Chapter 8
Recurrence Relations

Sequences are generally defined by specifying their general terms. Alternatively, a sequence may be defined by indicating a relation connecting its general term with one or more of the preceding terms. In other words, a sequence $\langle a_r \rangle$ may be defined by indicating a relation connecting its general term a_n with a_{n-1}, a_{n-2}, etc. Such a relation is called a *recurrence relation* for the sequence. The process of determining a_n from a recurrence relation is called "solving" of the relation. A value a_n that satisfies a recurrence relation is called its "general solution". If the values of some particular terms of the sequence are specified, then by making use of these values in the general solution we obtain the "particular solution" that uniquely determines the sequence.

In this chapter, we present some methods of solving some simple recurrence relations.

8.1 First-order Recurrence Relations

First, we consider for solution recurrence relations of the form

$$a_n = ca_{n-1} + f(n), \qquad \text{for } n \geq 1, \tag{1}$$

where c is a known constant and $f(n)$ is a known function. Such a relation is called a *linear recurrence relation of first-order with constant coefficient*. If $f(n) \equiv 0$, the relation is called *homogeneous*; otherwise, it is called *non-homogeneous* (or *inhomogeneous*).

The relation (1) can be solved in a trivial way. First, we note that this relation may be rewritten as (by changing n to $n+1$)

$$a_{n+1} = ca_n + f(n+1), \qquad \text{for } \; n \geq 0. \tag{2}$$

For $n = 0, 1, 2, 3, \ldots$, this relation yields, respectively,

$$\begin{aligned} a_1 &= ca_0 + f(1), \\ a_2 &= ca_1 + f(2) = c\{ca_0 + f(1)\} + f(2), \\ &= c^2 a_0 + cf(1) + f(2), \end{aligned}$$

$$a_3 = ca_2 + f(3) = c\{c^2a_0 + cf(1) + f(2)\} + f(3)$$
$$= c^3a_0 + c^2f(1) + cf(2) + f(3),$$

and so on. Examining these, we obtain, by induction,

$$a_n = c^na_0 + c^{n-1}f(1) + c^{n-2}f(2) + \cdots + cf(n-1) + f(n)$$
$$= c^na_0 + \sum_{k=1}^{n} c^{n-k}f(k), \qquad \text{for } n \geq 1 \tag{3}$$

This is the general solution of the recurrence relation (2) which is equivalent to the relation (1).

If $f(n) \equiv 0$, that is if the recurrence relation is homogeneous, the solution (3) becomes

$$a_n = c^na_0 \qquad \text{for } n \geq 1 \tag{4}$$

The solutions (3) and (4) yield particular solutions if a_0 is specified. The specified value of a_0 is called the *initial condition*.

Example 1 *Solve the recurrence relation $a_{n+1} = 4a_n$ for $n \geq 0$, given that $a_0 = 3$.*

► The given relation is homogeneous. Its general solution is (see expression (4))

$$a_n = 4^na_0 \qquad \text{for } n \geq 1. \tag{i}$$

It is given that $a_0 = 3$. Putting this into (i), we get

$$a_n = 3 \times 4^n \quad \text{for } n \geq 1. \tag{ii}$$

This is the particular solution of the given relation, satisfying the initial condition $a_0 = 3$. ■

Example 2 *Solve the recurrence relation $a_n = 7a_{n-1}$, where $n \geq 1$, given that $a_2 = 98$.*

► The given relation may be rewritten as $a_{n+1} = 7a_n$ for $n \geq 0$. The general solution of this homogeneous relation is

$$a_n = 7^na_0 \qquad \text{for } n \geq 1. \tag{i}$$

It is given that $a_2 = 98$. Using this in (i) we get $98 = a_2 = 7^2a_0$ so that $a_0 = 2$. Putting this into the general solution (i) we get the particular solution

$$a_n = 2 \times 7^n, \qquad \text{for } n \geq 1. \tag{ii}$$

This is the solution of the given relation under the condition $a_2 = 98$. ■

Example 3 *Solve the recurrence relation $a_n = na_{n-1}$ for $n \geq 1$, given that $a_0 = 1$.*

▶ From the given relation, we find that

$$a_1 = 1 \times a_0, \quad a_2 = 2a_1 = (2 \times 1)a_0,$$
$$a_3 = 3 \times a_2 = (3 \times 2 \times 1)a_0,$$
$$a_4 = 4 \times a_3 = (4 \times 3 \times 2 \times 1)a_0, \quad \text{and so on.}$$

Evidently, the general solution is (by induction)

$$a_n = (n!)a_0 \quad \text{for } n \geq 1.$$

Using the given initial condition $a_0 = 1$ in this, we get $a_n = n!$ as the required solution. ■

Example 4 *If a_n is a solution of the recurrence relation $a_{n+1} = ka_n$ for $n \geq 0$, and $a_3 = 153/49$ and $a_5 = 1377/2401$, what is k?*

▶ The general solution of the given relation is $a_n = k^n a_0$, for $n \geq 1$.

From this, we get $a_3 = k^3 a_0$ and $a_5 = k^5 a_0$, so that $a_5/a_3 = k^2$. Using the given values of a_3 and a_5 in this, we get

$$k^2 = \frac{1377}{2401} \cdot \frac{49}{153} = \frac{9}{49}.$$

Therefore, $k = \pm\frac{3}{7}$. ■

Example 5 *Find a_{12} if $a_{n+1}^2 = 5a_n^2$, where $a_n > 0$ for $n \geq 0$, given that $a_0 = 2$.*

▶ Setting $b_n = a_n^2$, the given relation reads $b_{n+1} = 5b_n$, whose general solution is $b_n = 5^n b_0$. Using $b_0 = a_0^2 = 4$ in this, we get $b_n = 4 \times 5^n$. Thus, we have $a_n^2 = 4 \times 5^n$. Since $a_n > 0$ for $n \geq 0$, this yields $a_n = 2(\sqrt{5})^n$ for $n \geq 0$. Therefore,

$$a_{12} = 2(\sqrt{5})^{12} = 2 \times 5^6 = 31,250.$$ ■

Example 6 *Solve the recurrence relation $a_n - 3a_{n-1} = 5 \times 7^n$, for $n \geq 1$, given that $a_0 = 2$.*

▶ The given relation may be rewritten as (by changing n to $n + 1$)

$$a_{n+1} = 3a_n + 5 \times 7^{n+1} \quad \text{for } n \geq 0$$
$$= 3a_n + f(n+1), \quad \text{where } f(n) = 5 \times 7^n.$$

The general solution of this non-homogeneous relation is (see expression (3))

$$a_n = 3^n a_0 + \sum_{k=1}^{n} 3^{n-k} f(k).$$

Substituting for a_0 and $f(k)$ in this we get

$$\begin{aligned}
a_n &= (2 \times 3^n) + \sum_{k=1}^{n} 3^{n-k} \times (5 \times 7^k) \\
&= (2 \times 3^n) + (5 \times 7 \times 3^{n-1}) \times \sum_{k=1}^{n} \left(\frac{7}{3}\right)^{k-1} \\
&= (2 \times 3^n) + (5 \times 7 \times 3^{n-1}) \times \frac{(7/3)^n - 1}{(7/3) - 1} \\
&= (2 \times 3^n) + (5 \times 7 \times 3^{n-1}) \times \frac{3}{4} \times \frac{7^n - 3^n}{3^n} \\
&= (2 \times 3^n) + \frac{1}{4}(5 \times 7)(7^n - 3^n) \\
&= \left(2 - \frac{35}{4}\right)3^n + \frac{5}{4}7^{n+1} = -\frac{27}{4}3^n + \frac{5}{4}7^{n+1} \\
&= \frac{5}{4}7^{n+1} - \frac{1}{4}3^{n+3}
\end{aligned}$$

This is the required solution.

Example 7 *Solve the recurrence relation $a_n - 3a_{n-1} = 5 \times 3^n$ for $n \geq 1$ given that $a_0 = 2$.*

▶ The given relation may be rewritten as

$$\begin{aligned}
a_{n+1} &= 3a_n + 5 \times 3^{n+1} \quad \text{for} \quad n \geq 0 \\
&= 3a_n + f(n+1), \quad \text{where} \quad f(n) = 5 \times 3^n
\end{aligned}$$

The general solution for this relation is

$$\begin{aligned}
a_n &= 3^n a_0 + \sum_{k=1}^{n} 3^{n-k} f(k) \\
&= 3^n a_0 + 3^{n-1} f(1) + 3^{n-2} f(2) + 3^{n-3} f(3) + \cdots + 3^0 f(n)
\end{aligned}$$

Substituting for a_0 and $f(n)$, $n = 1, 2, \ldots n$ in this, we get

$$\begin{aligned}
a_n &= 2 \times 3^n + 3^{n-1} \times (5 \times 3^1) + 3^{n-2} \times (5 \times 3^2) \\
&\quad + 3^{n-3} \times (5 \times 3^3) + \cdots + 3^0 \times (5 \times 3^n) \\
&= 2 \times 3^n + 5 \times (3^n + 3^n + 3^n + \cdots + 3^n) \quad (n \text{ terms}) \\
&= 2 \times 3^n + 5 \times (n3^n) \\
&= (2 + 5n)3^n
\end{aligned}$$

This is the required solution.

Example 8 *Find a recurrence relation and the initial condition for the sequence*

$$2, 10, 50, 250, \ldots$$

Hence find the general term of the sequence.

▶ The given sequence is $\langle a_r \rangle$, where $a_0 = 2$, $a_1 = 10$, $a_2 = 50$, $a_3 = 250, \ldots$
Evidently,

$$a_1 = 5a_0, \quad a_2 = 5a_1, \quad a_3 = 5a_2, \quad \text{and so on.}$$

From these, we readily note that the recurrence relation for the given sequence is $a_n = 5a_{n-1}$ for $n \geq 1$, with $a_0 = 2$ as the initial condition.

The solution of this relation is

$$a_n = 5^n a_0 = 5^n \times 2.$$

This is the general term of the given sequence. ■

Example 9 *Find the recurrence relation and the initial condition for the sequence*

$$0, 2, 6, 12, 20, 30, 42, \ldots$$

Hence find the general term of the sequence.

▶ Let the given sequence be $\langle a_r \rangle$. Then we note that

$$a_0 = 0, \; a_1 = 2, \quad a_1 - a_0 = 2,$$
$$a_2 = 6, \; a_2 - a_1 = 4, \quad a_3 = 12, \; a_3 - a_2 = 6,$$
$$a_4 = 20, \; a_4 - a_3 = 8, \quad a_5 = 30, \; a_5 - a_4 = 10,$$
$$a_6 = 42, \; a_6 - a_5 = 12, \quad \text{and so on.}$$

Evidently

$$a_n - a_{n-1} = 2n, \quad \text{or} \quad a_n = a_{n-1} + 2n, \quad \text{for } n \geq 1.$$

This is the recurrence relation for the given sequence, with $a_0 = 0$ as the initial condition.

From this recurrence relation, we note that (working back wards)

$$a_n - a_{n-1} = 2n$$
$$a_{n-1} - a_{n-2} = 2(n-1)$$
$$a_{n-2} - a_{n-3} = 2(n-2)$$
$$\cdots\cdots$$
$$\cdots\cdots$$
$$a_3 - a_2 = 2 \times 3$$

$$a_2 - a_1 = 2 \times 2$$
$$a_1 - a_0 = 2 \times 1$$

Adding all these, we get

$$\begin{aligned} a_n - a_0 &= 2\{n + (n-1) + (n-2) + \cdots + 3 + 2 + 1\} \\ &= 2 \cdot \frac{n(n+1)}{2} = n(n+1), \end{aligned}$$

or

$$a_n = n(n+1) + a_0 = n(n+1) + 0 = n^2 + n.$$

This is the general term of the given sequence.

Example 10 *Solve the recurrence relation*

$$a_n = 2a_{n/2} + (n-1) \quad \textit{for} \quad n = 2^k, \;\; k \geq 1,$$

given $a_1 = 0$.

▶ From the given recurrence relation, we obtain the following (successive) equations (with $n = 2^k$)

$$\begin{aligned} a_n - 2a_{n/2} &= (n-1) \\ a_{n/2} - 2a_{n/4} &= \left(\frac{n}{2} - 1\right) \\ a_{n/4} - 2a_{n/8} &= \left(\frac{n}{4} - 1\right) \\ &\vdots \\ a_{n/(2^{k-2})} - 2a_{n/(2^{k-1})} &= \left(\frac{n}{2^{k-2}} - 1\right) \\ a_{n/(2^{k-1})} - 2a_{n/(2^k)} &= \left(\frac{n}{2^{k-1}} - 1\right) \end{aligned}$$

These can be rewritten as

$$\begin{aligned} a_n - 2a_{n/2} &= (n-1) \\ 2a_{n/2} - 2^2 a_{n/4} &= (n-2) \\ 2^2 a_{n/4} - 2^3 a_{n/8} &= (n-2^2) \\ &\vdots \\ 2^{k-2} a_{n/(2^{k-2})} - 2^{k-1} a_{n/(2^{k-1})} &= (n - 2^{k-2}) \\ 2^{k-1} a_{n/(2^{k-1})} - 2^k a_{n/(2^k)} &= (n - 2^{k-1}) \end{aligned}$$

Adding all these equations, we obtain

$$a_n - 2^k a_{n/(2^k)} = (n-1) + (n-2) + (n-2^2) + \cdots + (n-2^{k-1})$$

Since $2^k = n$, we have $a_{n/(2^k)} = a_1 = 0$ (given), and the above expression becomes

$$\begin{aligned} a_n &= kn - (1 + 2 + 2^2 + \cdots + 2^{k-1}) \\ &= kn - \frac{(2^k - 1)}{2 - 1} = kn - (n-1) = 1 + (k-1)n \\ &= 1 + (\log_2 n - 1)n \end{aligned}$$

This is the solution of the given recurrence relation. ■

Example 11 *The number of virus affected files in a system is 1000 (to start with) and this increases 250% every two hours. Use a recurrence relation to determine the number of virus affected files in the system after one day.*

▶ In the beginning, the number of virus affected files is 1000. Let us denote this by a_0.

Let a_n denote the number of virus affected files after $2n$ hours. Then the number increases by $a_n \times \frac{250}{100}$ in the next two hours. Thus, after $2n + 2$ hours, the number is

$$\begin{aligned} a_{n+1} &= a_n + a_n \times \frac{250}{100} \\ &= a_n(1 + 2.5) = a_n(3.5). \end{aligned}$$

This is the recurrence relation for the number of virus affected files. Solving this relation, we get

$$a_n = (3.5)^n a_0 = 1000 \times (3.5)^n.$$

This gives the number of virus affected files after $2n$ hours. From this, we get (for $n = 12$)

$$a_{12} = 1000 \times (3.5)^{12} = 3379220508.$$

This is the number of virus affected files after one day (24 hours). ■

Example 12 *A person invests ₹10,000 at 10.5% interest (per year) compounded monthly. Find and solve the recurrence relation for the value of the investment at the end of n months. What is the value of the investment at the end of the first year? How long will it take to double the investment?*

▶ Since the annual interest is 10.5%, the monthly interest comes to $(10.5\%)/12 = 0.875\% = 0.00875$.

Let S_0 denote the investment made namely ₹10,000, and $S_1, S_2, \ldots, S_n$ denote the value of the investment at the end of n months.

Then

$$S_1 = S_0 + (0.00875)S_0 = (1.00875)S_0$$
$$S_2 = S_1 + (0.00875)S_1 = (1.00875)S_1$$
and $$S_n = S_{n-1} + (0.00875)S_{n-1} = (1.00875)S_{n-1}$$

This is the required recurrence relation. Solving this recurrence relation, we find that

$$S_n = (1.00875)^n S_0 = (1.00875)^n \times 10,000.$$

This gives the value at the end of n months.

Therefore, the value at the end of the first year is (in Rupees)

$$S_{12} = (1.00875)^{12} \times 10,000 \approx 11,102.$$

Next, we find that $S_n = 2S_0$ when

$$2S_0 = (1.00875)^n S_0, \quad \text{or} \quad 2 = (1.00875)^n,$$
or $$\log 2 = n \log(1.00875)$$
or $$n = \frac{\log 2}{\log(1.00875)} \approx 79.6.$$

Thus, the investment will be doubled in about 80 months (6 years and 8 months) time. ■

Example 13 *A bank pays a certain % of annual interest on deposits, compounding the interest once in 3 months. If a deposit doubles in 6 years and 6 months, what is the annual % of interest paid by the bank?*

▶ Let the annual rate of interest be $x\%$. Then, the quarterly rate of interest is

$$\left(\frac{x}{4}\right)\% = \left(\frac{x}{400}\right).$$

Let p_0 denote the deposit made (in Rs), and p_n denote the value of the deposit at the end of the n^{th} quarter. Then

$$p_{n+1} = p_n + \left(\frac{x}{400}\right)p_n = \left(1 + \frac{x}{400}\right)p_n \quad \text{for } n \geq 0 \tag{i}$$

This is the recurrence relation for the problem. The general solution of this homogeneous relation is

$$p_n = \left(1 + \frac{x}{400}\right)^n p_0 \quad \text{for } n \geq 1 \tag{ii}$$

From what is given, we have $p_n = 2p_0$ when $n = 26^{\dagger}$. Using this in (ii), we get

$$\left(1 + \frac{x}{400}\right)^{26} = 2$$

or

$$\log_e\left(1 + \frac{x}{400}\right) = \frac{\log_e 2}{26} = 0.02666$$

or

$$1 + \frac{x}{400} = e^{0.02666} = 1.027$$

or

$$x = 400 \times 0.027 = 10.8$$

Thus, the annual rate of interest paid by the bank is 10.8% (compounding the interest once in 3 months). ■

Example 14 *A person takes a loan of ₹ S with A as the interest-rate per month. The loan is to be paid back (with interest) in N months of time. If the person agrees to make a fixed payment of ₹ P at the end of every month until the loan (with interest) is cleared, what should P be?*

▶ Let p_n denote the amount still owed at the end of the n^{th} month (following the n^{th} payment). Then, at the end of the $(n+1)^{st}$ month, the amount still owed is

$$p_{n+1} = (p_n + Ap_n) - P = (1 + A)p_n - P \tag{i}$$

Here, P is the payment the person made at the end of the $(n+1)^{st}$ month.

Thus, (i) is the recurrence relation for p_n. The general solution of this nonhomogeneous relation is

$$p_n = (1 + A)^n p_0 + \sum_{k=1}^{n} (1 + A)^{n-k}(-P)$$

Since $p_0 = S$ (the loan taken), this gives

$$\begin{aligned} p_n &= S(1 + A)^n - P\{(1 + A)^{n-1} + (1 + A)^{n-2} + \cdots (1 + A) + 1\} \\ &= S(1 + A)^n - P \cdot \frac{(1 + A)^n - 1}{(1 + A) - 1} \end{aligned} \tag{ii}$$

† $n = 26$ corresponds to 6 years and 6 months.

At the end of the N^{th} month, the amount owed is Nil. Thus, $p_N = 0$. Using this in (ii), we get

$$0 = S(1+A)^N - \frac{P}{A}\{(1+A)^N - 1\}$$

This yields

$$P = \frac{AS(1+A)^N}{(1+A)^N - 1}.$$

■

Example 15 *Suppose that there are $n \geq 2$ persons at a party and that each of these persons shakes hands (exactly once) with all of the other persons present. Using a recurrence relation, find the number of hand shakes.*

► Let a_{n-2} denote the number of hand shakes among the $n \geq 2$ persons present. (If $n = 2$, the number of handshakes is 1; that is $a_0 = 1$). If a new person joins the party, he will shake hands with each of the n persons already present. Thus, the number of hand shakes increases by n when the number of persons changes to $n + 1$ from n. Thus,

$$a_{(n+1)-2} = a_{n-2} + n \quad \text{for } n \geq 2,$$

$$\text{or} \quad a_{m+1} = a_m + (m+2) \quad \text{for } m \geq 0, \quad \text{where} \quad m = n - 2$$

Setting $f(m) = m + 1$, this reads

$$a_{m+1} = a_m + f(m+1) \quad \text{for } m \geq 0$$

The general solution of this nonhomogeneous recurrence relation is (see expression (3))

$$a_m = (1^m \times a_0) + \sum_{k=1}^{m} 1^{m-k} f(k) = a_0 + \sum_{k=1}^{m} (k+1)$$

Since $a_0 = 1$, this becomes

$$\begin{aligned} a_m &= 1 + \{2 + 3 + 4 + \cdots + m + (m+1)\} \\ &= \frac{1}{2}(m+1)(m+2) \quad \text{for } m \geq 0 \end{aligned}$$

$$\text{or} \quad a_{n-2} = \frac{1}{2}(n-1)n \quad \text{for } n \geq 2.$$

This is the number of handshakes in the party when $n \geq 2$ persons are present.*

■

*This result agrees with the result got in Example 9, Section 4.3.

Example 16 *There are 3 pegs fixed vertically on a table, and n circular disks having holes at their centres and having increasing diameters are slipped onto one of these pegs, with the largest disk at the bottom. The disks are to be transferred, one at a time, onto another peg with the condition that at no time a larger disk is put on a smaller disk. Determine the number of moves for the transfer of all the n disks, so that at the end the disks are in their original order.*[†]

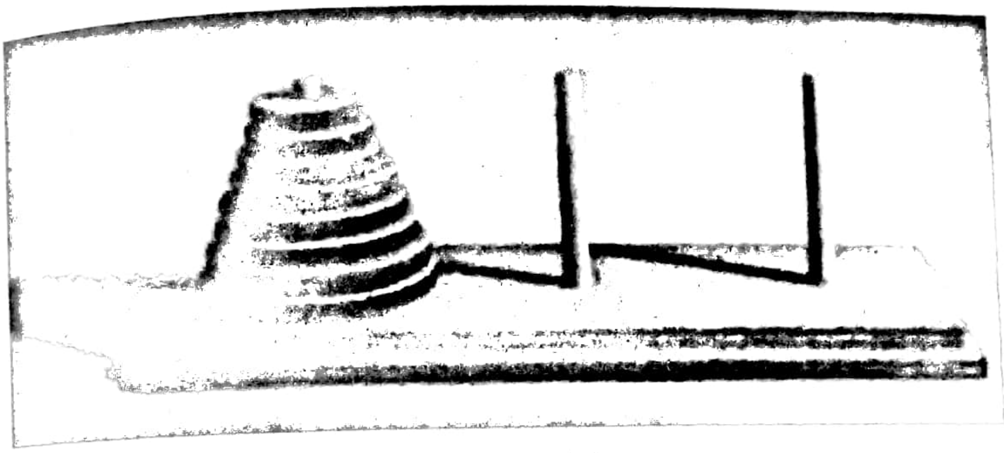

Figure 8.1

▶ Let a_n be the number of moves required to transfer n disks. Evidently, $a_0 = 0$. Let us denote the peg on which the disks are originally located as p_1. To effect the transfer, for $n \geq 1$, we first transfer the top $n-1$ disks to a vacant peg, say p_2, in the prescribed manner. This involves a_{n-1} moves. Then we transfer the n^{th} disk to the other vacant peg, say p_3. This involves 1 move. Lastly, we transfer the $n-1$ disks from peg p_2 to the peg p_3, in the prescribed manner. This involves a_{n-1} moves. Thus, the total number of moves involved in the transfer of n disks is

$$a_n = a_{n-1} + 1 + a_{n-1} = 2a_{n-1} + 1, \quad \text{for } n \geq 1,$$

or, equivalently

$$a_{n+1} = 2a_n + 1, \quad \text{for } n \geq 0.$$

The general solution for this nonhomogeneous recurrence relation is

$$\begin{aligned} a_n &= 2^n a_0 + \sum_{k=1}^{n} 2^{n-k} \cdot 1 = \sum_{k=1}^{n} 2^{n-k}, \quad \text{because } a_0 = 0. \\ &= 2^{n-1} + 2^{n-2} + \cdots + 2^2 + 2 + 1 = \frac{2^n - 1}{2-1} = 2^n - 1. \end{aligned}$$

This is the required number of moves. ■

[†] This problem is known as ***The Towers of Hanoi Problem***. Figure 8.1 corresponds to the case of 8 disks.

Remark: According to the result proved above $a_n = 3$ when $n = 2$. That is, three moves are required when there are two disks. If the disks are denoted by d_1 and d_2, where d_2 is the bigger disk, the three moves are: (i) d_1 from p_1 to p_2, (ii) d_2 from p_1 to p_3, and (iii) d_1 from p_2 to p_3.

For $n = 3$ we have $a_n = 7$, and for $n = 4$ we have $a_n = 15$. That is, 7 moves are required when there are 3 disks and 15 moves are required when there are 4 disks. The reader is urged to identify these moves.

Exercises

1. Solve the following recurrence relations:

   (i) $4a_n - 5a_{n-1} = 0, \quad n \geq 1, \quad a_0 = 1.$

   (ii) $3a_{n+1} - 4a_n = 0, \quad n \geq 0, \quad a_1 = 5.$

   (iii) $2a_n - 3a_{n-1} = 0, \quad n \geq 1, \quad a_4 = 81.$

   (iv) $a_{n+1} - 2a_n = 5, \quad n \geq 0, \quad a_0 = 1.$

   (v) $a_{n+1} = a_n + (2n+3), \quad n \geq 0, \quad a_0 = 1.$

   (vi) $a_n = a_{n-1} + 3n^2 + 3n + 1, \quad n \geq 1, \quad a_0 = 1.$

   (vii) $a_{n+1} - a_n = 3n^2 - n, \quad n \geq 0, \quad a_0 = 3.$

   (viii) $a_n - a_{n-1} = 3n^2, \quad n \geq 1, \quad a_0 = 7.$

   (ix) $a_n = a_{n-1} + n^3 \quad n \geq 1, \quad a_0 = 5.$

   (x) $a_n - a_{n-1} = \dfrac{1}{n(n+1)}, \quad n \geq 1, \quad a_0 = 1.$

   (xi) $a_{n+1} - 2a_n = 2^n, \quad n \geq 0, \quad a_0 = 1.$

2. Find the recurrence relation and the initial condition for each of the following sequences:

   (i) $3, 7, 11, 15, 19, \ldots$ (ii) $8, \dfrac{24}{7}, \dfrac{72}{49}, \dfrac{216}{343}, \ldots$

   (iii) $6, -18, 54, -162, \ldots$ (iv) $7, \dfrac{14}{5}, \dfrac{28}{25}, \dfrac{56}{125}, \ldots$

3. Solve the recurrence relation $a_n^2 - 2a_{n-1} = 0, \quad n \geq 1, \quad a_0 = 2.$ (Hint: Take $b_n = \log_2 a_n$.)

4. A person invests certain amount at 11% interest (per year) compounded annually. Howlong will it take to double the investment?

5. If a person invests ₹ 10,000 at 10% annual interest compounded quarterly, in how many months the money will become ₹ 15000?

6. Fifteen years ago, a person invested some amount at 14% annual interest compounded half-yearly. Now the amount has become ₹ 91,347. What was the amount he invested?

7. If a person invests ₹ 25,000 at 9% annual interest, find the amount he will get at the end of 5 years in each of the following situations:

   (i) The interest is compounded annually.

   (ii) The interest is compounded half-yearly.

   (iii) The interest is compounded quarterly.

   (iv) The interest is compounded monthly.

8. A person wishes to take a loan at 1% interest per month. He agrees to clear the loan with interest in 24 monthly instalments of ₹ 5000 (each). What is the loan amount he is eligible for?

9. On the first day of a new year, a person deposits ₹ S in an account that pays a% annual interest compounded monthly. At the beginning of each month he adds ₹ R to his account. If he continues to do this for the next n years (so that he makes $(12n-1)$ additional deposits of ₹ R) how much will his account be worth exactly n years after he opened it?

10. Use a recurrence relation to derive the formula for $\sum_{k=0}^{n} k^2$.

Answers

1. (i) $a_n = (5/4)^n$ (ii) $a_n = (15/4) \times (4/3)^n$ (iii) $a_n = \dfrac{3^n}{2^{n-4}}$

   (iv) $a_n = (6 \times 2^n) - 5$ (v) $a_n = (n+1)^2$ (vi) $a_n = 7 + \dfrac{1}{2}n(n+1)(2n+1)$

   (vii) $a_n = 3 + n(n-1)^2$ (viii) $a_n = (n+1)^3$ (ix) $a_n = \dfrac{1}{4}n^2(n+1)^2 + 5$

   (x) $a_n = 2 - \dfrac{1}{(n+1)}$ (xi) $a_n = 2^n + (n \times 2^{n-1})$

2. (i) $a_n = a_{n-1} + 4$, $n \geq 1$, and $a_0 = 3$ (ii) $a_n = (3/7)a_{n-1}$, $n \geq 1$ and $a_0 = 8$

   (iii) $a_n = -3a_{n-1}$, $n \geq 1$ and $a_0 = 6$. (iv) $a_n = (2/5)a_{n-1}$, $n \geq 1$ and $a_0 = 7$.

3. $a_n = 2$, $n \geq 0$. **4.** 6.64 years **5.** 51 months **6.** ₹ 12000.

7. (i) ₹ 38,465.60 (ii) ₹ 38, 824.25. (iii) ₹ 39,012.75 (iv) ₹ 39, 142.00

8. ₹ 1,06217. **9.** $(S + R/x)(1+x)^{12n} - (R/x)$, where $x = a/1200$.

8.2 Second-order Homogeneous Recurrence Relations

We now consider a method of solving recurrence relations of the form

$$c_n a_n + c_{n-1} a_{n-1} + c_{n-2} a_{n-2} = 0 \quad \text{for} \quad n \geq 2 \tag{1}$$

where c_n, c_{n-1} and c_{n-2} are real constants with $c_n \neq 0$. A relation of this types is called a *second-order linear homogeneous recurence relation with constant coefficients.*

We seek a solution of relation (1) in the form $a_n = ck^n$ where $c \neq 0$ and $k \neq 0$. Putting $a_n = ck^n$ in (1), we get

$$c_n ck^n + c_{n-1} ck^{n-1} + c_{n-2} ck^{n-2} = 0$$

or

$$c_n k^2 + c_{n-1} k + c_{n-2} = 0 \tag{2}$$

Thus, $a_n = ck^n$ is a solution of (1) if k satisfies the quadratic equation (2). This quadratic equation is called the ***auxiliary equation*** or the ***characteristic equation*** for the relation (1).

Now, the following three cases arise:

Case 1: The two roots k_1 and k_2 of equation (2) are real and distinct. Then we take

$$a_n = Ak_1^n + Bk_2^n, \tag{3}$$

where A and B are arbitrary real constants, as the general solution of the relation (1).

Case 2: The two roots k_1 and k_2 of equation (2) are real and equal, with k as the common value. Then we take

$$a_n = (A + Bn)k^n, \tag{4}$$

where A and B are arbitrary real constants, as the general solution of the relation (1).

Case 3: The two roots k_1 and k_2 of equation (2) are complex. Then k_1 and k_2 are complex conjugates of each other, so that if $k_1 = p + iq$, then $k_2 = p - iq$, and we take

$$a_n = r^n(A \cos n\theta + B \sin n\theta), \tag{5}$$

where A and B are arbitrary complex constants, $r = |k_1| = |k_2| = \sqrt{p^2 + q^2}$, and $\theta = \tan^{-1}(q/p)$, as the general solution of the relation (1).

It is not hard to verify that a_n given by expressions (3) – (5) satisfy the relation (1) in the respective cases. These expressions are called general solutions of the relation (1) in the sense that they contain two arbitrary constants A and B. These constants may be evaluated if a_n is specified for two particular values of n. If a_0 and a_1 are specified, the specified values are called the *initial conditions*.

Example 1 *Solve the recurrence relation*

$$a_n + a_{n-1} - 6a_{n-2} = 0 \quad \textit{for } n \geq 2,$$

given that $a_0 = -1$ *and* $a_1 = 8$.

▶ Here, the coefficients of a_n, a_{n-1} and a_{n-2} are $c_n = 1$, $c_{n-1} = 1$ and $c_{n-2} = -6$, respectively. Therefore, the characteristic equation is*

$$k^2 + k - 6 = 0 \quad \text{or} \quad (k+3)(k-2) = 0.$$

Evidently, the roots of this equation are $k_1 = -3$ and $k_2 = 2$ which are real and distinct. Therefore, the general solution of the given relation is **

$$a_n = A \times (-3)^n + B \times 2^n \tag{i}$$

where A and B are arbitrary constants. From this solution, we get $a_0 = A + B$, $a_1 = -3A + 2B$. Using the given values of a_0 and a_1, these become

$$-1 = A + B, \quad 8 = -3A + 2B.$$

Solving these, we get $A = -2$ and $B = 1$. Putting these into (i), we get

$$a_n = -2 \times (-3)^n + 2^n, \tag{ii}$$

This is the solution of the given relation, under the given initial conditions $a_0 = -1$ and $a_1 = 8$. ■

Example 2 *Solve the recurrence relation*

$$a_n = 3a_{n-1} - 2a_{n-2} \quad \textit{for } n \geq 2,$$

given that $a_1 = 5$ *and* $a_2 = 3$.

▶ Here, the coefficients of a_n, a_{n-1} and a_{n-2} are, respectively, $c_n = 1$, $c_{n-1} = -3$ and $c_{n-2} = 2$. Therefore, the characteristic equation is

$$k^2 - 3k + 2 = 0 \quad \text{or} \quad (k-2)(k-1) = 0$$

whose roots are $k_1 = 2$ and $k_2 = 1$. Therefore, the general solution for a_n is

$$a_n = A \times 2^n + B \times 1^n \tag{i}$$

where A and B are arbitrary constants. Using the given conditions $a_1 = 5$ and $a_2 = 3$ in this, we get

$$5 = 2A + B \quad \text{and} \quad 3 = 4A + B.$$

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*see expression (2) page 338.

**see solution (3) page 338.

Solving these, we get $A = -1$ and $B = 7$. Putting these into (i), we get

$$a_n = -2^n + 7$$

as the solution for the given relation under the given conditions. ∎

Example 3 *Solve the recurrence relation*

$$a_n - 6a_{n-1} + 9a_{n-2} = 0 \quad \text{for } n \geq 2,$$

given that $a_0 = 5$, $a_1 = 12$.

▶ The characteristic equation for the given relation is

$$k^2 - 6k + 9 = 0, \quad \text{or} \quad (k-3)^2 = 0$$

whose roots are $k_1 = k_2 = 3$. Therefore, the general solution for a_n is *

$$a_n = (A + Bn)3^n. \tag{i}$$

where A and B are arbitrary constants.

Using the given initial conditions $a_0 = 5$ and $a_1 = 12$ in (i), we get $5 = A$ and $12 = 3(A + B)$. Solving these we get $A = 5$ and $B = -1$. Putting these values in (i), we get

$$a_n = (5 - n)3^n. \tag{ii}$$

This is the solution of the given relation, under the given initial conditions. ∎

Example 4 *Solve the recurrence relation*

$$a_n = 2(a_{n-1} - a_{n-2}), \quad \text{for } n \geq 2,$$

given that $a_0 = 1$ *and* $a_1 = 2$.

▶ For the given relation, the characteristic equation is $k^2 - 2k + 2 = 0$ whose roots are

$$k = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$$

Therefore, the general solution for a_n is **

$$a_n = r^n[A\cos n\theta + B\sin n\theta], \tag{i}$$

where A and B are arbitrary constants, $r = |(1 \pm i)| = \sqrt{2}$, and $\tan\theta = 1/1 = 1$ which yields $\theta = \pi/4$. Thus,

$$a_n = (\sqrt{2})^n \left[A\cos\frac{n\pi}{4} + B\sin\frac{n\pi}{4}\right]. \tag{ii}$$

*See solution (4), page 338

**See solution (5) page 338

Using the given initial conditions $a_0 = 1$ and $a_1 = 2$ in this, we get

$$1 = A \quad \text{and} \quad 2 = \left(\sqrt{2}\right)\left[A\cos\frac{\pi}{4} + B\sin\frac{\pi}{4}\right] = A + B,$$

which yield $A = 1$ and $B = 1$. Putting these values of A and B in (ii) we get

$$a_n = \left(\sqrt{2}\right)^n\left[\cos\frac{n\pi}{4} + \sin\frac{n\pi}{4}\right]. \tag{iii}$$

This is the solution of the given relation under the given conditions. ■

Example 5 *Solve the recurrence relation*

$$D_n = bD_{n-1} - b^2 D_{n-2} \quad \textit{for } n \geq 3,$$

given $D_1 = b > 0$ and $D_2 = 0$.

▶ For the given relation, the characteristic equation is $k^2 - bk + b^2 = 0$, whose roots are

$$k = \frac{b \pm \sqrt{b^2 - 4b^2}}{2} = \frac{b}{2}(1 \pm i\sqrt{3})$$

Therefore, the general solution for D_n is

$$D_n = r^n[A\cos n\theta + B\sin n\theta], \tag{i}$$

where A and B are arbitrary constants, and

$$r = \left|\frac{b}{2}(1 \pm i\sqrt{3})\right| = \frac{b}{2}(\sqrt{1^2 + 3}) = b \quad \text{and} \quad \tan\theta = \frac{\sqrt{3}}{1} = \sqrt{3},$$

so that $\theta = \pi/3$. Thus, we have

$$D_n = b^n\left[A\cos\frac{n\pi}{3} + B\sin\frac{n\pi}{3}\right], \tag{ii}$$

Using the given conditions $D_1 = b > 0$, and $D_2 = 0$ in this, we get

$$b = b\left[A\cos\frac{\pi}{3} + B\sin\frac{\pi}{3}\right] \quad \text{and} \quad 0 = b^2\left[A\cos\frac{2\pi}{3} + B\sin\frac{2\pi}{3}\right]$$

which can be rewritten as

$$1 = \frac{1}{2}A + \frac{\sqrt{3}}{2}B \quad \text{and} \quad 0 = -\frac{1}{2}A + \frac{\sqrt{3}}{2}B.$$

Solving these, we get $A = 1$ and $B = 1/\sqrt{3}$. Putting these values of A and B into (ii), we get

$$D_n = b^n\left[\cos\frac{n\pi}{3} + \frac{1}{\sqrt{3}}\sin\frac{n\pi}{3}\right]. \tag{iii}$$

This is the solution of the given relation under the given conditions. ■

Example 6 *If $a_0 = 0$, $a_1 = 1$, $a_2 = 4$ and $a_3 = 37$ satisfy the recurrence relation*

$$a_{n+2} + ba_{n+1} + ca_n = 0 \quad \text{for } n \geq 0,$$

determine the constants b and c and then solve the relation for a_n.

▶ For $n = 0$ and $n = 1$, the given relation reads (respectively)

$$a_2 + ba_1 + ca_0 = 0 \quad \text{and} \quad a_3 + ba_2 + ca_1 = 0.$$

Substituting the given values of a_0, a_1, a_2 and a_3 in this, we get

$$4 + b + 0 = 0 \quad \text{and} \quad 37 + 4b + c = 0.$$

These yield $b = -4$ and $c = -21$.

With these values of b and c, the given recurrence relation reads

$$a_{n+2} - 4a_{n+1} - 21a_n = 0 \quad \text{for } n \geq 0,$$

or, equivalently,

$$a_n - 4a_{n-1} - 21a_{n-2} = 0 \quad \text{for } n \geq 2 \tag{i}$$

The characteristic equation for this relation is $k^2 - 4k - 21 = 0$ whose roots are $k_1 = 7$ and $k_2 = -3$. Therefore, the general solutions for a_n is

$$a_n = A \times 7^n + B \times (-3)^n \tag{ii}$$

where A and B are arbitrary constants.

Using the given conditions $a_0 = 0$ and $a_1 = 1$ in this, we get

$$0 = A + B \quad \text{and} \quad 1 = 7A - 3B$$

which yield $A = -B = \frac{1}{10}$. Putting these values into (ii), we get

$$a_n = \frac{1}{10}\{7^n - (-3)^n\}. \tag{iii}$$

This is the required solution. ∎

Example 7 *Solve the recurrence relation*

$$a_{n+2}^2 - 5a_{n+1}^2 + 4a_n^2 = 0 \quad \text{for } n \geq 0,$$

given $a_0 = 4$ and $a_1 = 13$.

▶ Let $b_n = a_n^2$. Then the given relation reads

$$b_{n+2} - 5b_{n+1} + 4b_n = 0, \quad n \geq 0,$$

or, equivalently,

$$b_n - 5b_{n-1} + 4b_{n-2} = 0, \quad n \geq 2.$$

The characteristic equation for this relation is $k^2 - 5k + 4 = 0$ whose roots are $k_1 = 4$ and $k_2 = 1$. Therefore, the general solution for b_n is

$$b_n = A \times 4^n + B \times 1^n \tag{i}$$

where A and B are arbitrary constants.

It is given that $a_0 = 4$ and $a_1 = 13$. These yield $b_0 = a_0^2 = 4^2 = 16$ and $b_1 = a_1^2 = 13^2 = 169$. Using these in (i), we get

$$16 = A + B \quad \text{and} \quad 169 = 4A + B.$$

Solving these, we get $A = 51$ and $B = -35$. Putting these into (i), we get

$$b_n = 51 \times 4^n - 35 \tag{ii}$$

From this, we get

$$a_n = \pm\sqrt{(51 \times 4^n - 35)}. \tag{iii}$$

This is the required solution for a_n. ■

Example 8 *Solve the recurrence relation**

$$F_{n+2} = F_{n+1} + F_n \quad \textit{for } n \geq 0,^* \quad \textit{given} \quad F_0 = 0, F_1 = 1.$$

▶ When rewritten, the given relation reads

$$F_n - F_{n-1} - F_{n-2} = 0 \quad \text{for } n \geq 2.$$

The characteristic equation for this is $k^2 - k - 1 = 0$ whose roots are

$$k = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1}{2}(1 \pm \sqrt{5}).$$

Accordingly, the general solution for F_n is

$$F_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n \tag{i}$$

*The recurrence relation considered here is known as the ***Fibonacci relation***. The sequence $\{F_r\}$ is known as the *Fibonacci sequence*. The terms $F_0, F_1, F_2, F_3, \ldots$ of this sequence are known as *Fibonacci numbers*. The result (ii) agrees with that proved in Example 7, Section 3.2.

where A and B are arbitrary constants. Using the given initial conditions $F_0 = 0$ and $F_1 = 1$ in this, we get

$$0 = A\left(\frac{1+\sqrt{5}}{2}\right)^0 + B\left(\frac{1-\sqrt{5}}{2}\right)^0 = A + B$$

$$1 = A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right)$$

Solving these, we get $A = -B = \frac{1}{\sqrt{5}}$. Putting these values of A and B into (i), we get

$$F_n = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\}$$

as the solution for F_n with $F_0 = 0$ and $F_1 = 1$. ∎

Example 9 *Solve the recurrence relation*

$$a_{n+2} = (a_{n+1})(a_n), \quad \textit{for} \quad n \geq 0$$

with $a_0 = 1$, $a_1 = 2$.

► Taking log on both sides, the given relation becomes

$$\log a_{n+2} = \log a_{n+1} + \log a_n, \quad n \geq 0.$$

Setting $b_n = \log a_n$, this becomes

$$b_{n+2} = b_{n+1} + b_n, \quad n \geq 0.$$

This is identical with the Fibonacci recurrence relation. Therefore, $b_n = F_n$ if the initial conditions of the Fibonacci numbers hold for b_n also; that is, if

$$0 = F_0 = b_0 = \log a_0 \quad \text{and} \quad 1 = F_1 = b_1 = \log a_1.$$

Since it is given that $a_0 = 1$ and $a_1 = 2$, these read $0 = \log 1$ and $1 = \log 2$. These conditions hold if the base of the logarithm is 2. Thus, we have

$$b_n = F_n \quad \text{if} \quad b_n = \log_2 a_n, \quad n \geq 0.$$

Consequently, the solution for a_n is

$$a_n = 2^{b_n} = 2^{F_n}.$$

∎

Example 10 *Find and solve a recurrence relation for the number of binary sequences of length $n \geq 1$ that have no consecutive 0's.*

► Let a_n denote the number of binary sequences of length $n \geq 1$, of the required type.

For $n = 1$, there exist two such sequences – one sequence consisting of one 0 and the other consisting of one 1. For $n = 2$, there exist three such sequences: 01, 11, 10. Thus, $a_1 = 2$ and $a_2 = 3$.

In a sequence of the desired type of length $n \geq 3$, the last entry may be 1 or 0. If the last entry is 1, then the preceding $n - 1$ entries form a sequence of the desired type of length $n - 1$; their number is a_{n-1}. If the last entry is 0, the preceding entry must be 1. The entries preceding this 1 form a sequence of length $(n - 2)$ of the desired type; their number is a_{n-2}.

All binary sequences of a given length must end with 1 or 0. Therefore, their number must be equal to the number of sequences which end with 1 plus the number of sequences which end with 0. Applying this result to the sequences that are being considered, we get

$$a_n = a_{n-1} + a_{n-2} \quad \text{for} \quad n \geq 3. \tag{i}$$

This is the required recurrence relation.

Therefore, (as in Example 8)

$$a_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n. \tag{ii}$$

Using the above-made observations that $a_1 = 2$ and $a_2 = 3$ in this, we get

$$2 = A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right) \quad \text{and} \quad 3 = A\left(\frac{1+\sqrt{5}}{2}\right)^2 + B\left(\frac{1-\sqrt{5}}{2}\right)^2.$$

These simplify to

$$4 = (A + B) + \sqrt{5}(A - B) \quad \text{and} \quad 6 = 3(A + B) + \sqrt{5}(A - B).$$

Solving these, we get $A + B = 1$ and $A - B = (3/\sqrt{5})$ which in turn yield

$$A = \frac{\sqrt{5}+3}{2\sqrt{5}}, \qquad B = \frac{\sqrt{5}-3}{2\sqrt{5}}$$

Substituting these into (ii), we get

$$a_n = \frac{1}{2\sqrt{5}}\left\{(\sqrt{5}+3)\left(\frac{1+\sqrt{5}}{2}\right)^n + (\sqrt{5}-3)\left(\frac{1-\sqrt{5}}{2}\right)^n\right\}. \tag{iii}$$

■

This gives the number of binary sequences of the desired type.

Example 11 *Consider the set $S_n = \{1, 2, 3, \ldots, n\}$ for $n \geq 1$. Let $S_0 = \Phi$ (null set). If a_n (for $n \geq 0$) is the number of subsets of S_n that do not contain consecutive integers, find a recurrence relation for a_n and hence determine a_n.*

► We recall that the set $S_0 = \Phi$ has only one subset, namely Φ itself. This subset does not contain consecutive integers. Therefore, $a_0 = 1$.

We have $S_1 = \{1\}$ which has two subsets (namely Φ and $\{1\}$). These subsets do not contain consecutive integers. Therefore $a_1 = 2$.

For $n \geq 2$, we have $S_n = \{1, 2, \ldots, n-1, n\}$. Let A be any subset of S_n. Then there are only the following two mutually exclusive possibilities:

(1) $n \in A$. Then $n - 1 \notin A$ and the subset $A - \{n\}$ of S_n would be counted in a_{n-2}.

(2) $n \notin A$. Then A would be counted in a_{n-1}.

Thus, we should have

$$a_n = a_{n-1} + a_{n-2} \quad \text{for} \quad n \geq 2, \quad \text{with} \quad a_0 = 1 \quad \text{and} \quad a_1 = 2$$

This is the recurrence relation for the problem.

Solving this recurrence relation, we obtain (as in Example 8).

$$a_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Using the initial conditions $a_0 = 1$ and $a_1 = 2$ in this, we get

$$1 = A + B, \quad 2 = A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right)$$

These give $A = \frac{1}{2} + \frac{3}{2\sqrt{5}}$; $B = \frac{1}{2} - \frac{3}{2\sqrt{5}}$.

Thus,

$$a_n = \frac{1}{2\sqrt{5}}\left\{(\sqrt{5}+3)\left(\frac{1+\sqrt{5}}{2}\right)^n + (\sqrt{5}-3)\left(\frac{1-\sqrt{5}}{2}\right)^n\right\}$$

This is the required a_n. ∎

Example 12 *Let a_n denote the number of n-letter sequences that can be formed using the letters A, B and C such that any nonterminal A has to be immediately followed by a B. Find the recurrence relation for a_n and solve it.*

▶ For $n = 1$, the possible sequences are the singleton sequences. These are 3 in number. Thus, $a_1 = 3$.

For $n = 2$, the possible sequences are: AB, BA, BB, BC, CA, CB, CC. Thus, $a_2 = 7$.

Let $n \geq 3$, and consider a sequence of the desired type. The first letter of this sequence can be A or cannot be A. In the former case, the second letter has to be B; consequently the first two positions are fixed, and the letters in the remaining positions correspond to $(n - 2)$ letter sequences of the desired type. Their number is a_{n-2}. In the latter case, there are two choices for the first place, and for each choice the letters in the remaining positions correspond to $(n - 1)$ letter sequences. Their number is $2 \times a_{n-1}$.

Thus,

$$a_n = a_{n-2} + 2a_{n-1}. \tag{i}$$

This is the recurrence relation for a_n.

The characteristic equation for the relation (i) is $k^2 - 2k - 1 = 0$ whose roots are

$$k = \frac{2 \pm \sqrt{4+4}}{2} = 1 \pm \sqrt{2}.$$

Therefore, the general solution for a_n is

$$a_n = A\left(1 + \sqrt{2}\right)^n + B\left(1 - \sqrt{2}\right)^n \tag{ii}$$

where A and B are arbitrary constants .

Using the fact that $a_1 = 3$ and $a_2 = 7$ in this, we get

$$3 = A\left(1 + \sqrt{2}\right) + B\left(1 - \sqrt{2}\right) \quad \text{and} \quad 7 = A\left(1 + \sqrt{2}\right)^2 + B\left(1 - \sqrt{2}\right)^2.$$

These are equivalent to

$$3 = (A + B) + \sqrt{2}(A - B) \quad \text{and} \quad 7 = 3(A + B) + 2\sqrt{2}(A - B)$$

Solving these, we obtain $A + B = 1$ and $A - B = \sqrt{2}$, so that $A = \left(1 + \sqrt{2}\right)/2$ and $B = \left(1 - \sqrt{2}\right)/2$. Putting these into (ii), we get

$$a_n = \frac{1}{2}\left[\left(1 + \sqrt{2}\right)^{n+1} + \left(1 - \sqrt{2}\right)^{n+1}\right] \tag{iii}$$

■

This is the solution for a_n, with $a_1 = 3$ and $a_2 = 7$.

Exercises

1. Solve the following recurrence relations:

(i) $a_{n+2} = 4(a_{n+1} - a_n)$, $n \geq 0$, $a_0 = 1$, $a_1 = 3$.

(ii) $a_n - 4a_{n-1} + 4a_{n-2} = 0$, $n \geq 2$, $a_0 = 5/2$, $a_1 = 8$.

(iii) $a_n = 5a_{n-1} + 6a_{n-2}$, $n \geq 2$, $a_0 = 1$, $a_1 = 3$.

(iv) $2a_n = 7a_{n-1} - 3a_{n-2}$, $n \geq 2$, $a_0 = 2$, $a_1 = 5$

(v) $2a_{n+2} - 11a_{n+1} + 5a_n = 0$, $n \geq 0$, $a_0 = 2$, $a_1 = -8$.

(vi) $a_{n+2} + a_n = 0$, $n \geq 0$, $a_0 = 0$, $a_1 = 3$.

(vii) $a_n = 4a_{n-1} - 4a_{n-2}$, $n \geq 3$, $a_1 = 1$, $a_2 = 3$

(viii) $a_n = 10a_{n-1} + 29a_{n-2}$, $n \geq 3$, $a_1 = 10$, $a_2 = 100$.

(ix) $a_n + 7a_{n-1} + 8a_{n-2} = 0$, $n \geq 2$, $a_0 = 2$, $a_1 = -7$.

(x) $a_n + 5a_{n-1} + 5a_{n-2} = 0$, $n \geq 2$, $a_0 = 0$, $a_1 = 2\sqrt{5}$.

2. Determine the constants b and c if $a_n = A + B \times 7^n$, $n \geq 0$, is the general solution of the relation $a_{n+2} + ba_{n+1} + ca_n = 0$, $n \geq 0$.

3. The Lucas numbers $L_0, L_1, L_2, \ldots$ are defined through the recurrence relation $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$ and $L_0 = 2$, $L_1 = 1$. Find L_n for $n \geq 0$.

4. A particle moves horizontally to the right. The distance the particle travels in the $(n+1)^{\text{st}}$ second is equal to twice the distance it has traveled in the n^{th} second. If a_n denotes the position of the particle at the start of the $(n+1)^{\text{st}}$ second, show, by using a recurrence relation, that $a_n = (4 \times 2^n) - 3$, $n \geq 0$.

5. For $n \geq 0$, let a_n count the number of ways a sequence of 1's and 2's will sum to n. By finding a recurrence relation for a_n, show that

$$a_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right].$$

Answers

1. (i) $a_n = 2^n + n\left(2^{n-1}\right)$

(ii) $a_n = (5/2)2^n + (3/2)n2^n$

(iii) $a_n = (3/7)(-1)^n + (4/7)6^n$

(iv) $a_n = (1/5)(8 \times 3^n + 2^{1-n})$

(v) $a_n = 4(1/2)^n - 2 \times 5^n$

(vi) $a_n = 3\sin(n\pi/2)$

(vii) $a_n = (n+1)2^{n-2}$

(viii) $a_n = \dfrac{5}{3\sqrt{6}}\left\{\left(5+3\sqrt{6}\right)^n - \left(5-3\sqrt{6}\right)^n\right\}$

(ix) $a_n = \left(\dfrac{-7+\sqrt{17}}{2}\right)^n + \left(\dfrac{-7-\sqrt{17}}{2}\right)^n$

(x) $a_n = 2\left(\dfrac{-5+\sqrt{5}}{2}\right)^n - 2\left(\dfrac{-5-\sqrt{5}}{2}\right)^n$

2. $b = -8, c = 7$

3. $L_n = \left(\dfrac{1+\sqrt{5}}{2}\right)^n + \left(\dfrac{1-\sqrt{5}}{2}\right)^n$