Chapter 2

Mathematical Logic-II

This chapter is a continuation of Chapter 1. Here, we present the topic of quantified statements and the methods of proof and disproof.
We presume that the reader has a basic knowledge of elementary Set Theory.

2.1 Open Statements; Quantifiers

In mathematical discussions, declarative sentences such as those given below are encountered:

(1) $x + 3 = 6$ (2) $x^2 < 10$ (3) x divides 4 (4) $x = \sqrt{2}$

As indicated in Section 1.1, these sentences are not propositions unless the symbol x is specified. Sentences of this kind are called **open statements** or **open sentences**, and the unspecified symbols, such as x in the sentences given above, are called **free variables**.

Consider the sentence (1) above, and the set of real numbers R. This sentence becomes a proposition if x is replaced by *any element* of R. For example, if x is replaced by 3, the sentence becomes a true proposition, and if x is replaced by 5, it becomes a false proposition. Here, we say R is a **universe** *(or universe of discourse)* for the variable x in the sentence (1). Similarly, R is a universe for x in sentences (2), (3) and (4) also. The set of all integers Z is also a universe for x in all the four sentences.

Open statements containing a variable x are denoted by $p(x)$, $q(x)$, etc. If U is the universe for the varible x in an open statement $p(x)$ and if $a \in \mathsf{U}$, then the proposition got by replacing x by a in $p(x)$ is denoted by $p(a)$.

Thus, if the set of all integers is the universe for x in the open statement $p(x) : x + 3 = 6$, then $p(2)$ is the proposition "$2 + 3 = 6$". We note that this proposition is false. Similarly, if $q(x)$ is the open statement "$x^2 < 10$" with the set of all real numbers as the universe for x, then $p(\sqrt{2})$ is the proposition "$2 < 10$" which is true.

It is to be emphasized that an open statement $p(x)$ becomes a proposition only when x is replaced by a chosen element of the universe. The truth or falsity of the proposition $p(a)$ depends upon the element a of the universe that is chosen to replace x.

Like compound propositions, compound open statements can be formed by using the logical connectives. Thus, $\neg p(x)$ is the negation of an open statement $p(x)$. Also, for the open statements $p(x)$ and $q(x)$, (i) $p(x) \wedge q(x)$ is the conjunction, (ii) $p(x) \vee q(x)$ is the disjunction, (iii) $p(x) \to q(x)$ is a conditional, and (iv) $p(x) \leftrightarrow q(x)$ is the biconditional. For a given universe and for a given element of the universe, the truth values of compound open statements are determined according to the same rules as those valid for compound propositions.

Example 1 *Suppose the universe consists of all integers. Consider the following open statements:*

$$p(x) : x \leq 3, \; q(x) : x + 1 \text{ is odd}, \; r(x) : x > 0.$$

Write down the truth values of the following:

(i) $p(2)$ (ii) $\neg\, q(4)$

(iii) $p(-1) \wedge q(1)$ (iv) $\neg\, p(3) \vee r(0)$

(v) $p(0) \rightarrow q(0)$ (vi) $p(1) \leftrightarrow \neg\, q(2)$

(vii) $p(4) \vee (q(1) \wedge r(2))$ (viii) $p(2) \wedge (q(0) \vee \neg\, r(2))$

► (i) $p(2)$ is the proposition "$2 \leq 3$" which is true.

(ii) $q(4)$ is the proposition "$4 + 1$" is odd which is true. Therefore, $\neg\, q(4)$ is false.

(iii) $p(-1)$ is the proposition "$-1 \leq 3$" which is true, and $q(1)$ is the proposition "$1 + 1$ is odd" which is false. Therefore $p(-1) \wedge q(1)$ is false.

(iv) $p(3)$ is true, so that $\neg\, p(3)$ is false and $r(0)$ is false. Therefore, $\neg\, p(3) \vee r(0)$ is false.

(v) $p(0)$ is true and $q(0)$ is true. Therefore, $p(0) \rightarrow q(0)$ is true

(vi) $p(1)$ is true and $q(2)$ is true. Therefore, $p(1) \leftrightarrow \neg\, q(2)$ is false.

(vii) $p(4)$ is false, $q(1)$ is false and $r(2)$ is true. Therefore, $q(1) \wedge r(2)$ is false, so that $p(4) \vee (q(1) \wedge r(2))$ is false.

(viii) $p(2)$ is true, $q(0)$ is true and $r(2)$ is true. Therefore, $q(0) \vee \neg\, r(2)$ is true, so that $p(2) \wedge (q(0) \vee \neg\, r(2))$ is true. ∎

Quantifiers

Consider the following propositions:

(1) All squares are rectangles.

(2) For every integer x, x^2 is a non-negative integer.

(3) Some determinants are equal to zero.

(4) There exists a real number whose square is equal to itself.

In these propositions, the words "all", "every", "some", "there exists" are associated with the idea of a *quantity*. Such words are called ***quantifiers***.

The propositions (1)–(4) considered above may be rewritten in alternative forms as explained below.

Let S denote the set of all squares. Then the proposition (1) may be rewritten as:

For all $x \in S$, x is a rectangle.

Symbolically, this is written as

$$\forall x \in S,\ p(x)$$

where the symbol $\forall$ denotes the phrase "for all", and $p(x)$ stands for the open statement "x is a rectangle."

Similarly, the proposition (2) may be symbolically written as $\forall x \in \mathsf{Z},\ q(x)$, where the symbol $\forall$ now denotes the phrase "for every", Z is the set of all integers and $q(x)$ is the open statement "x^2 is a non-negative integer".

Observe that the symbol $\forall$ has been used to denote the phrases "for all" and "for every". In logic, these phrases are regarded as equivalent phrases. The phrases "for each" and "for any" are also taken to be equivalent to these. The symbol $\forall$ is used to denote all of these phrases *. Each of these equivalent phrases is called the ***universal quantifier***.

Next, let us consider the proposition (3). If D now denotes the set of all determinants, then this proposition may be rewritten as:

For some $x \in D$, x is equal to zero.

Symbolically, this is written as

$$\exists x \in D,\ p(x)$$

where the symbol $\exists$ denotes the phrase "for some", and $p(x)$ stands for the open statement "x is equal to zero."

Similarly, the proposition (4) may be symbolically written as $\exists x \in \mathsf{R},\ q(x)$, where the symbol $\exists$ now denotes the phrase "there exists", R denotes the set of all real numbers and $q(x)$ stands for the open statement "x is a real number whose square is equal to itself."

Observe that the symbol $\exists$ has been used to denote the phrases "for some" and "there exists". In logic these two phrases are taken to be equivalent to one another. They are also taken to be equivalent to the phrase "for at least one". The symbol $\exists$ is used to denote all of these equivalent phrases. Each of these equivalent phrases is called the ***existential quantifier***.

A proposition involving the universal or the existential quantifier is called a ***quantified statement***. Thus, a quantified statement is a proposition of the form "$\forall x \in S,\ p(x)$" or "$\exists x \in S,\ p(x)$", where $p(x)$ is an open statement and S is the universe for x in $p(x)$. When the context tells what the universe is, the universe is not explicitly indicated; and a quantified statement is written as "$\forall x, p(x)$" or "$\exists x, p(x)$", as the case may be. The variable present in a quantified statement is called a ***bound variable*** —— it is bound by a quantifier.

There arise statements where the presence of a quantifier is understood (without being explicit). For example, the statement "If a triangle is equilateral then it is isoceles" actually

*Some authors use the notation ( ) for $\forall$. They write $\forall x, p(x)$ as $(x), p(x)$.

stands for the quantified statement " For every triangle T, if T is equilateral then T is isoceles" or "Every equilateral triangle is isoceles".

Similarly, when we say that "9 is the square of an integer" we actually mean that "9 is the square of some integer" or, equivalently, "There exists an integer whose square is 9".

Example 2 *For the universe of all integers, let*

$p(x)$: $x > 0$,

$q(x)$: x is even,

$r(x)$: x is a perfect square,

$s(x)$: x is divisible by 3,

$t(x)$: x is divisible by 7.

Write down the following quantified statements in symbolic form:

(i) *At least one integer is even.*

(ii) *There exists a positive integer that is even.*

(iii) *Some even integers are divisible by 3.*

(iv) *Every integer is either even or odd.*

(v) *If x is even and a perfect square, then x is not divisible by 3.*

(vi) *If x is odd or is not divisible by 7, then x is divisible by 3.*

► Using the definition of quantifiers, we find that the given statements read as follows in symbolic form:

(i) $\exists x, q(x)$ (ii) $\exists x, [p(x) \wedge q(x)]$

(iii) $\exists x, [q(x) \wedge s(x)]$ (iv) $\forall x, [q(x) \underline{\vee} \neg q(x)]$

(v) $\forall x, [\{q(x) \wedge r(x)\} \rightarrow \neg s(x)]$ (vi) $\forall x, [\{\neg q(x) \vee \neg t(x)\} \rightarrow s(x)]$

Truth value of a Quantified Statement

The following ***Rules*** are employed for determining the truth value of a quantified statement.

Rule 1: The statement "$\forall x \in S, p(x)$" is true only when $p(x)$ is true for each $x \in S$.

Rule 2: The statement "$\exists x \in S, p(x)$" is false only when $p(x)$ is false for every $x \in S$.

Accordingly, to infer that a proposition of the form "$\forall x \in S, p(x)$" is *false*, it is enough to exhibit *one* element a of S such that $p(a)$ is *false*. This element a is called a *counterexample*.

To infer that a proposition of the form "$\exists x \in S, p(x)$" is *true*, it is enough to exhibit *one* element a of S such that $p(a)$ is *true*.

Referring back to the quantified propositions (1)-(4) considered before*, we note that all of these propositions are *true* propositions.

It is obvious that the following propositions are *false*:

(5) All rectangles are squares.

(6) For every integer x, x^2 is a positive integer.

(7) The squares of some real numbers are negative.

Two Rules of Inference

As a consequence of the Rules 1 and 2 indicated above, we obtain the following *Rules of Inference*.

Rule 3: If an open statement $p(x)$ is known to be true for all x in a universe S and if $a \in S$, then $p(a)$ is true. (This is known as the ***Rule of Universal Specification***).

Rule 4: If an open statement $p(x)$ is proved to be true for any (arbitrary) x chosen from a set S, then the quantified statement $\forall x \in S, p(x)$ is true. (This is known as the ***Rule of Universal Generalization***).

Logical Equivalence

Two quantified statements are said to be *logically equivalent* whenever they have the same truth values in all possible situations.

The following results are easy to prove.

(i) $\forall x, [p(x) \wedge q(x)] \Leftrightarrow (\forall x, p(x)) \wedge (\forall x, q(x))$

(ii) $\exists x, [p(x) \vee q(x)] \Leftrightarrow (\exists x, p(x)) \vee (\exists x, q(x))$

(iii) $\exists x, [p(x) \rightarrow q(x)] \Leftrightarrow \exists x, [\neg\, p(x) \vee q(x)]$

In some situations, there arise quantified statements of the form "$\forall x, \neg p(x)$". Such a statement is taken to be logically equivalent to the statement "For no x, $p(x)$". Thus,

$$\forall x,\ \neg p(x) \Leftrightarrow \text{For no } x, p(x).$$

For example, the statement "For every integer x, x^2 is non-negative" is logically equivalent to the statement "For no integer x, x^2 is negative". Similarly, the statement "Every unit matrix is non-singular" is logically equivalent to the statement "No unit matrix is singular".

Rule for Negation of a Quantified Statement

Determining the negation of a quantified statement forms an important part of a logical argument. The following *Rule* is used in this regard.

Rule 5: To construct the negation of a quantified statement, change the quantifier from universal to existential and vice-versa, and also replace the open statement by its negation.

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That is:

$$\neg\{\forall x, p(x)\} \equiv \exists x, \{\neg p(x)\}$$

and

$$\neg\{\exists x, p(x)\} \equiv \forall x, \{\neg p(x)\}$$

To illustrate this rule, let us consider the quantified statement

"All equilateral triangles are isoceles",

In symbolic form, this quantified statement reads "$\forall x \in T, p(x)$", where T is the set of all equilateral triangles and $p(x)$ is the open statement "x is isosceles". According to the rule of negation stated above, the negation of this quantified statement is "$\exists x \in T, \neg p(x)$", In words, this reads

"For some equilateral triangle x, x is not isosceles" or, equivalently,

"Some equilateral triangles are not isosceles".

Thus, for the statement "All equilateral triangles are isosceles", the negation is "Some equilateral triangles are not isosceles".

As another example, consider the quantified statement:

"Some integers are even".

In symbolic form, this statement reads "$\exists x \in \mathsf{Z}, p(x)$", where Z is the set of all integers and $p(x)$: x is even. According to the rule of negation, the negation of this quantified statement is "$\forall x \in \mathsf{Z}, \neg p(x)$". In words, this reads

"For every integer x, x is not even", or, equivalently, "For no integer x, x is even".

These can also be rewritten as

"All integers are noneven" or "No integer is even".

Thus, for the proposition "Some integers are even", the negation is "All integers are non-even" or "No integer is even".

As one more example, consider the proposition

"No even integer is divisible by 7."

This proposition is logically equivalent to "For any even integer x, x is not divisible by 7". In symbolic form, this reads $\forall x \in E, \neg p(x)$. Here, E is the set of all even integers, and $p(x)$: x is divisible by 7.

According to the rule of negation, the negation of this quantified statement is $\exists x \in E, p(x)$. In words, this reads: "There exists an even integer x such that x is divisible by 7" or, equivalently, "Some even integer is divisible by 7".

Thus, for the proposition "No even integer is divisible by 7", the negation is

"Some even integer is divisible by 7."

Note: The above two examples illustrate the fact that for a statement of the form "For some x, $p(x)$", the negation can be expressed as "For no x, $p(x)$", and vice-versa.

Example 3 *Consider the open statements $p(x)$, $q(x)$, $r(x)$, $s(x)$, $t(x)$ of Example 2. Express each of the following symbolic statements in words and indicate its truth value.*

(i) $\forall x, [r(x) \to p(x)]$ (ii) $\exists x, [s(x) \wedge \neg q(x)]$

(iii) $\forall x, [\neg r(x)]$ (iv) $\forall x, [r(x) \vee t(x)]$

▸ (i) For any integer x, if x is a perfect square , then $x > 0$. — False (Take $x = 0$).

(ii) For some integer x, x is divisible by 3 and x is not even. (In other words: There is an integer which is divisible by 3 and which is not even,). — True (Take $x = 9$).

(iii) For any integer x, x is not a perfect square (In other words: No integer is a perfect square) — False.

(iv) For any integer x, x is a perfect square or x is divisible by 7. — False (Take $x = 8$). ■

Example 4 *Consider the following open statements with the set of all real numbers as the universe.*

$$p(x) : |x| > 3, \qquad q(x) : x > 3$$

*Find the truth value of the statement**

$$\forall x, [p(x) \to q(x)]. \tag{i}$$

Also, write down the converse, inverse and the contrapositive of this statement and find their truth values.

▸ We readily note that

$p(-4) \equiv |-4| > 3 \equiv 4 > 3$ is true, and $q(-4) \equiv -4 > 3$ is false.

Thus, $p(x) \to q(x)$ is false for $x = -4$. Accordingly, the given statement (i) is false. (We have got this result by taking $x = -4$ as a counterexample).

The *converse* of statement (i) is

$$\forall x, [q(x) \to p(x)] \tag{ii}$$

In words, this reads

"For every real number x, if $x > 3$ then $|x| > 3$", or equivalently,

"Every real number greater than 3 has its absolute value (magnitude) greater than 3".

This is a true statement.

*By a *statement* we mean here a *quantified statement*. The term "statement" is also used as an equivalent to the term "proposition".

Next, the *inverse* of the statement (i) is

$$\forall x, [\neg\, p(x) \rightarrow \neg\, q(x)] \tag{iii}$$

In words, this reads

"For every real number x, if $|x| \leq 3$, then $x \leq 3$", or, equivalently,

"If the magnitude of a real number x is less than or equal to 3, then the number x is less than or equal to 3.

Since the converse and inverse of a conditional are logically equivalent, the statements (ii) and (iii) have the same truth values. Thus, (iii) is a true statement.

Lastly, the *contrapositive* of the statement (i) is

$$\forall x, [\neg\, q(x) \rightarrow \neg\, p(x)] \tag{iv}$$

In words, this reads

"Every real number which is less than or equal to 3 has its magnitude less than or equal to 3".

Since a conditional and its contrapositive are logically equivalent, statements (i) and (iv) have the same truth value. Since (i) is a false statement, so is its contrapositive (iv). ∎

Example 5 *Consider the following open statements with the set of all real numbers as the universe.*

$$p(x) : x \geq 0, \qquad q(x) : x^2 \geq 0$$
$$r(x) : x^2 - 3x - 4 = 0, \qquad s(x) : x^2 - 3 > 0.$$

Determine the truth values of the following statements.

(i) $\exists x, p(x) \wedge q(x)$ (ii) $\forall x, p(x) \rightarrow q(x)$

(iii) $\forall x, q(x) \rightarrow s(x)$ (iv) $\forall x, r(x) \vee s(x)$

(v) $\exists x, p(x) \wedge r(x)$ (vi) $\forall x, r(x) \rightarrow p(x)$.

► (i) We note that there exists a real number x for which both of $p(x)$ and $q(x)$ are true; for instance $x = 1$. Therefore, $\exists x, p(x) \wedge q(x)$ is a true statement; its truth value is 1.

(ii) We note that, for every real number x, the statement $q(x)$ is true; that is $q(x)$ cannot be false for any real x. Hence $p(x) \rightarrow q(x)$ cannot be false for any real x. Therefore, $\forall x, p(x) \rightarrow q(x)$ is true; its truth value is 1.

(iii) We note that $s(x)$ is false and $q(x)$ is true for $x = 1$. Thus, $q(x) \rightarrow s(x)$ is false for $x = 1$, That is, the statement $q(x) \rightarrow s(x)$ is not always true. Accordingly $\forall x, q(x) \rightarrow s(x)$ is false; its truth value is 0.

(iv) We have $x^2 - 3x - 4 = (x-4)(x+1)$. Hence, $r(x)$ is true only for $x = 4$ or $x = -1$. As such, $r(x)$ and $s(x)$ are false for $x = 1$. Thus, $r(x) \vee s(x)$ is not always true. Accordingly, $\forall x, r(x) \vee s(x)$ is false; its truth value is 0.

(v) We note that, for $x = 4$, both of $p(x)$ and $r(x)$ are true. Therefore, $\exists x, p(x) \wedge r(x)$ is true; its truth value is 1.

(vi) We observe that $p(x)$ is false and $r(x)$ is true for $x = -1$. Hence $r(x) \to p(x)$ is false for $x = -1$. Thus, $r(x) \to p(x)$ is not always true. Accordingly, $\forall x, r(x) \to p(x)$ is false; its truth value is 0. ■

Example 6 *Let* $p(x) : x^2 - 7x + 10 = 0, \quad q(x) : x^2 - 2x - 3 = 0, \quad r(x) : x < 0.$

Determine the truth or falsity of the following statements when the universe U *contains only the integers 2 and 5. If a statement is false, provide a counterexample or explanation.*

(i) $\forall x, p(x) \to \neg r(x)$ (ii) $\forall x, q(x) \to r(x)$

(iii) $\exists x, q(x) \to r(x)$ (iv) $\exists x, p(x) \to r(x)$.

▶ Here, the universe is $\mathsf{U} = \{2, 5\}$.

We note that $x^2 - 7x + 10 \equiv (x-5)(x-2)$. Therefore, $p(x)$ is true for $x = 5$ and 2. That is, $p(x)$ is true for all $x \in \mathsf{U}$.

Further, $x^2 - 2x - 3 \equiv (x-3)(x+1)$. Therefore, $q(x)$ is true only for $x = 3$ and $x = -1$. Since $x = 3$ and $x = -1$ are not in the universe, $q(x)$ is false for all $x \in \mathsf{U}$.

Obviously, $r(x)$ is false for all $x \in \mathsf{U}$

Accordingly:

(i) Since $p(x)$ is true for all $x \in \mathsf{U}$ and $\neg r(x)$ is true for all $x \in \mathsf{U}$, the statement

$\forall x, p(x) \to \neg\, r(x)$ is true.

(ii) Since $q(x)$ is false for all $x \in \mathsf{U}$ and $r(x)$ is false for all $x \in \mathsf{U}$, the statement

$\forall x, q(x) \to r(x)$ is true.

(iii) Since $q(x)$ and $r(x)$ are false for $x = 2$, the statement $\exists x, q(x) \to r(x)$ is true.

(iv) Since $p(x)$ is true for all $x \in \mathsf{U}$ but $r(x)$ is false for all $x \in \mathsf{U}$, the statement $p(x) \to r(x)$ is false for every $x \in \mathsf{U}$. Consequently, $\exists x, p(x) \to r(x)$ is false. ■

Example 7 *Negate and simplify each of the following:*

(i) $\exists x, [p(x) \vee q(x)]$ (ii) $\forall x, [p(x) \wedge \neg q(x)]$

(iii) $\forall x, [p(x) \to q(x)]$ (iv) $\exists x, [\{p(x) \vee q(x)\} \to r(x)]$

► By using the rule of negation for quantified statements and the laws of logic, we find that

(i) $\neg\{\exists x, [p(x) \vee q(x)]\} \equiv \forall x, [\neg\{p(x) \vee q(x)\}]$

$$\equiv \forall x, [\neg p(x) \wedge \neg q(x)]$$

(ii) $\neg\{\forall x, [p(x) \wedge \neg q(x)]\} \equiv \exists x, [\neg\{p(x) \wedge \neg q(x)\}]$

$$\equiv \exists x, [\neg p(x) \vee q(x)]$$

(iii) $\neg\{\forall x, [p(x) \rightarrow q(x)]\} \equiv \exists x, [\neg\{p(x) \rightarrow q(x)\}]$

$$\equiv \exists x, [p(x) \wedge \neg q(x)]$$

(iv) $\neg\{\exists x, [(p(x) \vee q(x)) \rightarrow r(x)]\} \equiv \forall x, [\neg\{(p(x) \vee q(x)) \rightarrow r(x)\}]$

$$\Leftrightarrow \forall x, [\{p(x) \vee q(x)\} \wedge \neg r(x)]$$

Example 8 *Let the set* Z *of all integers be the universe. Consider the statements*

$$p(x) : 2x + 1 = 5 \ \text{ and } \ q(x) : x^2 = 9.$$

Obtain the negation of the quantified statement

$$\exists x \in \mathsf{Z}, [p(x) \wedge q(x)]$$

and express it in words.

► The negation of the given statement is

$$\forall x \in \mathsf{Z}, [\neg p(x) \vee \neg q(x)].$$

In words this reads:

For all integers x, $2x + 1 \neq 5$ or $x^2 \neq 9$.

Example 9 *Let the universe comprise of all integers. Given*

$$p(x) : x \ \text{ is odd}, \quad \text{and } \ q(x) : x^2 - 1 \text{ is even},$$

express the conditional

$$\text{For any } x, \text{ if } x \text{ is odd, then } x^2 - 1 \text{ is even}$$

in symbolic form and negate it.

▶ Let Z denote the set of all integers. Then, in symbolic form, the given conditional reads:

$$\forall x \in \mathsf{Z}, [p(x) \to q(x)].$$

The negation of this is

$$\exists x \in \mathsf{Z}, [p(x) \wedge \neg q(x)]$$

In words this reads:

For some integer x, x is odd and $x^2 - 1$ is not even. ■

Example 10 *Write down the following proposition in symbolic form and find its negation:*
"All integers are rational numbers and some rational numbers are not integers".

▶ Let

$p(x)$: x is a rational number. $q(x)$: x is an integer.

and Z : Set of all integers. Q : Set of all rational numbers.

Then, in symbolic form, the given proposition reads

$$\{\forall x \in \mathsf{Z},\ p(x)\} \wedge \{\exists x \in \mathsf{Q},\ \neg\, q(x)\}$$

The *negation* of this is

$$\neg\,\{\forall x \in \mathsf{Z},\ p(x)\} \vee \neg\{\exists x \in \mathsf{Q},\ \neg\, q(x)\}$$
$$\equiv \{\exists x \in \mathsf{Z},\ \neg\, p(x)\} \vee \{\forall x \in \mathsf{Q},\ q(x)\}$$

In words, this reads:

"Some integers are not rational numbers or every rational number is an integer."
Note that the given proposition is true and the negation is false. ■

Example 11 *Write down the following proposition in symbolic form, and find its negation:*
" If all triangles are right-angled, then no triangle is equiangular."

▶ Let T denote the set of all triangles. Also, let
$p(x)$: x is right-angled, $q(x)$: x is equiangular.
Then, in symbolic form, the given proposition reads

$$\{\forall x \in T, p(x)\} \to \{\forall x \in T,\ \neg\, q(x)\}$$

The *negation* of this is

$$\{\forall x \in T, p(x)\} \wedge \{\exists x \in T,\ q(x)\}$$

In words, this reads "All triangles are right-angled and some triangles are equiangular". ■

Example 12 *Write down the negation of each of the following statements.*

(i) *For all integers n, if n is not divisible by 2, then n is odd.*

(ii) *If k, m, n are any integers where $(k - m)$ and $(m - n)$ are odd, then $(k - n)$ is even.*

(iii) *For all real numbers x, if $|x - 3| < 7$, then $-4 < x < 10$.*

(iv) *If x is a real number where $x^2 > 16$, then $x < -4$ or $x > 4$.*

► Let Z denote the set of all integers and R denote the set of all real numbers. Then:

(i) The given statement is

$$\forall n \in Z, \quad \neg\, p(n) \to q(n)$$

where $p(n)$: n is divisible by 2, $q(n)$: n is odd.

Therefore, the negation of the given statement is

$$\exists n \in Z, \neg p(n) \wedge \neg q(n)$$

In words, this negation reads:

For some integer n, n is not divisible by 2 and n is not odd.

(ii) The given statement is

$$\forall k, m, n \in Z, [p(x) \wedge q(x)] \to r(x),$$

where $p(x)$: $k - m$ is odd, $q(x)$: $m - n$ is odd, $r(x)$: $k - n$ is even.

The negation of this is

$$\exists k, m, n \in Z, [p(x) \wedge q(x)] \wedge \neg r(x).$$

In words, this negation reads:

There exist integers k, m, n such that $k - m, m - n$ are odd and $k - n$ is not even.

(iii) The given statement is

$$\forall x \in R, p(x) \to q(x),$$

where $p(x)$: $|x - 3| < 7$, $q(x)$: $-4 < x < 10$; *i.e.*, $x \in (-4, 10)$.

The negation of this is

$$\exists x \in R, \quad p(x) \wedge \neg\, q(x); \text{ that is}$$

For some real number x, $|x - 3| < 7$ and $x \notin (-4, 10)$.

(iv) The given statement is

$$\forall x \in \mathbb{R}, \quad p(x) \to q(x) \vee r(x)$$

where $p(x) : x^2 > 16$, $q(x) : x < -4$, $r(x) : x > 4$.

The negation of this is

$\exists x \in \mathbb{R}, [p(x) \wedge \neg\, q(x) \wedge \neg\, r(x)]$; that is,

For some real number x, $x^2 > 16$ and $x \geq -4$ and $x \leq 4$. ■

Exercises

1. Let the universe be the set of all integers. Consider the following open statements:

$p(x) : x > 3, \quad q(x) : x + 1$ is even, $\quad r(x) : x \leq 0$.

Write down the truth values of the following:

(i) $p(2)$ (ii) $p(3) \vee \neg\, r(3)$

(iii) $\neg\, p(4) \wedge \neg\, q(5)$ (iv) $\neg\, \{p(2) \vee q(-3)\}$

(v) $\neg\, p(6) \wedge q(-6)$ (vi) $p(4) \to q(0) \wedge r(-1)$

(vii) $p(4) \leftrightarrow [q(1) \wedge r(0)]$ (viii) $[p(1) \wedge q(1)] \to \neg\, r(1)$

2. Write down the following propositions in symbolic form:

(i) Some integers are divisible by 5.

(ii) There exists a matrix whose transpose is itself.

(iii) All real number are complex numbers.

(iv) Every element of a group has an inverse.

(v) No real number is greater than its square.

(vi) At least one parallelogram is a rhombus.

(vii) There is an integer which is not a perfect square.

(viii) Every real number is rational or irrational but not both.

(ix) An equilateral triangle has three angles of 60°, and conversely.

(x) Every rational number is a real number and not every real number is a rational number.

3. Let Z: Set of integers, $p(x)$: x is even, and $q(x)$: x is a prime number, $x \in \mathbb{Z}$. Write down the following propositions in words:

(i) $\forall x,\ p(x)$ (ii) $\exists x,\ q(x)$ (iii) $\forall x,\ \neg\, q(x)$

(iv) $\exists x,\ p(x) \wedge q(x)$ (v) $[\forall x,\ p(x)] \to [\exists x, \neg\, q(x)]$

4. Let $p(x)$ be the open statement "$x^2 = 2x$" and $q(x)$ be the open statement "$x^3 = 4x$" with the set of all integers as the universe. Write down the truth values of the following quantified statements:

(i) $\forall x, p(x)$ (ii) $\exists x, q(x)$ (iii) $\forall x, \neg p(x)$

(iv) $\exists x, \neg q(x)$ (v) $\forall x, p(x) \wedge q(x)$ (vi) $\exists x, p(x) \wedge q(x)$

(vii) $\forall x, p(x) \vee q(x)$ (viii) $\exists x, p(x) \vee q(x)$

5. Find the truth value of each of the quantified statements considered in Worked Example 2. For each false statement, provide a counterexample.

6. Let

$$p(x) : x^2 - 8x + 15 = 0, \quad q(x) : x \text{ is odd}, \quad r(x) : x > 0$$

with the set of all integers as the universe. Determine the truth or falsity of each of the following statements. If a statement is false, give a counterexample:

(i) $\forall x, [p(x) \to q(x)]$ (ii) $\forall x, [q(x) \to p(x)]$

(iii) $\exists x, [p(x) \to q(x)]$ (iv) $\exists x, [q(x) \to p(x)]$

(v) $\exists x, [r(x) \to p(x)]$ (vi) $\forall x, [\neg q(x) \to \neg r(x)]$

(vii) $\exists x, [p(x) \to \{q(x) \wedge r(x)\}]$ (viii) $\forall x, [\{p(x) \vee q(x)\} \to r(x)]$

7. Write down the converse, inverse and contrapositive of each of the following statements for which the set of all real numbers is the universe. Also, indicate their truth values.

(i) $\forall x, [(x > 3) \to (x^2 > 9)]$

(ii) $\forall x, [\{(x^2 + 4x - 21) > 0\} \to \{(x > 3) \vee (x < -7)\}]$

8. Let $p(x) : |x| > 3$, $q(x) : x > 3$ and $r(x) : x < -3$ with the set of all real numbers as the universe. Prove that $\forall x, [p(x) \leftrightarrow \{r(x) \vee q(x)\}]$ is true.

9. Write down the negations of the following:

(i) $\{\forall x, p(x)\} \vee \{\forall x, \neg q(x)\}$ (ii) $\{\exists x, \neg p(x)\} \wedge \{\forall x, q(x)\}$

(iii) $\forall x, p(x) \to q(x)$ (iv) $\exists x, \neg p(x) \to q(x)$

(v) $\{\exists x, p(x)\} \to \{\exists x, \neg q(x)\}$ (vi) $\{\forall x, p(x)\} \to \{\exists x, q(x)\}$

10. Write down the negations of the following:

(i) All even numbers are multiples of 4.

(ii) At least one parallelogram is not a square.

(iii) No real number is greater than its square.

(iv) For all positive integers n, $(n^2 + 41n + 41)$ is a prime number.

(v) There is some integer k for which $12 = 3k$.

(vi) Some straight lines are parallel or all straight lines intersect.

(vii) All rational numbers are real and some real numbers are not rational.

(viii) For all real numbers x and y, if $x^2 > y^2$, then $x > y$.

(ix) There exist real numbers x and y such that x and y are rational but $x + y$ is irrational.

(x) There exist odd integers whose product is odd.

11. Prove the following logical equivalences:

(i) $\forall x, [p(x) \wedge q(x)] \Leftrightarrow \forall x, p(x) \wedge \forall x, q(x)$

(ii) $\exists x, [p(x) \vee q(x)] \Leftrightarrow \exists x, p(x) \vee \exists x, q(x)$

(iii) $\exists x, [p(x) \rightarrow q(x)] \Leftrightarrow \exists x, [\neg\, p(x) \vee q(x)]$

12. Prove the following logical equivalences:

(i) $\neg\, [\forall x, \neg p(x)] \Leftrightarrow \exists x, p(x)$

(ii) $\neg\, [\exists x, \neg\, p(x)] \Leftrightarrow \forall x, p(x)$

(iii) $\exists x, [p(x) \rightarrow q(x)] \Leftrightarrow \forall x, p(x) \rightarrow \exists x, q(x)$

(iv) $[\exists x, p(x) \rightarrow \forall x, q(x)] \Leftrightarrow \forall x, [p(x) \rightarrow q(x)]$

(v) $\forall x, [p(x) \wedge \{q(x) \wedge r(x)\}] \Leftrightarrow \forall x, [\{p(x) \wedge q(x)\} \wedge r(x)]$

13. Prove that the following are tautologies:

(i) $[\{\forall x, p(x)\} \vee \{\forall x, q(x)\}] \rightarrow \forall x, \{p(x) \vee q(x)\}$

(ii) $[\exists x, \{p(x) \wedge q(x)\}] \rightarrow [\{\exists x, p(x)\} \wedge \{\exists x, q(x)\}]$.

Answers

1. (i) False (ii) True (iii) False (iv) False
(v) False (vi) False (vii) True (viii) True

2. (i) $\exists x \in Z, p(x)$. Here, Z: set of all integers, and $p(x)$: x is divisible by 5

(ii) $\exists x \in M, p(x)$. Here, M is the set of all matrices, and $p(x)$: x is the transpose of itself.

(iii) $\forall x \in R, p(x)$. Here, R: set of all real numbers, and $p(x)$: x is a complex number.

(iv) $\forall x \in G, p(x)$. Here, G is a group, and $p(x)$: x has an inverse.

(v) $\forall x \in R, \neg p(x)$. Here, $p(x)$: x is greater than its square.

(vi) $\exists x \in S, p(x)$. Here, S is the set of all parallelograms, and $p(x)$: x is a rhombus.

(vii) $\exists x \in Z, \neg p(x)$. Here, Z is the set of all integers, and $p(x)$: x is a perfect square.

(viii) $\forall x \in R, p(x) \underline{\vee} q(x)$. Here, R is the set of all real numbers, $p(x)$: x is rational, $q(x)$: x is irrational.

(ix) $\forall x \in T, [p(x) \leftrightarrow q(x)]$. Here, T is the set of all triangles, $p(x)$: x is equilateral, $q(x)$: x has three angles of $60°$.

(x) $\{\forall x, [p(x) \rightarrow q(x)]\} \wedge [\neg \{\forall x, [q(x) \rightarrow p(x)]\}]$.
Here, $p(x)$: x is a rational number, and $q(x)$: x is a real number.

3. (i) Every integer is even. (ii) Some integers are prime numbers.

(iii) No integer is a prime number. (iv) Some integers are even and prime.

(v) If every integer is even then some integers are not prime.

4. (i) False (ii) True (iii) False (iv) true

(v) False (vi) True (vii) False (viii) True

5. (i) True (ii) True (iii) True (iv) True

(v) False ($x = 36$) (vi) False ($x = 11$)

6. (i) True (ii) False ($x = 7$) (iii) True

(iv) True (v) True (vi) False ($x = 2$)

(vii) True (viii) False ($x = -1$)

7. (i) converse : $\forall x, [(x^2 > 9) \rightarrow (x > 3)]$; False.

Inverse : $\forall x, [x \leq 3 \rightarrow (x^2 \leq 9)]$; False

contrapositive : $\forall x, [\{(x^2 \leq 9) \rightarrow (x \leq 3)]$; True

(ii) converse : $\forall x, [\{(x > 3) \vee (x < -7)\} \rightarrow (x^2 + 4x - 21) > 0]$; True.

Inverse : $\forall x, [\{(x^2 + 4x - 21) \leq 0\} \rightarrow \{(x \leq 3) \wedge (x \leq -7)\}]$; True

contrapositive : $\forall x, [\{(x \leq 3) \wedge (x \geq -7)\} \rightarrow (x^2 + 4x - 21) \leq 0]$; True

9. (i) $\{\exists x, \neg p(x)\} \wedge \{\exists x, q(x)\}$ (ii) $\{\forall x, p(x)\} \vee \{\exists x, \neg q(x)\}$

(iii) $\exists x, p(x) \wedge \neg q(x)$ (iv) $\forall x, \neg p(x) \wedge \neg q(x)$

(v) $\{\exists x, p(x)\} \wedge \{\forall x, q(x)\}$ (vi) $\{\forall x, p(x)\} \wedge \{\forall x, \neg q(x)\}$

10. (i) Some even numbers are not multiples of 4.

(ii) All parallelograms are squares.

(iii) At least one real number is greater than its square.

(iv) There is at least one positive integer n for which $n^2 + 41n + 41$ is not prime.

(v) For all integers k, $12 \neq 3k$. (or there is no integer for which $12 = 3k$).

(vi) All straight lines are non-parallel and some straight lines do not intersect.

(vii) Some rational numbers are not real or all real numbers are rational.

(viii) There exist real numbers x and y such that $x^2 > y^2$ and $x \leq y$.

(ix) For all real numbers x, y, if x and y are rational then $x + y$ is rational.

(x) The product of any two odd integers is even.

2.2 Logical Implication involving Quantifiers

A quantified statement P is said to logically imply a quantified statement Q if Q is true whenever P is true. Then we write $P \Rightarrow Q$.

Given a set of quantified statements $P_1, P_2, \ldots, P_n$ and Q, we say that Q is a *valid conclusion* from the premises $P_1, P_2, \ldots, P_n$ or that

$$P_1 \wedge P_2 \wedge \ldots \wedge P_n \rightarrow Q$$

is a ***valid argument*** if Q is true whenever each of $P_1, P_2 \ldots P_n$ is true, or equivalently if

$$P_1 \wedge P_2 \wedge \ldots \wedge P_n \Rightarrow Q.$$

The validity of an argument involving quantified statements is analysed on the basis of the Laws of Logic and the Rules of inference discussed in Chapter 1.

Example 1 *Prove the following:*

(i) $\forall x, p(x) \Rightarrow \exists x, p(x)$ (ii) $\forall x, [p(x) \vee q(x)] \Rightarrow \forall x, p(x) \vee \exists x, q(x)$

▶ Let S denote the universe. We find that:

(i)
$$\begin{aligned}\forall x, p(x) &\Rightarrow p(x) \text{ is true for every } x \in S\\ &\Rightarrow p(a) \text{ is true for } x = a \in S\\ &\Rightarrow p(x) \text{ is true for some } x \in S\\ &\Rightarrow \exists x, p(x).\end{aligned}$$

(ii)
$$\begin{aligned}\forall x, [p(x) \vee q(x)] &\Rightarrow p(x) \vee q(x) \text{ is true for every } x \in S\\ &\Rightarrow \{p(x) \text{ is true for every } x \in S\} \vee \{q(x) \text{ is true for every } x \in S\}\\ &\Rightarrow \forall x, p(x) \vee q(a) \text{ is true for } a \in S\\ &\Rightarrow \forall x, p(x) \vee \exists x, q(x)\end{aligned}$$

This proves the required result. ■

Example 2 *Prove that*

$$\exists x, [p(x) \wedge q(x)] \Rightarrow \exists x, p(x) \wedge \exists x, q(x).$$

Is the converse true?

► Let S denote the universe. We find that

$$\begin{aligned} \exists x, [p(x) \wedge q(x)] &\Rightarrow p(a) \wedge q(a), \text{ for some } a \in S \\ &\Rightarrow p(a), \text{ for some } a \in S \quad \text{and} \quad q(a), \text{ for some } a \in S \\ &\Rightarrow \exists x, p(x) \wedge \exists x, q(x). \end{aligned}$$

This proves the required implication.

Next, we observe that $\exists x, p(x) \Rightarrow p(a)$ for some $a \in S$ and $\exists x, q(x) \Rightarrow q(b)$ for some $b \in S$. Therefore,

$$\begin{aligned} \exists x, p(x) \wedge \exists x, q(x) &\Rightarrow p(a) \wedge q(b) \\ &\not\Leftrightarrow p(a) \wedge q(a), \text{ because } b \text{ need not be same as } a \end{aligned}$$

Thus,

$\exists x, [p(x) \wedge q(x)]$ need not be true when $\exists x, p(x) \wedge \exists x, q(x)$ is true. That is,

$$\exists x, p(x) \wedge \exists x, q(x) \not\Rightarrow \exists x, [p(x) \wedge q(x)]$$

Accordingly, the converse of the given implication is not necessarily true.

Note: As a consequence of the result proved above, it follows that

$$\exists x, [p(x) \wedge q(x)] \not\Leftrightarrow \exists x, p(x) \wedge \exists x, q(x)$$ ■

Example 3 *Prove that*

$$[\forall x, p(x) \vee \forall x, q(x)] \Rightarrow \forall x, [p(x) \vee q(x)]$$

Through a counterexample, show that the converse of this is not true.

► Let S denote the universe. Take *any* $a \in S$. Then $\forall x, [p(x) \vee q(x)]$ is true whenever $p(a) \vee q(a)$ is true;

i.e., whenever $p(a)$ is true or $q(a)$ is true,

i.e., whenever $p(x)$ is true for any x or $q(x)$ is true for any x,

i.e., whenever $\forall x, p(x)$ is true or $\forall x, q(x)$ is true,

i.e., whenever $\forall x, p(x) \vee \forall x, q(x)$ is true.

This means that

$$[\forall x, p(x) \vee \forall x, q(x)] \Rightarrow \forall x, [p(x) \vee q(x)]$$

For analysing the converse, let us consider the open statements $p(x): x^2 - 4 = 0$, $q(x): x^2 - 1 = 0$, with $S = \{1, 2\}$.

We find that for $x = 1$, $p(x)$ is false but $q(x)$ is true so that $p(x) \vee q(x)$ is true. For $x = 2$, $p(x)$ is true and $q(x)$ is false so that $p(x) \vee q(x)$ is true. Thus, for every $x \in S$, $p(x) \vee q(x)$ is true. That is, $\forall x, [p(x) \vee q(x)]$ is true.

But $p(x)$ is not true for every $x \in S$. That is, $\forall x, p(x)$ is false. Likewise, $\forall x, q(x)$ is false. Consequently, $[\forall x, p(x) \vee \forall x, q(x)]$ is false.

Thus,

$$\forall x, [p(x) \vee q(x)] \not\Rightarrow [\forall x, p(x) \vee \forall x, q(x)]$$

The converse of the given implication is therefore *not* true.

Remark: As a consequence of the result proved in this Example, we note that

$$\forall x, p(x) \vee \forall x, q(x) \not\Leftrightarrow \forall x, [p(x) \vee q(x)].$$ ■

Example 4 *Prove that the statement $\exists x, q(x)$ follows logically from the premises $\forall x, p(x) \to q(x)$ and $\exists x, p(x)$.*

▶ We note that

$$\exists x, p(x) \Rightarrow p(a) \text{ for some } a \text{ in the universe .}$$

Therefore,

$$\begin{aligned}[\exists x, p(x)] \wedge [\forall x, p(x) \to q(x)] &\Rightarrow p(a) \wedge [p(a) \to q(a)] \\ &\Rightarrow q(a), \text{ (by Modus Pones Rule)} \\ &\Rightarrow \exists x, q(x).\end{aligned}$$

This proves the required result. ■

Example 5 *Prove that the following argument is valid:*

All men are mortal.
Sachin is a man.
$\therefore$ *Sachin is mortal.*

▶ Let S denote the set of all men, $p(x)$ denote the statement "x is mortal", and a denote Sachin. Then, the given argument reads

$$\begin{array}{l} \forall x \in S, p(x) \\ \underline{\quad a \in S \quad} \\ \therefore \quad p(a) \end{array}$$

Since the statement $\forall x \in S, p(x)$ is true, and $a \in S$, it follows by the Rule of Universal Specification that $p(a)$ is true. Thus, the given argument is valid.

Aliter : The validity of the given argument can also be established by using a Venn diagram. Let the universal set U be the set of all mortal things. Then according to premises, $S \subseteq U$ and $a \in S$. Therefore, $a \in U$ (See Figure 2.1). This means that a is mortal. This establishes the validity of the given argument.

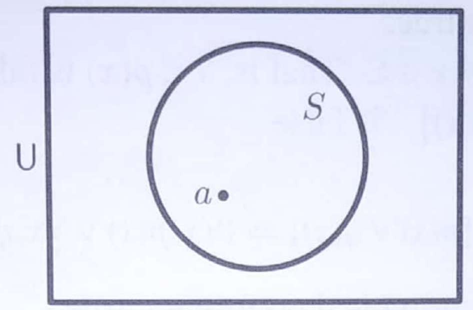

Figure 2.1

■

Example 6 *Find whether the following is a valid argument for which the universe is the set of all students.*

No Engineering student is bad in studies.

Anil is not bad in studies.

$\therefore$ *Anil is an Engineering student.*

► Let

$p(x)$: x is an engineering student, $q(x)$: x is bad in studies, a : Anil.

Then, the given argument reads

$$\begin{array}{l} \forall x, [p(x) \to \neg\, q(x)] \\ \neg\, q(a) \\ \hline \therefore \quad p(a) \end{array}$$

We note that

$$\forall x, [p(x) \to \neg q(x)] \Rightarrow [p(a) \to \neg q(a)], \text{ by the rule of universal specification.}$$

Therefore,

$$\{\forall x, [p(x) \to \neg\, q(x)]\} \wedge \neg\, q(a) \Rightarrow \{p(a) \to \neg\, q(a)\} \wedge \neg\, q(a)$$
$$\not\Rightarrow p(a)$$

because $p(a)$ can be false when both of $p(a) \to \neg\, q(a)$ and $\neg\, q(a)$ are true. As such, the given argument is not valid *.

Remark: The fact that the given argument is not valid can be seen from the following Venn diagram where S is the set of all students, E is the set of all engineering students and B is the set of all students who are bad in studies.

*A non-engineering student can also be not bad in studies!

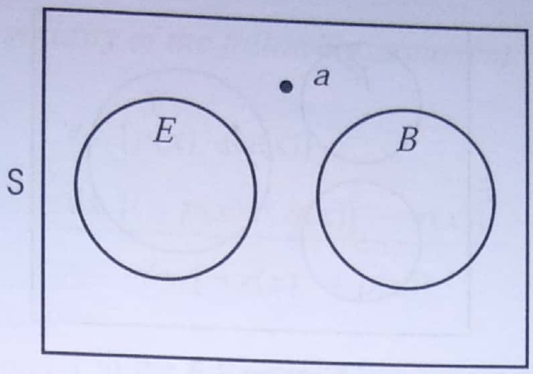

Figure 2.2

Example 7 *Find whether the following argument is valid:*

No engineering student of First or Second Semester studies Logic.
Anil is an engineering student who studies Logic.
$\therefore$ *Anil is not in Second Semester.*

▶ Let us take the universe to be the set of all engineering students, and let
$p(x)$: x is in First Semester, $q(x)$: x is in Second Semester,
$r(x)$: x studies logic, a: Anil.

Then the given argument reads

$$\frac{\begin{array}{c}\forall x, [\{p(x) \vee q(x)\} \to \neg\, r(x)] \\ r(a)\end{array}}{\therefore \quad \neg\, q(a).}$$

We note that

$\{\forall x, [\{p(x) \vee q(x)\} \to \neg\, r(x)]\} \Rightarrow \{p(a) \vee q(a)\} \to \neg\, r(a)$, by the Rule of Universal Specification.

Therefore,

$\{\forall x, [p(x) \vee q(x)] \to \neg\, r(x)\} \wedge r(a)$

$\Rightarrow \{[p(a) \vee q(a)] \to \neg\, r(a)\} \wedge r(a)$

$\Rightarrow r(a) \wedge \{r(a) \to \neg\, [p(a) \vee q(a)]\}$, using commutative law and contrapositive

$\Rightarrow \neg\, [p(a) \vee q(a)]$, by the Modus Pones Rule

$\Rightarrow \neg\, p(a) \wedge \neg\, q(a)$, by DeMorgan law

$\Rightarrow \neg\, q(a)$, by the Rule of conjuctive simplification.

This proves that the given argument is valid.

Illustration : The following Venn diagram illustrates the validity of the given argument. Here, E is the set of all engineering students, F is the set of all First semester students, S is the set of all Second semester students, L is the set of all students who study logic, and a denotes Anil.

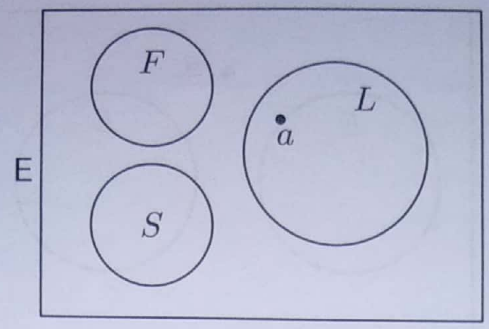

Figure 2.3

Example 8 *Prove that the following argument is valid:*

$$\begin{array}{l} \forall x, [p(x) \to q(x)] \\ \forall x, [q(x) \to r(x)] \\ \hline \therefore \quad \forall x, [p(x) \to r(x)] \end{array}$$

▶ Take *any a* from the universe. Then

$$\begin{aligned} &\{\forall x, [p(x) \to q(x)]\} \wedge \{\forall x, [q(x) \to r(x)]\} \\ &\qquad \Rightarrow \{p(a) \to q(a)\} \wedge \{q(a) \to r(a)\} \\ &\qquad \Rightarrow p(a) \to r(a), \text{ by the Rule of Syllogism} \\ &\qquad \Rightarrow \forall x, [p(x) \to r(x)], \text{by the Rule of Universal Generalization.} \end{aligned}$$

This proves that the given argument is valid. ■

Example 9 *Prove that the following argument is valid.*

$$\begin{array}{l} \forall x, [p(x) \to \{q(x) \wedge r(x)\}] \\ \forall x, [p(x) \wedge s(x)] \\ \hline \therefore \quad \forall x, [r(x) \wedge s(x)]. \end{array}$$

▶ Take *any x* from the universe. Then

$$\begin{aligned} &[p(x) \to \{q(x) \wedge r(x)\}] \wedge [p(x) \wedge s(x)] \\ &\qquad \Leftrightarrow p(x) \wedge [p(x) \to \{q(x) \wedge r(x)\}] \wedge s(x), \\ &\qquad\quad \text{by associative and commutative laws} \\ &\qquad \Rightarrow \{q(x) \wedge r(x)\} \wedge s(x), \text{ by the Modus Pones Rule} \\ &\qquad \Rightarrow \{r(x) \wedge s(x)\}, \text{ by the rule of conjuctive simplification.} \end{aligned}$$

In view of the Rule of Universal generalization, this proves that the given argument is valid. ■

Example 10 *Establish the validity of the following argument:*

$$\begin{array}{l} \forall x, [p(x) \vee q(x)] \\ \underline{\forall x, [\{\neg p(x) \wedge q(x)\} \to r(x)]} \\ \therefore \quad \forall x, [\neg r(x) \to p(x)]. \end{array}$$

▶ Suppose $\neg r(x)$ is true for *any* x in the universe. Then we find that

$$[\{\neg p(x) \wedge q(x)\} \to r(x)]$$

$$\Leftrightarrow \neg r(x) \to \{p(x) \vee \neg q(x)\}, \text{ using contrapositve and DeMorgan law.}$$

$$\Rightarrow p(x) \vee \neg q(x), \text{ by Modus Pones Rule.}$$

Therefore, for any x,

$$[p(x) \vee q(x)] \wedge [\{\neg p(x) \wedge q(x)\} \to r(x)]$$

$$\Rightarrow [p(x) \vee q(x)] \wedge [p(x) \vee \neg q(x)]$$

$$\Leftrightarrow p(x) \vee \{q(x) \wedge \neg q(x)\}, \text{ by distributive law}$$

$$\Leftrightarrow p(x), \quad \text{because } q(x) \wedge \neg q(x) \text{ is false}$$

Thus, when $\neg r(x)$ is true, the given premises imply $p(x)$. Therefore, $\neg r(x) \to p(x)$ is a true statement. In view of the Rule of Universal Generalization, this proves that the given argument is valid. ■

Example 11 *Find whether the following argument is valid:*

If a triangle has two equal sides, then it is isoceles.

If a triangle is isoceles, then it has two equal angles.

A certain triangle ABC does not have two equal angles.

∴ *The triangle ABC does not have two equal sides.*

▶ Let the universe be the set of all triangles, and let

$p(x)$: x has equal sides.

$q(x)$: x is isoceles.

$r(x)$: x has two equal angles.

Also, let c denote the triangle ABC.

Then, in symbols, the given argument reads as follows:

$$\begin{array}{l} \forall x, [p(x) \to q(x)] \\ \forall x, [q(x) \to r(x)] \\ \underline{\neg\, r(c)} \\ \therefore \quad \neg\, p(c) \end{array}$$

We note that

$$\begin{aligned} &\{\forall x, [p(x) \to q(x)]\} \wedge \{\forall x, [q(x) \to r(x)]\} \wedge \neg\, r(c) \\ &\qquad \Rightarrow \{\forall x, [p(x) \to r(x)]\} \wedge \neg\, r(c), \text{ see Example 8 above} \\ &\qquad \Rightarrow \{p(c) \to r(c)\} \wedge \neg\, r(c), \text{ by the Rule of Universal Specification} \\ &\qquad \Rightarrow \neg p(c), \text{ by Modus Tollens Rule.} \end{aligned}$$

This proves that the given argument is valid.

Example 12 *Prove that the following argument is valid:*

$$\begin{array}{l} \forall x, [p(x) \vee q(x)] \\ \exists x, \neg\, p(x) \\ \forall x, [\neg\, q(x) \vee r(x)] \\ \underline{\forall x, [s(x) \to \neg\, r(x)]} \\ \therefore \quad \exists x, \neg\, s(x) \end{array}$$

► We note that

$$\begin{aligned} &\{\forall x, [p(x) \vee q(x)]\} \wedge \{\exists x, \neg\, p(x)\} \\ &\qquad \Rightarrow \{p(a) \vee q(a)\} \wedge \{\neg p(a)\} \text{ for some } a \text{ in the universe.} \\ &\qquad \Rightarrow q(a), \text{ by the rule of disjunctive syllogism.} \end{aligned}$$

Therefore,

$$\begin{aligned} &\{\forall x, [p(x) \vee q(x)]\} \wedge \{\exists x, \neg p(x)\} \wedge \{\forall x, [\neg\, q(x) \vee r(x)]\} \\ &\qquad \Rightarrow q(a) \wedge \{\neg\, q(a) \vee r(a)\} \\ &\qquad \Rightarrow r(a), \text{ by the rule of disjunctive syllogism} \end{aligned}$$

Consequently,

$$\begin{aligned} &\{\forall x, [p(x) \vee q(x)]\} \wedge \{\exists x, \neg\, p(x)\} \wedge \{\forall x, [\neg\, q(x) \vee r(x)]\} \wedge \{\forall x, [s(x) \to \neg\, r(x)]\} \\ &\qquad \Rightarrow r(a) \wedge \{s(a) \to \neg\, r(a)\} \\ &\qquad \Rightarrow \neg\, s(a), \text{ by Modus Tollens Rule.} \\ &\qquad \Rightarrow \exists x, \neg\, s(x) \end{aligned}$$

This proves that the given argument is valid.

Exercises

1. Prove the following:

   (i) $\{\forall x, [p(x) \vee q(x)]\} \wedge \{\forall x, \neg\, p(x)\} \Rightarrow \exists x, q(x)$

   (ii) $\neg \{\forall x, [p(x) \wedge q(x)]\} \wedge [\forall x, p(x)] \Rightarrow \neg \{\forall x, q(x)\}$

   (iii) $\{\exists x, p(x) \rightarrow \forall x, q(x)\} \Rightarrow \forall x, [p(x) \rightarrow q(x)]$

   (iv) $\forall x, [p(x) \rightarrow q(x)] \Rightarrow [\forall x, p(x) \rightarrow \forall x, q(x)]$

2. Prove that the statement $\forall x, [p(x) \rightarrow \neg\, q(x)]$ is a valid conclusion from the statements

$$\exists x, [p(x) \wedge q(x)] \rightarrow \forall y,\ [r(y) \rightarrow s(y)] \text{ and } \exists y, [r(y) \wedge \neg\, s(y)].$$

3. Test the validity of the following arguments:

   (i) Some intelligent boys are lazy.
   Ravi is an intelligent boy.
   $\therefore$ Ravi is lazy.

   (ii) All squares have four sides.
   The qudrilateral $ABCD$ is not a square.
   $\therefore$ $ABCD$ does not have four sides.

   (iii) Every square is a rectangle.
   Every rectangle is a parallelogram.
   $\therefore$ Every square is a parallelogram.

   (iv) Some integers are powers of 3.
   All integers are rational numbers.
   $\therefore$ Some rational numbers are powers of 3.

   (v) No real numbers has negative square.
   All real numbers are complex numbers.
   Some complex numbers have negative squares.
   z is a number whose square is not negative.
   $\therefore$ z is a real number.

   (vi) All Engineering students study Physics.
   All Engineering students of Computer Science study Logic.
   Ravi is an Engineering student who does not study Logic.
   Sachin studies Logic but does not study Physics.
   $\therefore$ Ravi is not a student of Computer Science and Sachin is not an Engineering student.

Answers

3. (i) Not valid (ii) Not valid (iii) Valid (iv) Valid (v) Not valid (vi) Valid

2.3 Statements with more than one variable

Consider the following statements:

(1) $x - 2y$ is a positive integer. (2) $x + y - z = 0$.

These are open statements which contain more than one free variable. These become propositions if each variable is replaced by an element of a certain Universe. For example, if we take the set of all integers as the Universe and replace x and y in the statement (1) by 5 and -3 respectively, then this statement becomes the proposition "$5 - 2(-3)$ is a positive integer" (which is true). Similarly, if we take the set of all rational numbers as the Universe and replace x, y, z in the statement (2) by $1/2$, $1/4$, $1/4$, then the statement becomes the proposition "$1/2 + 1/4 - 1/4 = 0$" (which is false).

Open statements containing two variables x and y are usually denoted by $p(x, y)$, $q(x, y)$ etc and those with three variables x, y, z are denoted by $p(x, y, z)$, $q(x, y, z)$, etc. For an open statement with more than one variable, the Universe can be the same for all variables or can be different for different variables. For example, in the case of the open statement "$x - 2y$ is a positive integer" the set of all integers can be the Universe for both x and y, or the set of all integers can be the Universe for x and the set of all positive integers can be the Universe for y. Given an open statement, the Universe for each variable present therein is always specified before hand.

If U is the universe for x and V is the Universe for y in an open statement $p(x, y)$ and if $a \in$ U and $b \in$ V, then the proposition got by replacing x by a and y by b in $p(x, y)$ is denoted by $p(a, b)$.

Thus, if $p(x, y)$ is the open statement "$x - 2y$ is a positive integer" with Z as the Universe for both x and y, then $p(6, 4)$ is the proposition "$6 - (2 \times 4)$ is a positive integer". Similarly, $p(-4, 2)$ is the proposition "$-4 - (2 \times 2)$ is a positive integer".

Analogous notation is used in the cases where more than two free variables are involved.

Example 1 *Let $p(x, y)$ and $q(x, y)$ denote the following open statements.*

$$p(x, y) : x^2 \geq y, \quad q(x, y) : (x + 2) < y.$$

If the universe for both of x, y is the set of all real numbers, determine the truth value of each of the following statements:

(i) $p(2,4)$ (ii) $q(1,\pi)$

(iii) $p(-3,8) \wedge q(1,3)$ (iv) $p(1/2,1/3) \vee \neg\, q(-2,-3)$

(v) $p(2,2) \leftrightarrow q(1,1)$ (vi) $p(1,2) \leftrightarrow \neg\, q(3,8)$.

▸ We note that

(i) $p(2,4) \equiv 2^2 \geq 4$, which is true.

(ii) $q(1,\pi) \equiv (1+2) < \pi$, which is true.

(iii) $\{p(-3,8) \wedge q(1,3)\} \equiv [(-3)^2 \geq 8] \wedge [(1+2) < 3]$, which is false.

(iv) $\{p(1/2,1/3) \vee \neg\, q(-2,-3)\} \equiv \left[(1/2)^2 \geq (1/3)\right] \vee [(-2+2) \geq -3]$, which is true.

(v) $\{p(2,2) \leftrightarrow q(1,1)\} \equiv (2^2 \geq 2) \leftrightarrow \{(1+2) < 1\}$, which is false.

(vi) $\{p(1,2) \leftrightarrow \neg\, q(3,8)\} \equiv (1^2 \geq 2) \leftrightarrow (3+2 \geq 8)$, which is true. ■

Quantified Statements with more than one variable

When an open statement contains more than one free variable, quantification may be applied to each of the variables. Thus, if $p(x,y)$ is an open statement with variables x, y, we can have quantified statements of the following form:

(1) $\forall x, \forall y, p(x,y)$ (2) $\exists x, \exists y, p(x,y)$

(3) $\forall x, \exists y, p(x,y)$ (4) $\exists x, \forall y, p(x,y)$

In the above statements, x and y can have the same universe or different universes. When x and y have the *same universe*, the statements (1) and (2) are respectively rewritten as

$$(1)' \quad \forall x, y,\; p(x,y), \qquad (2)' \quad \exists x, y,\; p(x,y).$$

From the meaning of the quantifiers, the following results are readily obtained:

$$\forall x, \forall y, p(x,y) \Leftrightarrow \forall y, \forall x, p(x,y).$$
$$\exists x, \exists y, p(x,y) \Leftrightarrow \exists y, \exists x, p(x,y).$$

Let us analyse the quantified statement (3) in some detail. For this purpose, let us consider the open statement

$$p(x,y) : x + y = 1$$

with the set of all integers as the universe.

Then, the statement $\forall x, \exists y, p(x,y)$ reads:
"For every (each) integer x, there exists an integer y such that $x + y = 1$".

This statement carries the same meaning as the statement: "Given any integer x, we can find a corresponding integer y such that $x + y = 1$".

This is a true statement; because once we select *any* x, there does exist $y = 1 - x$ which meets the requirement $x + y = 1$.

On the other hand, the statement $\exists y, \forall x, p(x, y)$ reads:

"For some integer y and for all integers x, we have $x + y = 1$".

This is a false statement; because if this statement were to be true, then every integer x would be equal to $1 - y$ for some (fixed) integer y.

The above example illustrates the following important result:

$$\forall x, \exists y, p(x, y) \not\Leftrightarrow \exists y, \forall x, p(x, y).$$

Similarly,

$$\exists x, \forall y, p(x, y) \not\Leftrightarrow \forall y, \exists x, p(x, y).$$

Quantified statements involving more than two variables can be analyed similarly.

All rules applicable to quantified statements with one variable can be extended in a natural way to those involving more than one variable.

Example 2 *Let x and y denote integers. Consider the statement*

$$p(x, y) : x + y \text{ is even.}$$

Write down the following statements in words:

(i) $\forall x, \exists y, p(x, y)$ (ii) $\exists x, \forall y, p(x, y)$.

► In words, the required statements are

(i) With every integer x, there exists an integer y such that $x + y$ is even.

(ii) There exists an interger x such that $x + y$ is even for every integer y. ∎

Example 3 *Write down the following statements in symbolic form using quantifiers:*

(1) *Every real number has an additive inverse.*

(2) *The set of real numbers has a multiplicative identity.*

(3) *The integer 58 is equal to the sum of two perfect squares.*

► (1) The statement "Every real number has an additude inverse" is the same as:

"Given any real number x, there is a real number y such that $x + y = y + x = 0$".

In symbols, this reads

$$\forall x, \exists y, [x + y = y + x = 0].$$

Here, the set of all real numbers is the universe.

(2) The statement "The set of real numbers has a multiplicative identity" is the same as: "There exists a real number x such that $xy = yx = y$ for every y".

In symbols, this reads

$$\exists x, \forall y, [xy = yx = y].$$

Here, the set of all real numbers is the universe.

(3) The given statement is the same as "There exist integers m and n such that $58 = m^2 + n^2$."

In symbols, this reads

$$\exists m, \exists n, 58 = m^2 + n^2.$$

Here, the set of all integers is the universe. ■

Example 4 *Determine the truth value of each of the following quantified statements, the universe being the set of all non-zero integers.*

(i) $\exists x, \exists y, [xy = 1]$ (ii) $\exists x, \forall y, [xy = 1]$ (iii) $\forall x, \exists y, [xy = 1]$

(iv) $\exists x, \exists y, [(2x + y = 5) \wedge (x - 3y = -8)]$ (v) $\exists x, \exists y, [(3x - y = 17) \wedge (2x + 4y = 3)]$

▶ (i) True. (Take $x = 1, y = 1$).

(ii) False. (For a specified x, $xy = 1$ for every y is not true).

(iii) False. (For $x = 2$, there is no integer y such that $xy = 1$).

(iv) True. (Take $x = 1, y = 3$).

(v) False. (Equations $3x - y = 7$ and $2x + 4y = 3$ do not have a common integer solution). ■

Example 5 *Find the negation of the following quantified statement:*

$$\forall x, \exists y, [\{p(x, y) \wedge q(x, y)\} \rightarrow r(x, y)].$$

▶ By using the rule of negation for quantifiers and other rules, we find

$$\begin{aligned} \neg\, [\forall x, \exists y, \{p(x, y) \wedge q(x, y)\} \rightarrow r(x, y)] & \\ \Leftrightarrow \exists x, \neg\{\exists y, \{p(x, y) \wedge q(x, y)\} \rightarrow r(x, y)\} & \\ \Leftrightarrow \exists x, \forall y, \neg[\{p(x, y) \wedge q(x, y)\} \rightarrow r(x, y)] & \\ \Leftrightarrow \exists x, \forall y, [p(x, y) \wedge q(x, y) \wedge \neg r(x, y)] & \end{aligned}$$

■

Example 6 *Find the negation of each of the following quantified statements:*

(i) $\forall x, \forall y, [(x > y) \to ((x - y) > 0)]$

(ii) $\forall x, \forall y, [(x < y) \to \exists z, (x < z < y)]$

(iii) $\forall x, \forall y, [(|x| = |y|) \to (y = \pm x)]$

(iv) $[\forall x, \forall y, ((x < 0) \wedge (y > 0))] \to [\exists z, (xz > y)]$

▶ By using the rules for negation and other rules, we find that the following are the negations of the given statements:

(i) $\exists x, \exists y, [(x > y) \wedge ((x - y) \leq 0)]$

(ii) $\exists x, \exists y, [(x < y) \wedge \forall z, \{(x \geq z) \vee (z \geq y)\}]$

(iii) $\exists x, \exists y, [(|x| = |y|) \wedge (y \neq \pm x)]$

(iv) $[\forall x, \forall y, ((x < 0) \wedge (y < 0))] \wedge \forall z, \ (xz \leq y)$ ∎

Example 7 *Prove that the following argument is valid, where a and b are some particular members of the universe.*

$$\begin{array}{c} \forall x, \forall y, [p(x, y) \to q(x, y)] \\ \neg\, q(a, b) \\ \hline \therefore\ \exists x, \exists y, \{\neg\, p(x, y)\} \end{array}$$

▶ We note that

$$\begin{aligned} &\{\forall x, \forall y, [p(x, y) \to q(x, y)]\} \wedge \neg\, q(a, b) \\ &\qquad \Rightarrow \{p(a, b) \to q(a, b)\} \wedge \neg\, q(a, b) \\ &\qquad \Rightarrow \neg\, p(a, b), \text{ by the Modus Tollens Rule} \\ &\qquad \Rightarrow \exists x, \exists y, \{\neg p(x, y)\} \end{aligned}$$

This proves that the given argument is valid. ∎

Exercises

1. Identify the bound variables and free variables in the following statements:

(i) $\forall y, \exists z, \sin(x + y) = \cos(z - x)$

(ii) $\exists x, \exists y, \left[(x^2 + y^2) = z^2\right]$

2. Let $p(x, y)$ be the open statement "x divides y" and the set of all integers be the universe. Find whether the following are true or false:

(i) $p(3,7)$ (ii) $p(3,27)$ (iii) $\forall y,\ p(1,y)$ (iv) $\forall x,\ p(x,0)$

(v) $\forall x,\ p(x,x)$ (vi) $\forall y, \exists x,\ p(x,y)$ (vii) $\exists y, \forall x,\ p(x,y)$ (viii) $\forall x, \forall y,\ [p(x,y) \wedge p(y,x) \rightarrow (x = y)]$.

3. Let $p(x, y)$ be the open statement "y is a multiple of x" and the set of all positive integers be the universe. Find whether the following are true or false:

(i) $\forall x, \exists y,\ p(x,y)$ (ii) $\forall y, \exists x,\ p(x,y)$ (iii) $\exists x, \forall y,\ p(x,y)$ (iv) $\exists y, \forall x,\ p(x,y)$.

4. Find the negations of the following quantified statements:

(i) $\forall x, \exists y, [(p(x,y) \wedge q(x,y)) \rightarrow r(x,y)]$

(ii) $\exists x, \forall y, [p(x,y) \rightarrow (q(x,y) \vee r(x,y))]$

(iii) $\exists x, \exists y, [\{p(x,y) \vee q(x,y)\} \wedge \neg\, r(x,y)]$

5. Write down the negations of the following:

(i) $\exists x, \forall y,\ [(x < y) \wedge \{(x - y) \leq 0\}]$

(ii) $\forall x, \forall y,\ \left[\left(x^2 + y^2 = 0\right) \vee (x = 0 \wedge y = 0)\right]$

(iii) $\exists x, \exists y,\ [(x + y < 0) \rightarrow \{(x > 0) \vee (y \leq 0)\}]$

6. Prove the following:

(i) $\forall x, \forall y, p(x,y) \Rightarrow \exists y, \forall x, p(x,y)$ (ii) $\exists y, \forall x, p(x,y) \Rightarrow \forall x, \exists y, p(x,y)$

(iii) $\forall x, \exists y, p(x,y) \Rightarrow \exists y, \exists x, p(x,y)$ (iv) $\forall y, \exists x, p(x,y) \Rightarrow \exists x, \exists y, p(x,y)$

7. Prove that the following is a valid argument:

$$\begin{array}{c} \exists x,\ \exists y,\ p(x,y) \rightarrow q(x,y) \\ \forall x,\ y,\ \neg\, q(x,y) \\ \hline \therefore\ \exists x,\ y,\ [\neg\, p(x,y)] \end{array}$$

Answers

1. (i) x : free; y, z : bound (ii) x, y : bound; z free.

2. (i) False (ii) True (iii) True (iv) True (v) True (vi) True (vii) False (viii) True.

3. (i) True (ii) True (iii) True (iv) False.

4. (i) $\exists x, \forall y, [p(x, y) \wedge q(x, y) \wedge \neg r(x, y)]$

(ii) $\forall x, \exists y, [p(x, y) \wedge \{\neg q(x, y) \wedge \neg r(x, y)\}]$

(iii) $\forall x, \forall y, [\{\neg p(x, y) \wedge \neg q(x, y)\} \vee r(x, y)]$

5. (i) $\forall x, \exists y, [(x \geq y) \vee (x - y) > 0]$

(ii) $\exists x, \exists y, \left[\left(x^2 + y^2 \neq 0\right) \wedge (x \neq 0 \vee y \neq 0)\right]$

(iii) $\forall x, \forall y, [(x + y < 0) \wedge (x \leq 0) \wedge (y > 0)]$

2.4 Methods of Proof and Methods of Disproof

The propositions that commonly appear in mathematical discussions are conditionals of the form $p \to q$, where p and q are simple or compound propositions which may involve quantifiers as well *. Given such a conditional, the process of establishing that the conditional is *true* by using the rules / laws of logic and other known facts constitutes a ***proof*** of the conditional. The process of establishing that a proposition is *false* constitutes a ***disproof***.

Direct Proof

The direct method of proving a conditional $p \to q$ has the following lines of argument:

1. *Hypothesis:* First assume that p is true.
2. *Analysis:* Starting with the hypothesis and employing the rules / laws of logic and other known facts, infer that q is true.
3. *Conclusion:* $p \to q$ is true.

Example 1 *Give a direct proof of the statement:*

"The square of an odd integer is an odd integer".

► Here, the conditional to be proved is:

"If n is an odd integer, then n^2 is an odd integer."

Assume that n is an odd integer (hypothesis). Then, $n = 2k + 1$ for some integer k. Consequently,

$$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1.$$

We observe that the right hand side is not divisible by 2. Therefore, n^2 is not divisible by 2. This means that n^2 is an odd integer (conclusion). ∎

The given statement is thus proved by a direct proof.

*Propositions of major mathematical interest are called *Theorems*. The consequences which follow immediately from a theorem are termed *Corollaries*.

Example 2 *Prove that, for all integers k and l, if k and l are both odd, then $k + l$ is even and kl is odd.*

▶ Take *any* two integers k and l, and assume that both of these are odd (hypotheses).

Then, $k = 2m + 1$, $\quad l = 2n + 1$ for some integers m and n. Therefore,

$$k + l = (2m + 1) + (2n + 1) = 2(m + n + 1)$$

and $$kl = (2m + 1)(2n + 1) = 4mn + 2(m + n) + 1.$$

We observe that $k + l$ is divisible by 2 and kl is not divisible by 2. Therefore, $k + l$ is an even integer and kl is an odd integer.

Since k and l are arbitrary integers, the proof of the given statement (which is a direct proof) is complete (in view of the rule of universal generalization). ■

Indirect Proof

The reader may recall that a conditional $p \to q$ and its contrapositive $\neg q \to \neg p$ are logically equivalent. In some situations, given a conditional $p \to q$, a direct proof of the contrapositive $\neg q \to \neg p$ is easier. On the basis of this proof, we infer that the conditional $p \to q$ is true. This method of proving a conditional is called an *indirect method of proof*.

Example 3 *Let n be an integer. Prove that if n^2 is odd, then n is odd.*

▶ Here, the conditional to be proved is $p \to q$, where

$$p : n^2 \text{ is odd} \qquad \text{and} \qquad q : n \text{ is odd}.$$

We first prove that the contrapositive $\neg q \to \neg p$ is true.

Assume that $\neg q$ is true; that is, assume that n is not an odd integer. Then $n = 2k$, where k is an integer. Consequently, $n^2 = (2k)^2 = 2(2k^2)$, so that n^2 is not odd. That is, p is false, or equivalently, $\neg p$ is true. This proves the contrapositive statement $\neg q \to \neg p$.

This proof of the contrapositive $\neg q \to \neg p$ serves as an indirect proof of the given statement $p \to q$. ■

Example 4 *Give an indirect proof of the statement:*

"The product of two even integers is an even integer".

▶ The given statement is equivalent to the following statement:

"If a and b are even integers, then ab is an even integer".

Thus, the conditional to be proved is $p \to q$, where

p : a and b are even integers, and q : ab is an even integer.

We first prove that the contrapositive $\neg q \to \neg p$ is true.

Assume that $\neg q$ is true. That is, assume that ab is not an even integer. This means that ab is not divisible by 2. Hence a is not divisible by 2 and b is not divisible by 2. That is, a is not an even integer and b is not an even integer. This means that the proposition "a and b are even integers" is false. That is, p is false, or, equivalently, $\neg p$ is true. This proves the contrapositive statement: $\neg q \to \neg p$.

Consequently, it follows that the given statement $p \to q$ is true.

This completes an indirect proof of the given statement. ∎

Example 5 *Provide an indirect proof of the following statement:*

"For all positive real numbers x and y, if the product xy exceeds 25, then $x > 5$ or $y > 5$".

► Let x and y be *any two* positive real numbers. Then the given statement reads $p \to (q \vee r)$, where

$$p : xy > 25, \quad q : x > 5, \quad r : y > 5.$$

The contrapositive of this statement is $(\neg q \wedge \neg r) \to \neg p$. We will prove that this contrapositive is true.

Suppose $(\neg q \wedge \neg r)$ is true. Then $\neg q$ is true and $\neg r$ is true; that is, $x \leq 5$ and $y \leq 5$. This gives $xy \leq 25$, so that $\neg p$ is true. Thus, the contrapositive true.

This proof of the contrapositive serves as an indirect proof of the statement $p \to (q \vee r)$.

By virtue of the Rule of universal generalization, this indirect proof of $p \to (q \vee r)$ establishes the truth of the given statement. ∎

Example 6 *For each of the following statements, provide an indirect proof by stating and proving the contrapositive of the given statement.*

(i) *For all integers k and l, if kl is odd, then both k and l are odd.*

(ii) *For all integers k and l, if $k + l$ is even, then k and l are both even or both odd.*

► (i) The contrapositive of the given statement is: "For all integers k and l, if k is even or l is even, then kl is even.

We now prove this contrapositive.

For any integers k and l, assume that k is even. Then $k = 2m$ for some integer m, and $kl = (2m)l = 2(ml)$ which is evidently even. Similarly, if l is even, then $kl = k(2n) = 2kn$ for some integer n so that kl is even. This proves the contrapositive.

This proof of the contrapositive serves as an indirect proof of the given statement.

(ii) Here, the contrapositive of the given statement is: "For all integers k and l, if one of k and l is odd and the other is even, then $k + l$ is odd".

We now prove this contrapositive.

For any integers k and l, assume that, one of k and l is odd and the other is even. Suppose k is odd and l is even. Then $k = 2m+1$ and $l = 2n$ for some integers m and n. Consequently, $k+l = (2m+1)+2n = 2(m+n)+1$ which is evidently odd. Similarly, if k is even and l is odd, we find that $k+l$ is odd. This proves the contrapositive.

This proof of the contrapositive serves as an indirect proof of the given statement. ■

Example 7 *Let m and n be integers. Prove that $n^2 = m^2$ if and only if $m = n$ or $m = -n$.*

▶ Consider the propositions

$$p : n^2 = m^2, \quad q : m = n, \quad r : m = -n.$$

We have to prove that $p \leftrightarrow (q \vee r)$ is true.

First, assume that $q \vee r$ is true. Then $m = n$ or $m = -n$, so that $m^2 = n^2$; that is, p is true. Thus, $(q \vee r) \to p$ is true.

Next, assume that $\neg (q \vee r)$ is true; that is, $(\neg q) \wedge (\neg r)$ is true. Then q is false and r is false; that is, $m \neq n$ and $m \neq -n$. Then, $m^2 \neq n^2$; that is, $\neg p$ is true. This proves that $\neg (q \vee r) \to \neg p$ is true. Accordingly, the statement $p \to (q \vee r)$ is true.

Thus, we have proved that both of $(q \vee r) \to p$ and $p \to (q \vee r)$ are true. Hence $p \leftrightarrow (q \vee r)$ is true. ■

Proof by Contradiction

The indirect method of proof is equivalent to what is known as the *Proof by Contradiction**. The lines of argument in this method of proof of the statement $p \to q$ are as follows:

1. *Hypothesis:* Assume that $p \to q$ is false. That is, assume that p is *true* and q is *false*.
2. *Analysis:* Starting with the hypothesis that q is false and employing the rules of logic and other known facts, infer that p is false. This contradicts the assumption that p is true.
3. *Conclusion:* Because of the contradiction arrived in the analysis, we infer that $p \to q$ is true.

Example 8 *Provide a proof by contradiction of the following statement:*

For every integer n, if n^2 is odd, then n is odd.

▶ Let n be any integer. Then the given statement reads $p \to q$, where

$p : n^2$ is odd, and $q : n$ is odd.

Assume that $p \to q$ is *false*; that is, assume that p is *true* and q is *false*. Now, q is false means: n is even, so that $n = 2k$ for some integer k. This yields $n^2 = (2k)^2 = 4k^2$ from which it

*The method of proof by contradiction is traditionally called the ***Reductio ad absurdum*** (Reduction to absurdity). The method is based on the ***Rule of Contradiction*** stated in Section 1.3.

is evident that n^2 is even; that is, p is *false*. This contradicts the assumption that p is true. In view of this contradiction, we infer that the given conditional $p \to q$ is true (for any integer n). ∎

Example 9 *Prove the statement:*

"The square of an even integer is an even integer"

by the method of contradiction.

► Here, the statement to be proved can be put in the form $p \to q$, where

$$p : n \text{ is an even integer,} \qquad \text{and} \quad q : n^2 \text{ is an even integer.}$$

Assume that $p \to q$ is *false*; that is assume that p is *true* and q is *false*. Since q is false, $\neg q$ is true; that is, n^2 is not an even integer. Therefore, $n^2 = n \times n$ is not divisible by 2. This implies that n is not divisible by 2. That is, n is not an even integer. This means that p is *false*, which contradicts the assumption that p is true.

In view of this contradiction, we infer that the given proposition $p \to q$ is true. ∎

Example 10 *Prove that if m is an even integer, then m + 7 is an odd integer.*

► Here, the given statement is $p \to q$, where

$$p : m \text{ is even,} \qquad q : m + 7 \text{ is odd}$$

Assume that $p \to q$ is false; that is, assume that p is *true* and q is *false*. Since q is false, $m + 7$ is even. Hence, $m + 7 = 2k$ for some integer k. This yields

$$m = 2k - 7 = (2k - 8) + 1 = 2(k - 4) + 1$$

which shows that m is odd. This means that p is false, which contradicts the assumption that p is true. In view of this contradiction, we infer that the given statement $p \to q$ is true. ∎

Example 11 *Prove that, for all real numbers x and y, if $x + y \geq 100$, then $x \geq 50$ or $y \geq 50$.*

► Take *any* two real numbers x and y. Then the statement to be proved reads $p \to (q \vee r)$ where

$$p \equiv p(x, y) : x + y \geq 100, \quad q \equiv q(x) : x \geq 50, \quad r \equiv r(y) : y \geq 50.$$

Assume that p is true and $q \vee r$ is false. Since $q \vee r$ is false, q is false and r is false. This means that $x < 50$ and $y < 50$. This yields $x + y < 100$. Thus, p is false. This contradicts the assumption that p is true.

Hence, we infer that the given statement $p \to q$ is true. ∎

Example 12 *Prove that there is no rational number whose square is 2.*

► Let Q denote the set of all rational numbers. Then, the statement to be proved is equivalent to the statement $\forall x \in Q, p \to q$, where $p : x$ is a rational number, and $q : x^2 \neq 2$.

Take *any* $x \in Q$ Assume that $p \to q$ is false; that is, assume that p is true and q is false; that is, assume that x is a rational number and $x^2 = 2$. Since x is a rational number, $x = a/b$ for some integers a and b $(\neq 0)$ which have no common factors. Since $x^2 = 2$, this yields

$2 = a^2/b^2$ so that $a^2 = 2b^2$. Thus, a^2 is even. This implies that a is even; this means that $a = 2n$ for some integer n. Consequently, $2b^2 = a^2 = (2n)^2 = 4n^2$, so that b^2 is even and therefore b is even. We now have that a and b are both even and therefore have a common factor 2. This is a contradiction to the assumption that a and b have no common factors. Accordingly, our assumption must be wrong; as such, the conditional $p \to q$ is true. Since $p \to q$ is true for any $x \in \mathbb{Q}$, it follows that $\forall x \in \mathbb{Q}, p \to q$ is a true statement. This completes the proof.

Remark: The result proved in this example is equivalent to the result: $\sqrt{2}$ *is an irrational number.* ■

Example 13 *Give* (i) *a direct proof,* (ii) *an indirect proof, and* (iii) *proof by contradiction, for the following statement:*

"If n is an odd integer, then $n + 9$ is an even integer"

▶ (i) Direct proof : Assume that n is an odd integer. Then $n = 2k + 1$ for some integer k. This gives $n + 9 = (2k + 1) + 9 = 2(k + 5)$ from which it is evident that $n + 9$ is even. This establishes the truth of the given statement by a direct proof.

(ii) Indirect proof : Assume that $n + 9$ is not an even integer. Then $n + 9 = 2k + 1$ for some integer k. This gives $n = (2k + 1) - 9 = 2(k - 4)$, which shows that n is even. Thus, if $n+9$ is not even, then n is not odd. This proves the contrapositive of the given statement. This proof of the contrapositive serves as an indirect proof of the given statement.

(iii) Proof by contradiction : Assume that the given statement is false. That is, assume that n is odd and $n + 9$ is odd. Since $n + 9$ is odd, $n + 9 = 2k + 1$ for some integer k so that $n = (2k + 1) - 9 = 2(k - 4)$ which shows that n is even. This contradicts the assumption that n is odd. Hence the given statement must be true. ■

Proof by Exhaustion

Recall that a proposition of the form "$\forall x \in S, p(x)$" is true if $p(x)$ is true for every (each) x in S. If S consists of only a limited number of elements, we can prove that the statement "$\forall x \in S, p(x)$" is true by considering $p(a)$ for each a in S and verifying that $p(a)$ is true (in each case). Such a method of proof is called the *method of exhaustion**.

Example 14 *Prove that every even integer n with $2 \le n \le 26$ can be written as a sum of at most three perfect squares.*

▶ Let $S = \{2, 4, 6, \ldots 24, 26\}$. We have to prove that the statement: "$\forall x \in S, p(x)$" is true, where $p(x)$: x is a sum of at most three perfect squares.

*It is to be emphasised that this method works only when S has a limited number of elements and the verification of the truthness of $p(a)$ for each $a \in S$ is not hard.

We observe the following:

$$2 = 1^2 + 1^2 \qquad 16 = 4^2$$
$$4 = 2^2 \qquad 18 = 4^2 + 1^2 + 1^2$$
$$6 = 2^2 + 1^2 + 1^2 \qquad 20 = 4^2 + 2^2$$
$$8 = 2^2 + 2^2 \qquad 22 = 3^2 + 3^2 + 2^2$$
$$10 = 3^2 + 1^2 \qquad 24 = 4^2 + 2^2 + 2^2$$
$$12 = 2^2 + 2^2 + 2^2 \qquad 26 = 5^2 + 1^2$$
$$14 = 3^2 + 2^2 + 1^2$$

The above facts verify that each x in S is a sum of at most three perfect squares.

This establishes the required result. ∎

Proof of Existence

It was pointed out that a proposition of the form "$\exists x \in S, p(x)$" is *true* if any *one* element $a \in S$ such that $p(a)$ is *true* is exhibited. Hence, the best way of proving a proposition of the form "$\exists x \in S, p(x)$" is to exhibit the existence of one $a \in S$ such that $p(a)$ is true. This method of proof is called the *Proof of existence.*

Example 15 *Prove that there exists a real number x such that $x^3 + 2x^2 - 5x - 6 = 0$.*

► It is sufficient to exhibit *one* real number x such that $x^3 + 2x^2 - 5x - 6 = 0$. We check that $x = -1$ is one such real number. ∎

Example 16 *Prove that there exist positive integers m and n such that m, n and $m + n$ are all perfect squares.*

► We note that $m = 9$ and $n = 16$ are perfect squares and so is $m + n = 25$. This proves the required result. ∎

Disproof by Contradiction

Suppose we wish to disprove * a conditional $p \to q$. For this purpose, we start with the hypothesis that p is *true* and q is *true*, and end up with a contradiction. In view of the contradiction, we conclude that the conditional $p \to q$ is false. This method of disproving $p \to q$ is called *Disproof by Contradiction.*

Example 17 *Disprove the statement:*

"The sum of two odd integers is an odd integer".

► Here, the proposition to be *disproved* is $p \to q$, where

p : a and b are odd integers. and q : $a + b$ is an odd integer.

*Recall that *disproving* a proposition u means : proving that u is *false*.

Assume that p is true and q is true. Then

$$a = 2k_1 + 1, \; b = 2k_2 + 1 \quad \text{(i)}$$
$$a + b = 2k_3 + 1 \quad \text{(ii)}$$

for some integers k_1, k_2, k_3.

From (i), we get $a + b = 2(k_1 + k_2 + 1)$, which shows that $a + b$ is an even integer. This *contradicts* the assumption (ii).

In view of this contradiction, we infer that $p \to q$ is false. This *disproves* the given statement. ■

Disproof by Counterexample

Recall that a proposition of the form "$\forall x \in S, p(x)$" is *false* if any *one* element $a \in S$ such that $p(a)$ is *false* is exhibited. Hence the best way of *disproving* a proposition involving the universal quantifier is to exhibit just one case where the proposition is false. This method of disproof is called *Disproof by counterexample.*

A particular case where the proposition is false is called a *counterexample*.

Example 18 *Disprove the proposition: The product of any two odd integers is a perfect square.*

▶ We note that $m = 3$ and $n = 5$ are odd integers, but $mn = 15$ is not a perfect square. Thus, the given proposition is disproved, with $m = 3, n = 5$ serving as a counterexample. ■

Example 19 *Prove or disprove that if m and n are positive integers which are perfect squares, then m + n is a perfect square.*

▶ We note that $m = 9$ and $n = 4$ are perfect squares, but $m + n = 14$ is not a perfect square. Therefore, the given statement is not true; it is disproved through the counterexample $m = 9$, $n = 4$. ■

Example 20 *Prove or disprove that the sum of squares of any four non-zero integers is an even integer.*

▶ Here, the proposition is:

"For *any four* non-zero integers a, b, c, d,

$a^2 + b^2 + c^2 + d^2$ is an even integer".

We check that for $a = 1, b = 1, c = 1, d = 2$, the proposition is false. Thus, the given proposition is not a true proposition. The proposition is disproved through the counterexample $a = b = c = 1$ and $d = 2$. ■

Exercises

1. Give a direct proof for each of the following statements:

   (i) If m is an even integer, then $m + 7$ is an odd integer.

   (ii) The square of an even integer is an even integer.

   (iii) The sum of two odd integers is an even integer.

   (iv) For all integers k and l, if k and l are both even, then (i) $k + l$ is even, and (ii) kl is even.

   (v) For all positive integers m and n, if m and n are perfect squares, then mn is also a perfect square.

   (vi) If a, b, c are positive integers such that a divides b and b divides c, then a divides c.

   (vii) If an integer a is such that $a - 2$ is divisible by 3, then $a^2 - 1$ is divisible by 3.

   (viii) The sum of any five consecutive integers is divisible by 5.

2. Give an indirect proof for each of the following statements:

   (i) If m is an even integer, then $m + 7$ is an odd integer.

   (ii) For any real number x, if $x^2 > 0$, then $x \neq 0$.

   (iii) If x and y are integers such that xy is odd, then x and y are both odd.

   (iv) If x is a rational number and y is an irrational number, then $x + y$ is an irrational number.

   (v) If n is the product of two positive integers a and b, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

3. Prove the statements (i) - (v) of the above Exercise by the method of contradiction.

4. Prove the following using the direct method or the indirect method or the method of contradiction:

   (i) The sum of two prime numbers, each larger than 2, is not a prime number.

   (ii) An integer n is even if and only if n^2 is even.

   (iii) For all positive real numbers x and y, if the product xy exceeds 25, then $x > 5$ or $y > 5$.

5. Prove the following statements by the method of exhaustion.

   (i) Every even integer x such that $30 \leq x \leq 58$ can be written as a sum of at most three perfect squares.

   (ii) Every integer x such that $4 \leq x \leq 38$ can be written as a sum of two primes.

6. Disprove the following statements by the method of contradiction:

   (i) The square of an odd integer is an even integer.

   (ii) If the heights of two triangles are equal, then their areas are equal.

   (iii) The roots of a cubic equation with real coefficients can all be complex.

7. Disprove the following statements by providing a counterexample:

   (i) For all integers n, we have $n^2 \neq n$.

   (ii) For all integers m and n, $m^2 = n^2$ if and only if $m = n$.

   (iii) For any real number x, we have $x^3 > x^2$.

   (iv) There is no function which is equal to its derivative.

   (v) If two lines in space do not intersect, then they must be parallel.

Answers

7. (i) $n = 1$. (ii) $m = 2$, $n = -2$ (iii) $x = 1/2$ (iv) $f(x) = e^x$ (v) skew lines.