$ and $p \wedge (\neg p \wedge q)$. These truth tables are shown below in a combined form.

Scanned by CamScanner

p	q	$p \vee q$	$p \to (p \vee q)$	$\neg p$	$\neg p \wedge q$	$p \wedge (\neg p \wedge q)$
0	0	0	1	1	0	0
0	1	1	1	1	1	0
1	0	1	1	0	0	0
1	1	1	1	0	0	0

From the above table we note that, for all possible truth values of p and q, the compound proposition $p \to (p \vee q)$ is true and the compound proposition $p \wedge (\neg p \wedge q)$ is false. Therefore, $p \to (p \vee q)$ is a tautology and $p \wedge (\neg p \wedge q)$ is a contradiction. ∎

Example 4 *Show that the truth values of the following compound propositions are independent of the truth values of their components:*

(i) $\{p \wedge (p \to q)\} \to q$ (ii) $(p \to q) \leftrightarrow (\neg p \vee q)$

► Let us first prepare the truth tables for the given compound propositions. These are as shown below.

(i)

p	q	$p \to q$	$r = p \wedge (p \to q)$	$r \to q$
0	0	1	0	1
0	1	1	0	1
1	0	0	0	1
1	1	1	1	1

(ii)

p	q	$u = p \to q$	$\neg p$	$v = \neg p \vee q$	$u \leftrightarrow v$
0	0	1	1	1	1
0	1	1	1	1	1
1	0	0	0	0	1
1	1	1	0	1	1

The last columns in both of the above tables show that the truth value of each of the given compound propositions is 1 irrespective of what the truth values of its components are. This is what we had to show. Furthermore, we have proved that the given compound propositions are tautologies. ∎

Example 5 *Prove that, for any propositions p and q, the compound proposition*

$$[(\neg q) \wedge (p \to q)] \to (\neg p)$$

is a tautology.

Scanned by CamScanner

CENTRAL LIBRARY
15 OCT 2019
Acc. No.
Attavar, Mangalore - 575 002

► Let us prepare the following truth table:

p	q	$p \to q$	$r = [\neg q \wedge (p \to q)]$	$r \to (\neg p)$
0	0	1	1	1
0	1	1	0	1
1	0	0	0	1
1	1	1	0	1

We observe that the proposition $r \to \neg p$, where $r = [(\neg q) \wedge (p \to q)]$, is always true. Therefore, this proposition is a tautology. ■

Example 6 *Prove that, for any propositions p, q, r, the compound proposition*

$$[(p \to q) \wedge (q \to r)] \to (p \to r)$$

is a tautology.

► The following truth table proves the required result:

p	q	r	$p \to q$	$q \to r$	$(p \to q) \wedge (q \to r)$	$p \to r$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$
0	0	0	1	1	1	1	1
0	0	1	1	1	1	1	1
0	1	0	1	0	0	1	1
0	1	1	1	1	1	1	1
1	0	0	0	1	0	0	1
1	0	1	0	1	0	1	1
1	1	0	1	0	0	0	1
1	1	1	1	1	1	1	1

■

Example 7 *Prove that, for any propositions p, q, r, the compound proposition*

$$\{p \to (q \to r)\} \to \{(p \to q) \to (p \to r)\}$$

is a tautology.

Scanned by CamScanner

▶ The following truth table proves the required result.

p	q	r	$p \to q$	$q \to r$	$p \to r$	$p \to (q \to r)$	$(p \to q) \to (p \to r)$	$\{p \to (q \to r)\} \to \{(p \to q) \to (p \to r)\}$
0	0	0	1	1	1	1	1	1
0	0	1	1	1	1	1	1	1
0	1	0	1	0	1	1	1	1
0	1	1	1	1	1	1	1	1
1	0	0	0	1	0	1	1	1
1	0	1	0	1	1	1	1	1
1	1	0	1	0	0	0	0	1
1	1	1	1	1	1	1	1	1

Example 8 *Prove that, for any propositions p, q, r, the compound proposition*

$$[(p \vee q) \wedge \{(p \to r) \wedge (q \to r)\}] \to r$$

is a tautology.

▶ The following truth table proves the required result.

p	q	r	$p \vee q$	$p \to r$	$q \to r$	$(p \to r) \wedge (q \to r)$	$(p \vee q) \wedge \{(p \to r) \wedge (q \to r)\}$	$(p \vee q) \wedge \{(p \to r) \wedge (q - r)\} \to r$
0	0	0	0	1	1	1	0	1
0	0	1	0	1	1	1	0	1
0	1	0	1	1	0	0	0	1
0	1	1	1	1	1	1	1	1
1	0	0	1	0	1	0	0	1
1	0	1	1	1	1	1	1	1
1	1	0	1	0	0	0	0	1
1	1	1	1	1	1	1	1	1

Exercises

1. Determine whether the following are tautologies:

(1) $p \vee [\neg (p \wedge q)]$

(2) $(p \vee q) \vee \neg p$

(3) $p \to (p \wedge q)$

(4) $p \to (p \vee q)$

(5) $\{\neg (p \to q)\} \to (\neg q)$

Scanned by CamScanner

2. Prove that the following are tautologies:

(1) $(p \wedge q) \to p$ (2) $\neg (p \vee \neg q) \to \neg p$.

(3) $\neg p \to (p \to q)$ (4) $[p \wedge (p \to q)] \to q$

(5) $[(p \to q) \vee (p \to r)] \leftrightarrow [p \to (q \vee r)]$

(6) $[(p \to q) \wedge (p \to \neg q)] \leftrightarrow \neg p$

(7) $[(p \wedge \neg q) \to r] \to [p \to (q \vee r)]$

(8) $\neg (p \vee q) \vee [(\neg p) \wedge q] \vee p$

(9) $[(p \to r) \wedge (q \to r)] \to [(p \vee q) \to r]$

(10) $\{[(p \vee q) \to r] \wedge (\neg p)\} \to (q \to r)$

3. Find the possible truth values of p, q, r, s, t, for which the following are contradictions:

(1) $[(p \wedge q) \wedge r] \to (s \vee t)$ (2) $[p \wedge (q \wedge r)] \to (s \veebar t)$

Answers

1. (1), (2), (4), (5) : Yes (3): No.

3. (1) p, q, r are true; s and t are false (2) p, q, r are true; s and t are both true or both false.

1.2 Logical Equivalence

Two propositions u and v are said to be *logically equivalent* whenever u and v have the same truth value, or equivalently, the biconditional $u \leftrightarrow v$ is a tautology *

Then, we write $u \Leftrightarrow v$. Here the symbol $\Leftrightarrow$ stands for "logically equivalent to".

When the propositions u and v are not logically equivalent, we write $u \not\Leftrightarrow v$.

Logically equivalent propositions are treated as identical propositions. **

Example 1 *Let x be a specified positive integer. Consider the following propositions :*

p : x is an odd integer. *q : x is not divisible by 2.*

Are p and q logically equivalent?

► We note that p and q have the same truth values. As such, p and q are logically equivalent; that is, $p \Leftrightarrow q$. ■

*Here, u and v can be simple propositions or compound propositions. When u and v are compound propositions, they are logically equivalent if they have the same truth values for all possible combinations of the truth values of their corresponding components.

**For this reason, the symbol $\equiv$ is also used in place of $\Leftrightarrow$.

Scanned by CamScanner

Example 2 *For any two propositions p, q, prove that* $(p \to q) \Leftrightarrow (\neg p) \vee q$.

▶ We first prepare the following truth table:

p	q	$p \to q$	$\neg p$	$(\neg p) \vee q$
0	0	1	1	1
0	1	1	1	1
1	0	0	0	0
1	1	1	0	1

We observe that $p \to q$ and $(\neg p) \vee q$ have the same truth values for all possible truth values of p and$q$. Therefore, $(p \to q) \Leftrightarrow (\neg p) \vee q$. ∎

Example 3 *Prove that* $\{(p \to q) \to r\} \leftrightarrow \{(\neg p \vee q) \to r\}$ *is a tautology.*

▶ We have $(p \to q) \Leftrightarrow (\neg p) \vee q$. Therefore,

$$\{(p \to q) \to r\} \Leftrightarrow \{(\neg p \vee q) \to r\}$$

Consequently, $\{(p \to q) \to r\} \leftrightarrow \{(\neg p \vee q) \to r\}$ is a tautology. ∎

Example 4 *Prove that*

$$[(r \to s) \wedge \{(r \to s) \to (t \to u)\}] \to (\neg t \vee u)$$

is a tautology.

▶ Let p denote $r \to s$ and q denote $t \to u$. Since $(t \to u) \Leftrightarrow (\neg t \vee u)$, we have $q \Leftrightarrow \neg t \vee u$. Accordingly, the given proposition is logically equivalent to $\{p \wedge (p \to q)\} \to q$ which is a tautology.*

Therefore, the given proposition is also a tautology. ∎

Example 5 *Prove that, for any propositions p and q, the compound propositions* $p \veebar q$ *and* $(p \vee q) \wedge (\neg p \vee \neg q)$ *are logically equivalent.*

▶ Let us prepare the following truth table.

p	q	$p \vee q$	$p \veebar q$	$\neg p$	$\neg q$	$\neg p \vee \neg q$	$(p \vee q) \wedge (\neg p \vee \neg q)$
0	0	0	0	1	1	1	0
0	1	1	1	1	0	1	1
1	0	1	1	0	1	1	1
1	1	1	0	0	0	0	0

From columns 4 and 8 of the above truth table, we find that $p \veebar q$ and $(p \vee q) \wedge (\neg p \vee \neg q)$ have the same truth values for all possible truth values of p and q. Therefore,

$$(p \veebar q) \Leftrightarrow \{(p \vee q) \wedge (\neg p \vee \neg q)\}$$ ∎

*See Example 4, Section 1.1.2.

Scanned by CamScanner

Example 6 *Prove that, for any three propositions p, q, r,*

$$[p \to (q \wedge r)] \Leftrightarrow [(p \to q) \wedge (p \to r)]$$

▶ Let us prepare the following truth table.

p	q	r	$q \wedge r$	$p \to (q \wedge r)$	$p \to q$	$p \to r$	$(p \to q) \wedge (p \to r)$
0	0	0	0	1	1	1	1
0	0	1	0	1	1	1	1
0	1	0	0	1	1	1	1
0	1	1	1	1	1	1	1
1	0	0	0	0	0	0	0
1	0	1	0	0	0	1	0
1	1	0	0	0	1	0	0
1	1	1	1	1	1	1	1

Columns 5 and 8 of the above table show that $[p \to (q \wedge r)]$ and $[(p \to q) \wedge (p \to r)]$ have the same truth values in all possible situations. Therefore,

$$\{p \to (q \wedge r)\} \Leftrightarrow \{(p \to q) \wedge (p \to r)\}$$ ■

Example 7 *Prove that, for any three propositions p, q, r,*

$$[(p \vee q) \to r] \Leftrightarrow [(p \to r) \wedge (q \to r)]$$

▶ Let us prepare the following truth table:

p	q	r	$(p \vee q)$	$(p \vee q) \to r$	$p \to r$	$q \to r$	$(p \to r) \wedge (q \to r)$
0	0	0	0	1	1	1	1
0	0	1	0	1	1	1	1
0	1	0	1	0	1	0	0
0	1	1	1	1	1	1	1
1	0	0	1	0	0	1	0
1	0	1	1	1	1	1	1
1	1	0	1	0	0	0	0
1	1	1	1	1	1	1	1

Columns 5 and 8 of the table show that $(p \vee q) \to r$ and $(p \to r) \wedge (q \to r)$ have the same truth values in all possible situations. Therefore,

$$[(p \vee q) \to r] \Leftrightarrow [(p \to r)] \wedge (q \to r)$$ ■

Scanned by CamScanner

Example 8 *Examine whether the compound proposition*

$$[(p \vee q) \rightarrow r] \leftrightarrow [\neg r \rightarrow \neg(p \vee q)]$$

is a tautology.

► Let us prepare the following truth table:

p	q	r	$p \vee q$	$(p \vee q) \rightarrow r$	$\neg r$	$\neg(p \vee q)$	$\neg r \rightarrow \neg(p \vee q)$
0	0	0	0	1	1	1	1
0	0	1	0	1	0	1	1
0	1	0	1	0	1	0	0
0	1	1	1	1	0	0	1
1	0	0	1	0	1	0	0
1	0	1	1	1	0	0	1
1	1	0	1	0	1	0	0
1	1	1	1	1	0	0	1

We observe that $(p \vee q) \rightarrow r$ and $\neg r \rightarrow \neg(p \vee q)$ have the same truth values in all possible situations. Therefore, these propositions are logically equivalent. Thus,

$$[(p \vee q) \rightarrow r] \leftrightarrow [\neg r \rightarrow \neg(p \vee q)]$$

is a tautology. ■

Example 9 *Show that the compound propositions* $p \wedge (\neg q \vee r)$ *and* $p \vee (q \wedge \neg r)$ *are not logically equivalent.*

► Let us prepare the following truth table.

p	q	r	$\neg q$	$\neg r$	$\neg q \vee r$	$q \wedge \neg r$	$p \wedge (\neg q \vee r)$	$p \vee (q \wedge \neg r)$
0	0	0	1	1	1	0	0	0
0	0	1	1	0	1	0	0	0
0	1	0	0	1	0	1	0	1
0	1	1	0	0	1	0	0	0
1	0	0	1	1	1	0	1	1
1	0	1	1	0	1	0	1	1
1	1	0	0	1	0	1	0	1
1	1	1	0	0	1	0	1	1

From the last two rows we note that $p \wedge (\neg q \vee r)$ and $p \vee (q \wedge \neg r)$ do not have the same truth values in all possible situations. Therefore, they are not logically equivalent. ■

Scanned by CamScanner

Exercises

1. For any two propositions p and q, prove the following:

(1) $(p \to \neg q) \Leftrightarrow (q \to \neg p)$

(2) $\neg (p \to \neg q) \Leftrightarrow (p \wedge q)$

(3) $(p \vee q) \Leftrightarrow \neg (\neg p \wedge \neg q)$

(4) $(p \vee q) \wedge \neg (\neg p \wedge q) \Leftrightarrow p$

(5) $(p \leftrightarrow \neg q) \Leftrightarrow \neg p \leftrightarrow q$

(6) $[p \to (q \to p)] \Leftrightarrow [\neg p \to (p \to q)]$

(7) $\neg (p \leftrightarrow q) \Leftrightarrow (p \vee q) \wedge [\neg (p \wedge q)]$

2. For any propositions p, q, r, prove the following:

(1) $[(p \to q) \wedge (p \to r)] \Leftrightarrow p \to (q \wedge r)$

(2) $[(p \to q) \wedge (r \to q)] \Leftrightarrow [(p \vee r) \to q]$

(3) $[(p \to q) \wedge (p \to \neg q)] \Leftrightarrow \neg p$

(4) $[(p \to q) \to r] \Leftrightarrow [(p \wedge \neg r) \to \neg q]$

(5) $[p \wedge (\neg r \vee q \vee \neg q)] \vee [(r \vee t \vee \neg r) \wedge \neg q] \Leftrightarrow p \vee \neg q$

3. Prove the following logical equivalences:

(1) $[(p \leftrightarrow q) \wedge (q \leftrightarrow r) \wedge (r \leftrightarrow p)] \Leftrightarrow [(p \wedge q) \wedge (q \to r) \wedge (r \to p)]$

(2) $p \to (q \to r) \Leftrightarrow p \to (\neg q \vee r) \Leftrightarrow (p \wedge q) \to r \Leftrightarrow (p \to r) \vee (q \to r)$

(3) $(p \underline{\vee} q) \Leftrightarrow [(p \wedge \neg q) \vee (\neg p \wedge q)] \Leftrightarrow \neg (p \leftrightarrow q)$

(4) $[p \to (q \vee r)] \Leftrightarrow [(p \to q) \vee (p \to r)] \Leftrightarrow [\neg r \to (p \to q)] \Leftrightarrow [(p \wedge \neg q) \to r]$

4. Prove that each of the following is a tautology:

(1) $[p \vee (q \wedge r)] \vee \neg [p \vee (q \wedge r)]$

(2) $[p \wedge (p \to q)] \to q$

(3) $p \to [q \to (p \wedge q)]$

(4) $[(p \vee q) \wedge \neg (\neg p \wedge (\neg q \vee \neg r))] \vee (\neg p \wedge \neg q) \vee (\neg p \wedge \neg r)$

Scanned by CamScanner

1.2.1 The Laws of Logic

The following results, known as *the Laws of Logic*, follow from the definition of logical equivalence. In these laws, T_0 denotes a tautology and F_0 denotes a contradiction.

1. *Law of Double negation*

   For any proposition p, $(\neg \neg p) \Leftrightarrow p$

2. *Idempotent laws*

   For any proposition p,

   (a) $(p \vee p) \Leftrightarrow p$ (b) $(p \wedge p) \Leftrightarrow p$

3. *Identity Laws*

   For any proposition p,

   (a) $(p \vee F_0) \Leftrightarrow p$ (b) $(p \wedge T_0) \Leftrightarrow p$

4. *Inverse Laws*

   For any proposition p,

   (a) $(p \vee \neg p) \Leftrightarrow T_0$ (b) $(p \wedge \neg p) \Leftrightarrow F_0$

5. *Domination Laws*

   For any proposition p,

   (a) $(p \vee T_0) \Leftrightarrow T_0$ (b) $(p \wedge F_0) \Leftrightarrow F_0$

6. *Commutative Laws*

   For any two propositions p and q,

   (a) $(p \vee q) \Leftrightarrow (q \vee p)$ (b) $(p \wedge q) \Leftrightarrow (q \wedge p)$

7. *Absorption Laws*

   For any propositions p and q,

   (a) $[p \vee (p \wedge q)] \Leftrightarrow p$ (b) $[p \wedge (p \vee q)] \Leftrightarrow p$

8. *DeMorgan Laws*

   For any propositions p and q,

   (a) $\neg (p \vee q) \Leftrightarrow \neg p \wedge \neg q$ (b) $\neg (p \wedge q) \Leftrightarrow \neg p \vee \neg q$

Scanned by CamScanner

9. *Associative Laws*

For any three propositions p, q, r,

(a) $p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$ (b) $p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$

10. *Distributive Laws*

For any propositions p, q, r,

(a) $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$ (b) $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

The law of double negation and the idempotent and commutative laws are trivially true. Other laws can be verified or proved with the aid of the truth tables. As an illustration, we prove the laws 8(a) and 10(a).

Proof of 8(a): Let us prepare the truth tables for $\neg (p \vee q)$ and $\neg p \wedge \neg q$. The tables are shown below in a combined form:

p	q	$p \vee q$	$\neg (p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
0	0	0	1	1	1	1
0	1	1	0	1	0	0
1	0	1	0	0	1	0
1	1	1	0	0	0	0

From the above table, we note that $\neg (p \vee q)$ and $(\neg p \wedge \neg q)$ have the same truth values in all possible situations. Hence

$$\neg (p \vee q) \Leftrightarrow (\neg p \wedge \neg q)$$ •

Proof of 10(a): The truth values of $[p \vee (q \wedge r)]$ and $[(p \vee q) \wedge (p \vee r)]$ for all possible truth values of p, q, r are shown in the following table:

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
0	0	0	0	0	0	0	0
0	0	1	0	0	0	1	0
0	1	0	0	0	1	0	0
0	1	1	1	1	1	1	1
1	0	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	1	0	0	1	1	1	1
1	1	1	1	1	1	1	1

From columns 5 and 8 of the above table, we find that $[p \vee (q \wedge r)]$ and $[(p \vee q) \wedge (p \vee r)]$ have the same truth values in all possible situations. Therefore,

$$[p \vee (q \wedge r)] \Leftrightarrow [(p \vee q) \wedge (p \vee r)]$$ •

Scanned by CamScanner

Law for the Negation of a Conditional

Given a conditional $p \rightarrow q$, its negation is obtained by using the following *law*:

$$\neg (p \rightarrow q) \Leftrightarrow [p \wedge (\neg q)]$$

Proof: The following table gives the truth values of $\neg (p \rightarrow q)$ and $(p \wedge \neg q)$ for all possible truth values of p and q.

p	q	$p \rightarrow q$	$\neg (p \rightarrow q)$	$\neg q$	$p \wedge \neg q$
0	0	1	0	1	0
0	1	1	0	0	0
1	0	0	1	1	1
1	1	1	0	0	0

We note that $\neg (p \rightarrow q)$ and $(p \wedge \neg q)$ have the same truth values in all possible situations. Hence

$$\neg (p \rightarrow q) \Leftrightarrow p \wedge \neg q$$

Remark: As mentioned before, logically equivalent propositions are *treated* as identical propositions. Accordingly, in view of the laws indicated above, we have the following results (for any two propositions p and q):

(1) $\neg (p \vee q) \equiv (\neg p \wedge \neg q)$

(2) $\neg (p \wedge q) \equiv (\neg p \vee \neg q)$

(3) $\neg (p \rightarrow q) \equiv (p \wedge \neg q)$

(4) $(p \rightarrow q) \equiv \neg \neg (p \rightarrow q) \equiv \neg (p \wedge \neg q) \equiv \neg p \vee q$

For ready reference, the rules for finding the negations of compound propositions are summarized in the following table.

Table 1.8: Table for Negation of compound propositions

Proposition	Negation
$\neg p$	p
$p \wedge q$	$\neg p \vee \neg q$
$p \vee q$	$\neg p \wedge \neg q$
$p \rightarrow q$	$p \wedge \neg q$

Transitive and Substitution Rules

The laws of logic and the law for the negation of compound propositions are employed while analyzing the logical equivalence (or otherwise) of propositions. In addition to these laws, the following *Rules* are also employed.

Scanned by CamScanner

(1) If u, v, w are propositions such that $u \Leftrightarrow v$ and $v \Leftrightarrow w$, then $u \Leftrightarrow w$. (This is known as the *Transitive Rule*).

(2) Suppose that a compound proposition u is a tautology and p is a component of u. If we replace *each* occurrence of p in u by a proposition q, then the resulting compound proposition v is also a tautology. (This is called a *Substitution Rule*).

(3) Suppose that u is a compound proposition which contains a component p. Let q be a proposition such that $q \Leftrightarrow p$. Suppose we replace one or more occurrences of p by q and obtain a compound proposition v. Then $v \Leftrightarrow u$. (This is also a *Substitution Rule*).

Example 1 *Let x be a specified number. Write down the negation of the following conditional:*

"If x is an integer, then x is a rational number."

▶ The given conditional is $p \to q$, where

p : x is an integer. q : x is a rational number.

Therefore, according to the result $\neg\ (p \to q) \equiv (p \wedge (\neg\ q))$, the negation of the given proposition reads:

"x is an integer and x is not a rational number." ■

Example 2 *Let x be a specified number. Write down the negation of the following proposition:*

"If x is not a real number, then it is not a rational number and not an irrational number."

▶ Let

p : x is a real number. q : x is a rational number. r : x is an irrational number.

Then the given proposition reads: $\neg\, p \to (\neg\, q \wedge \neg\, r)$. Therefore,

$$\begin{aligned}\neg\,[\neg\, p \to (\neg\, q \wedge \neg\, r)] &\equiv \neg\, p \wedge [\neg\,(\neg\, q \wedge \neg\, r)]\\ &\equiv \neg\, p \wedge (\neg\neg\, q \vee \neg\neg\, r)\\ &\equiv \neg\, p \wedge (q \vee r)\end{aligned}$$

Thus, the negation of the given proposition is

"x is not a real number and it is a rational number or an irrational number". ■

Example 3 *Simplify the following compound propositions using the laws of logic:*

(i) $(p \vee q) \wedge [\neg\{(\neg\, p) \wedge q\}]$ (ii) $(p \vee q) \wedge [\neg\{(\neg\, p) \vee q\}]$ (iii) $\neg\,[\neg\{(p \vee q) \wedge r\} \vee \neg q]$

▶ (i)

$$\begin{aligned}&(p \vee q) \wedge [\neg\{(\neg\, p) \wedge q\}]\\ &\equiv (p \vee q) \wedge \{(\neg\neg\, p) \vee (\neg\, q)\}, && \text{by D'Morgan law}\\ &\equiv (p \vee q) \wedge \{p \vee (\neg q)\}, && \text{by law of double negation}\\ &\equiv p \vee \{q \wedge (\neg\, q)\}, && \text{by distributive law}\end{aligned}$$

Scanned by CamScanner

$$\equiv p \vee F_0, \quad \text{by inverse law}$$

$$\equiv p, \quad \text{by identity law.}$$

(ii) $(p \vee q) \wedge \neg\{(\neg p) \vee q\} \equiv (p \vee q) \wedge (p \wedge \neg q)$

$\equiv \{(p \vee q) \wedge p\} \wedge (\neg q)$, using Associative Law

$\equiv \{p \wedge (p \vee q)\} \wedge (\neg q)$, using Commutative Law

$\equiv p \wedge (\neg q)$, using Absorption Law

(iii) $\neg[\neg\{(p \vee q) \wedge r\} \vee \neg q] \equiv \neg[\neg\{((p \vee q) \wedge r) \wedge q\}]$, using Demorgan Law

$\equiv ((p \vee q) \wedge r) \wedge q$, using the Law of double negation

$\equiv (p \vee q) \wedge (q \wedge r)$, using Associative and Commutative laws

$\equiv \{(p \vee q) \wedge q\} \wedge r$, using Associative law

$\equiv q \wedge r$, using Absorption law ■

Example 4 *Prove the following logical equivalences without using truth tables:*

(i) $p \vee [p \wedge (p \vee q)] \Leftrightarrow p$ (ii) $[p \vee q \vee (\neg p \wedge \neg q \wedge r)] \Leftrightarrow (p \vee q \vee r)$

(iii) $[(\neg p \vee \neg q) \rightarrow (p \wedge q \wedge r)] \Leftrightarrow p \wedge q$

► (i) We have

$p \vee [p \wedge (p \vee q)] \Leftrightarrow p \vee p$, by an Absorption Law

$\Leftrightarrow p$, by an Idempotent Law.

(ii) We have

$\neg p \wedge \neg q \wedge r \Leftrightarrow (\neg p \wedge \neg q) \wedge r$, by Associative Law

$\Leftrightarrow \neg(p \vee q) \wedge r$, by DeMorgan Law.

Therefore,

$p \vee q \vee (\neg p \wedge \neg q \wedge r) \Leftrightarrow (p \vee q) \vee [\neg(p \vee q) \wedge r]$

$\Leftrightarrow [(p \vee q) \vee \neg(p \vee q)] \wedge [(p \vee q) \vee r]$, by Distribution Law

$\Leftrightarrow T_0 \wedge (p \vee q \vee r)$, by Inverse Law and Associative Law

$\Leftrightarrow p \vee q \vee r$, by Commutative and Identity Laws.

(iii) We have

$(\neg p \vee \neg q) \rightarrow (p \wedge q \wedge r) \Leftrightarrow \neg(\neg p \vee \neg q) \vee (p \wedge q \wedge r)$, because $u \rightarrow v \Leftrightarrow (\neg u \vee v)$

$\Leftrightarrow (p \wedge q) \vee [(p \wedge q) \wedge r]$, by DeMorgan Law and Associative Law

$\Leftrightarrow p \wedge q$, by Absorption Law ■

Scanned by CamScanner

Example 5 *Prove the following logical equivalences:*

(i) $[(p \vee q) \wedge (p \vee \neg q)] \vee q \Leftrightarrow p \vee q$ (ii) $(p \to q) \wedge [\neg q \wedge (r \vee \neg q)] \Leftrightarrow \neg (q \vee p)$

▶ (i) We have

$$\begin{aligned}(p \vee q) \wedge (p \vee \neg q) &\Leftrightarrow p \vee (q \wedge \neg q), \text{ by Distributive Law}\\ &\Leftrightarrow p \vee F_0, \text{ because } q \wedge \neg q \text{ is a contradiction}\\ &\Leftrightarrow p, \text{ by an Identity Law}\end{aligned}$$

Therefore, $[(p \vee q) \wedge (p \vee \neg q)] \vee q \Leftrightarrow p \vee q$

(ii) We have

$$\begin{aligned}(p \to q) \wedge [\neg q \wedge (r \vee \neg q)] &\Leftrightarrow (p \to q) \wedge [\neg q \wedge (\neg q \vee r)], \quad \text{by commutative law}\\ &\Leftrightarrow (p \to q) \wedge \neg q, \text{ by an absorption law}\\ &\Leftrightarrow \neg [(p \to q) \to q], \text{ because } \neg (u \to v) \Leftrightarrow u \wedge \neg v\\ &\Leftrightarrow \neg [\neg (p \to q) \vee q], \text{ because } u \to v \Leftrightarrow \neg u \vee v\\ &\Leftrightarrow \neg [(p \wedge \neg q) \vee q]\\ &\Leftrightarrow \neg [q \vee (p \wedge \neg q)], \text{ by commutative law}\\ &\Leftrightarrow \neg [(q \vee p) \wedge (q \vee \neg q)], \text{ by distribution law}\\ &\Leftrightarrow \neg [(q \vee p) \wedge T_0], \text{ because } q \vee \neg q \text{ is a tautology}\\ &\Leftrightarrow \neg (q \vee p), \text{ by an identity law.}\end{aligned}$$

This proves the required result. ■

Example 6 *Prove the following:*

(i) $p \to (q \to r) \Leftrightarrow (p \wedge q) \to r$ (ii) $[\neg p \wedge (\neg q \wedge r)] \vee (q \wedge r) \vee (p \wedge r) \Leftrightarrow r$.

▶(i) We have

$$\begin{aligned}p \to (q \to r) &\Leftrightarrow \neg p \vee (\neg q \vee r)\\ &\Leftrightarrow (\neg p \vee \neg q) \vee r\\ &\Leftrightarrow \neg (p \wedge q) \vee r\\ &\Leftrightarrow (p \wedge q) \to r\end{aligned}$$

Scanned by CamScanner

(ii) We have

$$[\neg p \wedge (\neg q \wedge r)] \Leftrightarrow (\neg p \wedge \neg q) \wedge r$$
$$\Leftrightarrow [\neg (p \vee q)] \wedge r \Leftrightarrow r \wedge [\neg (p \vee q)]$$

and
$$(q \wedge r) \vee (p \wedge r) \Leftrightarrow (r \wedge q) \vee (r \wedge p)$$
$$\Leftrightarrow r \wedge (q \vee p)$$
$$\Leftrightarrow r \wedge (p \vee q)$$

Therefore,

$$[\neg p \wedge (\neg q \wedge r) \vee (q \wedge r) \vee (p \wedge r)]$$
$$\Leftrightarrow \{r \wedge [\neg (p \vee q)]\} \vee \{r \wedge (p \vee q)\}$$
$$\Leftrightarrow r \wedge \{[\neg (p \vee q)] \vee (p \vee q)\}$$
$$\Leftrightarrow r \wedge T_0, \text{ because } [\neg (p \vee q)] \vee (p \vee q) \text{ is always true}$$
$$\Leftrightarrow r.$$

■

Example 7 *Prove the following result:*

$$\neg [\{(p \vee q) \wedge r\} \rightarrow \neg q] \Leftrightarrow \neg [\neg [(p \vee q) \wedge r] \vee \neg q] \Leftrightarrow q \wedge r$$

▶ We have

$$\neg [\{(p \vee q) \wedge r\} \rightarrow \neg q] \Leftrightarrow \neg [\neg \{(p \vee q) \wedge r\} \vee \neg q] \qquad (i^*)$$
$$\Leftrightarrow \neg \neg [\{(p \vee q) \wedge r\} \wedge q]$$
$$\Leftrightarrow (p \vee q) \wedge (r \wedge q),$$
$$\Leftrightarrow (p \vee q) \wedge (q \wedge r)$$
$$\Leftrightarrow [(p \vee q) \wedge q] \wedge r$$
$$\Leftrightarrow [q \wedge (q \vee p)] \wedge r$$
$$\Leftrightarrow q \wedge r \qquad (ii)$$

Results (i) and (ii) together establish the required result. ■

Example 8 *Prove that*

$$[(P \vee Q) \wedge \neg \{\neg P \wedge (\neg Q \vee \neg R)\}] \vee (\neg P \wedge \neg Q) \vee (\neg P \wedge \neg R)$$

is a tautology.

▶ Let w denote the given proposition. Then $w \equiv u \vee v$, where

$$u \equiv (P \vee Q) \wedge \neg \{\neg P \wedge (\neg Q \vee \neg R)\}$$

and
$$v \equiv (\neg P \wedge \neg Q) \vee (\neg P \wedge \neg R)$$

*using $u \rightarrow v \Leftrightarrow \neg u \vee v$

Scanned by CamScanner

By using the laws of logic, we find that

$$
\begin{aligned}
u &\Leftrightarrow (P \vee Q) \wedge \neg\{\neg P \wedge \neg (Q \wedge R)\}\\
&\Leftrightarrow (P \vee Q) \wedge \{P \vee (Q \wedge R)\}\\
&\Leftrightarrow P \vee \{Q \wedge (Q \wedge R)\} \Leftrightarrow P \vee \{(Q \wedge Q) \wedge R\}\\
&\Leftrightarrow P \vee (Q \wedge R)
\end{aligned}
$$

and

$$
\begin{aligned}
v &\Leftrightarrow \neg (P \vee Q) \vee \neg(P \vee R)\\
&\Leftrightarrow \neg \{(P \vee Q) \wedge (P \vee R)\}\\
&\Leftrightarrow \neg \{P \vee (Q \wedge R)\} \equiv \neg u
\end{aligned}
$$

Therefore, $w \equiv u \vee v \Leftrightarrow u \vee (\neg u).$

Since $u \vee (\neg u)$ is always true, it follows that the given compound proposition is a tautology. ■

Exercises

1. Verify the Identity, Inverse and domination laws through truth tables.

2. Prove the DeMorgan Law: $\neg (p \wedge q) \Leftrightarrow \neg p \vee \neg q$

3. Prove the Distributive Law: $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

4. Indicate the negations of the following compound propositions:

(1) 3 is a prime and 4 is even.

(2) $2 < 3$ or $4 < 3$.

(3) If it rains, then I do not drive the car.

(4) If $\sqrt{2}$ is rational, then $\sqrt{2} + 1$ is rational.

(5) If an odd integer is greater than 2 and less than 8, then it is a prime number.

(6) $(p \vee q) \wedge r$

(7) $p \rightarrow (q \rightarrow r)$

(8) $(p \wedge r) \rightarrow (q \vee r)$

(9) $(p \vee q) \rightarrow r$

(10) $(p \wedge q) \rightarrow r$

(11) $p \rightarrow (\neg q \wedge r)$

5. Rewrite the following conditionals in the form of disjunctions:

[Hint: $(p \rightarrow q) \Leftrightarrow (\neg p \vee q)$]

(1) If I dream of home, then I will work hard and earn money.

(2) If I am awake, then I will work on the computer or read a novel.

(3) If I study mathematics and discrete structures, then I will not fail in the examination.

6. Write down the negation of each of the conditionals given in the preceding exercise.

Scanned by CamScanner

Answers

4. (1) 3 is not a prime or 4 is not even.

   (2) $2 \geq 3$ and $4 \geq 3$.

   (3) It rains and I drive the car.

   (4) $\sqrt{2}$ is rational and $\sqrt{2}+1$ is not rational.

   (5) An odd integer is greater than 2 and less than 8, and it is not a prime number.

   (6) $(\neg p \wedge \neg q) \vee \neg r$ (7) $p \wedge (q \wedge \neg r)$

   (8) $(p \wedge r) \wedge (\neg q \wedge \neg r)$. (9) $(p \vee q) \wedge \neg r$

   (10) $(p \wedge q) \wedge \neg r$ (11) $p \wedge (q \vee \neg r)$

5. (1) I do not dream of home, or I will work hard and earn money.

   (2) I am not awake, or I will work on the computer or read a novel.

   (3) I do not study mathematics or discrete structures, or I will not fail in the examination.

6. (1) I dream of home, and I will not work hard or will not earn money.

   (2) I am awake, and I will not work on the computer and I will not read a novel.

   (3) I study mathematics and discrete structures, and I will fail in the examination.

1.2.2 Duality

Suppose u is a compound proposition that contains the connectives $\wedge$ and $\vee$. Suppose we replace each occurrence of $\wedge$ and $\vee$ in u by $\vee$ and $\wedge$ respectively. Also, if u contains T_0 and F_0 as components, suppose we replace each occurrence of T_0 and F_0 by F_0 and T_0 respectively. Then the resulting compound proposition is called the ***dual*** of u and is denoted by u^d.

For example, let

$$u : p \wedge (q \vee \neg r) \vee (s \wedge T_0).$$

Then, the dual of u is given by

$$u^d : p \vee (q \wedge \neg r) \wedge (s \vee F_0).$$

The following two results are of importance:

(1) $\left(u^d\right)^d \Leftrightarrow u$ (That is, the dual of the dual of u is logically equivalent to u).

(2) For any two propositions u and v, if $u \Leftrightarrow v$, then $u^d \Leftrightarrow v^d$. (This is known as the ***Principle of Duality***).

Scanned by CamScanner

Sahyadri College of Engineering & Management
CENTRAL LIBRARY
15 OCT 2019
Acc. No.
Adyar, Mangalore - 575 007

Example 1 *Write down the duals of the following propositions:*

(i) $\neg (p \vee q) \wedge [p \vee \neg (q \wedge \neg s)]$

(ii) $(p \wedge q) \vee [(\neg p \vee q) \wedge (\neg r \vee s)] \vee (r \wedge s)$

(iii) $(p \wedge \neg q) \vee (r \wedge T_0)$

(iv) $[(p \vee T_0) \wedge (q \vee F_0)] \vee [(r \wedge s) \wedge T_0]$

▶ By using the definition of the dual, we find that the duals are:

(i) $\neg (p \wedge q) \vee [p \wedge \neg (q \vee \neg s)]$

(ii) $(p \vee q) \wedge [(\neg p \wedge q) \vee (\neg r \wedge s)] \wedge (r \vee s)$

(iii) $(p \vee \neg q) \wedge (r \vee F_0)$

(iv) $[(p \wedge F_0) \vee (q \wedge T_0)] \wedge [(r \vee s) \vee F_0]$ ■

Example 2 *Write down the duals of the following propositions:*

(i) $p \to q$ (ii) $(p \to q) \to r$ (iii) $p \to (q \to r)$

▶ We recall that $(u \to v) \Leftrightarrow (\neg u \vee v)$. Therefore, by the principle of duality, we find that

(i) $(p \to q)^d \Leftrightarrow (\neg p \vee q)^d \equiv \neg p \wedge q$

(ii) $[(p \to q) \to r]^d \Leftrightarrow [\neg (p \to q) \vee r]^d$

$\Leftrightarrow [(p \wedge \neg q) \vee r]^d \equiv (p \vee \neg q) \wedge r$

(iii) $[p \to (q \to r)]^d \Leftrightarrow [\neg p \vee (q \to r)]^d$

$\Leftrightarrow [\neg p \vee (\neg q \vee r)]^d \equiv \neg p \wedge (\neg q \wedge r).$ ■

Example 3 *Prove that*

$$[(\neg p \vee q) \wedge (p \wedge (p \wedge q))] \Leftrightarrow p \wedge q$$

Hence deduce that

$$[(\neg p \wedge q) \vee (p \vee (p \vee q))] \Leftrightarrow p \vee q$$

▶ We have

$$\begin{aligned}(\neg p \vee q) \wedge (p \wedge (p \wedge q)) &\Leftrightarrow (\neg p \vee q) \wedge ((p \wedge p) \wedge q)\\ &\Leftrightarrow (\neg p \vee q) \wedge (p \wedge q)\\ &\Leftrightarrow [\neg p \wedge (p \wedge q)] \vee [q \wedge (p \wedge q)]\\ &\Leftrightarrow (F_0 \wedge q) \vee (q \wedge p) \Leftrightarrow F_0 \vee (p \wedge q)\\ &\Leftrightarrow p \wedge q\end{aligned}$$

Scanned by CamScanner

This proves the first of the required results. Taking the dual on both sides of this result, we get

$$(\neg p \wedge q) \vee (p \vee (p \vee q)) \Leftrightarrow p \vee q$$

This is the second of the required results. ■

Example 4 *Verify the principle of duality for the following logical equivalence:*

$$[\neg (p \wedge q) \to \neg p \vee (\neg p \vee q)] \Leftrightarrow (\neg p \vee q)$$

▶ The given logical equivalence is $u \Leftrightarrow v$, where

$$u = \neg(p \wedge q) \to (\neg p \vee (\neg p \vee q)) \quad \text{and} \quad v = \neg p \vee q$$

We note that

$$\begin{aligned} u &\Leftrightarrow \neg\neg (p \wedge q) \vee (\neg p \vee (\neg p \vee q)) \\ &\Leftrightarrow (p \wedge q) \vee (\neg p \vee (\neg p \vee q)) \end{aligned}$$

Therefore,

$$\begin{aligned} u^d &\Leftrightarrow (p \vee q) \wedge (\neg p \wedge (\neg p \wedge q)) \\ &\Leftrightarrow (p \vee q) \wedge (\neg p \wedge q) \\ &\Leftrightarrow [p \wedge (\neg p \wedge q)] \vee [q \wedge (\neg p \wedge q)] \\ &\Leftrightarrow (F_0 \wedge q) \vee (q \wedge \neg p) \\ &\Leftrightarrow F_0 \vee (q \wedge \neg p) \\ &\Leftrightarrow q \wedge \neg p \end{aligned}$$

Also,

$$v^d \Leftrightarrow \neg p \wedge q \Leftrightarrow q \wedge \neg p$$

We observe that

$$u^d \Leftrightarrow v^d.$$

This verifies the principle of duality for the given logical equivalence. ■

Exercises

1. Write down the duals of the following propositions :

(1) $q \to p$ (2) $(p \vee q) \wedge r$ (3) $(p \wedge q) \vee T_0$
(4) $p \to (q \wedge r)$ (5) $p \leftrightarrow q$ (6) $p \underline{\vee} q$

Scanned by CamScanner

2. Using the results $\neg (p \vee q) \Leftrightarrow \neg p \wedge \neg q$ and $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$, prove the following:

   (1) $\neg (p \wedge q) \Leftrightarrow \neg p \vee \neg q$ (2) $p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \vee (p \vee r)$.

3. Prove that $(\neg p \vee q) \wedge (p \wedge (p \wedge q)) \Leftrightarrow p \wedge q$.

   Hence deduce that $(\neg p \wedge q) \vee (p \vee (p \vee q)) \Leftrightarrow p \vee q$.

4. Verify the principle duality for the logical equivalence

$$(p \vee q) \wedge (\neg p \wedge (\neg p \wedge q)) \Leftrightarrow (\neg p \wedge q)$$

Answers

1 (1) $\neg q \wedge p$ (2) $(p \wedge q) \vee r$ (3) $(p \vee q) \wedge F_0$

(4) $\neg p \wedge (q \vee r)$ (5) $(\neg p \wedge q) \vee (\neg q \wedge p)$ (6) $(p \vee \neg q) \wedge (q \vee \neg p)$

1.2.3 The Connectives NAND and NOR

For any two propositions p and q, the DeMorgan Laws state that

$$\neg (p \wedge q) \Leftrightarrow \neg p \vee \neg q$$

and

$$\neg (p \vee q) \Leftrightarrow \neg p \wedge \neg q.$$

The compound proposition $\neg (p \wedge q)$ is read as "Not p and q" and is also denoted by $(p \uparrow q)$. The symbol $\uparrow$ is called the NAND connective. Here, NAND is a combination of *Not* and *and*.

The compound proposition $\neg (p \vee q)$ is read as "Not (p or q)" and is also denoted by $(p \downarrow q)$. The symbol $\downarrow$ is called the NOR connective. Here, NOR is a combination of *Not* and *or*.

Thus,

$$(p \uparrow q) = \neg (p \wedge q) \Leftrightarrow \neg p \vee \neg q$$

and

$$(p \downarrow q) = \neg (p \vee q) \Leftrightarrow \neg p \wedge \neg q.$$

Evidently, $(p \uparrow q)$ and $(p \downarrow q)$ are duals of each other. The combined truth table for these is shown below.

Table 1.9: Combined Truth Table for $(p \uparrow q)$ and $(p \downarrow q)$

p	q	$p \uparrow q$	$p \downarrow q$
0	0	1	1
0	1	1	0
1	0	1	0
1	1	0	0

Scanned by CamScanner

Example 1 *For any propositions p, q, prove the following:*

(i) $\neg (p \downarrow q) \Leftrightarrow (\neg p \uparrow \neg q)$ (ii) $\neg (p \uparrow q) \Leftrightarrow (\neg p \downarrow \neg q)$

► Using definition, we find that

$$\text{(i)} \quad \neg (p \downarrow q) \Leftrightarrow \neg \{\neg (p \vee q)\}$$
$$\Leftrightarrow \neg (\neg p \wedge \neg q) \Leftrightarrow (\neg p) \uparrow (\neg q)$$
$$\text{(ii)} \quad \neg(p \uparrow q) \Leftrightarrow \neg \{\neg (p \wedge q)\}$$
$$\Leftrightarrow \neg (\neg p \vee \neg q), \Leftrightarrow (\neg p) \downarrow (\neg q)$$

■

Example 2 *For any propositions p, q, r, prove the following:*

(i) $p \uparrow (q \uparrow r) \Leftrightarrow \neg p \vee (q \wedge r)$ (ii) $(p \uparrow q) \uparrow r \Leftrightarrow (p \wedge q) \vee \neg r$

(iii) $p \downarrow (q \downarrow r) \Leftrightarrow \neg p \wedge (q \vee r)$ (iv) $(p \downarrow q) \downarrow r \Leftrightarrow (p \vee q) \wedge \neg r$

► We find that

$$\text{(i)} \quad p \uparrow (q \uparrow r) \Leftrightarrow \neg[p \wedge (q \uparrow r)]$$
$$\Leftrightarrow \neg \{p \wedge [\neg (q \wedge r)]\} \Leftrightarrow \neg p \vee (q \wedge r)$$
$$\text{(ii)} \quad (p \uparrow q) \uparrow r \Leftrightarrow \neg[(p \uparrow q) \wedge r\} \Leftrightarrow \neg [\neg (p \wedge q) \wedge r]$$
$$\Leftrightarrow \neg \neg (p \wedge q) \vee \neg r \Leftrightarrow (p \wedge q) \vee \neg r$$
$$\text{(iii)} \quad p \downarrow (q \downarrow r) \Leftrightarrow \neg [p \vee (q \downarrow r)] \Leftrightarrow \neg [p \vee \neg(q \vee r)]$$
$$\Leftrightarrow \neg p \wedge \neg \neg (q \vee r) \Leftrightarrow \neg p \wedge (q \vee r)$$
$$\text{(iv)} \quad (p \downarrow q) \downarrow r \Leftrightarrow \neg [(p \downarrow q) \vee r] \Leftrightarrow \neg [\{\neg (p \vee q)\} \vee r]$$
$$\Leftrightarrow \neg \neg (p \vee q) \wedge \neg r \Leftrightarrow (p \vee q) \wedge \neg r$$

The required results are thus proved.

Remark: We observe that $[p \uparrow (q \uparrow r)] \not\Leftrightarrow [(p \uparrow q) \uparrow r]$ and $[p \downarrow (q \downarrow r)] \not\Leftrightarrow [(p \downarrow q) \downarrow r]$. That is, the connectives $\uparrow$ and $\downarrow$ are *not* associative.

■

Example 3 *Express the following propositions in terms of only* NAND *and only* NOR *connectives.*

(i) $\neg p$ (ii) $p \wedge q$ (iii) $p \vee q$ (iv) $p \rightarrow q$

►(i) For any proposition p, we have $p \wedge p \equiv p$ and $p \vee p \equiv p$. Therefore,

$$\neg p \Leftrightarrow \neg (p \wedge p) \Leftrightarrow (p \uparrow p).$$

Also, $$\neg p \Leftrightarrow \neg (p \vee p) \Leftrightarrow (p \downarrow p).$$

Scanned by CamScanner

(ii) We have

$$p \wedge q \Leftrightarrow \neg\neg (p \wedge q) \Leftrightarrow \neg (\neg p \vee \neg q)$$
$$\Leftrightarrow (\neg p \vee \neg q) \uparrow (\neg p \vee \neg q) \Leftrightarrow (p \uparrow q) \uparrow (p \uparrow q).$$

Also,

$$p \wedge q \Leftrightarrow \neg (\neg p) \wedge \neg (\neg q) \Leftrightarrow (\neg p) \downarrow (\neg q) \Leftrightarrow (p \downarrow p) \downarrow (q \downarrow q).$$

(iii) We have

$$p \vee q \Leftrightarrow \neg\neg p \vee \neg\neg q \Leftrightarrow (\neg p) \uparrow (\neg q) \Leftrightarrow (p \uparrow p) \uparrow (q \uparrow q).$$

Also,

$$p \vee q \Leftrightarrow \neg\neg (p \vee q) \Leftrightarrow \neg (\neg p \wedge \neg q) \Leftrightarrow \neg (p \downarrow q) \Leftrightarrow (p \downarrow q) \downarrow (p \downarrow q).$$

(iv) We have $p \rightarrow q \Leftrightarrow \neg (p \wedge \neg q) \Leftrightarrow p \uparrow (\neg q) \Leftrightarrow p \uparrow (q \uparrow q)$.

Also,

$$p \rightarrow q \Leftrightarrow (\neg p \vee q) \Leftrightarrow (\neg p \downarrow q) \downarrow (\neg p \downarrow q), \text{ using (iii) above}$$
$$\Leftrightarrow \{(p \downarrow p) \downarrow q\} \downarrow \{(p \downarrow p) \downarrow q\}.$$

■

Exercises

1. Prove that $(p \uparrow q) \Leftrightarrow (q \uparrow p)$, and $(p \downarrow q) \Leftrightarrow (q \downarrow p)$.

2. Prove that $[p \rightarrow (\neg p \rightarrow q)] \Leftrightarrow [p \uparrow (p \downarrow q)]$.

3. Prove the following:

   (i) $(p \uparrow q) \Leftrightarrow (\neg p \wedge q) \vee (p \wedge \neg q) \vee (\neg p \wedge \neg q)$.

   (ii) $(p \downarrow q) \Leftrightarrow (\neg p \vee q) \wedge (p \vee \neg q) \wedge (\neg p \vee \neg q)$.

4. Prove the following:

   (i) $(p \leftrightarrow q) \Leftrightarrow (r \uparrow s) \uparrow (r \uparrow s)$

   where $r = p \uparrow (q \uparrow q)$ and $s = q \uparrow (p \uparrow p)$

   (ii) $(p \leftrightarrow q) \Leftrightarrow (r \downarrow r) \downarrow (s \downarrow s)$

   where $r = \{(p \downarrow p) \downarrow q\} \downarrow \{(p \downarrow p) \downarrow q\}$

   and $s = \{(q \downarrow q) \downarrow p\} \downarrow \{(q \downarrow q) \downarrow p\}$

Scanned by CamScanner

1.2.4 Converse, Inverse and Contrapositive; Logical Implication

Consider a conditional $p \to q$. Then:

(1) $q \to p$ is called the ***converse*** of $p \to q$.

(2) $\neg p \to \neg q$ is called the ***inverse*** (or *opposite*) of $p \to q$.

(3) $\neg q \to \neg p$ is called the ***contrapositive*** of $p \to q$.

For example, let

p : 2 is an integer. q : 9 is a multiple of 3.

Then

$p \to q$: If 2 is an integer, then 9 is a multiple of 3.

The converse of this conditional is

$q \to p$: If 9 is a multiple of 3, then 2 is an integer.

The inverse is

$\neg p \to \neg q$: If 2 is not an integer, then 9 is not a multiple of 3.

The contrapositive is

$\neg q \to \neg p$: If 9 is not a multiple of 3, then 2 is not an integer.

The following table gives the truth values of $(p \to q)$, $(q \to p)$, $(\neg p) \to (\neg q)$ and $(\neg q) \to (\neg p)$ for all possible truth values of two arbitrary propositions p and q.

Table 1.10: Truth Table for Converse, Inverse, and Contrapositive

p	q	$\neg p$	$\neg q$	$p \to q$	$q \to p$	$\neg p \to \neg q$	$\neg q \to \neg p$
0	0	1	1	1	1	1	1
0	1	1	0	1	0	0	1
1	0	0	1	0	1	1	0
1	1	0	0	1	1	1	1

From this table, it is evident that $p \to q$ and $(\neg q) \to (\neg p)$ have the same truth values in all possible situations. Also, $q \to p$ and $(\neg p) \to (\neg q)$ have the same truth values in all possible situations. As such, we have the following two important results:

(1) *A conditional and its contrapositive are logically equivalent; that is, for any propositions p and q,*

$$(p \to q) \Leftrightarrow (\neg q) \to (\neg p).$$

(2) *The converse and the inverse of a conditional are logically equivalent; that is, for any propositions p and q,*

$$(q \to p) \Leftrightarrow (\neg p) \to (\neg q).$$

Scanned by CamScanner

Another important fact that we observe from the above truth table is that $p \to q$ and $q \to p$ do not have the same truth values (always). As such, $(p \to q) \not\Leftrightarrow (q \to p)$. Hence *if a conditional is true, its converse need not be true, and vice-versa*. The student is familiar with many such instances.

Logical Implication

From Section 1.1.1, we may recall that while defining $p \to q$ no restrictions were imposed on the choice of p and q. In other words, we can think of $p \to q$ for *any* two propositions p and q. As such, we may consider, for example, the propositions

p : *6 is a multiple of* 2, and q : 3 *is a prime number*,

and get the conditional

$p \to q$: *If* 6 *is a multiple of* 2, *then* 3 *is a prime number.*

We note that here p is true and q is true; hence $p \to q$ is true. But the question is: does this conditional $p \to q$ make any sense? The answer is: No. Because, there is no consistency in the statement $p \to q$ (all though it is logically true!).

As another example, consider the propositions

p : *4 is an odd number* and q : *Bengaluru is not in Karnataka*,

both of which are false. From these we get the conditional

$p \to q$: *If* 4 *is an odd number, then Bengaluru is not in Karnataka.*

This conditional makes no sense but it is logically true!

We do not deal with conditionals such as the ones considered in the above two examples. Our major interest lies in conditionals $p \to q$ where p and q are related in some way so that the truth value of q depends upon the truth value of p or vice-versa. Such conditionals are called *hypothetical (or implicative) statements*.

When a hypothetical statement $p \to q$ is such that q is true whenever p is true, we say that p (logically) *implies* q. This is symbolically written as $p \Rightarrow q$, the symbol $\Rightarrow$ denoting the word *implies*.

When a hypothetical statement $p \to q$ is such that q is not necessarily true whenever p is true, we say that p *does not imply* q. This is sybolically writtern as $p \not\Rightarrow q$, the symbol $\not\Rightarrow$ denoting the phrase *does not imply*.

Necessary and Sufficient Conditions

Consider two propositions p and q whose truth values are interrelated. Suppose that $p \Rightarrow q$. Then in order that q may be true it is *sufficient* that p is true. Also if p is true then it is *necessary* that q is true. In view of this interpretation, all of the following statements are taken to carry the same meaning:

(1) $p \Rightarrow q$. (2) p is sufficient for q. (3) q is necessary for p.

Scanned by CamScanner

For example, the logical implication "If a quadrilateral is a square, then it is rectangle" is taken to have the same meaning as "(The fact that) a quadrilateral is a square is a sufficient condition for it to be a rectangle" or as "(The fact that) a quadrilateral is a rectangle is a necessary condition for it to be a square".

For two propositions p and q, the following situations are possible:

(i) $p \Rightarrow q$, but $q \nRightarrow p$

(ii) $p \nRightarrow q$, but $q \Rightarrow p$

(iii) $p \Rightarrow q$, and $q \Rightarrow p$

In the first of the above cases, p is a sufficient but not a necessary condition for q. In the second case, p is a necessary but not a sufficient condition for q. Thus, a condition may be sufficient but not necessary, or it may be necessary but not sufficient. In the last case, p is a necessary and sufficient condition for q, and vice-versa*. This is often expressed as p *if and only if* q (or p *iff* q)**.

For example, let A denote a specified city. Consider the following propositions:

p: The city A is in Karnataka.

q: The city A is in India.

Here $p \Rightarrow q$, but $q \nRightarrow p$. Accordingly, p is a sufficient but not a necessary condition for q, and q is a necessary but not a sufficient condition for p.

As another example, consider a specified integer x, and let

p: The integer x is even.

q: The integer x is divisible by 2.

Obviously, $p \Rightarrow q$ and $q \Rightarrow p$. Thus, here, p is a necessary and sufficient condition for q and vice-versa; that is p iff q.

Example 1 *Write down the contrapositive of* $[p \to (q \to r)]$ *with*

(a) *only one occurrence of the connective* $\to$;

(b) *no occurrence of the connective* $\to$.

▶ We note that the contrapositive of $[p \to (q \to r)]$ is $[\neg (q \to r) \to (\neg p)]$.

We find that

$$\begin{aligned}[\neg (q \to r) \to (\neg p)] &\Leftrightarrow \neg \{\neg (q \to r)\} \vee \neg p \\ &\Leftrightarrow (q \to r) \vee \neg p \qquad \text{(i)} \\ &\Leftrightarrow (\neg q \vee r) \vee \neg p \qquad \text{(ii)}\end{aligned}$$

Expressions (i) and (ii) are the required representations. ■

*A necessary and suffieient condition is referred to as a *criterion*.

***iff* is an abrevation of *if and only if*.

Scanned by CamScanner

Example 2 *Prove the following:*

(i) $[p \wedge (p \to q)] \Rightarrow q$ (ii) $[(p \to q) \wedge \neg q] \Rightarrow \neg p$ (iii) $[(p \vee q) \wedge \neg p] \Rightarrow q$

▶ Let us first prepare the following truth table:

p	q	$\neg p$	$\neg q$	$p \vee q$	$p \to q$
0	0	1	1	0	1
0	1	1	0	1	1
1	0	0	1	1	0
1	1	0	0	1	1

(i) From the Table, we find that when both p and $p \to q$ are true then q is true. This proves that

$$[p \wedge (p \to q)] \Rightarrow q$$

(ii) From the Table, we find that when both $p \to q$ and $\neg q$ are true, then $\neg p$ is true. This proves that

$$[(p \to q) \wedge \neg q] \Rightarrow \neg p$$

(iii) From the Table, we find that when both $p \vee q$ and $\neg p$ are true, then q is true. This proves that

$$[(p \vee q) \wedge \neg p] \Rightarrow q.$$

■

Example 3 *Prove the following:*

(i) $[p \wedge (p \to q) \wedge r] \Rightarrow [(p \vee q) \to r]$ (ii) $\{[p \vee (q \vee r)] \wedge \neg q\} \Rightarrow p \vee r$

▶ (i) Let us prepare the following truth table:

p	q	r	$p \to q$	$p \vee q$	$(p \vee q) \to r$
0	0	0	1	0	1
0	0	1	1	0	1
0	1	0	1	1	0
0	1	1	1	1	1
1	0	0	0	1	0
1	0	1	0	1	1
1	1	0	1	1	0
1	1	1	1	1	1

Scanned by CamScanner

From the Table, we observe that when all of p, $p \to q$ and r are all true, then $(p \vee q) \to r$ is true. This proves that

$$[p \wedge (p \to q) \wedge r] \Rightarrow [(p \vee q) \to r]$$

(ii) Let us prepare the following truth table:

p	q	r	$p \vee (q \vee r)$	$\neg q$	$[p \vee (q \vee r)] \wedge \neg q$	$p \vee r$
0	0	0	0	1	0	0
0	0	1	1	1	1	1
0	1	0	1	0	0	0
0	1	1	1	0	0	1
1	0	0	1	1	1	1
1	0	1	1	1	1	1
1	1	0	1	0	0	1
1	1	1	1	0	0	1

From rows 2, 5 and 6 of the table we observe that if $[p \vee (q \vee r)] \wedge \neg q$ is true, then $p \vee r$ is true. This proves that

$$\{[p \vee (q \vee r)] \wedge \neg q\} \Rightarrow p \vee r. \quad \blacksquare$$

Exercises

1. State the converse, inverse and contrapositive of the following conditionals:

(1) If a quadrilateral is a parallelogram, then its diagonals bisect each other.

(2) If a real number x^2 is greater than zero, then x is not equal to zero.

(3) If a triangle is not isosceles, then it is not equilateral.

2. Write down the conditionals given in the previous exercise by using

(a) the 'necessary condition' language.

(b) the 'sufficient condition' language.

3. Write down the following statements in the 'necessary and sufficient condition' language.

(1) An integer is odd if and only if it is not divisible by 2.

(2) Two lines are parallel if and only if they are equidistant.

(3) A square matrix is non-singular if and only if its determinant is not zero.

Scanned by CamScanner

4. Prove the following:

(1) $p \wedge q \Rightarrow p \vee q$

(2) $p \Rightarrow p \vee q$

(3) $\neg p \Rightarrow p \rightarrow q$

(4) $q \Rightarrow (p \rightarrow q)$

(5) $p \wedge (q \vee r) \Rightarrow (p \wedge q) \vee r$

(6) $(p \rightarrow q) \Rightarrow p \rightarrow (p \wedge q)$

(7) $[(p \rightarrow q) \rightarrow q] \rightarrow q \Rightarrow p \vee q$

(8) $[(p \rightarrow q) \wedge (q \rightarrow r)] \Rightarrow [(p \wedge q) \rightarrow r]$

(9) $[(p \rightarrow r) \wedge (q \rightarrow r)] \Rightarrow [(p \vee q)] \rightarrow r$

(10) $(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r) \Rightarrow r$

(11) $\{[(p \wedge q) \rightarrow r] \wedge (\neg q) \wedge (p \rightarrow \neg r)\} \Rightarrow \neg p \vee \neg q$

(12) $[(p \rightarrow q) \wedge (r \rightarrow s) \wedge (p \vee r)] \Rightarrow (q \vee s)$

(13) $[(p \rightarrow q) \wedge (r \rightarrow s) \wedge (\neg q \vee \neg s)] \Rightarrow (\neg p \vee \neg r)$

Answers

1. (1) *converse :* If the diagonals of a quadrilateral bisect each other, then it is a parallelogram.

inverse : If a quadrilateral is not a parallelogram, then its diagonals do not bisect each other.

contrapositive : If the diagonals of a quadrilateral do not bisect each other, then it is not a parallelogram.

(2) *converse :* If a real number x is not equal to zero, then x^2 is greater than zero.

inverse : If a real number x^2 is not greater than zero, then x is equal to zero.

contrapositive : If a real number x is equal to zero, then x^2 is not greater than zero.

(3) *converse :* If a triangle is not equilateral, then it is not isosceles.

inverse : If a triangle is isosceles, then it is equilateral.

contrapositive : If a triangle is equilateral, then it is isosceles.

2. (a) (1) A necessary condition for a quadrilateral to be a parallelogram is that its diagonals bisect each other.

(2) A necessary condition for a real number x^2 to be greater than zero is that x is not equal to zero.

(3) For a triangle to be non-isosceles it is necessary that it is not equilateral.

(b) (1) A sufficient condition for the diagonals of a quadrilateral to bisect each other is that the quadrilateral is a parallelogram.

(2) For a real number x, the condition x^2 is greater than zero is sufficient for x to be not equal to zero.

(3) A sufficient condition for a triangle to be not equilateral is that it is not isosceles.

Scanned by CamScanner

3. (1) A necessary and sufficient condition that an integer is odd is that it is not divisible by 2.

(2) A necessary and sufficient condition that two lines are parallel is that they are equidistant.

(3) A necessary and sufficient condition that a square matrix is non-singular is that its determinant is not zero.

1.3 Rules of Inference

Consider a set of propositions $p_1, p_2, \ldots p_n$ and a proposition q. Then a compound proposition of the form

$$(p_1 \wedge p_2 \wedge p_3 \wedge \ldots p_n) \to Q$$

is called an ***argument***. Here, $p_1, p_2, \ldots p_n$ are called the *premises* of the argument and Q is called a *conclusion* of the argument.

It is a practice to write the above argument in the following tabular form:

$$\begin{array}{c} p_1 \\ p_2 \\ p_3 \\ \vdots \\ p_n \\ \hline \therefore \quad Q \end{array}$$

Here, the three-dot symbol stands for "Therefore".

The preceding argument is said to be *valid* if whenever each of the premises $p_1, p_2, \ldots p_n$ is *true*, then the conclusion Q is likewise *true*.

In other words, the argument

$$(p_1 \wedge p_2 \wedge p_3 \ldots \wedge p_n) \to Q$$

is valid when

$$(p_1 \wedge p_2 \wedge p_3 \wedge \cdots \wedge p_n) \Rightarrow Q.$$

It is to be emphasized that in an argument, the premises are always taken to be *true* whereas the conclusion may be true or false. The conclusion is *true* only in the case of a *valid argument*.

There exist rules of logic which can be employed for establishing the validity of arguments. These rules are called the *Rules of Inference*. Some of these rules are listed below.

(1) ***Rule of Conjunctive Simplification***

This rule states that, for any two propositions p and q, if $p \wedge q$ is true, then p is true. i.e.,

$$(p \wedge q) \Rightarrow p.$$

This rule follows from the definition of conjunction.

Scanned by CamScanner

(2) ***Rule of Disjunctive Amplification***

This rule states that, for any two propositions p and q, if p is true then $p \vee q$ is true; i.e.,

$$p \Rightarrow p \vee q.$$

This rule follows from the definition of disjunction.

(3) ***Rule of Syllogism***

This rule states that, for any three propositions p, q, r, if $p \to q$ is true and $q \to r$ is true, then $p \to r$ is true. ; i.e.,

$$\{(p \to q) \wedge (q \to r)\} \Rightarrow (p \to r).$$

This rule follows from the tautology $\{(p \to q) \wedge (q \to r)\} \to (p \to r)$ and is expressed in the following tabular form:

$$\begin{array}{rc} & p \to q \\ & q \to r \\ \hline \therefore & p \to r \end{array}$$

(4) ***Modus Pones**** ***(Rule of Detachment)***

This rule states that if p is true and $p \to q$ is true, then q is true; i.e.,

$$\{p \wedge (p \to q)\} \Rightarrow q.$$

For a proof of this rule, see Example 2 of Section 1.2.4. In tabular form, the rule reads thus:

$$\begin{array}{rr} & p \\ & p \to q \\ \hline \therefore & q \end{array}$$

(5) ***Modus Tollens*****

This rule states that if $p \to q$ is true and q is false, then p is false; i.e.,

$$\{(p \to q) \wedge \neg\, q\} \Rightarrow (\neg\, p).$$

For a proof of this rule, see Example 2 of Section 1.2.4. The rule is expressed in the following tabular form:

$$\begin{array}{rr} & p \to q \\ & \neg\, q \\ \hline \therefore & \neg\, p \end{array}$$

---

*"Modus Pones" is in Latin, and its English equivalent is "Method of Affirming".

**"Modus Tollens" is in Latin, and its English equivalent is "Method of Denying".

Scanned by CamScanner

(6) *Rule of Disjunctive Syllogism*

This rule states that if $p \vee q$ is true and p is false, then q is true; i.e.,

$$\{(p \vee q) \wedge \neg p\} \Rightarrow q$$

For a proof of this rule, see Example 2 of Section 1.2.4. The rule is expressed in the following tabular form:

$$\begin{array}{rr} & p \vee q \\ & \neg p \\ \hline \therefore & q \end{array}$$

(7) *Rule of Contradiction*

The rule states that if $\neg p \to F_0$ is true, then p is true; i.e.

$$(\neg p \to F_0) \Longrightarrow p.$$

Here, F_0 is a contradiction (namely a proposition which is always false). The truth table given below proves this rule.

p	F_0	$\neg p$	$\neg p \to F_0$	$(\neg p \to F_0) \to p$
0	0	1	0	1
1	0	0	1	1

The rule is expressed in the following form:

$$\begin{array}{c} \neg p \to F_0 \\ \hline \therefore\ p \end{array}$$

Remarks. The validity or otherwise of a given argument may be established with the aid of the above stated rules or their appropriate combinations. In this process, we can also employ the laws of logic, logical equivalences and/or tautologies. When there is an ambiguity or if there is no other avenue, appropriate truth tables are helpful.

Example 1 *Test whether the following is a valid argument.*

If Sachin hits a century, then he gets a free car.

Sachin hits a century.

$\therefore$ *Sachin gets a free car.*

► Let

p : Sachin hits a century. q : Sachin gets a free car.

Scanned by CamScanner

Then, the given argument reads

$$\begin{array}{rl} & p \to q \\ & p \\ \hline \therefore & q \end{array}$$

In view of the Modus Pones Rule, this is a valid argument. ■

Example 2 *Test whether the following is a valid argument:*

If Sachin hits a century, he gets a free car.
Sachin does not get a free car.
∴ *Sachin has not hit a century.*

▶ Let

p : Sachin hits a century. q : Sachin gets a free car

Then the given argument reads

$$\begin{array}{rl} & p \to q \\ & \neg q \\ \hline \therefore & \neg p \end{array}$$

In view of the Modus Tollens rule, the argument is valid. ■

Example 3 *Test whether the following is a valid argument:*

If Sachin hits a century, he gets a free car.
Sachin gets a free car.
∴ *Sachin has hit a century.*

▶ Let

p : Sachin hits a century. q : Sachin gets a free car.

Then, the given argument reads

$$\begin{array}{rl} & p \to q \\ & q \\ \hline \therefore & p \end{array}$$

We note that if $p \to q$ and q are true, there is no rule which asserts that p must be true. Indeed, p can be false when $p \to q$ and q are true. See the Table below.

p	q	$p \to q$	$(p \to q) \wedge q$
0	1	1	1

Thus, $[(p \to q) \wedge q] \to p$ is not a tautology. Therefore, the given argument is *not* a valid one. ■

Scanned by CamScanner

Example 4 *Test whether the following argument is valid:*

If I drive to work, then I will arrive tired.
I am not tired (when I arrive at work)
$\therefore$ *I do not drive to work.*

► Let
p : I drive to work. q : I arrive tired.
Then the given argument reads

$$\begin{array}{rr} & p \to q \\ & \neg q \\ \hline \therefore & \neg p \end{array}$$

In view of the Modus Tollens rule, this is a valid argument.

Example 5 *Test the validity of the following argument:*

I will become famous or I will not become a musician.
I will become a musician.
$\therefore$, *I will become famous.*

► Let
p : I will become famous. q : I will become a musician.
Then, the given argument reads

$$\begin{array}{rr} & p \vee \neg q \\ & q \\ \hline \therefore & p \end{array}$$

This argument is logically equivalent to

$$\begin{array}{rl} & q \to p \\ & q \\ \hline \therefore & p \end{array}$$

(because $p \vee \neg q \Leftrightarrow \neg q \vee p \Leftrightarrow q \to p$)

In view of the Modus Pones Rule, this argument is valid.

Example 6 *Test whether the following is a valid argument:*

If I study, then I do not fail in the examination.
If I do not fail in the examination, my father gifts a two-wheeler to me.
$\therefore$ *if I study then my father gifts a two-wheeler to me.*

► Let
p : I study. q : I do not fail in the examination.
r : My father gifts a two-wheeler to me.

Scanned by CamScanner

Then, the given argument reads

$$\begin{array}{rl} & p \to q \\ & q \to r \\ \hline \therefore & p \to r \end{array}$$

In view of the Rule of Syllogism, this is a valid argument. ■

Example 7 *Test the validity of the following argument:*

If Ravi goes out with friends, he will not study.

If Ravi does not study, his father becomes angry.

His father is not angry.

∴ Ravi has not gone out with friends.

▶ Let

p : Ravi goes out with friends.

q : Ravi does not study.

r : His father gets angry.

Then the given argument reads:

$$\begin{array}{rl} & p \to q \\ & q \to r \\ & \neg r \\ \hline \therefore & \neg p \end{array}$$

This argument is logically equivalent to

$$\begin{array}{rl} & p \to r \qquad \text{(using Rule of Syllogism)} \\ & \neg r \\ \hline \therefore & \neg p \end{array}$$

In view of Modus Tollens rule, this is a valid argument. ■

Example 8 *Test the validity of the following argument:*

If I study, I will not fail in the examination.

If I do not watch TV in the evenings, I will study.

I failed in the examination.

∴ I must have watched TV in the evenings.

▶ Let

p : I study. q : I fail in the examination.

r : I watch TV in the evenings.

Then, the given argument reads

$$\begin{array}{rl} & p \to \neg q \\ & \neg r \to p \\ & q \\ \hline \therefore & r \end{array}$$

This argument is logically equivalent to

$$\begin{array}{ll} q \to \neg p & \text{(because } (p \to \neg q) \Leftrightarrow (\neg\neg q \to \neg p)) \\ \neg p \to r & \text{(because } (\neg r \to p) \Leftrightarrow (\neg p \to r)) \\ \underline{q} & \\ \therefore \quad r & \end{array}$$

This is equivalent to

$$\begin{array}{ll} q \to r & \text{(using Rule of Syllogism)} \\ \underline{q} & \\ \therefore \quad r & \end{array}$$

This argument is valid, by the Modus Pones Rule.

Example 9 *Consider the following argument:*

I will get grade A in this course or I will not graduate.
If I do not graduate, I will join the army.
I got grade A.
∴ I will not join the army.
Is this a valid argument?

► Let

p : I get grade A in this course. q : I do not graduate.
r : I join the army.

Then the given argument reads

$$\begin{array}{l} p \vee q \\ q \to r \\ \underline{p} \\ \therefore \neg r \end{array}$$

This argument is logically equivalent to

$$\begin{array}{ll} \neg q \to p & \text{(because } p \vee q \equiv q \vee p \Leftrightarrow \neg q \to p) \\ \underline{\neg r \to \neg q} & \text{(using contrapositive)} \\ \therefore \; \neg r & \end{array}$$

This is logically equivalent to

$$\begin{array}{ll} \neg r \to p & \text{(using Rule of Syllogism)} \\ \underline{p} & \\ \therefore \neg r & \end{array}$$

This is *not* a valid argument (as in Example 3).

Scanned by CamScanner

Example 10 *Test the validity of the following arguments:*

(i) $p \wedge q$
$\underline{p \to (q \to r)}$
$\therefore \quad r$

(ii) p
$p \to \neg q$
$\underline{\neg q \to \neg r}$
$\therefore \quad \neg r$

(iii) $p \to r$
$\underline{q \to r}$
$\therefore \quad (p \vee q) \to r$

▶ (i) Since $p \wedge q$ is true, both p and q are true. Since p is true and $p \to (q \to r)$ is true, $q \to r$ has to be true. Since q is true and $q \to r$ is true, r has to be true. Hence the given argument is valid.

(ii) The premises $p \to \neg q$ and $\neg q \to \neg r$ together yield the premise $p \to \neg r$. Since p is true, this premise $(p \to \neg r)$ establishes that $\neg r$ is true. Hence the given argument is valid.

(iii) We note that

$$\begin{aligned}(p \to r) \wedge (q \to r) &\Leftrightarrow (\neg p \vee r) \wedge (\neg q \vee r)\\ &\Leftrightarrow (r \vee \neg p) \wedge (r \vee \neg q), \text{ by commutative law}\\ &\Leftrightarrow r \vee (\neg p \wedge \neg q), \text{ by distributive law}\\ &\Leftrightarrow \neg (p \vee q) \vee r, \text{ by commutative and DeMorgan laws}\\ &\Leftrightarrow (p \vee q) \to r.\end{aligned}$$

This logical equivalance shows that the given argument is valid. ■

Example 11 *Test whether the following arguments are valid:*

(i) $p \to q$
$r \to s$
$\underline{p \vee r}$
$\therefore \quad q \vee s$

(ii) $p \to q$
$r \to s$
$\underline{\neg q \vee \neg s}$
$\therefore \quad \neg (p \wedge r)$

▶ (i) We note that

$$\begin{aligned}(p \to q) \wedge (r \to s) \wedge (p \vee r) &\Leftrightarrow (p \to q) \wedge (r \to s) \wedge (\neg p \to r)\\ &\Leftrightarrow (p \to q) \wedge (\neg p \to r) \wedge (r \to s), \text{ by commutative law}\\ &\Rightarrow (p \to q) \wedge (\neg p \to s), \text{ using Rule of Syllogism}\\ &\Leftrightarrow (\neg q \to \neg p) \wedge (\neg p \to s), \text{ using contrapositive}\\ &\Rightarrow \neg q \to s, \text{ using Rule of Syllogism}\\ &\Leftrightarrow q \vee s\end{aligned}$$

This shows that the given argument is valid.

Scanned by CamScanner

(ii) We note that

$$(p \to q) \wedge (r \to s) \wedge (\neg q \vee \neg s) \Leftrightarrow (p \to q) \wedge (r \to s) \wedge (q \to \neg s)$$
$$\Rightarrow (p \to \neg s) \wedge (r \to s),$$
by commutative law and Rule of Syllogism
$$\Leftrightarrow (p \to \neg s) \wedge (\neg s \to \neg r), \text{ by using contrapositive}$$
$$\Rightarrow p \to \neg r, \text{ by Rule of Syllogism}$$
$$\Leftrightarrow \neg p \vee \neg r \Leftrightarrow \neg (p \wedge r)$$

This shows that the given argument is a valid argument. ■

Example 12 *Prove the validity of the following arguments:*

(i)
$$\begin{array}{l} p \to r \\ \neg p \to q \\ q \to s \\ \hline \therefore \quad \neg r \to s \end{array}$$

(ii)
$$\begin{array}{l} (\neg p \vee \neg q) \to (r \wedge s) \\ r \to t \\ \neg t \\ \hline \therefore \ p \end{array}$$

► (i) We note that

$$(p \to r) \wedge (\neg p \to q) \wedge (q \to s) \Rightarrow (p \to r) \wedge (\neg p \to s), \text{ by Rule of Syllogism}$$
$$\Leftrightarrow (\neg r \to \neg p) \wedge (\neg p \to s), \text{ using contrapositive}$$
$$\Rightarrow \neg r \to s, \text{ by Rule of Syllogism.}$$

This shows that the given argument is valid.

(ii) We note that

$$[(\neg p \vee \neg q) \to (r \wedge s)] \wedge (r \to t) \wedge (\neg t)$$
$$\Rightarrow [(\neg p \vee \neg q) \to (r \wedge s)] \wedge \neg r,$$
by Modus Tollens Rule
$$\Rightarrow [(\neg p \vee \neg q) \to (r \wedge s)] \wedge (\neg r \vee \neg s),$$
by the rule of disjunctive amplification
$$\Leftrightarrow [(\neg p \vee \neg q) \to (r \wedge s)] \wedge [\neg (r \wedge s)], \text{ by DeMorgan law}$$
$$\Rightarrow \neg (\neg p \vee \neg q), \text{ by the Modus Tollens Rule}$$
$$\Leftrightarrow p \wedge q, \text{ by DeMorgan law}$$
$$\Rightarrow p, \text{ by the Rule of conjunctive simplification}$$

This proves the validity of the given argument. ■

Scanned by CamScanner

Example 13 *Prove that the following are valid arguments:*

(i) $p \to (q \to r)$
$\neg q \to \neg p$
p
$\therefore\ r$

(ii) $\neg p \leftrightarrow q$
$q \to r$
$\neg r$
$\therefore\ p$

▶ (i) We find that

$$\begin{aligned}
&[p \to (q \to r)] \wedge [\neg q \to \neg p] \wedge p \\
&\quad \Leftrightarrow \{[p \to (q \to r)] \wedge p\} \wedge [\neg q \to \neg p] \\
&\quad \Rightarrow (q \to r) \wedge (\neg q \to \neg p), \text{ by the Modus Pones Rule} \\
&\quad \Leftrightarrow (q \to r) \wedge (p \to q), \text{ by contrapositive} \\
&\quad \Rightarrow p \to r, \text{ by Rule of Syllogism} \\
&\quad \Rightarrow r, \text{ because } p \text{ is true (premise).}
\end{aligned}$$

This proves that the given argument is valid.

(ii) We find that

$$\begin{aligned}
&(\neg p \leftrightarrow q) \wedge (q \to r) \wedge \neg r \\
&\quad \Leftrightarrow (\neg\ p \leftrightarrow q) \wedge \neg q, \text{ by Modus Tollens Rule} \\
&\quad \Leftrightarrow [(\neg p \to q) \wedge (q \to \neg p)] \wedge \neg q \\
&\quad \Rightarrow \{(\neg p \to q) \wedge \neg q\} \wedge (q \to \neg p) \\
&\quad \Rightarrow (\neg p \to q) \wedge \neg q, \text{ by Rule of conjunctive simplification} \\
&\quad \Rightarrow \neg\neg p, \text{ by Modus Tollens rule.} \\
&\quad \Leftrightarrow p
\end{aligned}$$

This proves that the given argument is a valid argument. ■

Example 14 *Prove that the following are valid arguments:*

(i) $p \to (q \to r)$
$p \vee \neg s$
q
$\therefore\ s \to r$

(ii) $p \to (q \wedge r)$
$r \to s$
$\neg (q \wedge s)$
$\therefore\ \neg p$

Scanned by CamScanner

► (i) Using appropriate rules (laws) we find that

$$
\begin{aligned}
[p \to (q \to r)] \wedge [p \vee \neg s] \wedge q &\Leftrightarrow [p \to (q \to r)] \wedge [s \to p] \wedge q \\
&\Leftrightarrow [s \to (q \to r)] \wedge q \\
&\Leftrightarrow [\neg s \vee (q \to r)] \wedge q \\
&\Leftrightarrow (\neg s \wedge q) \wedge (q \to r) \wedge q \\
&\Leftarrow (\neg s) \vee r, \\
&\Leftrightarrow s \to r
\end{aligned}
$$

This proves that the given argument is valid.

(ii) Using appropriate rules (laws) we find that

$$
\begin{aligned}
(r \to s) \wedge \{\neg(q \wedge s)\} &\Leftrightarrow (r \to s) \wedge \{\neg q \vee \neg s\} \\
&\Leftrightarrow (r \to s) \wedge \{s \to (\neg q)\} \\
&\Rightarrow r \to (\neg q) \\
&\Leftrightarrow \neg r \vee \neg q \Leftrightarrow (\neg (r \wedge q))
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\{p \to (q \wedge r)\} \wedge \{r \to s\} \wedge \{\neg(q \wedge s)\} &\Rightarrow \{p \to (q \wedge r)\} \wedge \{\neg(r \wedge q)\} \\
&\Leftrightarrow \{\neg p \vee (q \wedge r)\} \wedge \{\neg (q \wedge r)\} \\
&\Leftrightarrow \neg p \wedge \{\neg(q \wedge r)\} \vee [(q \wedge r) \wedge \{\neg(q \wedge r)\}] \\
&\Leftarrow \neg p \vee F_o \\
&\Rightarrow \neg p
\end{aligned}
$$

This proves that the given argument is valid.

Example 15 *Test the validity of the following arguments:*

(i)
$$
\begin{array}{l}
(\neg p \vee q) \to r \\
r \to (s \vee t) \\
\neg s \wedge \neg u \\
\neg u \to \neg t \\
\hline
\therefore \quad p
\end{array}
$$

(ii)
$$
\begin{array}{l}
p \to r \\
r \to s \\
t \vee \neg s \\
\neg t \vee u \\
\neg u \\
\hline
\therefore \quad \neg p
\end{array}
$$

Scanned by CamScanner

▶ (i) We find that

$$\begin{aligned}
&[(\neg p \vee q) \rightarrow r] \wedge [r \rightarrow (s \vee t)] \wedge (\neg s \wedge \neg u) \wedge (\neg u \rightarrow \neg t) \\
&\qquad \Rightarrow [(\neg p \vee q) \rightarrow (s \vee t)] \wedge \neg s \wedge \{\neg u \wedge (\neg u \rightarrow \neg t)\} \\
&\qquad\quad \text{by Rule of Syllogism and associative law} \\
&\qquad \Rightarrow [(\neg p \vee q) \rightarrow (s \vee t)] \wedge [\neg s \wedge \neg t], \text{ by Modus Pones Rule} \\
&\qquad \Leftrightarrow [(\neg p \vee q) \rightarrow (s \vee t)] \wedge [\neg (s \vee t)], \text{ using DeMorgan law} \\
&\qquad \Rightarrow \neg (\neg p \vee q), \text{ by Modus Tollens Rule} \\
&\qquad \Leftrightarrow p \wedge \neg q \\
&\qquad \Rightarrow p, \text{ by the Rule of conjunctive simplification.}
\end{aligned}$$

This proves that the given argument is valid.

(ii) Using appropriate rules (laws) we find that

$$\begin{aligned}
&(p \rightarrow r) \wedge (r \rightarrow s) \wedge (t \vee \neg s) \wedge (\neg t \vee u) \wedge (\neg u) \\
&\qquad \Rightarrow (p \rightarrow s) \wedge (s \rightarrow t) \wedge (t \rightarrow u) \wedge (\neg u) \\
&\qquad \Rightarrow (p \rightarrow u) \wedge (\neg u) \\
&\qquad \Rightarrow \neg p.
\end{aligned}$$

This proves that the given argument is valid. ■

Example 16 *Prove the validity of the following arguments:*

(i)		(ii)	
	$p \rightarrow q$		$u \rightarrow r$
	$q \rightarrow (r \wedge s)$		$(r \rightarrow s) \rightarrow (p \vee t)$
	$\neg r \vee (\neg t \vee u)$		$q \rightarrow (u \wedge s)$
	$p \wedge t$		$\neg t$
	$\therefore \quad u$		$\therefore \quad q \rightarrow p$

▶ (i) Using appropriate rules (laws) we note that

$$\begin{aligned}
&(p \rightarrow q) \wedge \{q \rightarrow (r \wedge s)\} \wedge \{\neg r \vee (\neg t \vee u)\} \wedge (p \wedge t) \\
&\qquad \Rightarrow \{p \rightarrow (r \wedge s)\} \wedge (p \wedge t) \wedge \{(\neg r \vee \neg t) \vee u\} \\
&\qquad \Leftrightarrow [\{p \rightarrow (r \wedge s)\} \wedge p] \wedge t \wedge [\{\neg (r \wedge t)\} \vee u] \\
&\qquad \Leftrightarrow (r \wedge s) \wedge t \wedge \{\neg (r \wedge t) \vee u\} \\
&\qquad \Leftrightarrow \{(r \wedge t) \wedge s\} \wedge \{\neg (r \wedge t) \vee u\}
\end{aligned}$$

Now, since the left hand side is true, it follows that r, t, s and $\neg (r \wedge t) \vee u$ must be true. Since r and t are true, $\neg (r \wedge t)$ is false. Consequently, u cannot be false; that is, u has to be true. This proves the validity of the given argument.

Scanned by CamScanner

(ii) Suppose q is true. Since $q \rightarrow (u \wedge s)$ is true(premise), u and s are to be true. Since u is true and $u \rightarrow r$ is true (premise), r has to be true. Since r and s are true, $r \rightarrow s$ is true. Consequently, since $(r \rightarrow s) \rightarrow (p \vee t)$ is true (premise), $p \vee t$ is true. Since $\neg t$ is true (premise), this yields p to be true.

Next, suppose q is false. Then, since $q \rightarrow (u \wedge s)$ is true (premise), $u \wedge s$ has to be false; that is, u is false and s is false. Since u is false and $u \rightarrow r$ is true (premise), r is false. Since r is false and s is false, $r \rightarrow s$ is true. Consequently, the premise $(r \rightarrow s) \rightarrow (p \vee t)$ yields that $p \vee t$ is true. Since $\neg t$ is true (premise), it follows that p is true.

Thus, p is true when q is true or otherwise. Hence $q \rightarrow p$ is true. This proves the validity of the given argument. ■

Example 17 *Test the validity of the following argument:*

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow s \\ r \rightarrow \neg s \\ \neg p \veebar r \\ \hline \therefore \quad \neg p \end{array}$$

▶ We note that

$$\begin{aligned}
(p \rightarrow q) \wedge (q \rightarrow s) \wedge (r \rightarrow \neg s) \wedge (\neg p \veebar r) & \\
&\Rightarrow (p \rightarrow s) \wedge (s \rightarrow \neg r) \wedge (\neg p \veebar r) \\
&\Rightarrow (p \rightarrow \neg r) \wedge (\neg p \veebar r) \\
&\Leftrightarrow (r \rightarrow \neg p) \wedge (r \veebar \neg p) \qquad \text{(i)}
\end{aligned}$$

Now, $r \veebar \neg p$ is true only in the following two possible cases:
(a) r is true and $\neg p$ is false.
(b) r is false and $\neg p$ is true.
In case (a), $r \rightarrow \neg p$ is false and in case (b), $r \rightarrow \neg p$ is true. Hence, it is only in case (b) that the RHS of (i) remains true. Thus, $\neg p$ is true is a valid conclusion. This means that the given argument is valid. ■

Example 18 *Establish the validity of the following argument using the Rules of Inference:*

$$\{p \wedge (p \rightarrow q) \wedge (s \vee r) \wedge (r \rightarrow \neg q)\} \rightarrow (s \vee t)$$

Scanned by CamScanner

▶ The given argument reads:

$$\begin{array}{l} p \\ p \to q \\ s \vee r \\ r \to \neg q \\ \hline \therefore \quad s \vee t \end{array}$$

We note that

$$\begin{aligned}(s \vee r) \wedge (r \to \neg q) &\Leftrightarrow (\neg s \to r) \wedge (r \to \neg q) \\ &\Rightarrow (\neg s \to \neg q), \text{ using Rule of Syllosism} \\ &\Leftrightarrow q \to s, \text{ using contrapositive}\end{aligned}$$

Therefore,

$$\begin{aligned}(p \to q) \wedge (s \vee r) \wedge (r \to \neg q) &\Rightarrow (p \to q) \wedge (q \to s) \\ &\Rightarrow (p \to s), \text{ using Rule of Syllogism}\end{aligned}$$

Consequently,

$$\begin{aligned}p \wedge (p \to q) \wedge (s \vee r) \wedge (r \to \neg g) &\Rightarrow p \wedge (p \to s) \\ &\Rightarrow s, \text{ by the Rule of detachment} \\ &\Rightarrow s \vee t, \text{ by the Rule of dijunative Amplification}\end{aligned}$$

This shows that the given argument is valid. ■

Example 19 *Show that the following argument is not valid :*

$$\begin{array}{l} p \\ p \vee q \\ q \to (r \to s) \\ t \to r \\ \hline \therefore \quad \neg s \to \neg t \end{array}$$

▶ Here p is true (premise) and $(p \vee q)$ is true (premise). Therefore, q may be true or false. Suppose q is false. Then, since $q \to (r \to s)$ is true (premise), $r \to s$ must be false. This means that r must be true and s must be false. Since r is true and $t \to r$ is true (premise), t may be true or false. Suppose t is true. Then $\neg t$ is false. Since s must be false, $\neg s$ is true. Consequently, $\neg s \to \neg t$ is false.

Thus, when q is false and t is true, the given conclusion does not follow from the given premises. As such, the given argument is *not* a valid argument. ■

Scanned by CamScanner

Example 20 *Show that $S \vee R$ is tautologically implied by $(P \vee Q) \wedge (P \rightarrow R) \wedge (Q \rightarrow S)$.*

▶ We note that

$$\begin{aligned}(P \vee Q) \wedge (P \rightarrow R) \wedge (Q \rightarrow S) \\ &\Leftrightarrow (\neg P \rightarrow Q) \wedge (Q \rightarrow S) \wedge (P \rightarrow R) \\ &\Rightarrow (\neg P \rightarrow S) \wedge (P \rightarrow R) \\ &\Leftrightarrow (\neg R \rightarrow \neg P) \wedge (\neg P \rightarrow S) \\ &\Rightarrow \neg R \rightarrow S \Leftrightarrow R \vee S \equiv (S \vee R)\end{aligned}$$

This proves the required result. ∎

Example 21 *Show that $R \vee S$ follows logically from the premises*

$C \vee D,\ (C \vee D) \rightarrow \neg H,\ \neg H \rightarrow (A \wedge \neg B)$ *and* $(A \wedge \neg B) \rightarrow R \vee S$.

▶ We find that

$$\begin{aligned}(C \vee D) \wedge \{(C \vee D) \rightarrow \neg H\} \wedge \{\neg H \rightarrow (A \wedge \neg B)\} \wedge \{(A \wedge \neg B) \rightarrow R \vee S\} \\ &\Rightarrow (C \vee D) \wedge \{(C \vee D) \rightarrow (R \vee S)\}, \text{ using the Rule of Syllogism (twice)} \\ &\Rightarrow R \vee S, \text{ using Modus Pones Rule}\end{aligned}$$

This proves the required result. ∎

Example 22 *Show that $R \wedge (P \vee Q)$ is a valid conclusion from the premises*

$P \vee Q,\quad Q \rightarrow R,\quad P \rightarrow M,$ *and* $\neg M$

▶ We note that

$$\begin{aligned}(P \vee Q) \wedge (Q \rightarrow R) \wedge (P \rightarrow M) \wedge (\neg M) \\ &\Rightarrow (\neg P \rightarrow Q) \wedge (Q \rightarrow R) \wedge (\neg P), \\ &\qquad \text{using } P \vee Q \Leftrightarrow \neg P \rightarrow Q \text{ and Modus Tollens Rule} \\ &\Rightarrow (\neg P \rightarrow R) \wedge (\neg P), \text{ using Rule of Syllogism} \\ &\Rightarrow R, \text{ using Modus Pones Rule} \\ &\Rightarrow R \wedge (P \vee Q), \text{ because } P \vee Q \text{ is true (premise)}\end{aligned}$$

This proves the required result. ∎

Example 23 *Prove that $R \rightarrow S$ is a valid conclusion from the premises $P \rightarrow (Q \rightarrow S)$, $\neg R \vee P$, and Q.*

▶ We note that

$$\begin{aligned}\{P \rightarrow (Q \rightarrow S)\} \wedge \{\neg R \vee P\} \wedge Q \\ &\Leftrightarrow \{P \rightarrow (Q \rightarrow S)\} \wedge (R \rightarrow P) \wedge Q \\ &\Leftrightarrow (R \rightarrow P) \wedge \{P \rightarrow (Q \rightarrow S)\} \wedge Q\end{aligned}$$

Scanned by CamScanner

$$\Rightarrow \{R \to (Q \to S)\} \wedge Q$$
$$\Leftrightarrow [\neg\neg \{R \to (Q \to S)\}] \wedge Q$$
$$\Leftrightarrow [\neg \{R \wedge \neg (Q \to S)\}] \wedge Q$$
$$\Leftrightarrow [\neg \{R \wedge Q \wedge \neg S\}] \wedge Q$$
$$\Leftrightarrow [\neg \{(R \wedge \neg S) \wedge Q\}] \wedge Q$$
$$\Rightarrow [\neg (R \wedge \neg S) \vee \neg Q] \wedge Q$$
$$\Rightarrow \neg (R \wedge \neg S), \text{ because } Q \text{ is true (premise)}$$
$$\Leftrightarrow \neg R \vee (\neg\neg S) \Leftrightarrow R \to S$$

This proves the required result. ■

Exercises

1. Test the validity of the following arguments:

(1) If a person is poor, he is unhappy.
If a person is unhappy, he dies young.
∴ Poor persons die young.

(2) If there is a strike by students, the examination will be postponed.
There was no strike by students.
∴ The examination was not postponed.

(3) If there is strike by students, the examination will be postponed.
The examination was not postponed.
∴ There was no strike by students.

(4) If I drive to work, then I will arrive tired.
I do not drive to work.
∴ I will not arrive tired.

(5) If I have talent and work hard, then I will become successful in life
If I become successful in life then I will be happy.
∴ If I will not be happy, then I did not work hard or I do not have talent.

(6) If Ravi studies, then he will pass in Discrete Mathematics paper.
If Ravi does not play cricket, then he will study.
Ravi failed in Discrete Mathematics paper
∴ Ravi played cricket.

Scanned by CamScanner

2. Prove that the following arguments are valid:

(1) $\neg p \to q$
$\neg q$
$\therefore p$

(2) $(p \wedge r) \to s$
p
$\therefore r \to s$

(3) $p \vee q$
$\neg p \vee r$
$\neg r$
$\therefore q$

(4) $p \to r$
$\neg q \to p$
$\neg r$
$\therefore q$

(5) p
$p \to \neg q$
$r \to q$
$\therefore \neg r$

(6) $p \to q$
$\neg q$
$\neg r$
$\therefore \neg (p \vee r)$

(7) $p \to q$
$r \to \neg q$
r
$\therefore \neg p$

(8) $p \to q$
$\neg r \to \neg q$
p
$\therefore r$

(9) $p \to q$
$\neg r \vee s$
$p \vee r$
$\therefore \neg q \to \neg s$

(10) $p \to (q \to r)$
$\neg q \to \neg p$
p
$\therefore r$

(11) $(p \wedge q) \to r$
$\neg q$
$p \to \neg r$
$\therefore \neg p \vee \neg q$

(12) $p \to q$
$r \to s$
$\neg q \vee \neg s$
$\therefore \neg p \vee \neg r$

(13) $p \to (q \to r)$
$p \vee s$
$t \to q$
$\neg s$
$\therefore \neg r \to \neg t$

(14) $p \wedge q$
$p \to (r \wedge q)$
$r \to (s \vee t)$
$\neg s$
$\therefore t$

(15) p
$p \to q$
$s \vee r$
$r \to \neg q$
$\therefore s$

(16) $(\neg p \vee q) \to r$
$r \to (s \vee t)$
$\neg s \wedge \neg u$
$\neg u \to \neg t$
$\therefore p$

3. Prove that the following arguments are *not* valid:

(1) $p \wedge \neg q$
$p \to (q \to r)$
$\therefore \neg r$

(2) $(p \wedge q) \to r$
$\neg q \vee r$
$\therefore p$

(3) p
$p \to r$
$p \to (q \vee r)$
$\neg q \vee \neg s$
$\therefore s$

(4) $p \leftrightarrow q$
$q \to r$
$r \vee \neg s$
$\neg s \to q$
$\therefore s$

Answers

1. (1), (3), (5), (6): Valid. (2), (4): Not valid

Scanned by CamScanner