MS
Chap 2

# Calculus of Variations

Functional:

consider a function of a form $f(x, y, y')$

where $y' = \frac{dy}{dx}$, $y = y(x)$

and $x \in (x_1, x_2)$

the integral $I(y)$ or $I = \int_{x_1}^{x_2} f(x, y, y')\,dx$ which

always give real value.

In order to find extreme value of $I(y)$ we apply Euler's eqn.

Euler's equation :-

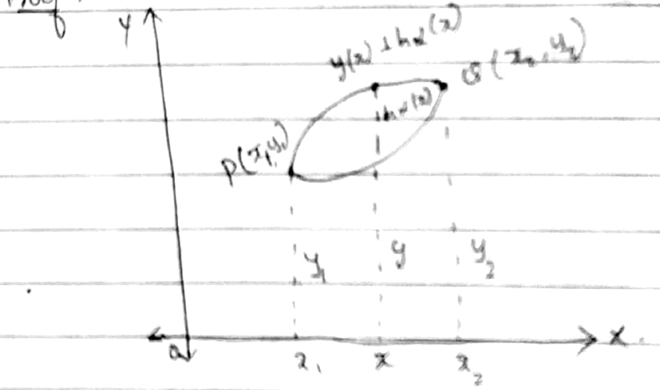
A necessary condition for the integral

$$I = \int_{x_1}^{x_2} f(x, y, y')\,dx \text{ where}$$

$y(x_1) = y_1$, $y(x_2) = y_2$ to be an extremum is that

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right] = 0$$

Proof:




Let $I$ be an extremum along the curve $y = y(x)$ passing through $P(x_1, y_1)$ & $Q(x_2, y_2)$.
Let $y = y(x) + h\alpha(x)$ — ① be the neighbouring curve of $y(x)$ joining these points so that have

$$\alpha(x_1) = 0, \quad \alpha(x_2) = 0 \quad \text{— ②}$$

when $h=0$ these two curves coincides thus making $I$ an extremum.

That is $I = \int_{x_1}^{x_2} f(x, y(x) + h\alpha(x), y'(x) + h\alpha'(x))\, dx$.

is an extremum when $h=0$.

This requires $\frac{dI}{dh} = 0$ when $h=0$ treat $I$ as the function of $h$.

$$\frac{dI}{dh} = \int_{x_1}^{x_2} \frac{\partial}{\partial h} f[x, y(x) + h\alpha(x), y'(x) + h\alpha'(x)]\, dx$$

$$\frac{dI}{dh} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial x}\frac{\partial x}{\partial h} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial h} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial h}\right] dx \quad -(3)$$

(using chain rule of Partial derivative).

'$x$' is independent of $h$ $\therefore \frac{\partial x}{\partial h} = 0$,

$$\frac{\partial y}{\partial h} = \frac{\partial}{\partial h}\left[y(x) + h\alpha(x)\right] = \alpha(x), \quad \frac{\partial y'}{\partial h} = \frac{\partial}{\partial h}\left[y'(x) + h\alpha'(x)\right] = \alpha'(x)$$

sub in (3) we get

$$\frac{dI}{dh} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y}\alpha(x) + \frac{\partial f}{\partial y'}\alpha'(x)\right] dx$$

$$\frac{dI}{dh} = \int_{x_1}^{x_2} \frac{\partial f}{\partial y}\alpha(x)\, dx + \int_{x_1}^{x_2} \frac{\partial f}{\partial y'}\alpha'(x)\, dx \quad -(4)$$

Keeping the 1st term of R.H.S of equ (4) as it is & integrating the 2nd term by parts. We get

$$\frac{dI}{dh} = \int_{x_1}^{x_2} \frac{\partial f}{\partial y}\cdot\alpha(x)\, dx + \left[\frac{\partial f}{\partial y'}\alpha(x)\right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \alpha(x)\frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right] dx$$

$$= \int_{x_1}^{x_2} \frac{\partial f}{\partial y}\alpha(x)\, dx + \frac{\partial f}{\partial y'}\left[\alpha(x_2) - \alpha(x_1)\right] - \int_{x_1}^{x_2} \alpha(x)\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) dx.$$

from (2) $\alpha(x_1) = 0$, $\alpha(x_2) = 0$

$$\frac{dI}{dh} = \int_{x_1}^{x_2} \frac{\partial f}{\partial y}\,\alpha(x)\,dx - \int_{x_1}^{x_2} \alpha(x)\,\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)dx.$$

$$\frac{dI}{dh} = \int_{x_1}^{x_2} \alpha(x)\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]dx \quad -(5)$$

to get we put $\frac{dI}{dh} = 0$

$$\int_{x_1}^{x_2} \alpha(x)\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]dx = 0$$

$$\Rightarrow \frac{\partial f}{\partial y} - \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right] = 0 \quad (\because \alpha(x) \text{ is arbitrary})$$

Note: 1) If $f$ is independent of $x$ then euler's equ becomes $f - y'\frac{\partial f}{\partial y'} = K$

2) If $f$ is independent of $y$ $\frac{\partial f}{\partial y'} = K$

Where 'K' is a constant.

3) The euler's equ $\frac{\partial f}{\partial y} - \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right] = 0$ is also expressed as $\delta I = 0$

## Variational Problems:

1. Find extreemal of $\int_{x_1}^{x_2} (y' + x^2y'^2)\,dx$.

→ Given $I = \int_{x_1}^{x_2} (y' + x^2y'^2)\,dx$, where $f = y' + x^2y'^2$

We have Euler's equ $\frac{\partial f}{\partial y} - \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right] = 0 \quad -(1)$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[y' + x^2y'^2\right]$$

$$\frac{\partial f}{\partial y} = 0$$

$$\frac{\partial f}{\partial y'} = \frac{\partial}{\partial y'}\left[y' + x^2 y'^2\right] = 1 + x^2 2y'$$

$\frac{\partial}{\partial x}\left(\frac{dy}{dx}\right)$

Sub in (1):

$$0 - \frac{d}{dx}\left[1 + 2x^2 y'\right] = 0$$

$$\frac{d}{dx}\left[1 + 2x^2 y'\right] = 0$$

(S.B. diff we will cancel $\frac{d}{dx}$) Integrating.

$$1 + 2x^2 y' = k$$

$$1 + 2x^2 \frac{dy}{dx} = k$$

$$2x^2 \frac{dy}{dx} = k - 1$$

Seperating Variable.

$$2dy = \frac{k-1}{x^2} dx.$$

Integrating both side.

$$2\int dy = (k-1)\int \frac{1}{x^2} dx \qquad \int 1/x^2 = -1/x$$

$$2y = (k-1)\left(-\frac{1}{x}\right) + c$$

$$2y = -\frac{(k-1)}{x} + c$$

$$(\div 2) \qquad y = \frac{1-k}{2x} + \frac{c}{2}$$

$$y = \frac{1-k}{2x} + c_1 \qquad \text{where } \frac{c}{2} = c_1 \text{ (constant)}$$

$$y = \frac{c_2}{x} + c_1 \qquad \text{where } c_2 = \frac{1-k}{2x}.$$

(1) Find the function 'y' which makes the integral to extremum

(2) $\int_{x_1}^{x_2} (1+xy'+xy'^2)\,dx$. Find the funct^n

$\rightarrow$. Given $I = \int_{x_1}^{x_2} (1+xy'+xy'^2)\,dx$  $f = (1+xy'+xy'^2)$

We have euler's eq^n $\frac{\partial f}{\partial y} - \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right] = 0$ — (1)

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}[1+xy'+xy'^2]$$

$$\frac{\partial f}{\partial y} = 0$$

$$\frac{\partial f}{\partial y'} = \frac{\partial}{\partial y'}[1+xy'+xy'^2]$$

$$= 0 + x + 2xy'$$

$$\frac{\partial f}{\partial y'} = x + 2xy'$$

Sub in (1).

$$0 - \frac{d}{dx}[x+2xy'] = 0.$$

$$\frac{d}{dx}[x+2xy'] = 0.$$

Integrating $x + 2xy' = k$.

$$x + 2x\frac{dy}{dx} = k.$$

$$\cancel{2x\frac{dy}{dx} = k - x}.$$

$$x\left(1 + 2\frac{dy}{dx}\right) = k.$$

$$1 + 2\frac{dy}{dx} = \frac{k}{x}.$$

$$2\frac{dy}{dx} = \frac{k}{x} - 1$$

$$2dy = \frac{k}{x}dx - 1dx.$$

$$2dy = \left(\frac{k}{x} - 1\right)dx.$$

Integrating both side:

$$2\int dy = \int \left(\frac{k}{x} - 1\right) dx.$$

$$2y = k\log x - x + c_1$$

$$\div 2 \quad y = \frac{k\log x}{2} - \frac{x}{2} + \frac{c_1}{2}.$$

$$\cancel{y = \frac{k\log x}{2} \frac{}{2} c_1}$$

$$y = c_2 \log x - c_3 x + c_4$$

where $c_2 = \frac{k}{2}$, $c_3 = \frac{1}{2}$, $c_4 = \frac{c_1}{2}$.

③ Find the extremum of the functional $\int_{x_1}^{x_2} (y^2 + y'^2 + 2ye^x) dx$.

→ $I = \int_{x_1}^{x_2} (y^2 + y'^2 + 2ye^x) dx$, $I = (y^2 + y'^2 + 2ye^x)$

Euler's equⁿ. $\frac{\partial f}{\partial y} - \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right] = 0$ — ①.

$$\frac{\partial f}{\partial y} = \frac{\partial (y^2 + y'^2 + 2ye^x)}{\partial y} = 2y + 2e^x.$$

$$= 2y + 2e^x.$$

$$\frac{\partial f}{\partial y'} = \frac{\partial (y^2 + y'^2 + 2ye^x)}{\partial y'} = 2y'$$

Sub in ①.

$$(2y + 2e^x) - \frac{d}{dx}(2y') = 0. \qquad \left(\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = y''\right)$$

$$2y + 2e^x - 2y'' = 0$$

$$(\div 2) \quad y + e^x - y'' = 0$$

$$[\times -1] \quad y'' - y = e^x$$

$$(D^2 - 1)y = e^x, \quad D = \frac{d}{dx}$$

A.E $m^2 - 1 = 0$

$m = \pm 1$

$$y_c = c_1 e^x + c_2 e^{-x}$$

$$y_p = \frac{1}{D^2-1} \cdot e^x$$

Put $D=1$ $\quad y_p = \frac{1}{0} e^x$

$$\therefore y_p = \frac{1}{2D} \cdot xe^x$$

Put $D=1$

$$y_p = \frac{1}{2} x e^x$$

$$\therefore y = y_c + y_p = C_1 e^x + C_2 e^{-x} + \frac{1}{2} xe^x$$

Thus $y = \left(C_1 + \frac{x}{2}\right)e^x + C_2 e^{-x}$.

(4) Find the extreemum of the function $\int_{x_1}^{x_2} (x^2y'^2 + 2y^2 + 2xy)$

→ Given $I = \int_{x_1}^{x_2} (x^2y'^2 + 2y^2 + 2xy)dx$, $f = (x^2y'^2 + 2y^2 + 2xy)$

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right] = 0.$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2y'^2 + 2y^2 + 2xy) = 0 + 4y + 2x = 4y + 2x$$

$$\frac{\partial f}{\partial y'} = 2x^2y'$$

Sub in (1).

$$4y + 2x - \frac{d}{dx}(2x^2y') = 0.$$

$$4y + 2x - 2[x^2y'' + y' \cdot 2x] = 0$$

$$4y + 2x - 2x^2y'' - 4xy' = 0$$

$\div 2$ $\quad 2y + x - x^2y'' - 2xy' = 0$

$$-x^2y'' - 2xy' + 2y = -x$$

$(\times -1)$ $\quad x^2y'' + 2xy' - 2y = x$

Cauchy's LDE.

Let $\log x = t$ $\quad e^t = x$

$$x^2 y'' = D(D-1)y$$

$$xy' = Dy \quad , \quad D = \frac{d}{dx}.$$

Subs in ②

$$D(D-1)y + 2Dy - 2y = e^t$$

$$(D^2 - D + 2D - 2)y = e^t$$

$$(D^2 + D - 2)y = e^t.$$

A.E $\quad m^2 + m - 2 = 0$

$$m = 1, -2$$

$$\therefore y_c = c_1 e^t + c_2 e^{-2t}$$

$$y_p = \frac{1}{D^2 + D - 2} \cdot e^t$$

Put $D = 1$

$$y_p = \frac{1}{0} e^t.$$

$$\therefore y_p = \frac{1}{2D+1} t \cdot e^t.$$

$D = 1$

$$y_p = \frac{1}{3} t \cdot e^t.$$

$$\therefore y = y_c + y_p.$$

$$y = c_1 e^t + c_2 e^{-2t} + \frac{te^t}{3}$$

Put $e^t = x$, $e^{-2t} = \frac{1}{x^2}$, $t = \log x$.

$$y = c_1 x + c_2 \frac{1}{x^2} + \log x \frac{x}{3}$$

$$y = c_1 x + \frac{c_2}{x} + \frac{x \log x}{3}$$

(5) Find the curve on which the functional $\int_0^1 [y'^2 + 12xy]\,dx$ with $y(0)=0,\ y(1)=1$ can be determined.

→ Given $f = y'^2 + 12xy$

Euler's eqn $\frac{\partial f}{\partial y} - \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right] = 0$ — (1)

$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}[y'^2 + 12xy] = 12x.$

$\frac{\partial f}{\partial y'} = 2y'$

Sub in (1).

$12x - \frac{d}{dx}[2y'] = 0$

$12x = 2\frac{d}{dx}[y']$

$12x = 2y''$

$6x = y''$ or $\frac{d^2y}{dx^2} = 6x.$

Integrate w.r.t x.

$\frac{dy}{dx} = \frac{\cancel{6}^{3}x^2}{\cancel{2}_{1}} + C_1$

$= 3x^2 + C_1$

Integration w.r.t x

$y = \frac{\cancel{3}x^3}{\cancel{3}} + C_1x + C_2$

$y = x^3 + C_1x + C_2$ — (2)

We find $C_1$ & $C_2$ using $y(0)=0,\ y(1)=1$

Put $x=0,\ y=0$ in (2)

$0 = 0 + C_2 \quad \therefore C_2 = 0.$

Put $x=1,\ y=1,\ C_2=0$ in (2).

$1 = 1 + C_1 = 0$

$C_1 = 0$

Sub in $C_1 = 0,\ C_2 = 0$ in (2) we get $y = x^3$

(6) Solve the variational problem $I = \delta\int_0^1 (x+y+y'^2)\,dx = 0$ with $y(0)=1$, $y(1)=2$

$\rightarrow$ $f = x+y+y'^2$

Euler eqn

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right] = 0 \quad \text{—(1)}$$

$$\frac{\partial f}{\partial y} = 1 \qquad \frac{\partial f}{\partial y'} = 2y'$$

Sub in (1).

$$1 - \frac{d}{dx}(2y') = 0.$$

$$1 = 2\frac{d}{dx}y'$$

$$1 = 2 \times y''$$

$$1 = 2\frac{d^2y}{dx^2} \Rightarrow y'' = \frac{1}{2}.$$

Integration w.r.t $x$.

$$\frac{dy}{dx} = \frac{1}{2}x + C_1$$

Integration w.r.t $x$.

$$y = \frac{1}{2} \times \frac{x^2}{2} + C_1x + C_2$$

$$y = \frac{x^2}{4} + C_1x + C_2 \quad \text{—(2)}$$

We find $C_1$ & $C_2$ using $y(0)=1$, $y(1)=2$.

Put $x=0$ $y=1$ in (2).

$$1 = 0+0+C_2$$

$$\therefore C_2 = 1$$

Put $x=1$ & $y=2$ in (2).

$$2 = \frac{1}{4} + C_1 + C_2.$$

$$2 = \frac{1}{4} + C_1 + 1$$

$$2 = \frac{5}{4} + C_1$$

$$2 - \frac{5}{4} = C_1$$

$$C_1 = 3/4$$

Sub $C_1$ & $C_2$ value in (2) we get.

$$y = \frac{x^2}{4} + \frac{3}{4}x + 1$$

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(7) S.T the extreemal of the functional $\int_0^1 y^2[3x(y'^2-1)+yy'^3]dx$ given that $y(0)=0$, $y(1)=2$ is the circle $x^2+y^2-5x=0$

$\rightarrow$ $f = y^2[3x(y'^2-1)+yy'^3] = 3xy^2y'^2-3xy^2+y^3y'^3$

euler's equⁿ $\frac{\partial f}{\partial y} - \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right]=0$.

$\frac{\partial f}{\partial y} = 3xy'^2\times 2y - 3x\times 2y + y'^3\times 3y^2$.

$\frac{\partial f}{\partial y} = 6xyy'^2 - 6xy + 3y^2y'^3$

$\frac{\partial f}{\partial y'} = \frac{\partial}{\partial y'}[3xy^2y'^2-3xy^2+y^3y'^3]$

$= 3xy^2\times 2y' - 0 + y^3\cdot 3y'^2$

$\frac{\partial f}{\partial y'} = 6xy^2y' + 3y^3y'^2$

sub in (1).

$[6xyy'^2-6xy+3y^2y'^3] - \frac{d}{dx}[6xy^2y'+3y^3y'^2]=0$

$6xyy'-6xy+3y^2y'^3 = \frac{d}{dx}[6xy^2y'+3y^3y'^2]$

$6xyy'-6xy+3y^2y'^3 = [6xy^2y''+6xy'\cdot 2yy'+y^2y'\cdot 6+3y^3\cdot 2y'y'' + y'^2 9y^2]$

$\left(\frac{d}{dx}(u,v,w) = uv\frac{dw}{dx} + uw\frac{dv}{dx} + vw\frac{du}{dx}\right)$

$6xyy'-6xy+3y^2y'^3 = 6xy^2y''+12xyy'^2+6y^2y'+6y^3y'y''+9y^2y'^2$

$\div 3y]$

$2xy'^2-2x+yy'^3 = 2xyy''+\cancel{4xy'^2}\,2xy'^2+2yy'+2y^2y'y''+3yy'^2$

$-2x+yy'^3 = 2xyy''+2xy'^2+2yy'+2y^2y'y''+3yy'$

$y[x+yy'][y'^2+1+yy'']=0$ —(2)

Given $y=0$

$\therefore x+yy'=0$, $y'^2+1+yy''=0$

We solve

$x+yy'=0$

$x=-yy'$

$x=-y\frac{dy}{dx}$

separating Variable.

$x\,dx=-y\,dy$

integrating

$$\frac{x^2}{2}=-\frac{y^2}{2}+c_1$$

$$\frac{x^2}{2}+\frac{y^2}{2}=c_1 \quad -(3)$$

Put $x=0$ & $y=0$

$0=c_1$

Note ⇒ eqⁿ does not satisfy the Given Conditions

$\therefore$ We solve $y'^2+1+yy''=0$.

$1+\frac{d}{dx}(yy')=0$

$\frac{d}{dx}(yy')=-1$

Integrating

$yy'=-x+c_2$

$y\frac{dy}{dx}=-x+c_2$.

~~$y\,dy=-c_2\,dx$.~~

solving $y\,dy=(-x+c_2)\,dx$.

integrating

$$\frac{y^2}{2}=-\frac{x^2}{2}+c_2x+c_3$$

$\times 2]$ $y^2=-x^2+2c_2x+2c_3 \quad -(4)$.

when $x=0, y=0, c_3=0$

$x=1, y=2, c_3=0$

$4=-1+2c_2+0$

$5=2c_2$

$c_2=5/2$

$\therefore$ become

$$y^2=-x^2+5x$$

$$x^2+y^2-5x=0$$

(8) Find the extremal of the functional $\int_{x_1}^{x_2}(y'^2-y^2+2y\sec x)\,dx$

$\rightarrow$ $f=y'^2-y^2+2y\sec x$.

We find the extremal using euler's eqn

$$\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)=0 \quad -(1)$$

$$\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}\left[y'^2-y^2+2y\sec x\right)=-2y+2\sec x$$

$$\frac{\partial f}{\partial y'}=\frac{\partial}{\partial y'}\left[y'^2-y^2+2y\sec x\right)=2y'$$

Sub in (1)

$$-2y+2\sec x-\frac{d}{dx}(2y')=0$$

$$\div 2]\quad -y+\sec x-\frac{d}{dx}(y')=0$$

$$-y+\sec x-y''=0$$

$$\sec x=y''+y$$

$$y''+y=\sec x.$$

$$(D^2+1)y=\sec x \qquad D=\frac{d}{dx}.$$

A.E $m^2+1=0$

$m=\pm i$

$$y_c=c_1\cos x+c_2\sin x$$

Let $y = A\cos x + B\sin x$ — ② be the solution where $A$ & $B$ are functions of $x$. We compute $A$ & $B$ using formula

$$A = -\int \frac{y_2 \phi(x)}{W} dx, \quad B = \int \frac{y_1 \phi(x)}{W} dx.$$

where $\phi(x) = \sec x$.

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1 \neq 0.$$

$$\therefore A = -\int \frac{\sin x \sec x}{1} dx = -\int \tan x \, dx$$

$$= -[\log \sec x] + K_1 = -\log \sec x + K_1$$

$$B = \int \frac{\cos x \sec x}{1} dx = \int dx = x + K_2.$$

Sub $A$ & $B$ in ②

$$y = [-\log \sec x + K_1] \cos x + (x + K_2) \sin x.$$

(9) Solve the Variational problem

① Solve $\delta \int_0^{\pi/2} (y^2 - y'^2) dx = 0$; $y(0) = 0$, $y(\pi/2) = 2$.

$\Rightarrow$ $f = y^2 - y'^2$ given $\delta I = 0$ — ①

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}[y^2 - y'^2] = 2y$$

$$\frac{\partial f}{\partial y'} = \frac{\partial}{\partial y'}[y^2 - y'^2] = -2y'$$

Sub in ①

$$2y - \frac{d}{dx}[-2y'] = 0$$

$$\div 2) \quad y + \frac{d}{dx}(y') = 0$$

$y + y'' = 0$

$y'' + y = 0$

$(D^2+1)y = 0, \; D = \frac{d}{dx}$

A.E $\quad m^2+1=0$

$m = \pm i$

$y = c_1 \cos x + c_2 \sin x \quad -②$

Now we find $c_1$ & $c_2$ using $y(0)=0, \; y(\pi/2) = 2$

Put $x=0, \; y=0$ in ②

$0 = c_1 \cos 0 + c_2 \sin 0$

$0 = c_1 \times 1 + 0$

$\therefore c_1 = 0$

Put $x = \frac{\pi}{2}, \; y = 2, \; c_1 = 0$ in ②

$\Rightarrow 2 = 0 \times \cos \frac{\pi}{2} + c_2 \sin \frac{\pi}{2}$

$2 = c_2 \times 1$

$\therefore c_2 = 2$

Sub $c_1 = 0, \; c_2 = 2$ in $-②$

$y = 0 \times \cos x + 2 \sin x$

$y = 2 \sin x$

⑩ S.T an extremal of $\int_{x_1}^{x_2} \frac{y'^2}{y^2} \, dx$ is in the form of

$y = ae^{bx}$

$\Rightarrow \; f = \frac{y'^2}{y^2}$ we find extremal of f using Euler's eqn

$\frac{\partial f}{\partial y} - \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right] = 0 \quad -①$

$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[\frac{y'^2}{y^2}\right] = y'^2 \frac{\partial}{\partial y}(y^{-2}) = y'^2(-2y^{-2-1}) = \frac{-2y'^2}{y^3}$

~~We find extremal of f using~~ ~~Eq~~

$$\frac{\partial f}{\partial y'} = \cancel{\frac{2y'}{y^2}} \quad \frac{1}{y^2}\frac{\partial}{\partial y'}(y'^2) = \frac{1}{y^2}2y'.$$

Sub in (1)

$$-\frac{2y'^2}{y^3} - \frac{d}{dx}\left[\frac{2y'}{y^2}\right] = 0$$

$\div 2]$

$$-\frac{y'^2}{y^3} - \left[\frac{y^2 \cdot y'' - y' \cdot 2yy'}{y^4}\right] = 0.$$

$\times y^3]$

$$-y'^2 - \left[\frac{y^2 \cdot y'' - y' \cdot 2yy'}{y}\right] = 0$$

$$-y'^2 - yy'' + 2y'^2 = 0$$

$$y'^2 - yy'' = 0$$

$$\Rightarrow \frac{d}{dx}\left(\frac{y}{y'}\right) = 0$$

$$\frac{d}{dx}\left(\frac{y}{y'}\right) = \frac{y'y' - yy''}{y'^2}$$

$$= y'^2 - yy''$$

Integrating

$$\frac{y}{y'} = c_1$$

$$y = y'c_1$$

$$y = c_1\frac{dy}{dx}$$

Seperating Variable

$$dx = c_1\frac{dy}{y}$$

Integrating $\int dx = c_1\int {}^1/_y\, dy$

$$x = c_1 \log y + c_2$$

$$x - c_2 = c_1 \log y$$

$$\log y = \frac{x - c_2}{c_1} = \frac{x}{c_1} - \frac{c_2}{c_1}$$

$$\log y = c_3 x - c_4 \qquad c_3 = \frac{1}{c_1},\ c_4 = \frac{c_2}{c_1}$$

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$$y = e^{c_3x - c_4}$$

$$y = e^{c_3x} e^{-c_4}$$

$$y = c_5 e^{c_3x}$$

## Applications of Calculus of Variation:

1- Geodesic : on a surface along which the distance when any 2 points of the surface is minimum.

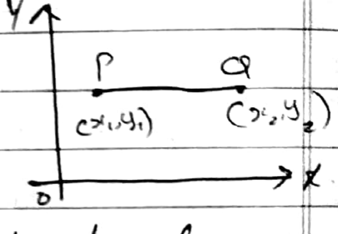
① P.T the shortest distance b/w 2 points in a plane is ~~around~~ along the straight line joining them OR P.T Geodesic on a plane are straight line.

→ $y = y(x)$ be the curve joining the point $(x_1, y_1)$ & $(x_2, y_2)$.

WKT the ~~arc~~ arc length b/w P & Q

$$S = \int_{x_1}^{x_2} \left(\frac{ds}{dx}\right) dx$$

$$I = S = \int_{x_1}^{x_2} \sqrt{1+y'^2}\, dx$$




Now we need to find a curve which represents equ of a straight line i.e it is possible ~~to find~~
i.e we need to find $y(x)$ such that I is minimum using Euler's equ

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right] = 0 \quad -①$$

here $f = \sqrt{1+y'^2}$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\sqrt{1+y'^2}\right) = 0$$

$$\frac{\partial f}{\partial y'} = \frac{\partial}{\partial y'}\left[\sqrt{1+y'^2}\right] = \frac{1}{\cancel{2}\sqrt{1+y'^2}} \cancel{2}y'$$

Sub in (1)

$$0 - \frac{d}{dx}\left[\frac{y'}{\sqrt{1+y'^2}}\right] = 0$$

$$\frac{d}{dx}\left[\frac{y'}{\sqrt{1+y'^2}}\right] = 0$$

Integrating

$$\frac{y'}{\sqrt{1+y'^2}} = C_1$$

$$y' = C_1\sqrt{1+y'^2}$$

Squaring

$$y'^2 = C_1^2(1+y'^2)$$

$$y'^2 = C_1^2 + C_1^2 y'^2$$

$$y'^2 - C_1^2 y'^2 = C_1^2$$

$$y'^2[1-C_1^2] = C_1^2$$

$$y'^2 = \frac{C_1^2}{1-C_1^2}$$

$$y' = \sqrt{\frac{C_1^2}{1-C_1^2}}$$

$$y' = \frac{C_1}{\sqrt{1-C_1^2}}$$

$$\frac{dy}{dx} = \frac{C_1}{\sqrt{1-C_1^2}}$$

Sep variable

$$dy = \frac{C_1}{\sqrt{1-C_1^2}}\,dx$$

$dy = C_2\,dx$ when $C_2 = \frac{C_1}{\sqrt{1-C_1^2}}$

Integrating $\int dy = \int C_2\,dx$

$$y = C_2 x + C_3$$

which is eqn of straight line

(Q) Find the Geodesic on a surface whose arc length on a surface is
$$S = \int_{x_1}^{x_2} \sqrt{x(1+y'^2)}\,dx$$

or Geodesic on a surface whose arc length $S = \int_{x_1}^{x_2} \sqrt{x(1+y'^2)}\,dx$ is a Parabola.

$\Rightarrow$ We have $I = \int_{x_1}^{x_2} \sqrt{x(1+y'^2)}\,dx$.

We find equ of the curve $y = y(x)$ such that $I$ is minimum using Euler eqn

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right] = 0 \quad \text{--- (1)}$$

where $f = \sqrt{x(1+y'^2)}$

$$\frac{\partial f}{\partial y} = \frac{\partial}{y}\left(\sqrt{x(1+y'^2)}\right) = 0$$

$$\frac{\partial f}{\partial y'} = \frac{\partial}{\partial y'}\left(\sqrt{x(1+y'^2)}\right) = \sqrt{x}\,\frac{\partial}{\partial y'}\left(\sqrt{(1+y'^2)}\right).$$

$$= \sqrt{x} \cdot \frac{1}{2\sqrt{(1+y'^2)}} \; 2y'$$

Sub in (1)

$$0 - \frac{d}{dx}\left[\frac{\sqrt{x}\cdot y'}{\sqrt{(1+y'^2)}}\right] = 0$$

$$\frac{d}{dx}\left[\frac{\sqrt{x}\cdot y'}{\sqrt{(1+y'^2)}}\right] = 0$$

Integrating

$$\frac{\sqrt{x}\;y'}{\sqrt{1+y'^2}} = c_1$$

$$\sqrt{x}\;y' = c_1\sqrt{1+y'^2}$$

Squaring,

$$xy'^2 = c_1^2(1+y'^2)$$

$$xy'^2 = c_1^2 + c_1^2 y'^2$$

$$xy'^2 - c_1^2 y'^2 = c_1^2$$

$$y'^2 [x - c_1^2] = c_1^2$$

$$y'^2 = \frac{c_1^2}{x - c_1^2}$$

$$y' = \sqrt{\frac{c_1^2}{x - c_1^2}} = \frac{c_1}{\sqrt{x - c_1^2}}$$

$$\frac{dy}{dx} = \frac{c_1}{\sqrt{x - c_1^2}}$$

sep variable

$$dy = c_1 \frac{dx}{\sqrt{x - c_1^2}}$$

integrating $\int dy = \int c_1 \frac{dx}{\sqrt{x - c_1^2}}$

$$y = c_1 \int \frac{1}{\sqrt{x - c_1^2}} \, dx.$$

$$y = c_1 \int \frac{1}{\sqrt{t}} \, dt = c_1 \int t^{-1/2} dt$$

$$= c_1 \left( \frac{t^{-1/2+1}}{-1/2+1} \right) + c_2$$

$$y = c_1 \frac{t^{1/2}}{1/2} + c_2$$

$$y = 2c_1 \sqrt{(x - c_1^2)} + c_2$$

$$y - c_2 = 2c_1 \sqrt{x - c_1^2}$$

$(y - c_2)^2 = 4c_1^2 (x - c_1^2)$ which is the parabola.

Imp (B) P.T catanary is the curve which when rotated about a line generates a surface of minimum area.

→ We have total surface area is ~~integration~~

$\int 2\pi y\, ds$

where the curve is rotating about x-axis

$\therefore I = \int_{x_1}^{x_2} 2\pi y \frac{ds}{dx} \cdot dx$. $\quad$ $\left( \frac{ds}{dx} = \sqrt{1+y'^2} \text{ (constant)} \right)$

$= 2\pi \int_{x_1}^{x_2} y\sqrt{1+y'^2}\, dx$.

where $f = \sqrt{1+y'^2}$

f is independent of x

$\therefore$ We find the equ of the catanary such that the surface area I is minimum

using Euler's equ

$f - y'\frac{\partial f}{\partial y'} = \text{Constant} = k$ —(1)

when $f = y\sqrt{1+y'^2}$

$\frac{\partial f}{\partial y'} = \frac{\partial}{\partial y'}\left[y\sqrt{1+y'^2}\right]$

$= y\left[\frac{1}{2\sqrt{1+y'^2}} \times 2y'\right]$

$= \frac{y \cdot y'}{\sqrt{1+y'^2}}$

sub in (1)

$\left[y\sqrt{1+y'^2} - \frac{yy'}{\sqrt{1+y'^2}}\right] = k$

$\cdot \div$ by $\sqrt{1+y'^2}$

$y(1+y'^2) - yy'^2 = k\sqrt{1+y'^2}$

$y + \cancel{yy'^2} - \cancel{yy'^2} = k\sqrt{1+y'^2}$

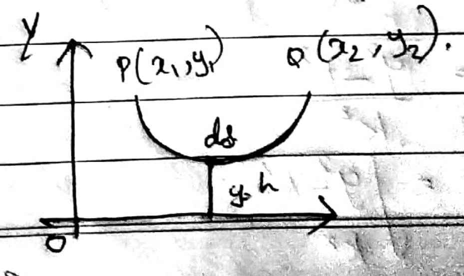
$y = k\sqrt{1+y'^2}$

$$y^2 = k^2(1+y'^2)$$
$$y^2 = k^2 + k^2 y'^2$$
$$\frac{y^2}{k^2} = 1 + y'^2$$
$$y'^2 = \frac{y^2}{k^2} - 1$$
$$y'^2 = \frac{y^2-k^2}{k^2}$$
$$y' = \sqrt{\frac{y^2-k^2}{k^2}}$$
$$y' = \frac{\sqrt{y^2-k^2}}{k}$$
$$\frac{dy}{dx} = \frac{\sqrt{y^2-k^2}}{k}$$
$$\int \frac{dy}{\sqrt{y^2-k^2}} = \int \frac{dx}{k}.$$
$$\cosh^{-1}\left(\frac{y}{k}\right) = \frac{1}{k}x + c,$$
$$\frac{y}{k} = \cosh\left[\frac{x}{k} + c_1\right]$$
$$y = k\cosh\left[\frac{x}{k} + c_1\right]$$
$$y = k\cosh\left[\frac{x + c_1 k}{k}\right]$$

Hanging Cable problem:

① A Heavy Cable hangs freely under Gravity b/w 2 fixed points. Show that the shape of the Cable is Catenary.

→



Let $p(x_1, y_1)$ & $Q(x_2, y_2)$ be 2 fixed points of the hanging Cable let us consider an elementary arc length $ds$ of the Cable. Let 'P' be the density of the Cable. Since

density = $\frac{\text{Mass}}{\text{unit length}}$

$$P = \frac{\text{mass}}{ds}$$

$\therefore$ Mass = $P ds$.

$g \to$ acceleration due to gravity then the potential energy of the element

P.E = mgh.

P.E = $(P ds) g (y)$

P.E = $P g y \, ds$.

The total potential energy of the Cable

$$I = \int_{x_1}^{x_2} (P g y) \frac{ds}{dx} \, x dx \quad (\div \text{ \& } \times \text{ by } dx)$$

$$\therefore I = \int_{x_1}^{x_2} P g y \sqrt{1+y'^2} \, dx.$$

$$I = P g \int_{x_1}^{x_2} y \sqrt{1+y'^2} \, dx.$$

Now we find extreem Value of I using euler's eqn.

$$f = y \sqrt{1+y'^2}$$

$$f - y' \frac{\partial f}{\partial y'} = k \quad \text{(1)}.$$

$$y\sqrt{1+y'^2} - y' \left[ \frac{y y'}{\sqrt{1+y'^2}} \right] = k.$$

$(\times \sqrt{1+y'^2})$ $\quad y(1+y'^2) - y y'^2 = k\sqrt{1+y'^2}$

$$y'^2 = \frac{y^2 - k^2}{k^2}$$

$$y' = \sqrt{\frac{y^2 - k^2}{k^2}}$$

$$y' = \frac{\sqrt{y^2-k^2}}{k}$$

$$\frac{dy}{\sqrt{y^2-k^2}} = \frac{dx}{k}$$

$$\int \frac{dy}{\sqrt{y^2-k^2}} = \int \frac{dx}{k}.$$

$$\cosh^{-1}\left(\frac{y}{k}\right) = \frac{1}{k}x + c_1$$

$$\frac{y}{k} = \cosh\left[\frac{1}{k}x + c_1\right]$$

$$= k\cosh\left[\frac{x}{k} + c_1\right]$$

$$\boxed{y = k\cosh\left(\frac{x + c_2}{k}\right)} \qquad c_2 = kc_1.$$