Numerical Methods-II and Calculus of Variation

Numerical Methods -II

In this method we solve second order differential equation of the form $\frac{d^2y}{dx^2} = f(x, y, y')$ with initial condition $y(x_0) = y_0$, $y'(x_0) = y_0'$ and substitute in (1) then we take $\frac{dy}{dx} = z$ and we get 2 simultaneous differential equation with initial condition $y(x_0) = y_0$, $z(x_0) = z_0$

I. Runge Kutta method

Runge Kutta method to solve (1) is

$k_1 = hf(x_0, y_0, z_0)$

$k_2 = hf(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}, z_0+\frac{l_1}{2})$

$k_3 = hf(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}, z_0+\frac{l_2}{2})$

$k_4 = hf(x_0+h, y_0+k_3, z_0+l_3)$

$l_1 = hg(x_0, y_0, z_0)$

$l_2 = hg(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}, z_0+\frac{l_1}{2})$

$l_3 = hg(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}, z_0+\frac{l_2}{2})$

$l_4 = hg(x_0+h, y_0+k_3, z_0+l_3)$

The value of y at x_1

$$y(x_1) = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$z(x_1) = z_0 + \frac{1}{6}[l_1 + 2l_2 + 2l_3 + l_4]$$

1) Using Runge Kutta method solve $\frac{d^2y}{dx^2} - x\left(\frac{dy}{dx}\right)^2 + y^2 = 0$ with initial conditions $x=0, y=1, y'=0$ for $x=0.2$ with initial conditions

find y $k_1, k_2, k_3, k_4,$
$y(x_1) =$

→ $\frac{d^2y}{dx^2} - x\left(\frac{dy}{dx}\right)^2 + y^2 = 0$ — (*)

Take $\frac{dy}{dx} = z$ — (1)

diff w.r.t 'x',

$\frac{d^2y}{dx^2} = \frac{dz}{dx}$

Sub in *, $\frac{dz}{dx} - xz^2 + y^2 = 0$

$\frac{dz}{dx} = xz^2 - y^2$

$f(x, y, z) = z$, $g(x, y, z) = xz^2 - y^2$

2)

→

$x_0 = 0$, $y_0 = 1$, $z_0 = 0$, $h = 0.2$

$0 - 1^2 = -1$

$k_1 = hf(x_0, y_0, z_0)$
$= 0.2 f(0,1,0)$
$= 0.2 \times 0$
$= 0$

$l_1 = hg(x_0, y_0, z_0)$
$= 0.2\, g(0,1,0)$
$= 0.2 \times -1$
$= -0.2$

$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2})$
$= 0.2 f(0.1, 1, -0.1)$
$= 0.2 \times -0.1$
$= -0.02$

$l_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2})$
$= 0.2 f(0.1, 1, -0.1)$
$= 0.2 \times -0.9990$
$= -0.1998$

$k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2})$
$= 0.2 f(0.1, 0.9900, -0.0999)$
$= 0.2 \times -0.0999$
$= -0.02$

$l_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2})$
$= 0.2 f(0.1, 0.9900, -0.0999)$
$= 0.2 \times -0.9791$
$= -0.1958$

~~$l_4 = hf(x_0 + h$~~

$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3)$
$= 0.2 f(0.2, 0.9800, -0.1958)$
$= 0.2 \times -0.1958$
$= -0.0392$

$\therefore y(0.2) = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$
$= 1 + \frac{1}{6}[0 + 2(-0.02) + 2(-0.02) + -0.0392]$
$= 1 + \frac{1}{6}(-0.1192)$
$= 0.9801$

2) Using Runge Kutta method solve the following differential equation at $x = 0.1$ given $\frac{d^2y}{dx^2} = x^3(y + \frac{dy}{dx})$, (*) $y(0) = 1$, $y'(0) = 0.5$ find y and y' at $x = 0.1$

→ Take $\frac{dy}{dx} = z$ —(1)

diff w.r.t 'x'

$\frac{d^2y}{dx^2} = \frac{dz}{dx}$

Sub in (*) , $\frac{dz}{dx} = x^3(y + z)$

$f(x,y,z) = z \qquad g(x,y,z) = x^2(y+z)$

$x_0 = 0 \quad y_0 = 1 \quad z_0 = 0.5 \quad h = 0.1$

$k_1 = hf(x_0, y_0, z_0)$
$= 0.1\, f(0, 1, 0.5)$
$= 0.1 \times 0.5$
$= \underline{0.05}$

$l_1 = hg(x_0, y_0, z_0)$
$= 0.1\, g(0, 1, 0.5)$
$= 0.1 \times 0$
$= \underline{0}$

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$
$= 0.1\, f(0.05, 1.0250, 0.5)$
$= 0.1 \times 0.5$
$= \underline{0.05}$

$l_2 = hg\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$
$= 0.1\, g(0.05, 1.0250, 0.5)$
$= 0.1 \times 0.0038$
$= \underline{0.0004}$

$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$
$= 0.1\, f(0.05, 1.0250, 0.5002)$
$= 0.1 \times 0.5002$
$= \underline{0.05}$

$l_3 = hg\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$
$= 0.1\, g(0.05, 1.0250, 0.5002)$
$= 0.1 \times 0.0038$
$= \underline{0.0004}$

$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3)$
$= 0.1\, f(0.1, 1.05, 0.5004)$
$= 0.1 \times 0.5004$
$= \underline{0.05}$

$l_4 = hg(x_0 + h, y_0 + k_3, z_0 + l_3)$
$= 0.1\, g(0.1, 1.05, 0.5004)$
$= 0.1 \times 0.0155$
$= \underline{0.0016}$

$y(0.1) = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$
$= 1 + \frac{1}{6}[0.05 + 2(0.05) + 2(0.05) + 0.05]$
$= 1 + \frac{1}{6}(0.3)$
$= \underline{1.05}$

$y'(0.1) = z(0.1) = z_0 + \frac{1}{6}[l_1 + 2l_2 + 2l_3 + l_4]$
$= 0.5 + \frac{1}{6}[0 + 2(0.0004) + 2(0.0004) + 0.00$
$= 0.5 + \frac{1}{6}(0.0032)$
$= \underline{0.5005}$

H.W $\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} - 2xy = 1$, $y(0)=1$, $y'(0)=1$, find y at $x=0.1$

II Milne's method (Corrector and Predictor Method)

$$y_4^{(P)} = y_0 + \frac{4h}{3}\left[2z_1 - z_2 + 2z_3\right]$$

$$z_4^{(P)} = z_0 + \frac{4h}{3}\left[2z_1' - z_2' + 2z_3'\right]$$

$$y_4^{(c)} = y_2 + \frac{h}{3}\left[z_2 + 4z_3 + z_4^{(P)}\right]$$

$$z_4^{(c)} = z_2 + \frac{h}{3}\left[z_2' + 4z_3' + z_4'\right]$$

1) Using Milne's method obtain an approximate solution at the point $x=0.8$ of the problem $\frac{d^2y}{dx^2} = 1 - 2y\frac{dy}{dx}$ given that $y(0)=0$, $y(0.2)=0.02$, $y(0.4)=0.0795$, $y(0.6)=0.1762$, $y'(0)=0$, $y'(0.2)=0.1996$, $y'(0.4)=0.3937$, $y'(0.6)=0.5689$

→ Take $\frac{dy}{dx} = z$ —(1)

diff w.r.t 'x'

$$\frac{d^2y}{dx^2} = \frac{dz}{dx} \text{ sub in } *$$

$$\frac{d^2y}{dx^2} = 1 - 2y\frac{dy}{dx} \quad -*$$

$$\frac{dz}{dx} = 1 - 2yz \quad -(2)$$

x	y	$y'=z$	$z' = 1-2yz$
$x_0=0$	$y_0=0$	$z_0=0$	$z_0'=1$
$x_1=0.2$	$y_1=0.02$	$z_1=0.1996$	$z_1'=0.9920$
$x_2=0.4$	$y_2=0.0795$	$z_2=0.3937$	$z_2'=0.9374$
$x_3=0.6$	$y_3=0.1762$	$z_3=0.5689$	$z_3'=0.7995$

Milne's Predictor Formula

$$y_4^{(P)} = y_0 + \frac{4h}{3}\left[2z_1 - z_2 + 2z_3\right]$$

$$= 0 + \frac{4(0.2)}{3}\left[2(0.1996) - 0.3937 + 2(0.5689)\right]$$

$$= 0.3049$$

$$z_4^{(P)} = z_0 + \frac{4h}{3}\left[2z_1' - z_2' + 2z_3'\right]$$

$$= 0 + \frac{4(0.2)}{3}\left[2(0.9920) - 0.9374 + 2\times 0.7995\right]$$

$$= \frac{4(0.2)}{3}(2.6456)$$

$$= \underline{0.7055}$$

$$y_4^{(c)} = y_2 + \frac{h}{3}\left[z_2 + 4z_3 + z_4^{(P)}\right]$$

$$= 0.0795 + \frac{0.2}{3}\left[0.3937 + 4(0.5689) + 0.7055\right]$$

$$= \underline{0.3045}$$

$$z_4^{(c)} = z_2 + \frac{h}{3}\left[z_2' + 4z_3' + z_4'\right]$$

$$z_4' = 1 - 2y_4^{(c)} z_4^{(P)}$$

$$= 1 - 2(0.3045)(0.7055)$$

$$= \underline{0.5704}$$

$$= 0.3937 + \frac{0.2}{3}\left[0.9374 + 4(0.7995) + 0.5704\right]$$

$$= \underline{0.7074}$$

$$\therefore y(0.8) = \underline{0.3045}$$

2) Using Milne's method obtain an approximate solution at the point $x = 0.4$ of the problem $\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} - 6y = 0$ given $y(0) = 1$, $y'(0) = 0.1$, $y(0.1) = \ldots$, $y'(0.1) = 0.6955$, $y(0.2) = 1.1380$, $y'(0.2) = 1.258$, $y(0.3) = 1.\ldots$, $y'(0.3) = 1.873$. Also evaluate $y'(0.4)$

→ Take $\frac{dy}{dx} = z$ —(1) $\qquad \frac{d^2y}{dx^2} + 3x\frac{dy}{dx} - 6y = 0$ —(*

diff w.r.t x

$$\frac{d^2y}{dx^2} = \frac{dz}{dx}$$

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sub in * $\frac{dz}{dx} + 3xz - 6y = 0$

$$\frac{dz}{dx} = 6y - 3xz \quad -(2)$$

x	y	$y' = z$	$z' = 6y - 3xz$
$x_0 = 0$	$y_0 = 1$	$z_0 = 0.1$	$z_0' = 6$
$x_1 = 0.1$	$y_1 = 1.0399$	$z_1 = 0.6955$	$z_1' = 6.0308$
$x_2 = 0.2$	$y_2 = 1.1380$	$z_2 = 1.258$	$z_2' = 6.0732$
$x_3 = 0.3$	$y_3 = 1.2987$	$z_3 = 1.873$	$z_3' = 6.1065$

Milne's Predictor Method

$$y_4^{(P)} = y_0 + \frac{4h}{3}\left[2z_1 - z_2 + 2z_3\right]$$

$$= 1 + \frac{4(0.1)}{3}\left[2(0.6955) - 1.258 + 2(1.873)\right]$$

$$= \underline{1.5172}$$

$$z_4^{(P)} = z_0 + \frac{4h}{3}\left[2z_1' - z_2' + 2z_3'\right]$$

$$= 0.1 + \frac{4(0.1)}{3}\left[2(6.0308) - 6.0732 + 2(6.1065)\right]$$

$$= \underline{2.5269}$$

Milne's Corrector Formula

$$y_4^{(C)} = y_2 + \frac{h}{3}\left[z_2 + 4z_3 + z_4^{(P)}\right]$$

$$= 1.1380 + \frac{0.1}{3}\left[1.258 + 4(1.873) + 2.5269\right]$$

$$= \underline{1.5139}$$

$$z_4^{(C)} = z_2 + \frac{h}{3}\left[z_2' + 4z_3' + z_4'\right]$$

$$z_4' = 6y_4^{(C)} - 3x_4 z_4^{(P)}$$

$$= 6(1.5139) - 3(0.4)(2.5269)$$

$$= 9.0834 - 3.0323$$

$$= \underline{6.0511}$$

$$z_4^{(C)} = 1.258 + \frac{0.1}{3}\left[6.0732 + 4(6.1065) + 6.0511\right]$$

$$= \overline{2.4763}$$

$\therefore y(0.4) = \underline{1.5139}$ and $y'(0.4) = 2.4763$

sub in * $\frac{dz}{dx} + 3xz - 6y = 0$

$$\frac{dz}{dx} = 6y - 3xz \quad -(2)$$

x	y	$y' = z$	$z' = 6y - 3xz$
$x_0 = 0$	$y_0 = 1$	$z_0 = 0.1$	$z_0' = 6$
$x_1 = 0.1$	$y_1 = 1.0399$	$z_1 = 0.6955$	$z_1' = 6.0308$
$x_2 = 0.2$	$y_2 = 1.1380$	$z_2 = 1.258$	$z_2' = 6.0732$
$x_3 = 0.3$	$y_3 = 1.2987$	$z_3 = 1.873$	$z_3' = 6.1065$

Milne's Predictor Method

$$y_4^{(P)} = y_0 + \frac{4h}{3}\left[2z_1 - z_2 + 2z_3\right]$$

$$= 1 + \frac{4(0.1)}{3}\left[2(0.6955) - 1.258 + 2(1.873)\right]$$

$$= \underline{1.5172}$$

$$z_4^{(P)} = z_0 + \frac{4h}{3}\left[2z_1' - z_2' + 2z_3'\right]$$

$$= 0.1 + \frac{4(0.1)}{3}\left[2(6.0308) - 6.0732 + 2(6.1065)\right]$$

$$= \underline{2.5269}$$

Milne's Corrector Formula

$$y_4^{(C)} = y_2 + \frac{h}{3}\left[z_2 + 4z_3 + z_4^{(P)}\right]$$

$$= 1.1380 + \frac{0.1}{3}\left[1.258 + 4(1.873) + 2.5269\right]$$

$$= \underline{1.5139}$$

$$z_4^{(C)} = z_2 + \frac{h}{3}\left[z_2' + 4z_3' + z_4'\right]$$

$$z_4' = 6y_4^{(C)} - 3x_4 z_4^{(P)}$$

$$= 6(1.5139) - 3(0.4)(2.5269)$$

$$= 9.0834 - 3.0323$$

$$= \underline{6.0511}$$

$$z_4^{(C)} = 1.258 + \frac{0.1}{3}\left[6.0732 + 4(6.1065) + 6.0511\right]$$

$$= \underline{2.4763}$$

$\therefore y(0.4) = \underline{1.5139}$ and $y'(0.4) = \underline{2.4763}$

3) Apply Milne's Method to compute $y(1.4)$

given $2\frac{d^2y}{dx^2} = 4x + \frac{dy}{dx}$ — *

x	1	1.1	1.2	1.3
y	2	2.2156	2.4649	2.7514
y'	2	2.3178	2.6725	3.0657

→ Take $\frac{dy}{dx} = z$ —(1)

diff w.r.t x

$$\frac{d^2y}{dx^2} = \frac{dz}{dx}$$

sub in * $\quad 2\frac{dz}{dx} = 4x + z$

$$\frac{dz}{dx} = \frac{4x+z}{2} \quad -(2)$$

x	y	$y' = z$	$z' = \frac{4x+z}{2} = 2x + \frac{z}{2}$
$x_0 = 1$	$y_0 = 2$	$z_0 = 2$	$z_0' = 3$
$x_1 = 1.1$	$y_1 = 2.2156$	$z_1 = 2.3178$	$z_1' = 3.3589$
$x_2 = 1.2$	$y_2 = 2.4649$	$z_2 = 2.6725$	$z_2' = 3.7363$
$x_3 = 1.3$	$y_3 = 2.7514$	$z_3 = 3.0657$	$z_3' = 4.1329$

Milne's Predictor Formula

$$y_4^{(P)} = y_0 + \frac{4h}{3}[2z_1 - z_2 + 2z_3]$$

$$= 2 + \frac{4(0.1)}{3}[2(2.3178) - 2.6725 + 2(3.0657)]$$

$$= 3.0793$$

$$z_4^{(P)} = z_0 + \frac{4h}{3}[2z_1' - z_2' + 2z_3']$$

$$= 2 + \frac{4(0.1)}{3}[2(3.3589) - 3.7363 + 2(4.1329)]$$

$$= 3.4996$$

Milne's Corrector Formula

$$y_4^{(c)} = y_2 + \frac{h}{3}[z_2 + 4z_3 + z_4^{(P)}]$$

$$= 2.4649 + \frac{0.1}{3}[2.6725 + 4(3.0657) + 3.4996]$$

$$= 3.0794$$

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$\therefore y(1.4) = 3.0794$

Calculus of Variation

It is a powerful technique for the solution of the problems in dynamic of rigid body, optimization of orbits and vibration problems. It concerns with finding maximum or minimum values of definite integrals.

Functional:

Consider the curve $y = y(x)$ where $y(x_1) = y_1$ $y(x_2) = y_2$ such that for the function $\phi(x, y, y')$, $I = \int_{x_1}^{x_2} \phi(x, y, y')\,dx$ —(1) is a stationary value or an extremum. The integral such as (1) which assumes a definite value of the type $y = y(x)$ is called a functional.

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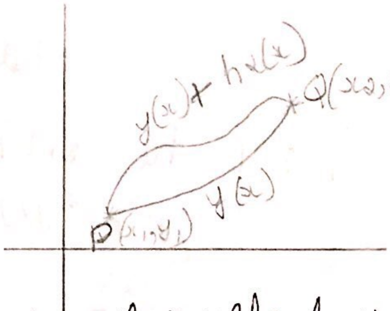
Euler's Theorem

A necessary condition for $I = \int_{x_1}^{x_2} \phi(x, y, y')\,dx$ to be an extremum is that $\frac{\partial \phi}{\partial y} - \frac{d}{dx}\left(\frac{\partial \phi}{\partial y'}\right) = 0$

This is called Euler's equation.



Let I be an extremum along some curve $y = y(x)$ passing through $P(x_1, y_1)$ and $Q(x_2, y_2)$. Let $y = y(x) + h\alpha(x)$ be the neighbouring curve joining these points so that we must have $\alpha(x_1) = 0$ at P and $\alpha(x_2) = 0$ at Q — (1)

when $h = 0$, these curves coincide thus making I extremum. i.e $I = \int_{x_1}^{x_2} \phi(x, y(x) + h\alpha(x), y'(x) + h\alpha'(x))dx$

is an extremum at $h = 0$

This requires $\frac{dI}{dh} = 0$ when $h = 0$, treating I to be function of h.

Using Leibnitz rule for differentiation under integral sign we have

$$\frac{dI}{dh} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial x} \frac{\partial x}{\partial h} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial h} + \frac{\partial f}{\partial y'} \frac{\partial y'}{\partial h} \right) dx$$

but x is independent of h, and $\frac{\partial y'}{\partial h} = \alpha'(x)$

Hence $\frac{\partial x}{\partial h} = 0$. Also $\frac{\partial y}{\partial h} = \alpha(x)$

$$\therefore \frac{dI}{dh} = \int_{x_1}^{x_2} \left[0 + \frac{\partial f}{\partial y} \alpha(x) + \frac{\partial f}{\partial y'} \alpha'(x) \right] dx \quad -(3)$$

Keeping the first term in the RHS of as it is and integrating the second term by parts we get

$$\frac{dI}{dh} = \int_{x_1}^{x_2} \frac{\partial f}{\partial y} \alpha(x) dx + \left[\frac{\partial f}{\partial y} \alpha'(x) \right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \alpha(x) \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) dx$$

$$\frac{dI}{dh} = \int_{x_1}^{x_2} \frac{\partial f}{\partial y} \alpha(x) dx + \left[\frac{\partial f}{\partial y'} [\alpha(x_2) - \alpha(x_1)] \right] - \int_{x_1}^{x_2} \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) dx$$

From (1) $\alpha(x_1) = \alpha(x_2) = 0$

$$= \frac{dI}{dh} = \int_{x_1}^{x_2} \alpha(x) \left[\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) \right] dx$$

From (2) $\frac{dI}{dh} = 0$

$$\Rightarrow \frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) = 0 \quad [\because \alpha(x) \neq 0$$

This is the required Euler's equation

1) Find the extremal of the functional

$$\int_{x_1}^{x_2} [y' + x^2 (y')^2] dx$$

$$\Rightarrow I = \int_{x_1}^{x_2} [y' + x^2 (y')^2] dx$$

By Euler's equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) = 0$$

$$f(x, y, y') = y' + x^2 (y')^2$$

$$\frac{\partial f}{\partial y} = 0$$

$$\frac{\partial f}{\partial y'} = 1 + x^2 \, 2y'$$

$$\therefore\ 0 - \frac{d}{dx}\left(1 + 2x^2 y'\right) = 0$$

$$\frac{d}{dx}\left(1 + 2x^2 y'\right) = 0$$

Integrate w.r.t x

$$1 + 2x^2 y' = C$$

$$2x^2 y' = C - 1$$

$$y' = \frac{C-1}{2x^2}$$

$$y' = \frac{C-1}{2} \cdot \frac{1}{x^2}$$

Integrate w.r.t x

$$y = \frac{C-1}{2} \frac{x^{-1}}{-1} + C_2$$

$$y = C_1 x^{-1} + C_2$$

$$y = \frac{C_1}{x} + C_2$$

2) Find the extremal of the functional

$$\int_{x_1}^{x_2} \left[1 + xy' + x(y')^2\right] dx$$

$\rightarrow$

$$I = \int_{x_1}^{x_2} \left(1 + xy' + x(y')^2\right) dx$$

By Euler's equation $\quad f(x, y, y') = 1 + xy' + x(y')^2$

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

$$\frac{\partial f}{\partial y} = 0$$

$$\frac{\partial f}{\partial y'} = x + 2xy'$$

$$0 - \frac{d}{dx}\left(x + 2xy'\right) = 0$$

$$\frac{d}{dx}\left(x + 2xy'\right) = 0$$

Integrate w.r.t x

$$x + 2xy' = C$$

$$y' = \frac{C - x}{2x}$$

$$= \frac{C}{2x} - \frac{x}{2x}$$

$$\frac{dy}{dx} = \frac{C}{2x} - \frac{1}{2}$$

Integrate w.r.t x

$$y = \frac{c}{2}\log x - \frac{1}{2}x + C_2$$

3) Find the extremal of the functional

$$\int_a^t [y^2 + (y')^2 + 2ye^x]\,dx$$

$\Rightarrow$ $f(x,y,y') = y^2 + (y')^2 + 2ye^x$

$$\frac{\partial f}{\partial y} = 2y + 2e^x$$

$$\frac{\partial f}{\partial y'} = 2y'$$

By Euler's Equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

$$2y + 2e^x - \frac{d}{dx}(2y') = 0$$

$$2y + 2e^x - 2y'' = 0$$

$$y + e^x - y'' = 0$$

$$y'' - y = e^x$$

$$(D^2 - 1)y = e^x$$

A.E $(m^2 - 1) = 0$

$$m^2 = 1$$

$$m = \pm 1$$

$$y_c = C_1e^x + C_2e^{-x}$$

$$y_p = \frac{1}{D^2-1}e^x$$

$$= x \cdot \frac{1}{2D}e^x$$

$$= x\frac{1}{2(1)}e^x = \frac{xe^x}{2}$$

(Type I $\frac{1}{f(D)}e^{ax}$

$\frac{1}{f(a)}e^{ax}$

if $f(a) = 0$ then

$= x\frac{1}{f'(D)}e^{ax}$)

$\therefore$ $y = y_c + y_p$

$$= C_1e^x + C_2e^{-x} + \frac{xe^x}{2}$$

4) Find the extremal of the functional $\int_0^{\pi/2} [y^2 - (y')^2 - 2y\sin x]\,dx$ under the condition

$y(0) = y\left(\frac{\pi}{2}\right) = 0$

$\Rightarrow f(x, y, y') = y^2 - (y')^2 - 2y\sin x$

$\frac{\partial f}{\partial y} = 2y - 2\sin x$

$\frac{\partial f}{\partial y'} = -2y'$

By Euler's Equation

$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$

$(2y - 2\sin x) - \frac{d}{dx}(-2y') = 0$

$2y - 2\sin x + 2y'' = 0$

$y - \sin x + y'' = 0$

$y'' + y = \sin x$

$(D^2+1)\, y = \sin x$

A.E $m^2+1 = 0$

$m^2 = -1$

$m = \pm i$

$y_c = e^{0x}[C_1 \cos x + C_2 \sin x]$

$y_c = C_1 \cos x + C_2 \sin x$

(type II:

$\frac{1}{f(D)} \sin ax, \cos ax$

$D^2 \rightarrow -a^2$

$f(a^2) = 0$

$= x \cdot \frac{1}{f'(D)} \sin ax$

$\frac{1}{D} \sin x = \int \sin x$

$= -\cos x$)

$y_p = \frac{1}{D^2+1} \sin ax$

$= D^2 \rightarrow -1$

$= x \frac{1}{2D} \sin x$

$= x \frac{1}{2}\left(\frac{1}{D} \sin x\right)$

$y_p = \frac{-x}{2} \cos x$

$\therefore y = y_c + y_p$

$y = C_1 \cos x + C_2 \sin x - \frac{x}{2} \cos x$

given: $y(0) = y\left(\frac{\pi}{2}\right) = 0$

$0 = C_1 \cos 0 + C_2 \sin 0 - \frac{0}{2} \cos 0$

$0 = C_1$

$0 = c_1 \cos\frac{\pi}{2} + c_2 \sin\frac{\pi}{2} - \frac{\pi}{4}\cos\frac{\pi}{2}$

$0 = 0 + c_2 - 0$

$c_2 = 0$

$\therefore y = \frac{-x}{2}\cos x$

5) Find the extremal of functional $\int_{x_1}^{x_2} [(y')^2 - y^2 + 2y\sec x]\,dx$

$\rightarrow f(x, y, y') = (y')^2 - y^2 + 2y\sec x$

$\frac{\partial f}{\partial y} = -2y + 2\sec x$

$\frac{\partial f}{\partial y'} = 2y'$.

By Euler's Equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

$$(-2y + 2\sec x) - \frac{d}{dx}(2y') = 0$$

$$-2y + 2\sec x - 2y'' = 0$$

$$-y + \sec x - y'' = 0$$

$$y'' + y = \sec x$$

$$(D^2+1)y = \sec x$$

$$m^2 + 1 = 0$$

$$m^2 = -1$$

$$m = \pm i$$

$$y_c = e^{0x}(c_1\cos x + c_2\sin x)$$

$$y_c = c_1\cos x + c_2\sin x$$

(variation of parame

$y_p = Ay_1 + By_2$

$A = -\int \frac{y_2}{W}\phi(x)$

$B = \int \frac{y_1}{W}\phi(x)dx$

$\left(y_p = \frac{1}{D^2+1}\right)X$

$y_1 = \cos x$ $\quad$ $y_2 = \sin x$

$y_1' = -\sin x$ $\quad$ $y_2' = \cos x$

$W = y_1y_2' - y_2y_1' = \cos x\cos x - \sin x(-\sin x)$ $\quad$ $\therefore W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$= \cos^2 x + \sin^2 x$

$= 1$.

$$A = -\int \frac{y_2}{W}\phi(x)\,dx$$

$$= -\int \frac{\sin x}{1} \sec x\,dx$$

$$= -\int \frac{\sin x}{\cos x}\,dx$$

$$= -\int \tan x\,dx$$

$$= -\log \sec x + C_1$$

$$B = \int \frac{y_1}{W}\phi(x)\,dx$$

$$= \int \frac{\cos x}{1} \sec x\,dx$$

$$= \int \frac{\cos x}{\cos x}\,dx$$

$$= \int dx$$

$$= x + C_2$$

$$y = A y_1 + B y_2$$

$$\therefore y = (-\log \sec x + C_1)\cos x + (x + C_2)\sin x$$

6) Solve the variational $\int_1^2 [x^2(y')^2 + 2y(x+y)]\,dx$

$y(1) = y(2) = 0$

$\rightarrow$ $f(x, y, y') = x^2(y')^2 + 2y(x+y)$

$$\frac{\partial f}{\partial y} = 2x + 4y$$

$$\frac{\partial f}{\partial y'} = 2x^2 y'$$

By Euler's Equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

$$2x + 4y - \frac{d}{dx}(2x^2 y') = 0$$

$$2x + 4y - 2(x^2 y'' + y' 2x) = 0$$

$$2x + 4y - 2x^2 y'' - 4xy' = 0$$

$$2x + 4y - 2x^2 y'' - 2xy' = 0$$

$$x + 2y - x^2 y'' - 2xy' = 0$$

$$x^2 y'' + 2xy' - 2y = x$$

This is Cauchy's D·E

Take $t = \log x \Rightarrow e^t = x$

$x^2y'' = D(D-1)y$

$xy' = Dy$

$(D^2-D)y + 2(Dy) - 2y = e^t$

$D^2y - Dy + 2Dy - 2y = e^t$

$D^2y + Dy - 2y = e^t$

$(D^2+D-2)y = e^t$

A·E $\quad m^2+m-2=0$ $\quad m^2+2m-m-2=0$

$(m^2+m=2$ $\quad m(m+2)-(m+2)=0$

$m(m+1)=2$ $\quad (m-1)=0 \quad m+2=0$

$m=2 \quad m+1=2$ $\quad m=1 \quad m=-2$

$m=1$)x

$$y_c = C_1e^{-2t} + C_2e^t$$

$$y_p = \frac{1}{D^2+D-2}e^t$$

$$= t \cdot \frac{1}{2D+1}e^t$$

$$= \frac{t}{2+1}e^t$$

$$= \frac{te^t}{3}$$

$\therefore y = y_c + y_p$

$$= C_1e^{-2t} + C_2e^t + \frac{te^t}{3}$$

$$y = C_1x^{-2} + C_2x + \frac{(\log x)x}{3}$$

given $y(1)=0$

$$0 = C_1\cdot 1 + C_2\cdot 1 + \frac{1(\log 1)}{3}$$

$$0 = C_1 + C_2 + 0$$

$$C_1 + C_2 = 0$$

$$C_1 = -C_2$$

given $y(2)=0$

$$0 = C_1\frac{1}{2^2} + C_2\cdot 2 + \frac{2(\log 2)}{3}$$

$$0 = \frac{c_1}{4} + 2c_2 + \frac{2(\log 2)}{3}$$

$$-\frac{c_2}{4} + 2c_2 = -\frac{2\log 2}{3}$$

$$\frac{-c_2 + 8c_2}{4} = -\frac{2\log 2}{3}$$

$$\frac{7c_2}{4} = -\frac{2\log 2}{3}$$

$$c_2 = -\frac{2(\log 2)4}{3\times 7}$$

$$c_2 = \frac{-8(\log 2)}{21} = -0.2641$$

$$c_1 = -c_2 = \frac{8\log 2}{21} = +0.2641$$

$$\therefore\ y = \frac{0.2641}{x^2} - 0.2641x + \frac{x\log x}{3}$$

7) S.T the equation of the curve joining the points $(1,0)$ and $(2,1)$ for which $I = \int_1^2 \frac{1}{x}\sqrt{1+(y')^2}$ is an extremum is a circle

$\rightarrow$ $(x-a)^2 + (y-b)^2 = r^2$

$$f(x,y,y') = \frac{1}{x}\sqrt{1+(y')^2}$$

$$\frac{\partial f}{\partial y} = 0$$

$$\frac{\partial f}{\partial y'} = \frac{1}{x}\frac{1}{2\sqrt{1+(y')^2}}\cdot 2y'$$

$$= \frac{y'}{x\sqrt{1+(y')^2}}$$

$$\frac{1}{\sqrt{x}} = \frac{1}{x^{\frac{1}{2}}}$$

$$\frac{d}{dx}x^{-\frac{1}{2}}$$

By Euler's Equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

$$0 - \frac{d}{dx}\left(\frac{y'}{x\sqrt{1+(y')^2}}\right) = 0$$

$$\frac{d}{dx}\left(\frac{y'}{x\sqrt{1+(y')^2}}\right) = 0$$

Integrate w.r.t x,

$$\frac{y'}{x\sqrt{1+(y')^2}} = C_1$$

$$y' = C_1 x\sqrt{1+(y')^2}$$

$$(y')^2 = \left(C_1 x\sqrt{1+y'^2}\right)^2$$

$$(y')^2 = C_1^2 x^2\left[1+(y')^2\right]$$

$$(y')^2 = C_1^2 x^2 + C_1^2 x^2 (y')^2$$

$$(y')^2 - C_1^2 x^2 (y')^2 = C_1^2 x^2$$

$$(y')^2\left[1-C_1^2x^2\right] = C_1^2 x^2$$

$$(y')^2 = \frac{C_1^2 x^2}{1-C_1^2x^2}$$

$$y' = \frac{C_1 x}{\sqrt{1-C_1^2x^2}}$$

Integrate

$$y = \int \frac{C_1 x}{\sqrt{1-C_1^2x^2}}\,dx$$

put $1-C_1^2x^2 = t$

$$-C_1^2\, 2x = \frac{dt}{dx}$$

$$dt = -C_1^2\, 2x\, dx$$

$$-\frac{dt}{2C_1} = C_1 x\, dx$$

$$y = \int \frac{-\frac{dt}{2C_1}}{\sqrt{t}}$$

$$y = -\frac{1}{2C_1}\int \frac{dt}{\sqrt{t}} \qquad \frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}$$

$$y = -\frac{1}{2C_1}\,\frac{t^{\frac{1}{2}}}{\frac{1}{2}}$$

$$y = -\frac{1}{C_1}t^{\frac{1}{2}} + C_2$$

$$y = -\frac{\sqrt{t}}{C_1} + C_2$$

$$y = -\frac{\sqrt{1-C_1^2x^2}}{C_1} + C_2$$

(given $y(1) = 0$)

$$0 = -\frac{1}{c_1}\sqrt{1-c_1^2} + c_2$$ ~~$y(2)=1$~~

Ie

$$c_2 = \frac{1}{c_1}\sqrt{1-c_1^2}$$

$$c_1c_2 = \sqrt{1-c_1^2}$$

$$c_1^2c_2^2 = 1-c_1^2$$

$$c_1^2 = \frac{1-c_1^2}{c_2} \quad /\times$$

$$(y-c_2) = -\frac{1}{c_1}\sqrt{1-c_1^2x^2}$$

$$(y-c_2)^2 = \frac{1}{c_1^2}(1-c_1^2x^2)$$

$$(y-c_2)^2 = \left(\frac{1}{c_1^2} - x^2\right)$$

$$x^2 + (y-c_2)^2 = \frac{1}{c_1^2} \quad -(*)$$

At $(1,0)$

$$1 + (-c_2)^2 = \frac{1}{c_1^2} \quad (*)$$

$$1 + c_2^2 = \frac{1}{c_1^2} \quad -(1)$$

At $(2,1)$

$$4 + (1-c_2)^2 = \frac{1}{c_1^2} \quad -(2)$$

$$1 + c_2^2 = 4 + (1-c_2)^2$$

$$1 + c_2^2 = 4 + 1 + \cancel{c_2^2} - 2c_2$$

$$\cancel{1} + \cancel{c_2^2} = 4 + \cancel{1}$$

$$2c_2 = 4$$

$$c_2 = \frac{4}{2} = 2$$

From (1) $\quad 1 + 2^2 = \frac{1}{c_1^2}$

$$1 + 4 = \frac{1}{c_1^2}$$

$$5 = \frac{1}{c_1^2}$$

$$c_1^2 = \frac{1}{5}$$

$$c_1 = \frac{1}{\sqrt{5}}$$

$\therefore$ The rigid circle is $x^2 + (y-2)^2 = \frac{[illegible]}{5}$

8) Find the curves on which the function

$\int_0^1 [(y')^2 + 12xy]\,dx \quad y(0)=0,\ y(1)=1$ can be extremised

$\rightarrow \quad f = (y')^2 + 12xy$

$\frac{\partial f}{\partial y} = 12x$

$\frac{\partial f}{\partial y'} = 2y'$

By Euler's Formula

$\frac{\partial f}{\partial y} - \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right] = 0$

$12x - \frac{d}{dx}\left[2y'\right] = 0$

$12x - 2y'' = 0$

$12x = 2y''$

$y'' = 6x$

integrate w.r.t x

$y' = \frac{6x^2}{2} + C_1$

integrate w.r.t x

$y = \frac{6x^3}{2*3} + C_1x + C_2$

$y = x^3 + C_1x + C_2$

when $x=0,\ y=0$

$0 = 0 + C_2 \Rightarrow C_2 = 0$

when $x=1,\ y=1$

$1 = 1 + C_1$

$C_1 = 0$

$\therefore y = x^3$

9) $\int_0^{\frac{\pi}{2}} [(y')^2 - y^2 + 2xy]\,dx$ with $y(0) = y(\frac{\pi}{2}) = 0$

$\rightarrow \quad f = (y')^2 - y^2 + 2xy$

$\frac{\partial f}{\partial y} = -2y + 2x$

$\frac{\partial f}{\partial y'} = 2y'$

By Euler's formula

$\frac{\partial f}{\partial y} - \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right] = 0$

$$-2y+2x-\frac{d}{dx}[2y']=0$$

$$-2y+2x-2y''=0$$

$$x=y+y''$$

$$y+y''=x$$

A.E $m^2+1=0$

$$m^2=-1$$

$$m=\pm i$$

$$y_c=C_1\cos x+C_2\sin x$$

$$y_p=\frac{x}{D^2+1}$$

$$1+D^2 \overline{) \; x \;} \; x$$
$$\underline{x+0}$$
$$0$$

$$y_p=x$$

$$y=C_1\cos x+C_2\sin x+x$$

when $x=0, y=0$

$$\boxed{0=C_1}$$

when $x=\frac{\pi}{2}, y=0$

$$0=C_2+\frac{\pi}{2}$$

$$C_2=-\frac{\pi}{2}$$

Geodesics

Geodesics on a surface is a curve along which the distance between any two points of the surface is minimum.

1) S.T The geodesics on a plane are straight line
or P.T the distance b/w 2 points in a plane is along the straight line joining them.

→ Let $y(x)$ be the curve joining the points $P(x_1,y_1)$ and $Q(x_2,y_2)$ in XOY Plane. Let S be the distance arc length of the curve connecting them, then

$$S=\int_{x_1}^{x_2} ds=\int_{x_1}^{x_2}\frac{ds}{dx}dx$$

$$S=\int_{x_1}^{x_2}\sqrt{1+(y')^2}\,dx$$

S will be minimum by Euler's theorem
∴ By Euler's equation

$$\frac{\partial f}{\partial y}-\frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right]=0$$

$$0-\frac{d}{dx}\left[\frac{1}{2\sqrt{1+(y')^2}}\cdot 2y'\right]=0$$

integrate w.r.t x $\quad \frac{y'}{\sqrt{1+(y')^2}}=C_1$

$\sqrt{1+(y')^2}$

Squaring

$$(y')^2 = c_1^2\left(1+(y')^2\right)$$

$$(y')^2 = c_1^2 + c_1^2 (y')^2$$

$$(y')^2 \left(1 - c_1^2\right) = c_1^2$$

$$(y')^2 = \frac{c_1^2}{1-c_1^2}$$

$$y' = \frac{c_1}{\sqrt{1-c_1^2}} \qquad \left[\because \frac{c_1}{\sqrt{1-c_1^2}} = \cos t = m\right]$$

$$y' = m$$

integrate w.r.t x

$y = mx + c$, which is a straight line

Hence proved.

2) P.T catenary is a curve which, when rotated about the line (x-axis) generates a surface of minimum area

NOTE: If f is independent of x The Euler's equation is given

$$f - y'\frac{\partial f}{\partial y'} = \text{const}$$

→ The total surface area is given by $\int_{x_1}^{x_2} 2\pi y \, ds$, where the curve is rotated about the x-axis

$$A = 2\pi \int_{x_1}^{x_2} y \frac{ds}{dx} dx = 2\pi \int_{x_1}^{x_2} y\sqrt{1+(y')^2}\, dx$$

A will have minimum surface area, if it satisfies Euler's equation

$\because f(x, y, y') = y\sqrt{1+(y')^2}$ is independent of x

we have Euler's equation given by

$$f - y'\frac{\partial f}{\partial y'} = \text{const} = c$$

$$y\sqrt{1+(y')^2} - y' \cdot y \cdot \frac{1}{2\sqrt{1+(y')^2}} \cdot 2y' = c$$

$$\frac{y\left[1+(y')^2\right] - y(y')^2}{\sqrt{1+(y')^2}} = c$$

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$$y = C\sqrt{1+(y')^2}$$

Squaring

$$y^2 = C^2\left(1+(y')^2\right)$$