$$f(x) = \frac{2}{\pi} \int_0^\infty F_c(u) \cos ux \, du.$$

ii) Fourier sine transform. denoted by $F_s(u)$

$$F_s(u) = \int_0^\infty f(x) \sin ux \, dx$$

inverse fourier sine transform.

$$f(x) = \frac{2}{\pi} \int_0^\infty F_s(u) \sin ux \, du$$

Linearity Property:

If $c_1, c_2 \ldots c_n$ are the constants

$$F\left[c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x)\right] =$$

$$c_1 F(f_1(x)) + c_2 F(f_2(x)) + \cdots + c_n F(f_n(x))$$

1) Find the Fourier transform of the function

$$f(x) = \begin{cases} 1, & \text{for } |x| \le a \\ 0, & \text{for } |x| > a \end{cases} \qquad \int_0^\infty \frac{\sin x}{x} dx$$

$|x| \le a$

$-x \le a$

$x \le a$

$x \ge -a$

$\cancel{x}$

$-a \le x \le a$

$|x| > a$

$x > a$

$-x > a$

$x < -a$

$$f(x) = \begin{cases} 1 & , -a \le x \le a \\ 0 & , \text{otherwise} \end{cases}$$

Fourier transform,

$$F(u) = \int_{-\infty}^{\infty} f(x)\, e^{iux}\, dx$$

$$= \int_{-\infty}^{-a} 0\, dx + \int_{-a}^{a} 1 \cdot e^{iux}\, dx + \int_{a}^{\infty} 0\, dx$$

$$= \int_{-a}^{a} e^{iux}\, dx$$

$$= \left[\frac{e^{iux}}{iu}\right]_{-a}^{a} \Rightarrow \frac{1}{iu}\left[e^{iua} - e^{-iua}\right]$$

$$\left[\begin{aligned} e^{i\theta} &= \cos\theta + i\sin\theta \\ e^{-i\theta} &= \cos\theta - i\sin\theta \end{aligned}\right]$$

$$= \frac{1}{iu}\left[\cancel{\cos ua} + i\sin ua - \cancel{\cos ua} + i\sin ua\right]$$

$$= \frac{1}{\cancel{i}u}\left[2\cancel{i}\sin ua\right]$$

$$F(u) = \frac{2\sin ua}{u}$$

ii) Inverse Fourier transform

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(u)\, e^{-iux}\, du$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2\sin ua}{u}\, e^{-iux}\, du$$

put $x = 0$, then $f(0) = 1$

$$f(0) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin ua}{u}\, e^{0}\, du$$

$$1 = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin ua}{u}\, du$$

$f(u) = \frac{\sin ux}{u}$ is an even function

For, $f(-u) = \frac{\sin(-u)x}{-u} = \frac{\cancel{+}\sin ua}{\cancel{+}u} = \frac{\sin ua}{u}$

$$\therefore\ 1 = \frac{1}{\pi} \times 2 \cdot \int_{0}^{\infty} \frac{\sin ua}{u}\, du$$ as it is even $= 2\int_0^a f(x)\,dx$

$$\frac{\pi}{2} = \int_{0}^{\infty} \frac{\sin ua}{u}\, du$$

Take $a = 1$

$$\pi/2 = \int_{0}^{\infty} \frac{\sin u}{u}\, du$$

$$= \int_{0}^{\infty} \frac{\sin x}{x}\, dx$$

2) Find the Fourier transform of $f(x)$,

$$f(x) = \begin{cases} 1-x^2, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}$$

and Hence evaluate $\int_0^\infty \left(\frac{x\cos x - \sin x}{x^3}\right) dx$ ii) $\int_0^\infty \frac{x\cos x - \sin x}{x^3} \cos\left(\frac{x}{2}\right) dx$

$$f(x) = \begin{cases} 1-x^2, & -1 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

$x \leq 1$

$-x \leq 1$

$x \geq -1$

Fourier transform,

$$F(u) = \int_{-\infty}^{\infty} f(x)\, e^{iux}\, dx$$

$$= \int_{-\infty}^{-1} 0\, dx + \int_{-1}^{1} (1-x^2)\, e^{iux}\, dx + \int_{1}^{\infty} 0\, dx$$

$$= \int_{-1}^{1} (1-x^2) \cdot e^{iux}\, dx$$

$$= \left[(1-x^2)\frac{e^{iux}}{iu} - (0-2x)\frac{e^{iux}}{(iu)^2} + (0-2)\frac{e^{iux}}{(iu)^3}\right]_{-1}^{1}$$

$$= \left[0 + 2 \cdot \frac{e^{iu}}{(iu)^2} - \frac{2e^{iu}}{(iu)^3}\right] - \left[0 - 2\frac{e^{-iu}}{(iu)^2} - 2\frac{e^{-iu}}{(iu)^3}\right]$$

$$= \left[\frac{2e^{iu}}{(iu)^2} - \frac{2e^{iu}}{(iu)^3} + \frac{2 \cdot e^{-iu}}{(iu)^2} + \frac{2 \cdot e^{-iu}}{(iu)^3}\right]$$

$$= \frac{2}{(iu)^2}\left[e^{iu} - \frac{e^{iu}}{(iu)} + e^{-iu} + \frac{e^{-iu}}{(iu)}\right]$$

$$\frac{2}{i^2 u^2}\left[\frac{iu\, e^{iu} - e^{iu}}{iu} + \frac{iu\, e^{-iu} - e^{-iu}}{iu}\right]$$

$$= \frac{2}{i^2 u^2}\left[e^{iu} + e^{-iu}\right] - \frac{2}{i^3 u^3}\left[e^{iu} - e^{-iu}\right]$$

$$= \frac{2}{i^2 u^2}\left(\cos u + i\sin u + \cos u - i\sin u\right) - \frac{2}{i^3 u^3}\left[\cancel{\cos u} + i\sin u - \cancel{\cos u} + i\sin u\right]$$

$$= \frac{2}{i^2 u^2}\left(2\cos u\right) - \frac{2}{i^3 u^3}\left(2 i\sin u\right)$$

$$= -\frac{4\cos u}{i^2 u^2} - \frac{4\sin u}{i^2 u^3} \qquad (i^2 = -1)$$

$$F(u) = \frac{4\left(\sin u - u\cos u\right)}{u^3}$$ ✓

II) By Inverse Fourier transform

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)\, e^{-iux}\, du$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4(\sin u - u\cos u)}{u^3}\, e^{-iux}\, du$$

Put $x = 0$ then $f(0) = 1 - x^2 \Rightarrow 1 - 0^2$
$= \underline{\underline{1}}$

$$1 = \frac{4}{2\pi}\int_{-\infty}^{\infty} \left(\frac{\sin u - u\cos u}{u^3}\right) du$$

$$f(u) = -\frac{2}{\pi}\int_{-\infty}^{\infty} \frac{u\cos u - \sin u}{u^3}\, du$$

$$f(u) = \frac{u\cos u - \sin u}{u^3}$$

$$= \frac{-u\cos(-u) - \sin(-u)}{(-u)^3} = \frac{-u\cos u + \sin u}{-u^3} = \frac{-u\cos u + \sin u}{-u^3}$$

$$= \cancel{-}\left[\frac{u\cos u - \sin u}{\cancel{-}u^3}\right]$$

$\therefore$ $f(u)$ is even function

$$\therefore\ 1 = -\frac{2}{\pi} \times 2\int_0^{\infty}\left[\frac{u\cos u - \sin u}{u^3}\right] du$$

$$-\frac{\pi}{4} = \int_0^{\infty} \frac{u\cos u - \sin u}{u^3} \qquad \text{Take } u = x$$

$$-\pi/4 = \int_0^{\infty} \frac{x\cos x - \sin x}{x^3}$$

I ii) Put $x = 1/2$, Then $f(1/2) = 1 - (1/2)^2 = 1 - 1/4 = 3/4$

$$e^{-iux} \Rightarrow \cos ux - i\sin ux$$

$$3/4 = -\frac{2}{\pi}\int_{-\infty}^{\infty} \frac{u\cos u - \sin u}{u^3} \cdot e^{-iu/2}\, du \qquad \cos u/2 - i\sin u/2$$

comparing the real parts both the sides

$$-3/4(\pi/2) = \int_{-\infty}^{\infty} \frac{u\cos u - \sin u}{u^3} \cos u/2\, du$$

$$f(-u) = \frac{-u\cos u + \sin u}{-u^3} = \cancel{-}\left[\frac{u\cos u - \sin u}{\cancel{-}u^3}\right]$$

$\because$ $f(u)$ is even function

$$-\frac{3\pi}{8} = 2\int_0^{\infty}\left[\frac{u\cos u - \sin u}{u^3}\right]\cos\frac{u}{2}\, du$$

$$= -\frac{3\pi}{16} = \int_0^\infty \frac{x\cos x - \sin x}{x^3} \cos\left(\frac{x}{2}\right) dx$$

3. Find the Fourier transform of the function

$$f(x) = \begin{cases} 1 - |x| , & |x| \leq 1 \\ 0 , & |x| > 1 \end{cases}$$ & Hence

prove that $\int_0^\infty \frac{\sin^2 t}{t^2} dt = \pi/2$

Rewrite the function and Hence prove the given function.

$$= f(x) = \begin{cases} 1 - |x| , & -1 \leq x \leq 1 \\ 0 , & \text{otherwise} \end{cases}$$

By fourier transform

$$F(u) = \int_{-\infty}^{\infty} f(x) \cdot e^{iux} dx$$

$$= \int_{-\infty}^{-1} 0 \, dx + \int_{-1}^{1} (1 - |x|) e^{iux} dx + \int_{1}^{\infty} 0 \, dx$$

$$= \int_{-1}^{1} (1 - |x|) e^{iux} dx$$

$\therefore 1 - (-x)$ $\quad$ $1 - x$

$$= \int_{-1}^{0} (1+x) e^{iux} dx + \int_{0}^{1} (1-x) e^{iux} dx$$

$$= \left[(1+x) \frac{e^{iux}}{iu} - (0+1) \frac{e^{iux}}{(iu)^2} \right]_{-1}^{0} + \left[(1-x) \frac{e^{iux}}{iu} - (0-1) \frac{e^{iux}}{(iu)^2} \right.$$

$$\left[\left(\cancel{\frac{1}{iu}} - \frac{1}{(iu)^2}\right) - \left[\cancel{(1+1)}\ 0 - \frac{e^{-iu}}{(iu)^2}\right]\right] +$$

$$\left[\left(0 + \frac{e^{iu}}{(iu)^2}\right) - \left(\frac{1}{iu} + \frac{1}{(iu)^2}\right)\right]$$

$$= \left[\cancel{\frac{1}{iu}} - \frac{1}{(iu)^2} + \frac{e^{-iu}}{(iu)^2} + \frac{e^{iu}}{(iu)^2} - \cancel{\frac{1}{iu}} - \frac{1}{(iu)^2}\right]$$

$$= \left[\frac{e^{-iu}}{(iu)^2} + \frac{e^{iu}}{(iu)^2} - \frac{2}{(iu)^2}\right]$$

$$= \frac{1}{(iu)^2}\left[e^{-iu} + e^{iu} - 2\right]$$

$$= \frac{1}{(iu)^2}\left[\cos u - i\cancel{\sin u} + \cos u + i\cancel{\sin u} - 2\right]$$

$$= \frac{1}{(iu)^2}\left[2\cos u - 2\right]$$

$$= \frac{2}{(iu)^2}\left[\cos u - 1\right] = -\frac{2}{u^2}\left[\cos u - 1\right]$$

$$= \frac{2}{u^2}\left[1 - \cos u\right]$$

$i^2 = -1$

$$= \frac{4\sin^2(u/2)}{u^2}$$

$$\frac{1 - \cos x}{2} = \sin^2 x/2$$

$$1 - \cos x = 2\sin^2 x/2$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(u) \cdot e^{-iux}\, du$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4\ \sin^2(u/2)}{u^2}\cancel{[1-\cos u]} \cdot e^{-iux}\, du$$

$$= \frac{1}{\cancel{2}\pi}\int_{-\infty}^{\infty} \frac{2\cancel{4}}{u^2}\sin^2(u/2)\, e^{-iux}\, du$$

$$|-|x||$$

$f(x) =$

put $x = 0$, $\quad f(0) = 1 - |0|$
$= 1$

$$1 = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\sin^2 u/2}{u^2} \cdot e^{0} \, du$$

$$\therefore f(0) = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\sin^2 u/2}{u^2} \, du$$

$$\pi/2 = \int_{-\infty}^{\infty} \frac{\sin^2 u/2}{u^2} \, du$$

$$f(u) = \frac{\sin^2 u/2}{u^2}$$

$$f(-u) = \frac{[\sin(-u/2)]^2}{u^2} = \frac{\sin^2(u/2)}{u^2}$$

which is even $= f(u)$

when $u = 0$, $t = 0$
$u = \infty$, $t = \infty$

$$= \pi/2 = 2 \cdot \int_{0}^{\infty} \frac{\sin^2 u/2}{u^2} \, du$$

put $u/2 = t$ $\Rightarrow \boxed{\frac{du}{2} = dt}$ $\boxed{u^2 = (2t)^2}$

$$\pi/\cancel{4}_2 = \int_{0}^{\infty} \frac{\sin^2 t}{\cancel{4}_{\cancel{2}} t^2} \cancel{2} dt$$

$$\pi/2 = \int_{0}^{\infty} \frac{\sin^2 t}{t^2} \, dt$$

4. ... of the function

$$f(x) = x e^{-|x|}$$

$$F(u) = \int_{-\infty}^{\infty} f(x) e^{iux}\, dx$$

$$= \int_{-\infty}^{\infty} x \cdot e^{-|x|} e^{iux}\, dx$$

By Fourier transform,

$$F(u) = \int_{-\infty}^{\infty} f(x) e^{iux}\, dx$$

$$= \int_{-\infty}^{0} x e^{x} e^{iux}\, dx + \int_{0}^{\infty} x \cdot e^{-x} e^{iux}\, dx$$

$$= \int_{-\infty}^{0} x \cdot e^{(1+iu)x}\, dx + \int_{0}^{\infty} x\, e^{-(1-iu)x}\, dx$$

$$\left[x \cdot \frac{e^{(1+iu)x}}{1+iu} - (1) \cdot \frac{e^{(1+iu)x}}{(1+iu)^2} \right]_{-\infty}^{0} +$$

$$\left[x \cdot \frac{e^{-(1-iu)x}}{-(1-iu)} - (1) \cdot \frac{e^{-(1-iu)x}}{+(1-iu)^2} \right]_{0}^{\infty}$$

$x=1,\ e^{1}=2.5$
$x=2 = e^{2}$
$x=\infty = e^{0}$

$$= \left[\left(0 - \frac{1}{(1+iu)^2}\right) - (0 - 0) \right] +$$

$$\left[(0 - 0) - \left(0 - \frac{1}{(1-iu)^2}\right) \right]$$

$$= \left[-\frac{1}{(1+iu)^2} + \frac{1}{(1+iu)^2} \right]$$

$$= \frac{-(1-iu)^2 + (1+iu)^2}{(1+iu)^2(1-iu)^2}$$

$$= \frac{-(1+i^2u^2-2ui)+(1+i^2u^2+2ui)}{[(1+iu)(1-iu)]^2}$$

$$= \frac{4ui}{(1^2-i^2u^2)^2} = \frac{4ui}{(1+u^2)^2}$$

H.W

5) i) $f(x) = \begin{cases} x^2, & |x| \le a \\ 0, & |x| > a \end{cases}$

6) ii) $f(x) = e^{-|x|}$

5) $F(u) = \int_{-\infty}^{\infty} f(x)\, e^{iux}\, dx$

$f(x) = \begin{cases} x^2, & -a \le x \le a \\ 0, & \text{otherwise} \end{cases}$

$$\Rightarrow \int_{-\infty}^{-\alpha} a\cdot e^{iux}\,dx + \int_{-\alpha}^{\alpha} x^2 e^{iux}\,dx + \int_{\alpha}^{\infty} a\cdot e^{iux}\cdot dx$$

$$= \left[a\cdot \frac{e^{iux}}{iu}\right]_{-\infty}^{-\alpha} + \left[x^2\cdot\frac{e^{iux}}{(iu)} - 2x\cdot\frac{e^{iux}}{(iu)^2} + 2\frac{e^{iux}}{(iu)^3}\right]_{-\alpha}^{\alpha}$$

$$+ \left[a\cdot\frac{e^{iux}}{iu}\right]_{\alpha}^{\infty}$$

$$= \cancel{\left[a\cdot\frac{e^{iu\alpha}}{iu}\right]} + \left[\left[\alpha^2\cdot\frac{e^{iu\alpha}}{iu} - 2\alpha\frac{e^{iu\alpha}}{(iu)^2} + 2\cdot\frac{e^{iu\alpha}}{(iu)^3}\right] -\right.$$

$$\left.\left[\alpha^2\frac{e^{-iu\alpha}}{(iu)} + 2\alpha\frac{e^{-iu\alpha}}{(iu)^2} + 2\frac{e^{-iu\alpha}}{(iu)^3}\right]\right] + \cancel{\left[0 - \frac{a\,e^{iu\alpha}}{iu}\right]}$$

$$\Rightarrow \alpha^2\cdot\frac{e^{iu\alpha}}{iu} - 2\alpha\frac{e^{iu\alpha}}{(iu)^2} + 2\cdot\frac{e^{iu\alpha}}{(iu)^3} - \alpha^2\cdot\frac{e^{-iu\alpha}}{iu} +$$

$$2\alpha\frac{e^{-iu\alpha}}{(iu)^2} + 2\cdot\frac{e^{-iu\alpha}}{(iu)^3}.$$

$$\Rightarrow \frac{\alpha^2}{iu}\left[e^{iu\alpha} - e^{-iu\alpha}\right] + \frac{2\alpha}{(iu)^2}\left[e^{-iu\alpha} - e^{iu\alpha}\right]$$

$$+ \frac{2}{(iu)^3}\left[e^{iu\alpha} + e^{-iu\alpha}\right]$$

$$\Rightarrow \frac{\alpha^2}{iu}\left[\cancel{\cos}u\alpha + i\sin u\alpha - (\cancel{\cos}u\alpha - i\sin u\alpha)\right] + \frac{2\alpha}{(iu)^2}$$

$$\left[\cancel{\cos}u\alpha - i\sin u\alpha - (\cancel{\cos}u\alpha + i\sin u\alpha)\right] + \frac{2\alpha}{(iu)^3}\Big[$$

$$(\cos u\alpha + i\sin \cancel{u\alpha} + \cos u\alpha - i\sin\cancel{u\alpha}\Big]$$

$$= 1/u^3\left[2(\alpha^2u^2 - 2)\sin\alpha u + 4\alpha u\cos\alpha u\right.$$

6) $f(x) = e^{-|x|}$

$$F(u) = \int_{-\infty}^{\infty} f(x)\, e^{+iux}\, dx$$

$$= \int_{-\infty}^{\infty} e^{-|x|}\, e^{+iux}\, dx$$

$$= \int_{-\infty}^{0} e^{-(-x)} \cdot e^{+iux}\, dx + \int_{0}^{\infty} e^{-x} \cdot e^{+iux}\, dx$$

$$= \int_{-\infty}^{0} e^{x} e^{iux}\, dx + \int_{0}^{\infty} e^{-x} e^{iux}\, dx$$

$$= \int_{-\infty}^{0} e^{(1+iu)x}\, dx + \int_{0}^{\infty} e^{-(1-iu)x}\, dx$$

$$\Rightarrow \left[\frac{e^{(1+iu)x}}{1+iu}\right]_{-\infty}^{0} + \left[\frac{e^{-(1-iu)x}}{-(1-iu)}\right]_{0}^{\infty}$$

$$\Rightarrow \left[\frac{1}{1+iu} - 0\right] + \left[0 + \frac{1}{(1-iu)}\right]$$

$$\Rightarrow \left[\frac{1}{1+iu} + \frac{1}{(1-iu)}\right]$$

$$\Rightarrow \left[\frac{(1 - \cancel{iu}) + (1 + \cancel{iu})}{(1+iu)(1-iu)}\right]$$

$$= \left[\frac{2}{1^2 - (iu)^2}\right] \Rightarrow \frac{2}{1^2 + u^2}$$

Find The Fourier series of $e^{-a^2x^2}$, $a>0$ hence deduce that $e^{-x^2/2}$ is self reciprocal with in respect of fourier transform.

By Fourier transform,

$$F(u) = \int_{-\infty}^{\infty} f(x)\, e^{iux}\, dx$$

$$= \int_{-\infty}^{\infty} e^{-a^2x^2} e^{iux}\, dx$$

$$= \int_{-\infty}^{\infty} e^{-a^2x^2 + iux}\, dx$$

$$= \int_{-\infty}^{\infty} e^{-a^2\left(x^2 - \frac{iux}{a^2}\right)} \cdot dx \qquad -a^2\left[x^2 - \frac{iux}{a^2}\right]$$

Taking power

$$\left[-a^2\left[x^2 - \frac{iux}{a^2}\right]\right]$$

$a^2 - 2ab + b^2$

$$-a^2\left[\left[x^2 - 2x\left[\frac{iu}{2a^2}\right] + \left(\frac{iu}{2a^2}\right)^2\right] - \frac{i^2u^2}{4a^4}\right]$$

$$-a^2\left[\left(x - \frac{iu}{2(a)^2}\right)^2 + \frac{u^2}{4a^4}\right] \qquad i^2 = -1$$

$$\left[-a^2\left(x - \frac{iu}{2a^2}\right)^2 - \frac{u^2}{4a^2}\right]$$ multiply a^2 inside

$$= \int_{-\infty}^{\infty} e^{-a^2\left(x - \frac{iu}{2a^2}\right)^2} \cdot e^{-u^2/4a^2}\, dx$$

$$= e^{-u^2/4a^2} \int_{-\infty}^{\infty} e^{-a^2\left(x - \frac{iu}{2a^2}\right)^2}\, dx$$

Put $a\left(x - \frac{iu}{2a^2}\right) = t$

$$a\,dx = dt$$

when $x = \infty$ $\quad t = \infty$

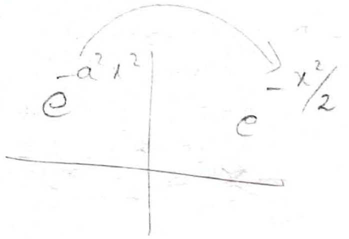
$x = -\infty$ $\quad t = -\infty$

$$\therefore F(u) = e^{-u^2/4a^2} \int_{-\infty}^{\infty} e^{-t^2} \cdot \frac{dt}{a}$$

$$\boxed{\int e^{-x^2} dx = \sqrt{\pi}}$$

$$= \frac{e^{-u^2/4a^2}}{a} \int_{-\infty}^{\infty} e^{-t^2} dt$$

$$F\left(e^{-a^2x^2}\right) = \frac{e^{-u^2/4a^2}}{a} \sqrt{\pi}$$



$a^2 = 1/2$

Put $a^2 = 1/2$

$$F\left(e^{-x^2/2}\right) = \frac{e^{\frac{-u^2}{4\times 1/2}} \sqrt{\pi}}{1/\sqrt{2}}$$

$$= \sqrt{2\pi}\, e^{-u^2/2}$$

$\Rightarrow$ Fourier transform of $e^{-x^2/2}$ in constant time $e^{-u^2/2}$

$\Rightarrow$ $e^{-u^2/2}$ is a self reciprocal for w.r.t Fourier transform.

8) Find the infinite Fourier Cosine Transform of e^{-x^2}.

By Fourier Cosine transform,

$$F_c(u) = \int_0^\infty f(x)\cos ux\,dx$$

$$F_c(u) = \int_0^\infty e^{-x^2}\cos ux\,dx \quad \text{--- (1)}$$

Leibnitzs rule of differentiating under integral sign.

$$\frac{d}{du}F_c(u) = \int_0^\infty \frac{\partial}{\partial u}\left(e^{-x^2}\cos ux\right)dx$$

$$F_c'(u) = \int_0^\infty e^{-x^2}(-\sin ux \cdot x)\,dx$$

$$= \int_0^\infty \left(-x e^{-x^2}\right)(\sin ux)\,dx$$

$\because \int -x e^{-x^2}\,dx$

$-x^2 = t$

$-2x\,dx = dt$

$x\,dx = -\frac{1}{2}dt$

$\int e^t \frac{dt}{2}$

$= \frac{1}{2}\int e^t\,dt$

$= \frac{1}{2}e^{-x^2}$

$$= \sin ux\left(\tfrac{1}{2}e^{-x^2}\right)\Big|_0^\infty - \int_0^\infty \tfrac{1}{2}e^{-x^2}\cos ux \cdot u\,dx$$

$$= (0-0) - \tfrac{1}{2}u\int_0^\infty e^{-x^2}\cos ux\,dx$$

$$F_c'(u) = -\tfrac{1}{2}u\,F_c(u) \quad \text{from (1)}$$

$$\frac{F_c'(u)}{F_c(u)} = -\tfrac{1}{2}u$$

Integrate w.r.t U, $\frac{du}{U} = \log u$

$$\log F_c(u) = -\frac{1}{2}\frac{U^2}{2} + k$$

Antilog

$$= e^{-U^2/4 + k}$$

$$= e^{-U^2/4} \cdot e^{k}$$

$$F_c(u) = k \cdot e^{-U^2/4} \quad \text{——— (2)}$$

Put $U = 0$ in (1) & (2)

In (1) $F_c(0) = \int_0^\infty e^{-x^2} \cos 0 \, dx$

$$= \int_0^\infty e^{-x^2} dx = \sqrt{\pi}/2$$

standard value

In (2) $F_c(0) = k \cdot e^0 = k$

$$= k = \sqrt{\pi}/2$$

Sub in (2)

$$F_c(u) = \frac{\sqrt{\pi}}{2} e^{-U^2/4}$$

a) Find the Fourier Cosine transform of

$$f(x) = \begin{cases} 4x & 0 < x < 1 \\ 4-x & 1 < x < 4 \\ 0 & x > 4 \end{cases}$$

$$F_c(u) = \int_0^1 4x \cos ux \, dx + \int_1^4 (4-x) \cos ux \, dx + \int_4^\infty 0 \, dx$$

$$= \left[4x \cdot \left(\frac{\sin ux}{u}\right) - 4\left(-\frac{\cos ux}{u^2}\right) \right]_0^1 +$$

$$\left[(4-x)\left(\frac{\sin ux}{u}\right) - (0-1)\left(-\frac{\cos ux}{u^2}\right) \right]_1^4$$

$$= \left[\left(4 \frac{\sin u}{u} + 4 \frac{\cos u}{u^2}\right) - \left(0 + \frac{4}{u^2}\right) \right] +$$

$$\left[\left(0 - \frac{\cos 4u}{u^2}\right) - \left(3 \cdot \frac{\sin u}{u} - \frac{\cos u}{u^2}\right) \right]$$

$$= \left[4 \left[\frac{\sin u}{u} + \cancel{\frac{\cos u}{u^2}} - \frac{1}{u^2} \right] - \frac{\cos 4u}{u^2} - \frac{3 \sin u}{u} - \cancel{\frac{\cos u}{u^2}} \right]$$

$$= \frac{\sin u}{u} + 3 \frac{\cos u}{u^2} - \frac{4}{u^2} - \frac{\cos 4u}{u^2}$$

$$= \cancel{\frac{\sin u}{u}} +$$

10 Find the Sine transform of $f(x) = e^{-|x|}$ and hence evaluate $\int_0^\infty \frac{x \sin mx}{1+x^2} dx$, $m > 0$

= By fourier sine transform

$$F_s(u) = \int_0^\infty f(x) \sin ux \, dx$$

$$= \int_0^\infty e^{-|x|} \sin ux \, dx$$ bcz 0 - ∞ only positive x

$$= \int_0^\infty e^{-x} \sin ux \, dx \quad \rightarrow \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx)$$

(bcz x is linear).

$$= \quad a = -1, \quad b = u$$

$$= \left[\frac{e^{-x}}{(-1)^2 + u^2} (-1 \sin ux - u \cos ux) \right]_0^\infty$$

$$= \frac{1}{1^2 + u^2} \left[0 - e^0 (0 - u \cos 0) \right]$$

$$= \frac{1}{1+u^2} [u] = \frac{u}{1+u^2}$$

By inverse fourier sine transform

$$f(x) = \frac{2}{\pi} \int_0^\infty F_s(u) \sin ux \, du$$

$$= \frac{2}{\pi} \int_0^\infty \frac{u}{1+u^2} \sin ux \, du$$

Put $x = m$,

$$f(m) = e^{-|m|}, \quad m>0$$

$$= e^{-m}$$

$$f(m) = \frac{2}{\pi}\int_0^\infty \frac{u}{1+u^2} \sin mu \, du$$

$$\frac{\pi e^{-m}}{2} = \int_0^\infty \frac{x}{1+x^2} \sin mx \, dx$$

11. Find the Fourier sine transform of the function $\frac{e^{-ax}}{x}$, $a>0$.

By fourier sine transfn

$$F_s(u) = \int_0^\infty f(x) \cdot \sin ux \, dx$$

$$= \int_0^\infty \frac{e^{-ax}}{x} \cdot \sin ux \, dx \quad \text{——— (1)}$$

By libnitz rule differentiating under integral ~~By~~ sign.

$$\frac{d}{du}(F_s(u)) = \int_0^\infty \frac{\partial}{\partial u}\left(\frac{e^{-ax}}{x} \cdot \sin ux\right) dx$$

$$F_s'(u) = \int_0^\infty \frac{e^{-ax}}{\cancel{x}} \cos ux \cdot \cancel{x} \, dx$$

$$= \int_0^\infty e^{-ax} \cos ux \, dx$$

$a =$ $\quad$ $a = -a$ $\quad$ $b = u$

$$\Rightarrow \frac{e^{-ax}}{(-a)^2 + u^2}\left(-a \cos ux + u \sin ux\right)\Big|_0^{\infty}$$

$$\Rightarrow \frac{1}{a^2+u^2}\left[0 - e^0\left[-a + 0\right]\right]$$

$$\Rightarrow F_s'(u) = \frac{a}{a^2+u^2}$$

integrate on both the sides :

$$F_s(u) = a \int_0^{u} \frac{1}{a^2+u^2}\, du$$

$$= \cancel{a} \times \frac{1}{\cancel{a}} \tan^{-1}\left(\frac{u}{a}\right) + k$$

$$= \tan^{-1} u/a + k \quad \text{—— (2)}$$

Put $u = 0$ in (1) & (2)

From (1) $F_s(u) = 0$

From (2) $F_s(u) = 0 + k$ $\quad \} \; k = 0$

$$\therefore F_s(u) = \tan^{-1} u/a$$

12. Show that $x \cdot e^{-x^2/2}$ is self reciprocal under sine transform.

By Fourier sine transform

$$F_s(u) = \int_0^\infty f(x) \sin ux \, dx$$

$$= \int_0^\infty x \cdot e^{-x^2/2} \sin ux \, dx$$

By parts $\xrightarrow{\text{Integration}}$ method,

$\int x e^{-x^2/2} dx$

$-x^2/2 = t$

$-\frac{2x}{2} dx = dt$

$= -x dx = dt$

$= -\int e^t dt$

$= -e^t dt$

$= -e^{-x^2/2}$

$$= \int_0^\infty x e^{-x^2/2} \sin ux \, dx$$

$$= \sin ux \left(-e^{-x^2/2}\right)\Big|_0^\infty - \int_0^\infty \left(-e^{-x^2/2}\right) \cos ux \cdot u \, dx$$

$\int uv \, dx \quad u \int v - \int \int v \cdot du$

$$= (0 - 0) + u \int_0^\infty e^{-x^2/2} \cos ux \, dx.$$

$$F_s(u) = u \int_0^\infty e^{-x^2/2} \cos ux \, dx$$

Consider $F_s(u) = uI$ — * dilt under integral sign

use libnitz rule

$$\frac{d(F_s(u))}{du} = \int_0^\infty \frac{\partial}{\partial u} \left(e^{-x^2/2} \cos ux\right) dx.$$

$$\cancel{F_s(u) = u \int_0^\infty e^{-x^2/2} \cdot u(-\sin ux) \cdot x \, dx}$$

$$\cancel{\int_0^\infty e^{-x^2/2} (u \cdot \sin ux \cdot x + \cos ux) \, dx}$$

$$I = \int_0^\infty e^{-x^2/2} \cos ux \, dx \quad \text{———} \quad (1)$$

$$\frac{d}{du}(F_s(u)) = \int_0^\infty e^{-x^2/2} \frac{\partial}{\partial u}(\cos ux) \, dx$$

$$\int_0 e^{-x^2/2} \left(-\sin ux \cdot x\right) dx$$

$$= -\int_0^\infty \overset{\mathrm{I}}{\sin ux} \; \overset{\mathrm{II}}{\left(x \cdot e^{-x^2/2}\right)} \, dx$$

integration by parts

$$= -\left[\sin ux \left(-e^{-x^2/2}\right)\Big|_0^\infty + \int_0^\infty \left(-e^{-x^2/2}\right) \cdot \cos ux \cdot u \, dx\right]$$

$$I' = (0-0) - u \int_0^\infty e^{-x^2/2} \cos ux \, dx$$

$$I' = -u I$$

$$\frac{I'}{I} = -u$$

Integrate w.r.t u,

$$\log I = -u^2/2 + k$$

$$I = e^{-u^2/2} + k$$

$$I = e^{-u^2/2} \cdot e^{k}$$

$$I = k e^{-u^2/2} \quad \text{——— (2)}$$

put $u = 0$ in (1) & (2)

$$I = \int_0^\infty e^{-x^2/2} \cdot dx = \sqrt{\pi/2}$$

$$I = k \cdot e^0 = k$$

$$k = \sqrt{\pi/2}$$

$\therefore\ I = \frac{\sqrt{\pi}}{2} e^{-u^2/2}$

In $(*)$

$F_s(u) = u\left(\frac{\sqrt{\pi}}{2} e^{-u^2/2}\right)$

$F_s(u) = \frac{\sqrt{\pi}}{2} u \cdot e^{-u^2/2}$

$F_s\left(x e^{-x^2/2}\right) = \frac{\sqrt{\pi}}{2} u e^{-u^2/2}$

$x \cdot e^{-x^2/2}$ is a self reciprocal for w.r.t Fourier sine transform.

13 Obtain The Fourier sine transform of

$$f(x) = \begin{cases} x & 0 < x < 1 \\ 2-x & 1 < x < 2 \\ 0 & x > 2 \end{cases}$$

= By Fourier sine transform

$$\Rightarrow \int_0^1 x \sin ux\, dx + \int_1^2 (2-x) \sin u\, dx + \int_2^\infty 0\, dx.$$

$$\Rightarrow \left[x\left(\frac{-\cos ux}{u}\right) - 1\left(\frac{-\sin ux}{u^2}\right)\right]_0^1 + \left[(2-x)\left(-\frac{\cos ux}{u}\right) + (0-1)\frac{\sin ux}{u^2}\right]_1^2$$

$$\Rightarrow \left[\frac{-\cos u}{u} + \frac{\sin u}{u^2} - (0+0)\right] + \left[\left(0 - \frac{\sin 2u}{u^2}\right) - \left(-\frac{\cos u}{u} - \frac{\sin u}{u^2}\right)\right]$$

$$\cancel{\frac{-\cos u}{u}} + \frac{\sin u}{u^2} - \frac{\sin 2u}{u^2} + \cancel{\frac{\cos u}{u}} + \frac{\sin u}{u^2}$$

$$= \frac{1}{u^2}\left[2 \sin u - \sin 2u\right]$$

14. obtain F Cosine transform $f(x) = \frac{1}{1+x^2}$

Fourier Cosine Transform

$$F_c(u) = \int_0^\infty f(x) \cos ux \, dx \, .$$

$$F_c(u) = \int_0^\infty \frac{1}{1+x^2} \cos ux \, dx \quad \text{——— (1)}$$

By Leibnitz rule,

$$\frac{d}{du} F_c(u) = \int_0^\infty \frac{\partial}{\partial u}\left(\frac{1}{1+x^2} \cos ux\right) dx$$

$$F_c'(u) = \int_0^\infty \frac{1}{1+x^2} (-\sin ux \cdot x) \, dx$$

$\times$ly by $\frac{x}{x}$ $$= -\int_0^\infty \frac{x^2}{x(1+x^2)} \sin ux \, dx$$

-1 on numer $$= -\int_0^\infty \frac{(1+x^2)-1}{x(1+x^2)} \sin ux \, dx$$

$$= -\int_0^\infty \frac{1}{x} \sin ux \, dx + \int_0^\infty \frac{1}{x(1+x^2)} \sin ux \, dx$$

$$\left[\because \int_0^\infty \frac{\sin ax}{x} \, dx = \pi/2\right]$$

$$F_c'(u) = -\pi/2 + \int_0^{\infty} \frac{1}{x(1+x^2)} \sin ux \, dx \quad \text{——— (2)}$$

Again apply libnitz rule under inital sign

$$\frac{d}{du}(F_c'(u)) = 0 + \int_0^{\infty} \frac{\partial}{\partial u}\left(\frac{1}{x(1+x^2)} \sin ux\right) dx$$

$$F_c''(u) = \int_0^{\infty} \frac{1}{\cancel{x}(1+x^2)} \cos ux \cdot \cancel{x} \, dx$$

$$= \int_0^{\infty} \frac{1}{1+x^2} \cos ux \, dx$$

$$F_c''(u) = F_c(u)$$

$$F_c''(u) - F_c(u) = 0$$

$$\frac{d^2y}{dx^2} + y = 0$$

$$y'' - y = 0$$

$$A.E \quad m^2 - 1 = 0$$

$$m = \pm 1$$

$$y_c = c_1 e^{x} + c_2 e^{-x}$$

$$A.E : m^2 - 1 = 0$$

$$m^2 = 1$$

$$m = \pm 1$$

$$\therefore F_c(u) = c_1 e^{u} + c_2 e^{-u} \quad \text{——— (3)}$$

$$F_c'(u) = c_1 e^{u} - c_2 e^{-u} \quad \text{——— (4)}$$

put $u = 0$ in (1) & (3)

In (1) $$F_c(0) = \int_0^{\infty} \frac{1}{1+x^2} dx = \tan^{-1} x \Big|_0^{\infty}$$

$$= \tan^{-1}\infty - \tan^{-1} 0$$

$$= \pi/2 - 0 = \pi/2$$

In (3), $$F_c(0) = c_1 e^0 + c_2 e^0$$

$$= c_1 + c_2$$

$$\pi/$$

put $u=0$ in ② & ④

In ②, $F_c'(0) = -\pi/2 + 0 = -\pi/2$

In ④, $F_c'(0) = c_1 - c_2$

$$\boxed{c_1 - c_2 = -\pi/2}$$

$$c_1 + c_2 = \pi/2$$
$$c_1 - c_2 = -\pi/2$$
$$2c_1 = 0 \Rightarrow c_1 = 0$$

$\therefore c_2 = \pi/2$

$\therefore F_c(u) = 0 + \pi/2\, e^{-u}$

$$F_c(u) = \pi/2\, e^{-u}$$

15. Obtain the Sine transform,

H.W $\quad f(x) = \dfrac{1}{x(1+x^2)}$

16. Solve the integral equation $\int_0^\infty f(\theta)\cos\alpha\theta\, d\theta$

$$= \begin{cases} 1-\alpha, & 0 \le \alpha \le 1 \\ 0, & \alpha > 1 \end{cases}$$

Hence evaluate $\int_0^\infty \dfrac{\sin^2 t}{t^2}\, dt$

= ⊛ is equivalent to

$$F_c(\alpha) = \begin{cases} 1-\alpha, & 0 \le \alpha \le 1 \\ 0, & \alpha > 1 \end{cases}$$

By inverse fourier cosine transform,

$$f(x) = \frac{2}{\pi} \int_0^\infty F_c(u) \cos ux \, du$$

$$f(\theta) = \frac{2}{\pi} \int_0^\infty F_c(\alpha) \cos \alpha\theta \, d\alpha$$

$$= \frac{2}{\pi} \left[\int_0^1 (1-\alpha) \cos\alpha\theta \, d\alpha + \int_1^\infty 0 \, d\alpha \right]$$

$$= \frac{2}{\pi} \left[(1-\alpha) \frac{\sin\alpha\theta}{\theta} - (-1)\left(-\frac{\cos\alpha\theta}{\theta^2}\right) \right]_0^1 + 0$$

$$= \frac{2}{\pi} \left[\left(0 - \frac{\cos\theta}{\theta^2}\right) - \left(0 - \frac{1}{\theta^2}\right) \right]$$

$$f(\theta) = \frac{2}{\pi\theta^2} (1 - \cos\theta) = \frac{2}{\pi\theta^2} (2\sin^2 \theta/2)$$

$$= 4 \frac{\sin^2 \theta/2}{\pi\theta^2}$$

By Fourier Cosine transform,

$$F_c(u) = \int_0^\infty f(x) \cos ux \, dx$$

$$F_c(\alpha) = \int_0^\infty f(\theta) \cos\alpha\theta \, d\theta$$

According to the qtn reduce

$$= \int_0^\infty \frac{4 \sin^2 \theta/2}{\pi\theta^2} \cos\alpha\theta \, d\theta$$

Put $\alpha = 0$, $F_c(0) = 1 - 0 = 1$

$$I = \frac{4}{\pi} \int_0^\infty \frac{\sin^2 \theta/2}{\theta^2} \cos\theta \, d\theta$$

put $t = \theta/2$ $\quad dt = d\theta/2$

when $\theta = 0$, $t = 0$

$\theta = \infty$, $t = \infty$

$$\therefore I = \frac{4}{\pi} \int_0^\infty \frac{\sin^2 t}{(2t)^2} \cdot 2dt$$

$$I = \frac{2}{\pi} \int_0^\infty \frac{\sin^2 t}{t^2} dt$$

$$\pi/2 = \int_0^\infty \frac{\sin^2 t}{t^2} dt$$

Z Transform

Z tansform plays a role in discrete analysis.

Z transform has many properties similar to Those of laplace transform.

Main difference is not on function