Module-4

Numerical Solution of Ordinary Differential Equation

In this module we use numerical techniques to solve first order first degree differential equation.

Taylor's Series Method

Consider the D.E $\frac{dy}{dx} = f(x,y)$ with initial condition $y(x_0) = y_0$, then by taylor series method

$$y(x_{n+1}) = y(x_n) + \frac{h}{1!} y'(x_n) + \frac{h^2}{2!} y''(x_n) + \frac{h^3}{3!} y'''(x_n) + \ldots\ldots$$

Problem

1) Employee Taylor Series Method to find the value of y at $x=0.1$. Given $\frac{dy}{dx} = x - y^2$, $y(0) = 1$

$y(0) = 1$ find y at $x = 0.1$

0 0.1

$h = 0.1$

If they don't mention how much terms we should perform
Perform 4 terms

→ $y(0) = 1$

$x_0 = 0 \quad y_0 = 1$

$y' = x - y^2, \quad h = 0.1$

By Taylor Series

$$y(x_1) = y(x_0) + h\,y'(x_0) + \frac{h^2}{2!} y''(x_0) + \frac{h^3}{3!} y'''(x_0) + \frac{h^4}{4!} y^{iv}(x_0)$$

$y' = x - y^2 \qquad y'(x_0) = y'(0) = x_0 - y_0^2 = 0 - 1^2 = -1$

$y'' = 1 - 2yy' \qquad y''(x_0) = y''(0) = 1 - 2y_0 y_0' = 1 - 2 \times 1 \times (-1)$
$= 1 + 2 = \underline{3}$

$y''' = -2[y \cdot y'' + y' \cdot y'] = -2[y \cdot y'' + (y')^2]$

$y'''(x_0) = y'''(0) = -2[y_0 \cdot y_0'' + (y_0')^2]$
$= -2[1.3 + (-1)^2]$
$= -2[3+1]$
$= \underline{-8}$

$y^{iv} = -2[y \cdot y''' + y'' \cdot y' + 2y' \cdot y''] = -2[y y''' + 3y''y']$

$y^{iv}(x_0) = y^{iv}(0) = -2[1.(-8) + 3(3)(-1)]$
$= -2[-8 - 9]$
$= -2[-17]$
$= \underline{34}$

$\therefore y(0.1) = y(0) + hy'(0) + \frac{h^2}{2} y''(0) + \frac{h^3}{6} y'''(0) + \frac{h^4}{24} y^{iv}(0)$

$\approx 1 + 0.1(-1) + \frac{(0.1)^2}{2}(3) + \frac{(0.1)^3}{6}(-8) + \frac{(0.1)^4}{24}(34)$

$\approx 1 - 0.1 + 0.0150 - 0.0013 + 0.0001$

≈ 0.9138

Module - 3

H.W 7) $y_{n+2} + 2y_{n+1} + y_n = 36 \qquad y_0 = y_1 = 0$

Applying Z-Transform on both sides

$Z_T(y_{n+2}) + 2Z_T(y_{n+1}) + Z_T(y_n) = Z_T(36)$

$z^2\left[\bar{y}(z) - y_0 - \frac{y_1}{z}\right] + 2z\left[\bar{y}(z) - y_0\right] + \bar{y}(z) = 36\, Z_T(1)$

$\bar{y}(z)\left[z^2 + 2z + 1\right] = 36 \cdot \frac{z}{z-1}$

$\bar{y}(z)(z+1)^2 = 36 \frac{z}{z-1}$

$\bar{y}(z) = \frac{36 \cdot z}{(z-1)(z+1)^2}$

$\bar{y}(z) = A\frac{z}{z-1} + B\frac{z}{z+1} + C\frac{-z}{(z+1)^2}$

$36z = Az(z+1)^2 + Bz(z-1)(z+1) + C(-z)(z-1)$

$z = -1 \qquad -36 = C(-1)(-1-1)$

$-36 = C(-1)(-2)$

$-36 = C \cdot 2$

$C = \frac{-36}{2} = -18$

$C = -18$

$z = 1 \qquad 36 = A\,1\,(1+1)^2$

$36 = A\,2^2$

$A = \frac{36}{4}$

$A = 9$

~~Coef~~ Comparing the coefficient of z^3

$0 = A + B$

$B = -A$

$= -9$

$\bar{y}(z) = 9\frac{z}{z-1} - 9\frac{z}{z+1} - 18\left(\frac{-z}{(z+1)^2}\right)$

Applying inverse

$Z_T^{-1}(\bar{y}(z)) = 9Z_T^{-1}\left(\frac{z}{z-1}\right) - 9Z_T^{-1}\left(\frac{z}{z+1}\right) - 18Z_T^{-1}\left(\frac{-z}{(z+1)^2}\right)$

$y_n = 9 - 9(-1)^n - 18(-1)^n n$

$y_n = 9\left[1 - (-1)^n - 2(-1)^n n\right]$

Module-4

2) Employee Taylor series method obtain the value of y at $x=0.1$, $x=0.2$: $\frac{dy}{dx} = 2y + 3e^x$, $y(0)=1$ upto 4^{th} degree

$\Rightarrow$ By Taylor series

$$y(x_1) = y(x_0) + h\,y'(x_0) + \frac{h^2}{2!}y''(x_0) + \frac{h^3}{3!}y'''(x_0) + \frac{h^4}{4!}y^{IV}(x_0)$$

$y(0) = 1$, i.e. x_0, y_0

$x_0 = 0$ $y_0 = 1$

i) $h = 0.1$

$y' = 2y + 3e^x$ $\quad y'(0) = 2y_0 + 3e^{x_0} = 2(1) + 3e^0 = 5$

$y'' = 2y' + 3e^x$ $\quad y''(0) = 2y_0' + 3e^{x_0} = 2(5) + 3 = 13$

$y''' = 2y'' + 3e^x$ $\quad y'''(0) = 2y_0'' + 3e^{x_0} = 2(13) + 3 = 26 + 3 = 29$

$y^{IV} = 2y''' + 3e^x$ $\quad y^{IV}(0) = 2y_0''' + 3e^{x_0} = 2(29) + 3e^0 = 58 + 3 = 61$

at $x = 0.1$, $h = 0.1$

$$\therefore y(0.1) = y(0) + 0.1\,y'(0) + \frac{(0.1)^2}{2}y''(0) + \frac{(0.1)^3}{6}y'''(0) + \frac{(0.1)^4}{24}y^{IV}(0)$$

$$= 1 + 0.1(5) + \frac{(0.1)^2}{2}\times 13 + \frac{(0.1)^3}{6}(29) + \frac{(0.1)^4}{24}(61)$$

$$= 1 + 0.5 + 0.065 + 0.0048 + 0.0003$$

$$= 1.5701$$

ii) $x = 0.2$, $h = 0.2$

$$\therefore y(0.2) = y(0) + 0.2\,y'(0) + \frac{(0.2)^2}{2}y''(0) + \frac{(0.2)^3}{6}y'''(0) + \frac{(0.2)^4}{24}y^{IV}(0)$$

$$= 1 + 0.2(5) + \frac{(0.2)^2}{2}(13) + \frac{(0.2)^3}{6}(29) + \frac{(0.2)^4}{24}(61)$$

$$= 1 + 1 + 0.2600 + 0.0387 + 0.0041$$

$$= 2.3028$$

3) Use Taylor series method to solve $5x\frac{dy}{dx} + y^2 - 2 = 0$, $y(4) = 4$ at $x = 4.1$

$\Rightarrow$ $x_0 = 4$ $\quad y_0 = 4$ $\quad x = 4.1$ $\quad h = 0.1$

$$\frac{dy}{dx} = \frac{2-y^2}{5x}$$

$$y' = \frac{2-y^2}{5x} \qquad y'(x_0) = \frac{2-4^2}{5\times4} = \frac{2-16}{20} = \frac{-14}{20} = -0.7$$

$$5xy' = 2-y^2$$

$$5(xy''+y'.1) = \cancel{0}-2yy'$$

$$5xy''+5y' = \cancel{0}-2yy'$$

$$y'' = \frac{\cancel{0}-2yy'-5y'}{5x} \qquad y''(x_0) = \frac{\cancel{0}-2(4)(-0.7)-5(-0.7)}{5\times4}$$

$$= \cancel{0.55}\ 0.4550$$

$$5(xy'''+y'')+5y'' = -2(yy''+y'y')$$

$$5xy'''+10y'' = -2yy''-2(y')^2$$

at $y'''(x_0)$

$$5(4)y'''+10(0.4550) = -2(4)(0.4550)-2(-0.7)^2$$

$$20y'''+4.5500 = -3.6400-0.9800$$

$$20y''' = -\frac{\cancel{4.0541}\ 4.6200 - \cancel{0.9800}\ 4.5500}{20}$$

$$y''' = -0.4585$$

Diff again

$$5(xy^{iv}+y'''.1)+10y''' = -2(yy'''+y''y')-2.2y'y''$$

$$5xy^{iv}+5y'''+10y''' = -2yy'''-\cancel{2}y''y'-4y'y''$$

$$5xy^{iv}+15y''' = -2yy'''-6y''y'$$

at (x_0, y_0).

$$5(4)y^{iv}+15(-0.4585) = -2(4)(-0.4585)-6(0.4550)(-0.7)$$

$$20y^{iv}-6.8775 = +3.6680+1.9110$$

$$y^{iv} = 0.6228$$

By taylor's series

$$y(4.1) = y(x_0)+hy'(x_0)+\frac{h^2}{2!}y''(x_0)+\frac{h^3}{3!}y'''(x_0)+\frac{h^4}{4!}y^{iv}(x_0)$$

$$= 4+(0.1)(-0.7)+\frac{(0.1)^2}{2}(0.4550)+\frac{(0.1)^3}{6}(-0.4585)$$

$$+\frac{(0.1)^4}{24}(0.6228)$$

Scanned with CamScanner

$= 4 - 0.07 + 0.0023 - 0.0001 + 0$

$= 3.9322$

4) Use Taylor Series method solve $\frac{dy}{dx} = x^2y - 1$ $y(0) = 1$ at $x = 0.1$ and $x = 0.2$

$\rightarrow$ $x_0 = 0$ $y_0 = 1$ ~~at 0.1~~

$y' = x^2y - 1$

(i) $x = 0.1$ $h = 0.1$

$y' = x^2y - 1$ $y'(0) = 0 - 1 = -1$

By Taylor series

$$y(x_n) = y(x_0) + hy'(x_0) + \frac{h^2}{2!}y''(x_0) + \frac{h^3}{3!}y'''(x_0) + \frac{h^4}{4!}y^{iv}(x_0)$$

i) $x = 0.1$ $h = 0.1$

$y' = x^2y - 1$ $y'(0) = 0 - 1 = -1$

$y'' = x^2y' + y2x$ $y''(0) = 0$

$y''' = (x^2y'' + y'2x) + 2(xy' + y)$

$= x^2y'' + 2xy' + 2xy' + 2y$

$= x^2y'' + 4xy' + 2y$ $y'''(0) = 2(1) = 2$

$y^{iv} = x^2y''' + y''2x + 4(xy'' + y') + 2y'$

$= x^2y''' + 2xy'' + 4xy'' + 4y' + 2y'$

$= x^2y''' + 6xy'' + 6y'$ $y^{iv}(0) = 6(-1) = -6$

$$y(0.1) = y(0) + (0.1)y'(0) + \frac{(0.1)^2}{2!}y''(0) + \frac{(0.1)^3}{3!}y'''(0) + \frac{(0.1)^4}{4!}y^{iv}(0)$$

$$= 1 + 0.1(-1) + \frac{(0.1)^2}{2}.0 + \frac{(0.1)^3}{\cancel{6}\,3}(\cancel{2}\,1) + \frac{(0.1)^4}{\cancel{24}\,4}(-\cancel{6}\,1)$$

$= 1 - 0.1 + 0.0003 - 0$

$= 0.9003$

ii) $x = 0.2$ $h = 0.2$

$$y(0.2) = y(0) + (0.2)y'(0) + \frac{(0.2)^2}{2!}y''(0) + \frac{(0.2)^3}{3!}y'''(0) + \frac{(0.2)^4}{4!}y^{iv}(0)$$

$$= 1 + 0.2(-1) + 0 + \frac{(0.2)^3}{\cancel{6}\,3}(\cancel{2}\,1) + \frac{(0.2)^4}{\cancel{24}\,4}(-\cancel{6}\,1)$$

$= 1 - 0.2 + 0.0027 - 0.0004 = 0.8023$

H.W Use Taylor Series to obtain a power series in $(x-4)$ for equation $\frac{dy}{dx} = \frac{1}{x^2+y}$ upto third degree, given $y(4)=4$ and find $y(4.1)$

$y(4)=4$

$x_0 = 4 \quad y_0 = 4 \quad h = x - x_0 = x - 4 = 4.1 - 4 = 0.1$

By Taylor Series

$$y(x_1) = y(x_0) + h\,y'(x_0) + \frac{h^2}{2!}y''(x_0) + \frac{h^3}{3!}y'''(x_0)$$

$$y' = \frac{1}{x^2+y} \qquad y'(4) = \frac{1}{4^2+4} = \frac{1}{16+4} = \frac{1}{20} = 0.05$$

$(x^2+y)\,y' = 1$

$$y'' = \frac{(x^2+y)0 - 1(2x+y')}{(x^2+y)^2}$$

$$y'' = \frac{-(2x+y')}{(x^2+y)^2} \qquad y''(4) = \frac{-(2\times4+0.05)}{(4^2+4)^2} = \frac{-8.05}{400} = -0.0201$$

$$y''' = \frac{(x^2+y)^2(-2-y'') + (2x+y')2(x^2+y).(2x+y')}{(x^2+y)^4}$$

$$= \frac{\cancel{(x^2+y)}\left[(x^2+y)(-2-y'') + (2x+y')^2 2 \cancel{(x^2+y)}\right]}{(x^2+y)^4}$$

$$y''' = \frac{(x^2+y)(-2-y'') + 2(2x+y')^2}{(x^2+y)^3}$$

$$y'''(4) = \frac{(4^2+4)(-2+0.0201) + 2(2\cdot4+0.05)^2}{(4^2+4)^3}$$

$$= \frac{-39.5980 + 12.0050}{8000}$$

$$= -0.0034$$

$$\therefore y(4.1) = y(4) + h\,y'(4) + \frac{h^2}{2!}y''(4) + \frac{h^3}{3!}y'''(4)$$

$$= y(4) + (x-4)y'(4) + \frac{(x-4)^2}{2!}y''(4) + \frac{(x-4)^3}{3!}y'''(4)$$

$$= y(4) + (x-4)\,0.05 + \frac{(x-4)^2}{2}(-0.0201) + \frac{(x-4)^3}{6}(-0.0034)$$

$$= 4 + 0.1(0.05) + \frac{(0.1)^2}{2}(-0.0201) + \frac{(0.1)^3}{6}(-0.0034)$$

$$= 4.0050 - 0.0001 - 0$$

$$= 4.0049$$

Modified Euler's method

Note: Consider the initial value problem $\frac{dy}{dx} = f(x,y)$ $y(x_0) = y_0$ we need to find y at $x_1 = x_0 + h$ we first obtain $y(x_1) = y_1$ by applying euler's formula and this value is regarded as initial approximation and usually denoted by $y_1^{(0)}$ and it is called predicted value of y_1

Euler's formula given by $y_1^{(0)} = y_0 + hf(x_0, y_0)$

Since the accuracy is poor in this formula this value y_1 is successively improved to the desired degree of accuracy by Modified Euler's formula Where the successive approximation is denoted by

$y_1^{(1)}, y_1^{(2)}, y_1^{(3)}, \ldots$

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right]$$

1) Given $\frac{dy}{dx} = 1 + \frac{y}{x}$, $y = 2$ at $x = 1$ Find the approximate value of y at $x = 1.4$. apply Modified Euler's method

$\rightarrow$ $f(x,y) = 1 + \frac{y}{x}$ $\quad x_0 = 1 \quad y_0 = 2 \quad h = 0.4$

Euler's formula

$$y_1^{(0)} = y_0 + hf(x_0, y_0)$$
$$= 2 + 0.4f(1, 2)$$
$$= 2 + 0.4 \times 3$$
$$= 2 + 1.2$$
$$= 3.2$$

$f(1,2) = 1 + \frac{2}{1}$
$= 1 + 2$
$= \underline{3}$

By modified Euler's formula

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$
$$= 2 + \frac{0.4}{2}\left[3 + 3.2857\right]$$
$$= 2 + 1.2571$$
$$= \underline{3.2571}$$

$f(x_1, y_1^{(0)})$
$f(1.4, 3.2)$
$1 + \frac{3.2}{1.4}$
$1 + 2.2857$
$\underline{3.2857}$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$
$$= 2 + \frac{0.4}{2}(3 + 3.3265) = \underline{3.2653}$$

$f(x_1, y_1^{(1)})$
$f(1.4, 3.2571)$
$1 + \frac{3.2571}{1.4}$
$\underline{3.3265}$

$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right]$ $\quad f(1.4, 3.2653)$ $\quad 1 + \frac{3.2653}{1.4}$ $\quad \underline{3.3324}$

$= 2 + \frac{0.4}{2}\left[3 + 3.3324\right]$

$= \underline{3.2665}$

$y_1^{(4)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(3)})\right]$ $\quad f(1.4, 3.2665)$ $\quad \underline{3.3332}$

$= 2 + \frac{0.4}{2}\left[3 + 3.3332\right]$

$= \underline{3.2666}$

$\boxed{\therefore y(1.4) = 3.2666}$

2) Using Modified Euler's method solve the initial value problem $\frac{dy}{dx} = \frac{y-x}{y+x}$, $y(0)=1$ at $x=0.1$ carry out 3 modifications with step size 0.1

$\Rightarrow \quad f(x,y) = \frac{y-x}{y+x} \quad x_0 = 0 \quad y_0 = 1 \quad x_1 = 0.1 \quad h = 0.1$

Euler's formula $\quad f(0,1)$ $\quad \frac{1-0}{1+0} = \frac{1}{1} = 1$

$y_1^{(0)} = y_0 + h f(x_0, y_0)$

$= 1 + 0.1 f(0,1)$

$= 1 + 0.1$

$= \underline{1.1}$

~~Using~~ By Modified Euler's formula $\quad f(0.1, 1.1)$ $\quad \frac{1.1-0.1}{1.1+0.1} = \frac{1}{1.2}$ $\quad 0.8333$

$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$

$= 1 + \frac{0.1}{2}\left[1 + 0.8333\right]$

$= \underline{1.0917}$

$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$ $\quad f(0.1, 1.0917)$ $\quad \frac{1.0917-0.1}{1.0917+0.1}$ $\quad 0.8322$

$= 1 + \frac{0.1}{2}\left[1 + 0.8322\right]$

$= \underline{1.0916}$

$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right]$ $\quad f(0.1, 1.0916)$ $\quad 0.8322$

$= 1 + \frac{0.1}{2}\left[1 + 0.8322\right]$

$= \underline{1.0916}$

$\boxed{\therefore y(0.1) = 1.0916}$

3/ Using modified Euler's method solve the initial value problem $\frac{dy}{dx} = \frac{1}{x+y}$, $y(0)=1$ at $x=1$ in the steps of length $h=0.5$, carry out 2 modifications in each step size.

$\rightarrow$ $f(x,y) = \frac{1}{x+y}$ $\quad x_0 = 0 \quad y_0 = 1$

Stage 1 : $h = 0.5 \quad x_0 = 0 \quad y_0 = 1$

Euler's formula:

$$y_1^{(0)} = y_0 + h f(x_0, y_0)$$
$$= 1 + 0.5 f(0,1)$$
$$= 1 + 0.5$$
$$= 1.5$$

$f(0,1)$
$\frac{1}{0+1} = \frac{1}{1}$

By modified Euler's formula

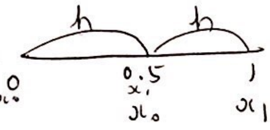
$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1, y_1^{(0)})\right]$$
$$= 1 + \frac{0.5}{2}\left[1 + 0.5\right]$$
$$= \underline{1.3750}$$

$f(0.5, 1.5)$
$\frac{1}{0.5+1.5} = \cancel{0.6667}\ 0.5$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1, y_1^{(1)})\right]$$
$$= 1 + \frac{0.5}{2}\left[1 + 0.5333\right]$$
$$= \underline{1.3833}$$

$f(0.5, 1.3750)$
$\frac{1}{0.5+1.3750}$
$= 0.5333$

$\therefore y(0.5) = 1.3833$



Stage 2: $x_0 = 0.5 \quad y_0 = 1.3833 \quad x_1 = 1$
$h = 0.5$

Euler's formula

$$y_1^{(0)} = y_0 + h f(x_0, y_0)$$
$$= 1.3833 + 0.5 f(0.5, 1.3833)$$
$$= 1.3833 + 0.5(0.5310)$$
$$= \underline{1.6488}$$

$f(0.5, 1.3833)$
$\frac{1}{0.5+1.3833}$

By modified Euler's formula

$$y_1^{(1)} = 1.3833 + \frac{0.5}{2}\left[0.5310 + f(1, 1.6488)\right]$$
$$= \underline{1.6104}$$

$\frac{1}{1+1.6488}$

$y_1^{(2)} = 1.3833 + \frac{0.5}{2}[0.5310 + f(1, 1.6104)]$ $\quad \frac{1}{1+1.6104} = 0.3831$

$= 1.3833 + \frac{0.5}{2}[0.5310 + 0.3831]$

$= 1.6118$

$\boxed{\therefore y(1) = 1.6118}$

4) Using Modified Euler's Method solve $\frac{dy}{dx} = x+y$ at $x = 0.2$, $y(0) = 1$

$y_0 = 1$ $\quad h = 0.2$ $\quad x_1 = 0.2$

→ $f(x, y) = x+y$ $\quad x_0 = 0$

Euler's formula

$y_1^{(0)} = y_0 + h f(x_0, y_0)$ $\quad f(0,1) = 0+1 = 1$

$= 1 + 0.2 f(0,1)$

$= 1 + 0.2$

$= 1.2$

By Modified Euler's formula $\quad f(0.2, 1.2)$ $\quad 1.2 + 0.2$ $\quad 1.4$

$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(0)})]$

$= 1 + \frac{0.2}{2}[1 + 1.4]$

$= 1.2400$

$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})]$ $\quad f(0.2, 1.24)$ $\quad 0.2 + 1.24$ $\quad 1.44$

$= 1 + \frac{0.2}{2}[1 + 1.44]$

$= 1 + \frac{0.2}{2}(2.44)$

$= 1.2440$

$y_1^{(3)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(2)})]$ $\quad f(0.2, 1.2440)$ $\quad 1.2440 + 0.2 = 1.4440$

$= 1 + \frac{0.2}{2}[1 + 1.4440]$

$= 1 + \frac{0.2}{2}[2.4440]$

$= 1.2444$

Runge Kutta Method of 4^{th} Order

Consider initial value problem $\frac{dy}{dx} = f(x,y)$
$y(x_0) = y_0$ we need to find an approximate solution of y at $x_1 = x_0 + h$. where h is the step size.

Runge Kutta Method of 4^{th} Order is given by

$$y_1 = y_0 + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$

where $k_1 = hf(x_0, y_0)$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = hf\left(x_0 + h, y_0 + k_3\right)$$

1) Using Fourth order Runge Kutta method solve the initial value problem $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$, $y(0) = 1$ at $x = 0.2$

→ $f(x,y) = \frac{y^2 - x^2}{y^2 + x^2}$ $\quad x_0 = 0 \quad y_0 = 1 \quad h = 0.2$

$f(0,1) = \frac{1^2 - 0^2}{1^2 + 0} = 1$

$$k_1 = hf(x_0, y_0)$$
$$= 0.2\, f(0,1)$$
$$= 0.2 \times 1$$
$$= \underline{0.2}$$

$f(0.1, 1.1) = \frac{1.1^2 - 0.1^2}{1.1^2 + 0.1^2} = 0.9836$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$
$$= 0.2\, f\left(\left(0 + \frac{0.2}{2}\right), \left(1 + \frac{0.2}{2}\right)\right)$$
$$= 0.2\, f(0.1, 1.1)$$
$$= 0.2 \times 0.9836$$
$$= \underline{0.1967}$$

$f(0.1, 1.0984) = \frac{1.0984^2 - 0.1^2}{1.0984^2 + 0.1^2} = 0.9836$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$
$$= 0.2\, f\left(0.1, 1 + \frac{0.1967}{2}\right)$$
$$= 0.2\, f(0.1, 1.0984)$$
$$= 0.2 \times 0.9836$$
$$= \underline{0.1967}$$

$f(0.2, 1.1967)$
0.9457

$$k_4 = hf(x_0 + h, y_0 + k_3)$$
$$= 0.2\, f(0.2, 1 + 0.1967)$$
$$= 0.2\, f(0.2, 1.1967)$$

$= 0.2 \times 0.9457$

$= 0.1891$

$y(0.2) = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$= 1 + \frac{1}{6}[0.2 + 2\times 0.1967 + 2\times 0.1967 + 0.1891]$

$= 1 + \frac{1}{6}(1.1759)$

$= 1 + 0.1960$

$= 1.1960$

2) Apply Runge Kutta method of 4^{th} order to find the approximate value of y for $x = 0.1$ if $10\frac{dy}{dx} = x^2 + y^2$ given $y(0) = 1$

→

$10\frac{dy}{dx} = x^2 + y^2$

$\frac{dy}{dx} = \frac{x^2 + y^2}{10}$

$f(x,y) = \frac{x^2 + y^2}{10}$ $\quad x_0 = 0 \quad y_0 = 1 \quad h = 0.1$

$f(0,1) = \frac{0^2 + 1^2}{10} = \frac{1}{10}$

$k_1 = h f(x_0, y_0)$

$= 0.1 \times f(0,1)$

$= 0.1 \times 0.1$

$= 0.01$

$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$

$= 0.1\, f\left(0 + \frac{0.1}{2}, 1 + \frac{0.01}{2}\right)$

$= 0.1\, f(0.05, 1.0050)$

$= 0.1 \times 0.1013$

$= 0.0101$

$k_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$

$= 0.1\, f\left(0 + \frac{0.1}{2}, 1 + \frac{0.0101}{2}\right)$

$= 0.1\, f(0.05, 1.0051)$

$= 0.1 \times 0.1013$

$= 0.0101$

$k_4 = h f(x_0 + h, y_0 + k_3)$

$= 0.1\, f(0.1, 1 + 0.0101)$

$= 0.1\, f(0.1, 1.0101)$

$= 0.1 \times 0.1030$

$= \underline{0.0103}$

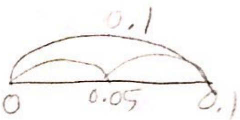
$$y(0.1) = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= 1 + \frac{1}{6}[0.01 + 2\times 0.0101 + 2\times 0.0101 + 0.0103]$$

$$= 1 + \frac{1}{6}(0.0607)$$

$$= \underline{1.0101}$$

3) Given $\frac{dy}{dx} + y - x^2 = 0$, $y(0) = 1$ Find the approximate value of y at $x = 0.1$ by taking $h = 0.05$ by Runge Kutta Method



→ $\frac{dy}{dx} = x^2 - y$ $\quad x_0 = 0 \quad y_0 = 1$

Stage I: $h = 0.05$

$k_1 = h f(x_0, y_0)$ $\qquad f(0,1) = 0^2 - 1 = -1$

$= 0.05\, f(0, 1)$

$= 0.05 \times -1$

$= \underline{-0.05}$

$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$

$= 0.05\, f(0.0250, 0.9750)$

$= 0.05 \times -0.9744$

$= \underline{-0.0487}$

$k_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$

$= 0.05\, f(0.0250, 0.9757)$

$= 0.05 \times -0.9751$

$= \underline{-0.0488}$

$k_4 = h f(x_0 + h, y_0 + k_3)$

$= 0.05\, f(0.05, 1 - 0.0488)$

$= 0.05 \times -0.9487$

$= \underline{-0.0474}$

$$y(0.05) = y_0 + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$

$$= 1 + \frac{1}{6}\left[-0.05 + 2(-0.0487) + 2\times(-0.0488) + 0.04\right.$$

$$= 1 + \frac{1}{6}(-0.2924)$$

$$= \underline{0.9513}$$

Stage II : $x_0 = 0.05$ $y_0 = 0.9513$ $h = 0.05$

$$k_1 = hf(x_0, y_0)$$

$$= 0.05\, f(0.05, 0.9513)$$

$$= 0.05 \times -0.9488$$

$$= \underline{-0.0474}$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= 0.05\, f(0.0750, 0.9240)$$

$$= 0.05 \times -0.9220$$

$$= \underline{-0.0461}$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= 0.05\, f(0.0750, 0.9283)$$

$$= 0.05 \times -0.9227$$

$$= \underline{-0.0461}$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$= 0.05\, f(0.1, 0.9052)$$

$$= 0.05 \times -0.8952$$

$$= \underline{-0.0448}$$

$$y(0.1) = y_0 + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$

$$= 0.9513 + \frac{1}{6}\left[-0.0474 + 2\times -0.0461 + 2\times -0.0461 - 0.0448\right]$$

$$= 0.9513 + \frac{1}{6}(-0.2766)$$

$$= \underline{0.9052}$$

Scanned with CamScanner

H.W $\dfrac{dy}{dx} = 3x + \dfrac{y}{2}$, $y(0)=1$ find y at $x=0.2$ by Runge - Kutta method

$\rightarrow$ $\dfrac{dy}{dx} = 3x + \dfrac{y}{2}$ $\quad x_0=0 \quad y_0=1 \quad h=0.2$

$$k_1 = hf(x_0, y_0)$$
$$= 0.2\, f(0,1)$$
$$= 0.2 \times 0.5$$
$$= \underline{0.1}$$

$$f(0,1) = 0 + \frac{1}{2} = 0.5$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$
$$= 0.2\, f\left(\frac{0.2}{2}, 1 + \frac{0.1}{2}\right)$$
$$= 0.2\, f(0.1, 1.05)$$
$$= 0.2 \times 0.8250$$
$$= \underline{0.1650}$$

$$f(0.1, 1.05) = 3 \times 0.1 + \frac{1.05}{2}$$
$$= 0.3 + 0.5250$$
$$= \underline{0.8250}$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$
$$= 0.2\, f\left(0.1, 1 + \frac{0.165}{2}\right)$$
$$= 0.2\, f(0.1, 1.0825)$$
$$= 0.2 \times 0.8413$$
$$= \underline{0.1683}$$

$$f(0.1, 1.0825) = 3 \times 0.1 + \frac{1.0825}{2}$$
$$= \underline{0.8413}$$

$$k_4 = hf\left(x_0 + h, y_0 + k_3\right)$$
$$= 0.2\, f(0.2, 1 + 0.1683)$$
$$= 0.2\, f(0.2, 1.1683)$$
$$= \cancel{0.1684}\ 0.2 \times 1.1842$$
$$= \underline{0.2368}$$

$$\cancel{f(0.1, 1.0842) = 3 \times 0.1 + \frac{1.0842}{2} = 0.3 + 0.5421 = 0.8421}$$

$$f(0.2, 1.1683) = 3 \times 0.2 + \frac{1.1683}{2}$$
$$= \underline{1.1842}$$

$$y(0.2) = y_0 + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$
$$= 1 + \frac{1}{6}\left[0.1\cancel{650} + 2(0.1650) + 2(0.1683) + 0.2368\right]$$
$$= 1 + \frac{1}{6}(1.0034)$$
$$= \underline{1.1672}$$

Corrector and Predictor Method

Consider a differential equation $y' = f(x, y)$, with the set of four predetermined value of y. $y(x_0) = y_0$, $y(x_1) = y_1$, $y(x_2) = y_2$, $y(x_3) = y_3$ where x_0, x_1, x_2, x_3 are equally spaced values of x with width h. The predictor and the corrector formula to compute y at x_4, is $y(x_0+4h) = y(x_4) = y_4$

I) Milne's Predictor & Corrector Formula

$$y_4^{(P)} = y_0 + \frac{4h}{3}\left[2y_1' - y_2' + 2y_3'\right]$$

$$y_4^{(C)} = y_2 + \frac{h}{3}\left[y_2' + 4y_3' + y_4'\right]$$

II) Adam's Bashforth Predictor & Corrector Formula

$$y_4^{(P)} = y_3 + \frac{h}{24}\left[55y_3' - 59y_2' + 37y_1' - 9y_0'\right]$$

$$y_4^{(C)} = y_3 + \frac{h}{24}\left[9y_4' + 19y_3' - 5y_2' + y_1'\right]$$

1) Using Milne's method find y at $x = 0.8$ given that $\frac{dy}{dx} = x - y^2$, $y(0) = 0$, $y(0.2) = 0.02$, $y(0.4) = 0.0795$, $y(0.6) = 0.1762$

→ $y' = x - y^2$ $h = 0.2$

x	y	$y' = x - y^2$
$x_0 = 0$	$y_0 = 0$	$y_0' = 0$
$x_1 = 0.2$	$y_1 = 0.02$	$y_1' = 0.1996$
$x_2 = 0.4$	$y_2 = 0.0795$	$y_2' = 0.3937$
$x_3 = 0.6$	$y_3 = 0.1762$	$y_3' = 0.5688$
$x_4 = 0.8$		

Milne's Predictor

$$y_4^{(P)} = y_0 + \frac{4h}{3}\left[2y_1' - y_2' + 2y_3'\right]$$

$$= 0 + \frac{4(0.2)}{3}\left[2(0.1996) - 0.3937 + 2(0.5688)\right]$$

$$= 0.3048$$

Milne's Correction Formula

$$y_4^{(c)} = y_2 + \frac{h}{3}[y_2' + 4y_3' + y_4']$$

$$y_4' = x_4 - [y_4^{(P)}]^2$$
$$= 0.8 - (0.3048)^2$$
$$= 0.7071$$

$$\therefore y_4^{(c)} = 0.3937 + \frac{0.2}{3}[0.3937 + 4(0.5688) + 0.7071]$$
$$= 0.3046$$

2) Given $\frac{dy}{dx} = xy + y^2$ $y(0) = 1$, $y(0.1) = 1.1169$, $y(0.2) = 1.2773$, $y(0.3) = 1.5049$ find $y(0.4)$ find y with Milne's Predictor & corrector method by applying correction formula twice.

→ $y = xy + y^2$, $h = 0.1$

x	y	$y' = xy + y^2$
$x_0 = 0$	$y_0 = 1$	$y_0' = 1$
$x_1 = 0.1$	$y_1 = 1.1169$	$y_1' = 1.3592$
$x_2 = 0.2$	$y_2 = 1.2773$	$y_2' = 1.8870$
$x_3 = 0.3$	$y_3 = 1.5049$	$y_3' = 2.7162$

$x_4 = 0.4$

Milne's Predictor

$$y_4^{(P)} = y_0 + \frac{4h}{3}[2y_1' - y_2' + 2y_3']$$
$$= 1 + \frac{4(0.1)}{3}[2(1.3592) - 1.8870 + 2(2.7162)]$$
$$= 1 + 0.8352$$
$$= 1.8352$$

Milne's Corrector Formula

$$y_4^{(c)} = y_2 + \frac{h}{3}[y_2' + 4y_3' + y_4']$$
$$y_4' = x_4 y_4^{(P)} + [y_4^{(P)}]^2 = 0.4 \times 1.8352 + (1.8352)^2$$
$$= 4.1020$$

$$y_4^{(c)} = 1.2773 + \frac{0.1}{3}[1.8870 + 4(2.7162) + 4.1020]$$

$= 1.8391$

Applying Corrector Formula Twice

$y_4' = x_4 y_4^{(c)} + [y_4^{(c)}]^2$

$= 0.4(1.8391) + (1.8391)^2$

$= 4.1179$

$y_4^{(c)} = y_2 + \frac{h}{3}[y_2' + 4y_3' + y_4']$

$= 1.2773 + \frac{0.1}{3}[1.8870 + 4(2.7162) + 4.1179]$

$= 1.2773 + 0.5623$

$y(0.4) = 1.8396$

3) Given that $\frac{dy}{dx} = x^2(1+y)$ find y at 1.4 by Adam's Bashforth method using the data given below.

x	1	1.1	1.2	1.3
y	1	1.233	1.548	1.979

→ $y' = x^2(1+y)$ $h = 0.1$

x	y	$y' = x^2(1+y)$
$x_0 = 1$	$y_0 = 1$	$y_0' = 2$
$x_1 = 1.1$	$y_1 = 1.233$	$y_1' = 2.7019$
$x_2 = 1.2$	$y_2 = 1.548$	$y_2' = 3.6691$
$x_3 = 1.3$	$y_3 = 1.979$	$y_3' = 5.0345$
$x_4 = 1.4$		

By Adam's Bashforth Predictor

$y_4^{(P)} = y_3 + \frac{h}{24}[55y_3' - 59y_2' + 37y_1' - 9y_0']$

$= 1.979 + \frac{0.1}{24}[55(5.0345) - 59(3.6691) + 37(2.7019) - 9(2)]$

$= 1.979 + \frac{0.1}{24}(142.3909)$

$= 1.979 + 0.5933$

$= 2.5723$

correction Formula

$$y_4^{(c)} = y_3 + \frac{h}{24}\left[9y_4' + 19y_3' - 5y_2' + y_1'\right]$$

$$\cancel{= 1.979 + \frac{0.1}{24}\left[9\right.}$$

$$y_4' = x_4^2(1 + y_4^{(P)})$$

$$= 1.4^2(1 + 2.5723)$$

$$= 7.0017$$

$$= 1.979 + \frac{0.1}{24}\left[9(7.0017) + 19(5.0345) - 5(3.6691) + 2.7019\right]$$

$$= 2.5749$$

4) Given $\frac{dy}{dx} = x^2 + \frac{y}{2}$, $y(1) = 2$, $y(1.1) = 2.2156$, $y(1.2) = 2.4649$, $y(1.3) = 2.7514$. Find $y(1.4)$ by Adam's Bashforth method.

$\rightarrow$ $y' = x^2 + \frac{y}{2}$ $\quad h = 0.1$

x	y	$y' = x^2 + \frac{y}{2}$
$x_0 = 1$	$y_0 = 2$	$y_0' = 2$
$x_1 = 1.1$	$y_1 = 2.2156$	$y_1' = 2.3178$
$x_2 = 1.2$	$y_2 = 2.4649$	$y_2' = 2.6725$
$x_3 = 1.3$	$y_3 = 2.7514$	$y_3' = 3.0657$
$x_4 = 1.4$		

Adam's Bashforth method

Predictor

$$y_4^{(P)} = y_3 + \frac{h}{24}\left[55y_3' - 59y_2' + 37y_1' - 9y_0'\right]$$

$$= 2.7514 + \frac{0.1}{24}\left[55(3.0657) - 59(2.6725) + 37(2.3178) - 9(2)\right]$$

$$= 3.0793$$

corrector

$$y_4^{(c)} = y_3 + \frac{h}{24}\left[9y_4' + 19y_3' - 5y_2' + y_1'\right]$$

$$y_4' = 1.4^2 + \frac{3.0793}{2} = 3.4997$$

Scanned with CamScanner