FOURIER TRANSFORMS

3.1 Infinite Fourier transform (Complex Fourier transform) and inverse Fourier transform

The *infinite Fourier transform* or simply the *Fourier transform* of a real valued function $f(x)$ is defined by

$$F[f(x)] = \int_{-\infty}^{\infty} f(x)e^{iux}\,dx \qquad \ldots (1)$$

provided the integral exists. On integration we obtain a function of u which is usually denoted by $F(u)$ or $\hat{f}(u)$.

The inverse Fourier transform of $F(u)$ denoted by $F^{-1}[F(u)]$ or $F^{-1}[\hat{f}(u)]$ is defined by the integral

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)e^{-iux}\,du \qquad \ldots (2)$$

On integration we obtain a function of x. That is

$$f(x) = F^{-1}[F(u)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)e^{-iux}\,du$$

Note : (1) *The definitions are deduced from the Fourier integral*

$$f(x) = \frac{1}{2\pi}\int_{u=-\infty}^{\infty}\int_{t=-\infty}^{\infty} f(t)e^{iu(t-x)}\,dt\,du$$

(2) *In view of the term* e^{iux} *present in the definition of the Fourier transform, it is also called the* ***Complex Fourier transform.***

3.2 Properties of Fourier transform

1. *Linearity property*

If $c_1, c_2, \ldots c_n$ *are constants then*

$$F[c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x)]$$
$$= c_1 F[f_1(x)] + c_2 F[f_2(x)] + \cdots + c_n F[f_n(x)]$$

Proof : By the definition,

$$F[c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x)]$$

$$= \int_{-\infty}^{\infty} [c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x)] e^{iux}\, dx$$

$$= c_1 \int_{-\infty}^{\infty} f_1(x) e^{iux}\, dx + c_2 \int_{-\infty}^{\infty} f_2(x) e^{iux}\, dx + \cdots + c_n \int_{-\infty}^{\infty} f_n(x) e^{iux}\, dx$$

$$= c_1 F[f_1(x)] + c_2 F[f_2(x)] + \cdots + c_n F[f_n(x)]$$

2. *Change of scale property*

If $F[f(x)] = \hat{f}(u)$, *then* $F[f(ax)] = \frac{1}{a}\hat{f}\left(\frac{u}{a}\right)$

Proof : By the definition,

$$F[f(ax)] = \int_{-\infty}^{\infty} f(ax) e^{iux}\, dx$$

Put, $ax = t \quad \therefore\ dx = dt/a$ and t also varies from $-\infty$ to ∞.

Now, $F[f(ax)] = \int_{-\infty}^{\infty} f(t)\, e^{iut/a} \frac{dt}{a}$

ie., $F[f(ax)] = \frac{1}{a}\int_{-\infty}^{\infty} f(t)\, e^{i\frac{u}{a}t}\, dt = \frac{1}{a}\hat{f}\left(\frac{u}{a}\right)$

Thus, $F[f(ax)] = \frac{1}{a}\hat{f}\left(\frac{u}{a}\right)$

3. *Shifting property*

If $F[f(x)] = \hat{f}(u)$ *then* $F[f(x-a)] = e^{iua}\hat{f}(u)$

Proof : By the definition we have,

$$\hat{f}(u) = F[f(x)] = \int_{-\infty}^{\infty} f(x) e^{iux}\, dx$$

Hence, $F[f(x-a)] = \int_{-\infty}^{\infty} f(x-a) e^{iux}\, dx$

Put, $x - a = t \quad \therefore\ dx = dt$, t also varies from $-\infty$ to ∞.

Now, $F[f(x-a)] = \int_{-\infty}^{\infty} f(t)\, e^{iu(t+a)}\, dt = e^{iua} \int_{-\infty}^{\infty} f(t) e^{iut}\, dt = e^{iua} \hat{f}(u)$

Thus, $F[f(x-a)] = e^{iua} \hat{f}(u)$

4. *Modulation property*

If $F[f(x)] = \hat{f}(u)$ *then* $F[f(x)\cos ax] = \frac{1}{2}\left[\hat{f}(u+a) + \hat{f}(u-a)\right]$

Proof : By definition

$$F[f(x)] = \int_{-\infty}^{\infty} f(x) e^{iux}\, dx$$

$$\therefore \quad F[f(x)\cos ax] = \int_{-\infty}^{\infty} f(x) \cos ax\, e^{iux}\, dx$$

ie.,

$$= \int_{-\infty}^{\infty} f(x) \cdot \frac{e^{iax} + e^{-iax}}{2} \cdot e^{iux}\, dx$$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty} f(x) e^{i(u+a)x}\, dx + \int_{-\infty}^{\infty} f(x) e^{i(u-a)x}\, dx\right]$$

$$= \frac{1}{2}\left[\hat{f}(u+a) + \hat{f}(u-a)\right]$$

Thus, $F[f(x)\cos ax] = \frac{1}{2}\left[\hat{f}(u+a) + \hat{f}(u-a)\right]$

3.3 Fourier cosine and Fourier sine transforms
Inverse Fourier cosine and Inverse Fourier sine transforms

If $f(x)$ is defined for all positive values of x, we define the following.

$$F_c[f(x)] = \int_0^{\infty} f(x) \cos ux\, dx = F_c(u) \cdots$$ *Fourier cosine transform*

$$f(x) = \frac{2}{\pi} \int_0^{\infty} F_c(u) \cos ux\, du \cdots$$ *Inverse Fourier cosine transform*

$$F_s[f(x)] = \int_0^{\infty} f(x) \sin ux\, dx = F_s(u) \cdots$$ *Fourier sine transform*

$$f(x) = \frac{2}{\pi}\int_0^\infty F_s(u)\sin ux\,du \cdots \qquad \textit{Inverse Fourier sine transform}$$

Note : *The following properties concerning Fourier cosine and Fourier sine transforms can easily be established as in the case of Fourier transform.*

1. *Linearity Property*

$$F_c[c_1 f_1(x)+c_2 f_2(x)+\cdots+c_n f_n(x)]$$
$$= c_1 F_c[f_1(x)]+c_2 F_c[f_2(x)]+\cdots+c_n F_c[f_n(x)]$$

2. *Change of scale property*

If $F_c[f(x)] = F_c(u)$ then $F_c[f(ax)] = \frac{1}{a}F_c\left(\frac{u}{a}\right)$

These two properties continue to hold good in the case of Fourier sine transform also.

3. *Modulation properties*

If $F_s[f(x)] = F_s(u)$ *and* $F_c[f(x)] = F_c(u)$ *then*

(i) $F_s[f(x)\cos ax] = \frac{1}{2}\left[F_s(u+a)+F_s(u-a)\right]$

(ii) $F_s[f(x)\sin ax] = \frac{1}{2}\left[F_c(u-a)-F_c(u+a)\right]$

(iii) $F_c[f(x)\cos ax] = \frac{1}{2}\left[F_c(u+a)+F_c(u-a)\right]$

(iv) $F_c[f(x)\sin ax] = \frac{1}{2}\left[F_s(u+a)-F_s(u-a)\right]$

Proof :

(i) $F_s[f(x)\cos ax] = \int_0^\infty f(x)\cos ax\cdot\sin ux\,dx$

$$= \int_0^\infty f(x)\cdot\frac{1}{2}\left[\sin(u+a)x+\sin(u-a)x\right]dx$$

$$= \frac{1}{2}\left[\int_0^\infty f(x)\sin(u+a)x\,dx+\int_0^\infty f(x)\sin(u-a)x\,dx\right]$$

$$F_s[f(x)\cos ax] = \frac{1}{2}\left[F_s(u+a)+F_s(u-a)\right]$$

(ii) $F_s[f(x)\sin ax] = \int_0^\infty f(x)\sin ax \cdot \sin ux\, dx$

$$= \int_0^\infty f(x)\cdot\frac{1}{2}[\cos(u-a)x - \cos(u+a)x]dx$$

$$= \frac{1}{2}\left[\int_0^\infty f(x)\cos(u-a)x\,dx - \int_0^\infty f(x)\cos(u+a)x\,dx\right]$$

$$F_s[f(x)\sin ax] = \frac{1}{2}\left[F_c(u-a) - F_c(u+a)\right]$$

(iii) $F_c[f(x)\cos ax] = \int_0^\infty f(x)\cos ax \cdot \cos ux\, dx$

$$= \int_0^\infty f(x)\cdot\frac{1}{2}[\cos(u+a)x + \cos(u-a)x]dx$$

$$= \frac{1}{2}\left[\int_0^\infty f(x)\cos(u+a)x\,dx + \int_0^\infty f(x)\cos(u-a)x\,dx\right]$$

$$F_c[f(x)\cos ax] = \frac{1}{2}\left[F_c(u+a) + F_c(u-a)\right]$$

(iv) $F_c[f(x)\sin ax] = \int_0^\infty f(x)\sin ax \cdot \cos ux\, dx$

$$= \int_0^\infty f(x)\cdot\frac{1}{2}[\sin(a+u)x + \sin(a-u)x]dx$$

$$= \frac{1}{2}\left[\int_0^\infty f(x)\sin(u+a)x\,dx - \int_0^\infty f(x)\sin(u-a)x\,dx\right]$$

$$F_c[f(x)\sin ax] = \frac{1}{2}\left[F_s(u+a) - F_s(u-a)\right]$$

Definitions at a glance - Infinite Fourier transforms

Type	Transform	Inverse Transform
Fourier transform	$\int_{-\infty}^{\infty} f(x)e^{iux}\,dx = F(u)$	$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)\,e^{-iux}\,du = f(x)$
Fourier cosine transform	$\int_{0}^{\infty} f(x)\cos ux\,dx = F_c(u)$	$\frac{2}{\pi}\int_{0}^{\infty} F_c(u)\cos ux\,du = f(x)$
Fourier sine transform	$\int_{0}^{\infty} f(x)\sin ux\,dx = F_s(u)$	$\frac{2}{\pi}\int_{0}^{\infty} F_s(u)\sin ux\,du = f(x)$

Note : Definitions in the alternative/equivalent form

Type	Transform	Inverse transform
Fourier transform	$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{iux}\,dx = F(u)$	$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(u)\,e^{-iux}\,du = f(x)$
Fourier cosine transform	$\sqrt{2/\pi}\int_{0}^{\infty} f(x)\cos ux\,dx = F_c(u)$	$\sqrt{2/\pi}\int_{0}^{\infty} F_c(u)\cos ux\,du = f(x)$
Fourier sine transform	$\sqrt{2/\pi}\int_{0}^{\infty} f(x)\sin ux\,dx = F_s(u)$	$\sqrt{2/\pi}\int_{0}^{\infty} F_s(u)\sin ux\,du = f(x)$

WORKED PROBLEMS

[1] *Find the complex Fourier transform of the function*

$$f(x) = \begin{cases} 1 & for\ |x| \le a \\ 0 & for\ |x| > a \end{cases} \quad \text{Hence evaluate } \int_{0}^{\infty} \frac{\sin x}{x}\,dx$$ **[Dec. 2017, 18]**

☞ Complex Fourier transform of $f(x)$ is given by

$$F(u) = \int_{x=-\infty}^{\infty} f(x)e^{iux}\,dx$$

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$$F(u) = \int_{x=-a}^{a} 1 \cdot e^{iux}\, dx, \quad \text{since} \quad f(x) = \begin{cases} 1 & \text{for } -a \le x \le a \\ 0 & \text{otherwise} \end{cases}$$

$$F(u) = \left[\frac{e^{iux}}{iu}\right]_{x=-a}^{a} = \frac{1}{iu}\left\{ e^{iua} - e^{-iua} \right\}$$

$$F(u) = \frac{1}{iu}\{(\cos au + i\sin au) - (\cos au - i\sin au)\}$$

$$= \frac{1}{iu}(2i\sin au) = \frac{2\sin au}{u}$$

Thus, $\boxed{F(u) = \frac{2\sin au}{u}}$

Let us evaluate $\displaystyle\int_0^\infty \frac{\sin x}{x}\, dx$

We have obtained, $F(u) = \dfrac{2\sin au}{u}$

Inverse Fourier transform is $\displaystyle\frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)e^{-iux}\, du = f(x)$

ie., $$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\sin au}{u} e^{-iux}\, du = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin au}{u} e^{-iux}\, du$$

Now, let us put $x = 0$.
Since $x = 0$ is a point of continuity of $f(x)$, the value of $f(x)$ at $x = 0$ being $f(0) = 1$ because $f(x) = 1$ for $|x| \le a$.

Hence, $\displaystyle\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin au}{u}\, du = 1$, since $e^0 = 1$

ie., $\displaystyle\frac{2}{\pi}\int_0^{\infty} \frac{\sin au}{u}\, du = 1$, since $\dfrac{\sin au}{u}$ is an even function of u.

$$\therefore \quad \int_0^{\infty} \frac{\sin au}{u}\, du = \frac{\pi}{2}$$

Putting $a = 1$, $\int_0^\infty \frac{\sin u}{u} du = \frac{\pi}{2}$

Thus by changing u to x, we have $\boxed{\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}}$

[2] *Find the complex Fourier transform of the function*

$$f(x) = \begin{cases} x, & |x| \le \alpha \\ 0, & |x| > \alpha \end{cases}$$ *where α is a positive constant.*

☞ $F(u) = \int_{x=-\infty}^{\infty} f(x) e^{iux} dx$ and by data $f(x) = x$ for $|x| \le \alpha$.

$$= \int_{-\alpha}^{\alpha} x \cdot e^{iux} dx$$

$$= \left[x \cdot \frac{e^{iux}}{iu} - 1 \cdot \frac{e^{iux}}{i^2 u^2} \right]_{-\alpha}^{\alpha}, \text{ by Bernoulli's rule.}$$

$$= \frac{1}{iu} \left[x e^{iux} \right]_{-\alpha}^{\alpha} - \frac{1}{i^2 u^2} \left[e^{iux} \right]_{-\alpha}^{\alpha}.$$

$$= \frac{-i}{u} \left\{ \alpha e^{iu\alpha} - (-\alpha) e^{-iu\alpha} \right\} + \frac{1}{u^2} \left\{ e^{iu\alpha} - e^{-iu\alpha} \right\}$$

Also, $e^{iu\alpha} = \cos u\alpha + i \sin u\alpha$, $e^{-iu\alpha} = \cos u\alpha - i \sin u\alpha$.

$\therefore \quad e^{iu\alpha} + e^{-iu\alpha} = 2\cos u\alpha, \quad e^{iu\alpha} - e^{-iu\alpha} = 2i \sin u\alpha$

Hence, $F(u) = \frac{-i(2\alpha \cos \alpha u)}{u} + \frac{i(2 \sin \alpha u)}{u^2}$

Thus, $$\boxed{F(u) = 2i \left(\frac{\sin \alpha u}{u^2} - \frac{\alpha \cos \alpha u}{u} \right)}$$

[3] *If* $f(x) = \begin{cases} 1 - x^2, & |x| < 1 \\ 0, & |x| \ge 1 \end{cases}$

find the Fourier transform of $f(x)$ and hence find the value of

(i) $\displaystyle\int_0^\infty \frac{x\cos x - \sin x}{x^3}\,dx$ **(ii)** $\displaystyle\int_0^\infty \frac{x\cos x - \sin x}{x^3}\cos\left(\frac{x}{2}\right)dx$

[June 2017, Dec. 16, 18]

☞ $$F(u) = \int_{x=-\infty}^{\infty} f(x)e^{iux}\,dx = \int_{-1}^{1}(1-x^2)e^{iux}\,dx,$$

$\because$ $f(x) = 0$ for $|x| \geq 1$ and $1-x^2$ for $|x|<1$.

$\therefore$ $F(u) = \left[(1-x^2)\dfrac{e^{iux}}{iu} - (-2x)\dfrac{e^{iux}}{i^2u^2} + (-2)\dfrac{e^{iux}}{i^3u^3}\right]_{x=-1}^{1}$, by Bernoulli's rule.

$$= \frac{-i}{u}\left[(1-x^2)e^{iux}\right]_{x=-1}^{1} - \frac{2}{u^2}\left[xe^{iux}\right]_{x=-1}^{1} - \frac{2i}{u^3}\left[e^{iux}\right]_{x=-1}^{1}$$

$$\left(i^2 = -1,\ \frac{1}{i} = -i,\ \frac{1}{i^3} = i\right)$$

$$F(u) = \frac{-i}{u}(0-0) - \frac{2}{u^2}\left\{1\cdot e^{iu} - (-1)e^{-iu}\right\} - \frac{2i}{u^3}(e^{iu} - e^{-iu})$$

$$= -\frac{2}{u^2}(e^{iu} + e^{-iu}) - \frac{2i}{u^3}(e^{iu} - e^{-iu})$$

But, $e^{iu} = \cos u + i\sin u,\ e^{-iu} = \cos u - i\sin u$

$\therefore$ $e^{iu} + e^{-iu} = 2\cos u,\ e^{iu} - e^{-iu} = 2i\sin u$

Hence, $F(u) = \dfrac{-4\cos u}{u^2} + \dfrac{4\sin u}{u^3}$

Thus, $\boxed{F(u) = 4\left(\dfrac{\sin u - u\cos u}{u^3}\right)}$

Let us evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{x\cos x - \sin x}{x^3}\,dx$

By inverse Fourier transform, we have,

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)e^{-iux}\,du \qquad \ldots (1)$$

If $x = 0 : f(x) = 1 - 0^2 = 1$ at $x = 0$. By putting $x = 0$ in the integral and using the expression of $F(u)$ we get

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} 4\left(\frac{\sin u - u\cos u}{u^3}\right)e^0\,du = f(0) = 1.$$

$$\int_{-\infty}^{\infty}\frac{\sin u - u\cos u}{u^3}\,du = \frac{2\pi}{4} = \frac{\pi}{2}$$

If u is changed to $-u$, the expression $\dfrac{\sin u - u\cos u}{u^3}$ becomes

$\dfrac{\sin(-u)-(-u)\cos(-u)}{(-u)^3} = \dfrac{\sin u - u\cos u}{u^3}$ itself. Therefore the function is even and hence the integral from $-\infty$ to ∞ is twice the integral from 0 to ∞.

$$\therefore \quad 2\int_0^{\infty}\frac{\sin u - u\cos u}{u^3}\,du = \frac{\pi}{2} \quad \text{or} \quad \int_0^{\infty}\frac{u\cos u - \sin u}{u^3}\,du = -\frac{\pi}{4}$$

Changing u to x we get $\boxed{\displaystyle\int_0^{\infty}\frac{x\cos x - \sin x}{x^3}\,dx = -\frac{\pi}{4}}$

Next, let us evaluate $\displaystyle\int_0^{\infty}\frac{x\cos x - \sin x}{x^3}\cos\left(\frac{x}{2}\right)dx$

Putting $x = 1/2$ in (1) we have,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} 4\left(\frac{\sin u - u\cos u}{u^3}\right)e^{\frac{-iu}{2}}\,du = f\left(\frac{1}{2}\right) = 1-\left(\frac{1}{2}\right)^2 = \frac{3}{4}$$

ie., $$\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{\sin u - u\cos u}{u^3}\left(\cos\frac{u}{2} - i\sin\frac{u}{2}\right)du = \frac{3}{4}$$

Equating the real parts on both sides we get,

$$\int_{-\infty}^{\infty}\frac{\sin u - u\cos u}{u^3}\cos\frac{u}{2}\,du = \frac{3\pi}{8}$$

ie., $$2\int_0^{\infty}\frac{\sin u - u\cos u}{u^3}\cos\frac{u}{2}\,du = \frac{3\pi}{8},$$ since the integrand is even.

Dividing by 2, changing the sign and writing x in place of u we get,

$$\boxed{\int_0^\infty \frac{x\cos x - \sin x}{x^3}\cos\frac{x}{2}\,dx = \frac{-3\pi}{16}}$$

[4] *Find the Fourier transform of $f(x) = e^{-|x|}$* **[June , Dec. 2018]**

☞ Fourier transform of $f(x)$ is given by $F(u) = \int_{-\infty}^{\infty} f(x)e^{iux}\,dx$

Here, $f(x) = e^{-|x|} = \begin{cases} e^{-x} & \text{for } x \geq 0 \\ e^{x} & \text{for } x < 0 \end{cases}$

$\therefore$ $$F(u) = \int_{-\infty}^{0} e^{x}\,e^{iux}\,dx + \int_{0}^{\infty} e^{-x}\,e^{iux}\,dx$$

$$F(u) = \int_{-\infty}^{0} e^{(1+iu)x}\,dx + \int_{0}^{\infty} e^{-(1-iu)x}\,dx$$

$$= \left[\frac{e^{(1+iu)x}}{1+iu}\right]_{x=-\infty}^{0} + \left[\frac{e^{-(1-iu)x}}{-(1-iu)}\right]_{x=0}^{\infty}$$

$$= \left[\frac{1}{1+iu} - 0\right] + \left[0 - \frac{1}{-(1-iu)}\right]$$

$$= \frac{1}{1+iu} + \frac{1}{1-iu} = \frac{2}{1-i^2u^2} = \frac{2}{1+u^2}$$

Thus, $$\boxed{F(u) = \frac{2}{1+u^2}}$$

[5] *Find the Fourier transform of*

$$f(x) = \begin{cases} 1-|x| & \textit{for } |x| \leq 1 \\ 0 & \textit{for } |x| > 1 \end{cases}$$ *and hence deduce that* $$\int_0^\infty \frac{\sin^2 t}{t^2}\,dt = \frac{\pi}{2}$$

[June 2018]

☞ $$F[f(x)] = \int_{-\infty}^{\infty} f(x)e^{iux}\,dx = \int_{-1}^{1} \{1-|x|\}\,e^{iux}\,dx$$

$$F[f(x)] = \int_{-1}^{0}[1-(-x)]e^{iux}\,dx + \int_{0}^{1}[1-(+x)]e^{iux}\,dx$$

$$= \int_{-1}^{0}(1+x)e^{iux}\,dx + \int_{0}^{1}(1-x)e^{iux}\,dx$$

$$F[f(x)] = \left[(1+x)\frac{e^{iux}}{iu} - (1)\frac{e^{iux}}{(iu)^2}\right]_{-1}^{0} + \left[(1-x)\frac{e^{iux}}{iu} - (-1)\frac{e^{iux}}{(iu)^2}\right]_{0}^{1}$$

$$= \frac{1}{iu}(1-0) + \frac{1}{u^2}(1-e^{-iu}) + \frac{1}{iu}(0-1) - \frac{1}{u^2}(e^{iu}-1)$$

$$= \frac{2}{u^2} - \frac{1}{u^2}(e^{iu}+e^{-iu}) = \frac{2}{u^2} - \frac{2\cos u}{u^2} = \frac{2(1-\cos u)}{u^2}$$

Thus, $$\boxed{F[f(x)] = \frac{4\sin^2(u/2)}{u^2} = F(u)}$$

Now, $f(x) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} F(u)e^{-iux}\,du$, being the inverse Fourier transform.

ie., $$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{4\sin^2(u/2)}{u^2}\,e^{-iux}\,du$$

ie., $$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\sin^2(u/2)}{(u/2)^2}\,e^{-iux}\,du$$

Putting $x = 0$ we have $f(0) = 1$ by the definition of $f(x)$.

$$\therefore \quad 1 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\sin^2(u/2)}{(u/2)^2}\,du$$

Put $u/2 = t \therefore du = 2dt$ and t also varies from $-\infty$ to ∞.

Hence, $$1 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\sin^2 t}{t^2}\cdot 2\,dt = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin^2 t}{t^2}\,dt$$

or $1 = \dfrac{1}{\pi}\cdot 2\displaystyle\int_{0}^{\infty}\frac{\sin^2 t}{t^2}\,dt$, since the integrand is even.

Thus, $$\boxed{\int_{0}^{\infty}\frac{\sin^2 t}{t^2}\,dt = \frac{\pi}{2}}$$

[6] *Find the Fourier transform of*

$$f(x)=\begin{cases} x^2, & |x|<a \\ 0, & |x|>a \end{cases}$$ *where 'a' is a positive constant.*

☞ Fourier transform of $f(x)$ is given by $F(u)=\int_{-\infty}^{\infty} f(x)e^{iux}\,dx$

Since $f(x)=x^2$ for $|x|<a$ by data we have,

$$F(u)=\int_{-a}^{a} x^2 e^{iux}\,dx.$$

$$=\left[(x^2)\frac{e^{iux}}{iu}-(2x)\frac{e^{iux}}{i^2u^2}+(2)\frac{e^{iux}}{i^3u^3}\right]_{-a}^{a}$$

$$=\frac{1}{iu}(a^2e^{iua}-a^2e^{-iua})+\frac{2}{u^2}(a\,e^{iua}+a\,e^{-iua})-\frac{2}{iu^3}(e^{iua}-e^{-iua})$$

$$=\frac{a^2}{iu}(2i\sin au)+\frac{2a}{u^2}(2\cos au)-\frac{2}{iu^3}(2i\sin au)$$

$$=\frac{2a^2\sin au}{u}+\frac{4a\cos au}{u^2}-\frac{4\sin au}{u^3}$$

Thus, $$\boxed{F(u)=\frac{1}{u^3}\left[2(a^2u^2-2)\sin au+4au\cos au\right]}$$

[7] *Find the Fourier transform of* $f(x)=x\,e^{-|x|}$

☞ We have, $F(u)=F[f(x)]=\int_{-\infty}^{\infty} f(x)e^{iux}\,dx$

$$f(x)=x\,e^{-|x|}=\begin{cases} x\,e^{x} & \text{for } x<0 \\ x\,e^{-x} & \text{for } x>0 \end{cases}$$

$$F(u)=\int_{-\infty}^{0} f(x)e^{iux}\,dx+\int_{0}^{\infty} f(x)e^{iux}\,dx$$

$$F(u) = \int_{-\infty}^{0} x\, e^{x}\, e^{iux}\, dx + \int_{0}^{\infty} x\, e^{-x}\, e^{iux}\, dx$$

$$F(u) = \int_{-\infty}^{0} x\, e^{(1+iu)x}\, dx + \int_{0}^{\infty} x\, e^{-(1-iu)x}\, dx$$

Applying Bernoulli's rule to each of the integrals,

$$F(u) = \left[(x)\frac{e^{(1+iu)x}}{(1+iu)} - (1)\frac{e^{(1+iu)x}}{(1+iu)^2}\right]_{-\infty}^{0} + \left[(x)\frac{e^{-(1-iu)x}}{-(1-iu)} - (1)\frac{e^{-(1-iu)x}}{(1-iu)^2}\right]_{0}^{\infty}$$

The first and third terms vanish.

$$F(u) = \frac{-1}{(1+iu)^2}(1-0) - \frac{1}{(1-iu)^2}(0-1)$$

$$= \frac{1}{(1-iu)^2} - \frac{1}{(1+iu)^2} = \frac{(1+iu)^2 - (1-iu)^2}{(1+u^2)^2}$$

Thus, $$\boxed{F(u) = \frac{4iu}{(1+u^2)^2}}$$

[8] *Find the Fourier transform of*

$$f(x) = \begin{cases} 1 + (x/a), & -a < x < 0 \\ 1 - (x/a), & 0 < x < a \\ 0 & \text{otherwise} \end{cases}$$

☞ $$F(u) = F[f(x)] = \int_{-\infty}^{\infty} f(x)\, e^{iux}\, dx$$

ie., $$F(u) = \int_{-a}^{a} f(x)\, e^{iux}\, dx = \int_{-a}^{0} f(x)\, e^{iux}\, dx + \int_{0}^{a} f(x)\, e^{iux}\, dx$$

$$F(u) = \int_{-a}^{0}\left(1 + \frac{x}{a}\right) e^{iux}\, dx + \int_{0}^{a}\left(1 - \frac{x}{a}\right) e^{iux}\, dx$$

$$= \left[\left(1 + \frac{x}{a}\right)\frac{e^{iux}}{iu} - \left(\frac{1}{a}\right)\frac{e^{iux}}{i^2u^2}\right]_{-a}^{0} + \left[\left(1 - \frac{x}{a}\right)\frac{e^{iux}}{iu} - \left(\frac{-1}{a}\right)\frac{e^{iux}}{i^2u^2}\right]_{0}^{a}$$

$$F(u) = \frac{1}{iu}(1-0) + \frac{1}{au^2}(1-e^{-iua}) + \frac{1}{iu}(0-1) - \frac{1}{au^2}(e^{iua}-1)$$

$$= \frac{1}{au^2}\left[2-\left(e^{iau}+e^{-iau}\right)\right] = \frac{2(1-\cos au)}{au^2}$$

Thus, $\boxed{F(u) = \frac{4\sin^2(au/2)}{au^2}}$

[9] *Find the complex Fourier transform of* $e^{-a^2x^2}, a>0$. *Hence deduce that* $e^{-x^2/2}$ *is self reciprocal in respect of the complex Fourier transform.*

☞ $$F(u) = F[f(x)] = \int_{-\infty}^{\infty} f(x)e^{iux}\,dx$$

ie., $$F(u) = \int_{-\infty}^{\infty} e^{-a^2x^2}\cdot e^{iux}\,dx = \int_{-\infty}^{\infty} e^{-a^2\left(x^2-\frac{iux}{a^2}\right)}\,dx$$

$$= \int_{-\infty}^{\infty} e^{-a^2\left(x^2-2x\cdot\frac{iu}{2a^2}+\frac{i^2u^2}{4a^4}-\frac{i^2u^2}{4a^4}\right)}\,dx$$

$$= \int_{-\infty}^{\infty} e^{-a^2\left(x-\frac{iu}{2a^2}\right)^2}\cdot e^{-u^2/4a^2}\,dx$$

Put, $a\left(x-\frac{iu}{2a^2}\right) = t \quad \therefore \quad dx = \frac{dt}{a}$ and t also varies from $-\infty$ to ∞.

Now, $F(u) = e^{-u^2/4a^2}\int_{-\infty}^{\infty} e^{-t^2}\cdot\frac{dt}{a}$ and we know that $\int_{-\infty}^{\infty} e^{-t^2}\,dt = \sqrt{\pi}$

Thus, $\boxed{F(u) = \frac{\sqrt{\pi}}{a}e^{-u^2/4a^2}}$

Now taking, $a^2 = 1/2$ we have,

$$F(u) = F\left[e^{-x^2/2}\right] = \frac{\sqrt{\pi}}{(1/\sqrt{2})}e^{-u^2/2} = \sqrt{2\pi}\,e^{-u^2/2}$$

It can be seen that the Fourier transform of $e^{-x^2/2}$ is a constant times $e^{-u^2/2}$. The function $e^{-x^2/2}$ and $e^{-u^2/2}$ are same but for the change in the variable. Hence we conclude that $e^{-x^2/2}$ is self reciprocal under complex Fourier transform.

[10] *Find the inverse Fourier transform of* e^{-u^2}

☞ We have the inverse Fourier transform

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)e^{-iux}\,du$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-u^2}\cdot e^{-iux}\,du = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-(u^2+iux)}\,du$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\left(u^2+2\cdot u\cdot\frac{ix}{2}+\frac{i^2x^2}{4}-\frac{i^2x^2}{4}\right)}\,du$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\left(u+\frac{ix}{2}\right)^2}\cdot e^{-x^2/4}\,du$$

Put, $u+\dfrac{ix}{2} = t \quad \therefore\ du = dt$ and t varies from $-\infty$ to ∞.

Hence, $f(x) = \dfrac{e^{-x^2/4}}{2\pi}\displaystyle\int_{-\infty}^{\infty} e^{-t^2}\,dt = \dfrac{e^{-x^2/4}}{2\pi}\sqrt{\pi} = \dfrac{e^{-x^2/4}}{2\sqrt{\pi}}$

Thus the required inverse Fourier transform is $\boxed{\dfrac{e^{-x^2/4}}{2\sqrt{\pi}}}$

[11] *Find the Fourier sine and cosine transforms of*

$$f(x) = \begin{cases} x, & 0 < x < 2 \\ 0, & else\ where \end{cases}$$

☞ The Fourier sine and cosine transforms of $f(x)$ are given by,

$$F_s(u) = \int_0^{\infty} f(x)\sin ux\,dx \text{ and } F_c(u) = \int_0^{\infty} f(x)\cos ux\,dx$$

$$\therefore \quad F_s(u) = \int_0^2 x \sin ux \, dx$$

$$= \left[x \cdot \frac{-\cos ux}{u} - 1 \cdot \frac{-\sin ux}{u^2} \right]_0^2, \text{ by Bernoulli's rule.}$$

$$F_s(u) = -\frac{1}{u}[x \cos ux]_0^2 + \frac{1}{u^2}[\sin ux]_0^2$$

$$= -\frac{1}{u}(2\cos 2u - 0) + \frac{1}{u^2}(\sin 2u - \sin 0)$$

Thus, $\boxed{F_s(u) = \dfrac{\sin 2u - 2u\cos 2u}{u^2}}$

Also, $F_c(u) = \int_0^2 x \cos ux \, dx$

$$= \left[x \frac{\sin ux}{u} - 1 \cdot \frac{-\cos ux}{u^2} \right]_0^2 \text{ by Bernoulli's rule.}$$

$$= \frac{1}{u}[x \sin ux]_0^2 + \frac{1}{u^2}[\cos ux]_0^2$$

$$= \frac{1}{u}(2\sin 2u - 0) + \frac{1}{u^2}(\cos 2u - \cos 0)$$

$$= \frac{2\sin 2u}{u} + \frac{\cos 2u - 1}{u^2}$$

Thus, $\boxed{F_c(u) = \dfrac{2u\sin 2u + \cos 2u - 1}{u^2}}$

[12] *Find the Fourier sine and cosine transforms of* $f(x) = e^{-\alpha x}$, $\alpha > 0$.

[June 2018]

☞ Fourier sine and cosine transforms are given by

$$F_s(u) = \int_0^\infty f(x) \sin ux \, dx \quad \text{and} \quad F_c(u) = \int_0^\infty f(x) \cos ux \, dx$$

$$\therefore \quad F_s(u) = \int_0^\infty e^{-\alpha x} \sin ux \, dx$$

$$F_s(u) = \left[\frac{e^{-\alpha x}}{(-\alpha)^2 + u^2}(-\alpha \sin ux - u \cos ux)\right]_0^\infty$$

by using the standard formula,

$$\int e^{ax} \sin bx\, dx = \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2 + b^2}$$

But, $e^{-\alpha x} \to 0$ as $x \to \infty$, $e^0 = 1$, $\cos 0 = 1$, and $\sin 0 = 0$.

Thus, $\boxed{F_s(u) = \dfrac{u}{\alpha^2 + u^2}}$

Also, $F_c(u) = \int_0^\infty f(x) \cos ux\, dx = \int_0^\infty e^{-\alpha x} \cos ux\, dx$

$$= \left[\frac{e^{-\alpha x}}{(-\alpha)^2 + u^2}(-\alpha \cos ux + u \sin ux)\right]_{x=0}^\infty = \frac{\alpha}{\alpha^2 + u^2}$$

by using $\int e^{ax} \cos bx\, dx = \dfrac{e^{ax}(a \cos bx + b \sin bx)}{a^2 + b^2}$

Thus, $\boxed{F_c(u) = \dfrac{\alpha}{\alpha^2 + u^2}}$

[13] *Obtain the Fourier cosine transform of the function* **[Dec. 2018]**

$$f(x) = \begin{cases} 4x, & 0 < x < 1 \\ 4 - x, & 1 < x < 4 \\ 0, & x > 4 \end{cases}$$

☞ Fourier cosine transform is given by

$$F_c(u) = \int_0^\infty f(x) \cos ux\, dx$$

$$= \int_0^1 f(x) \cos ux\, dx + \int_1^4 f(x) \cos ux\, dx + \int_4^\infty f(x) \cos ux\, dx$$

$$\therefore \quad F_c(u) = \int_0^1 4x\cos ux\,dx + \int_1^4 (4-x)\cos ux\,dx + \int_4^\infty 0\cdot\cos ux\,dx$$

Applying Bernoulli's rule to the integrals we have,

$$F_c(u) = \left[4x\cdot\frac{\sin ux}{u} - 4\frac{-\cos ux}{u^2}\right]_0^1 + \left[(4-x)\frac{\sin ux}{u} - (-1)\frac{-\cos ux}{u^2}\right]_1^4 + 0$$

$$= \frac{4}{u}\left[x\sin ux\right]_0^1 + \frac{4}{u^2}\left[\cos ux\right]_0^1 + \frac{1}{u}\left[(4-x)\sin ux\right]_1^4 - \frac{1}{u^2}\left[\cos ux\right]_1^4$$

$$= \frac{4}{u}(\sin u - 0) + \frac{4}{u^2}(\cos u - 1) + \frac{1}{u}(0 - 3\sin u) - \frac{1}{u^2}(\cos 4u - \cos u)$$

$$= \frac{4}{u}\sin u + \frac{4}{u^2}\cos u - \frac{4}{u^2} - \frac{3}{u}\sin u - \frac{1}{u^2}\cos 4u + \frac{1}{u^2}\cos u$$

Thus, $$\boxed{F_c(u) = \frac{1}{u}\sin u + \frac{5\cos u - 4}{u^2} - \frac{1}{u^2}\cos 4u}$$

Note : Similar problem

Find the Fourier cosine transform of $f(x) = \begin{cases} x & for\ \ 0<x<1 \\ 2-x & for\ \ 1<x<2 \\ 0 & for\ \ x>2 \end{cases}$

[June & Dec 2017]

[14] *Find the infinite Fourier cosine transform of* e^{-x^2}

☞ Fourier cosine transform is given by

$$F_c(u) = \int_0^\infty f(x)\cos ux\,dx$$

$$F_c(u) = \int_0^\infty e^{-x^2}\cos ux\,dx \qquad \ldots(1)$$

[We cannot evaluate the integral directly and hence proceed as follows. The process involved is called differentiation under the integral sign.]

Differentiating w.r.t. u we have,

$$\frac{dF_c}{du} = \int_0^\infty \frac{\partial}{\partial u}\left(e^{-x^2}\cos ux\right)dx$$

$$\frac{dF_c}{du} = \int_0^\infty e^{-x^2}(-\sin ux \cdot x)\,dx = \frac{1}{2}\int_0^\infty \sin ux\{e^{-x^2}(-2x)\}\,dx$$

or $$2\frac{dF_c}{du} = \int_0^\infty \sin ux\{e^{-x^2}(-2x)\}\,dx$$

Integrating RHS by parts we have,

$$2\frac{dF_c}{du} = \left[\sin ux(e^{-x^2})\right]_0^\infty - \int_0^\infty e^{-x^2}(\cos ux \cdot u)\,dx$$

But, $e^{-x^2} \to 0$ as $x \to \infty$ and $\sin 0 = 0$.

$\therefore$ $$2\frac{dF_c}{du} = (0-0) - u\int_0^\infty e^{-x^2}\cos ux\,dx \quad \text{or} \quad 2\frac{dF_c}{du} = -uF_c$$

ie., $$2\frac{dF_c}{F_c} = -u\,du$$

or $$\frac{dF_c}{F_c} = -\frac{u}{2}du$$ and integration yields,

$$\log F_c = \frac{-u^2}{4} + \log k,$$ where $\log k$ is a constant.

ie., $$\log\left(\frac{F_c}{k}\right) = \frac{-u^2}{4} \quad \text{or} \quad \frac{F_c}{k} = e^{-u^2/4}$$

Hence we have, $F_c(u) = k\,e^{-u^2/4}$ . . . (2)

To find k let us put $u = 0$ in (1) and (2).

(1) gives. $$F_c(0) = \int_0^\infty e^{-x^2}\cos 0\,dx = \int_0^\infty e^{-x^2}\,dx$$

But, $\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$ and hence $F_c(0) = \frac{\sqrt{\pi}}{2}$

Now putting $u = 0$ in (2), $F_c(0) = k\,e^0 = k$. From these we get $k = \sqrt{\pi}/2$.

Thus by substituting the value of k in (2) we have

$$\boxed{F_c(u) = (\sqrt{\pi}/2)\, e^{-u^2/4}}$$

[15] *Find the Fourier sine transform of* $f(x) = e^{-|x|}$ *and hence evaluate*

$$\int_0^\infty \frac{x \sin mx}{1+x^2}\,dx, m>0.$$ **[Dec. 2018]**

☞ Fourier sine transform is given by

$$F_s(u) = \int_0^\infty f(x)\sin ux\,dx$$

$$F_s(u) = \int_0^\infty e^{-|x|}\sin ux\,dx = \int_0^\infty e^{-x}\sin ux\,dx, \text{ since } |x| = x,\ x>0.$$

$$F_s(u) = \left[\frac{e^{-x}(-1\sin ux - u\cos ux)}{(-1)^2+u^2}\right]_0^\infty$$

But, $e^{-x} \to 0$ as $x \to \infty, e^0 = 1, \cos 0 = 1, \sin 0 = 0$.

Thus, $F_s(u) = \dfrac{u}{1+u^2}$

By inverse Fourier sine transform we have,

$$\frac{2}{\pi}\int_0^\infty F_s(u)\sin ux\,du = f(x)$$

ie., $$\int_0^\infty \frac{u}{1+u^2}\sin ux\,du = \frac{\pi}{2}f(x)$$

Putting $x = m$ where $m > 0$ we have $f(x) = e^{-|m|} = e^{-m}$

$$\therefore \quad \int_0^\infty \frac{u \sin mu}{1+u^2}\,du = \frac{\pi}{2}e^{-m}$$

Thus by changing the variable u to x, $\boxed{\int_0^\infty \frac{x\sin mx}{1+x^2}\,dx = \frac{\pi}{2}e^{-m}}$

[16] *If the Fourier sine transform of $f(x)$ is given by $F_s(u) = (\pi/2)e^{-2u}$ find the function $f(x)$.*

☞ By data, $F_s(u) = (\pi/2)e^{-2u}$,

By inverse Fourier sine transform,

$$f(x) = \frac{2}{\pi}\int_0^\infty F_s(u)\sin ux\,du$$

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\pi}{2}e^{-2u}\sin ux\,du$$

But, $\int e^{ax}\sin bx\,dx = \dfrac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}$

$$\therefore \quad f(x) = \left[\frac{e^{-2u}(-2\sin xu - x\cos xu)}{(-2)^2+x^2}\right]_{u=0}^{\infty}$$

$$= \frac{-1}{4+x^2}\left[e^{-2u}(2\sin xu + x\cos xu)\right]_{u=0}^{\infty}$$

But, $e^{-2u} \to 0$ as $u \to \infty$, $e^0 = 1$, $\cos 0 = 1$, $\sin 0 = 0$.

Hence, $f(x) = \dfrac{-1}{(4+x^2)}\cdot\{0 - x\} = \dfrac{x}{(4+x^2)}$

Thus, $$\boxed{f(x) = \frac{x}{(4+x^2)}}$$

[17] *Solve the integral equation*

$$\int_0^\infty f(\theta)\cos\alpha\theta\,d\theta = \begin{cases} 1-\alpha, & 0 \le \alpha \le 1 \\ 0, & \alpha > 1 \end{cases} \quad \textit{and hence evaluate} \int_0^\infty \frac{\sin^2 t}{t^2}dt$$

☞ Here we have to find $f(\theta)$ and we shall consider the inverse Fourier cosine transform with $F(\alpha) = \begin{cases} 1-\alpha, & 0 \le \alpha \le 1 \\ 0, & \alpha > 1 \end{cases}$

$$f(\theta) = \frac{2}{\pi}\int_0^\infty F(\alpha)\cos\alpha\theta\,d\alpha$$

$$f(\theta) = \frac{2}{\pi}\int_{\alpha=0}^{1}(1-\alpha)\cos\alpha\theta\, d\alpha, \text{ since } F(\alpha)=0 \text{ for } \alpha>1$$

$$= \frac{2}{\pi}\left[(1-\alpha)\frac{\sin\alpha\theta}{\theta} - (-1)\left(\frac{-\cos\alpha\theta}{\theta^2}\right)\right]_{\alpha=0}^{1}$$

$$= \frac{2}{\pi\theta}\left[(1-\alpha)\sin\alpha\theta\right]_{\alpha=0}^{1} - \frac{2}{\pi\theta^2}\left[\cos\alpha\theta\right]_{\alpha=0}^{1}$$

$$f(\theta) = \frac{2}{\pi\theta}[0-0] - \frac{2}{\pi\theta^2}[\cos\theta - 1]$$

ie., $$f(\theta) = \frac{2(1-\cos\theta)}{\pi\theta^2} = \frac{2\cdot 2\sin^2(\theta/2)}{\pi\theta^2} = \frac{4\sin^2(\theta/2)}{\pi\theta^2}$$

Hence we now have $\displaystyle\int_0^\infty \frac{4\sin^2(\theta/2)}{\pi\theta^2}\cos\alpha\theta\, d\theta = F(\alpha)$

ie., $$\int_0^\infty \frac{\sin^2(\theta/2)}{(\theta/2)^2}\cos\alpha\theta\, d\theta = \pi F(\alpha)$$

Putting $\theta/2 = t$, $d\theta = 2\,dt$ and t varies from 0 to ∞.

$\therefore$ $$\int_0^\infty \frac{\sin^2 t}{t^2}\cos(2\alpha t)\cdot 2\,dt = \pi F(\alpha)$$

ie., $$\int_0^\infty \frac{\sin^2 t}{t^2}\cos(2\alpha t)\,dt = \frac{\pi}{2}F(\alpha)$$

Putting $\alpha = 0$, $F(\alpha) = 1-0 = 1$

Thus, $$\boxed{\int_0^\infty \frac{\sin^2 t}{t^2}\,dt = \frac{\pi}{2}}$$

[18] *Find the Fourier sine transform of* $\dfrac{e^{-ax}}{x}$, $a>0$ **[June 2017, Dec. 18]**

☞ We have $F_s(u) = \displaystyle\int_0^\infty f(x)\sin ux\,dx$ and let $f(x) = \dfrac{e^{-ax}}{x}$

ie., $$F_s(u) = \int_0^\infty \frac{e^{-ax}}{x} \sin ux\, dx \qquad \ldots (1)$$

We cannot evaluate this integral directly and hence we proceed as follows.

$$\therefore \quad \frac{d}{du}[F_s(u)] = \int_0^\infty \frac{e^{-ax}}{x} \frac{\partial}{\partial u}(\sin ux)\, dx = \int_0^\infty \frac{e^{-ax}}{x}\, x\cos ux\, dx$$

$$\frac{d}{du}[F_s(u)] = \int_0^\infty e^{-ax} \cos ux\, dx$$

$$= \left[\frac{e^{-ax}}{a^2+u^2}(-a\cos ux + u\sin ux)\right]_{x=0}^\infty = \frac{1}{a^2+u^2}(0+a) = \frac{a}{a^2+u^2}$$

Hence, $\frac{d}{du}[F_s(u)] = \frac{a}{a^2+u^2}$ and by integrating w.r.t u we get,

$$F_s(u) = \tan^{-1}(u/a) + c$$

To evaluate c, let us put $u = 0 \quad \therefore F_s(0) = \tan^{-1}(0) + c$

But, $F_s(0) = 0$ from (1) and hence $c = 0$.

Thus, $$\boxed{F_s(u) = \tan^{-1}(u/a)}$$

[19] *Show that* $x e^{-x^2/2}$ *is self reciprocal under the Fourier sine transform.*

☞ By the definition, $F_s[f(x)] = \int_0^\infty f(x)\sin ux\, dx = F_s(u)$

Now, $$F_s\left[x e^{-x^2/2}\right] = \int_0^\infty x e^{-x^2/2} \sin ux\, dx$$

$$= \int_0^\infty \sin ux (x e^{-x^2/2})\, dx.$$

Integrating by parts we have,

$$F_s\left[x e^{-x^2/2}\right] = \left[\sin ux(-e^{-x^2/2})\right]_{x=0}^\infty - \int_0^\infty (-e^{-x^2/2})\, u\cos ux\, dx$$

$$F_s(u) = 0 + u\int_0^\infty e^{-x^2/2}\cos ux\,dx$$

$$\therefore \quad F_s(u) = u\int_0^\infty e^{-x^2/2}\cos ux\,dx \qquad \ldots (1)$$

We cannot evaluate the integral directly and hence proceed as follows.

Let, $$\phi(u) = \int_0^\infty e^{-x^2/2}\cos ux\,dx. \qquad \ldots (2)$$

$$\therefore \quad \phi'(u) = \int_0^\infty e^{-x^2/2}\frac{\partial}{\partial u}(\cos ux)\,dx$$

$$\phi'(u) = \int_0^\infty e^{-x^2/2}(-x\sin ux)\,dx$$

$$\phi'(u) = \int_0^\infty \sin ux\,(-x\,e^{-x^2/2})\,dx$$

Now, integrating by parts we have,

$$\phi'(u) = \left[\sin ux\,(e^{-x^2/2})\right]_{x=0}^{\infty} - \int_0^\infty e^{-x^2/2}(u\cos ux)\,dx$$

$$= 0 - u\int_0^\infty e^{-x^2/2}\cos ux\,dx$$

ie., $$\phi'(u) = -u\,\phi(u) \quad \text{or} \quad \frac{\phi'(u)}{\phi(u)} = -u$$

$$\therefore \quad \int\frac{\phi'(u)}{\phi(u)}\,du = -\int u\,du + c$$

ie., $$\log\phi(u) = (-u^2/2) + c \quad \text{or} \quad \phi(u) = e^{(-u^2/2)+c}$$

Hence, $\phi(u) = k\,e^{-u^2/2}$ where $k = e^c$

To evaluate c, let us put $u = 0$.

$\therefore \quad \phi(0) = k.$ But, $\phi(0) = \int_0^\infty e^{-x^2/2}\, dx$ from (2).

Put, $\quad x/\sqrt{2} = t \quad \therefore\ dx = \sqrt{2}\, dt$

Now, $\phi(0) = \int_{t=0}^{\infty} e^{-t^2}\sqrt{2}\, dt$ But $\int_{t=0}^{\infty} e^{-t^2}\, dt = \dfrac{\sqrt{\pi}}{2}$ (*Standard integral*)

Hence, $\qquad \phi(0) = \sqrt{\pi/2} = k.$

We now have, $\phi(u) = \sqrt{\pi/2}\; e^{-u^2/2}$

Also we have from (1), $F_s(u) = u\,\phi(u) = \sqrt{\pi/2}\; u\, e^{-u^2/2}$

ie., $\quad \int_0^\infty x\, e^{-x^2/2} \sin ux\, dx = \sqrt{\pi/2}\; u\, e^{-u^2/2} = \text{const.}\,(u\, e^{-u^2/2})$

We note that $u\, e^{-u^2/2}$ is of the same form as $x\, e^{-x^2/2}$

Thus $x\, e^{-x^2/2}$ is self reciprocal under the Fourier sine transform.

[20] *Find the inverse Fourier sine transform of* $\hat{f}_s(\alpha) = \dfrac{1}{\alpha} e^{-a\alpha},\ a > 0$ **[Dec 2016]**

☞ By data, $\hat{f}_s(\alpha) = \dfrac{e^{-a\alpha}}{\alpha}$

ie., $\quad F_s[f(x)] = \int_0^\infty f(x) \sin \alpha x\, dx = \hat{f}_s(\alpha)$ and hence

$f(x) = \dfrac{2}{\pi}\int_0^\infty \hat{f}_s(\alpha) \sin \alpha x\, d\alpha$, being the inverse Fourier sine transform.

ie., $\quad f(x) = \dfrac{2}{\pi}\int_0^\infty \dfrac{e^{-a\alpha}}{\alpha} \sin \alpha x\, d\alpha = \dfrac{2}{\pi}\phi(x)\ (say)$

where , $\phi(x) = \int_0^\infty \dfrac{e^{-a\alpha}}{\alpha} \sin \alpha x\, d\alpha$

We cannot evaluate the integral directly and hence proceed as follows.

$$\therefore \quad \phi'(x) = \int_0^\infty \frac{e^{-a\alpha}}{\alpha} \cos \alpha x \cdot \alpha \, d\alpha = \int_0^\infty e^{-a\alpha} \cos \alpha x \, d\alpha$$

$$\phi'(x) = \left[\frac{e^{-a\alpha}}{a^2 + x^2} (-a \cos \alpha x + x \sin \alpha x \right]_{\alpha = 0}^{\infty}$$, by a standard formula.

$$\phi'(x) = \frac{1}{a^2 + x^2} \{ 0 - (-a) \} = \frac{a}{a^2 + x^2}$$

$$\therefore \quad \phi(x) = \int \frac{a}{a^2 + x^2} dx + c$$

ie., $\phi(x) = \tan^{-1}(x/a) + c$

To evaluate c let us put $x = 0$.

$$\phi(0) = \tan^{-1}(0) + c \quad \text{or} \quad 0 = 0 + c \;\therefore c = 0, \text{ since } \phi(0) = 0.$$

Hence, $\phi(x) = \tan^{-1}(x/a)$ and we have, $f(x) = \frac{2}{\pi} \phi(x)$

Thus, $$\boxed{f(x) = \frac{2}{\pi} \tan^{-1}(x/a)}$$

[21] *Find the Fourier cosine transform of* $f(x) = \dfrac{1}{1 + x^2}$

☞ By the definition, $F_c[f(x)] = \int_0^\infty f(x) \cos ux \, dx = F_c(u)$

ie., $$F_c(u) = \int_0^\infty \frac{1}{1 + x^2} \cos ux \, dx \quad \ldots (1)$$

We cannot evaluate the RHS directly and hence we proceed as follows.

$$\frac{d F_c(u)}{du} = F_c'(u) = \int_0^\infty \frac{1}{1 + x^2} (-\sin ux) x \, dx$$

ie., $$F_c'(u) = -\int_0^\infty \frac{x}{1 + x^2} \sin ux \, dx$$

(We cannot evaluate RHS now also and hence modify the integrand)

$$F_c'(u) = -\int_0^\infty \frac{x^2}{x(1+x^2)} \sin ux \, dx$$

$$= -\int_0^\infty \frac{(1+x^2)-1}{x(1+x^2)} \sin ux \, dx$$

$$F_c'(u) = -\int_0^\infty \frac{\sin ux}{x} \, dx + \int_0^\infty \frac{\sin ux}{x(1+x^2)} \, dx$$

We note that $\int_0^\infty \frac{\sin ux}{x} \, dx = \frac{\pi}{2}$ and hence we have

$$F_c'(u) = -\frac{\pi}{2} + \int_0^\infty \frac{\sin ux}{x(1+x^2)} \, dx \qquad \ldots (2)$$

Differentiating again w.r.t u we have,

$$F_c''(u) = \int_0^\infty \frac{\cos ux \cdot x}{x(1+x^2)} \, dx = \int_0^\infty \frac{\cos ux}{1+x^2} \, dx$$

$\therefore$ $F_c''(u) = F_c(u)$, by using (1).

or $F_c''(u) - F_c(u) = 0$ and this is a second order DE of the form

$(D^2 - 1) F_c(u) = 0$, where $D = \frac{d}{du}$

AE is $m^2 - 1 = 0 \;\therefore\; m = 1, -1$

The general solution is given by

$$F_c(u) = c_1 e^u + c_2 e^{-u} \qquad \ldots (3)$$

We shall find c_1 and c_2.

We have, $F_c(0) = c_1 + c_2$ from (3).

But from (1), $F_c(0) = \int_0^\infty \frac{1}{1+x^2} \, dx = \left[\tan^{-1} x\right]_0^\infty = \frac{\pi}{2}$

So we have, $c_1 + c_2 = \pi/2 \qquad \ldots (4)$

Also from (3), $F_c'(u) = c_1 e^u - c_2 e^{-u}$ and hence $F_c'(0) = c_1 - c_2$

From (2), $F_c'(0) = -\pi/2 + 0 = -\pi/2$

Hence, we have, $c_1 - c_2 = -\pi/2$ . . . (5)

By solving (4) and (5) we get, $c_1 = 0$ and $c_2 = \pi/2$.

Using these values in (3) we have $F_c(u) = (\pi/2)e^{-u}$

Thus the required, $\boxed{F_c\left[\frac{1}{1+x^2}\right] = \frac{\pi}{2}e^{-u}}$

[22] *Find the Fourier sine transform of* $f(x) = \dfrac{1}{x(1+x^2)}$

☞ $F_s[f(x)] = \int_0^\infty f(x)\sin ux\,dx = F_s(u)$

(This problem is similar to the previous one)

ie., $$F_s(u) = \int_0^\infty \frac{\sin ux}{x(1+x^2)}\,dx \qquad \text{. . .(1)}$$

We cannot evaluate the integral directly and hence we proceed as follows.

$$\therefore \quad F_s'(u) = \int_0^\infty \frac{x\cos ux}{x(1+x^2)}\,dx = \int_0^\infty \frac{\cos ux}{(1+x^2)}\,dx \qquad \text{. . . (2)}$$

Differentiating again w.r.t u,

$$F_s''(u) = \int_0^\infty \frac{-x\sin ux}{(1+x^2)}\,dx = -\int_0^\infty \frac{x^2\sin ux}{x(1+x^2)}\,dx$$

ie., $$F_s''(u) = -\int_0^\infty \frac{(1+x^2)-1}{x(1+x^2)}\sin ux\,dx$$

$$F_s''(u) = -\int_0^\infty \frac{\sin ux}{x}\,dx + \int_0^\infty \frac{\sin ux}{x(1+x^2)}\,dx$$

ie., $$F_s''(u) = -\frac{\pi}{2} + F_s(u)$$

or $$F_s''(u) - F_s(u) = \frac{-\pi}{2}$$

ie., $[D^2-1]F_s(u) = -\pi/2$ where $D = \dfrac{d}{du}$

AE is $m^2-1=0 \therefore m = \pm 1$

CF is given by $c_1 e^u + c_2 e^{-u}$

Also, $PI = \dfrac{-\pi/2}{D^2-1} = \dfrac{-\pi/2}{-1} = \dfrac{\pi}{2}$

The general solution is given by CF + PI

ie., $F_s(u) = c_1 e^u + c_2 e^{-u} + \pi/2$ . . . (3)

We need to find c_1 and c_2.

$F_s'(u) = c_1 e^u - c_2 e^{-u}$ . . . (4)

Putting $u = 0$ in (3) and (4) we have

$$F_s(0) = c_1 + c_2 + \pi/2 \text{ and } F_s'(0) = c_1 - c_2$$

But, $F_s(0) = 0$ from (1) and $F_s'(0) = \displaystyle\int_0^\infty \frac{1}{1+x^2}dx$ from (2).

ie., $F_s'(0) = [\tan^{-1}x]_0^\infty = \pi/2$

We have the system of equations

$$c_1 + c_2 + \pi/2 = 0 \text{ and } c_1 - c_2 = \pi/2$$

By solving we get $c_1 = 0$, $c_2 = -\pi/2$ and we substitute these values in (3).

Thus, $\boxed{F_s(u) = \pi/2 \cdot (1-e^{-u})}$

[23] *Find the function $f(x)$ whose Fourier cosine transform is given by*

$$F(\alpha) = \begin{cases} a-(\alpha/2), & 0 \le \alpha < 2a \\ 0, & \alpha > 2a \end{cases}$$

☞ By the definition $f(x) = \dfrac{2}{\pi}\displaystyle\int_0^\infty F(\alpha)\cos\alpha x \, d\alpha$

$$\therefore \quad f(x) = \frac{2}{\pi}\int_0^{2a}\left(a - \frac{\alpha}{2}\right)\cos\alpha x \, d\alpha$$

Applying Bernoulli's rule,

$$f(x) = \frac{2}{\pi}\left[\left(a-\frac{\alpha}{2}\right)\frac{\sin\alpha x}{x} - \left(\frac{-1}{2}\right)\left(\frac{-\cos\alpha x}{x^2}\right)\right]_{\alpha=0}^{2a}$$

$$= \frac{2}{\pi}\left[(0-0) - \frac{1}{2x^2}(\cos 2ax - 1)\right]$$

Thus, $$\boxed{f(x) = \frac{1}{\pi x^2}(1-\cos 2ax) = \frac{2\sin^2 ax}{\pi x^2}}$$

[24] *Solve the integral equation :* $\int_0^\infty f(x)\cos\alpha x\,dx = e^{-a\alpha}$

☞ We have by inverse Fourier cosine transform,

$$f(x) = \frac{2}{\pi}\int_0^\infty F(\alpha)\cos\alpha x\,d\alpha \text{ and } F(\alpha) = e^{-a\alpha} \text{ by data.}$$

$$\therefore \quad f(x) = \frac{2}{\pi}\int_0^\infty e^{-a\alpha}\cos\alpha x\,d\alpha$$

$$= \frac{2}{\pi}\left[\frac{e^{-a\alpha}}{(-a)^2+x^2}(-a\cos\alpha x + x\sin\alpha x)\right]_{\alpha=0}^{\infty}$$

Thus, $$\boxed{f(x) = \frac{2a}{\pi(a^2+x^2)}}$$

[25] *Find the Fourier cosine transform of* e^{-ax} *and hence deduce the Fourier cosine transform of* $x e^{-ax}$. *Further evaluate* $\int_0^\infty \frac{\cos\lambda x}{x^2+a^2}dx$

☞ We have $F_c[e^{-ax}] = \frac{a}{a^2+u^2}$ *(Refer Problem - [12])*

ie., $$\int_0^\infty e^{-ax}\cos ux\,dx = \frac{a}{a^2+u^2} \qquad \ldots (1)$$

Differentiating (1) w.r.t a on both sides we get,

$$\int_0^\infty e^{-ax}(-x)\cos ux\,dx = \frac{(a^2+u^2)(1)-2a^2}{(a^2+u^2)^2} = \frac{u^2-a^2}{(a^2+u^2)^2}$$

or $$\int_0^\infty (x e^{-ax})\cos ux\,dx = \frac{a^2-u^2}{(a^2+u^2)^2}$$

That is, $$\boxed{F_c\left[x e^{-ax}\right] = \frac{a^2-u^2}{a^2+u^2}}$$

Further, $$F_c\left[e^{-ax}\right] = \frac{a}{a^2+u^2}$$

$$\Rightarrow \quad e^{-ax} = \frac{2}{\pi}\int_0^\infty \frac{a}{a^2+u^2}\cos ux\,du,$$ by inverse cosine transform.

or $$\int_0^\infty \frac{\cos ux}{a^2+u^2}\,du = \frac{\pi}{2a}e^{-ax}$$

Let us change, x to λ and u to x.

Thus, $$\boxed{\int_0^\infty \frac{\cos\lambda x}{a^2+x^2}\,dx = \frac{\pi}{2a}e^{-a\lambda}}$$

[26] *Find the Fourier transform of* $f(x) = \begin{cases} e^{-x}, & x>0 \\ -e^{x}, & x<0 \end{cases}$

☞ $$F[f(x)] = \int_{-\infty}^{\infty} f(x)e^{iux}\,dx$$

$$= \int_{-\infty}^{0} -e^{x}e^{iux}\,dx + \int_0^\infty e^{-x}e^{iux}\,dx$$

$$= \int_{-\infty}^{\infty} -e^{(1+iu)x}\,dx + \int_0^\infty e^{-(1-iu)x}\,dx$$

$$= -\left[\frac{e^{(1+iu)x}}{1+iu}\right]_{-\infty}^{0} + \left[\frac{e^{-(1-iu)x}}{-(1-iu)}\right]_0^\infty$$

$$F[f(x)] = \frac{-1}{1+iu} + \frac{1}{1-iu} = \frac{-1+iu+1+iu}{1+u^2}$$

Thus, $$\boxed{F[f(x)] = \frac{2iu}{1+u^2}}$$

[27] *Find the Fourier sine and cosine transforms of* $2e^{-3x} + 3e^{-2x}$

☞ $$F_s[f(x)] = \int_0^\infty f(x)\sin ux\,dx$$

$$= 2\int_0^\infty e^{-3x}\sin ux\,dx + 3\int_0^\infty e^{-2x}\sin ux\,dx$$

$$F_s[f(x)] = 2\left[\frac{e^{-3x}}{9+u^2}(-3\sin ux - u\cos ux)\right]_0^\infty + 3\left[\frac{e^{-2x}}{4+u^2}(-2\sin ux - u\cos ux)\right]_0^\infty$$

Thus, $$\boxed{F_s[f(x)] = \frac{2u}{9+u^2} + \frac{3u}{4+u^2} = u\left[\frac{2}{9+u^2} + \frac{3}{4+u^2}\right].}$$

$$F_c[f(x)] = 2\int_0^\infty e^{-3x}\cos ux\,dx + 3\int_0^\infty e^{-2x}\cos ux\,dx$$

$$F_c[f(x)] = 2\left[\frac{e^{-3x}}{9+u^2}(-3\cos ux + u\sin ux)\right]_0^\infty + 3\left[\frac{e^{-2x}}{4+u^2}(-2\cos ux + u\sin ux)\right]_0^\infty$$

Thus, $$\boxed{F_c[u] = \frac{6}{9+u^2} + \frac{6}{4+u^2} = 6\left[\frac{1}{9+u^2} + \frac{1}{4+u^2}\right]}$$

[28] *Solve the following integral equation :*

$$\int_0^\infty f(x)\sin\alpha x\,dx = \begin{cases} 10, & 0 \le \alpha < 1 \\ 20, & 1 \le \alpha < 2 \\ 0, & \alpha > 2 \end{cases}$$

☞ If $\int_0^\infty f(x)\sin\alpha x\,dx = F_s(\alpha)$ then we have to find $f(x)$ where,

$$f(x)=\frac{2}{\pi}\int_0^\infty F_s(\alpha)\sin\alpha x\,d\alpha$$

$$f(x)=\frac{2}{\pi}\left[\int_0^1 10\sin\alpha x\,d\alpha+\int_1^2 20\sin\alpha x\,d\alpha+\int_2^\infty 0\cdot\sin\alpha x\,d\alpha\right]$$

$$=\frac{2}{\pi}\left\{\left[\frac{-10\cos\alpha x}{x}\right]_0^1+\left[\frac{-20\cos\alpha x}{x}\right]_1^2\right\}$$

$$=\frac{-20}{\pi x}\left\{[\cos\alpha x]_0^1+2[\cos\alpha x]_1^2\right\}$$

$$=\frac{-20}{\pi x}(\cos x-1+2\cos 2x-2\cos x)=\frac{-20}{\pi x}(-1-\cos x+2\cos 2x)$$

Thus, $$\boxed{f(x)=\frac{20}{\pi x}(1+\cos x-2\cos 2x)}$$

ASSIGNMENT

Find the Fourier transform of the following functions

1. $f(x)=\begin{cases}x, & |x|\le a\\ 0, & |x|>a\end{cases}$

2. $f(x)=\begin{cases}a^2-x^2, & |x|<a\\ 0, & |x|>a\end{cases}$ and hence show that $\int_0^\infty\frac{\sin x-x\cos x}{x^3}dx=\frac{\pi}{4}$

3. $f(x)=\begin{cases}1+x, & -1<x\le 0\\ 1-x, & 0\le x<1\\ 0 & \text{otherwise}\end{cases}$

4. $f(x)=e^{-4x^2}$

5. $f(x)=xe^{-a|x|}$

Find the Fourier sine and cosine transform of the following functions.

6. $f(x)=e^{-2x}$

7. $f(x)=xe^{-2x}$

8. $$f(x) = \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

Solve the following integral equations

9. $$\int_0^\infty f(x) \sin \alpha x \, dx = \begin{cases} 1 - \alpha, & 0 \le \alpha < 1 \\ 0, & \alpha > 1 \end{cases}$$

10. $$\int_0^\infty f(x) \sin \alpha x \, dx = \begin{cases} 1, & 0 < \alpha < 1 \\ 2, & 1 < \alpha < 2 \\ 0, & \alpha > 2 \end{cases}$$

ANSWERS

1. $\dfrac{\sin au - au \cos au}{u^2}$

2. $\dfrac{4}{u^3}(\sin au - au \cos au)$

3. $\dfrac{\sin^2(u/2)}{(u/2)}$

4. $(\sqrt{\pi}/2)\, e^{-u^2/16}$

5. $\dfrac{4iau}{(a^2 + u^2)^2}$

6. $\dfrac{u}{u^2 + 4}$ and $\dfrac{2}{u^2 + 4}$

7. $\dfrac{4u}{(u^2 + 4)^2}$ and $\dfrac{4 - u^2}{(u^2 + 4)^2}$

8. $\dfrac{\sin u \sin^2(u/2)}{(u/2)^2}$ and $\dfrac{\cos u \sin^2(u/2)}{(u/2)^2}$

9. $\dfrac{2(x - \sin x)}{\pi x^2}$

10. $\dfrac{2}{\pi x}(1 + \cos x - 2\cos 2x)$

Z - TRANSFORMS

We introduce *Difference Equations* based on the concept of finite differences whose general / complete solution can be obtained in a manner analogous to the method of solving linear differential equations with constant coefficients.

We discuss ***Z-transforms*** in detail and also the solution of difference equations using Z-transforms.

3.4 Difference Equations

Consider a function $y=f(x)$. Let $x_0, x_1 = x_0+h, x_2 = x_1+h, \cdots x_n = x_{n-1}+h$, be a set of equidistant values of x, distant h. Let the corresponding values of $y=f(x)$ be $y_0 = f(x_0), y_1 = f(x_1), y_2 = f(x_2), \cdots y_n = f(x_n)$.

The forward difference of $f(x)$ denoted by $\Delta f(x)$ is defined as follows.

$$\Delta f(x) = f(x+h) - f(x)$$

Now, if $x = x_0$: $\Delta f(x_0) = f(x_0+h) - f(x_0) = f(x_1) - f(x_0)$

or $\Delta y_0 = y_1 - y_0$. Similarly we have $\Delta y_1 = y_2 - y_1, \Delta y_2 = y_3 - y_2$ etc.,

In general we have, $\Delta y_n = y_{n+1} - y_n$

Further, $\Delta^2 y_n = \Delta(\Delta y_n) = \Delta(y_{n+1}) - \Delta(y_n)$

$$= (y_{n+2} - y_{n+1}) - (y_{n+1} - y_n) = y_{n+2} - 2y_{n+1} + y_n \text{ etc.}$$

Suppose that,

$$\Delta y_n = 1(say),\ \Delta^2 y_n + \Delta y_n = 0(say)$$

We obtain equations of the form

$$y_{n+1} - y_n = 1,\ y_{n+2} - y_{n+1} = 0$$

In fact, these type of equations are referred to as **Difference Equations.**

Definition : A **Difference Equation** is a relationship between the differences of an unknown function (*dependent variable* y) at several values of the independent variable.

An equation of the form,

$$a_r y_{n+r} + a_{r-1} y_{n+r-1} + a_{r-2} y_{n+r-2} + \cdots + a_2 y_{n+2} + a_1 y_{n+1} + a_0 y_n = \phi(n)$$

where n take the values 0, 1, 2, 3, . . . and $a_r, a_{r-1}, \ldots a_0$ are all constants is called a *linear difference equation of order r.*

In other words we can say that a difference equation is a relationship interms of the values $y_{n+r}, y_{n+r-1}, \cdots y_{n+1}, y_n$.

Finding the sequence y_n constitutes a solution of the difference equation.

It may be noted that the general solution of a difference equation contains arbitrary constants equal to the order of the difference equation. (*Analogous to ODE*)

3.5 Introduction to Z-transforms

We are acquainted with Laplace transforms and Fourier transforms whose basic definition is in the form of a definite integral in which the integrand is involved with two parameters. The resulting integral whenever it exists will be a function of a single parameter.

Z - transforms operates on the sequences of functions of a single variable defined for non negative integral values of the variable. This transform has number of properties similar to that of Laplace transforms.

Difference equations arises in situations with the data consisting of only a set of values of an unknown function (*discrete values*). Just as Laplace transforms and Fourier transforms serves as a tool to solve some types of ordinary and partial differential equations, Z *transforms serves as a tool to solve difference equations.*

Z - transforms play an important role in the analysis and representations of discrete time linear shift invariant systems. It has applications in control system of engineering and also in some advanced statistical problems.

3.51 Definition of Z - transform

If $u_n = f(n)$ defined for all n = 0, 1, 2, 3, . . . and $u_n = 0$ for $n < 0$ then the ***Z-transform*** of u_n denoted by $Z_T(u_n)$ is defined by

$$Z_T(u_n) = \sum_{n=0}^{\infty} u_n z^{-n} \qquad \ldots (1)$$

Whenever the series on the RHS of (1) converges, it will be a function of z and we write,

$$Z_T(u_n) = \bar{u}(z) \qquad \cdots (2)$$

Further (2) can be written in the equivalent form

$$Z_T^{-1}[\bar{u}(z)] = u_n \qquad \cdots (3)$$

This is called the **Inverse *Z-transform*.**

Note : *Notation* $Z(u_n), Z^{-1}[\bar{u}(z)]$ *is also used.*

Notation $U(z)$ *for* $\bar{u}(z)$ *is also used.*

Property : $Z_T(n^k) = -z\dfrac{d}{dz} Z_T(n^{k-1})$ *where k is a positive integer.*

Proof : Consider RHS

That is, $\displaystyle -z\frac{d}{dz}Z_T(n^{k-1}) = -z\frac{d}{dz}\sum_{n=0}^{\infty} n^{k-1} z^{-n}$

$$= -z\sum_{n=0}^{\infty} n^{k-1}(-n)z^{-n-1}$$

$$= \sum_{n=0}^{\infty} n^k z^{-n} = Z_T(n^k) = \text{LHS}$$

Thus we have proved that, $Z_T(n^k) = -z\dfrac{d}{dz}Z_T(n^{k-1})$

3.52 Z-transform of some standard functions (Standard Z transforms)

1. $Z_T(k^n)$ 2. $Z_T(1)$ 3. $Z_T(n)$ 4. $Z_T(n^2)$ 5. $Z_T(n^3)$

1. By the definition, $\displaystyle Z_T(k^n) = \sum_{n=0}^{\infty} k^n z^{-n} = \sum_{n=0}^{\infty}(k/z)^n$

$$= 1 + (k/z) + (k/z)^2 + \cdots$$

The series on the RHS is a geometric series of the form $1 + r + r^2 + \cdots$ whose sum to infinity $1/1-r$ where $r = k/z$.

$$\therefore \quad Z_T(k^n) = \frac{1}{1-(k/z)}$$

Thus, $Z_T(k^n) = \dfrac{z}{z-k}$

2. Putting $k = 1$ in the previous result we have $k^n = 1$

Thus, $Z_T(1) = \dfrac{z}{z-1}$

Remark : *The result can easily be established independently also.*
Results (3), (4), (5) are established by using the property,

$$Z_T(n^k) = -z\frac{d}{dz}Z_T(n^{k-1})$$

We obtain the required Z-transforms by taking $k = 1$, $k = 2$, $k = 3$.

3. When $k = 1$, $Z_T(n) = -z\dfrac{d}{dz}Z_T(n^0) = -z\dfrac{d}{dz}Z_T(1)$

ie., $$Z_T(n) = -z\frac{d}{dz}\left(\frac{z}{z-1}\right) = -z\left\{\frac{(z-1)-z}{(z-1)^2}\right\}$$

Thus, $Z_T(n) = \dfrac{z}{(z-1)^2}$

4. When $k = 2$, $Z_T(n^2) = -z\dfrac{d}{dz}Z_T(n)$

ie., $$Z_T(n^2) = -z\frac{d}{dz}\left[\frac{z}{(z-1)^2}\right]$$

$$= -z\left\{\frac{(z-1)^2 - z\cdot 2(z-1)}{(z-1)^4}\right\}$$

$$= -z(z-1)\left\{\frac{z-1-2z}{(z-1)^4}\right\} = \frac{-z(-z-1)}{(z-1)^3}$$

Thus, $Z_T(n^2) = \dfrac{z(z+1)}{(z-1)^3} = \dfrac{z^2+z}{(z-1)^3}$

Note : *Find* $Z_T(n^2)$ **[June 2017]**

5. When $k = 3$, $Z_T(n^3) = -z\dfrac{d}{dz}Z_T(n^2)$

ie., $$Z_T(n^3) = -z\frac{d}{dz}\left[\frac{z^2+z}{(z-1)^3}\right]$$

$$Z_T(n^3) = -z\left\{\frac{(z-1)^3(2z+1)-(z^2+z)\cdot 3(z-1)^2}{(z-1)^6}\right\}$$

$$= -z(z-1)^2\left\{\frac{(z-1)(2z+1)-3(z^2+z)}{(z-1)^6}\right\}$$

$$Z_T(n^3) = \frac{-z(-z^2-4z-1)}{(z-1)^4} = \frac{z^3+4z^2+z}{(z-1)^4}$$

Thus, $Z_T(n^3) = \dfrac{z^3+4z^2+z}{(z-1)^4}$

We proceed to give some properties / rules associated with Z-transform. Proofs are given for the benefit of readers

3.53 Linearity property

Statement : *If u_n and v_n be any two discrete valued functions then* $Z_T(c_1 u_n + c_2 v_n) = c_1 Z_T(u_n) + c_2 Z_T(v_n)$ *where c_1, c_2 are constants.*

Proof : We have by the definition,

$$Z_T(c_1 u_n + c_2 v_n) = \sum_{n=0}^{\infty}(c_1 u_n + c_2 v_n)z^{-n}$$

$$= c_1\sum_{n=0}^{\infty}u_n z^{-n} + c_2\sum_{n=0}^{\infty}v_n z^{-n}$$

Thus, $Z_T(c_1 u_n + c_2 v_n) = c_1 Z_T(u_n) + c_2 Z_T(v_n)$

3.54 Damping rule (property)

Statement : *If* $Z_T(u_n) = \bar{u}(z)$ *then*

(i) $Z_T(k^n u_n) = \bar{u}(z/k)$ **(ii)** $Z_T(k^{-n} u_n) = \bar{u}(kz)$

Proof : (i) $Z_T(k^n u_n) = \sum_{n=0}^{\infty}(k^n u_n)z^{-n}$

$$Z_T\left[k^n u_n\right] = \sum_{n=0}^{\infty}u_n(z/k)^{-n} = \bar{u}(z/k)$$

Thus, $Z_T(k^n u_n) = \bar{u}(z/k)$

(ii) $$Z_T(k^{-n}u_n) = \sum_{n=0}^{\infty}(k^{-n}u_n)z^{-n}$$

$$= \sum_{n=0}^{\infty} u_n (kz)^{-n} = \bar{u}(kz)$$

Thus, $Z_T(k^{-n}u_n) = \bar{u}(kz)$

- **Some applications of damping rule**

We have already obtained $Z_T(n)$, $Z_T(n^2)$, $Z_T(n^3)$ and we can obtain $Z_T(k^n n)$, $Z_T(k^n n^2)$, $Z_T(k^n n^3)$ by using damping rule.

(i) $Z_T(k^n n) = \{Z_T(n)\}_{z \to (z/k)}$ where $Z_T(n) = \dfrac{z}{(z-1)^2}$

$$\therefore \quad Z_T(k^n n) = \frac{(z/k)}{(z/k-1)^2}$$

Thus, $Z_T(k^n u_n) = \dfrac{kz}{(z-k)^2}$

(ii) We have, $Z_T(n^2) = \dfrac{z^2+z}{(z-1)^3}$

$$\therefore \quad Z_T(k^n n^2) = \{Z_T(n^2)\}_{z \to (z/k)} = \frac{(z/k)^2 + (z/k)}{(z/k-1)^3}$$

Thus, $Z_T(k^n n^2) = \dfrac{kz^2 + k^2 z}{(z-k)^3}$

(iii) We have, $Z_T(n^3) = \dfrac{z^3 + 4z^2 + z}{(z-1)^4}$

$$\therefore \quad Z_T(k^n n^3) = \{Z_T(n^3)\}_{z \to (z/k)} = \frac{(z/k)^3 + 4(z/k)^2 + (z/k)}{(z/k-1)^4}$$

Thus, $Z_T(k^n n^3) = \dfrac{kz^3 + 4k^2 z^2 + k^3 z}{(z-k)^4}$

List of standard Z-transforms

1. $Z_T(1) = \dfrac{z}{z-1}$	2. $Z_T(k^n) = \dfrac{z}{z-k}$
3. $Z_T(n) = \dfrac{z}{(z-1)^2}$	4. $Z_T(k^n n) = \dfrac{kz}{(z-k)^2}$
5. $Z_T(n^2) = \dfrac{z^2+z}{(z-1)^3}$	6. $Z_T(k^n n^2) = \dfrac{kz^2+k^2z}{(z-k)^3}$
7. $Z_T(n^3) = \dfrac{z^3+4z^2+z}{(z-1)^4}$	8. $Z_T(k^n n^3) = \dfrac{kz^3+4k^2z^2+k^3z}{(z-k)^4}$

3.55 Shifting rule (property)

1. ***Right shifting rule***

If $Z_T(u_n) = \bar{u}(z)$ *then* $Z_T(u_{n-k}) = z^{-k}\,\bar{u}(z)$ *where* $k > 0$.

Proof : By the definition,

$$Z_T(u_{n-k}) = \sum_{n=0}^{\infty} u_{n-k}\, z^{-n}$$

Since $u_n = 0$ for $n < 0$ in the general context, we have $u_{n-k} = 0$ for $n = 0, 1, 2, \cdots (k-1)$.

$$\therefore \quad Z_T(u_{n-k}) = \sum_{n=k}^{\infty} u_{n-k}\, z^{-n}$$

$$= u_0 z^{-k} + u_1 z^{-(k+1)} + u_2 z^{-(k+2)} + \cdots$$

ie., $$= z^{-k}(u_0 + u_1 z^{-1} + u_2 z^{-2} + \cdots) = z^{-k}\sum_{n=0}^{\infty} u_n z^{-n} = z^{-k}\,\bar{u}(z)$$

Thus, $Z_T(u_{n-k}) = z^{-k}\,\bar{u}(z)$

2. *Left shifting rule*

$$Z_T(u_{n+k}) = z^k\left[\bar{u}(z) - u_0 - u_1 z^{-1} - u_2 z^{-2} - \cdots u_{k-1} z^{-(k-1)}\right]$$

or

$$Z_T(u_{n+k}) = z^k\left[\bar{u}(z) - \sum_{r=0}^{k-1} u_r z^{-r}\right]$$

Proof : $Z_T(u_{n+k}) = \sum_{n=0}^{\infty} u_{n+k} z^{-n} = z^k \sum_{n=0}^{\infty} u_{n+k} z^{-(n+k)}$

ie., $Z_T(u_{n+k}) = z^k\left[u_k z^{-k} + u_{k+1} z^{-(k+1)} + u_{k+2} z^{-(k+2)} + \cdots\right]$

$$= z^k\Big[\left(u_0 + u_1 z^{-1} + u_2 z^{-2} + \cdots + u_{k-1} z^{-(k-1)} + u_k z^{-k} + u_{k+1} z^{-(k+1)} + \cdots\right)$$

$$-\left(u_0 + u_1 z^{-1} + u_2 z^{-2} + \cdots + u_{k-1} z^{-(k-1)}\right)\Big]$$

$$= z^k\left[\sum_{n=0}^{\infty} u_n z^{-n} - \left(u_0 + u_1 z^{-1} + u_2 z^{-2} + \cdots + u_{k-1} z^{-(k-1)}\right)\right]$$

Thus, $Z_T(u_{n+k}) = z^k\left[\bar{u}(z) - u_0 - u_1 z^{-1} - u_2 z^{-2} - \cdots - u_{k-1} z^{-(k-1)}\right]$

Note : *Some particular cases.*

1. $Z_T(u_{n+1}) = z\left[\bar{u}(z) - u_0\right]$
2. $Z_T(u_{n+2}) = z^2\left[\bar{u}(z) - u_0 - u_1 z^{-1}\right]$
3. $Z_T(u_{n+3}) = z^3\left[\bar{u}(z) - u_0 - u_1 z^{-1} - u_2 z^{-2}\right]$ etc.

WORKED PROBELMS

[29] *Find the Z-transforms of the following.*

(i) e^{-an} **(ii)** $e^{-an}\, n$ **(iii)** $e^{-an}\, n^2$

☞ **(i)** $e^{-an} = (e^{-a})^n = k^n$ (*say*) where $k = e^{-a}$

We have, $Z_T(k^n) = \dfrac{z}{z-k}$

Thus, $$\boxed{Z_T(e^{-an}) = \frac{z}{z - e^{-a}}}$$

(ii) We have, $Z_T(n) = \dfrac{z}{(z-1)^2}$

$$\therefore \quad Z_T(k^n n) = \left\{\frac{z}{(z-1)^2}\right\}_{z\to(z/k)} = \frac{(z/k)}{(z/k-1)^2}$$

$$ie., \quad Z_T(k^n n) = \frac{kz}{(z-k)^2}$$

Thus by taking $k = e^{-a}$ we obtain,

$$\boxed{Z_T(e^{-an} n) = \frac{e^{-a} z}{(z-e^{-a})^2}}$$

(iii) We have, $Z_T(n^2) = \dfrac{z^2+z}{(z-1)^3}$

$$\therefore \quad Z_T(k^n n^2) = \left\{\frac{z^2+z}{(z-1)^3}\right\}_{z\to(z/k)} = \frac{(z/k)^2+(z/k)}{(z/k-1)^3}$$

$$ie., \quad Z_T(k^n n^2) = \frac{kz^2+k^2z}{(z-k)^3} = \frac{kz(z+k)}{(z-k)^3}$$

Thus by taking $k = e^{-a}$, we obtain

$$\boxed{Z_T(e^{-an} n^2) = \frac{e^{-a} z(z+e^{-a})}{(z-e^{-a})^3}}$$

[30] *Obtain the Z-transform of* $\cos n\theta$ *and* $\sin n\theta$. *Hence deduce Z-transforms of the following.*

(i) $k^n \cos n\theta$ **(ii)** $k^n \sin n\theta$ **(iii)** $e^{-an}\cos n\theta$ **(iv)** $e^{-an}\sin n\theta$

☞ We know that, $e^{in\theta} = \cos n\theta + i\sin n\theta$

We can write, $e^{in\theta} = (e^{i\theta})^n = k^n$ where $k = e^{i\theta}$

We have, $Z_T(k^n) = \dfrac{z}{z-k}$, k being $e^{i\theta}$

$$\therefore \quad Z_T(e^{in\theta}) = \frac{z}{z-e^{i\theta}} = \frac{z(z-e^{-i\theta})}{(z-e^{-i\theta})(z-e^{i\theta})}$$

$$= \frac{z[z-(\cos\theta - i\sin\theta)]}{z^2 - z(e^{i\theta}+e^{-i\theta})+1}$$

$$= \frac{z[(z-\cos\theta)+i\sin\theta]}{z^2-2z\cos\theta+1}$$

ie., $$Z_T(\cos n\theta + i\sin n\theta) = \frac{z[(z-\cos\theta)+i\sin\theta]}{z^2-2z\cos\theta+1}$$

or $$Z_T(\cos n\theta) + iZ_T(\sin n\theta) = \left[\frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}\right] + i\left[\frac{z\sin\theta}{z^2-2z\cos\theta+1}\right]$$

Equating the real and imaginary parts we get

$$Z_T(\cos n\theta) = \frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}$$

$$Z_T(\sin n\theta) = \frac{z\sin\theta}{z^2-2z\cos\theta+1}$$

Now, suppose $Z_T(\cos n\theta) = \bar{u}(z)$ & $Z_T(\sin n\theta) = \bar{v}(z)$ then by damping rule,

$$Z_T(k^n\cos n\theta) = \bar{u}(z/k) \text{ and } Z_T(k^n\sin n\theta) = \bar{v}(z/k)$$

$$\therefore \quad Z_T(k^n\cos n\theta) = \frac{(z/k)(z/k-\cos\theta)}{(z/k)^2 - 2\cdot z/k\cdot\cos\theta+1}$$

Thus, $$\boxed{Z_T(k^n\cos n\theta) = \frac{z(z-k\cos\theta)}{z^2-2kz\cos\theta+k^2}} \quad \ldots \text{(i)}$$

Also, $$Z_T(k^n\sin n\theta) = \frac{z/k\cdot\sin\theta}{(z/k)^2-2\cdot z/k\cdot\cos\theta+1}$$

Thus, $$\boxed{Z_T(k^n\sin n\theta) = \frac{kz\sin\theta}{z^2-2kz\cos\theta+k^2}} \quad \ldots \text{(ii)}$$

By taking $k = e^{-a}$ in (i) and (ii) we obtain the required results (iii) and (iv) as follows.

$$Z_T(e^{-an}\cos n\theta) = \frac{z(z-e^{-a}\cos\theta)}{z^2-2e^{-a}z\cos\theta+e^{-2a}} \quad \ldots (iii)$$

$$Z_T(e^{-an}\sin n\theta) = \frac{e^{-a}z\sin\theta}{z^2-2e^{-a}z\cos\theta+e^{-2a}} \quad \ldots (iv)$$

Note : Similar problem

Find the Z-transform of (i) $a^n \sin n\theta$ (ii) $a^{-n}\cos n\theta$ **[Dec 2018]**

[31] *Find the Z - transform of* $(\cos\theta + i\sin\theta)^n$

☞ We know that $\cos\theta + i\sin\theta = e^{i\theta}$

$\therefore \quad (\cos\theta+i\sin\theta)^n = (e^{i\theta})^n = k^n\,(say)$ where $k = e^{i\theta}$.

We have, $Z_T(k^n) = \dfrac{z}{z-k}$

ie., $\quad Z_T(e^{in\theta}) = \dfrac{z}{z-e^{i\theta}}$

Thus, $\boxed{Z_T\left[(\cos\theta+i\sin\theta)^n\right] = \dfrac{z}{z-e^{i\theta}}}$

[32] *Find the Z - transform of* $(n+1)^2$

☞ $Z_T[(n+1)^2] = Z_T(n^2+2n+1)$

$$= Z_T(n^2)+2Z_T(n)+Z_T(1)$$

$$= \frac{z^2+z}{(z-1)^3}+2\cdot\frac{z}{(z-1)^2}+\frac{z}{z-1}$$

$$= \frac{z^2+z+2z(z-1)+z(z-1)^2}{(z-1)^3}$$

Thus, $\boxed{Z_T\left[(n+1)^2\right] = \dfrac{z^3+z^2}{(z-1)^3}}$

[33] *Obtain the Z - transform of cosh nθ and sinh nθ.* **[June 2017]**

☞ $$\cos h\, n\theta = \frac{1}{2}(e^{n\theta} + e^{-n\theta}) = \frac{1}{2}\{(e^{\theta})^n + (e^{-\theta})^n\}$$

ie., $$\cos h\, n\theta = \frac{1}{2}\{p^n + q^n\}(say) \text{ where, } p = e^{\theta} \text{ and } q = e^{-\theta}$$

Now, $$Z_T(\cos h\, n\theta) = \frac{1}{2}\{Z_T(p^n) + Z_T(q^n)\}$$

$$= \frac{1}{2}\left\{\frac{z}{z-p} + \frac{z}{z-q}\right\}$$

$$= \frac{z}{2}\left\{\frac{1}{z-e^{\theta}} + \frac{1}{z-e^{-\theta}}\right\}$$

$$= \frac{z}{2}\left\{\frac{z-e^{-\theta}+z-e^{\theta}}{(z-e^{\theta})(z-e^{-\theta})}\right\}$$

$$= \frac{z}{2}\left\{\frac{2z-(e^{\theta}+e^{-\theta})}{z^2 - z(e^{\theta}+e^{-\theta})+1}\right\}$$

$$= \frac{z}{2}\left\{\frac{2z - 2\cos h\theta}{z^2 - 2z\cos h\,\theta + 1}\right\}$$

Thus, $$\boxed{Z_T(\cos h\, n\,\theta) = \frac{z(z-\cos h\,\theta)}{z^2 - 2z\cos h\,\theta + 1}}$$

Next, $$\sin h\, n\,\theta = \frac{e^{n\theta} - e^{-n\theta}}{2}$$

Proceeding on the same lines as before, we have,

$$Z_T(\sin h\, n\,\theta) = \frac{z}{2}\left\{\frac{1}{z-e^{\theta}} - \frac{1}{z-e^{-\theta}}\right\}$$

$$= \frac{z}{2}\left\{\frac{z-e^{-\theta}-z+e^{\theta}}{z^2 - 2z\cos h\,\theta + 1}\right\}$$

$$= \frac{z}{2} \cdot \frac{(e^{\theta} - e^{-\theta})}{z^2 - 2z\cosh\theta + 1}$$

$$= \frac{z}{2} \cdot \frac{2\sinh\theta}{z^2 - 2z\cosh\theta + 1}$$

Thus, $$\boxed{Z_T(\sinh n\theta) = \frac{z\sinh\theta}{z^2 - 2z\cosh\theta + 1}}$$

[34] *Obtain the Z - transform of u_{n+1}, u_{n+2} from the basic definition. Also give the Z - transform of u_{n+k}.*

☞ $$Z_T(u_{n+1}) = \sum_{n=0}^{\infty} u_{n+1} z^{-n} = z\sum_{0}^{\infty} \frac{1}{z} u_{n+1} z^{-n}$$

$$= z\sum_{0}^{\infty} u_{n+1} z^{-(n+1)}$$

$$= z\left\{u_1 z^{-1} + u_2 z^{-2} + \cdots\right\}$$

$$= z\left\{(u_0 + u_1 z^{-1} + u_2 z^{-2} + \cdots) - u_0\right\}$$

$$= z\left\{\sum_{n=0}^{\infty} u_n z^{-n} - u_0\right\}$$

Thus, $$\boxed{Z_T(u_{n+1}) = z[\bar{u}(z) - u_0]} \qquad \cdots (1)$$

Next, $$Z_T(u_{n+2}) = \sum_{n=0}^{\infty} u_{n+2} z^{-n} = z^2 \sum_{0}^{\infty} \frac{1}{z^2} u_{n+2} z^{-n}$$

$$Z_T(u_{n+2}) = z^2 \sum_{0}^{\infty} u_{n+2} z^{-(n+2)}$$

$$= z^2\left\{u_2 z^{-2} + u_3 z^{-3} + \cdots\right\}$$

$$= z^2\left\{(u_0 + u_1 z^{-1} + u_2 z^{-2} + u_3 z^{-3} + \cdots) - u_0 - u_1 z^{-1}\right\}$$

$$= z^2\left\{\sum_{n=0}^{\infty} u_n z^{-n} - u_0 - u_1 z^{-1}\right\}$$

Thus, $\boxed{Z_T(u_{n+2}) = z^2\left[\bar{u}(z) - u_0 - u_1 z^{-1}\right]}$ . . . (2)

Similarly, $Z_T(u_{n+3}) = z^3\left[\bar{u}(z) - u_0 - u_1 z^{-1} - u_2 z^{-2}\right]$ . . . (3)

In general we can write,

$$\boxed{Z_T(u_{n+k}) = z^k\left[\bar{u}(z) - u_0 - u_1 z^{-1} - u_2 z^{-2} - \cdots - u_{k-1} z^{-(k-1)}\right]}$$

[35] *Starting from the definition of the Z - transform find the Z - transform of* 1.

☞ By the definition, $Z_T(u_n) = \sum_{n=0}^{\infty} u_n z^{-n}$

$$\therefore \quad Z_T(1) = \sum_0^{\infty} z^{-n} = \sum_0^{\infty} \frac{1}{z^n}$$

$$= 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots \text{ is a geometric series.}$$

Hence, $$Z_T(1) = \frac{1}{1-(1/z)} = \frac{z}{z-1}$$

Thus, $$\boxed{Z_T(1) = \frac{z}{z-1}}$$

[36] *Find the Z - transform of* $2n + \sin(n\pi/4) + 1$ **[Dec 2016]**

☞ Let $u_n = 2n + \sin(n\pi/4) + 1$

$$\therefore \quad Z_T(u_n) = 2Z_T(n) + Z_T[\sin(n\pi/4)] + Z_T(1)$$

ie., $$Z_T(u_n) = \frac{2z}{(z-1)^2} + Z_T[\sin(n\pi/4)] + \frac{z}{z-1} \quad \text{. . . (1)}$$

We shall find, $Z_T[\sin(n\pi/4)]$

We have, $e^{in\pi/4} = \cos(n\pi/4) + i\sin(n\pi/4)$

But, $e^{in\pi/4} = (e^{i\pi/4})^n = k^n\,(say)$ where $k = e^{i\pi/4}$

We know that, $Z_T(k^n) = \dfrac{z}{z-k}$

$$\therefore \quad Z_T(e^{in\pi/4}) = \frac{z}{z - e^{i\pi/4}}$$

ie., $$Z_T[\cos(n\pi/4) + i\sin(n\pi/4)] = \frac{2}{z - \{\cos(\pi/4) + i\sin(\pi/4)\}}$$

$$= \frac{z}{z - \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}}$$

$$Z_T[\cos(n\pi/4) + i\sin(n\pi/4)] = \frac{z\left[\left(z - \frac{1}{\sqrt{2}}\right) + \frac{i}{\sqrt{2}}\right]}{\left(z - \frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2}}$$

$$\therefore \quad Z_T[\cos(n\pi/4) + i\sin(n\pi/4)] = \frac{z\left[\left(z - \frac{1}{\sqrt{2}}\right) + \frac{i}{\sqrt{2}}\right]}{z^2 - \sqrt{2}\,z + 1}$$

Equating the imaginary parts on both sides we get,

$$Z_T[\sin(n\pi/4)] = \frac{z/\sqrt{2}}{z^2 - \sqrt{2}\,z + 1} = \frac{z}{\sqrt{2}(z^2 - \sqrt{2}\,z + 1)}$$

We substitute this result in (1).

Thus, $$\boxed{Z_T(u_n) = \frac{2z}{(z-1)^2} + \frac{z}{\sqrt{2}(z^2 - \sqrt{2}\,z + 1)} + \frac{z}{z-1}}$$

where $u_n = 2n + \sin(n\pi/4) + 1$

[37] *Using* $Z_T(n^2) = \frac{z^2 + z}{(z-1)^3}$ *show that* $Z_T[(n+1)^2] = \frac{z^3 + z^2}{(z-1)^3}$

☞ Let $u_n = n^2$ and we have $Z_T(u_n) = Z_T(n^2) = \bar{u}(z)$.

Consider the property, $Z_T(u_{n+1}) = z[\bar{u}(z) - u_0]$ ... (1)

Since, $u_n = n^2$, $u_{n+1} = (n+1)^2$ and $u_0 = 0$. Hence (1) becomes,

$$Z_T[(n+1)^2] = z\left[\frac{z^2+z}{(z-1)^3} - 0\right]$$

Thus, $$\boxed{Z_T[(n+1)^2] = \frac{z^3+z^2}{(z-1)^3}}$$

[38] *Show that* $Z_T\left[\frac{1}{n!}\right] = e^{1/z}$. *Hence find* $Z_T\left[\frac{1}{(n+1)!}\right]$ *and* $Z_T\left[\frac{1}{(n+2)!}\right]$

☞ By the definition,

$$Z_T\left[\frac{1}{n!}\right] = \sum_{n=0}^{\infty}\frac{1}{n!}z^{-n} = 1 + \frac{z^{-1}}{1!} + \frac{z^{-2}}{2!} + \frac{z^{-3}}{3!} + \cdots$$

But, $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ and here we have $x = z^{-1}$

Thus, $$\boxed{Z_T\left[\frac{1}{n!}\right] = e^{z^{-1}} = e^{1/z}}$$

We have the properties,

$$Z_T(u_{n+1}) = z[\bar{u}(z) - u_0] \qquad \ldots (1)$$

$$Z_T(u_{n+2}) = z[\bar{u}(z) - u_0 - u_1 z^{-1}] \qquad \ldots (2)$$

Let, $u_n = \frac{1}{n!}$ $\therefore$ $Z_T(u_n) = \bar{u}(z) = e^{1/z}$

Also, $u_0 = \frac{1}{0!} = 1$ and $u_1 = \frac{1}{1!} = 1$

Thus by using these results in (1) and (2) we obtain,

$$\boxed{\begin{aligned} Z_T\left[\frac{1}{(n+1)!}\right] &= z[e^{1/z} - 1] \\ Z_T\left[\frac{1}{(n+2)!}\right] &= z[e^{1/z} - 1 - z^{-1}] \end{aligned}}$$

[39] *Find the Z - transform of* $\cos(n\pi/2+\pi/4)$

☞ Let $u_n = \cos(n\pi/2+\pi/4)$

$$= \cos(n\pi/2)\cos(\pi/4) - \sin(n\pi/2)\sin(\pi/4)$$

ie., $u_n = \dfrac{1}{\sqrt{2}}[\cos(n\pi/2) - \sin(n\pi/2)]$

$$\therefore \quad Z_T(u_n) = \frac{1}{\sqrt{2}}\left[Z_T\cos(n\pi/2) - Z_T\sin(n\pi/2)\right] \qquad \ldots (1)$$

Consider, $e^{i(n\pi/2)} = (e^{i\pi/2})^n = k^n\,(say)$ where $k = e^{i\pi/2}$.

We know that, $Z_T(k^n) = \dfrac{z}{z-k}$ and hence we have,

$$Z_T(e^{in\pi/2}) = \frac{z}{z-e^{i\pi/2}} = \frac{z}{z-\cos(\pi/2)-i\sin(\pi/2)} = \frac{z}{z-i}$$

ie., $Z_T(e^{in\pi/2}) = \dfrac{z(z+i)}{(z-i)(z+i)} = \dfrac{z^2+iz}{z^2+1}$

ie., $Z_T[\cos(n\pi/2)+i\sin(n\pi/2)] = \dfrac{z^2}{z^2+1} + i\dfrac{z}{z^2+1}$

$\Rightarrow \quad Z_T[\cos(n\pi/2)] = \dfrac{z^2}{z^2+1}$ and $Z_T[\sin(n\pi/2)] = \dfrac{z}{z^2+1}$

We substitute these results in (1).

Thus, $$\boxed{Z_T(u_n) = \frac{1}{\sqrt{2}}\left[\frac{z^2}{z^2+1} - \frac{z}{z^2+1}\right] = \frac{z(z-1)}{\sqrt{2}(z^2+1)}}$$

where $u_n = \cos(n\pi/2+n\pi/4)$

Remark : *It is advisible to remember the following results*

$$Z_T[\sin(n\pi/2)] = \frac{z}{z^2+1} \quad and \quad Z_T[\cos(n\pi/2)] = \frac{z^2}{z^2+1}$$

Note : Similar Problem

Find the Z - transform of $\cos(n\pi/2+\alpha)$ **[Dec 2017]**

[40] *Find the Z - transform of* $\sin(3n+5)$ **[June 2018]**

☞ Let $u_n = \sin(3n+5) = \sin 3n \cos 5 + \cos 3n \sin 5$

$$\therefore \quad Z_T(u_n) = \cos 5\, Z_T(\sin 3n) + \sin 5\, Z_T(\cos 3n) \qquad \ldots (1)$$

Consider, $e^{i(3n)} = (e^{3i})^n = k^n$ (*say*) where $k = e^{3i}$

We know that, $Z_T(k^n) = \dfrac{z}{z-k}$

ie., $$Z_T(e^{3in}) = \frac{z}{z-e^{3i}} = \frac{z}{(z-\cos 3)-i\sin 3}$$

$$= \frac{z[(z-\cos 3)+i\sin 3]}{(z-\cos 3)^2+\sin^2 3}$$

ie., $$Z_T(\cos 3n + i\sin 3n) = \frac{z(z-\cos 3)}{z^2-2z\cos 3+1} + i\frac{z\sin 3}{z^2-2z\cos 3+1}$$

$$\Rightarrow \quad Z_T(\cos 3n) = \frac{z(z-\cos 3)}{z^2-2z\cos 3+1} \text{ and } Z_T(\sin 3n) = \frac{z\sin 3}{z^2-2z\cos 3+1}$$

Substituting these results in (1) we get,

$$Z_T(u_n) = \cos 5\cdot\frac{z\sin 3}{z^2-2z\cos 3+1} + \sin 5\cdot\frac{z(z-\cos 3)}{z^2-2z\cos 3+1}$$

$$= \frac{z(\sin 3\cos 5-\cos 3\sin 5)+z^2\sin 5}{z^2-2z\cos 3+1}$$

$$= \frac{z(-\sin 2)+z^2\sin 5}{z^2-2z\cos 3+1}$$

Thus, $\boxed{Z_T(u_n) = \dfrac{z(z\sin 5-\sin 2)}{z^2-2z\cos 3+1}}$ where $u_n = \sin(3n+5)$

[41] *Find the Z - transform of* $\cos h(n\pi/2+\theta)$ **[Dec 2018]**

☞ Let $u_n = \cos h(n\pi/2+\theta) = \dfrac{1}{2}\left[e^{(n\pi/2+\theta)} + e^{-(n\pi/2+\theta)}\right]$

ie., $\quad u_n = \frac{1}{2}\left[e^{\theta} e^{n\pi/2} + e^{-\theta} e^{-n\pi/2}\right]$

$$\therefore \quad Z_T(u_n) = \frac{1}{2}\left[e^{\theta} Z_T(e^{n\pi/2}) + e^{-\theta} Z_T(e^{-n\pi/2})\right] \qquad \ldots(1)$$

We have, $\quad Z_T(k^n) = \frac{z}{z-k}$

Taking, $k = e^{\pi/2}$ and $e^{-\pi/2}$ we have,

$$Z_T(e^{n\pi/2}) = \frac{z}{z - e^{\pi/2}} \text{ and } Z_T(e^{-n\pi/2}) = \frac{z}{z - e^{-\pi/2}}$$

Hence (1) becomes,

$$Z_T(u_n) = \frac{1}{2}\left[e^{\theta} \cdot \frac{z}{z - e^{\pi/2}} + e^{-\theta} \cdot \frac{z}{z - e^{-\pi/2}}\right]$$

$$= \frac{z}{2}\left[\frac{e^{\theta}(z - e^{-\pi/2}) + e^{-\theta}(z - e^{\pi/2})}{(z - e^{\pi/2})(z - e^{-\pi/2})}\right]$$

$$= \frac{z}{2}\left[\frac{z(e^{\theta} + e^{-\theta}) - \{e^{(\pi/2-\theta)} + e^{-(\pi/2-\theta)}\}}{z^2 - z(e^{\pi/2} + e^{-\pi/2}) + 1}\right]$$

$$= \frac{z}{2}\left[\frac{2z\cos h\theta - 2\cos h(\pi/2 - \theta)}{z^2 - 2z\cos h(\pi/2) + 1}\right]$$

Thus, $\boxed{Z_T(u_n) = \frac{z^2 \cos h\theta - z\cos h(\pi/2 - \theta)}{z^2 - 2z\cos h(\pi/2) + 1}}$ where $u_n = \cos(n\pi/2 + n\theta)$

[42] *Prove that* $Z_T(e^{-an} u_n) = \bar{u}(e^a z)$ *given that* $Z_T(u_n) = \bar{u}(z)$

☞ By the definition,

$$Z_T(e^{-an} u_n) = \sum_{n=0}^{\infty} e^{-an} u_n z^{-n} = \sum_{0}^{\infty} u_n (e^a)^{-n} z^{-n}$$

$$= \sum_{0}^{\infty} u_n (e^a z)^{-n} = \bar{u}(e^a z)$$

Thus, $\boxed{Z_T(e^{-an} u_n) = \bar{u}(e^a z)}$

Note : 1. *This result can be deduced from the damping rule by taking $k = e^a$*
2. *This result is also referred to as the exponential shifting rule.*

[43] *If* $u_n = (1/2)^n$, *show that* $Z_T(u_n) = \dfrac{2z}{2z-1}$ *from the definition.*

☞ By the definition, $Z_T\left[\,(1/2)^n\right] = \sum_0^\infty (1/2)^n z^{-n}$

ie., $$Z_T\left[\,(1/2)^n\right] = \sum_0^\infty (2z)^{-n} = 1 + (2z)^{-1} + (2z)^{-2} + (2z)^{-3} + \cdots$$

$$= 1 + (1/2z) + (1/2z)^2 + (1/2z)^3 + \cdots$$

The series in RHS being a geometric series we have,

$$Z_T\left[\,(1/2)^n\right] = \frac{1}{1-(1/2z)} = \frac{2z}{2z-1}$$

Thus, $$\boxed{Z_T\left[\,(1/2)^n\right] = \frac{2z}{2z-1}}$$

[44] *Find the Z - transform of the unit step sequence* $u(n) = \begin{cases} 0, & n<0 \\ 1, & n \ge 0 \end{cases}$

☞ $$Z_T[u(n)] = \sum_{n=0}^\infty u(n) z^{-n}$$

$$= \sum_{n=0}^\infty 1 \cdot z^{-n}$$

$$= 1 + (1/z) + (1/z)^2 + (1/z)^3 + \cdots,$$ is a geometric series.

$\therefore$ $$Z_T[u(n)] = \frac{1}{1-(1/z)} = \frac{z}{z-1}$$

Thus, $$\boxed{Z_T[u(n)] = \frac{z}{z-1}}$$

[45] *Find the Z - transform of the unit impulse sequence* $\delta(n) = \begin{cases} 1, & n=0 \\ 0, & n \ne 0 \end{cases}$

☞ $$Z_T[\delta(n)] = \sum_{n=0}^\infty \delta(n) z^{-n}$$

$$Z_T[\delta(n)] = \delta(0)z^0 + \delta(1)z^{-1} + \delta(2)z^{-2} + \cdots$$
$$= 1 + 0 + 0 + \cdots = 1$$

Thus, $$\boxed{Z_T[\delta(n)] = 1}$$

[46] *If* $Z_T(u_n) = \bar{u}(z)$, *show that* $Z_T(n\,u_n) = -z\dfrac{d}{dz}[\bar{u}(z)]$

☞ $$Z_T(n\,u_n) = \sum_{n=0}^{\infty}(n\,u_n)\cdot z^{-n}$$
$$= -z\sum_{n=0}^{\infty} -n\,u_n\, z^{-n-1}$$
$$Z_T(n\,u_n) = -z\sum_{n=0}^{\infty} u_n \frac{d}{dz}(z^{-n})$$
$$= -z\sum_{n=0}^{\infty}\frac{d}{dz}(u_n\, z^{-n})$$
$$= -z\frac{d}{dz}\sum_{n=0}^{\infty} u_n\, z^{-n} = -z\frac{d}{dz}[\bar{u}(z)]$$

Thus, $$\boxed{Z_T(n\,u_n) = -z\frac{d}{dz}[\bar{u}(z)] = -z\frac{d}{dz}[Z_T(u_n)]}$$

Remark : *In general we can show that*

$$Z_T[n^k u_n] = (-1)^k z^k \frac{d^k}{dz^k}[\bar{u}(z)] \text{ where } k \text{ is a positive integer.}$$

[47] *Find the Z - transform of* $n\cos n\theta$

☞ Let $u_n = \cos n\theta$

We have, $Z_T(u_n) = Z_T(\cos n\theta) = \dfrac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1} = \bar{u}(z)$ *(Refer* [27])

We also have the property

$$Z_T(n\,u_n) = -z\frac{d}{dz}[\bar{u}(z)]$$

Hence, $Z_T(n\cos n\theta) = -z\dfrac{d}{dz}\left[\dfrac{z^2 - z\cos\theta}{z^2 - 2z\cos\theta + 1}\right]$

ie., $= -z\left\{\dfrac{(z^2 - 2z\cos\theta + 1)(2z - \cos\theta) - (z^2 - z\cos\theta)(2z - 2\cos\theta)}{(z^2 - 2z\cos\theta + 1)^2}\right\}$

$$= \frac{-z}{(z^2 - 2z\cos\theta + 1)^2}\left\{\begin{matrix} 2z^3 - 4z^2\cos\theta + 2z - z^2\cos\theta + 2z\cos^2\theta - \cos\theta \\ -(2z^3 - 2z^2\cos\theta - 2z^2\cos\theta + 2z\cos^2\theta \end{matrix}\right\}$$

$$= \frac{-z}{(z^2 - 2z\cos\theta + 1)^2}(2z - z^2\cos\theta - \cos\theta)$$

Thus, $\boxed{Z_T(n\cos n\theta) = \dfrac{z^3\cos\theta + z\cos\theta - 2z^2}{(z^2 - 2z\cos\theta + 1)^2}}$

[48] *Find* $Z_T\left[\dfrac{1}{n+1}\right]$

☞ $Z_T\left[\dfrac{1}{n+1}\right] = \displaystyle\sum_{n=0}^{\infty}\frac{1}{n+1}z^{-n}$

$$= 1 + \frac{1}{2}z^{-1} + \frac{1}{3}z^{-2} + \frac{1}{4}z^{-3} + \cdots$$

$$= 1 + \frac{1}{2z} + \frac{1}{3z^2} + \frac{1}{4z^3} + \cdots$$

or $Z_T\left[\dfrac{1}{n+1}\right] = z\left[\dfrac{1}{z} + \dfrac{1}{2z^2} + \dfrac{1}{3z^3} + \dfrac{1}{4z^4} + \cdots\right]$ . . . (1)

We have, $\log(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \cdots$

$\therefore$ $\log(1-x) = -x - \dfrac{x^2}{2} - \dfrac{x^3}{3} - \dfrac{x^4}{4} - \cdots$

or $-\log(1-x) = x + \dfrac{x^2}{2} + \dfrac{x^3}{3} + \dfrac{x^4}{4} + \cdots$

By taking $x = 1/z$ we have,

$$-\log\left(1-\frac{1}{z}\right) = \frac{1}{z}+\frac{1}{2z^2}+\frac{1}{3z^3}+\frac{1}{4z^4}+\cdots$$

or $$-\log\left(\frac{z-1}{z}\right) = \frac{1}{z}+\frac{1}{2z^2}+\frac{1}{3z^3}+\frac{1}{4z^4}+\cdots$$

or $$\log\left(\frac{z}{z-1}\right) = \frac{1}{z}+\frac{1}{2z^2}+\frac{1}{3z^3}+\frac{1}{4z^4}+\cdots \quad \ldots (2)$$

We use (2) in the RHS of (1).

Thus, $$\boxed{Z_T\left[\frac{1}{n+1}\right] = z\log\left(\frac{z}{z-1}\right)}$$

[49] *Find* $Z_T\left[\dfrac{1}{(n+1)(n+2)}\right]$

☞ Let $u_n = \dfrac{1}{(n+1)(n+2)}$

$$\frac{1}{(n+1)(n+2)} = \frac{1}{(n+1)}-\frac{1}{n+2}\text{, by partial fractions.}$$

Now, $$Z_T(u_n) = Z_T\left[\frac{1}{n+1}\right]-Z_T\left[\frac{1}{n+2}\right] \quad \ldots (1)$$

But, $$Z_T\left[\frac{1}{n+1}\right] = z\log\left(\frac{z}{z-1}\right) \quad \text{(Refer Problem - [48])} \quad \ldots (2)$$

Next, $$Z_T\left[\frac{1}{n+2}\right] = \sum_{n=0}^{\infty}\frac{1}{n+2}z^{-n}$$

$$= \frac{1}{2}+\frac{z^{-1}}{3}+\frac{z^{-2}}{4}+\cdots$$

ie., $$Z_T\left[\frac{1}{n+2}\right] = \frac{1}{2}+\frac{x}{3}+\frac{x^2}{4}+\cdots \quad \text{where} \quad x = z^{-1} = \frac{1}{z}$$

$$Z_T\left[\frac{1}{n+2}\right] = \frac{1}{x^2}\left(\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\cdots\right)$$

$$Z_T\left[\frac{1}{n+2}\right] = \frac{1}{x^2}\{-\log(1-x)-x\} \qquad \textit{(Refer Problem - [48])}$$

$$= -z^2\log\left(1-\frac{1}{z}\right)-z$$

$$\therefore \quad Z_T\left[\frac{1}{n+2}\right] = -z^2\log\left(\frac{z-1}{z}\right)-z \qquad \ldots (3)$$

Using (2) and (3) in (1) we have,

$$Z_T(u_n) = z\log\left(\frac{z}{z-1}\right)+z^2\log\left(\frac{z-1}{z}\right)+z$$

ie., $$Z_T(u_n) = z\log\left(\frac{z}{z-1}\right)-z^2\log\left(\frac{z}{z-1}\right)+z$$

Thus, $$\boxed{Z_T\left[\frac{1}{(n+1)(n+2)}\right] = z\log\left(\frac{z}{z-1}\right)\{1-z\}+z}$$

[50] *Find* (i) $Z_T\left[{}^{(n+p)}C_p\right]$ (ii) $Z_T\left[a^n \cdot {}^{(n+p)}C_p\right]$

☞ (i) $$Z_T\left[{}^{(n+p)}C_p\right] = \sum_{n=0}^{\infty} {}^{(n+p)}C_p\, z^{-n}$$

$$= {}^{p}C_p + {}^{(1+p)}C_p\, z^{-1} + {}^{(2+p)}C_p\, z^{-2} + \cdots$$

Using the fundamental property of combinations,

${}^{n}C_r = {}^{n}C_{n-r}$ and also ${}^{n}C_n = 1$ we have,

$$Z_T\left[{}^{(n+p)}C_p\right] = 1 + {}^{(1+p)}C_1\, z^{-1} + {}^{(2+p)}C_2\, z^{-2} + \cdots$$

$$Z_T\left[{}^{(n+p)}C_p\right] = 1+(1+p)z^{-1}+\frac{(2+p)(1+p)}{2!}z^{-2}+\cdots \qquad \ldots (1)$$

We have, $$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\cdots$$

$$\therefore \quad (1-x)^{-n} = 1+nx+\frac{n(n+1)}{2!}x^2+\cdots$$

Taking $x = z^{-1}$ and $n = 1 + p$ in this expansion we have

$$(1 - z^{-1})^{-(1+p)} = 1 + (1+p) z^{-1} + \frac{(1+p)(2+p)}{2!} \cdot z^{-2} + \cdots$$

Using this result in the RHS of (1) we get,

$$Z_T\left[{}^{(n+p)}C_p\right] = (1 - z^{-1})^{-(1+p)} = \left(\frac{z-1}{z}\right)^{-(1+p)}$$

Thus, $$\boxed{Z_T\left[{}^{(n+p)}C_p\right] = \left(\frac{z}{z-1}\right)^{1+p}}$$

(ii) We have the damping rule,

$$Z_T(a^n \cdot u_n) = \bar{u}(z/a) \text{ where } Z_T(u_n) = \bar{u}(z)$$

Now, $$Z_T\left[a^n \cdot {}^{n+p}C_p\right] = \left[\frac{z/a}{(z/a) - 1}\right]^{1+p}$$

Thus, $$\boxed{Z_T\left[a^n \cdot {}^{(n+p)}C_p\right] = \left[\frac{z}{z-a}\right]^{1+p}}$$

3.6 Initial Value Theorem

Statement : *If* $Z_T(u_n) = \bar{u}(z)$ *then* $\underset{z \to \infty}{lt}\, \bar{u}(z) = u_0$

Proof : We have by the definition,

$$Z_T(u_n) = \sum_{n=0}^{\infty} u_n z^{-n}$$

ie., $$\bar{u}(z) = u_0 + u_1 z^{-1} + u_2 z^{-2} + + u_3 z^{-3} + \cdots \quad \ldots (1)$$

$$\therefore \quad \underset{z\to\infty}{lt}\, \bar{u}(z) = \underset{z\to\infty}{lt}\left[u_0 + \frac{u_1}{z} + \frac{u_2}{z^2} + \frac{u_3}{z^3} + \cdots\right] = u_0 + 0 + 0 + \cdots = u_0$$

Thus, $\underset{z\to\infty}{lt}\, \bar{u}(z) = u_0$

Remarks : *Similarly we can also obtain other initial values* $u_1, u_2, \ldots$ *as follows. We have from* (1),

$$\therefore \quad \bar{u}(z) - u_0 = \frac{u_1}{z} + \frac{u_2}{z^2} + \frac{u_3}{z^3} + \cdots$$

or $$z[\bar{u}(z) - u_0] = u_1 + \frac{u_2}{z} + \frac{u_3}{z^2} + \cdots$$

$$\therefore \quad \underset{z \to \infty}{lt}\, z[\bar{u}(z) - u_0] = u_1$$

Also, $$\bar{u}(z) - u_0 - \frac{u_1}{z} = \frac{u_2}{z^2} + \frac{u_3}{z^3} + \cdots$$

$$\therefore \quad \underset{z \to \infty}{lt}\, z^2 \left\{ \bar{u}(z) - u_0 - \frac{u_1}{z} \right\} = u_2$$

3.7 Final Value Theorem

Statement : If $Z_T(u_n) = \bar{u}(z)$ *then,* $\underset{z \to 1}{lt}\, [(z-1)\,\bar{u}(z)] = \underset{n \to \infty}{lt}\, u_n$

Proof : We have the result,

$$Z_T(u_{n+1}) = z[\bar{u}(z) - u_0] \qquad \ldots (1)$$

Also, $$Z_T(u_n) = \bar{u}(z) \qquad \ldots (2)$$

Now, (1) – (2) will give us,

$$Z_T(u_{n+1}) - Z_T(u_n) = z\bar{u}(z) - zu_0 - \bar{u}(z)$$

or $$Z_T(u_{n+1} - u_n) = (z-1)\bar{u}(z) - zu_0 \; .$$

ie., $$\sum_{n=0}^{\infty} (u_{n+1} - u_n) z^{-n} = (z - `1)\bar{u}(z) - zu_0$$

Now taking limit as $z \to 1$ we have,

$$\underset{z \to 1}{lt} \sum_{n=0}^{\infty} (u_{n+1} - u_n) \frac{1}{z^n} = \underset{z \to 1}{lt} [\, z-1)\bar{u}(z) - zu_0 \,]$$

ie., $$\sum_{n=0}^{\infty} (u_{n+1} - u_n) = \underset{z \to 1}{lt} [(z-1)\bar{u}(z)] - u_0$$

or $$\underset{z \to 1}{lt} [(z-1)\bar{u}(z)] = u_0 + \sum_{n=0}^{\infty} (u_{n+1} - u_n)$$

$$\underset{z\to 1}{lt}\left[(z-1)\bar{u}(z)\right] = u_0 + \underset{m\to\infty}{lt}\sum_{n=0}^{m}(u_{n+1}-u_n)$$

$$= u_0 + \underset{m\to\infty}{lt}\{(u_1-u_0)+(u_2-u_1)+(u_3-u_2)+\ldots$$

$$+(u_m-u_{m-1})+(u_{m+1}-u_m$$

$$= u_0 + \underset{m\to\infty}{lt}(-u_0+u_{m+1})$$

$$= \underset{m\to\infty}{lt}\, u_{m+1}$$

It should be noted that as $m\to\infty$, $(m+1)$ also tends ∞ and hence RHS equivalent to $\underset{(m+1)\to\infty}{lt} u_{m+1}$ which is further equivalent to $\underset{n\to\infty}{lt} u_n$

Thus we have proved that,

$$\underset{z\to 1}{lt}\left[(z-1)\bar{u}(z)\right] = \underset{n\to\infty}{lt}\, u_n$$

WORKED PROBLEMS

[51] *If* $\bar{u}(z) = \dfrac{2z^2+3z+12}{(z-1)^4}$, *find the value of* u_0, u_1, u_2, u_3.

☞ We have by the definition of Z - transform,

$$\bar{u}(z) = u_0 + u_1 z^{-1} + u_2 z^{-2} + u_3 z^{-3} + u_4 z^{-4} + \cdots$$

ie., $$\bar{u}(z) = u_0 + \frac{u_1}{z} + \frac{u_2}{z^2} + \frac{u_3}{z^3} + \frac{u_4}{z^4} + \cdots$$

Therefore we have,

$$u_0 = \underset{z\to\infty}{lt}\,\bar{u}(z) \quad \ldots($$

$$u_1 = \underset{z\to\infty}{lt}\, z[\bar{u}(z)-u_0] \quad \ldots($$

$$u_2 = \underset{z\to\infty}{lt}\, z^2\left[\bar{u}(z)-u_0-\frac{u_1}{z}\right] \quad \ldots($$

$$u_3 = \underset{z\to\infty}{lt}\, z^3\left[\bar{u}(z)-u_0-\frac{u_1}{z}-\frac{u_2}{z^2}\right] \quad \ldots($$

From (1), $u_0 = \underset{z\to\infty}{lt} \dfrac{2z^2+3z+12}{(z-1)^4}$

$$= \underset{z\to\infty}{lt} \frac{z^2(2+3/z+12/z^2)}{z^4(1-1/z)^4}$$

$$= \underset{z\to\infty}{lt} \frac{1}{z^2} \cdot \frac{(2+3/z+12/z^2)}{(1-1/z)^4} = 0 \cdot \frac{2}{1} = 0$$

$\therefore \quad u_0 = 0$

From (2), $u_1 = \underset{z\to\infty}{lt}\ z \cdot \dfrac{2z^2+3z+12}{(z-1)^4}$, since $u_0 = 0$

$$= \underset{z\to\infty}{lt}\ z^3 \cdot \frac{(2+3/z+12/z^2)}{z^4(1-1/z)^4}$$

$$= \underset{z\to\infty}{lt} \frac{1}{z} \cdot \frac{(2+3/z+12/z^2)}{(1-1/z)^4} = 0$$

$\therefore \quad u_1 = 0$

From (3), $u_2 = \underset{z\to\infty}{lt}\ z^2 \left[\dfrac{2z^2+3z+12}{(z-1)^4}\right]$ since $u_0 = 0 = u_1$.

$$= \underset{z\to\infty}{lt}\ z^4 \cdot \frac{(2+3/z+12/z^2)}{z^4(1-1/z)^4} = \frac{2+0+0}{(1-0)^4} = 2$$

$\therefore \quad u_2 = 2$

From (4), $u_3 = \underset{z\to\infty}{lt}\ z^3 \left[\dfrac{2z^2+3z+12}{(z-1)^4} - \dfrac{2}{z^2}\right]$

(We need to simplify here)

$$u_3 = \underset{z\to\infty}{lt}\ z^3 \left[\frac{2z^4+3z^3+12z^2-2(z^4-4z^3+6z^2-4z+1)}{z^2(z-1)^4}\right]$$

$$u_3 = \underset{z\to\infty}{lt}\ z\left[\frac{11z^3+8z-2}{(z-1)^4}\right]$$

$$= \underset{z\to\infty}{lt}\ z^4 \cdot \frac{(11+8/z^2-2/z^3)}{z^4(1-1/z)^4} = 11$$

$\therefore \quad u_3 = 11$

Thus, $\boxed{u_0 = 0,\ u_1 = 0,\ u_2 = 2 \text{ and } u_3 = 11}$

[52] *Given,* $Z_T(u_n) = \dfrac{2z^2+3z+4}{(z-3)^3}$, $|z|>3$ *show that* $u_1 = 2$, $u_2 = 21$, $u_3 = 139$

☞ [*As in Problem-53, results* (1) *to* (4) *need to be given*]

$$u_0 = \underset{z\to\infty}{lt}\ \frac{2z^2+3z+4}{(z-3)^3}$$

$$= \underset{z\to\infty}{lt}\ \frac{z^2(2+3/z+4/z^2)}{z^3(1-3/z)^3}$$

$$= \underset{z\to\infty}{lt}\ \frac{1}{z}\cdot\frac{(2+3/z+4/z^2)}{(1-3/z)^3} = 0$$

$\therefore \quad u_0 = 0$

$$u_1 = \underset{z\to\infty}{lt}\ z\left[\frac{2z^2+3z+4}{(z-3)^3}\right] \text{ since } u_0 = 0.$$

$$= \underset{z\to\infty}{lt}\ z^3\cdot\frac{(2+3/z+4/z^2)}{z^3(1-3/z)^3} = 2$$

$\therefore \quad u_1 = 2$

$$u_2 = \underset{z\to\infty}{lt}\ z^2\left[\frac{2z^2+3z+4}{(z-3)^3} - 0 - \frac{2}{z}\right]$$

$$= \underset{z\to\infty}{lt}\ z^2\left[\frac{2z^3+3z^2+4z-2(z^3-9z^2+27z-27)}{z(z-3)^3}\right]$$

$$u_2 = \lim_{z\to\infty} \frac{z(21z^2 - 50z + 54)}{(z-3)^3}$$

$$= \lim_{z\to\infty} \frac{z^3(21 - 50/z + 54/z^2)}{z^3(1-3/z)^3} = 21$$

$\therefore \quad u_2 = 21$

$$u_3 = \lim_{z\to\infty} z^3\left[\frac{2z^2+3z+4}{(z-3)^3} - \frac{2}{z} - \frac{21}{z^2}\right]$$

$$= \lim_{z\to\infty} z^3\left[\frac{z^3(2z^2+3z+4) - 2z^2(z-3)^3 - 21z(z-3)^3}{z^3(z-3)^3}\right]$$

$$= \lim_{z\to\infty}\left[\frac{(2z^5+3z^4+4z^3) - (2z^2+21z)(z^3 - 9z^2 + 27z - 27)}{(z-3)^3}\right]$$

$$= \lim_{z\to\infty}\left[\frac{(2z^5+3z^4+4z^3) - (2z^5+3z^4 - 135z^3 + 513z^2 - 567z)}{(z-3)^3}\right]$$

$$= \lim_{z\to\infty}\left[\frac{139z^3 - 513z^2 + 567z}{(z-3)^3}\right]$$

$$= \lim_{z\to\infty} \frac{z^3(139 - 513/z + 567/z^2)}{z^3(1-3/z)^3} = 139$$

$\therefore \quad u_3 = 139$

Thus we have proved that $\boxed{u_1 = 2,\ u_2 = 21 \text{ and } u_3 = 139}$

[53] *Given* $Z_T(u_n) = \dfrac{z}{z-1} + \dfrac{z}{z^2+1}$ *obtain the Z - transform of* u_{n+2}

☞ We have, $Z_T(u_{n+2}) = z^2\left[\bar{u}(z) - u_0 - \dfrac{u_1}{z}\right]$ ... (1)

We shall first compute u_0 and u_1.

$$u_0 = \lim_{z\to\infty} \bar{u}(z)$$

$$u_0 = \underset{z\to\infty}{lt}\left[\frac{z}{z-1} + \frac{z}{z^2+1}\right]$$

$$= \underset{z\to\infty}{lt}\left[\frac{z}{z(1-1/z)} + \frac{z}{z^2(1+1/z^2)}\right]$$

$$= \underset{z\to\infty}{lt}\left[\frac{1}{1-1/z} + \frac{1}{z}\cdot\frac{1}{(1+1/z^2)}\right] = 1+0 = 1$$

$\therefore \quad u_0 = 1$

$$u_1 = \underset{z\to\infty}{lt}\, z[\bar{u}(z) - u_0]$$

$$= \underset{z\to\infty}{lt}\, z\left[\frac{z}{z-1} + \frac{z}{z^2+1} - 1\right]$$

$$= \underset{z\to\infty}{lt}\, z\left[\frac{z^3+z+z^2-z-(z^3-z^2+z-1)}{(z-1)(z^2+1)}\right]$$

$$= \underset{z\to\infty}{lt}\, z\left[\frac{2z^2-z+1}{(z-1)(z^2+1)}\right]$$

$$= \underset{z\to\infty}{lt}\, \frac{z^3(2-1/z+1/z^2)}{z(1-1/z)\,z^2(1+1/z^2)} = 2$$

$\therefore \quad u_1 = 2$

Now, from (1) we have,

$$Z_T(u_{n+2}) = z^2\left[\frac{z}{z-1} + \frac{z}{z^2+1} - 1 - \frac{2}{z}\right]$$

$$= z^2\left[\frac{z^3+z^2}{(z-1)(z^2+1)} - \frac{(z+2)}{z}\right]$$

$$= \frac{z^2(z^2-z+2)}{(z-1)(z^2+1)(z)},$$ on simplification.

Thus,

$$\boxed{Z_T(u_{n+2}) = \frac{z(z^2-z+2)}{(z-1)(z^2+1)}}$$

ASSIGNMENT

1. Show that the Z - transform of n^4 is $\dfrac{z^4+11z^3+11z^2+z}{(z-1)^5}$

2. Find the Z - transform of

   (i) e^{an} (ii) $e^{an}\cdot n$ (iii) $e^{an}\cdot n^2$

3. Obtain the Z - transform of

   (i) $(n+2)^2$ (ii) $(n+1)^3$ (iii) k^{n+4}

4. Show that the Z-transform of $\dfrac{a^n}{n!}e^{-a}$ is $e^{a/z}$

5. Show that

   (i) $Z_T(a^n \cosh n\theta) = \dfrac{z(z-a\cosh\theta)}{z^2-2az\cosh\theta+a^2}$

   (ii) $Z_T(a^n \sinh n\theta) = \dfrac{az\sinh\theta}{z^2-2az\cosh\theta+a^2}$

6. Show that the Z-transform of $\cos(n\pi/8+\theta)$ is

   $$\frac{z^2\cos\theta - z\cos(\pi/8-\theta)}{z^2-2z\cos(\pi/8)+1}$$

7. Starting from the basic definition of Z-transform show that

   $$Z_T[(1/3)^n] = \frac{3z}{3z-1}$$

8. Show that $Z_T[\sin(n+1)\theta] = \dfrac{z^2\sin\theta}{z^2-2z\cos\theta+1}$

9. If $\bar{u}(z) = \dfrac{2z^2+5z+14}{(z-1)^4}$ show that $u_0 = 0 = u_1$, $u_2 = 2$ and $u_3 = 13$

10. If $Z_T(u_n) = \dfrac{5z^2+3z+12}{(z-1)^4}$ show that $u_2 = 5$ and $u_3 = 23$

ANSWERS

2. (i) $\dfrac{z}{z-e^a}$ (ii) $\dfrac{z e^a}{(z-e^a)^2}$ (iii) $\dfrac{e^a(z+e^a)}{(z-e^a)^3}$

3. (i) $\dfrac{z^2+z}{(z-1)^3}+\dfrac{4z}{(z-1)^2}+\dfrac{4z}{z-1}$

(ii) $\dfrac{z^3+4z^2+z}{(z-1)^4}+\dfrac{3(z^2+z)}{(z-1)^2}+\dfrac{3z}{(z-1)^2}+\dfrac{z}{z-1}$

(iii) $\dfrac{k^4 z}{z-k}$

3.8 Inverse Z - Transforms

We have already stated that, if $Z_T(u_n)=\bar{u}(z)$ then $Z_T^{-1}[\bar{u}(z)]=u_n$ is called the inverse Z-transform of $\bar{u}(z)$.

List of standard inverse Z - transforms

1. $Z_T^{-1}\left[\dfrac{z}{z-1}\right]=1$

2. $Z_T^{-1}\left[\dfrac{z}{z-k}\right]=k^n$

3. $Z_T^{-1}\left[\dfrac{z}{(z-1)^2}\right]=n$

4. $Z_T^{-1}\left[\dfrac{kz}{(z-k)^2}\right]=k^n\cdot n$

5. $Z_T^{-1}\left[\dfrac{z^2+z}{(z-1)^3}\right]=n^2$

6. $Z_T^{-1}\left[\dfrac{kz^2+k^2z}{(z-k)^3}\right]=k^n\cdot n^2$

7. $Z_T^{-1}\left[\dfrac{z^3+4z^2+z}{(z-1)^4}\right]=n^3$

8. $Z_T^{-1}\left[\dfrac{kz^3+4k^2z^2+k^3z}{(z-k)^4}\right]=k^n\cdot n^3$

9. $Z_T^{-1}\left[\dfrac{z}{z^2+1}\right]=\sin(n\pi/2)$

10. $Z_T^{-1}\left[\dfrac{z^2}{z^2+1}\right]=\cos(n\pi/2)$

Type - 1.

Inverse Z-transform of rational algebraic functions by partial fractions method.

Step by step working procedure for problems.

Step-1 Given $\bar{u}(z) = \dfrac{f(z)}{g(z)}$ we need to express $g(z)$ in terms of non repeated linear factors only.

Step-2 We consider $\dfrac{\bar{u}(z)}{z}$ in the form of a proper fraction and resolve into partial fractions.

Step-3 We multiply by z to have $\bar{u}(z)$ involving various terms of the form $c \cdot (z/z-k)$, c being a constant.

Step-4 Finally we compute the inverse Z-transform of these terms resulting in the required $Z_T^{-1}[\bar{u}(z)]$

Important Note : *If $g(z)$ involves repeated linear factors of the form : $(z-k)^2, (z-k)^3, (z-k)^4 \cdots$ we need to take it into account the corresponding terms in the numerator : kz, (kz^2+k^2z), $(kz^3+4k^2z^2+k^3z), \cdots$ (by refering into inverse Z-transform) respectively and express $u(z)$ suitably with terms multiplied by $A, B, C, \ldots$*

We compute $A, B, C, \ldots$ and find $Z_T^{-1}[\bar{u}(z)]$

WORKED PROBLEMS

[54] *Find the inverse Z-transform of* $\dfrac{z}{(z-1)(z-2)}$ **[June 2018]**

☞ Let $\bar{u}(z) = \dfrac{z}{(z-1)(z-2)}$

$$\therefore \quad \frac{\bar{u}(z)}{z} = \frac{1}{(z-1)(z-2)}$$

Let $\dfrac{1}{(z-1)(z-2)} = \dfrac{A}{z-1} + \dfrac{B}{z-2}$ or $1 = A(z-2) + B(z-1)$

Put $z = 1 : 1 = A(-1)$ $\therefore\ A = -1$

Put $z = 2 : 1 = B(1)$ $\therefore\ B = 1$

Hence, $\dfrac{\bar{u}(z)}{z} = \dfrac{-1}{z-1} + \dfrac{1}{z-2}$ or $\bar{u}(z) = \dfrac{-z}{z-1} + \dfrac{z}{z-2}$

$$\Rightarrow \quad Z_T^{-1}[\bar{u}(z)] = Z_T^{-1}\left[\frac{z}{z-2}\right] - Z_T^{-1}\left[\frac{z}{z-1}\right]$$

We have, $Z_T^{-1}\left[\dfrac{z}{z-k}\right] = k^n$

Thus, $\boxed{Z_T^{-1}[\bar{u}(z)] = u_n = 2^n - 1}$

Note : Similar Problems : *Find the inverse Z - transform :*

(i) $\dfrac{z}{(z-2)(z-3)}$ **[Dec 2017]** (ii) $\dfrac{z}{z^2+7z+10}$ **[Dec 2016]**

Remark : *It may be easily seen that in* (ii) $z^2 + 7z + 10 = (z+2)(z+5)$.

[55] *Find* $Z_T^{-1}\left[\dfrac{5z}{(2-z)(3z-1)}\right]$

☞ Let $\bar{u}(z) = \dfrac{5z}{(2-z)(3z-1)}$

$$\therefore \quad \frac{\bar{u}(z)}{z} = \frac{5}{(2-z)(3z-1)}$$

Let $\dfrac{5}{(2-z)(3z-1)} = \dfrac{A}{2-z} + \dfrac{B}{3z-1}$

or $5 = A(3z-1) + B(2-z)$

Put $z = 2 : 5 = A(5)$ $\therefore\ A = 1$

Put $z = 1/3 : 5 = B(2-1/3)$ or $5 = B(5/3)$ $\therefore\ B = 3$

Hence, $\dfrac{\bar{u}(z)}{z} = \dfrac{1}{2-z} + \dfrac{3}{3z-1}$

or $\quad \bar{u}(z) = \dfrac{-z}{z-2} + \dfrac{3z}{3(z-1/3)}$

$\Rightarrow \quad Z_T^{-1}[\bar{u}(z)] = -Z_T^{-1}\left[\dfrac{z}{z-2}\right] + Z_T^{-1}\left[\dfrac{z}{z-1/3}\right]$

Thus, $\quad \boxed{Z_T^{-1}[\bar{u}(z)] = u_n = -2^n + (1/3)^n}$

[58] *Compute the inverse z-transform of* $\dfrac{3z^2+2z}{(5z-1)(5z+2)}$ **[June 2017, Dec. 18]**

☞ Let, $\bar{u}(z) = \dfrac{3z^2+2z}{(5z-1)(5z+2)}$

$\therefore \quad \dfrac{\bar{u}(z)}{z} = \dfrac{3z+2}{(5z-1)(5z+2)}$

Let, $\quad \dfrac{3z+2}{(5z-1)(5z+2)} = \dfrac{A}{5z-1} + \dfrac{B}{5z+2}$

or $\quad 3z+2 = A(5z+2) + B(5z-1)$

Put $\quad z = 1/5 : 13/5 = A(3) \quad \therefore A = 13/15$

Put $\quad z = -2/5 : 4/5 = B(-3) \quad \therefore B = -4/15$

Hence, $\quad \dfrac{\bar{u}(z)}{z} = \dfrac{13}{15}\dfrac{1}{5z-1} - \dfrac{4}{15}\dfrac{1}{5z+2}$

or $\quad \bar{u}(z) = \dfrac{13}{75}\dfrac{z}{z-(1/5)} - \dfrac{4}{75}\dfrac{z}{z+(2/5)}$

$\Rightarrow \quad Z_T^{-1}[\bar{u}(z)] = \dfrac{13}{75}Z_T^{-1}\left[\dfrac{z}{z-(1/5)}\right] - \dfrac{4}{75}Z_T^{-1}\left[\dfrac{z}{z+(2/5)}\right]$

Thus, $\quad \boxed{Z_T^{-1}[\bar{u}(z)] = u_n = \dfrac{1}{75}\left\{13(1/5)^n - 4(-2/5)^n\right\}}$

[59] *Obtain the inverse Z-transform of* $\dfrac{2z^2+3z}{(z+2)(z-4)}$

☞ Let $\quad \bar{u}(z) = \dfrac{2z^2+3z}{(z+2)(z-4)}$

$$\therefore \quad \frac{\bar{u}(z)}{z} = \frac{2z+3}{(z+2)(z-4)}$$

Let, $\dfrac{2z+3}{(z+2)(z-4)} = \dfrac{A}{z+2} + \dfrac{B}{z-4}$

or $\quad 2z+3 = A(z-4) + B(z+2)$

Put $\quad z = -2 : -1 = A(-6) \quad \therefore \; A = 1/6$

Put $\quad z = 4 \quad : 11 = B(6) \qquad \therefore \; B = 11/6$

Hence, $\dfrac{\bar{u}(z)}{z} = \dfrac{1}{6} \cdot \dfrac{1}{z+2} + \dfrac{11}{6} \cdot \dfrac{1}{z-4}$

or $\quad \bar{u}(z) = \dfrac{1}{6} \cdot \dfrac{z}{z+2} + \dfrac{11}{6} \cdot \dfrac{z}{z-4}$

$$\Rightarrow \quad Z_T^{-1}[\bar{u}(z)] = \frac{1}{6} Z_T^{-1}\left[\frac{z}{z+2}\right] + \frac{11}{6} Z_T^{-1}\left[\frac{z}{z-4}\right]$$

Thus, $\boxed{Z_T^{-1}[\bar{u}(z)] = u_n = \frac{1}{6}\{(-2)^n + 11(4)^n\}}$

[58] *Find* $Z_T^{-1}\left[\dfrac{8z-z^3}{(4-z)^3}\right]$

☞ [*We have to note that the denominator has repeated linear factors*]

Let $\quad \bar{u}(z) = \dfrac{8z-z^3}{(4-z)^3} = \dfrac{z^3-8z}{(z-4)^3}$

We have, $Z_T^{-1}\left[\dfrac{z}{z-4}\right] = 4^n$, $Z_T^{-1}\left[\dfrac{4z}{(z-4)^2}\right] = 4^n \cdot n$

and $\quad Z_T^{-1}\left[\dfrac{4z^2+16z}{(z-4)^3}\right] = 4^n \cdot n^2$

We resolve $\bar{u}(z)$ as follows. (*Note this important step*)

Let, $\dfrac{z^3-8z}{(z-4)^3} = A \cdot \dfrac{z}{z-4} + B \cdot \dfrac{4z}{(z-4)^2} + C \cdot \dfrac{4z^2+16z}{(z-4)^3} \qquad \ldots (1)$

or $$\frac{z^3-8z}{(z-4)^3} = \frac{Az(z-4)^2+4Bz(z-4)+4Cz(z+4)}{(z-4)^3}$$

or $$z^2-8 = A\,(z-4)^2+4B(z-4)+4C(z+4)$$

Put $z=4 : 8 = 4C(8) \quad \therefore\ C = 1/4$

Equating the coefficient of z^2 on both sides we have, $A = 1$

Also by equating the coefficient of z on both sides we have,

$$-8A+4B+4C = 0$$

ie., $-8+4B+1 = 0 \qquad \therefore\ B = 7/4$

Substituting the values of A , B , C in (1) we have,

$$\frac{z^3-8z}{(z-4)^3} = \frac{z}{z-4}+\frac{7}{4}\cdot\frac{4z}{(z-4)^2}+\frac{1}{4}\cdot\frac{4z^2+16z}{(z-4)^3}$$

$$\Rightarrow \quad Z_T^{-1}\left[\frac{z^3-8z}{(z-4)^3}\right] = Z_T^{-1}\left[\frac{z}{z-4}\right]+\frac{7}{4}Z_T^{-1}\left[\frac{4z}{(z-4)^2}\right]+\frac{1}{4}Z_T^{-1}\left[\frac{4z^2+16z}{(z-4)^3}\right]$$

That is, $Z_T^{-1}[\bar{u}(z)] = u_n = 4^n+\frac{7}{4}4^n\cdot n+\frac{1}{4}4^n n^2$

Thus, $$\boxed{Z_T^{-1}[\bar{u}(z)] = u_n = \frac{4^n}{4}(4+7n+n^2) = 4^{n-1}(4+7n+n^2)}$$

[59] *Given* $U(z) = \dfrac{4z^2-2z}{z^3-5z^2+8z-4}$ *, find* u_n.

☞ $U(z)$ or $\bar{u}(z) = \dfrac{4z^2-2z}{z^3-5z^2+8z-4}$ by data.

We shall factorize the denominator first.

$$\begin{aligned} z^3-5z^2+8z-4 &= (z^3-5z^2+4z)+(4z-4) \\ &= z(z^2-5z+4)+4(z-1) \\ &= z(z-1)(z-4)+4(z-1) \\ &= (z-1)(z^2-4z+4) \\ &= (z-1)(z-2)^2 \end{aligned}$$

We have, $\bar{u}(z) = \dfrac{4z^2 - 2z}{(z-1)(z-2)^2}$

We have, $Z_T^{-1}\left[\dfrac{z}{z-1}\right] = 1,\ Z_T^{-1}\left[\dfrac{z}{z-2}\right] = 2^n,\ Z_T^{-1}\left[\dfrac{2z}{(z-2)^2}\right] = 2^n \cdot n$

We resolve $\bar{u}(z)$ as follows.

$$\bar{u}(z) = \frac{4z^2 - 2z}{(z-1)(z-2)^2} = A\cdot\frac{z}{z-1} + B\cdot\frac{z}{z-2} + C\cdot\frac{2z}{(z-2)^2} \qquad \ldots(1)$$

or $$\frac{4z^2 - 2z}{(z-1)(z-2)^2} = \frac{Az(z-2)^2 + Bz(z-1)(z-2) + 2Cz(z-1)}{(z-1)(z-2)^2}$$

or $\quad 4z - 2 = A(z-2)^2 + B(z-1)(z-2) + 2C(z-1)$

Put $\quad z = 1 : 2 = A(1) \quad \therefore\ A = 2$

Put $\quad z = 2 : 6 = 2C(1) \quad \therefore\ C = 3$

Equating the coefficient of z^2 on both sides we have,

$$A + B = 0 \qquad \therefore\ B = -2$$

Substituting the values of A, B, C in (1) and taking inverse we have,

$$Z_T^{-1}[\bar{u}(z)] = 2Z_T^{-1}\left[\frac{z}{z-1}\right] - 2Z_T^{-1}\left[\frac{z}{z-2}\right] + 3Z_T^{-1}\left[\frac{2z}{(z-2)^2}\right]$$

$$= 2\cdot 1 - 2\cdot 2^n + 3\cdot 2^n\cdot n$$

Thus, $$\boxed{Z_T^{-1}[\bar{u}(z)] = u_n = 2 - 2^{n+1} + 3n\cdot 2^n}$$

[60] *Find the inverse Z-transform of* $\dfrac{z^3 - 20z}{(z-2)^3(z-4)}$

☞ Let $\bar{u}(z) = \dfrac{z^3 - 20z}{(z-2)^3(z-4)}$

We have,

$$Z_T^{-1}\left[\frac{z}{z-2}\right] = 2^n, \quad Z_T^{-1}\left[\frac{2z}{(z-2)^2}\right] = 2^n\, n$$

$$Z_T^{-1}\left[\frac{2z^2+4z}{(z-2)^3}\right]=2^n\cdot n^2,\quad Z_T^{-1}\left[\frac{z}{z-4}\right]=4^n$$

We resolve $\bar{u}(z)$ as follows.

$$\bar{u}(z)=\frac{z^3-20z}{(z-2)^3(z-4)}=A\cdot\frac{z}{z-2}+B\cdot\frac{2z}{(z-2)^2}+C\cdot\frac{2z^2+4z}{(z-2)^3}+D\cdot\frac{z}{z-4}\cdots\ldots(1)$$

$$\frac{z^3-20z}{(z-2)^3(z-4)}=\frac{Az(z-2)^2(z-4)+2Bz(z-2)(z-4)+C(2z^2+4z)(z-4)+Dz(z-2)^3}{(z-2)^3(z-4)}$$

or $z^2-20=A(z-2)^2(z-4)+2B(z-2)(z-4)+C(2z+4)(z-4)+D(z-2)^3$

Put $z=2:\ -16=-16C \quad \therefore\ C=1$

Put $z=4:\ -4=D(8) \quad \therefore\ D=-1/2$

Equating the coefficient of z^3 on both sides we have,

$A+D=0 \quad \therefore\ A=1/2$

Put $z=0:-20=A(4)(-4)+2B(8)+C(-16)+D(-8)$

ie., $-20=-8+16B-16+4\ \therefore\ B=0$

Substituting the values of A, B, C, D in (1) and taking inverse we have,

$$Z_T^{-1}[\bar{u}(z)]=\frac{1}{2}Z_T^{-1}\left[\frac{z}{z-2}\right]+Z_T^{-1}\left[\frac{2z^2+4z}{(z-2)^3}\right]-\frac{1}{2}Z_T^{-1}\left[\frac{z}{z-4}\right]$$

$$=\frac{1}{2}\cdot 2^n+2^n\cdot n^2-\frac{1}{2}\cdot 4^n$$

Thus, $\boxed{Z_T^{-1}[\bar{u}(z)]=u_n=2^{n-1}+2^n\cdot n^2-2^{2n-1}}$

Type-2. Power series Method

Step by step working procedure for problems

Step-1 The given $\bar{u}(z)$ is modified into a suitable form so as to expand it as an infinite series by using some of the standard expansions like

(i) $(1+x)^n=1+nx+\dfrac{n(n-1)}{2!}x^2+\cdots$

(ii) $\log(1+x)=x-\dfrac{x^2}{2}+\dfrac{x^3}{3}-\dfrac{x^4}{4}+\cdots$

(iii) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

Step-2 $\bar{u}(z)$ is put in the form $\sum_{n=0}^{\infty} u_n z^{-n}$

That is $\bar{u}(z) = Z_T(u_n)$

Step-3 The required $Z_T^{-1}[\bar{u}(z)]$ will be u_n

WORKED PROBLEMS

[61] *Find the inverse Z-transform of*

(a) $\log\left(\frac{z}{z+1}\right)$ (b) $z\log\left(\frac{z}{z+1}\right)$

☞ **(a)** Let, $\bar{u}(z) = \log\left(\frac{z}{z+1}\right)$

ie., $\bar{u}(z) = \log\left[\frac{z}{z\left(1+\frac{1}{z}\right)}\right] = -\log\left(1+\frac{1}{z}\right)$

We have, $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

$\therefore$ $\bar{u}(z) = -\left\{\left(\frac{1}{z}\right) - \frac{1}{2}\left(\frac{1}{z^2}\right) + \frac{1}{3}\left(\frac{1}{z^3}\right) - \frac{1}{4}\left(\frac{1}{z^4}\right) + \cdots\right\}$

ie., $\bar{u}(z) = -\frac{1}{z} + \frac{1}{2}\left(\frac{1}{z^2}\right) - \frac{1}{3}\left(\frac{1}{z^3}\right) + \frac{1}{4}\left(\frac{1}{z^4}\right) - \cdots$

ie., $\bar{u}(z) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}\frac{1}{z^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^{-n} = \sum_{n=0}^{\infty} u_n z^{-n}$

Thus, $$\boxed{u_n = Z_T^{-1}[\bar{u}(z)] = \begin{cases} \frac{(-1)^n}{n} & \text{if } n \geq 1 \\ 0 & \text{if } n = 0 \end{cases}}$$

(b) Let, $\bar{u}(z) = z\log\left(\dfrac{z}{z+1}\right)$

$$\bar{u}(z) = z\left\{-\frac{1}{z}+\frac{1}{2}\left(\frac{1}{z^2}\right)-\frac{1}{3}\left(\frac{1}{z^3}\right)+\frac{1}{4}\left(\frac{1}{z^4}\right)-\cdots\right\} \text{ as in } (a).$$

$$= -1+\frac{1}{2}\left(\frac{1}{z}\right)-\frac{1}{3}\left(\frac{1}{z^2}\right)+\frac{1}{4}\left(\frac{1}{z^3}\right)-\cdots$$

$$\bar{u}(z) = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n+1}z^{-n} = \sum_{n=0}^{\infty}u_n z^{-n}$$

Thus, $$\boxed{u_n = Z_T^{-1}[\bar{u}(z)] = \frac{(-1)^{n+1}}{n+1}}$$

[62] *Obtain the inverse Z-transform of* $\dfrac{z}{(z+1)^2}$ *by power series expansion*

☞ Let, $\bar{u}(z) = \dfrac{z}{(z+1)^2}$

ie., $$\bar{u}(z) = \frac{z}{z^2\left(1+\frac{1}{z}\right)^2} = \frac{1}{z}\left(1+\frac{1}{z}\right)^{-2}$$

We have, $$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\cdots$$

$$\therefore \quad (1+x)^{-2} = 1-2x+3x^2-4x^3+\cdots$$

Now, $$\bar{u}(z) = \frac{1}{z}\left\{1-2\cdot\frac{1}{z}+3\cdot\frac{1}{z^2}-4\cdot\frac{1}{z^3}+\cdots\right\}$$

$$\bar{u}(z) = \frac{1}{z}-2\frac{1}{z^2}+3\frac{1}{z^3}-4\frac{1}{z^4}+\cdots$$

ie., $$\bar{u}(z) = \sum_{n=0}^{\infty}(-1)^{n-1}n\,z^{-n} = \sum_{n=0}^{\infty}u_n z^{-n}$$

Thus, $$\boxed{u_n = Z_T^{-1}[\bar{u}(z)] = (-1)^{n-1}n}$$

[63] *Find the inverse Z-transform of* $\dfrac{z^2}{(z-k)^2}$ *by power series method.*

☞ Let, $\bar{u}(z) = \dfrac{z^2}{(z-k)^2}$

ie., $\bar{u}(z) = \dfrac{z^2}{z^2\left(1-\dfrac{k}{z}\right)^2} = \left(1-\dfrac{k}{z}\right)^{-2}$

We have, $(1-x)^{-2} = 1+2x+3x^2+4x^3+\cdots$

$\therefore$ $\bar{u}(z) = 1+2\left(\dfrac{k}{z}\right)+3\left(\dfrac{k}{z}\right)^2+4\left(\dfrac{k}{z}\right)^3+\cdots$

ie., $\bar{u}(z) = \sum\limits_{n=0}^{\infty}(n+1)\left(\dfrac{k}{z}\right)^n = \sum\limits_{n=0}^{\infty}[(n+1)k^n]z^{-n} = \sum\limits_{n=0}^{\infty}u_n z^{-n}$

Thus, $\boxed{u_n = Z_T^{-1}[\bar{u}(z)] = (n+1)k^n}$

[64] *Prove the following* : (a) $Z_T^{-1}[e^{1/z}] = \dfrac{1}{n!}$ (b) $Z_T^{-1}[z(e^{1/z}-1)] = \dfrac{1}{(n+1}$

☞ (a) $e^{1/z} = 1+\dfrac{1}{1!}\left(\dfrac{1}{z}\right)+\dfrac{1}{2!}\dfrac{1}{z^2}+\dfrac{1}{3!}\dfrac{1}{z^3}+\cdots$

ie., $e^{1/z} = \sum\limits_{n=0}^{\infty}\dfrac{1}{n!}z^{-n} = Z_T\left[\dfrac{1}{n!}\right]$

Thus, $\boxed{Z_T^{-1}[e^{1/z}] = \dfrac{1}{n!}}$

(b) $z(e^{1/z}-1) = z\left(\dfrac{1}{1!}\cdot\dfrac{1}{z}+\dfrac{1}{2!}\dfrac{1}{z^2}+\dfrac{1}{3!}\dfrac{1}{z^3}+\cdots\right)$

ie., $z(e^{1/z}-1) = \dfrac{1}{1!}+\dfrac{1}{2!}\dfrac{1}{z}+\dfrac{1}{3!}\dfrac{1}{z^2}+\cdots$

ie., $$z(e^{1/z}-1)=\sum_{n=0}^{\infty}\frac{1}{(n+1)!}z^{-n}=Z_T\left[\frac{1}{(n+1)!}\right]$$

Thus, $$Z_T^{-1}[z(e^{1/z}-1)]=\frac{1}{(n+1)!}$$

ASSIGNMENT

Find the inverse Z-transform of the following functions

1. $\dfrac{2z^2+3z}{(z+2)(z-4)}$
2. $\dfrac{z^2}{z^2-(a+b)z+ab}$ where $a \neq b$
3. $\dfrac{z}{2z^2+z-3}$
4. $\dfrac{8z^2}{(2z-1)(4z-1)}$
5. $\dfrac{z^2}{z^2-7z+12}$
6. $\dfrac{5z^2-2z}{(z-1)^4}$
7. $\dfrac{3z}{z^2-3z+2}$
8. $\dfrac{z}{z^2+11z+24}$
9. Given $\dfrac{\bar{u}(z)}{z}=\dfrac{2}{z+1}-\dfrac{3}{z-4}+\dfrac{4}{z-2}$, find u_n
10. Find the inverse Z-transform of $\left(\dfrac{z}{z-2}\right)^2$ by power series method.

ANSWERS

1. $\frac{1}{6}(-2)^n+\frac{11}{6}(4)^n$
2. $\dfrac{a^{n+1}-b^{n+1}}{a-b}$, $a \neq b$, $a>b$
3. $\frac{1}{5}\{1-(-3/2)^n\}$
4. $2(1/2)^n-(1/4)^n$
5. $4^{n+1}-3^{n+1}$
6. $\frac{n}{2}(n^2+2n-3)$
7. $3(2^n-1)$
8. $\frac{1}{5}\{(-3)^n-(-8)^n\}$
9. $2(-1)^n-3(4)^n+4(2)^n$
10. $(n+1)2^n$

3.9 Application of Z-transforms to solve difference equations

Step by step working procedure for problems

Step-1 We take Z-transforms on both sides of the given difference equation.

Step-2 We use the known expressions for the Z-transform for the terms like u_{n+2}, u_{n+1}.

Step-3 We obtain $\bar{u}(z) = Z_T(u_n)$ as a function of z

Step-4 The required solution $u_n = Z_T^{-1}[\bar{u}(z)]$

Note - 1. *If the initial values $u_0, u_1, \cdots$ are not given we get the* ***general solution*** *of the given difference equation. If the vlaues $u_0, u_1, \cdots$ are given specifically, we use them in the expressions of Z-transforms of u_{n+2}, u_{n+1} and obtain $\bar{u}(z) = Z_T(u_n)$. Further, the solution $u_n = Z_T^{-1}[\bar{u}(z)]$ so obtained will be the* ***particular solution*** *of the given difference equation.*

Note-2 ***Remember the following results in both ways.***

[Z-transform and Inverse Z-transform]

1. $Z_T(1) = \dfrac{z}{z-1}$	2. $Z_T(k^n) = \dfrac{z}{z-k}$
3. $Z_T(n) = \dfrac{z}{(z-1)^2}$	4. $Z_T(k^n n) = \dfrac{kz}{(z-k)^2}$
5. $Z_T(n^2) = \dfrac{z^2+z}{(z-1)^3}$	6. $Z_T(k^n n^2) = \dfrac{kz^2+k^2z}{(z-k)^3}$
7. $Z_T(n^3) = \dfrac{z^3+4z^2+4z}{(z-1)^4}$	8. $Z_T(k^n n^3) = \dfrac{kz^3+4k^2z^2+k^3z}{(z-k)^3}$
9. $Z_T[\sin(n\pi/2)] = \dfrac{z}{z^2+1}$	10. $Z_T(\cos(n\pi/2)] = \dfrac{z^2}{z^2+1}$

Note-3

1. $Z_T(u_{n+1}) = z[\bar{u}(z) - u_0]$ 2. $Z_T(u_{n+2}) = z^2[\bar{u}(z) - u_0 - u_1 z^{-1}]$

WORKED PROBLEMS

[65] *Solve by using Z-transforms :* $u_{n+2} - 5u_{n+1} + 6u_n = 0$

☞ Taking Z-transforms on both sides of the given equation we have,

$Z_T(u_{n+2}) - 5Z_T(u_{n+1}) + 6Z_T(u_n) = Z_T(0)$

ie., $z^2[\bar{u}(z) - u_0 - u_1 z^{-1}] - 5z[\bar{u}(z) - u_0] + 6\bar{u}(z) = 0$

ie., $[z^2 - 5z + 6]\,\bar{u}(z) - u_0(z^2 - 5z) - u_1 z = 0$

ie., $[z^2 - 5z + 6]\,\bar{u}(z) = u_0(z^2 - 5z) + u_1 z$

or $$\bar{u}(z) = u_0 \cdot \frac{z^2 - 5z}{z^2 - 5z + 6} + u_1 \cdot \frac{z}{z^2 - 5z + 6}$$

$$\Rightarrow \quad Z_T^{-1}[\bar{u}(z)] = u_0 Z_T^{-1}\left[\frac{z^2 - 5z}{(z-2)(z-3)}\right] + u_1 Z_T^{-1}\left[\frac{z}{(z-2)(z-3)}\right] \quad \ldots (1)$$

Let, $$p(z) = \frac{z^2 - 5z}{(z-2)(z-3)}$$

Further, let $$\frac{p(z)}{z} = \frac{z-5}{(z-2)(z-3)} = \frac{A}{z-2} + \frac{B}{z-3}$$

or $z - 5 = A(z-3) + B(z-2)$

Put $z = 3 : -2 = B(1) \quad \therefore B = -2$

Put $z = 2 : -3 = A(-1) \quad \therefore A = 3$

Hence, $$p(z) = 3 \cdot \frac{z}{z-2} - 2\frac{z}{z-3}$$

$$\Rightarrow \quad Z_T^{-1}[p(z)] = 3Z_T^{-1}\left[\frac{z}{z-2}\right] - 2Z_T^{-1}\left[\frac{z}{z-3}\right]$$

ie., $$Z_T^{-1}\left[\frac{z^2 - 5z}{z^2 - 5z + 6}\right] = 3 \cdot 2^n - 2 \cdot 3^n \quad \ldots (2)$$

Next, let $$q(z) = \frac{z}{(z-2)(z-3)}$$

Further, let $\dfrac{q(z)}{z} = \dfrac{1}{(z-2)(z-3)} = \dfrac{C}{z-2} + \dfrac{D}{z-3}$

or $\quad 1 = C(z-3) + D(z-2)$

Put $\quad z = 2 : 1 = C(-1) \quad \therefore C = -1$

Put $\quad z = 3 : 1 = D(1) \quad \therefore D = 1$

Hence, $q(z) = \dfrac{-z}{z-2} + \dfrac{z}{z-3}$

$$\Rightarrow \quad Z_T^{-1}[q(z)] = -Z_T^{-1}\left[\frac{z}{z-2}\right] + Z_T^{-1}\left[\frac{z}{z-3}\right]$$

ie., $$Z_T^{-1}\left[\frac{z}{z^2-5z+6}\right] = -2^n + 3^n \qquad \ldots (3)$$

Using (2) and (3) in (1) we have,

$$Z_T^{-1}[\bar{u}(z)] = u_0\{3\cdot 2^n - 2\cdot 3^n\} + u_1\{-2^n + 3^n\}$$

ie., $\quad u_n = (3u_0 - u_1)2^n + (-2u_0 + u_1)3^n$

Let $\quad c_1 = 3u_0 - u_1$ and $c_2 = -2u_0 + u_1$

Thus, $\boxed{u_n = c_1 \cdot 2^n + c_2 \cdot 3^n,}$ where c_1 and c_2 are arbitrary constants is the general solution of the given difference equation.

[66] *Solve the difference equation $u_{n+2} + u_n = 0$, by using Z-transforms.*

☞ Taking Z-transforms on both sides of the given equation we have,

$$Z_T(u_{n+2}) + Z_T(u_n) = Z_T(0)$$

ie., $\quad z^2[\bar{u}(z) - u_0 - u_1 z^{-1}] + \bar{u}(z) = 0$

ie., $\quad [z^2+1]\bar{u}(z) = u_0 z^2 + u_1 z$

or $\quad \bar{u}(z) = u_0 \cdot \dfrac{z^2}{z^2+1} + u_1 \cdot \dfrac{z}{z^2+1}$

$$\Rightarrow \quad Z_T^{-1}[\bar{u}(z)] = u_0 Z_T^{-1}\left[\frac{z^2}{z^2+1}\right] + u_1 Z_T^{-1}\left[\frac{z}{z^2+1}\right]$$

Thus, $\boxed{u_n = u_0 \cos(n\pi/2) + u_1 \sin(n\pi/2)}$

where u_0 and u_1 are arbitrary constants is the required general solution of the given difference equation.

[67] *Solve by using Z-transforms:* $y_{n+2} - 4y_n = 0$, *given that* $y_0 = 0$ *and* $y_1 = 2$

[Dec 2016]

☞ Taking Z-transforms on both sides of the given equation we have,

$$Z_T(y_{n+2}) - 4Z_T(y_n) = Z_T(0)$$

ie., $\quad z^2[\bar{y}(z) - y_0 - y_1 z^{-1}] - 4\bar{y}(z) = 0$

ie., $\quad [z^2 - 4]\,\bar{y}(z) - 2z = 0$, by using the given values.

or $\quad \bar{y}(z) = \dfrac{2z}{z^2 - 4}$

Let $\quad \dfrac{\bar{y}(z)}{z} = \dfrac{2}{(z-2)(z+2)} = \dfrac{A}{z-2} + \dfrac{B}{z+2}$

or $\quad 2 = A(z+2) + B(z-2)$

Put $\quad z = 2 : 2 = A(4) \quad \therefore \quad A = 1/2$

Put $\quad z = -2 : 2 = B(-4) \quad \therefore \quad B = -1/2$

Hence, $\quad \bar{y}(z) = \dfrac{1}{2}\dfrac{z}{z-2} - \dfrac{1}{2}\dfrac{z}{z+2}$

$$\Rightarrow \quad Z_T^{-1}[\bar{y}(z)] = \frac{1}{2}\left\{Z_T^{-1}\left[\frac{z}{z-2}\right] - Z_T^{-1}\left[\frac{z}{z+2}\right]\right\}$$

ie., $$y_n = \frac{1}{2}\{2^n - (-2)^n\} = \frac{2^n}{2} + \frac{(-2)^n}{-2}$$

Thus, $\boxed{y_n = 2^{n-1} + (-2)^{n-1}}$ is the required particular solution.

[68] *Solve by using Z-transforms :* $y_{n+1} + \dfrac{1}{4}y_n = \left(\dfrac{1}{4}\right)^n \ (n \geq 0), y_0 = 0$

[June 2018]

☞ Taking Z-transforms on both sides of the given equation we have,

$$Z_T(y_{n+1}) + \frac{1}{4} Z_T(y_n) = Z_T\left[\left(\frac{1}{4}\right)^n\right]$$

ie., $$z[\,\bar{y}(z) - y_0\,] + \frac{1}{4}\,\bar{y}(z) = \frac{z}{z - \frac{1}{4}}$$

ie., $$\left[z + \frac{1}{4}\right]\bar{y}(z) = \frac{z}{z - \frac{1}{4}}$$, by using the given value.

or $$\bar{y}(z) = \frac{z}{\left(z - \frac{1}{4}\right)\left(z + \frac{1}{4}\right)}$$

Let $$\frac{\bar{y}(z)}{z} = \frac{1}{\left(z - \frac{1}{4}\right)\left(z + \frac{1}{4}\right)} = \frac{A}{z - \frac{1}{4}} + \frac{B}{z + \frac{1}{4}}$$

or $$1 = A\left(z + \frac{1}{4}\right) + B\left(z - \frac{1}{4}\right)$$

Put $z = 1/4$: $1 = A(1/2)$ $\therefore$ $A = 2$

Put $z = -1/4$: $1 = B(-1/2)$ $\therefore$ $B = -2$

Hence, $$\bar{y}(z) = 2\cdot\frac{z}{z - \frac{1}{4}} - 2\cdot\frac{z}{z + \frac{1}{4}}$$

$$\Rightarrow \quad Z_T^{-1}[\bar{y}(z)] = 2\left\{Z_T^{-1}\left[\frac{z}{z - \frac{1}{4}}\right] - Z_T^{-1}\left[\frac{z}{z + \frac{1}{4}}\right]\right\}$$

Thus, $\boxed{y_n = 2\left\{\left(\frac{1}{4}\right)^n - \left(-\frac{1}{4}\right)^n\right\}}$ is the required particular solution.

[69] *Solve :* $u_{n+2} - 3u_{n+1} + 2u_n = 1$ *by using Z-transforms.*

☞ Taking Z-transforms on both sides of the given equation we have,

$$Z_T(u_{n+2}) - 3Z_T(u_{n+1}) + 2Z_T(u_n) = Z_T(1)$$

ie., $$z^2[\bar{u}(z) - u_0 - u_1 z^{-1}] - 3z[\bar{u}(z) - u_0] + 2\bar{u}(z) = \frac{z}{z-1}$$

ie., $$[z^2 - 3z + 2]\bar{u}(z) - u_0(z^2 - 3z) - u_1 z = \frac{z}{z-1}$$

ie., $$[(z-1)(z-2)]\bar{u}(z) = u_0(z^2 - 3z) + u_1 z + \frac{z}{z-1}$$

or $$\bar{u}(z) = u_0 \cdot \frac{z^2 - 3z}{(z-1)(z-2)} + u_1 \cdot \frac{z}{(z-1)(z-2)} + \frac{z}{(z-1)^2(z-2)}$$

ie., $$\bar{u}(z) = u_0 \cdot p(z) + u_1 \cdot q(z) + r(z)(say) \quad \ldots (1)$$

We shall find the inverse Z-transform of $p(z)$, $q(z)$ and $r(z)$.

Consider, $$p(z) = \frac{z^2 - 3z}{(z-1)(z-2)}$$

Let $$\frac{p(z)}{z} = \frac{z-3}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$$

or $$z - 3 = A(z-2) + B(z-1)$$

Put $z = 1$: $-2 = A(-1)$ $\quad \therefore \; A = 2$

Put $z = 2$: $-1 = B(1)$ $\quad \therefore \; B = -1$

$$\therefore \quad Z_T^{-1}[p(z)] = 2Z_T^{-1}\left[\frac{z}{z-1}\right] - Z_T^{-1}\left[\frac{z}{z-2}\right]$$

ie., $$Z_T^{-1}[p(z)] = 2 \cdot 1 - 2^n = 2 - 2^n \quad \ldots (2)$$

Consider, $$q(z) = \frac{z}{(z-1)(z-2)}$$ (*Refer Problem-54*)

$$Z_T^{-1}[q(z)] = 2^n - 1 \quad \ldots (3)$$

Consider, $$r(z) = \frac{z}{(z-1)^2(z-2)}$$

Let. $$\frac{z}{(z-1)^2(z-2)} = C\cdot\frac{z}{z-1} + D\cdot\frac{z}{(z-1)^2} + E\cdot\frac{z}{z-2}$$

or $$1 = C(z-1)(z-2) + D(z-2) + E(z-1)^2$$

Put $z = 1$: $1 = D(-1)$ $\therefore$ $D = -1$

Put $z = 2$: $1 = E(1)$ $\therefore$ $E = 1$

Equating the coefficient of z^2 on both sides we get,

$$C + E = 0 \qquad \therefore\ C = -1$$

Now, $$Z_T^{-1}[r(z)] = -Z_T^{-1}\left[\frac{z}{z-1}\right] - Z_T^{-1}\left[\frac{z}{(z-1)^2}\right] + Z_T^{-1}\left[\frac{z}{z-2}\right]$$

ie., $$Z_T^{-1}[r(z)] = -1 - n + 2^n \qquad \cdots (4)$$

With reference to (1) we have,

$$Z_T^{-1}[\bar{u}(z)] = u_0\cdot Z_T^{-1}[p(z)] + u_1 Z_T^{-1}[q(z)] + Z_T^{-1}[r(z)]$$

Using (2), (3) and (4) in the RHS we have,

$$Z_T^{-1}[\bar{u}(z)] = u_0(2-2^n) + u_1(2^n - 1) - 1 - n + 2^n$$

ie., $$u_n = (2u_0 - u_1 - 1) + (-u_0 + u_1 + 1)2^n - n$$

Let us denote, $c_1 = 2u_0 - u_1 - 1$ and $c_2 = -u_0 + u_1 + 1$ where c_1 and c_2 are arbitrary constants.

Thus, $\boxed{u_n = c_1 + c_2\cdot 2^n - n}$ is the required solution.

[70] *Solve by using Z-transforms :* $y_{n+2} + 2y_{n+1} + y_n = n$ *with* $y_0 = 0 = y_1$

☞ Taking Z-transforms on both sides of the given equation we have,

$$Z_T(y_{n+2}) + 2Z_T(y_{n+1}) + Z_T(y_n) = Z_T(n)$$

ie., $$z^2[\bar{y}(z) - y_0 - y_1 z^{-1}] + 2z[\bar{y}(z) - y_0] + \bar{y}(z) = \frac{z}{(z-1)^2}$$

ie., $$[z^2 + 2z + 1]\,\bar{y}(z) = \frac{z}{(z-1)^2},$$ by using the given values.

or $\quad \bar{y}(z) = \dfrac{z}{(z-1)^2(z+1)^2}$

Let, $\dfrac{z}{(z-1)^2(z+1)^2} = A\cdot\dfrac{z}{z-1} + B\cdot\dfrac{z}{(z-1)^2} + C\cdot\dfrac{z}{(z+1)} + D\cdot\dfrac{z}{(z+1)^2}$ ...(1)

or $\quad 1 = A(z-1)(z+1)^2 + B(z+1)^2 + C(z-1)^2(z+1) + D(z-1)^2$

Put $\quad z = 1 \quad : \quad 1 = B(4) \quad \therefore\ B = 1/4$

Put $\quad z = -1 \quad : \quad 1 = D(4) \quad \therefore\ D = 1/4$

Equating the coefficient of z^3 on both sides we get,

$A + C = 0 \quad$ or $\quad C = -A$

Put $\quad z = 0 : 1 = -A + B + C + D$

ie., $\quad 1 = C + 1/4 + C + 1/4 \quad$ or $\quad 1/2 = 2C \quad \therefore\ C = 1/4$. Also $A = -1/4$

Substituting, $A = -1/4,\ B = C = D = 1/4$ in (1) and taking the inverse Z-transform we have,

$$Z_T^{-1}[\bar{y}(z)] = -\frac{1}{4}Z_T^{-1}\left[\frac{z}{z-1}\right] + \frac{1}{4}Z_T^{-1}\left[\frac{z}{(z-1)^2}\right]$$

$$+\frac{1}{4}Z_T^{-1}\left[\frac{z}{z+1}\right] + \frac{1}{4}Z_T^{-1}\left[\frac{z}{(z+1)^2}\right]$$

ie., $\quad y_n = -\dfrac{1}{4}\cdot 1 + \dfrac{1}{4}n + \dfrac{1}{4}(-1)^n + \dfrac{1}{4}\cdot(-1)(-1)^n n$

$$= \frac{1}{4}\{(n-1) - (-1)^n(n-1)\}$$

Thus, $\boxed{y_n = \dfrac{(n-1)}{4}[1-(-1)^n]}$ is the required solution.

[71] *Solve by using Z-transforms :* $u_{n+2} - 5u_{n+1} - 6u_n = 2^n$

☞ Taking Z-transforms on both sides of the given equation we have,

$$Z_T(u_{n+2}) - 5Z_T(u_{n+1}) - 6Z_T(u_n) = Z_T(2^n)$$

ie., $z^2[\bar{u}(z)-u_0-u_1 z^{-1}]-5z[\bar{u}(z)-u_0]-6\bar{u}(z) = \dfrac{z}{z-2}$

ie., $[z^2-5z-6]\bar{u}(z)-u_0(z^2-5z)-u_1 z = \dfrac{z}{z-2}$

ie., $(z-6)(z+1)\bar{u}(z) = u_0(z^2-5z)+u_1 z+\dfrac{z}{z-2}$

or $\bar{u}(z) = u_0\cdot\dfrac{z^2-5z}{(z-6)(z+1)}+u_1\dfrac{z}{(z-6)(z+1)}+\dfrac{z}{(z-2)(z-6)(z+1)}$

ie., $\bar{u}(z) = u_0\cdot p(z)+u_1\cdot q(z)+r(z)$ (say) ... (1)

We shall find the inverse Z-transform of $p(z)$, $q(z)$ and $r(z)$.

Consider, $p(z) = \dfrac{z^2-5z}{(z-6)(z+1)}$

Let, $\dfrac{p(z)}{z} = \dfrac{z-5}{(z-6)(z+1)} = \dfrac{A}{z-6}+\dfrac{B}{z+1}$ or $z-5 = A(z+1)+B(z-6)$

Put $z=6$: $1 = A(7)$ $\therefore$ $A = 1/7$

Put $z=-1$: $-6 = B(-7)$ $\therefore$ $B = 6/7$

Hence, $Z_T^{-1}[p(z)] = \dfrac{1}{7}Z_T^{-1}\left[\dfrac{z}{z-6}\right]+\dfrac{6}{7}Z_T^{-1}\left[\dfrac{z}{z+1}\right]$

ie., $Z_T^{-1}[p(z)] = \dfrac{1}{7}(6)^n+\dfrac{6}{7}(-1)^n$ ... (2)

Consider, $q(z) = \dfrac{z}{(z-6)(z+1)}$

Let, $\dfrac{q(z)}{z} = \dfrac{1}{(z-6)(z+1)} = \dfrac{C}{z-6}+\dfrac{D}{z+1}$

or $1 = C(z+1)+D(z-6)$

Put $z=6$: $1 = C(7)$ $\therefore$ $C = 1/7$

Put $z=-1$: $1 = D(-7)$ $\therefore$ $D = -1/7$

Hence, $$Z_T^{-1}[q(z)] = \frac{1}{7} Z_T^{-1}\left[\frac{z}{z-6}\right] - \frac{1}{7} Z_T^{-1}\left[\frac{z}{z+1}\right]$$

ie., $$Z_T^{-1}[q(z)] = \frac{1}{7}(6)^n - \frac{1}{7}(-1)^n \qquad \ldots (3)$$

Consider, $$r(z) = \frac{z}{(z-2)(z-6)(z+1)}$$

Let, $$\frac{r(z)}{z} = \frac{1}{(z-2)(z-6)(z+1)} = \frac{E}{z-2} + \frac{F}{z-6} + \frac{G}{z+1}$$

or $\quad 1 = E(z-6)(z+1) + F(z-2)(z+1) + G(z-2)(z-6)$

Put $\quad z = 2 \quad : \quad 1 = E(-12) \quad \therefore \; E = -1/12$

Put $\quad z = 6 \quad : \quad 1 = F(28) \quad \therefore \; F = 1/28$

Put $\quad z = -1 \quad : \quad 1 = G(21) \quad \therefore \; G = 1/21$

Hence, $$Z_T^{-1}[r(z)] = \frac{-1}{12} Z_T^{-1}\left[\frac{z}{z-2}\right] + \frac{1}{28} Z_T^{-1}\left[\frac{z}{z-6}\right] + \frac{1}{21} Z_T^{-1}\left[\frac{z}{z+1}\right]$$

ie.; $$Z_T^{-1}[r(z)] = \frac{-1}{12}(2)^n + \frac{1}{28}(6)^n + \frac{1}{21}(-1)^n \qquad \ldots (4)$$

With reference to (1) we have,

$$Z_T^{-1}[\bar{u}(z)] = u_0 \cdot Z_T^{-1}[p(z)] + u_1 \cdot Z_T^{-1}[q(z)] + Z_T^{-1}[r(z)]$$

Using (2), (3) and (4) in the RHS we have,

$$Z_T^{-1}[\bar{u}(z)] = u_0\left\{\frac{1}{7}(6)^n + \frac{6}{7}(-1)^n\right\} + u_1\left\{\frac{1}{7}(6)^n - \frac{1}{7}(-1)^n\right\} + \left\{\frac{-1}{12}(2)^n + \frac{1}{28}(6)^n + \frac{1}{21}(-1)^n\right\}$$

$$Z_T^{-1}[\bar{u}(z)] = u_n = \left[\frac{u_0}{7} + \frac{u_1}{7} + \frac{1}{28}\right](6)^n + \left[\frac{6u_0}{7} - \frac{u_1}{7} + \frac{1}{21}\right](-1)^n - \frac{2^n}{12}$$

Let us denote, $c_1 = u_0/7 + u_1/7 + 1/28$ and $c_2 = 6u_0/7 - u_1/7 + 1/21$, where c_1 and c_2 are arbitrary constants.

Thus, $\boxed{u_n = c_1(6)^n + c_2(-1)^n - \frac{2^n}{12}}$ is the required solution.

[72] *Solve the difference equation* $y_{n+2} - 6y_{n+1} + 9y_n = 3^n$ *by using Z-transforms.*

☞ Taking Z-transforms on both sides of the given equation we have,

$$Z_T(y_{n+2}) - 6Z_T(y_{n+1}) + 9Z_T(y_n) = Z_T(3^n)$$

ie., $$z^2[\bar{y}(z) - y_0 - y_1 z^{-1}] - 6z[\bar{y}(z) - y_0] + 9\bar{y}(z) = \frac{z}{z-3}$$

ie., $$[z^2 - 6z + 9]\bar{y}(z) - y_0(z^2 - 6z) - y_1 z = \frac{z}{z-3}$$

ie., $$(z-3)^2 \cdot \bar{y}(z) = y_0(z^2 - 6z) + y_1 z + \frac{z}{z-3}$$

or $$\bar{y}(z) = y_0 \cdot \frac{(z^2 - 6z)}{(z-3)^2} + y_1 \cdot \frac{z}{(z-3)^2} + \frac{z}{(z-3)^3}$$

ie., $$\bar{y}(z) = y_0 \cdot p(z) + y_1 \cdot q(z) + r(z) \; (say) \qquad \ldots(1)$$

We shall find the inverse Z-transform of $p(z)$, $q(z)$ and $r(z)$.

Consider, $$p(z) = \frac{z^2 - 6z}{(z-3)^2}$$

We take note that, $Z_T(3^n) = \dfrac{z}{z-3}$ and $Z_T(3^n \cdot n) = \dfrac{3z}{(z-3)^2}$

Let $$p(z) = \frac{z^2 - 6z}{(z-3)^2} = A \cdot \frac{z}{z-3} + B \cdot \frac{3z}{(z-3)^2}$$

or $$z - 6 = A(z-3) + 3B$$

Put $z = 3$: $-3 = 3B$ $\quad \therefore \; B = -1$

Also, $-3A + 3B = -6$ $\quad \therefore \; A = 1$

Hence, $$Z_T^{-1}[p(z)] = Z_T^{-1}\left[\frac{z}{z-3}\right] - Z_T^{-1}\left[\frac{3z}{(z-3)^2}\right]$$

ie., $$Z_T^{-1}[p(z)] = 3^n - 3^n \cdot n \qquad \ldots(2)$$

Consider, $q(z) = \dfrac{z}{(z-3)^2} = \dfrac{1}{3}\cdot\dfrac{3z}{(z-3)^2}$

$\Rightarrow \quad Z_T^{-1}[q(z)] = \dfrac{1}{3}Z_T^{-1}\left[\dfrac{3z}{(z-3)^2}\right]$

ie., $\quad Z_T^{-1}[q(z)] = \dfrac{1}{3}(3^n n)$ ... (3)

Consider, $r(z) = \dfrac{z}{(z-3)^2}$

We take note that, $Z_T(3^n n^2) = \dfrac{3z^2+9z}{(z-3)^3}$

Let, $r(z) = \dfrac{z}{(z-3)^3} = C\cdot\dfrac{z}{z-3} + D\cdot\dfrac{3z}{(z-3)^2} + E\cdot\dfrac{3z^2+9z}{(z-3)^3}$

or $\quad 1 = C(z-3)^2 + 3D(z-3) + E(3z+9)$

Put $\quad z = 3 \ : 1 = E(18) \qquad \therefore \ E = 1/18$

Equating the coefficient of z^2 and z on both sides we get,

$$C = 0 \ \text{ and } \ -6C + 3D + 3E = 0 \qquad \therefore \ D = -1/18$$

Now, $Z_T^{-1}[r(z)] = -\dfrac{1}{18}Z_T^{-1}\left[\dfrac{3z}{(z-3)^2}\right] + \dfrac{1}{18}Z_T^{-1}\left[\dfrac{3z^2+9z}{(z-3)^3}\right]$

ie., $\quad Z_T^{-1}[r(z)] = -\dfrac{1}{18}(3^n\cdot n) + \dfrac{1}{18}(3^n\cdot n^2)$ ... (4)

With reference to (1) we have,

$$Z_T^{-1}[\bar{y}(z)] = y_0\, Z_T^{-1}[p(z)] + y_1\, Z_T^{-1}[q(z)] + Z_T^{-1}[r(z)]$$

Using (2), (3) and (4) in the RHS we have,

$$y_n = y_0(3^n - 3^n\cdot n) + y_1\cdot\frac{1}{3}(3^n n) - \frac{1}{18}(3^n\cdot n) + \frac{1}{18}(3^n\cdot n^2)$$

ie., $\quad y_n = y_0\, 3^n + \left(-y_0 + \dfrac{1}{3}y_1\right)(3^n n) + \dfrac{3^n}{18}(n^2 - n)$

We denote $c_1 = y_0$ and $c_2 = -y_0 + (y_1/3)$, c_1 and c_2 are arbitrary constants.

$$\therefore \quad y_n = c_1(3^n) + c_2(3^n n) + \frac{3^n}{3^2 \cdot 2} n(n-1)$$

Thus, $\boxed{y_n = (c_1 + c_2 n)3^n + \frac{3^{n-2}}{2} n(n-1)}$ is the required solution.

[73] *Solve the difference equation* $y_{n+2} + 6y_{n+1} + 9y_n = 2^n$ *with* $y_0 = y_1 = 0$ *using Z-transforms.* **[Dec 2018]**

☞ Taking Z-transforms on both sides of the given equation we have,

$$Z_T(y_{n+2}) + 6Z_T(y_{n+1}) + 9Z_T(y_n) = Z_T(2^n)$$

ie., $$z^2[\bar{y}(z) - y_0 - y_1 z^{-1}] + 6z[\bar{y}(z) - y_0] + 9\bar{y}(z) = \frac{z}{z-2}$$

ie., $$[z^2 + 6z + 9]\bar{y}(z) = \frac{z}{z-2}$$, by using the initial values.

or $$\bar{y}(z) = \frac{z}{(z-2)(z+3)^2}$$

Let $$\frac{z}{(z-2)(z+3)^2} = A \cdot \frac{z}{z-2} + B \cdot \frac{z}{z+3} + C \cdot \frac{z}{(z+3)^2}$$

or $$1 = A(z+3)^2 + B(z-2)(z+3) + C(z-2)$$

Put $z = 2 \; : 1 = A(25) \quad \therefore \; A = 1/25$

Put $z = -3 \; : 1 = C(-5) \quad \therefore \; C = -1/5$

Equating the coefficient of z^2 on both sides we get, $0 = A + B \quad \therefore \; B = -1/25$

Hence, $$\bar{y}(z) = \frac{1}{25} \cdot \frac{z}{z-2} - \frac{1}{25} \frac{z}{z+3} - \frac{1}{5} \frac{z}{(z+3)^2}$$

or $$\bar{y}(z) = \frac{1}{25} \cdot \frac{z}{z-2} - \frac{1}{25} \cdot \frac{z}{z+3} - \frac{1}{5} \cdot \frac{1}{-3} \frac{-3z}{(z+3)^2}$$

$$\Rightarrow \quad Z_T^{-1}[\bar{y}(z)] = \frac{1}{25}Z_T^{-1}\left[\frac{z}{z-2}\right] - \frac{1}{25}Z_T^{-1}\left[\frac{z}{z+3}\right] + \frac{1}{15}Z_T^{-1}\left[\frac{-3z}{(z+3)^2}\right]$$

ie., $$y_n = \frac{1}{25}(2)^n - \frac{1}{25}(-3)^n + \frac{1}{15}(-3)^n \cdot n$$

Thus $\boxed{y_n = \frac{1}{5}\left\{\frac{1}{5}(2)^n - \frac{1}{5}(-3)^n + \frac{1}{3}(-3)^n \cdot n\right\}}$ is the required solution.

[74] *The equation* $u_{n+2} - 2u_{n+1} + u_n = 3n+5$ *is satisfied by a sequence* u_n. *Find the sequence using Z-transforms.*

☞ Taking Z-transforms on both sides of the given equation we have,

$$Z_T(u_{n+2}) - 2Z_T(u_{n+1}) + Z_T(u_n) = 3Z_T(n) + 5Z_T(1)$$

ie., $$z^2[\bar{u}(z) - u_0 - u_1 z^{-1}] - 2z[\bar{u}(z) - u_0] + \bar{u}(z) = 3\cdot\frac{z}{(z-1)^2} + 5\cdot\frac{z}{z-1}$$

ie., $$[z^2 - 2z + 1]\bar{u}(z) - u_0(z^2 - 2z) - u_1 z = 3\frac{z}{(z-1)^2} + 5\frac{z}{(z-1)}$$

ie., $$(z-1)^2\,\bar{u}(z) = u_0(z^2 - 2z) + u_1 z + 3\frac{z}{(z-1)^2} + 5\frac{z}{(z-1)}$$

or $$\bar{u}(z) = u_0\cdot\frac{z^2-2z}{(z-1)^2} + u_1\cdot\frac{z}{(z-1)^2} + 3\frac{z}{(z-1)^4} + 5\frac{z}{(z-1)^3}$$

or $$\bar{u}(z) = u_0\cdot\frac{z^2-2z}{(z-1)^2} + u_1\cdot\frac{z}{(z-1)^2} + \frac{5z^2-2z}{(z-1)^4}$$

ie., $$\bar{u}(z) = u_0\cdot p(z) + u_1\cdot q(z) + r(z) \text{ (say)} \qquad \ldots(1)$$

We shall find the inverse Z-transforms of $p(z)$, $q(z)$ and $r(z)$.

Consider, $$p(z) = \frac{z^2-2z}{(z-1)^2} = A\cdot\frac{z}{z-1} + B\cdot\frac{z}{(z-1)^2}$$

or $$z - 2 = A(z-1) + B$$

$$\Rightarrow \quad A = 1,\ -A + B = -2 \quad \therefore\ B = -1$$

Hence, $Z_T^{-1}[p(z)] = Z_T^{-1}\left[\dfrac{z}{z-1}\right] - Z_T^{-1}\left[\dfrac{z}{(z-1)^2}\right]$

ie., $Z_T^{-1}[p(z)] = 1 - n$ . . . (2)

Also, $Z_T^{-1}[q(z)] = Z_T^{-1}\left[\dfrac{z}{(z-1)^2}\right] = n$ . . . (3)

Consider, $r(z) = \dfrac{5z^2 - 2z}{(z-1)^4}$

We also take note that,

$$Z_T^{-1}\left[\frac{z^2+z}{(z-1)^3}\right] = n^2 \text{ and } Z_T^{-1}\left[\frac{z^3+4z^2+z}{(z-1)^4}\right] = n^3$$

Let

$$r(z) = \frac{5z^2-2z}{(z-1)^4} = C\cdot\frac{z}{z-1} + D\cdot\frac{z}{(z-1)^2} + E\cdot\frac{z^2+z}{(z-1)^3} + F\cdot\frac{z^3+4z^2+z}{(z-1)^4}$$

or $5z - 2 = C(z-1)^3 + D\cdot(z-1)^2 + E(z+1)(z-1) + F(z^2+4z+1)$

Put $z = 1$: $3 = F(6)$ $\therefore$ $F = 1/2$

Equating the coefficient of z^3, z^2 and z on both sides we get,

$$C = 0,\ -3C + D + E + F = 0 \text{ and } 3C - 2D + 4F = 5$$

By solving we get, $D = -3/2$ and $E = 1$.

Hence, $Z_T^{-1}[r(z)] = \dfrac{-3}{2} Z_T^{-1}\left[\dfrac{z}{(z-1)^2}\right]$

$$+ Z_T^{-1}\left[\frac{z^2+z}{(z-1)^3}\right] + \frac{1}{2} Z_T^{-1}\left[\frac{z^3+4z^2+4z}{(z-1)^4}\right]$$

ie., $Z_T^{-1}[r(z)] = \dfrac{-3}{2}\cdot n + n^2 + \dfrac{1}{2} n^3$

ie., $Z_T^{-1}[r(z)] = \dfrac{n}{2}(n^2 + 2n - 3)$ . . .(4)

With reference to (1) we have,

$$Z_T^{-1}[\bar{u}(z)] = u_0 Z_T^{-1}[p(z)] + u_1 Z_T^{-1}[q(z)] + Z_T^{-1}[r(z)]$$

Using (2), (3) and (4) in the RHS we have,

$$Z_T^{-1}[\bar{u}(z)] = u_0(1-n) + u_1(n) + \frac{n}{2}(n^2 + 2n - 3)$$

$$Z_T^{-1}[\bar{u}(z)] = u_n = u_0 + \left(-u_0 + u_1 - \frac{3}{2}\right)n + \frac{n(n^2+2n)}{2}.$$

We denote, $c_1 = u_0$ and $c_2 = -u_0 + u_1 - (3/2)$, c_1 and c_2 are arbitrary constants.

Thus, $\boxed{u_n = c_1 + c_2 n + \frac{n^2(n+2)}{2}}$ is the required sequence.

[75] *Find the response of the system* $y_{n+2} - 4y_{n+1} + 3y_n = u_n$ *with* $y_0 = 0$, $y_1 = 1$. *and* $u_n = 1$ *for* $n = 0, 1, 2, 3, \ldots$ *by Z-transform method.*

☞ The given equation is $y_{n+2} - 4y_{n+1} + 3y_n = 1$.

Taking Z-transforms on both sides we have,

$$Z_T(y_{n+2}) - 4Z_T(y_{n+1}) + 3Z_T(y_n) = Z_T(1)$$

ie., $$z^2[\bar{y}(z) - y_0 - y_1 z^{-1}] - 4z[\bar{y}(z) - y_0] + 3\bar{y}(z) = \frac{z}{z-1}$$

ie., $$[z^2 - 4z + 3]\,\bar{y}(z) - z = \frac{z}{z-1}$$

ie., $$(z-1)(z-3)\bar{y}(z) = z + \frac{z}{(z-1)}$$

or $$\bar{y}(z) = \frac{z}{(z-1)(z-3)} + \frac{z}{(z-1)^2(z-3)} \qquad \ldots(1)$$

ie., $$\bar{y}(z) = p(z) + q(z)\,(say)$$

Consider, $$p(z) = \frac{z}{(z-1)(z-3)}$$

Let, $\dfrac{p(z)}{z} = \dfrac{1}{(z-1)(z-3)} = \dfrac{A}{z-1} + \dfrac{B}{z-3}$

or $1 = A(z-3) + B(z-1)$

Put $z = 1 : 1 = A(-2) \therefore A = -1/2$

Put $z = 3 : 1 = B(2) \therefore B = 1/2$

Hence, $Z_T^{-1}[p(z)] = \dfrac{-1}{2} Z_T^{-1}\left[\dfrac{z}{z-1}\right] + \dfrac{1}{2} Z_T^{-1}\left[\dfrac{z}{z-3}\right]$

ie., $Z_T^{-1}[p(z)] = \dfrac{-1}{2} \cdot 1 + \dfrac{1}{2}(3^n)$ . . . (2)

Consider, $q(z) = \dfrac{z}{(z-1)^2(z-3)}$

Let $q(z) = \dfrac{z}{(z-1)^2(z-3)} = C \cdot \dfrac{z}{z-1} + D \cdot \dfrac{z}{(z-1)^2} + E \cdot \dfrac{z}{z-3}$

or $1 = C(z-1)(z-3) + D(z-3) + E(z-1)^2$

Put $z = 1 : 1 = D(-2) \therefore D = -1/2$

Put $z = 3 : 1 = E(4) \therefore E = 1/4$

Also we must have $C + E = 0 \therefore C = -1/4$

Hence, $Z_T^{-1}[q(z)] = -\dfrac{1}{4} Z_T^{-1}\left[\dfrac{z}{z-1}\right] - \dfrac{1}{2} Z_T^{-1}\left[\dfrac{z}{(z-1)^2}\right] + \dfrac{1}{4} Z_T^{-1}\left[\dfrac{z}{z-3}\right]$

ie., $Z_T^{-1}[q(z)] = \dfrac{-1}{4} \cdot 1 - \dfrac{1}{2} \cdot n + \dfrac{1}{4} \cdot 3^n$ . . . (3)

With reference to (1) we have,

$Z_T^{-1}[\bar{y}(z)] = Z_T^{-1}[p(z)] + Z_T^{-1}[q(z)]$

Using (2) and (3) in the RHS we have,

$$Z_T^{-1}[\bar{y}(z)] = -\frac{1}{2}+\frac{3^n}{2}-\frac{1}{4}-\frac{n}{2}+\frac{3^n}{4}$$

ie., $$y_n = -\frac{3}{4}+\frac{3}{4}\cdot 3^n - \frac{n}{2}$$

Thus, $\boxed{y_n = \frac{1}{4}[-3+3^{n+1}-2n]}$ is the required response.

ASSIGNMENT

Solve the following difference equations using Z-transforms

1. $y_{n+2} - y_{n+1} - 2y_n = 0$
2. $y_{n+2} - 6y_{n+1} + 9y_n = 0$
3. $y_{n+2} - 9y_n = 0$ given that $y_0 = 0,\ y = 2$
4. $y_{n+2} - 5y_{n+1} - 6y_n = 4^n$
5. $y_{n+2} - 2y_{n+1} + y_n = 2^n$; $y_0 = 2$ and $y_1 = 1$
6. $u_{n+2} - u_n = (n-1)$ given that $u_0 = 1$ and $u_1 = 2$
7. $u_{n+2} - 4u_{n+1} + 3u_n = 5^n$
8. $u_{n+2} + 4u_{n+1} + 3u_n = 3^n$ with $u_0 = 0$ and $u_1 = 1$
9. $u_{n+2} + 6u_{n+1} + 9u_n = 2^n$ with $u_0 = 0 = u_1$
10. Find the response of the system $y_{n+2} - 5y_{n+1} + 6y_n = u_n$ with

    $y_0 = 0,\ y_1 = 1$ and $u_n = 1$ for $n = 0, 1, 2, 3, \ldots$

ANSWERS

1. $y_n = c_1(2)^n + c_2(-1)^n$

2. $y_n = (c_1 + c_2 n)3^n$

3. $y_n = 3^{n-1} + (-3)^{n-1}$

4. $y_n = c_1(6)^n + c_2(-1)^n - (4^n/10)$

5. $y_n = 1 - 2n + 2^n$

6. $u_n = \frac{1}{9}[9(2)^n - (-2)^n + 1 - 3^n]$

7. $u_n = c_1 + c_2(3)^n + 5^n/8$

8. $u_n = \frac{1}{24}[9(-1)^n - 10(-3)^n + 3^n]$

9. $u_n = \frac{1}{25}[2^n - (-3)^n + \frac{5n}{3}(-3)^n]$

10. $u_n = \frac{1}{2}[1 - 2^{n+2} + 3^{n+1}]$

TRANSFORM CALCULUS, FOURIER SERIES & NUMERICAL TECHNIQUES

MODULE - 4

We have studied various analytical (theoretical) methods of solving differential equations, applicable only to equations in some specific form. But the differential equations arising out of many physical problems do not belong to a specific form and some times analytical solution may not even exist. In some cases it may be very difficult to solve by analytical methods. In such cases ***Numerical Methods*** *assumes importance and computers help in many numerical methods for obtaining the result to the highest degree of accuracy.*

We discuss five Numerical Methods.