FOURIER SERIES

2.0 Convergence & Divergence of infinite series of positive terms (Recap)

If u_n is a function of n defined for all integral values of n, an expression of the form $u_1 + u_2 + u_3 + \cdots + u_n + \cdots$ containing infinite number of terms is called an ***infinite series*** usually denoted by $\sum_{n=1}^{\infty} u_n$ or simply $\sum u_n$.

u_n is called the n^{th} term or the general term of the infinite series. The sum of the first n terms of the series is denoted by s_n. That is,

$$s_n = u_1 + u_2 + u_3 + \cdots + u_n$$

A series $\sum u_n$ is said to be ***convergent*** if $\lim_{n\to\infty} s_n = l$, where l is a finite quantity and $\sum u_n$ is said to be ***divergent*** if $\lim_{n\to\infty} s_n = \pm\infty$.

Illustrative Examples

Example - 1

Let us consider the geometric series : $a + ar + ar^2 + \cdots$

$$s_n = \frac{a(1-r^n)}{1-r} \text{ if } r < 1 \qquad \ldots (1)$$

and $$s_n = \frac{a(r^n-1)}{r-1} \text{ if } r > 1 \qquad \ldots (2)$$

Now, if $|r|<1, r^n \to 0$ as $n \to \infty$ and from (1) we have,

$$\lim_{n\to\infty} s_n = \frac{a}{1-r}(1-0) = \frac{a}{1-r} \text{ which is a finite quantity.}$$

Hence we conclude that the geometric series is convergent for $|r| < 1$.

Next if $r > 1$, we have from (2),

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a(r^n-1)}{r-1} = \infty, \text{ since } r^n \to \infty \text{ when } r > 1.$$

Hence we conclude that the geometric series is divergent for $r > 1$.

Example - 2

Let us consider the series : $1+2+3+\cdots+n+\cdots$

$$s_n = 1+2+3+\cdots+n = \sum n = \frac{n(n+1)}{2}$$

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{n(n+1)}{2} = \infty$$

Hence we conclude that the series is divergent.

Example - 3

Let us consider the series : $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots$

The n^{th} term $u_n = \frac{1}{n(n+1)}$

Further, $u_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$ by partial fractions.

Now, $s_n = u_1 + u_2 + u_3 + \cdots + u_n$

ie., $s_n = \left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)$

or $s_n = 1 - \frac{1}{n+1}$

$\therefore \quad \lim_{n\to\infty} s_n = \lim_{n\to\infty}\left[1-\frac{1}{n+1}\right] = 1-0 = 1$, a finite qty.

Hence we conclude that the series is convergent.

2.1 Periodic functions

A real valued function $f(x)$ is said to be ***periodic of period T*** if $f(x+T) = f(x), T > 0$.

k (constant), $\sin x$, $\cos x$ are periodic functions of period 2π as we know from trigonometry that

$$\sin(x+2\pi) = \sin x, \quad \cos(x+2\pi) = \cos x$$

Also if $f(x) = k$ then $f(x+2\pi) = k$

Further we also have a **property** stating that *A linear combination of periodic functions having period T is also periodic of period T.*

2.2 Trigonometric series and Euler's formulae

The function k, $\cos nx$, $\sin nx$ ($n = 1, 2, 3, \cdots$) are all periodic functions of period 2π. Taking the constant $k = a_0/2$, the linear combination of all the periodic functions is of the form :

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where, a_0, a_n, b_n $(n = 1, 2, 3, \cdots)$ are all constants is called a **Trigonometric series.** Hence any function $f(x)$ expressible as trigonometric series of the above form must also be periodic with period 2π.
We shall assume that $f(x)$ is defined in an interval of length 2π, say $(c, c + 2\pi)$ and be considered as period with period 2π. Then we have,

$$a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\,dx, \quad a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx, \quad b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx$$

The expressions for finding a_0, a_n, b_n are called *Euler's formulae.* These can be established with the help of the following basic formulae.

1. $\displaystyle\int_c^{c+2\pi} \cos nx\,dx = 0 = \int_c^{c+2\pi} \sin nx\,dx$, where n is a positive integer.

2. $\displaystyle\int_c^{c+2\pi} \cos mx \cos nx\,dx = 0 = \int_c^{c+2\pi} \sin mx \sin nx\,dx$

   where m and n are positive integers, $m \neq n$

3. $\displaystyle\int_c^{c+2\pi} \sin mx \cos nx\,dx = 0$, where m and n are positive integers.

4. $\displaystyle\int_c^{c+2\pi} \cos^2 nx\,dx = \pi = \int_c^{c+2\pi} \sin^2 nx\,dx$, where n is a positive integer.

Proof of Euler's formulae

Let, $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \cdots (1)$$

Integrating (1) w.r.t x from c to $c+2\pi$,

$$\int_c^{c+2\pi} f(x)\,dx = \int_c^{c+2\pi} \frac{a_0}{2}\,dx + \sum_{n=1}^{\infty} a_n \int_c^{c+2\pi} \cos nx\,dx + \sum_{n=1}^{\infty} b_n \int_c^{c+2\pi} \sin nx\,dx$$

$$= \frac{a_0}{2}[x]_c^{c+2\pi} + 0 + 0, \text{ by using (1).}$$

ie.,
$$\int_c^{c+2\pi} f(x)\,dx = \frac{a_0}{2}[c+2\pi-c] = a_0\,\pi$$

$$\therefore \quad a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\,dx \qquad \ldots (2)$$

Next, taking the expanded form of (1), multiplying by $\cos nx$ and integrating w.r.t x from c to $c+2\pi$ we have,

$$\int_c^{c+2\pi} f(x)\cos nx\,dx = \frac{a_0}{2}\int_c^{c+2\pi} \cos nx\,dx + a_1\int_c^{c+2\pi} \cos nx \cdot \cos x\,dx$$

$$+ a_2\int_c^{c+2\pi} \cos nx \cos 2x\,dx + \cdots + a_n\int_c^{c+2\pi} \cos^2 nx\,dx$$

$$+ b_1\int_c^{c+2\pi} \sin x \cos nx\,dx + b_2\int_c^{c+2\pi} \sin 2x \cos nx\,dx + \cdots$$

Using the basic results appropriately onto the RHS of the above we have,

$$\int_c^{c+2\pi} f(x)\cos nx\,dx = 0 + 0 + \cdots + a_n\pi + 0 + 0 + \cdots = a_n\,\pi$$

$$\therefore \quad a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx \qquad \ldots (3)$$

Similarly, multiplying the expanded form of (1) by $\sin nx$ and proceeding on the same lines we obtain,

$$b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx \qquad \ldots (4)$$

Thus we have established ***Euler's formulae.***

Remark : *The constant term in the series* (1) is taken as $a_0/2$ *so as to make the formula derived for* a_n *valid for the particular case* $n = 0$ *as well as for any positive integer* n.

2.3 Dirichlet's Conditions and Fourier Series of period 2π

Suppose we form the trigonometric series from $f(x)$ defined in $(c, c + 2\pi)$ with the help of ***Euler's formulae,*** we cannot conclude that the series will converge to $f(x)$. We can only say that, when $f(x)$ is of the form (1) the coefficients of the terms in the series are given by the formulae (2), (3), (4). We now proceed to state the conditions known as ***Dirichlet's conditions*** under which the expansion of $f(x)$ as a trigonometric series will converge to $f(x)$ at every point of continuity.

1. $f(x)$ is single valued and finite in the interval $(c, c + 2\pi)$
2. $f(x)$ is periodic with period 2π
3. $f(x)$ has only a finite number of discontinuities in $(c, c + 2\pi)$
4. $f(x)$ has at the most a finite number of maxima and minima in $(c, c + 2\pi)$

Thus we can say that, if $f(x)$ is defined in $(c, c + 2\pi)$ and satisfies Dirichlet's conditions, then the trigonometric series (1) is called as the ***Fourier Series*** of $f(x)$ in $(c, c + 2\pi)$. The constants a_0, a_n, b_n as given by (2), (3), (4) respectively are called ***Fourier coefficients.***

Remark : ***If*** $f(x)$ ***is discontinuous at*** x, *then the series* (1) *converges to* $\frac{1}{2}[f(x^+) + f(x^-)]$ *where* $f(x^+), f(x^-)$ *are respectively right hand and left hand limits of* $f(x)$ *given by*

$$f(x^+) = \underset{h\to 0}{lt} f(x+h), \; f(x^-) = \underset{h\to 0}{lt} f(x-h), \; h>0$$

However at the end points, $f(x)$ *converges to* $\frac{1}{2}[f(c) + f(c+2\pi)]$

Note : ***Bernoulli's generalized rule of integration by parts***

While finding the Fourier coefficients, in most of the problems we have to perform integration of a product with the first function as a polynomial in x. In such cases Bernoulli's rule as given below will be highly helpful.

$$\int u\,v\,dx = u\int v\,dx - u'\iint v\,dx\,dx + u''\iiint v\,dx\,dx\,dx - \cdots$$

Here are a few illustrative examples.

1. $\int x e^{3x}\,dx = x\left(\dfrac{e^{3x}}{3}\right) - (1)\left(\dfrac{e^{3x}}{9}\right)$

2. $\int (x + x^2)\cos nx\,dx$

$$= (x + x^2)\left(\frac{\sin nx}{n}\right) - (1 + 2x)\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)$$

The following integrals will be useful in problems

1. $\int e^{ax}\cos bx\,dx = \dfrac{e^{ax}}{a^2 + b^2}(a\cos bx + b\sin bx)$

2. $\int e^{ax}\sin bx\,dx = \dfrac{e^{ax}}{a^2 + b^2}(a\sin bx - b\cos bx)$

The following values will have frequent reference in problems when n is an integer.

1. $\sin n\pi = 0$ 2. $\cos n\pi = (-1)^n$

In particular, $\cos(2n)\pi = 1$, $\cos(2n+1)\pi = -1$.

That is to say that, $\cos n\pi = +1$ *when* n *is even and is equal to* -1 *when* n *is odd.*

Remark : *It is also possible to deduce a particular series from the Fourier series of the given* $f(x)$ *in a given interval. We have to substitute a suitable value for* x *in the given interval. Noramlly we take the values to be either of the end points or the middle point. The resulting series will be equal to the value as discussed in convergence.*

WORKED PROBLEMS

[1] *Obtain the Fourier series of* $f(x) = \dfrac{\pi - x}{2}$ *in* $0 < x < 2\pi$. *Hence deduce that*

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\cdots = \frac{\pi}{4}$$ **[Dec. 2016]**

☞ The Fourier series of $f(x)$ having period 2π is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \ldots (1)$$

where, $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx,\ a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx,\ b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$

Now, $a_0 = \frac{1}{\pi}\int_0^{2\pi} \frac{\pi - x}{2}\,dx = \frac{1}{2\pi}\left[\pi x - \frac{x^2}{2}\right]_0^{2\pi} = \frac{1}{2\pi}\{(2\pi^2 - 2\pi^2) - 0\} = 0$

$a_0 = 0$

$a_n = \frac{1}{\pi}\int_0^{2\pi} \frac{\pi - x}{2}\cos nx\,dx.$ Applying Bernoulli's rule,

$$a_n = \frac{1}{2\pi}\left[(\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(\frac{-\cos nx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{-1}{2\pi n^2}[\cos nx]_0^{2\pi}, \quad \text{since, } \sin 2n\pi = 0 = \sin 0$$

$$= \frac{-1}{2\pi n^2}[\cos 2n\pi - \cos 0] = 0, \quad \text{since, } \cos 2n\pi = 1 = \cos 0$$

$a_n = 0$

$b_n = \frac{1}{\pi}\int_0^{2\pi} \frac{\pi - x}{2}\sin nx\,dx.$ Again by Bernoulli's rule,

$$b_n = \frac{1}{2\pi}\left[(\pi - x)\left(\frac{-\cos nx}{n}\right) - (-1)\left(\frac{-\sin nx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{-1}{2\pi n}[(\pi - x)\cos nx]_0^{2\pi} + 0$$

$$= \frac{-1}{2\pi n}(-\pi\cos 2n\pi - \pi\cos 0) = \frac{-1}{2\pi n}(-\pi - \pi) = \frac{1}{n}$$

$b_n = 1/n$

Thus by substituting the values of a_0, a_n, b_n in (1), the Fourier series is given by,

$$\boxed{f(x) = \frac{\pi - x}{2} = \sum_{n=1}^{\infty} \frac{1}{n}\sin nx}$$

To deduce the required series we put $x = \pi/2$ in the Fourier series of $f(x)$.

[*Note that at* $x = 0$ *or* 2π, $x = \pi$ *RHS of the Fourier series becomes zero and hence we try* $x = \pi/2 \in (0, 2\pi)$]

$$\therefore \quad f(\pi/2) = \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi}{2}\right)$$

ie.,

$$\frac{\pi - (\pi/2)}{2} = \frac{\sin(\pi/2)}{1} + \frac{\sin \pi}{2} + \frac{\sin(3\pi/2)}{3} + \frac{\sin 2\pi}{4} + \frac{\sin(5\pi/2)}{5} + \cdots$$

Thus,

$$\boxed{\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots}$$

[2] *Obtain the Fourier series for the function* x^2 *in* $-\pi \le x \le \pi$ *and hence deduce that*

(i) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$

(ii) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$

(iii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}$

☛ The Fourier series of $f(x)$ having period 2π is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \ldots (1)$$

where the Fourier coefficients a_0, a_n, b_n are given by Euler's formulae.

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2\, dx$$

$$a_0 = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{1}{3\pi}\left\{\pi^3 - (-\pi)^3\right\} = \frac{2\pi^3}{3\pi} = \frac{2\pi^2}{3}$$

$$a_0/2 = \pi^2/3$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos nx\, dx$$

$$a_n = \frac{1}{\pi}\left[(x^2)\left(\frac{\sin nx}{n}\right) - (2x)\left(\frac{-\cos nx}{n^2}\right) + (2)\left(\frac{-\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{2}{\pi n^2}\left[x \cos nx\right]_{-\pi}^{\pi}, \text{ since } \sin n\pi = 0.$$

$$= \frac{2}{\pi n^2}\{\pi \cos n\pi - (-\pi)\cos n\pi\}, \text{ since } \cos(-n\pi) = \cos n\pi.$$

$$= \frac{2}{\pi n^2} \cdot 2\pi \cos n\pi$$

$$a_n = \frac{4(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin nx \, dx$$

$$b_n = \frac{1}{\pi}\left[(x^2)\left(\frac{-\cos nx}{n}\right) - (2x)\left(\frac{-\sin nx}{n^2}\right) + (2)\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$b_n = \frac{1}{n}\left\{\frac{-1}{\pi}(\pi^2 \cos n\pi - \pi^2 \cos n\pi) + 0 + \frac{2}{n^3}(\cos n\pi - \cos n\pi)\right\}$$

$$b_n = 0$$

Thus by substituting the values of a_0, a_n, b_n in (1) the Fourier series is given by

$$\boxed{f(x) = x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nx} \quad \ldots (2)$$

Deductions : Putting $x = 0$ in (2) we get,

$$f(0) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos 0$$

ie., $$0 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}, \text{ since } f(0) = 0^2 = 0; \ \cos 0 = 1$$

ie., $$-\frac{\pi^2}{3} = 4\left(\frac{-1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \cdots\right)$$

$$\therefore \quad \boxed{\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots} \quad \ldots \text{(i)}$$

Again, putting $x = \pi$ in (2) we get

$$f(\pi) = \frac{\pi^2}{3} + \sum_1^\infty \frac{4(-1)^n}{n^2} \cos n\pi$$

$$ie., \quad \pi^2 = \frac{\pi^2}{3} + 4\sum_1^\infty \frac{4(-1)^n}{n^2}(-1)^n, \text{ since } f(\pi) = \pi^2 \text{ and } \cos n\pi = (-1)^n$$

$$ie., \quad \pi^2 - \frac{\pi^2}{3} = 4\sum_1^\infty \frac{1}{n^2} \text{ or } \frac{2\pi^2}{3} = 4\sum_1^\infty \frac{1}{n^2}, \text{ since } (-1)^{2n} = +1$$

$$\therefore \quad \boxed{\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots} \quad \ldots \text{(ii)}$$

Now by adding (i) and (ii) we get,

$$\frac{\pi^2}{12} + \frac{\pi^2}{6} = 2 \cdot \frac{1}{1^2} + 2 \cdot \frac{1}{3^2} + 2 \cdot \frac{1}{5^2} + \cdots$$

$$ie., \quad \frac{\pi^2}{4} = 2\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$$

$$\therefore \quad \boxed{\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots} \quad \ldots \text{(iii)}$$

[3] *If* $f(x) = x(2\pi - x)$ *in* $0 \le x \le 2\pi$, *show that*

$$f(x) = \frac{2\pi^2}{3} - 4\left(\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \cdots\right)$$

Hence deduce that $\dfrac{\pi^2}{8} = \dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \cdots$

☞ The Fourier series of $f(x)$ having period 2π is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^\infty a_n \cos nx + \sum_{n=1}^\infty b_n \sin nx \quad \ldots (1)$$

Now, $a_0 = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} f(x)\,dx = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} (2\pi x - x^2)\,dx$

$$a_0 = \frac{1}{\pi}\left[\pi x^2 - \frac{x^3}{3}\right]_0^{2\pi} = \frac{1}{\pi}\left(4\pi^3 - \frac{8\pi^3}{3}\right) = \frac{4\pi^2}{3}$$

$$\therefore \quad a_0/2 = 2\pi^2/3$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi}(2\pi x - x^2)\cos nx\,dx$$

$$a_n = \frac{1}{\pi}\left[(2\pi x - x^2)\left(\frac{\sin nx}{n}\right) - (2\pi - 2x)\left(\frac{-\cos nx}{n^2}\right) + (-2)\left(\frac{-\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[0 + \frac{1}{n^2}(2\pi - 2x)\cos nx + 0\right]_0^{2\pi} = \frac{1}{\pi n^2}(-2\pi\cos 2n\pi - 2\pi\cos 0)$$

$$\therefore \quad a_n = \frac{-4}{n^2} \qquad (\cos 2n\pi = 1 = \cos 0)$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi}(2\pi x - x^2)\sin nx\,dx$$

$$b_n = \frac{1}{\pi}\left[(2\pi x - x^2)\left(\frac{-\cos nx}{n}\right) - (2\pi - 2x)\left(\frac{-\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$= \frac{1}{\pi}\left[\frac{-2}{n^3}\cos nx\right]_0^{2\pi}$, as the first two terms are zero, since $(2\pi x - x^2)$ is zero at $x = 0, 2\pi$ and $\sin 2n\pi = 0 = \sin 0$.

$$b_n = \frac{-2}{\pi n^3}(\cos 2n\pi - \cos 0) = \frac{-2}{\pi n^3}(1-1) = 0$$

$$\therefore \quad b_n = 0$$

Thus by substituting the values of a_0, a_n, b_n in (1) we have,

$$\boxed{f(x) = 2\pi x - x^2 = \frac{2\pi^2}{3} + \sum_{n=1}^{\infty} \frac{-4}{n^2}\cos nx}$$

Expanding RHS by putting $n = 1, 2, 3, \ldots$ we get,

$$f(x)=\frac{2\pi^2}{3}-4\left(\frac{\cos x}{1^2}+\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}+\cdots\right) \quad \ldots (2)$$

To deduce the required series we shall first put $x = 0$ in (2).

Since $f(x) = 2\pi x - x^2$ in $0 \le x \le 2\pi$, $f(0) = 0$ and hence (2) becomes

$$0=\frac{2\pi^2}{3}-4\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right) \text{ or } \frac{-2\pi^2}{3}=-4\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right)$$

$$\therefore \quad \frac{\pi^2}{6}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots \quad \ldots (3)$$

Again putting $x = \pi$ in (2), $f(x) = f(\pi) = 2\pi(\pi) - \pi^2 = \pi^2$ and hence (2) becomes,

$$\pi^2=\frac{2\pi^2}{3}-4\left(\frac{1}{1^2}\cos\pi+\frac{1}{2^2}\cos 2\pi+\frac{1}{3^2}\cos 3\pi+\cdots\right)$$

$$\pi^2-\frac{2\pi^2}{3}=-4\left(\frac{-1}{1^2}+\frac{1}{2^2}+\frac{-1}{3^2}-\cdots\right) \text{ or } \frac{\pi^2}{3}=4\left(\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\cdots\right)$$

$$\therefore \quad \frac{\pi^2}{12}=\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\cdots \quad \ldots (4)$$

Now adding (3) and (4) we obtain,

$$\frac{\pi^2}{6}+\frac{\pi^2}{12}=2\left(\frac{1}{1^2}+\frac{1}{3^2}+\cdots\right) \text{ or } \frac{\pi^2}{4}=2\left(\frac{1}{1^2}+\frac{1}{3^2}+\cdots\right)$$

Thus, $$\boxed{\frac{\pi^2}{8}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots}$$

[4] *Obtain the Fourier series to represent e^{-ax} from $x = -\pi$ to $x = \pi$. Hence derive the series for $\dfrac{\pi}{\sinh \pi}$*

☞ The Fourier series of $f(x)$ having period 2π is given by

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx \quad \ldots (1)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

Now, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\,dx$

$$= \frac{1}{\pi}\left[\frac{e^{-ax}}{-a}\right]_{-\pi}^{\pi} = \frac{-1}{a\pi}\left[e^{-a\pi} - e^{a\pi}\right] = \frac{1}{a\pi}\left[e^{a\pi} - e^{-a\pi}\right]$$

$$\therefore \quad \frac{a_0}{2} = \frac{1}{a\pi}\left(\frac{e^{a\pi} - e^{-a\pi}}{2}\right) \text{ or } \frac{a_0}{2} = \frac{\sinh a\pi}{a\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\cos nx\,dx.$$

We have the standard formula,

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + n\sin bx)$$

$$\therefore \quad a_n = \frac{1}{\pi}\left[\frac{e^{-ax}}{(-a)^2+n^2}(-a\cos nx + n\sin nx)\right]_{-\pi}^{\pi}$$

$$= \frac{-a}{\pi(a^2+n^2)}\left[e^{-ax}\cos nx\right]_{-\pi}^{\pi}, \text{ since } \sin n\pi = 0 = \sin(-n\pi)$$

$$a_n = \frac{-a}{\pi(a^2+n^2)}\left\{e^{-a\pi}\cos n\pi - e^{a\pi}\cdot\cos(-n\pi)\right\}$$

$$= \frac{-a\cos n\pi}{\pi(a^2+n^2)}\left(e^{-a\pi} - e^{a\pi}\right) = \frac{a(-1)^n}{\pi(a^2+n^2)}\left(e^{a\pi} - e^{-a\pi}\right)$$

$$\therefore \quad a_n = \frac{2a(-1)^n \sinh a\pi}{\pi(a^2+n^2)}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\sin nx\,dx.$$

We have the standard formula,

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$$

$$\therefore \quad b_n = \frac{1}{\pi}\left[\frac{e^{-ax}}{(-a)^2+n^2}(-a\sin nx - n\cos nx)\right]_{-\pi}^{\pi}$$

$$= \frac{-n}{\pi(a^2+n^2)}\left[e^{-ax}\cos nx\right]_{-\pi}^{\pi}$$

The function to be evaluated between the limits $-\pi$ to π is same as in a_n.

$$\therefore \quad b_n = \frac{2n(-1)^n\sinh a\pi}{\pi(a^2+n^2)}$$

Substituting the values of a_0, a_n, b_n in (1) the Fourier series is given by

$$f(x) = \frac{\sin h\,a\pi}{a\pi} + \sum_{n=1}^{\infty}\frac{2a(-1)^n\sin h\,a\pi}{\pi(a^2+n^2)}\cos nx + \sum_{n=1}^{\infty}\frac{2n(-1)^n\sin h\,a\pi}{\pi(a^2+n^2)}\sin n$$

Thus, $$\boxed{e^{-ax} = \frac{\sin h\,a\pi}{a\pi}\left\{1+\sum_{1}^{\infty}\frac{2a^2(-1)^n}{a^2+n^2}\cos nx + \sum_{1}^{\infty}\frac{2an(-1)^n}{a^2+n^2}\sin nx\right\}}$$

To deduce the series we shall put $a = 1$, $x = 0$ in the Fourier series.

$$e^0 = \frac{\sin h\,\pi}{\pi}\left\{1+\sum_{1}^{\infty}\frac{2(-1)^n}{1+n^2}\right\} \quad \text{since } \cos 0 = 1,\ \sin 0 = 0$$

ie., $$1 = \frac{\sin h\,\pi}{\pi}\left\{1+2\sum_{1}^{\infty}\frac{2(-1)^n}{1+n^2}\right\} \quad \text{or} \quad \frac{\pi}{\sin h\,\pi} = 1+2\sum_{1}^{\infty}\frac{(-1)^n}{1+n^2}$$

ie., $$\frac{\pi}{\sin h\,\pi} = 1+2\left(\frac{-1}{1+1^2}+\frac{1}{1+2^2}-\frac{1}{1+3^2}+\frac{1}{1+4^2}-\cdots\right)$$

$$= 1-1+2\left(\frac{1}{5}-\frac{1}{10}+\frac{1}{17}-\cdots\right)$$

Thus, $$\boxed{\frac{\pi}{\sin h\,\pi} = 2\left(\frac{1}{5}-\frac{1}{10}+\frac{1}{17}-\cdots\right)}$$

[5] *Find a Fourier series in* $(-\pi, \pi)$ *to represent the following functions.*

(a) $f(x) = x - x^2$ **[Dec. 2016]** **(b)** $f(x) = x + x^2$

Hence deduce that,

(i) $\dfrac{\pi^2}{12} = \dfrac{1}{1^2} - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \dfrac{1}{4^2} + \cdots$

(ii) $\dfrac{\pi^2}{6} = \dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \dfrac{1}{4^2} + \cdots$

(iii) $\dfrac{\pi^2}{8} = \dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \dfrac{1}{7^2} + \cdots$

☞ (a) Period of $f(x) = \pi - (-\pi) = 2\pi$ and the Fourier series of period 2π is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx, \text{ where}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx,\ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx,\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\,dx = \frac{1}{\pi}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left\{\left(\frac{\pi^2}{2} - \frac{\pi^2}{2}\right) - \left(\frac{\pi^3}{3} - \frac{-\pi^3}{3}\right)\right\} = \frac{-2\pi^2}{3}$$

$\therefore \quad a_0/2 = -\pi^2/3$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[(x - x^2)\left(\frac{\sin nx}{n}\right) - (1 - 2x)\left(\frac{-\cos nx}{n^2}\right) + (-2)\left(\frac{-\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi n^2}\left[(1 - 2x)\cos nx\right]_{-\pi}^{\pi} = \frac{1}{\pi n^2}\{(1 - 2\pi)\cos n\pi - (1 + 2\pi)\cos n\pi\}$$

ie., $$a_n = \frac{-4}{n^2}\cos n\pi = -\frac{4}{n^2}(-1)^n$$

$$\therefore \quad a_n = \frac{4(-1)^{n+1}}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(x-x^2)\left(\frac{-\cos nx}{n}\right)-(1-2x)\left(\frac{-\sin nx}{n^2}\right)+(-2)\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-1}{n}\left\{(\pi-\pi^2)\cos n\pi-(-\pi-\pi^2)\cos n\pi-\frac{2}{n^3}(\cos n\pi-\cos n\pi)\right\}\right]$$

$$b_n = -\frac{1}{\pi n}(2\pi\cos n\pi) = \frac{-2}{n}(-1)^n$$

$$\therefore \quad b_n = \frac{2}{n}(-1)^{n+1}$$

The required Fourier series is given by

$$\boxed{x-x^2 = \frac{-\pi^2}{3}+4\sum_1^{\infty}\frac{(-1)^{n+1}}{n^2}\cos nx+2\sum_1^{\infty}\frac{(-1)^{n+1}}{n}\sin nx} \quad \ldots (1$$

To deduce the series, first let us put $x = 0$ which is a point of the interva $(-\pi, \pi)$. Hence (1) becomes

$$0 = -\frac{\pi^2}{3}+4\sum_1^{\infty}\frac{(-1)^{n+1}}{n^2}\cdot 1+0$$

ie., $$\frac{\pi^2}{3} = 4\left[\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots\right]$$

or $$\boxed{\frac{\pi^2}{12} = \frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots} \quad \ldots (i$$

Next, let us put $x = \pi$ in (1). Since $f(x) = x-x^2$ is defined in $-\pi < x < \pi$ th value of $f(x)$ at $x = \pi$ being $f(\pi)$ is given by $\frac{1}{2}[f(-\pi)+f(\pi)]$

which being $\frac{1}{2}[(-\pi-\pi^2)+(\pi-\pi^2)] = -\pi^2$

Hence (1) becomes,

$$-\pi^2 = \frac{-\pi^2}{3}+4\sum_1^\infty \frac{(-1)^{n+1}}{n^2}\cos n\pi + 0$$

ie., $$-\pi^2+\frac{\pi^2}{3} = 4\sum_1^\infty \frac{(-1)^{2n+1}}{n^2}, \text{ since } \cos n\pi = (-1)^n$$

ie., $$-\frac{\pi^2}{6} = \frac{-1}{1^2}-\frac{1}{2^2}-\frac{1}{3^2}-\frac{1}{4^2}-\cdots$$

or $$\boxed{\frac{\pi^2}{6} = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots} \quad \ldots \text{(ii)}$$

Adding (i) and (ii) we obtain

$$\boxed{\frac{\pi^2}{8} = \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots} \quad \ldots \text{(iii)}$$

(b) Fourier series of $f(x) = x+x^2$ in $-\pi<x<\pi$ on similar lines can be obtained in the form

$$\boxed{x+x^2 = \frac{\pi^2}{3}+4\sum_1^\infty \frac{(-1)^n}{n^2}\cos nx + 2\sum_1^\infty \frac{(-1)^{n+1}}{n}\sin nx}$$

All the three series can be deduced in a similar way.

[6] *If* $f(x) = \begin{cases} x & \text{in } 0 \le x \le \pi \\ 2\pi - x & \text{in } \pi \le x \le 2\pi \end{cases}$ *show that the Fourier series of* $f(x)$ *in* $[0,2\pi]$ *is* $\frac{\pi}{2}-\frac{4}{\pi}\left[\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+\cdots\right]$ *and hence deduce that* $\frac{\pi^2}{8} = \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots$

☛ The Fourier series of $f(x)$ having period 2π is given by

$$f(x) = \frac{a_0}{2}+\sum_{n=1}^\infty a_n\cos nx + \sum_{n=1}^\infty b_n \sin nx.$$

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Now, $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\left\{\int_0^{\pi} f(x)\,dx + \int_{\pi}^{2\pi} f(x)\,dx\right\}$

ie., $a_0 = \frac{1}{\pi}\left\{\int_0^{\pi} x\,dx + \int_{\pi}^{2\pi}(2\pi - x)\,dx\right\}$

$$= \frac{1}{\pi}\left\{\left[\frac{x^2}{2}\right]_0^{\pi} + \left[2\pi x - \frac{x^2}{2}\right]_{\pi}^{2\pi}\right\}$$

$$a_0 = \frac{1}{\pi}\left\{\left(\frac{\pi^2}{2} - 0\right) + (4\pi^2 - 2\pi^2) - \left(2\pi^2 - \frac{\pi^2}{2}\right)\right\} = \frac{1}{\pi}(\pi^2) = \pi$$

$\therefore \quad a_0/2 = \pi/2$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left\{\int_0^{\pi} f(x)\cos nx\,dx + \int_{\pi}^{2\pi} f(x)\cos nx\,dx\right\}$$

$$a_n = \frac{1}{\pi}\left\{\int_0^{\pi} x\cos nx\,dx + \int_{\pi}^{2\pi}(2\pi - x)\cos nx\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\left[x\left(\frac{\sin nx}{n}\right) - (1)\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}\right.$$

$$\left. + \left[(2\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(\frac{-\cos nx}{n^2}\right)\right]_{\pi}^{2\pi}\right\}$$

where we have applied Bernoulli's rule to each of the integrals.

$$a_n = \frac{1}{\pi n^2}\left\{[\cos nx]_0^{\pi} - [\cos nx]_{\pi}^{2\pi}\right\}, \text{ since } \sin n\pi = 0 = \sin 0.$$

$$= \frac{1}{\pi n^2}\{(\cos n\pi - 1) - (1 - \cos n\pi)\}, \text{ since } \cos 0 = 1 = \cos 2n\pi$$

$$= \frac{1}{\pi n^2}(-2 + 2\cos n\pi) = \frac{-2}{\pi n^2}(1 - \cos n\pi)$$

$$\therefore \quad a_n = \frac{-2}{\pi n^2}\{1-(-1)^n\}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left\{\int_0^{\pi} f(x)\sin nx\,dx + \int_{\pi}^{2\pi} f(x)\sin nx\,dx\right\}$$

$$b_n = \frac{1}{\pi}\left\{\int_0^{\pi} x\sin nx\,dx + \int_{\pi}^{2\pi}(2\pi - x)\sin nx\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\left[x\left(\frac{-\cos nx}{n}\right) - (1)\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi}\right.$$

$$\left. + \left[(2\pi - x)\left(\frac{-\cos nx}{n}\right) - (-1)\left(\frac{-\sin nx}{n^2}\right)\right]_{\pi}^{2\pi}\right\}$$

$$= \frac{-1}{\pi n}\left\{[x\cos nx]_0^{\pi} + 0 + [(2\pi - x)\cos nx]_{\pi}^{2\pi} - 0\right\}$$

$$b_n = \frac{-1}{\pi n}\{(\pi\cos n\pi - 0) + (0 - \pi\cos n\pi)\} = 0$$

$$\therefore \quad b_n = 0$$

The Fourier series representation of $f(x)$ is given by

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{-2}{\pi n^2}\{1-(-1)^n\}\cos nx$$

But, $$1-(-1)^n = \begin{cases} 1-(-1) = 2 & \text{if } n \text{ is odd} \\ 1-(+1) = 0 & \text{if } n \text{ is even} \end{cases}$$

$$f(x) = \frac{\pi}{2} - \frac{2}{\pi}\sum_{n=1,3,5\ldots}^{\infty} \frac{1}{n^2}\cdot 2\cos nx$$

Thus, $$\boxed{f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right]}$$

To deduce the series, let us put $x = 0$.

Then $f(x) = 0$ since $f(x) = x$ in $0 \le x \le \pi$.

Hence the Fourier series becomes,

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right] \quad \text{or} \quad \frac{-\pi}{2} = \frac{-4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

Thus,
$$\boxed{\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots}$$

[7] *Obtain the Fourier series for the function*

$$f(x) = \begin{cases} -\pi & in \ -\pi < x < 0 \\ x & in \ \ 0 < x < \pi \end{cases}$$ *Hence deduce that,*

the sum of the reciprocal squares of the odd integers is equal to $\pi^2/8$. **[June 2018]**

☞ $f(x)$ is defined in $(-\pi, \pi)$ and the Fourier series of $f(x)$ having period 2π is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \ldots (1)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left\{\int_{-\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx\right\}$$

$$a_0 = \frac{1}{\pi}\left\{\int_{-\pi}^{0} -\pi\,dx + \int_{0}^{\pi} x\,dx\right\} = \frac{1}{\pi}\left\{-\pi[x]_{-\pi}^{0} + \left[\frac{x^2}{2}\right]_0^{\pi}\right\}$$

$$a_0 = \frac{1}{\pi}\left\{-\pi(0-(-\pi)) + \left(\frac{\pi^2}{2} - 0\right)\right\}$$

$$a_0 = \frac{1}{\pi}\left(-\pi^2 + \frac{\pi^2}{2}\right) = \frac{1}{\pi}\left(\frac{-\pi^2}{2}\right) = -\frac{\pi}{2}$$

$\therefore \qquad a_0/2 = -\pi/4$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{0} f(x)\cos nx\,dx + \int_{0}^{\pi} f(x)\cos nx\,dx\right\}$$

$$= \frac{1}{\pi}\left\{ \int_{-\pi}^{0} -\pi \cos nx\, dx + \int_{0}^{\pi} x \cos nx\, dx \right\}$$

$$= \frac{1}{\pi}\left\{ -\pi\left[\frac{\sin nx}{n}\right]_{-\pi}^{0} + \left[x \cdot \frac{\sin nx}{n}\right]_{0}^{\pi} - \left[1 \cdot \frac{-\cos nx}{n^2}\right]_{0}^{\pi} \right\}$$

where the integration is carried out by Bernoulli's rule in the second integral.

$$a_n = \frac{1}{\pi n^2}[\cos nx]_0^{\pi} = \frac{1}{\pi n^2}(\cos n\pi - \cos 0), \text{ since } \sin n\pi = 0 = \sin 0.$$

$$a_n = \frac{1}{\pi n^2}\{(-1)^n - 1\} \text{ or } a_n = \frac{-1}{\pi n^2}\{1-(-1)^n\}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx$$

$$= \frac{1}{\pi}\left\{ \int_{-\pi}^{0} f(x) \sin nx\, dx + \int_{0}^{\pi} f(x) \sin nx\, dx \right\}$$

$$= \frac{1}{\pi}\left\{ \int_{-\pi}^{0} -\pi \sin nx\, dx + \int_{0}^{\pi} x \sin nx\, dx \right\}$$

$$= \frac{1}{\pi}\left\{ -\pi\left[\frac{-\cos nx}{n}\right]_{-\pi}^{0} + \left[x \cdot \frac{-\cos nx}{n}\right]_{0}^{\pi} - \left[1 \cdot \frac{-\sin nx}{n^2}\right]_{0}^{\pi} \right\}$$

$$= \frac{1}{\pi n}\left\{ \pi[\cos nx]_{-\pi}^{0} - [x \cos nx]_0^{\pi} \right\}, \text{ since } \sin n\pi = 0 \sin 0.$$

$$= \frac{1}{\pi n}\{ \pi(\cos 0 - \cos n\pi) - (\pi \cos n\pi - 0) \}$$

$$b_n = \frac{\pi}{\pi n}\{1 - \cos n\pi - \cos n\pi\} = \frac{1}{n}(1 - 2\cos n\pi)$$

$$\therefore \quad b_n = \frac{1}{n}\{1 - 2(-1)^n\}$$

Substituting the values of a_0, a_n, b_n in (1), the Fourier series is given by

$$\boxed{f(x) = \frac{-\pi}{4} + \sum_{n=1}^{\infty} \frac{-1}{\pi n^2}\{1-(-1)^n\}\cos nx + \sum_{n=1}^{\infty} \frac{1}{n}\{1-2(-1)^n)\}\sin nx}$$

To deduce the required series let us put $x = 0$ in the Fourier series.

It should be observed from the given $f(x)$ that $x = 0$ is a point of discontinuity and hence the series converges to

$$\frac{1}{2}\left[f(0^+)+f(0^-)\right] = \frac{1}{2}[0+(-\pi)] = \frac{-\pi}{2}$$

Because to the right of 0, in $(0,\pi)$, $f(x) = x$ and $f(0^+) = 0$. Also to the left of 0, in $(-\pi,0)$, $f(x) = -\pi$ and $f(0^-) = -\pi$

Hence the Fourier series becomes

$$-\frac{\pi}{2} = -\frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{-1}{\pi n^2}\{1-(-1)^n\}, \text{ since } \cos 0 = 1, \sin 0 = 0.$$

ie., $$-\frac{\pi}{2}+\frac{\pi}{4} = -\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\{1-(-1)^n\}$$

ie., $$-\frac{\pi}{4} = -\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\{1-(-1)^n\} \text{ or } \frac{\pi^2}{4} = \sum_{1}^{\infty}\frac{1}{n^2}\{1-(-1)^n\}$$

But $$1-(-1)^n = \begin{cases} 1-(+1) = 0 & \text{when } n \text{ is even} \\ 1-(-1) = 2 & \text{when } n \text{ is odd} \end{cases}$$

Hence we get, $$\frac{\pi^2}{4} = \sum_{n=1,3,,5\cdots}^{\infty} \frac{1}{n^2}(2) \text{ or } \frac{\pi^2}{8} = \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots$$

Thus the sum of the reciprocal squares of the odd integers is $\boxed{\pi^2/8.}$

[8] *Obtain the Fourier series for the function*

$$f(x) = \begin{cases} 0 & in\ -\pi \le x \le 0 \\ \sin x & in\ \ 0 \le x \le \pi \end{cases} \quad \textit{Deduce that,}$$

(i) $\dfrac{1}{1\cdot 3}+\dfrac{1}{3\cdot 5}+\dfrac{1}{5\cdot 7}+\cdots = \dfrac{1}{2}$ **(ii)** $\dfrac{1}{1\cdot 3}-\dfrac{1}{3\cdot 5}+\dfrac{1}{5\cdot 7}-\cdots = \dfrac{\pi-2}{4}$

☞ $f(x)$ is defined in $[-\pi, \pi]$ and the Fourier series of period 2π is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \ldots (1)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left\{\int_{-\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx\right\}$$

$$a_0 = \frac{1}{\pi}\left\{\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi}\sin x\,dx\right\} = \frac{1}{\pi}\int_{0}^{\pi}\sin x\,dx$$

$$a_0 = \frac{1}{\pi}\left[-\cos x\right]_0^{\pi} = \frac{-1}{\pi}(\cos\pi - \cos 0) = -\frac{1}{\pi}(-1-1) = \frac{2}{\pi}$$

$\therefore \quad a_0/2 = 1/\pi$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left\{\int_{-\pi}^{0} 0\cdot\cos nx\,dx + \int_{0}^{\pi}\sin x\cdot\cos nx\,dx\right\}$$

ie., $$a_n = \frac{1}{\pi}\int_{0}^{\pi}\sin x\cos nx\,dx.$$

Using, $\sin A\cos B = \frac{1}{2}\{\sin(A+B)+\sin(A-B)\}$, we have,

$$a_n = \frac{1}{\pi}\int_{0}^{\pi}\frac{1}{2}\{\sin(x+nx)+\sin(x-nx)\}\,dx$$

$$= \frac{1}{2\pi}\int_{0}^{\pi}\{\sin(1+n)x+\sin(1-n)x\}\,dx$$

$$= \frac{1}{2\pi}\int_{0}^{\pi}\{\sin(n+1)x-\sin(n-1)x\}\,dx$$

$$= \frac{1}{2\pi}\left[\frac{-\cos(n+1)x}{n+1}+\frac{\cos(n-1)x}{n-1}\right]_0^{\pi} \quad \text{where } n \neq 1$$

$$= \frac{1}{2\pi}\left[\frac{-1}{n+1}\{\cos(n+1)\pi-\cos 0\}+\frac{1}{(n-1)}\{\cos(n-1)\pi-\cos 0\}\right]$$

Using, $\cos 0 = 1, \cos k\pi = (-1)^k$ and rearranging,

$$a_n = \frac{1}{2\pi}\left\{\left(\frac{1}{n+1} - \frac{1}{n-1}\right) + \left(\frac{-1}{n+1}(-1)^{n+1} + \frac{1}{n-1}(-1)^{n-1}\right)\right\}$$

$$a_n = \frac{1}{2\pi}\left\{\frac{-2}{n^2-1} + \frac{(-1)^{n+2}}{n+1} + \frac{(-1)^{n-1}}{n-1}\right\}$$

$$= \frac{1}{2\pi}\left\{\frac{-2}{n^2-1} + (-1)^n\left(\frac{(-1)^2}{n+1} + \frac{(-1)^{-1}}{n-1}\right)\right\}$$

$$= \frac{1}{2\pi}\left\{\frac{-2}{n^2-1} + (-1)^n\left(\frac{1}{n+1} - \frac{1}{n-1}\right)\right\} \quad \because \ (-1)^{-1} = 1/-1 = -1$$

$$= \frac{1}{2\pi}\left\{\frac{-2}{n^2-1} + (-1)^n \frac{-2}{n^2-1}\right\}$$

$$a_n = \frac{-1}{\pi(n^2-1)}\{1 + (-1)^n\} \quad \text{where } n \neq 1$$

Now we shall find a_n when $n = 1$. That is to find a_1.

We have, $a_n = \frac{1}{\pi}\int_0^{\pi} \sin x \cos nx\, dx$

Putting $n = 1, a_1 = \frac{1}{\pi}\int_0^{\pi} \sin x \cos x\, dx = \frac{1}{\pi}\int_0^{\pi} \frac{\sin 2x}{2}\, dx$

ie., $\quad a_1 = \frac{1}{2\pi}\left[\frac{-\cos 2x}{2}\right]_0^{\pi} = \frac{-1}{4\pi}(\cos 2\pi - \cos 0) = \frac{-1}{4\pi}(1-1) = 0$

$\therefore \quad a_1 = 0$ and $a_n = \frac{-1}{\pi(n^2-1)}\{1 + (-1)^n\}$, for $n \neq 1$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx = \frac{1}{\pi}\left\{\int_{-\pi}^{0} 0 \cdot \sin nx\, dx + \int_0^{\pi} \sin x \sin nx\, dx\right\}$$

ie., $\quad b_n = \frac{1}{\pi}\int_0^{\pi} \sin x \sin nx\, dx$

Using, $\sin A \sin B = \frac{1}{2}\{\cos(A-B) - \cos(A+B)\}$,

$$b_n = \frac{1}{\pi}\int_0^{\pi}\frac{1}{2}\{\cos(x-nx) - \cos(x+nx)\}\,dx$$

$$= \frac{1}{2\pi}\int_0^{\pi}\{\cos(1-n)x - \cos(1+n)x\}\,dx$$

$$= \frac{1}{2\pi}\int_0^{\pi}\{\cos(n-1)x - \cos(n+1)x\}\,dx, \text{ since } \cos(-\theta) = \cos\theta.$$

$$= \frac{1}{2\pi}\left[\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right]_0^{\pi} \text{ where } n \neq 1$$

$b_n = 0\ (n \neq 1)$ since $\sin k\pi = 0$ for integral values of k.

Now we shall find b_n when $n = 1$. That is to find b_1.

We have, $b_n = \frac{1}{\pi}\int_0^{\pi} \sin x \sin nx\,dx$

Putting $n = 1$, $b_1 = \frac{1}{\pi}\int_0^{\pi} \sin x \cdot \sin x\,dx = \frac{1}{\pi}\int_0^{\pi} \sin^2 x\,dx$

ie., $b_1 = \frac{1}{\pi}\int_0^{\pi}\frac{1}{2}(1-\cos 2x)\,dx = \frac{1}{2\pi}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi}$.

$$= \frac{1}{2\pi}(\pi - 0) = \frac{1}{2}, \text{ since } \sin 2\pi = 0 = \sin 0.$$

$\therefore\quad b_1 = 1/2,\ b_n = 0$ for $n \neq 1$

Let us consider (1) in the form,

$$f(x) = \frac{a_0}{2} + a_1\cos x + \sum_{n=2}^{\infty} a_n \cos nx + b_1 \sin x + \sum_{n=2}^{\infty} b_n \sin nx$$

Thus by substituting the values of the Fourier coefficients, the required Fourier series is given by

$$\boxed{f(x) = \frac{1}{\pi} + \sum_{n=2}^{\infty} \frac{-1}{\pi(n^2-1)}\{1+(-1)^n\}\cos nx + \frac{1}{2}\sin x}$$

To deduce the required series we shall first put $x = 0$.

$$\therefore \quad f(0) = \frac{1}{\pi} + \sum_{2}^{\infty} \frac{-1}{\pi(n^2-1)}\{1+(-1)^n\}\cdot 1 + 0$$

Also, $f(0) = 0$ as we have $f(x) = 0$ in $-\pi \le x \le 0$

ie., $$0 = \frac{1}{\pi} + \sum_{2}^{\infty} \frac{-1}{\pi(n^2-1)}\{1+(-1)^n\}$$

ie., $$\frac{-1}{\pi} = \frac{-1}{\pi}\sum_{2}^{\infty} \frac{1}{n^2-1}\{1+(-1)^n\}$$

ie., $$1 = \sum_{2}^{\infty} \frac{1}{n^2-1}\{1+(-1)^n\}$$

But, $$1+(-1)^n = \begin{cases} 1+1 = 2 & \text{when } n \text{ is even} \\ 1-1 = 0 & \text{when } n \text{ is odd} \end{cases}$$

$$\therefore \quad 1 = \sum_{n=2,4,6\cdots}^{\infty} \frac{1}{n^2-1}(2) \text{ or } \frac{1}{2} = \sum_{n=2,4,\cdots}^{\infty} \frac{1}{n^2-1}$$

Thus, $$\boxed{\frac{1}{2} = \frac{1}{3} + \frac{1}{15} + \frac{1}{35} + \cdots \text{ or } \frac{1}{2} = \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots}$$

Next let us put $x = \pi/2$ in the Fourier series.

$$\therefore \quad f\left(\frac{\pi}{2}\right) = \frac{1}{\pi} + \sum_{n=2}^{\infty} \frac{-1}{\pi(n^2-1)}\{1+(-1)^n\}\cos\frac{n\pi}{2} + \frac{1}{2}\cdot 1$$

Also, $f(\pi/2) = \sin(\pi/2) = 1$, since $f(x) = \sin x$ in $0 \le x \le \pi$

ie., $$1 = \frac{1}{\pi} + \sum_{n=2}^{\infty} \frac{-1}{\pi(n^2-1)}\{1+(-1)^n\}\cos\frac{n\pi}{2} + \frac{1}{2}$$

ie., $$1 - \frac{1}{2} - \frac{1}{\pi} = \frac{-1}{\pi}\sum_{n=2}^{\infty} \frac{1}{n^2-1}\{1+(-1)^n\}\cos\frac{n\pi}{2}$$

ie., $$\frac{\pi-2}{2\pi} = \frac{-1}{\pi}\sum_{n=2}^{\infty} \frac{1}{n^2-1}\{1+(-1)^n\}\cos\frac{n\pi}{2}$$

i.e., $$\frac{\pi-2}{2} = -\sum_{n=2,4,6\cdots}^{\infty} \frac{1}{n^2-1}(2)\cos\frac{n\pi}{2}$$

i.e., $$\frac{\pi-2}{4} = -\left\{\frac{1}{3}\cos\pi + \frac{1}{15}\cos 2\pi + \frac{1}{35}\cos 3\pi + \cdots\right\}$$

But, $\cos\pi = -1 = \cos 3\pi,\ \cos 2\pi = 1.$

$$\therefore\ \frac{\pi-2}{4} = -\left(-\frac{1}{3}+\frac{1}{15}-\frac{1}{35}+\cdots\right) = \frac{1}{3}-\frac{1}{15}+\frac{1}{35}-\cdots$$

Thus, $$\boxed{\frac{\pi-2}{4} = \frac{1}{1\cdot 3}-\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}-\cdots}$$

[9] *An alternating current after passing through a rectifier has the form*

$$I = \begin{cases} I_0 \sin\theta & \text{for } 0<\theta\le\pi \\ 0 & \text{for } \pi<\theta\le 2\pi \end{cases}$$ [Dec. 2018]

where I_0 is the maximum current. Express I as a Fourier series in $(0, 2\pi)$

☞ The Fourier series of period of 2π is given by

$$I = f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\theta + \sum_{n=1}^{\infty} b_n \sin n\theta \qquad \ldots (1)$$

We have to find the Fourier coefficients by using Euler's formulae for the interval $(0, 2\pi)$ where

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\,d\theta,\ a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta\,d\theta,\ b_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta\,d\theta$$

Each of the above integrals after splitting into two integrals in the range $(0,\pi)$, $(\pi, 2\pi)$ and substituting for $f(\theta)$ will give us

$$a_0 = \frac{I_0}{\pi}\int_0^{\pi}\sin\theta\,d\theta,\ a_n = \frac{I_0}{\pi}\int_0^{\pi}\sin\theta\cos n\theta\,d\theta,\ b_n = \frac{I_0}{\pi}\int_0^{\pi}\sin\theta\sin n\theta\,d\theta$$

These integrals are same as in the previous problem and hence the Fourier coefficients are as follows.

$$a_0 = \frac{2I_0}{\pi},\ a_n = \frac{-I_0}{\pi(n^2-1)}\{1+(-1)^n\} \text{ if } n\ne 1,\ a_1 = 0$$

$$b_n = 0 \text{ if } n \neq 1 \text{ and } b_1 = \frac{I_0}{2}$$

Thus by substituting these values in (1) the required Fourier series is given by

$$\boxed{I = f(\theta) = \frac{I_0}{\pi} + \sum_{n=2}^{\infty} \frac{-I_0}{\pi(n^2-1)}\{1+(-1)^n\}\cos n\theta + \frac{I_0}{2}\sin\theta}$$

[10] *Obtain the Fourier series for*

$$f(x) = \begin{cases} -k & in\ (-\pi, 0) \\ +k & in\ (0, \pi) \end{cases} \quad \textit{Hence deduce that, } \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

☞ The Fourier series of $f(x)$ defined in $(-\pi, \pi)$ having period 2π is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \ldots (1)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left\{\int_{-\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx\right\}$$

$$a_0 = \frac{1}{\pi}\left\{\int_{-\pi}^{0} -k\,dx + \int_{0}^{\pi} k\,dx\right\} = \frac{k}{\pi}\left\{[-x]_{-\pi}^{0} + [x]_0^{\pi}\right\}$$

$$a_0 = \frac{k}{\pi}\{-(0+\pi)+(\pi-0)\} = 0$$

$$\therefore \quad a_0/2 = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left\{\int_{-\pi}^{0} f(x)\cos nx\,dx + \int_{0}^{\pi} f(x)\cos nx\,dx\right\}$$

$$a_n = \frac{1}{\pi}\left\{\int_{-\pi}^{0} -k\cos nx\,dx + \int_{0}^{\pi} k\cos nx\,dx\right\}$$

$$= \frac{k}{\pi}\left\{-\left[\frac{\sin nx}{n}\right]_{-\pi}^{0} + \left[\frac{\sin nx}{n}\right]_0^{\pi}\right\} = 0, \text{ since, } \sin 0 = 0 = \sin n\pi.$$

$$\therefore \quad a_n = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left\{\int_{-\pi}^{0} f(x)\sin nx\,dx + \int_{0}^{\pi} f(x)\sin nx\,dx\right\}$$

$$b_n = \frac{1}{\pi}\left\{\int_{-\pi}^{0} -k\sin nx\,dx + \int_{0}^{\pi} k\sin nx\,dx\right\}$$

$$b_n = \frac{k}{\pi}\left\{\left[\frac{\cos nx}{n}\right]_{-\pi}^{0} + \left[\frac{-\cos nx}{n}\right]_{0}^{\pi}\right\}$$

$$= \frac{k}{\pi n}\{(1-\cos n\pi) - (\cos n\pi - 1)\} = \frac{k}{\pi n}(2-2\cos n\pi)$$

$$\therefore \quad b_n = \frac{2k}{\pi n}\{1-(-1)^n\}$$

Thus the required Fourier series is given by

$$\boxed{f(x) = \sum_{n=1}^{\infty} \frac{2k}{\pi n}\{1-(-1)^n\}\sin nx}$$

But, $$1-(-1)^n = \begin{cases} 1-(+1) = 0 & \text{if } n \text{ is even} \\ 1-(-1) = 2 & \text{if } n \text{ is odd} \end{cases}$$

$$\therefore \quad f(x) = \frac{4k}{\pi}\sum_{n=1,3,5\ldots}^{\infty} \frac{1}{n}\sin nx$$

$$f(x) = \frac{4k}{\pi}\left\{\frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \frac{\sin 4x}{4} + \cdots\right\}$$

To deduce the series let us put $x = \pi/2$.

Then $f(x) = k$ since $f(x) = k$ in $0 < x < \pi$.

Hence the Fourier series becomes,

$$k = \frac{4k}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \cdots\right), \quad \text{since} \quad \sin\left(\frac{3\pi}{2}\right) = -1, \ \sin\left(\frac{5\pi}{2}\right) = 1$$

Thus, $$\boxed{\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots}$$

[11] *Find the Fourier series of*

$$f(x)=\begin{cases}1+\dfrac{2x}{\pi} & in\ -\pi<x\le 0\\ 1-\dfrac{2x}{\pi} & in\ \ 0\le \pi<\pi\end{cases}\qquad \textit{Hence deduce that, } \frac{\pi^2}{8}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots$$

☞ The Fourier series of $f(x)$ defined in $(-\pi, \pi)$ having period 2π is given by

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx=\frac{1}{\pi}\left\{\int_{-\pi}^{0}f(x)\,dx+\int_{0}^{\pi}f(x)\,dx\right\}$$

$$a_0=\frac{1}{\pi}\left\{\int_{-\pi}^{0}\left(1+\frac{2x}{\pi}\right)dx+\int_{0}^{\pi}\left(1-\frac{2x}{\pi}\right)dx\right\}$$

$$=\frac{1}{\pi}\left\{\left[x+\frac{x^2}{\pi}\right]_{-\pi}^{0}+\left[x-\frac{x^2}{\pi}\right]_{0}^{\pi}\right\}$$

$$a_0=\frac{1}{\pi}\{0-(-\pi+\pi)+(\pi-\pi)-0\}=0$$

$\therefore \quad a_0/2=0$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx$$

$$=\frac{1}{\pi}\left\{\int_{-\pi}^{0}f(x)\cos nx\,dx+\int_{0}^{\pi}f(x)\cos nx\,dx\right\}$$

$$=\frac{1}{\pi}\left\{\int_{-\pi}^{0}\left(1+\frac{2x}{\pi}\right)\cos nx\,dx+\int_{0}^{\pi}\left(1-\frac{2x}{\pi}\right)\cos nx\,dx\right\}$$

$$=\frac{1}{\pi}\left\{\left[\left(1+\frac{2x}{\pi}\right)\left(\frac{\sin nx}{n}\right)-\left(\frac{2}{\pi}\right)\left(\frac{-\cos nx}{n^2}\right)\right]_{-\pi}^{0}\right.$$

$$\left.+\left[\left(1-\frac{2x}{\pi}\right)\left(\frac{\sin nx}{n}\right)-\left(\frac{-2}{\pi}\right)\left(\frac{-\cos nx}{n^2}\right)\right]_{0}^{\pi}\right\}$$

But, $\sin n\pi = 0 = \sin 0$.

$$a_n = \frac{2}{\pi^2 n^2}\{(1-\cos n\pi)-(\cos n\pi - 1)\} = \frac{2}{\pi^2 n^2}(2-2\cos n\pi)$$

$$\therefore \quad a_n = \frac{4}{\pi^2 n^2}\{1-(-1)^n\}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left\{\int_{-\pi}^{0} f(x)\sin nx\,dx + \int_{0}^{\pi} f(x)\sin nx\,dx\right\}$$

$$b_n = \frac{1}{\pi}\left\{\int_{-\pi}^{0}\left(1+\frac{2x}{\pi}\right)\sin nx\,dx + \int_{0}^{\pi}\left(1-\frac{2x}{\pi}\right)\sin nx\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\left[\left(1+\frac{2x}{\pi}\right)\left(\frac{-\cos nx}{n}\right)-\left(\frac{2}{\pi}\right)\left(\frac{-\sin nx}{n^2}\right)\right]_{-\pi}^{0}\right.$$

$$\left. + \left[\left(1-\frac{2x}{\pi}\right)\left(\frac{-\cos nx}{n}\right)-\left(\frac{-2}{\pi}\right)\left(\frac{-\sin nx}{n^2}\right)\right]_{0}^{\pi}\right\}$$

$$= \frac{-1}{\pi n}\left\{\left[\left(1+\frac{2x}{\pi}\right)(\cos nx)\right]_{-\pi}^{0} + \left[\left(1-\frac{2x}{\pi}\right)(\cos nx)\right]_{0}^{\pi}\right\}$$

$$= \frac{-1}{\pi n}\{1+\cos n\pi - \cos n\pi - 1\} = 0$$

$$\therefore \quad b_n = 0$$

Thus the required Fourier series is given by

$$\boxed{f(x) = \sum_{n=1}^{\infty}\frac{4}{\pi^2 n^2}\{1-(-1)^n\}\cos nx}$$

But, $1-(-1)^n = \begin{cases} 1-(+1) = 0 & \text{if } n \text{ is even} \\ 1-(-1) = 2 & \text{if } n \text{ is odd} \end{cases}$

$$\therefore \quad f(x) = \frac{8}{\pi^2}\sum_{n=1,3,5\ldots}^{\infty}\frac{\cos nx}{n^2}$$

or $$f(x) = \frac{8}{\pi^2}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos x}{5^2} + \cdots\right)$$

Now putting $x = 0$, $f(x) = 1$. The Fourier series becomes,

$$1 = \frac{8}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$$

Thus, $$\boxed{\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots}$$

2.4 Fourier series of even and odd functions in the interval $(-\pi, \pi)$

Definition : A function $f(x)$ is said to be **even** if $f(-x) = f(x)$ and **od**
if $f(-x) = -f(x)$

For example : $x^2, x^4, x^6, \ldots \cos x$ are even functions and $x, x^3, x^5, \ldots \sin x$ ar
odd functions.

Property - 1 : The product of two even functions and that of two odd function
is always even whereas the product of an even and an odd function is alway
odd.

Property - 2 : $$\int_{-a}^{+a} \phi(x)\,dx = \begin{cases} 2\int_0^a \phi(x)\,dx & \text{if } \phi(x) \text{ is even} \\ 0 & \text{if } \phi(x) \text{ is odd} \end{cases}$$

Now, suppose the periodic function $f(x)$ is defined in the interval $(-\pi, \pi$
then the Fourier coefficients are given by

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx,\ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx,\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

Let us examine the following two cases.

Case - i : $f(x)$ is an even function

$f(x)\cos nx$ being the products of two even functions is also even an
$f(x)\sin nx$ being the product of an even and odd function is odd [*Property*-1]
Now applying property-2 to these integrals we have,

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx,\ a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx,\ b_n = 0$$

Case - ii : $f(x)$ is an odd function

$f(x)\cos nx$ will be an odd function and $f(x)\sin nx$ will be an even function.

Now by property - 2 we have,

$$a_0 = 0,\ a_n = 0,\ b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

Thus we can conclude that when $x \in (-\pi, \pi)$ if $f(x)$ is even $b_n = 0$ and if $f(x)$ is odd $a_0 = 0,\ a_n = 0$

Note : If $f(x) = \begin{cases} \phi(x) & in\ -\pi < x < 0 \\ \psi(x) & in\ \ 0 < x < \pi \end{cases}$

we say that $f(x)$ is even if $\phi(-x) = \psi(x)$ and $f(x)$ is odd if $\phi(-x) = -\psi(x)$

Examples

1. $f(x) = \begin{cases} -x & \text{in } -\pi < x < 0 \\ +x & \text{in } \ 0 < x < \pi \end{cases}$ is an even function

2. $f(x) = \begin{cases} 1 + \dfrac{2x}{\pi} & \text{in } -\pi < x < 0 \\ 1 - \dfrac{2x}{\pi} & \text{in } \ 0 < x < \pi \end{cases}$ is an even function

3. $f(x) = \begin{cases} x - \dfrac{\pi}{2} & \text{in } -\pi < x < 0 \\ x + \dfrac{\pi}{2} & \text{in } \ 0 < x < \pi \end{cases}$ is an odd function because

   if $\phi(x) = x - \dfrac{\pi}{2}$ then $\phi(-x) = -x - \dfrac{\pi}{2} = -\left(x + \dfrac{\pi}{2}\right) = -\psi(x)$

4. $f(x) = \begin{cases} -k & \text{in } -\pi < x < 0 \\ +k & \text{in } \ 0 < x < \pi \end{cases}$ is an odd function because

   if $\phi(x) = -k,$ then $\phi(-x) = -k = -\psi(x)$

5. $f(x) = |x|$ is an even function, since $|-x| = |x|$

The results are tabulated for ready reference where $f(x)$ is defined in $(-\pi, \pi)$.

Nature & condition of $f(x)$	a_0	a_n	b_n
Even function $f(-x) = f(x)$	$\frac{2}{\pi}\int_0^\pi f(x)dx$	$\frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx$	0
Odd function $f(-x) = -f(x)$	0	0	$\frac{2}{\pi}\int_0^\pi f(x)\sin nx\,dx$

Remark : *We have already worked problems to find the Fourier series of $f(x)$ directly by applying Euler's formulae when the interval of x is $(-\pi, \pi)$ or $(0, 2\pi)$. However, when the interval of x is $(-\pi, \pi)$ and also if $f(x)$ is either even or odd we can use the results as tabulated above to find the Fourier coefficients thus making the computation work much easier.*

WORKED PROBLEMS

[*As a matter of comparison we shall obtain the Fourier coefficients in three of the already worked problems using the concept of event and odd functions*]

Referring to Problem - [2] : $f(x) = x^2$ in $-\pi \le x \le \pi$

$f(x) = x^2$ and $f(-x) = (-x)^2 = x^2 = f(x)$. Hence $f(x)$ is even.

So we have,

$$a_0 = \frac{2}{\pi}\int_0^\pi f(x)dx,\ a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx,\ b_n = 0$$

$$a_0 = \frac{2}{\pi}\int_0^\pi x^2\,dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^\pi = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^\pi x^2\cos nx\,dx.$$

Applying Bernoulli's rule,

$$a_n = \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - (2x)\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^\pi$$

$$a_n = \frac{4}{\pi n^2}\left[x \cos nx \right]_0^\pi = \frac{4}{\pi n^2}(\pi \cos n\pi - 0) = \frac{4(-1)^n}{n^2}$$

Referring to Problem - [10] :

$$f(x) = \begin{cases} -k & \text{in } -\pi < x < 0 \\ +k & \text{in } 0 < x < \pi \end{cases}$$

Let $f(x) = \begin{cases} \phi(x) & \text{in } -\pi < x < 0 \\ \psi(x) & \text{in } 0 < x < \pi \end{cases}$ where,

$\phi(x) = -k$, $\psi(x) = +k$. Now $\phi(-x) = -k = -\psi(x)$

Therefore $f(x)$ is odd.

Hence we have, $a_0 = 0, a_n = 0$, $b_n = \frac{2}{\pi}\int_0^\pi f(x) \sin nx\, dx$

Now, $b_n = \frac{2}{\pi}\int_0^\pi k \sin nx\, dx = \frac{2k}{\pi}\left[\frac{-\cos nx}{n}\right]_0^\pi = \frac{-2k}{\pi n}(\cos n\pi - 1)$

$$b_n = \frac{2k}{\pi n}\{1-(-1)^n\}$$

Referring to Problem - [11] : $f(x) = \begin{cases} 1 + \frac{2x}{\pi} & \text{in } -\pi < x < 0 \\ 1 - \frac{2x}{\pi} & \text{in } 0 < x < \pi \end{cases}$

If $\phi(x) = 1 + \frac{2x}{\pi}$ and $\psi(x) = 1 - \frac{2x}{\pi}$ in $f(x)$, we have

$\phi(-x) = 1 - \frac{2x}{\pi} = \psi(x)$. Therefore $f(x)$ is even.

Hence we have , $a_0 = \frac{2}{\pi}\int_0^\pi f(x)\, dx$, $a_n = \frac{2}{\pi}\int_0^\pi f(x) \cos nx\, dx$, $b_n = 0$

Now, $a_0 = \frac{2}{\pi}\int_0^\pi \left(1 - \frac{2x}{\pi}\right) dx = \frac{2}{\pi}\left[x - \frac{x^2}{\pi}\right]_0^\pi = 0$; $a_0 = 0$

$$a_n = \frac{2}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)\cos nx\,dx.$$

Applying Bernoulli's rule,

$$a_n = \frac{2}{\pi}\left[\left(1-\frac{2x}{\pi}\right)\left(\frac{\sin nx}{n}\right)-\left(\frac{-2}{\pi}\right)\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{-4}{\pi^2 n^2}[\cos nx]_0^{\pi}$$

$$a_n = \frac{-4}{\pi^2 n^2}(\cos n\pi - 1) = \frac{4}{\pi^2 n^2}\{1-(-1)^n\}$$

[12] *Obtain the Fourier series for* $f(x) = \sin(mx)$ *in the range* $(-\pi, \pi)$ *when m is neither zero nor an integer.*

☞ The Fourier series of $f(x)$ having period 2π is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \ldots (1)$$

We shall check $f(x) = \sin(mx)$ for even or odd nature.

$$f(-x) = \sin(-mx) = -\sin mx = -f(x)$$

$\therefore$ $f(-x) = -f(x)$ and hence $f(x)$ is odd.

Consequently, $a_0 = 0$, $a_n = 0$.

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi}\sin mx \sin nx\,dx$$

Using, $\sin A \sin B = \frac{1}{2}\{\cos(A-B)-\cos(A+B)\}$ we get,

$$b_n = \frac{2}{\pi}\int_0^{\pi}\frac{1}{2}\{\cos(m-n)x - \cos(m+n)x\}\,dx$$

$$= \frac{1}{\pi}\left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n}\right]_0^{\pi}$$

Note : *It is important to observe that* $\sin k\pi = 0$ *only when k is an integer. since m is not an integer by data,* $\sin(m-n)\pi, \sin(m+n)\pi$ *are not equal to zero and the simplification is carried out as follows.*

$$b_n = \frac{1}{\pi}\left[\frac{1}{m-n}\{\sin(m-n)\pi - \sin 0\} - \frac{1}{m+n}\{\sin(m+n)\pi - \sin 0\}\right]$$

$$= \frac{1}{\pi}\left\{\frac{1}{m-n}\sin(m\pi - n\pi) - \frac{1}{m+n}\sin(m\pi + n\pi)\right\}$$

$$= \frac{1}{\pi}\left\{\frac{1}{m-n}(\sin m\pi\cos n\pi - \cos m\pi \sin n\pi)\right.$$

$$\left. - \frac{1}{m+n}(\sin m\pi \cos n\pi + \cos m\pi \sin n\pi)\right\}$$

$$= \frac{1}{\pi}\left\{\sin m\pi \cos n\pi\left(\frac{1}{m-n} - \frac{1}{m+n}\right)\right\}$$

(*Here* $\sin n\pi = 0$, *since* $n = 1, 2, \ldots$)

$$b_n = \frac{1}{\pi}\left\{\sin m\pi \cos n\pi\left(\frac{2n}{m^2-n^2}\right)\right\}$$

$$\therefore \quad b_n = \frac{2n(-1)^n \sin m\pi}{\pi(m^2-n^2)}$$

Thus by substituting the values of a_0, a_n, b_n in (1), the Fourier series is given by

$$\boxed{\sin(mx) = \sum_{n=1}^{\infty} \frac{2n(-1)^n \sin m\pi}{\pi(m^2-n^2)} \sin nx}$$

[13] *Obtain the Fourier series in* $(-\pi, \pi)$ *for* $f(x) = x\cos x$

☞ The Fourier series of $f(x)$ having period 2π is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \ldots (1)$$

We shall check $f(x) = x\cos x$ for even or odd nature.

$$f(-x) = -x\cos(-x) = -x\cos x$$

$$\therefore \quad f(-x) = -f(x) \text{ and hence } f(x) \text{ is odd.}$$

Consequently, $a_0 = 0$, $a_n = 0$.

$$b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^\pi x\cos x\sin nx\,dx$$

ie., $$b_n = \frac{2}{\pi}\int_0^\pi x\sin nx\cos x\,dx \qquad \ldots (2$$

Using, $\sin A\cos B = \frac{1}{2}\{\sin(A+B)+\sin(A-B)\}$ we have,

$$b_n = \frac{2}{\pi}\int_0^\pi x\cdot\frac{1}{2}\{\sin(nx+x)+\sin(nx-x)\}\,dx$$

$$= \frac{1}{\pi}\left\{\int_0^\pi x\sin(n+1)x\,dx + \int_0^\pi x\sin(n-1)x\,dx\right\}$$

Applying Bernoulli's rule to each of the integrals,

$$b_n = \frac{1}{\pi}\left[x\cdot-\frac{\cos(n+1)x}{n+1} - 1\cdot-\frac{\sin(n+1)x}{(n+1)^2}\right]_0^\pi$$

$$+\frac{1}{\pi}\left[x\cdot-\frac{\cos(n-1)x}{n-1} - 1\cdot-\frac{\sin(n-1)x}{(n-1)^2}\right]_0^\pi, \ (n \neq 1$$

$$= \frac{1}{\pi}\left\{\frac{-1}{n+1}[\pi\cos(n+1)\pi - 0] - \frac{1}{n-1}[\pi\cos(n-1)\pi - 0]\right\}$$

Here, $\sin(n+1)\pi = 0 = \sin(n-1)\pi$, since $n = 1, 2, 3, \ldots$

$$b_n = -\left\{\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1}\right\}$$

$$= -\left\{\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1}\right\}$$

$$= (-1)^n\left\{\frac{1}{n+1} + \frac{1}{n-1}\right\} = \frac{(-1)^n\,2n}{n^2-1}$$

$$b_n = \frac{2n(-1)^n}{n^2-1} \quad (n \neq 1)$$

$\therefore$

We shall now find b_n when $n = 1$. That is to find b_1.
Let us consider b_n as given by (2) :

$$b_n = \frac{2}{\pi}\int_0^\pi x \sin nx \cos x \, dx.$$ Putting $n = 1$ we get,

$$b_1 = \frac{2}{\pi}\int_0^\pi x \sin x \cos x \, dx = \frac{2}{\pi}\int_0^\pi x \frac{\sin 2x}{2} \, dx$$

$$b_1 = \frac{1}{\pi}\int_0^\pi x \sin 2x \, dx.$$

$$b_1 = \frac{1}{\pi}\left[x\left(\frac{-\cos 2x}{2}\right) - (1)\left(\frac{-\sin 2x}{4}\right)\right]_0^\pi$$

$$= \frac{-1}{2\pi}[x \cos 2x]_0^\pi \quad \text{since} \quad \sin 2\pi = 0 = \sin 0.$$

$$= \frac{-1}{2\pi}(\pi \cos 2\pi - 0) = \frac{-1}{2}, \quad \text{since} \quad \cos 2\pi = 1.$$

$\therefore \quad b_1 = -1/2$

We shall write (1) in the form,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_1 \sin x + \sum_{n=2}^{\infty} b_n \sin nx$$

Thus by substituting the values for the Fourier coefficients we have,

$$\boxed{x \cos x = \frac{-1}{2}\sin x + \sum_{n=2}^{\infty} \frac{2n(-1)^n}{n^2-1} \sin nx}$$

[14] *Expand the function $f(x) = x \sin x$ as a Fourier series in the interval $-\pi \leq x \leq \pi$. Deduce that,*

$$\frac{1}{1 \cdot 3} - \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} - \cdots = \frac{\pi - 2}{4}$$

☞ The Fourier series of $f(x)$ having period 2π is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \dots (1)$$

We shall check $f(x) = x \sin x$ for even or odd nature.

$$f(-x) = (-x)\sin(-x) = (-x)(-\sin x) = x \sin x$$

$\therefore \quad f(-x) = f(x)$ and hence $f(x)$ is even. Consequently $b_n = 0$.

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx\,;\ a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x \sin x\,dx.$$

$$a_0 = \frac{2}{\pi}\left[\,x(-\cos x) - (1)(-\sin x)\right]_0^{\pi}.\ \text{But } \sin\pi = 0 = \sin 0$$

$$a_0 = \frac{-2}{\pi}(\pi\cos\pi - 0) = 2$$

$\therefore \quad a_0/2 = 1$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos nx\,dx \quad \dots (2)$$

Using, $\sin A \cos B = \frac{1}{2}\{\sin(A+B) + \sin(A-B)\}$, we have,

$$a_n = \frac{2}{\pi}\int_0^{\pi} x \cdot \frac{1}{2}\{\sin(x+nx) + \sin(x-nx)\}\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x \cdot \{\sin(1+n)x + \sin(1-n)x\}\,dx.$$

But, $\sin(1-n)x = \sin\{-(n-1)x\} = -\sin(n-1)x$

$$\therefore \quad a_n = \frac{1}{\pi}\int_0^{\pi} x \sin(n+1)x\,dx - \frac{1}{\pi}\int_0^{\pi} x \sin(n-1)x\,dx$$

Applying Bernoulli's rule to each of the integrals,

$$a_n = \frac{1}{\pi}\left[x\cdot\frac{-\cos(n+1)x}{n+1} - (1)\cdot\frac{-\sin(n+1)x}{(n+1)^2}\right]_0^\pi$$

$$-\frac{1}{\pi}\left[x\cdot\frac{-\cos(n-1)x}{n-1} - (1)\cdot\frac{-\sin(n-1)x}{(n-1)^2}\right]_0^\pi, \ (n \neq 1)$$

But, $\sin(n+1)\pi = 0 = \sin(n-1)\pi;\ \sin 0 = 0.$

$$a_n = \frac{1}{\pi}\left[\frac{-1}{n+1}x\cos(n+1)x\right]_0^\pi + \frac{1}{\pi}\left[\frac{x\cos(n-1)x}{n-1}\right]_0^\pi$$

$$= \frac{1}{\pi}\left\{\frac{-1}{n+1}[\pi\cos(n+1)\pi - 0]\right\} + \frac{1}{\pi}\left\{\frac{1}{n-1}[\pi\cos(n-1)\pi - 0]\right\}$$

$$= -\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1}$$

$$= -1\cdot\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1}, \text{ since } \cos k\pi = (-1)^k$$

$$= \frac{(-1)^{n+2}}{n+1} + \frac{(-1)^{n-1}}{n-1} = (-1)^n\left\{\frac{(-1)^2}{n+1} + \frac{(-1)^{-1}}{n-1}\right\}.$$

But, $(-1)^{-1} = 1/-1 = -1$

$$a_n = (-1)^n\left\{\frac{1}{n+1} - \frac{1}{n-1}\right\} = (-1)^n\cdot\frac{-2}{n^2-1} = \frac{2(-1)^{n+1}}{n^2-1}$$

$$\therefore \quad a_n = \frac{2(-1)^{n+1}}{n^2-1} \text{ where } n \neq 1$$

We shall now find a_n when $n = 1$. That is to find a_1.
Let us consider a_n as given by (2). Putting $n = 1$ we have,

$$a_1 = \frac{2}{\pi}\int_0^\pi x\sin x\cos x\,dx = \frac{2}{\pi}\int_0^\pi x\frac{\sin 2x}{2}dx$$

$$a_1 = \frac{1}{\pi}\int_0^\pi x\sin 2x\,dx = -\frac{1}{2} \text{ (As in } b_1 \text{ of Problem - [13])}$$

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$\therefore \quad a_1 = -1/2$

Substituting $b_n = 0$ in (1) we have, $f(x) = \dfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

ie., $\quad f(x) = \dfrac{a_0}{2} + a_1 \cos x + \sum_{n=2}^{\infty} a_n \cos nx$

Now substituting the values of $a_0/2$, a_1, a_n the required Fourier series is given by

$$\boxed{x \sin x = 1 - \frac{1}{2}\cos x + \sum_{n=2}^{\infty} \frac{2(-1)^{n+1}}{n^2 - 1} \cos nx}$$

To deduce the series, let us put $x = \pi/2$.

$$\therefore \quad \frac{\pi}{2}\sin\frac{\pi}{2} = 1 - \frac{1}{2}\cos\frac{\pi}{2} + 2\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n^2-1}\cos\frac{n\pi}{2}$$

ie., $\quad \dfrac{\pi}{2} = 1 + 2\sum_{n=2}^{\infty} \dfrac{(-1)^{n+1}}{n^2-1}\cos\dfrac{n\pi}{2}$, since $\sin\dfrac{\pi}{2} = 1, \cos\dfrac{\pi}{2} = 0$

ie., $\quad \dfrac{\pi}{2} - 1 = 2\left(\dfrac{-1}{3}\cos\pi + \dfrac{1}{8}\cos\dfrac{3\pi}{2} - \dfrac{1}{15}\cos 2\pi + \cdots\right)$

ie., $\dfrac{\pi - 2}{2} = 2\left(\dfrac{1}{3} - \dfrac{1}{15} + - \cdots\right)$, since $\cos\pi = -1, \cos 2\pi = 1, \cos(3\pi/2) = 0$

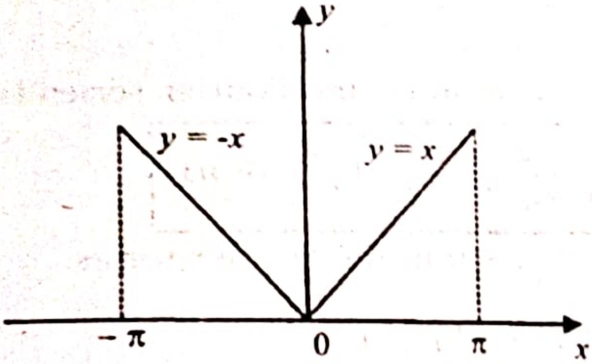
Thus, $$\boxed{\frac{\pi-2}{4} = \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \cdots}$$

[15] *Sketch the graph of the function* $f(x) = |x|$ *in* $-\pi \le x \le \pi$ *and obtain its Fourier series. Hence deduce that* $\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \cdots = \dfrac{\pi^2}{8}$

☞ $f(x) = |x|$ in $-\pi \le x \le \pi$ means that the function must be positive in the given interval which consists of negative values and positive values. Hence the given $f(x)$ may be split into the form,

$$f(x) = \begin{cases} -x & \text{in } -\pi \le x \le 0 \\ +x & \text{in } 0 \le x \le \pi \end{cases}$$

The equations $y = x$ and $y = -x$ represent straight lines through the origin with slopes 1, −1 (*Lines subtending* 45^0, 135^0 *with x-axis*) and the graph is as follows.



The Fourier series of $f(x)$ having period 2π is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \ldots (1)$$

We shall check $f(x) = |x|$ for even or odd nature.

$$f(-x) = |-x| = |x| = f(x)$$

$\therefore \quad f(-x) = f(x)$ and hence $f(x)$ is even. Consequently $b_n = 0$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx\,; \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

Here, $f(x) = |x| = x \quad \text{for} \quad x \in (0, \pi)$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \frac{1}{\pi}(\pi^2 - 0) = \pi.$$

$\therefore \quad a_0/2 = \pi/2$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx.$$

Applying Bernoulli's rule,

$$a_n = \frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - (1)\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi n^2}\left[\cos nx\right]_0^\pi, \text{ since } \sin n\pi = 0 = \sin 0.$$

$$= \frac{2}{\pi n^2}(\cos n\pi - \cos 0) = \frac{2}{\pi n^2}(\cos n\pi - 1) = \frac{-2}{\pi n^2}(1 - \cos n\pi)$$

$$a_n = \frac{-2}{\pi n^2}\{1-(-1)^n\}$$

Substituting the values of a_0, a_n, b_n in (1) the Fourier series is given by

$$\boxed{f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{-2}{\pi n^2}\{1-(-1)^n\}\cos nx}$$

To deduce the series let us put $x = 0$ in the Fourier series.

$$f(0) = \frac{\pi}{2} - \frac{2}{\pi}\sum_{1}^{\infty}\frac{1}{n^2}\{1-(-1)^n\}\cdot 1$$

ie.,
$$0 = \frac{\pi}{2} - \frac{2}{\pi}\sum_{1}^{\infty}\frac{1}{n^2}\{1-(-1)^n\}$$

$$-\frac{\pi}{2} = -\frac{2}{\pi}\sum_{1}^{\infty}\frac{1}{n^2}\{1-(-1)^n\} \text{ or } \frac{\pi^2}{4} = \sum_{1}^{\infty}\frac{1}{n^2}\{1-(-1)^n\}.$$

But,
$$1-(-1)^n = \begin{cases} 1-(+1) = 0 & \text{if } n \text{ is even} \\ 1-(-1) = 2 & \text{if } n \text{ is odd} \end{cases}$$

Hence we get,
$$\frac{\pi^2}{4} = \sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^2}(2)$$

Thus,
$$\boxed{\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots}$$

[16] *Find the Fourier series in $(-\pi, \pi)$ for the function $f(x)$ defined by*

$$f(x) = \begin{cases} \pi + x & in \quad (-\pi, -\pi/2) \\ \pi/2 & in \quad (-\pi/2, \pi/2) \\ \pi - x & in \quad (\pi/2, \pi) \end{cases}$$

☞ $f(x)$ is defined in $(-\pi, \pi)$ and the Fourier series of period 2π is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \ldots (1)$$

We shall check $f(x)$ for even or odd nature by writing $f(x)$ as follows.

Interval of x	$(-\pi, -\pi/2)$	$(-\pi/2, 0)$	$(0, \pi/2)$	$(\pi/2, \pi)$
$f(x)$	$\pi + x$	$\pi/2$	$\pi/2$	$\pi - x$

ie., $$f(x) = \begin{cases} \phi(x) & \text{in } (-\pi, 0) \\ \psi(x) & \text{in } (0, \pi) \end{cases}$$

where, $\phi(x) = \pi + x$ or $\pi/2$ and $\psi(x) = \pi - x$ or $\pi/2$

$\therefore$ $\phi(-x) = \pi - x$ or $\pi/2 = \psi(x)$ and hence we conclude that $f(x)$ is even.

Consequently $b_n = 0$.

$$a_0 = \frac{2}{\pi}\int_0^\pi f(x)\,dx, \quad a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx$$

$$a_0 = \frac{2}{\pi}\left\{\int_0^{\pi/2} f(x)\,dx + \int_{\pi/2}^{\pi} f(x)\,dx\right\} = \frac{2}{\pi}\left\{\int_0^{\pi/2}\frac{\pi}{2}\,dx + \int_{\pi/2}^{\pi}(\pi - x)\,dx\right\}$$

$$a_0 = \frac{2}{\pi}\left\{\frac{\pi}{2}[\,x\,]_0^{\pi/2} + \left[\pi x - \frac{x^2}{2}\right]_{\pi/2}^{\pi}\right\}$$

$$= \frac{2}{\pi}\left\{\frac{\pi}{2}\left(\frac{\pi}{2} - 0\right) + \left(\pi^2 - \frac{\pi^2}{2}\right) - \left(\frac{\pi^2}{2} - \frac{\pi^2}{8}\right)\right\} = \frac{2}{\pi}\cdot\frac{3\pi^2}{8} = \frac{3\pi}{4}$$

$$\therefore \quad a_0/2 = 3\pi/8$$

$$a_n = \frac{2}{\pi}\left\{\int_0^{\pi/2} f(x)\cos nx\,dx + \int_{\pi/2}^{\pi} f(x)\cos nx\,dx\right\}$$

$$= \frac{2}{\pi}\left\{\int_0^{\pi/2}\frac{\pi}{2}\cos nx\,dx + \int_{\pi/2}^{\pi}(\pi - x)\cos nx\,dx\right\}$$

$$= \frac{2}{\pi}\left\{\frac{\pi}{2}\left[\frac{\sin nx}{n}\right]_0^{\pi/2} + \left[(\pi - x)\frac{\sin nx}{n}\right]_{\pi/2}^{\pi} - \left[(-1)\cdot - \frac{\cos nx}{n^2}\right]_{\pi/2}^{\pi}\right\}$$

$$= \frac{2}{\pi}\left\{\frac{\pi}{2n}\left(\sin\frac{n\pi}{2} - 0\right) + \frac{1}{n}\left(0 - \frac{\pi}{2}\sin\frac{n\pi}{2}\right) - \frac{1}{n^2}\left(\cos n\pi - \cos\frac{n\pi}{2}\right)\right.$$

$$= \frac{-2}{\pi n^2}\left(\cos n\pi - \cos\frac{n\pi}{2}\right)$$

$$\therefore \quad a_n = \frac{-2}{\pi n^2}\left\{(-1)^n - \cos\frac{n\pi}{2}\right\}$$

Thus by substituting the values of a_0, a_n, b_n in (1), the required Fourier seri is given by

$$\boxed{f(x) = \frac{3\pi}{8} + \sum_{n=1}^{\infty} \frac{-2}{\pi n^2}\left\{(-1)^n - \cos\frac{n\pi}{2}\right\}\cos nx}$$

2.5 Even and odd nature of $f(x)$ in $(0, 2\pi)$ & associated Fourier serie

$f(x)$ is said to be even if $f(2\pi - x) = f(x)$ and odd if $f(2\pi - x) = -f(x)$

We note that $\cos x$ is an even function since $\cos(2\pi - x) = \cos x$ and sin is an odd function since $\sin(2\pi - x) = -\sin x$.

Further we have the standard integral property :

$$\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{if } f(x) \text{ is even} \\ 0 & \text{if } f(x) \text{ is odd} \end{cases}$$

If $f(x)$ is a periodic function of period 2π defined in $(0, 2\pi)$ then the Fourier coefficients are given by

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx, \quad a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx, \quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

Let us examine the following two cases.

Case - i : $f(x)$ is an even function.

$f(x)\cos nx$ being the product of two even functions is also even and $f(x)\sin nx$ being the product of an even and an odd function is odd.

The Fourier coefficients with the application of the integral property becomes

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx, \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx, \quad b_n = 0$$

Case - ii : $f(x)$ is an odd function.
$f(x)\cos nx$ will be an odd function and $f(x)\sin nx$ will be an even function. Accordingly the Fourier coefficients with the application of the integral property becomes

$$a_0 = 0,\ a_n = 0,\ b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

Thus we can conclude that when $x \in (0, 2\pi)$, if $f(x)$ is even $b_n = 0$ and if $f(x)$ is odd $a_0 = 0,\ a_n = 0$.

Further it should be observed that the result in respect of the Fourier coefficients involving $f(x)$ defined in $(0, 2\pi)$ are same as in the case of $f(x)$ defined in $(-\pi, \pi)$ where $f(x)$ is even or odd in the relevant interval.

Also it may be noted that if $f(x) = \begin{cases} \phi(x) & \text{in } 0 < x < \pi \\ \psi(x) & \text{in } \pi < x < 2\pi \end{cases}$

We say that $f(x)$ is even if $\phi(2\pi - x) = \psi(x)$ and $f(x)$ is odd if $\phi(2\pi - x) = -\psi(x)$.

WORKED PROBLEMS

[*As a matter of comparision we briefly discuss three of the already worked problems using the concept of even and odd functions*]

Referring to Problem - [1] : $f(x) = \dfrac{\pi - x}{2}$ in $(0, 2\pi)$

$$f(2\pi - x) = \frac{\pi - (2\pi - x)}{2} = \frac{-\pi + x}{2} = \frac{-(\pi - x)}{2} = -f(x)$$

$\therefore$ $f(x)$ is odd in $(0, 2\pi)$ and hence $a_0 = 0,\ a_n = 0$.

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} \frac{\pi - x}{2}\sin nx\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} (\pi - x)\sin nx\,dx = \frac{1}{n},\ \text{on integration.}$$

Referring to Problem - [3] : $f(x) = x(2\pi - x)$ in $0 \le x \le 2\pi$

$$f(2\pi - x) = (2\pi - x)(2\pi - \overline{2\pi - x}) = (2\pi - x)x = f(x)$$

$\therefore$ $f(x)$ is even in $(0, 2\pi)$ and hence $b_n = 0$.

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx, \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi}(2\pi x - x^2)\,dx = \frac{4\pi^2}{3}, \text{ on integration.}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi}(2\pi x - x^2)\cos nx\,dx = \frac{-4}{n^2}, \text{ on integration.}$$

Referring to Problem - [6] : $f(x) = \begin{cases} x, & 0 \le x \le \pi \\ 2\pi - x, & \pi \le x \le 2\pi \end{cases}$

Let $f(x) = \begin{cases} \phi(x) = x & \text{in } 0 \le x \le \pi \\ \psi(x) = 2\pi - x & \text{in } \pi \le x \le 2\pi \end{cases}$

$$\phi(2\pi - x) = 2\pi - x = \psi(x)$$

$\therefore$ $f(x)$ is even in $(0, 2\pi)$ and hence $b_n = 0$.

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx, \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx = \pi, \text{on integration.}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx = \frac{-2}{\pi n^2}\{1 - (-1)^n\}, \text{ on integration.}$$

[17] *Obtain the Fourier series representation in* $(0, 2\pi)$ *of the function* $f(x)$ *defined by*

$$f(x) = \begin{cases} x^2 & in\ (0, \pi) \\ -(2\pi - x)^2 & in\ (\pi, 2\pi) \end{cases}$$

☞ In the given $f(x)$, let $\phi(x) = x^2$ and $\psi(x) = -(2\pi - x)^2$

Now, $\phi(2\pi - x) = (2\pi - x)^2 = -\psi(x)$

$f(x)$ is odd in $(0, 2\pi)$ and hence $a_0 = 0,\ a_n = 0$.

$$\therefore \quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} x^2 \sin nx\,dx$$

$$b_n = \frac{2}{\pi}\left[x^2\left(\frac{-\cos nx}{n}\right) - (2x)\left(\frac{-\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{-1}{n}(\pi^2\cos n\pi - 0) + 0 + \frac{2}{n^3}(\cos n\pi - 1)\right]$$

$$b_n = \frac{2\pi}{n}(-1)^{n+1} + \frac{4}{\pi n^3}\{(-1)^n - 1\}$$

Thus the Fourier series representation of $f(x)$ is given by

$$\boxed{f(x) = \sum_{n=1}^{\infty}\left[\frac{2\pi}{n}(-1)^{n+1} + \frac{4}{\pi n^3}\{(-1)^n - 1\}\right]\sin nx.}$$

[18] *An alternating current after passing through a full wave rectifier has the form $I = I_0|\sin t|,\ 0 < t < 2\pi$ where I_0 is the maximum current. Express I as a Fourier series and hence deduce that* $\frac{1}{2} = \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots$

☞ By data, $I = f(t) = I_0|\sin t|$ in $(0, 2\pi)$

ie., $$I = f(t) = \begin{cases} I_0 \sin t & \text{in } (0,\pi) \\ -I_0 \sin t & \text{in } (\pi, 2\pi) \end{cases}$$

since, sin$t$ is positive if $0 < t < \pi$ and negative if $\pi < t < 2\pi$.

Let, $$I = f(t) = \begin{cases} \phi(t) = I_0 \sin t & \text{in } (0,\pi) \\ \psi(t) = -I_0 \sin t & \text{in } (\pi, 2\pi) \end{cases}$$

$$\phi(2\pi - t) = I_0 \sin(2\pi - t) = -I_0 \sin t = \psi(t)$$

$\therefore$ $f(t)$ is even in $(0, 2\pi)$ and $b_n = 0$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(t)\,dt, \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(t)\cos nt\,dt$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} I_0 \sin t\,dt = \frac{2I_0}{\pi}\left[-\cos t\right]_0^{\pi} = \frac{-2I_0}{\pi}(-1-1) = \frac{4I_0}{\pi}$$

$$\therefore \quad a_0/2 = 2I_0/\pi$$

$$a_n = \frac{2I_0}{\pi}\int_0^{\pi} \sin t \cos nt\,dt$$

Referring to Problem - [8] for the integration process we have,

$$a_n = \frac{-2I_0}{\pi(n^2-1)}\left\{1+(-1)^n\right\} \text{ for } n \neq 1 \text{ and } a_1 = 0.$$

We have Fourier series in the form,

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt$$

ie., $$f(t) = \frac{2I_0}{\pi} + \sum_{n=2}^{\infty} \frac{-2I_0}{\pi(n^2-1)}\left\{1+(-1)^n\right\}\cos nt$$

Thus, $$\boxed{f(t) = \frac{2I_0}{\pi} - \frac{4I_0}{\pi}\sum_{n=2,4,6\cdots}^{\infty} \frac{\cos nt}{n^2-1}}$$

To deduce the series let us put $t = 0$.

$$f(0) = \frac{f(0)+f(2\pi)}{2} = \frac{0+0}{2} = 0$$

The Fourier series becomes,

$$0 = \frac{2I_0}{\pi} - \frac{4I_0}{\pi}\sum_{n=2,4,6\cdots}^{\infty} \frac{1}{n^2-1}$$

ie., $$\frac{-2I_0}{\pi} = \frac{-4I_0}{\pi}\left[\frac{1}{3}+\frac{1}{15}+\frac{1}{35}+\cdots\right]$$

Thus, $$\boxed{\frac{1}{2} = \frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\cdots}$$

[9] *Obtain the Fourier series expansion of the function*

$$f(x) = \begin{cases} x & in \quad 0 < x < \pi \\ x - 2\pi & in \quad \pi < x < 2\pi \end{cases}$$

Hence deduce that, $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots$.

☞ The Fourier series of period 2π is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

In the given $f(x)$, let $\phi(x) = x$ and $\psi(x) = x - 2\pi$

$$\phi(2\pi - x) = 2\pi - x = -(x - 2\pi) = -\psi(x)$$

$\therefore$ $f(x)$ is odd in $(0, 2\pi)$ and hence $a_0 = 0$, $a_n = 0$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx = \frac{2}{\pi}\int_0^{\pi} x \sin nx \, dx$$

$$b_n = \frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - (1)\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$b_n = \frac{-2}{\pi n}\pi \cos n\pi = \frac{-2}{n}(-1)^n$$

$$\therefore \quad b_n = \frac{2(-1)^{n+1}}{n}$$

Thus the required Fourier series is given by

$$\boxed{f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin nx}$$

To deduce the series let us put $x = \pi/2$. Then $f(x) = \pi/2$ since $f(x) = x$ in $(0, \pi)$. Hence the Fourier series becomes

$$\frac{\pi}{2} = 2\sum_{1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi}{2}$$

Expanding and noting that $\sin(3\pi/2) = -1$, $\sin(5\pi/2) = 1$, we get

$$\boxed{\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots}$$

[20] *If* $f(x) = \left(\frac{\pi - x}{2}\right)^2$ *in* $0 < x < 2\pi$, *show that*

$$f(x) = \frac{\pi^2}{12} + \sum_{1}^{\infty} \frac{\cos nx}{n^2}.$$ *Hence deduce that*

(i) $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$

(ii) $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$

(iii) $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$

☞ The Fourier series of period 2π is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Consider, $f(x) = \left(\frac{\pi - x}{2}\right)^2$

$$f(2\pi - x) = \left(\frac{\pi - \overline{2\pi - x}}{2}\right)^2 = \left(\frac{x - \pi}{2}\right)^2 = \left(\frac{\pi - x}{2}\right)^2 = f(x)$$

$\therefore$ $f(x)$ is even in $(0, 2\pi)$ and hence $b_n = 0$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx, \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi}\frac{(\pi - x)^2}{4}\,dx = \frac{1}{2\pi}\left[\frac{(\pi - x)^3}{-3}\right]_0^{\pi} = \frac{-1}{6\pi}(0 - \pi^3) = \frac{\pi^2}{6}$$

$\therefore$ $a_0/2 = \pi^2/12$

$$a_n = \frac{2}{\pi}\int_0^\pi \frac{(\pi-x)^2}{4}\cos nx\,dx = \frac{1}{2\pi}\int_0^\pi (\pi-x)^2\cos nx\,dx$$

Applying Bernoulli's rule we get,

$$a_n = \frac{1}{2\pi}\left[(\pi-x)^2\left(\frac{\sin nx}{n}\right)-(-2)(\pi-x)\left(\frac{-\cos nx}{n^2}\right)+(2)\left(\frac{-\sin nx}{n^3}\right)\right]_0^\pi$$

$$= \frac{-1}{\pi n^2}\left[(\pi-x)\cos nx\right]_0^\pi = \frac{-1}{\pi n^2}(0-\pi) = \frac{1}{n^2}$$

$\therefore \quad a_n = 1/n^2$

Thus the required Fourier series is given by

$$\boxed{f(x) = \left(\frac{\pi-x}{2}\right)^2 = \frac{\pi^2}{12}+\sum_1^\infty \frac{\cos nx}{n^2}}$$

To deduce the series, we first put $x = 0$

$$f(0) = \frac{f(0)+f(2\pi)}{2} = \frac{(\pi/2)^2+(-\pi/2)^2}{2} = \frac{\pi^2}{4}$$

Hence the Fourier series becomes,

$$\frac{\pi^2}{4} = \frac{\pi^2}{12}+\sum_1^\infty \frac{1}{n^2} \quad \text{or} \quad \frac{\pi^2}{4}-\frac{\pi^2}{12} = \sum_1^\infty \frac{1}{n^2}$$

Thus, $$\boxed{\frac{\pi^2}{6} = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots} \qquad \ldots (1)$$

Next let us put $x = \pi$ in the Fourier series.

$$\therefore \quad 0 = \frac{\pi^2}{12}+\sum_1^\infty \frac{\cos n\pi}{n^2} \quad \text{or} \quad \frac{-\pi^2}{12} = \sum_1^\infty \frac{(-1)^n}{n^2}$$

Thus, $$\boxed{\frac{\pi^2}{12} = \frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots} \qquad \ldots (ii)$$

Adding (i) and (ii) we obtain

$$\boxed{\frac{\pi^2}{8} = \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots} \qquad \ldots (iii)$$

Note : Similar Problem

Expand in Fourier series $f(x) = (\pi - x)^2$ *over the interval* $0 \le x \le 2\pi$.

[Dec 2017, 18

2.6 Fourier series of arbitrary period

A function $f(x)$ need not always be defined in an interval of length 2π only When the length of the interval is other than 2π, we shall denote it by $2l$ A general interval of length $2l$ be $(c, c+2l)$. It is important to note that the sine and cosine functions of the form $\sin\left(\frac{\pi x}{l}\right)$ and $\cos\left(\frac{\pi x}{l}\right)$ are periodic functions of period $2l$. It is justified as follows.

Let, $F(x) = \sin\left(\frac{\pi x}{l}\right)$; $G(x) = \cos\left(\frac{\pi x}{l}\right)$

Then, $F(x+2l) = \sin\left[\frac{\pi}{l}(x+2l)\right] = \sin\left(\frac{\pi x}{l} + 2\pi\right)$

$$= \sin\left(\frac{\pi x}{l}\right) = F(x)$$

Similarly, $G(x+2l) = G(x)$.

As already discussed in the article 2.3, the trigonometric series is of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

If $f(x)$ is defined in $(c, c+2l)$ satisfies Direchlet's condition then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

is called as the ***Fourier series of arbitrary period 2l.***

Proceeding on similar lines as in the article 2.3, we can establish Euler's formulae for the Fourier coefficients in the form

$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx,\quad a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx,$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

Step by step working procedure for problems

Step - 1 If the period of the given function is other than 2π we first equate the period to $2l$ and obtain the value l.

Step - 2 We then write the appropriate Fourier series and compute a_0, a_n, b_n associated with it.

Step - 3 However if $f(x)$ is defined in an interval of the form $(-l, l)$ or $(0, 2l)$ we can compute a_0, a_n, b_n using the concept of even and odd functions taking the following table into consideration.

Nature & Condition of $f(x)$	a_0	a_n	b_n
Even function $f(-x) = f(x)$ or $f(2l-x) = f(x)$	$\frac{2}{l}\int_0^l f(x)\,dx$	$\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$	0
Odd function $f(-x) = -f(x)$ or $f(2l-x) = -f(x)$	0	0	$\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$

WORKED PROBLEMS

[21] *Obtain the Fourier series of* $f(x) = |x|$ in $(-l, l)$.

Hence show that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$

☞ The period of $f(x) = l - (-l) = 2l$ and the Fourier series of period $2l$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \qquad \ldots (1)$$

We shall check $f(x) = |x|$ for even or odd nature.

$$f(-x) = |-x| = |x| = f(x)$$

Hence $f(x)$ is even and consequently $b_n = 0$.

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx,\quad a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$$a_0 = \frac{2}{l}\int_0^l x\,dx = \frac{2}{l}\left[\frac{x^2}{2}\right]_0^l = \frac{1}{l}(l^2 - 0) = l$$

$\therefore \quad a_0/2 = l/2$

$a_n = \dfrac{2}{l}\displaystyle\int_0^l x\cos\frac{n\pi x}{l}\,dx$. Applying Bernoulli's rule,

$$a_n = \frac{2}{l}\left[x\cdot\frac{\sin\frac{n\pi x}{l}}{(n\pi/l)} - (1)\cdot -\frac{\cos\frac{n\pi x}{l}}{(n\pi/l)^2}\right]_0^l.$$

$$= \frac{2}{l}\frac{l^2}{n^2\pi^2}\left[\cos\frac{n\pi x}{l}\right]_0^l = \frac{2l}{n^2\pi^2}(\cos n\pi - 1),\ \text{since, } \sin n\pi = 0 = \sin 0$$

$$\therefore \quad a_n = \frac{-2l}{n^2\pi^2}\{1-(-1)^n\}$$

Thus the required Fourier series is given by

$$\boxed{f(x) = \frac{l}{2} + \sum_{n=1}^{\infty}\frac{-2l}{n^2\pi^2}\{1-(-1)^n\}\cos\frac{n\pi x}{l}}$$

To deduce the series we shall put $x = 0$ which gives $f(x) = 0$ and the Fourier series becomes,

$$0 = \frac{l}{2} - \frac{2l}{\pi^2}\sum_1^{\infty}\frac{1}{n^2}\{1-(-1)^n\}\cdot 1$$

ie., $$\frac{-l}{2} = \frac{-2l}{\pi^2}\sum_1^{\infty}\frac{1}{n^2}\{1-(-1)^n\} \text{ or } \frac{\pi^2}{4} = \sum_{n=1}^{\infty}\frac{1}{n^2}\{1-(-1)^n\}$$

But, $$1-(-1)^n = \begin{cases} 1-(+1) = 0 & \text{when } n \text{ is even} \\ 1-(-1) = 2 & \text{when } n \text{ is odd} \end{cases}$$

$$\therefore \quad \frac{\pi^2}{4} = \sum_{n=1,3,5,\cdots}^{\infty} \frac{1}{n^2}\cdot 2 \quad \text{or} \quad \boxed{\frac{\pi^2}{8} = \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots}$$

Note : Similar Problem

Obtain the Fourier series expansion of $f(x) = |x|$ on $(-\pi,\pi)$ and hence deduce that $\dfrac{\pi^2}{8} = \dfrac{1}{1^2}+\dfrac{1}{3^2}+\dfrac{1}{5^2}+\cdots$ **[June 2017, Dec. 18]**

[22] *Obtain the Fourier series to represent $f(x) = x - x^2$ in $-1<x<1$.*

☞ Here period of $f(x) = 1-(-1) = 2 \quad \therefore \; 2l = 2$ or $l = 1$.

The Fourier series of $f(x)$ having period 2 is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x) + \sum_{n=1}^{\infty} b_n \sin(n\pi x)$$

Since the interval is $(-1,1)$ let us check the given function for even or odd nature.

$f(x) = x - x^2 \; \therefore \; f(-x) = -x-(-x)^2 = -x - x^2$ which is neither equal to $f(x)$ nor equal to $-f(x)$. So $f(x)$ is neither even nor odd.

We shall find the Fourier coefficients by Euler's formulae.

$$a_0 = \frac{1}{1}\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1}(x-x^2)\,dx = \left[\frac{x^2}{2}-\frac{x^3}{3}\right]_{-1}^{1}$$

ie., $$a_0 = \left(\frac{1}{2}-\frac{1}{3}\right)-\left(\frac{1}{2}+\frac{1}{3}\right) = \frac{-2}{3}$$

$$\therefore \quad a_0/2 = -1/3$$

$$a_n = \frac{1}{1}\int_{-1}^{1} f(x)\cos(n\pi x)\,dx$$

$a_n = \displaystyle\int_{-1}^{1}(x-x^2)\cos(n\pi x)\,dx$. Applying Bernoulli's rule,

$$\therefore \quad a_n = \left[(x-x^2)\frac{\sin(n\pi x)}{n\pi} - (1-2x)\cdot -\frac{\cos(n\pi x)}{n^2\pi^2} + (-2)\cdot -\frac{\sin(n\pi x)}{n^3\pi^3}\right]_{-1}^{1}$$

$$= \frac{1}{n^2\pi^2}\left[(1-2x)\cos(n\pi x)\right]_{-1}^{1}, \text{ since } \sin n\pi = 0.$$

$$= \frac{1}{n^2\pi^2}\{-\cos n\pi - 3\cos n\pi\} = \frac{-4\cos n\pi}{n^2\pi^2} = \frac{-4(-1)^n}{n^2\pi^2}$$

$$\therefore \quad a_n = \frac{4(-1)^{n+1}}{n^2\pi^2}$$

$$b_n = \frac{1}{1}\int_{-1}^{1} f(x)\sin(n\pi x)\,dx = \int_{-1}^{1}(x-x^2)\sin(n\pi x)\,dx$$

$$= \left[(x-x^2)\cdot\frac{-\cos(n\pi x)}{n\pi} - (1-2x)\cdot\frac{-\sin(n\pi x)}{n^2\pi^2} + (-2)\frac{\cos(n\pi x)}{n^3\pi^3}\right]_{-1}^{1}$$

$$b_n = \frac{-1}{n\pi}\{0-(-2\cos n\pi)\} - \frac{2}{n^3\pi^3}(\cos n\pi - \cos n\pi) = \frac{-2}{n\pi}(-1)^n$$

$$\therefore \quad b_n = \frac{2}{n\pi}(-1)^{n+1}$$

Thus the required Fourier series is given by

$$\boxed{f(x) = \frac{-1}{3} + \frac{4}{\pi^2}\sum_{1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos n\pi x + \frac{2}{\pi}\sum_{1}^{\infty}\frac{(-1)^{n+1}}{n}\sin n\pi x}$$

[23] *Draw the graph of the function*

$$f(x) = \begin{cases} \pi x & in \quad 0 \le x \le 1 \\ \pi(2-x) & in \quad 1 \le x \le 2 \end{cases}$$

and also show that the Fourier expansion of the function $f(x)$ *is*

$$\frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos \pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \cdots\right)$$ **[June 2017, Dec. 18**

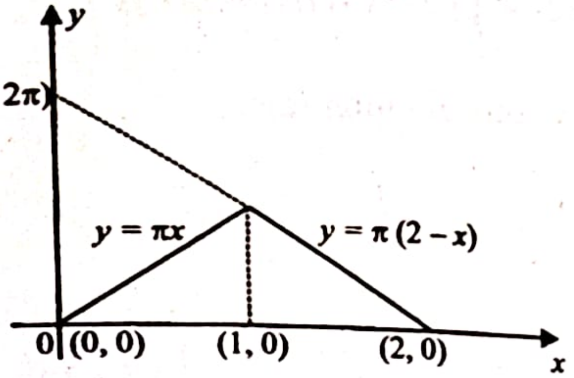
☞ Graph of $f(x) = y$

$f(x) = \pi x$ or $y = \pi x$ is a line passing through the origin in [0, 1

$f(x) = \pi(2-x)$ or $y = \pi(2-x)$ or $\pi x + y = 2\pi$ or $\frac{x}{2} + \frac{y}{2\pi} =$

in [1, 2], which is a straight line passing through (2, 0) and (0, 2π

The graph is as follows.



Here $f(x)$ is defined in [0, 2] and period of $f(x) = 2-0 = 2$.

$\therefore \quad 2l = 2$ or $l = 1$.

The Fourier series of $f(x)$ having period 2 is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x) + \sum_{n=1}^{\infty} b_n \sin(n\pi x)$$

$$a_0 = \frac{1}{1}\int_0^2 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx$$

$$a_0 = \int_0^1 \pi x\,dx + \int_1^2 \pi(2-x)\,dx$$

$$a_0 = \pi\left\{\left[\frac{x^2}{2}\right]_0^1 + \left[2x - \frac{x^2}{2}\right]_1^2\right\}$$

$$= \pi\left\{\left(\frac{1}{2} - 0\right) + (4-2) - \left(2 - \frac{1}{2}\right)\right\} = \pi$$

$$\therefore \quad a_0/2 = \pi/2$$

$$a_n = \frac{1}{1}\int_0^2 f(x)\cos(n\pi x)\,dx$$

$$= \int_0^1 f(x)\cos(n\pi x)\,dx + \int_1^2 f(x)\cos(n\pi x)\,dx$$

$$= \int_0^1 \pi x\cos(n\pi x)\,dx + \int_1^2 \pi(2-x)\cos(n\pi x)\,dx$$

$$= \pi\left\{\int_0^1 x\cos(n\pi x)\,dx + \int_1^2 (2-x)\cos(n\pi x)\,dx\right\}$$

Applying Bernoulli's rule to both the integrals,

$$a_n = \pi\left\{\left[x\cdot\frac{\sin(n\pi x)}{n\pi} - 1\cdot -\frac{\cos(n\pi x)}{n^2\pi^2}\right]_0^1 + \right.$$

$$\left.\left[(2-x)\frac{\sin(n\pi x)}{n\pi} - (-1)\cdot -\frac{\cos(n\pi x)}{n^2\pi^2}\right]_1^2\right.$$

$$= \frac{\pi}{n^2\pi^2}\left\{[\cos n\pi x]_0^1 - [\cos n\pi x]_1^2\right\},\ \text{since, } \sin n\pi = 0 = \sin 0.$$

$$= \frac{1}{n^2\pi}(\cos n\pi - \cos 0 - \cos 2n\pi + \cos n\pi).\ \text{But, } \cos 2n\pi = 1 = \cos($$

$$a_n = \frac{1}{n^2\pi}(-2+2\cos n\pi)$$

$$\therefore \quad a_n = \frac{-2}{\pi n^2}\{1-(-1)^n\}$$

$$b_n = \frac{1}{1}\int_0^2 f(x)\sin(n\pi x)\,dx$$

$$= \int_0^1 f(x)\sin(n\pi x)\,dx + \int_1^2 f(x)\sin(n\pi x)\,dx$$

$$= \int_0^1 \pi x\sin(n\pi x)\,dx + \int_1^2 \pi(2-x)\sin(n\pi x)\,dx$$

$$= \pi\left\{\left[x\cdot\frac{-\cos(n\pi x)}{n\pi} - (1)\cdot\frac{-\sin(n\pi x)}{n^2\pi^2}\right]_0^1\right.$$

$$\left. + \left[(2-x)\cdot\frac{-\cos(n\pi x)}{n\pi} - (-1)\cdot\frac{-\sin(n\pi x)}{n^2\pi^2}\right]_1^2\right.$$

$$= \frac{-\pi}{n\pi}\left\{ [x\cos(n\pi x)]_0^1 + [(2-x)\cos(n\pi x)]_1^2 \right\}$$

$$= \frac{-1}{n}\{(\cos n\pi - 0) + (0 - \cos n\pi)\} = 0$$

$$b_n = 0$$

$\therefore$ The required Fourier series is given by

$$f(x) = \frac{\pi}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty}\left\{\frac{1-(-1)^n}{n^2}\right\}\cos(n\pi x)$$

But, $$1-(-1)^n = \begin{cases} 1-(+1) = 0 & \text{if } n \text{ is even} \\ 1-(-1) = 2 & \text{if } n \text{ is odd} \end{cases}$$

Hence, $$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1,3,5,\cdots}^{\infty}\frac{\cos(n\pi x)}{n^2}$$

Thus, $$\boxed{f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos \pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \cdots\right)}$$

Now putting $x = 0$, we have $f(x) = 0$ since $f(x) = \pi x$ in $[0, 1]$.
The Fourier series becomes

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right) \quad \text{or} \quad -\frac{\pi}{2} = \frac{-4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$$

Thus, $$\boxed{\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots}$$

Aliter : *(Using the concept of even and odd functions)*
$f(x)$ is defined in $(0, 2)$ which is of the form $(0, 2l)$.
In the given $f(x)$ if $\phi(x) = \pi x$ and $\psi(x) = \pi(2-x)$,

$$\phi(2l - x) = \phi(2-x) = \pi(2-x) = \psi(x)$$

$\therefore$ $f(x)$ is even in $(0, 2)$ and hence $b_n = 0$

$$a_0 = \frac{2}{1}\int_0^1 f(x)\,dx, \quad a_n = \frac{2}{1}\int_0^1 f(x)\cos(n\pi x)\,dx$$

$$a_0 = 2\int_0^1 \pi x\, dx = \pi, \text{ on integration.}$$

$$a_n = 2\int_0^1 \pi x \cos(n\pi x)\, dx$$

$$= 2\pi \int_0^1 x\cos(n\pi x)\, dx = \frac{-2}{\pi n^2}\left\{1-(-1)^n\right\}, \text{ on integration.}$$

[24] *Obtain the Fourier series for the function*

$$f(x) = \begin{cases} 1+\dfrac{4x}{3} & in \quad \dfrac{-3}{2}<x\le 0 \\ 1-\dfrac{4x}{3} & in \quad 0\le x<\dfrac{3}{2} \end{cases}$$

Hence deduce that $\dfrac{\pi^2}{8} = \dfrac{1}{1^2}+\dfrac{1}{3^2}+\dfrac{1}{5^2}+\cdots$

☞ $f(x)$ is defined in the interval $(-3/2, 3/2)$.

∴ period of $f(x) = 3/2-(-3/2) = 3$. $2l = 3$ or $l = 3/2$.

We shall check $f(x)$ for even or odd nature.

If $\phi(x) = 1+\dfrac{4x}{3}$, $\phi(-x) = 1-\dfrac{4x}{3} = \psi(x)$

∴ $f(x)$ is an even function. Consequently $b_n = 0$.

The Fourier series of $f(x)$ having period 3 is given by

$$f(x) = \frac{a_0}{2}+\sum_1^\infty a_n \cos\left(\frac{n\pi x}{3/2}\right)+\sum_1^\infty b_n \sin\left(\frac{n\pi x}{3/2}\right)$$

ie., $$f(x) = \frac{a_0}{2}+\sum_1^\infty a_n \cos\left(\frac{2n\pi x}{3}\right)+\sum_1^\infty b_n \sin\left(\frac{2n\pi x}{3}\right)$$

Since $f(x)$ is even we have,

$$a_0 = \frac{2}{3/2}\int_0^{3/2} f(x)\, dx, \text{ since } a_0 = \frac{2}{l}\int_0^l f(x)\, dx$$

ie., $a_0 = \frac{4}{3}\int_0^{3/2}\left(1-\frac{4x}{3}\right)dx$

$$= \frac{4}{3}\left[x-\frac{2x^2}{3}\right]_0^{3/2} = \frac{4}{3}\left\{\left(\frac{3}{2}-\frac{2}{3}\cdot\frac{9}{4}\right)-0\right\} = 0$$

$\therefore \quad a_0/2 = 0$

$$a_n = \frac{2}{3/2}\int_0^{3/2} f(x)\cos\left(\frac{2n\pi x}{3}\right)dx \text{ since, } a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$$a_n = \frac{4}{3}\int_0^{3/2}\left(1-\frac{4x}{3}\right)\cos\left(\frac{2n\pi x}{3}\right)dx.$$ Applying Bernoulli's rule,

$$a_n = \frac{4}{3}\left[\left(1-\frac{4x}{3}\right)\frac{\sin\frac{2n\pi x}{3}}{2n\pi/3} - \left(\frac{-4}{3}\right)\cdot -\frac{\cos\frac{2n\pi x}{3}}{(2n\pi/3)^2}\right]_0^{3/2}$$

$$= \frac{4}{3}\cdot\frac{-4}{3}\cdot\frac{9}{4n^2\pi^2}\left[\cos\left(\frac{2n\pi x}{3}\right)\right]_0^{3/2} = \frac{-4}{n^2\pi^2}(\cos n\pi - 1)$$

$$a_n = \frac{4}{n^2\pi^2}\{1-(-1)^n\} \text{ or } a_n = 8/n^2\pi^2 \text{ where } n = 1, 3, 5, \ldots$$

Thus the required Fourier series is given by

$$\boxed{f(x) = \frac{8}{\pi^2}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^2}\cos\left(\frac{2n\pi x}{3}\right)}$$

Putting $x = 0$ we get $f(x) = 1$. The Fourier series becomes

$$1 = \frac{8}{\pi^2}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^2} \quad \text{or} \quad \boxed{\frac{\pi^2}{8} = \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots}$$

[25] *Obtain the Fourier series for $f(x) = e^{-x}$ in the interval $0 < x < 2$.*

☞ The period of $f(x) = 2-0 = 2 \quad \therefore \quad 2l = 2$ or $l = 1$.

The relevant Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_1^\infty a_n \cos n\pi x + \sum_1^\infty b_n \sin n\pi x \qquad \ldots (1)$$

$$a_0 = \int_0^2 f(x)\,dx, \quad a_n = \int_0^2 f(x)\cos n\pi x\,dx, \quad b_n = \int_0^2 f(x)\sin n\pi x\,dx$$

(*In each of the above integrals* $1/l = 1/1 = 1$)

$$a_0 = \int_0^2 e^{-x}\,dx = \left[-e^{-x}\right]_0^2 = -(e^{-2}-1) = 1 - \frac{1}{e^2} = \frac{e^2-1}{e^2}$$

$\therefore \quad a_0/2 = (e^2-1)/2e^2$

$$a_n = \int_0^2 e^{-x}\cos n\pi x\,dx$$

We have, $\int e^{ax}\cos bx\,dx = \dfrac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$

$$a_n = \left[\frac{e^{-x}}{1+n^2\pi^2}(-\cos n\pi x + n\pi \sin n\pi x)\right]_0^2$$

$$a_n = \frac{-1}{1+n^2\pi^2}\left[e^{-x}\cos n\pi x\right]_0^2 \quad \text{since} \quad \sin 2n\pi = 0 = \sin 0.$$

$$= \frac{-1}{1+n^2\pi^2}\left\{e^{-2}\cos 2n\pi - 1\right\} = \frac{-1}{1+n^2\pi^2}\left(\frac{1}{e^2} - 1\right)$$

$$\therefore \quad a_n = \frac{e^2-1}{e^2(1+n^2\pi^2)}$$

$$b_n = \int_0^2 e^{-x}\sin n\pi x\,dx$$

We have, $\int e^{ax}\sin bx\,dx = \dfrac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$

$$b_n = \left[\frac{e^{-x}}{1+n^2\pi^2}(-\sin n\pi x - n\pi\cos n\pi x)\right]_0^2$$

$$= \frac{-n\pi}{1+n^2\pi^2}\left[e^{-x}\cos n\pi x\right]_0^2 = \frac{-n\pi}{1+n^2\pi^2}\left(\frac{1}{e^2}-1\right)$$

$$\therefore \quad b_n = \frac{n\pi(e^2-1)}{e^2(1+n^2\pi^2)}$$

Thus by substituting the values of a_0, a_n, b_n in (1), the Fourier series is given by

$$\boxed{f(x) = \frac{e^2-1}{2e^2} + \sum_1^\infty \frac{e^2-1}{e^2(1+n^2\pi^2)}\cos n\pi x + \sum_1^\infty \frac{n\pi(e^2-1)}{e^2(1+n^2\pi^2)}\sin n\pi x}$$

[26] *Find the Fourier series of the periodic function defined by* $f(x) = 2x - x^2$ *in the interval* $0 < x < 3$. [June 2018]

☞ The period of $f(x) = 3 - 0 = 3 \quad \therefore \; 2l = 3$ or $l = 3/2$

The Fourier series of period $2l$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^\infty a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^\infty b_n \sin\frac{n\pi x}{l}$$

The relevant Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_1^\infty a_n\cos\frac{2n\pi x}{3} + \sum_1^\infty b_n \sin\frac{2n\pi x}{3} \qquad \ldots (1)$$

We shall find Fourier coefficients from Euler's formulae for the interval $(0, 3)$ with reference to the Fourier series (1). That is,

$$a_0 = \frac{2}{3}\int_0^3 f(x)\,dx, \quad a_n = \frac{2}{3}\int_0^3 f(x)\cos\frac{2n\pi x}{3}\,dx, \quad b_n = \frac{2}{3}\int_0^3 f(x)\sin\frac{2n\pi x}{3}\,dx$$

In each of the above integrals $1/l = 1/(3/2) = 2/3$

$$a_0 = \frac{2}{3}\int_0^3 (2x - x^2)\,dx$$

$$= \frac{2}{3}\left[x^2 - \frac{x^3}{3}\right]_0^3 = \frac{2}{3}\{(9-9)-0\} = 0$$

$$\therefore \quad a_0/2 = 0$$

$$a_n = \frac{2}{3}\int_0^3 (2x - x^2)\cos\frac{2n\pi x}{3}\,dx$$

Applying Bernoulli's rule,

$$a_n = \frac{2}{3}\left[(2x-x^2)\frac{\sin\frac{2n\pi x}{3}}{(2n\pi/3)} - (2-2x)\cdot\frac{-\cos\frac{2n\pi x}{3}}{(2n\pi/3)^2} + (-2)\cdot\frac{-\sin\frac{2n\pi x}{3}}{(2n\pi/3)^3}\right]_0^3$$

$$= \frac{2}{3}\,\frac{9}{4n^2\pi^2}\left[(2-2x)\cos\frac{2n\pi x}{3}\right]_0^3$$

The first and third terms vanish since, $\sin 2n\pi = 0 = \sin 0$.

$$a_n = \frac{3}{2n^2\pi^2}\{(2-6)\cos 2n\pi - (2-0)\cos 0\} = \frac{3}{2n^2\pi^2}(-4-2)$$

$\therefore \quad a_n = -9/n^2\pi^2$

$$b_n = \frac{2}{3}\int_0^3 (2x - x^2)\sin\frac{2n\pi x}{3}\,dx$$ Applying Bernoulli's rule,

$$b_n = \frac{2}{3}\left[(2x-x^2)\cdot\frac{-\cos\frac{2n\pi x}{3}}{(2n\pi/3)} - (2-2x)\cdot\frac{-\sin\frac{2n\pi x}{3}}{(2n\pi/3)^2} + (-2)\cdot\frac{\cos\frac{2n\pi x}{3}}{(2n\pi/3)^3}\right]_0^3$$

$$= \frac{2}{3}\left[\frac{-3}{2n\pi}\left\{(2x-x^2)\cos\frac{2n\pi x}{3}\right\} - \frac{54}{8n^3\pi^3}\left\{\cos\frac{2n\pi x}{3}\right\}\right]_0^3$$

$$= \frac{2}{3}\left[\frac{-3}{2n\pi}\{(6-9)\cos 2n\pi - 0\} - \frac{54}{8n^3\pi^3}\{\cos 2n\pi - \cos 0\}\right]$$

$$= \frac{2}{3}\left\{\frac{-3}{2n\pi}(-3)\right\} = \frac{3}{n\pi}$$

$\therefore \quad b_n = 3/n\pi$

Thus by substituting the values of a_0, a_n, b_n in (1) the required Fourier series is given by

$$\boxed{f(x) = \sum_{n=1}^{\infty} \frac{-9}{n^2\pi^2}\cos\frac{2n\pi x}{3} + \sum_{n=1}^{\infty} \frac{3}{n\pi}\sin\frac{2n\pi x}{3}}$$

[27] *Obtain the Fourier series for the function* $f(x) = 2x - x^2$ *in* $0 \le x \le 2$

☞ Comparing the given interval (0, 2) with (0, 2l) we have,

$2l = 2$ or $l = 1$. By data, $f(x) = x(2-x)$

$$f(2l-x) = f(2-x) = (2-x)(2-\overline{2-x}) = (2-x)x = f(x)$$

$\therefore$ $f(x)$ is even in (0, 2) and hence $b_n = 0$.

$$a_0 = \frac{2}{1}\int_0^1 f(x)\,dx\,,\quad a_n = \frac{2}{1}\int_0^1 f(x)\cos n\pi x\,dx$$

$$a_0 = 2\int_0^1 (2x - x^2)\,dx = 2\left[x^2 - \frac{x^3}{3}\right]_0^1 = 2\left[\left(1-\frac{1}{3}\right)-0\right] = \frac{4}{3}$$

$\therefore$ $a_0/2 = 2/3$

$$a_n = 2\int_0^1 (2x - x^2)\cos(n\pi x)\,dx.$$ Applying Bernoulli's rule,

$$a_n = 2\left[(2x-x^2)\frac{\sin(n\pi x)}{n\pi} - (2-2x)\cdot\frac{-\cos(n\pi x)}{n^2\pi^2} + (-2)\cdot\frac{-\sin(n\pi x)}{n^3\pi^3}\right]_0^1$$

$$= \frac{2}{n^2\pi^2}\left[(2-2x)\cos(n\pi x)\right]_0^1 = \frac{2}{n^2\pi^2}\left[0 - 2\cos 0\right] = \frac{-4}{n^2\pi^2}$$

$\therefore$ $a_n = -4/n^2\pi^2$

The Fourier series of period 2 is represented by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(n\pi x) + \sum_{n=1}^{\infty} b_n\sin(n\pi x)$$

Thus the required Fourier series is given by

[28] *Obtain the Fourier series of the saw-tooth function* $f(t) = Et/T$ *for* $0<t<T$ *given that* $f(t+T) = f(t)$ *for all* $t > 0$

☞ We have $2l = T$ or $l = T/2$ and the associated Fourier series of period $2l = T$ is given by,

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{T/2}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{T/2}\right)$$

ie., $$f(t) = \frac{a_0}{2} + \sum_{1}^{\infty} a_n \cos\left(\frac{2n\pi t}{T}\right) + \sum_{1}^{\infty} b_n \sin\left(\frac{2n\pi t}{T}\right)$$

We compute Fourier coefficients by Euler's formulae.

$$a_0 = \frac{1}{T/2}\int_0^T f(t)\,dt,\ a_n = \frac{1}{T/2}\int_0^T f(t)\cos\left(\frac{2n\pi t}{T}\right)dt$$

$$b_n = \frac{1}{T/2}\int_0^T f(t)\sin\left(\frac{2n\pi t}{T}\right)dt$$

$$a_0 = \frac{2}{T}\int_0^T \frac{Et}{T}\,dt = \frac{2E}{T^2}\left[\frac{t^2}{2}\right]_0^T = \frac{E}{T^2}(T^2 - 0) = E$$

$\therefore$ $a_0/2 = E/2$

$$a_n = \frac{2}{T}\int_0^T \frac{Et}{T}\cos\frac{2n\pi t}{T}\,dt = \frac{2E}{T^2}\int_0^T t\cos\frac{2n\pi t}{T}\,dt$$

$$a_n = \frac{2E}{T^2}\left[t\cdot\frac{\sin\dfrac{2n\pi t}{T}}{(2n\pi/T)} - 1\cdot\frac{-\cos\dfrac{2n\pi t}{T}}{(2n\pi/T)^2}\right]_0^T$$

$$= \frac{2E}{T^2}\,\frac{T^2}{4n^2\pi^2}(\cos 2n\pi - \cos 0) = 0$$

$\therefore$ $a_n = 0$

$$b_n = \frac{2}{T}\cdot\frac{E}{T}\int_0^T t\sin\frac{2n\pi t}{T}\,dt$$

$$= \frac{2E}{T^2}\left[t\cdot\frac{-\cos\frac{2n\pi t}{T}}{(2n\pi/T)} - 1\cdot\frac{-\sin\frac{2n\pi t}{T}}{(2n\pi/T)^2}\right]_0^T$$

$$= \frac{-E}{n\pi T}\left[t\cos\frac{2n\pi t}{T}\right]_0^T = \frac{-E}{n\pi T}(T\cos 2n\pi - 0) = \frac{-E}{n\pi}$$

$\therefore \quad b_n = -E/n\pi$

Thus the required Fourier series is given by

$$\boxed{f(t) = \frac{E}{2} - \frac{E}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{2n\pi t}{T}}$$

[29] *If* $f(x) = \begin{cases} 2-x & in\ 0 \le x \le 4 \\ x-6 & in\ 4 \le x \le 8 \end{cases}$

Express $f(x)$ *as a Fourier series and hence deduce that* $\dfrac{\pi^2}{8} = \displaystyle\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$

☞ Comparing the given interval $(0, 8)$ with $(0, 2l)$ we have $2l = 8$ or $l = 4$. The Fourier series having period 8 is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{4}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{4}\right)$$

In the given $f(x)$, let $\phi(x) = 2-x$, $\psi(x) = x-6$

Now, $\phi(2l-x) = \phi(8-x) = 2-(8-x) = x-6 = \psi(x)$

$\therefore \quad f(x)$ is even in $(0, 8)$ and hence $b_n = 0$.

$$a_0 = \frac{1}{2}\int_0^4 f(x)\,dx, \quad a_n = \frac{2}{4}\int_0^4 f(x)\cos\left(\frac{n\pi x}{4}\right)dx$$

$$a_0 = \frac{1}{2}\int_0^4 (2-x)\,dx = \frac{1}{2}\left[2x - \frac{x^2}{2}\right]_0^4 = \frac{1}{2}[(8-8)-0] = 0$$

$\therefore \quad a_0/2 = 0$

$a_n = \dfrac{1}{2}\displaystyle\int_0^4 (2-x)\cos\frac{n\pi x}{4}\,dx$ Applying Bernoulli's rule,

$$a_n = \frac{1}{2}\left[(2-x)\cdot\frac{\sin\frac{n\pi x}{4}}{(n\pi/4)} - (-1)\cdot\frac{-\cos\frac{n\pi x}{4}}{(n\pi/4)^2}\right]_0^4$$

$$= \frac{-8}{n^2\pi^2}\left[\cos\frac{n\pi x}{4}\right]_0^4 = \frac{-8}{n^2\pi^2}\{\cos n\pi - 1\}$$

$$a_n = \frac{8}{n^2\pi^2}\{1-(-1)^n\} \text{ or } a_n = \frac{16}{n^2\pi^2} \text{ where } n = 1, 3, 5, \ldots$$

Thus the required Fourier series is given by

$$\boxed{f(x) = \sum_{n=1,3,5,\ldots}^{\infty} \frac{16}{n^2\pi^2}\cos\left(\frac{n\pi x}{4}\right)}$$

To deduce the series we put $x = 0$.

$f(x) = 2-0 = 2$ and the Fourier series becomes

$$2 = \frac{16}{\pi^2}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^2}\cdot 1 \text{ or } \frac{\pi^2}{8} = \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots$$

Equivalently we have $\boxed{\frac{\pi^2}{8} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}}$

[30] *Find the Fourier expansion of the function $f(x)$ defined by*

$$f(x) = \begin{cases} 0 & in \quad -2<x<-1 \\ 2 & in \quad -1<x<1 \\ 0 & in \quad 1<x<2 \end{cases}$$

☞ $f(x)$ is defined in $(-2, 2)$ and period of $f(x) = 2-(-2) = 4$

We have, $2l = 4$ or $l = 2$. The associated Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2} \qquad \ldots (1)$$

The given $f(x)$ can be written as follows.

Interval of x	$(-2,-1)$	$(-1,0)$	$(0,1)$	$(1,2)$
$f(x)$	0	2	2	0

Let, $f(x) = \begin{cases} \phi(x) & \text{in } (-2,0) \\ \psi(x) & \text{in } (0,2) \end{cases}$

where $\phi(x) = 0$ or 2, $\psi(x) = 2$ or 0. Obviously $\phi(-x) = \psi(x)$

$\therefore$ $f(x)$ is even and consequently $b_n = 0$.

$$a_0 = \frac{2}{2}\int_0^2 f(x)\,dx, \quad a_n = \frac{2}{2}\int_0^2 f(x)\cos\left(\frac{n\pi x}{2}\right)dx$$

$$a_0 = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx = \int_0^1 2\,dx + 0 = [\,2x\,]_0^1 = 2$$

$\therefore$ $a_0/2 = 1$

$$a_n = \int_0^1 f(x)\cos\frac{n\pi x}{2}dx + \int_1^2 f(x)\cos\frac{n\pi x}{2}dx$$

$$= \int_0^1 2\cos\frac{n\pi x}{2}dx + 0$$

$$a_n = 2\left[\frac{\sin\frac{n\pi x}{2}}{(n\pi/2)}\right]_0^1 = \frac{4}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

Thus the required Fourier series is given by

$$\boxed{f(x) = 1 + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi}{2}\right)\cos\frac{n\pi x}{2}}$$

ASSIGNMENT

Find the Fourier series of the following functions over the indicated intervals.

1. $f(x) = -1 + x;\ -\pi < x < \pi$
2. $f(x) = x^2;\ 0 < x < 2\pi$
3. $f(x) = \pi^2 - x^2;\ -\pi \le x \le \pi$. Deduce that

   (a) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - + \cdots = \frac{\pi^2}{12}$ (b) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$
4. $f(x) = \frac{x}{12}(\pi^2 - x^2);\ -\pi < x < \pi$.

5. $f(x) = x\sin x;\ (0, 2\pi)$. Deduce that

(a) $\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \cdots = \frac{\pi - 2}{4}$

(b) $\sum_{n=1}^{\infty} \frac{1}{n^2 - 1} = \frac{3}{4}$

6. $f(x) = \begin{cases} x - \frac{\pi}{2} & \text{in } (-\pi, 0) \\ x + \frac{\pi}{2} & \text{in } (0, \pi) \end{cases}$

7. $f(x) = \begin{cases} \pi - x & \text{in } 0 \le x \le \pi \\ x - \pi & \text{in } \pi \le x \le 2\pi \end{cases}$

8. $f(x) = \begin{cases} -\cos x & \text{in } (-\pi, 0) \\ \cos x & \text{in } (0, \pi) \end{cases}$

9. $f(x) = \begin{cases} -1 & \text{in } -\pi < x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{in } 0 < x < \pi \end{cases}$

Hence deduce that, $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots$

10. $f(x) = l^2 + x^2$ in $-l \le x \le l$.

Deduce that $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$

11. $f(x) = 1 - 2|x|$ in $-1 \le x \le 1$

12. $f(x) = x^2 - x$ in $(-2, 2)$

13. $f(x) = x - x^2$ in $(0, 1)$.

Deduce that $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$

14. $f(x) = \begin{cases} \frac{l}{2} - x & \text{in } (-l, 0) \\ \frac{l}{2} + x & \text{in } (0, l) \end{cases}$

Deduce that $\dfrac{\pi^2}{8} = \dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \cdots$

15. $$f(x) = \begin{cases} x^2 & \text{in } (0,1) \\ -(2-x)^2 & \text{in } (1,2) \end{cases}$$

16. $$f(x) = \begin{cases} 8 & \text{in } 0<x<2 \\ -8 & \text{in } 2<x<4 \end{cases}$$

17. $$f(x) = \begin{cases} 0 & \text{in} -2<x\le -1 \\ 1+x & \text{in } -1<x<0 \\ (1-x) & \text{in } 0\le x<1 \\ 0 & \text{in } 1\le x<2 \end{cases}$$

18. Prove that if $-\pi \le x \le \pi$ and a is not an integer

$$\cos ax = \frac{2a}{\pi}\sin a\pi\left\{\frac{1}{2a^2} + \sum_{1}^{\infty}\frac{(-1)^{n+1}}{n^2-a^2}\cos nx\right\}$$

Hence show that $\dfrac{1-a\pi\cot a\pi}{2a^2} = \displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2-a^2}$

19. If $0<x<2\pi$ show that $x\cos x = \pi\cos x - \dfrac{1}{2}\sin x - 2\displaystyle\sum_{n=2}^{\infty}\frac{n}{n^2-1}\sin nx$

20. Show that Fourier series representation in $(0, 2T)$ of the half wave rectifier $f(t)$ defined by

$$f(t) = \begin{cases} E\sin\dfrac{\pi t}{T} & \text{in } (0,T) \\ 0 & \text{in } (T,2T) \end{cases} \quad \text{is}$$

$$f(t) = \frac{E}{\pi} + \sum_{n=2,4,6,\ldots}^{\infty}\frac{2E}{\pi(n^2-1)}\cos\frac{n\pi t}{T} + \frac{E}{2}\sin\frac{\pi t}{T}$$

ANSWERS

1. $$-1 + 2\sum_{1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx$$

2. $$\frac{4\pi^2}{3} + 4\sum_{1}^{\infty}\frac{\cos nx}{n^2} - 4\pi\sum_{1}^{\infty}\frac{\sin nx}{n}$$

3. $\dfrac{2\pi^2}{3} + 4\sum_1^{\infty} \dfrac{(-1)^{n+1}}{n^2} \cos nx$

4. $\sum_1^{\infty} \dfrac{(-1)^{n+1}}{n^3} \sin nx$

5. $-1 - \dfrac{1}{2}\cos x + 2\sum_2^{\infty} \dfrac{\cos nx}{n^2 - 1} + \pi \sin x$

6. $\sum_1^{\infty} \dfrac{1 + 3(-1)^{n+1}}{n} \sin nx$

7. $\dfrac{\pi}{2} + \dfrac{4}{\pi} \sum_{n = 1,3,5,\ldots}^{\infty} \dfrac{1}{n^2} \cos nx$

8. $\dfrac{8}{\pi} \sum_1^{\infty} \dfrac{n}{n^2 - 1} \sin nx$

9. $\dfrac{4}{\pi} \sum_1^{\infty} \dfrac{\sin nx}{n}$

10. $\dfrac{4l^2}{3} + \dfrac{4l^2}{\pi^2} \sum_1^{\infty} \dfrac{(-1)^n}{n^2} \cos \dfrac{n\pi x}{l}$

11. $\dfrac{4}{\pi^2} \sum_1^{\infty} \dfrac{1}{n^2} \{1 - (-1)^n\} \cos n\pi x$

12. $\dfrac{4}{3} + \dfrac{16}{\pi^2} \sum_1^{\infty} \dfrac{(-1)^n}{n^2} \cos \dfrac{n\pi x}{2} + \dfrac{4}{\pi} \sum_1^{\infty} \dfrac{(-1)^n}{n} \sin \dfrac{n\pi x}{2}$

13. $\dfrac{1}{6} - \dfrac{1}{\pi^2} \sum_1^{\infty} \dfrac{1}{n^2} \cos 2n\pi x$

14. $l - \dfrac{4l}{\pi^2} \sum_{n = 1,3,5,\ldots}^{\infty} \dfrac{1}{n^2} \cos \dfrac{n\pi x}{l}$

15. $\sum_1^{\infty} \left[\dfrac{2(-1)^{n+1}}{n\pi} - \dfrac{4}{n^3\pi^3} \{1 - (-1)^n\} \right] \sin(n\pi x)$

16. $\dfrac{32}{\pi} \sum_1^{\infty} \dfrac{1}{n^2} \sin \dfrac{n\pi x}{2}$

17. $\dfrac{1}{4} + \dfrac{4}{\pi^2} \sum_1^{\infty} \dfrac{1}{n^2} \left\{1 - \cos \dfrac{n\pi}{2}\right\} \cos \dfrac{n\pi x}{2}$

2.7 Half Range Fourier Series

In an interval of length $2l$, we have seen that in general a periodic function of x will have Fourier expansion containing cosine terms and sine terms. Many times it becomes necessary to have the expansion containing only cosine terms or only sine terms.

To achieve this, the function must be defined in the interval of the form $(0, l)$ which is to be regarded as half the interval. We then extend the definition to the other half in such a manner that the function becomes even or odd. This will result in cosine series only or sine series only.

Case - (i) : ***For cosine series***

$$f(x) = \begin{cases} \phi(x) & \text{in } (0,l) \text{ as given} \\ \phi(-x) & \text{in } (-l,0)\ [\textit{assumed}] \text{ to make } f(x) \text{ even} \end{cases}$$

Case - (ii) : ***For sine series***

$$f(x) = \begin{cases} \phi(x) & \text{in } (0,l) \text{ as given} \\ -\phi(-x) & \text{in } (-l,0)\ [\textit{assumed}] \text{ to make } f(x) \text{ odd} \end{cases}$$

We have already seen that in the case - (i) $b_n = 0$ and in the case-(ii) $a_0 = 0$, $a_n = 0$.

$\therefore$ We have in the case - (i), $f(x) = \dfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\dfrac{n\pi x}{l}$ . . . (1)

where, $a_0 = \dfrac{2}{l}\int_0^l f(x)\,dx$, $a_n = \dfrac{2}{l}\int_0^l f(x)\cos\dfrac{n\pi x}{l}\,dx$

Also in the case - (ii), $f(x) = \sum_{n=1}^{\infty} b_n \sin\dfrac{n\pi x}{l}$ . . . (2)

where, $b_n = \dfrac{2}{l}\int_0^l f(x)\sin\dfrac{n\pi x}{l}\,dx$

The series (1) which contains only cosine terms is called the *cosine half range Fourier series* for $f(x)$ in $(0, l)$ and the series (2) which contains only sine terms is called the ***sine half range Fourier series*** for $f(x)$ in $(0, l)$.

Similar consideration hold good for $(0, \pi)$ as it is a particular case when $l = \pi$.

The following table summarizes the theory discussed and will be useful for working problems.

$f(x)$ in	Required Series	Series	Fourier Coefficients
$(0, l)$	Cosine series	$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$	$a_0 = \frac{2}{l} \int_0^l f(x)\,dx$ $a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l}\,dx$
$(0, l)$	Sine series	$\sum_1^{\infty} b_n \sin \frac{n\pi x}{l}$	$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l}\,dx$
$(0, \pi)$	Cosine series	$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$	$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\,dx$ $a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\,dx$
$(0, \pi)$	Sine series	$\sum_1^{\infty} b_n \sin nx$	$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx\,dx$

Illustrations : (i) Suppose we want to find the cosine half range Fourier series for $f(x) = l - x$ in $(0, l)$. Treating this as $\phi(x)$ in $(0, l)$ we take,

$\psi(x) = \phi(-x)$ in $(-l, 0)$ so that we have,

$$f(x) = \begin{cases} \phi(x) = l - x & \text{in } (0, l) \text{ as given} \\ \psi(x) = l + x & \text{in } (-l, 0) \text{ assumed} \end{cases}$$

It may be observed that $f(x)$ is even in $(-l, l)$. Consequently $b_n = 0$ and the Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_1^{\infty} a_n \cos \frac{n\pi x}{l} \qquad \ldots (1)$$

where, $a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{l}\int_0^l (l-x)\,dx$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx = \frac{2}{l}\int_0^l (l-x)\cos\frac{n\pi x}{l}\,dx$$

The series represented by (1) on substitution of the values of a_0 and a_n is the required cosine half range Fourier series for $f(x)$ in $(0, l)$.

(ii) Suppose we want to find the sine half range Fourier series for $f(x) = x^2$ in $0 < x < \pi$. Treating this as $\phi(x)$ in $(0, \pi)$ we take

$\psi(x) = -\phi(-x)$ in $(-\pi, 0)$ so that we have,

$$f(x) = \begin{cases} \phi(x) = x^2 & \text{in } (0,\pi) \text{ as given} \\ \psi(x) = -(-x)^2 = -x^2 & \text{in } (-\pi,0) \text{ assumed} \end{cases}$$

It may be observed that $f(x)$ is odd in $(-\pi, \pi)$. Consequently $a_0 = 0$, $a_n = 0$ and the Fourier series is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \qquad \ldots (2)$$

where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} x^2 \sin nx\,dx$

The series represented by (2) on substitution of the value of b_n is the required sine half range series.

Note : *In problems of half range Fourier series we can directly assume the appropriate series along with the expressions for the Fourier coefficients as summarized in the table before.*

WORKED PROBLEMS

[31] *Obtain the sine half range Fourier series of* $f(x) = x^2$ *in* $0 < x < \pi$ **[Dec. 2017]**

☞ The sine half range Fourier series of the function $f(x)$ in $(0, \pi)$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad \text{where, } b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

We have, $b_n = \frac{2}{\pi}\int_0^{\pi} x^2 \sin nx\,dx$. Applying Bernoulli's rule,

$$b_n = \frac{2}{\pi}\left[x^2 \cdot -\frac{\cos nx}{n} - 2x \cdot -\frac{\sin nx}{n^2} + 2 \cdot \frac{\cos nx}{n^3}\right]_0^\pi$$

$$= \frac{2}{\pi}\left\{\frac{-1}{n}[x^2 \cos nx]_0^\pi + 0 + \frac{2}{n^3}[\cos nx]_0^\pi\right\}$$

$$= \frac{2}{\pi}\left\{\frac{-1}{n}(\pi^2 \cos n\pi - 0) + \frac{2}{n^3}(\cos n\pi - 1)\right\}$$

$$b_n = \frac{2}{\pi}\left\{\frac{(-1)^{n+1}\pi^2}{n} - \frac{2}{n^3}[1-(-1)^n]\right\}$$

Thus the required sine half range Fourier series is given by

$$\boxed{f(x) = \sum_{n=1}^{\infty} \frac{2}{\pi}\left\{\frac{(-1)^{n+1}\pi^2}{n} - \frac{2}{n^3}[1-(-1)^n]\right\}\sin nx}$$

[32] *Expand* $f(x) = 2x - 1$ *as a cosine half range Fourier series in* $0 < x < 1$

☞ Comparing the given interval (0, 1) with (0, l) we have $l = 1$. The corresponding cosine half range Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x$$

where, $a_0 = \frac{2}{1}\int_0^1 f(x)\,dx$, $a_n = \frac{2}{1}\int_0^1 f(x)\cos n\pi x\, dx$

$$a_0 = 2\int_0^1 (2x-1)\,dx = 2\left[x^2 - x\right]_0^1 = 0 \quad \therefore a_0 = 0$$

$$a_n = 2\int_0^1 (2x-1)\cos n\pi x\, dx$$

$$= 2\left[(2x-1)\frac{\sin n\pi x}{n\pi} - (2)\cdot\frac{-\cos n\pi x}{n^2\pi^2}\right]_0^1$$

$$= \frac{4}{n^2\pi^2}[\cos n\pi x]_0^1 = \frac{4}{n^2\pi^2}(\cos n\pi - 1)$$

$$\therefore \quad a_n = \frac{-4}{n^2\pi^2}\{1-(-1)^n\}$$

Thus the required cosine half range Fourier series is given by

$$\boxed{f(x) = \sum_1^{\infty} \frac{-4}{n^2\pi^2}\{1-(-1)^n\}\cos n\pi x}$$

[33] *Show that the sine half range series for the function* $f(x) = lx - x^2$ *in* $0 < x < l$

is $\dfrac{8l^2}{\pi^3}\displaystyle\sum_0^{\infty}\frac{1}{(2n+1)^3}\sin\left(\frac{2n+1}{l}\right)\pi x$ **[June 2018]**

☞ The sine half range Fourier series of $f(x)$ in $(0, l)$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \quad \text{where,}$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\int_0^l (lx - x^2)\sin\frac{n\pi x}{l}\,dx$$ Applying Bernoulli's rule,

$$= \frac{2}{l}\left[(lx-x^2)\cdot\frac{-\cos\frac{n\pi x}{l}}{(n\pi/l)} - (l-2x)\cdot\frac{-\sin\frac{n\pi x}{l}}{(n\pi/l)^2} + (-2)\frac{\cos\frac{n\pi x}{l}}{(n\pi/l)^3}\right]_0^l$$

$$= \frac{-4}{l}\frac{l^3}{n^3\pi^3}\left[\cos\frac{n\pi x}{l}\right]_0^l$$

($lx - x^2$ *is zero at* $x = 0, l$ *and* $\sin n\pi = 0 = \sin 0$)

$$b_n = \frac{-4l^2}{n^3\pi^3}(\cos n\pi - 1) \quad \text{or} \quad b_n = \frac{4l^2}{n^3\pi^3}\{1-(-1)^n\}$$

The sine half range Fourier series is given by

$$f(x) = \frac{4l^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}\{1-(-1)^n\}\sin\frac{n\pi x}{l}$$

But, $$1-(-1)^n = \begin{cases} 1-(+1) = 0 & \text{when } n \text{ is even} \\ 1-(-1) = 2 & \text{when } n \text{ is odd} \end{cases}$$

$$\therefore \quad f(x) = \frac{4l^2}{\pi^3} \sum_{n=1,3,5,\ldots}^{\infty} \frac{2}{n^3} \sin\frac{n\pi x}{l}$$

ie., $$f(x) = \frac{8l^2}{\pi^3} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^3} \sin\frac{n\pi x}{l}$$

But, 1, 3, 5, . . . are odd numbers represented in general as $(2n+1)$ where $n = 0, 1, 2, 3, \ldots$ Thus we have,

$$\boxed{f(x) = \frac{8l^2}{\pi^3} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} \sin\left(\frac{2n+1}{l}\right)\pi x}$$

[34] *Show that* $\dfrac{l}{2} - x = \dfrac{l}{\pi} \displaystyle\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{2n\pi x}{l}\right),\ 0 < x < l$

☞ From the given answer and the given interval of x it is evident that we have to find the sine half range Fourier series of $f(x) = \dfrac{l}{2} - x$ in $(0, l)$.

The series is represented by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \quad \text{where, } b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

ie., $$b_n = \frac{2}{l}\int_0^l \left(\frac{l}{2} - x\right)\sin\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{2}{l}\left[\left(\frac{l}{2} - x\right)\cdot\frac{-\cos\frac{n\pi x}{l}}{(n\pi/l)} - (-1)\cdot\frac{-\sin\frac{n\pi x}{l}}{(n\pi/l)^2}\right]_0^l$$

$$= \frac{-2}{l}\frac{l}{n\pi}\left[\left(\frac{l}{2} - x\right)\cos\frac{n\pi x}{l}\right]_0^l, \text{ since } \sin n\pi = 0 = \sin 0.$$

$$= \frac{-2}{n\pi}\left(-\frac{l}{2}\cos n\pi - \frac{l}{2}\right) = \frac{1}{n\pi}(\cos n\pi + 1)$$

$$\therefore \quad b_n = \frac{1}{n\pi}\{1+(-1)^n\}$$

The sine half range Fourier series is given by

$$f(x) = \frac{l}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\{1+(-1)^n\}\sin\frac{n\pi x}{l}$$

But, $1+(-1)^n = \begin{cases} 1+1 = 2 & \text{when } n \text{ is even} \\ 1-1 = 0 & \text{when } n \text{ is odd} \end{cases}$

$$\therefore \quad f(x) = \frac{l}{\pi}\sum_{n=2,4,6,\ldots}^{\infty}\frac{2}{n}\sin\frac{n\pi x}{l}$$

But 2, 4, 6, . . . are even numbers represented in general as $2n$ where $n = 1, 2, 3, \ldots$ and hence we can write the series replacing n by $2n$ in the form,

$$\boxed{f(x) = \frac{l}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{2n\pi x}{l}}$$

[35] *Find the cosine half range series for* $f(x) = x(l-x);\ 0 \le x \le l$

☞ The cosine half range Fourier series of $f(x)$ in $[0, l]$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} \quad \text{where,}$$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx\,; \quad a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$$a_0 = \frac{2}{l}\int_0^l (lx - x^2)\,dx = \frac{2}{l}\left[l\frac{x^2}{2} - \frac{x^3}{3}\right]_0^l$$

$$= \frac{2}{l}\left(\frac{l^3}{2} - \frac{l^3}{3}\right) = \frac{l^2}{3}$$

$$\therefore \quad a_0/2 = l^2/6$$

$$a_n = \frac{2}{l}\int_0^l (lx - x^2)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[(lx - x^2)\frac{\sin\frac{n\pi x}{l}}{(n\pi/l)} - (l-2x)\cdot\frac{-\cos\frac{n\pi x}{l}}{(n\pi/l)^2} + (-2)\cdot\frac{-\sin\frac{n\pi x}{l}}{(n\pi/l)^3}\right]_0^l$$

$$= \frac{2}{l}\frac{l^2}{n^2\pi^2}\left[(l-2x)\cos\frac{n\pi x}{l}\right]_0^l \text{ since first and third terms vanish.}$$

$$= \frac{2l}{n^2\pi^2}(-l\cos n\pi - l) = \frac{-2l^2}{n^2\pi^2}(\cos n\pi + 1)$$

$$\therefore \quad a_n = \frac{-2l^2}{n^2\pi^2}\{1+(-1)^n\}$$

Thus the required cosine half range series is given by

$$\boxed{f(x) = \frac{l^2}{6} - \frac{2l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\{1+(-1)^n\}\cos\frac{n\pi x}{l}}$$

Note : Similar problem

Expand the function $f(x) = x(\pi - x)$ *over the interval* $(0, \pi)$ *in half range Fourier cosine series.* **[Dec. 2018]**

[36] *Obtain the half range cosine series for the function*

$$f(x) = \sin\left(\frac{m\pi x}{l}\right) \text{ where } m \text{ is a positive integer over the interval } (0, l)$$

☞ We have, $f(x) = \dfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\dfrac{n\pi x}{l}$ where,

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx \text{ and } a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$$\therefore \quad a_0 = \frac{2}{l}\int_0^l \sin\frac{m\pi x}{l}\,dx$$

$$= \frac{2}{l}\cdot\frac{l}{m\pi}\left[-\cos\frac{m\pi x}{l}\right]_0^l = \frac{-2}{m\pi}(\cos m\pi - 1) = \frac{-2}{m\pi}\{(-1)^m - 1\}$$

$$\frac{a_0}{2} = \frac{1}{m\pi}\{1-(-1)^m\}$$

$$a_n = \frac{2}{l}\int_0^l \sin\frac{m\pi x}{l}\cdot\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\int_0^l \frac{1}{2}\left\{\sin(m+n)\frac{\pi x}{l}+\sin(m-n)\frac{\pi x}{l}\right\}dx$$

$$= \frac{1}{l}\left[\frac{-l}{(m+n)\pi}\cos(m+n)\frac{\pi x}{l}-\frac{l}{(m-n)\pi}\cos(m-n)\frac{\pi x}{l}\right]_0^l,\ m \neq n$$

$$= \frac{1}{\pi}\left[\frac{-1}{(m+n)}\{\cos(m+n)\pi-1\}-\frac{1}{(m-n)}\{\cos(m-n)\pi-1\}\right]$$

$$= \frac{-1}{\pi}\left[\left\{\frac{-1}{m+n}-\frac{1}{m-n}\right\}+\frac{1}{m+n}\{\cos m\pi\cos n\pi-\sin m\pi\sin n\pi\}\right.$$
$$\left.+\frac{1}{m-n}\{\cos m\pi\cos n\pi+\sin m\pi\sin n\pi\}\right]$$

$$= \frac{-1}{\pi}\left[-\left\{\frac{1}{(m+n)}+\frac{1}{(m-n)}\right\}+\cos m\pi\cos n\pi\left\{\frac{1}{(m+n)}+\frac{1}{(m-n)}\right\}\right]$$

where we have, $\sin m\pi = 0 = \sin n\pi$.

$$a_n = \frac{1}{\pi}\left[\frac{2m}{m^2-n^2}\{1-\cos m\pi\cos n\pi\}\right]$$

$$\therefore \quad a_n = \frac{2m}{\pi(m^2-n^2)}\{1+(-1)^{m+n+1}\} \text{ where } m \neq n.$$

If $m = n$, $a_n = \frac{2}{l}\int_0^l \sin\frac{n\pi x}{l}\cos\frac{n\pi x}{l}\,dx$

ie., $a_n = \frac{1}{l}\int_0^l \sin\frac{2n\pi x}{l}\,dx = \frac{1}{l}\left[\frac{-l}{2n\pi}\cos\frac{2n\pi x}{l}\right]_0^l$

$$= -\frac{1}{2n\pi}(\cos 2n\pi - 1) = 0 \text{ since, } \cos 2n\pi = +1$$

$\therefore \quad a_n = 0$ when $m = n$.

Thus the required cosine half range Fourier series when $m \neq n$ is given by

$$\boxed{f(x) = \frac{1}{m\pi}\{1-(-1)^m\} + \sum_{n=1}^{\infty} \frac{2m}{\pi(m^2-n^2)}\{1+(-1)^{m+n+1}\}\cos\frac{n\pi x}{l}}$$

[37] *Find a cosine series for* $f(x) = (x-1)^2, 0 \le x \le 1$ **[June 2018]**

☞ Comparing the given interval [0, 1] with half range [0, l] we have $l = 1$. The corresponding cosine half range Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x \text{ where,}$$

$$a_0 = \frac{2}{1}\int_0^1 f(x)\,dx\,;\; a_n = \frac{2}{1}\int_0^1 f(x)\cos n\pi x\,dx$$

$$a_0 = 2\int_0^1 (x-1)^2\,dx = 2\left[\frac{(x-1)^3}{3}\right]_0^1 = \frac{2}{3}\{0-(-1)^3\} = \frac{2}{3}$$

$\therefore \quad a_0/2 = 1/3$

$$a_n = 2\int_0^1 (x-1)^2 \cos n\pi x\,dx$$

$$= 2\left[(x-1)^2 \cdot \frac{\sin n\pi x}{n\pi} - 2(x-1)\cdot -\frac{\cos n\pi x}{n^2\pi^2} + 2\cdot -\frac{\sin n\pi x}{n^3\pi^3}\right]_0^1$$

$$= \frac{4}{n^2\pi^2}[(x-1)\cos n\pi x]_0^1 = \frac{4}{n^2\pi^2}\{0-(-1)\} = \frac{4}{n^2\pi^2}$$

$$a_n = 4/n^2\pi^2$$

Thus the required cosine half range Fourier series is given by

$$\boxed{f(x) = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi x}$$

Note : *By putting* $x = 0$ *and* $x = 1$ *in the series we can respectively deduce the series*

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots;\; \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots$$

Adding these we also obtain $\dfrac{\pi^2}{8} = \dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \cdots$

[38] *Find the cosine half range series of $f(x) = x\sin x$ in $0<x<\pi$. Deduce that*

$$\frac{1}{1\cdot 3}-\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}-\cdots=\frac{\pi-2}{4} \quad or \quad 1+\frac{2}{1\cdot 3}-\frac{2}{3\cdot 5}+\frac{2}{5\cdot 7}-\cdots=\frac{\pi}{2}$$ **[June 2017]**

☞ The cosine half range series of $f(x)$ in half the range $(0, \pi)$ is given by

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx \text{ where,}$$

$$a_0=\frac{2}{\pi}\int_0^{\pi} f(x)\,dx;\quad a_n=\frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

ie., $$a_0=\frac{2}{\pi}\int_0^{\pi} x\sin x\,dx;\quad a_n=\frac{2}{\pi}\int_0^{\pi} x\sin x\cos nx\,dx$$

Referring to problem-[14] for the integration process, the required series is given by

$$\boxed{f(x)=1-\frac{1}{2}\cos x+2\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n^2-1}\cos nx}$$

Now putting, $x=\pi/2$, $f(x)=\pi/2\sin(\pi/2)=\pi/2$

The Fourier series becomes

$$\frac{\pi}{2}=1+2\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n^2-1}\cos\frac{n\pi}{2}$$

$$\frac{\pi}{2}-1=2\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n^2-1}\cos\frac{n\pi}{2} \quad \text{or} \quad \frac{\pi-2}{2}=2\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n^2-1}\cos\frac{n\pi}{2}$$

On expanding the RHS and using $\cos\pi=-1=\cos 3\pi=\cos 5\pi\cdots$

$\cos(3\pi/2)=0=\cos(5\pi/2)\cdots$ we get,

$$\frac{\pi-2}{4}=\frac{1}{3}-\frac{1}{15}+\frac{1}{35}-\cdots,$$

$$\boxed{\frac{\pi-2}{4}=\frac{1}{1\cdot 3}-\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}-\cdots,}$$

Multiplying by 2 and transposing 1 on the RHS we get,

$$\boxed{\frac{\pi}{2} = 1 + \frac{2}{1\cdot 3} - \frac{2}{3\cdot 5} + \frac{2}{5\cdot 7} - \cdots}$$

[39] *If* $f(x) = \begin{cases} x & \text{in } 0 < x < \dfrac{\pi}{2} \\ \pi - x & \text{in } \dfrac{\pi}{2} < x < \pi \end{cases}$ *show that,*

(i) $f(x) = \dfrac{4}{\pi}\left[\sin x - \dfrac{\sin 3x}{3^2} + \dfrac{\sin 3x}{5^2} - \cdots\right]$

(ii) $f(x) = \dfrac{\pi}{4} - \dfrac{2}{\pi}\left[\dfrac{\cos 2x}{1^2} + \dfrac{\cos 6x}{3^2} + \dfrac{\cos 10x}{5^2} + \cdots\right]$

☞ $f(x)$ is defined in $(0, \pi)$ and we need to find the Fourier sine half range series and cosine half range series separately.

(i) The sine half range series is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad \text{where,} \quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx$$

ie., $$b_n = \frac{2}{\pi}\left\{\int_0^{\pi/2} x \sin nx\, dx + \int_{\pi/2}^{\pi} (\pi - x)\sin nx\, dx\right\}$$

Applying Bernoulli's rule to each of the integrals we have,

$$b_n = \frac{2}{\pi}\left\{\left[x\cdot\frac{-\cos nx}{n} - (1)\cdot\frac{-\sin nx}{n^2}\right]_0^{\pi/2}\right.$$

$$\left. + \left[(\pi - x)\cdot\frac{-\cos nx}{n} - (-1)\cdot\frac{-\sin nx}{n^2}\right]_{\pi/2}^{\pi}\right\}$$

$$= \frac{2}{\pi}\left\{\frac{-1}{n}\left[x\cos nx\right]_0^{\pi/2} + \frac{1}{n^2}\left[\sin nx\right]_0^{\pi/2}\right.$$

$$\left. - \frac{1}{n}\left[(\pi - x)\cos nx\right]_{\pi/2}^{\pi} - \frac{1}{n^2}\left[\sin nx\right]_{\pi/2}^{\pi}\right\}$$

$$= \frac{2}{\pi}\left\{\frac{-1}{n}\left(\frac{\pi}{2}\cos\frac{n\pi}{2} - 0\right) + \frac{1}{n^2}\left(\sin\frac{n\pi}{2} - 0\right)\right.$$

$$\left. - \frac{1}{n}\left(0 - \frac{\pi}{2}\cos\frac{n\pi}{2}\right) - \frac{1}{n^2}\left(0 - \sin\frac{n\pi}{2}\right)\right\}$$

$$= \frac{2}{\pi}\left\{\frac{-\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2} + \frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2}\right\}$$

$$= \frac{2}{\pi}\cdot\frac{2}{n^2}\sin\frac{n\pi}{2}$$

$$\therefore \quad b_n = \frac{4}{\pi n^2}\sin\frac{n\pi}{2}$$

The required sine half range series is given by

$$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin nx$$

$$f(x) = \frac{4}{\pi}\left\{\frac{1}{1^2}\sin\frac{\pi}{2}\sin x + \frac{1}{2^2}\sin\pi\sin 2x + \frac{1}{3^2}\sin\frac{3\pi}{2}\sin 3x + \cdots\right\}$$

Thus, $\boxed{f(x) = \frac{4}{\pi}\left\{\sin x - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \cdots\right\}}$

(ii) **The cosine half range Fourier series of $f(x)$ in $(0, \pi)$ is given by**

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{where,}$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx\,; \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$a_0 = \frac{2}{\pi}\left\{\int_0^{\pi/2} x\,dx + \int_{\pi/2}^{\pi}(\pi - x)\,dx\right\}$$

$$= \frac{2}{\pi}\left\{\left[\frac{x^2}{2}\right]_0^{\pi/2} + \left[\pi x - \frac{x^2}{2}\right]_{\pi/2}^{\pi}\right\}$$

$$a_0 = \frac{2}{\pi}\left\{\left(\frac{\pi^2}{8}-0\right)+\left(\pi^2-\frac{\pi^2}{2}\right)-\left(\frac{\pi^2}{2}-\frac{\pi^2}{8}\right)\right\} = \frac{\pi}{2}$$

$\therefore \quad a_0/2 = \pi/4$

$$a_n = \frac{2}{\pi}\left\{\int_0^{\pi/2} x\cos nx\,dx + \int_{\pi/2}^{\pi}(\pi-x)\cos nx\,dx\right\}$$

Applying Bernoulli's rule to each of the integrals,

$$a_n = \frac{2}{\pi}\left\{\left[x\cdot\frac{\sin nx}{n}-(1)\cdot-\frac{\cos nx}{n^2}\right]_0^{\pi/2}\right.$$

$$\left.+\left[(\pi-x)\frac{\sin nx}{n}-(-1)\cdot-\frac{\cos nx}{n^2}\right]_{\pi/2}^{\pi}\right\}$$

$$= \frac{2}{\pi}\left\{\frac{1}{n}\left[x\sin nx\right]_0^{\pi/2}+\frac{1}{n^2}\left[\cos nx\right]_0^{\pi/2}\right.$$

$$\left.+\frac{1}{n}\left[(\pi-x)\sin nx\right]_{\pi/2}^{\pi}-\frac{1}{n^2}\left[\cos nx\right]_{\pi/2}^{\pi}\right\}$$

$$= \frac{2}{\pi}\left\{\frac{1}{n}\left(\frac{\pi}{2}\sin\frac{n\pi}{2}-0\right)+\frac{1}{n^2}\left(\cos\frac{n\pi}{2}-1\right)\right.$$

$$\left.+\frac{1}{n}\left(0-\frac{\pi}{2}\sin\frac{n\pi}{2}\right)-\frac{1}{n^2}\left(\cos n\pi-\cos\frac{n\pi}{2}\right)\right\}$$

$$= \frac{2}{\pi}\left\{\frac{\pi}{2n}\sin\frac{n\pi}{2}+\frac{1}{n^2}\left(\cos\frac{n\pi}{2}-1-\cos n\pi+\cos\frac{n\pi}{2}\right)-\frac{\pi}{2n}\sin\frac{n\pi}{2}\right\}$$

$$= \frac{2}{\pi n^2}\left(-1-\cos n\pi+2\cos\frac{n\pi}{2}\right) = \frac{-2}{\pi n^2}\left\{1+(-1)^n-2\cos\frac{n\pi}{2}\right\}$$

But, $\quad 1+(-1)^n = \begin{cases} 2 \text{ when } n \text{ is even} \\ 0 \text{ when } n \text{ is odd}\end{cases}$

$$a_n = \frac{-2}{\pi n^2}\left(2-2\cos\frac{n\pi}{2}\right), \text{ where } n = 2,4,6\cdots$$

$$a_n = \frac{-4}{\pi n^2}\left(1-\cos\frac{n\pi}{2}\right),\ n = 2,4,6\cdots$$

But, $$1-\cos\frac{n\pi}{2} = \begin{cases} 1-(-1) = 2 & \text{where } n = 2,6,10,\cdots \\ 1-(+1) = 0 & \text{where } n = 4,8,12,\cdots \end{cases}$$

$$\therefore \quad a_n = \frac{-4}{\pi n^2}(2) \text{ or } a_n = \frac{-8}{\pi n^2} \text{ where } n = 2,6,10,\cdots$$

The required cosine half range Fourier series is given by

$$f(x) = \frac{\pi}{4} - \frac{8}{\pi}\sum_{n=2,6,10,\cdots}^{\infty}\frac{1}{n^2}\cos nx$$

ie., $$f(x) = \frac{\pi}{4} - \frac{8}{\pi}\left(\frac{1}{2^2}\cos 2x + \frac{1}{6^2}\cos 6x + \frac{1}{10^2}\cos 10x + \cdots\right)$$

$$= \frac{\pi}{4} - \frac{8}{\pi}\cdot\frac{1}{2^2}\left(\frac{1}{1^2}\cos 2x + \frac{1}{3^2}\cos 6x + \frac{1}{5^2}\cos 10x + \cdots\right)$$

Thus, $$\boxed{f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left(\frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \frac{\cos 10x}{5^2} + \cdots\right)}$$

[40] *Obtain the sine half range series of*

$$f(x) = \begin{cases} \frac{1}{4} - x & in\ \ 0 < x < \frac{1}{2} \\ x - \frac{3}{4} & in\ \ \frac{1}{2} < x < 1 \end{cases}$$ **[Dec 2016]**

☞ $f(x)$ is defined in $(0, 1)$. Comparing with half range $(0, l)$ we have $l = 1$. The corresponding sine half range series is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x \quad \text{where,} \quad b_n = \frac{2}{1}\int_0^1 f(x)\sin n\pi x\, dx$$

ie., $$b_n = 2\left\{\int_0^{1/2}\left(\frac{1}{4}-x\right)\sin n\pi x\, dx + \int_{1/2}^{1}\left(x-\frac{3}{4}\right)\sin n\pi x\, dx\right\}$$

Applying Bernoulli's rule to each of the integrals,

$$b_n = 2\left\{\left[\left(\frac{1}{4}-x\right)\cdot\frac{-\cos n\pi x}{n\pi}-(-1)\cdot\frac{-\sin n\pi x}{n^2\pi^2}\right]_0^{1/2}\right.$$

$$\left.+\left[\left(x-\frac{3}{4}\right)\cdot\frac{-\cos n\pi x}{n\pi}-1\cdot\frac{-\sin n\pi x}{n^2\pi^2}\right]_{1/2}^{1}\right\}$$

$$= 2\left\{\frac{-1}{n\pi}\left[\left(\frac{1}{4}-x\right)\cos n\pi x\right]_0^{1/2}-\frac{1}{n^2\pi^2}[\sin n\pi x]_0^{1/2}\right.$$

$$\left.-\frac{1}{n\pi}\left[\left(x-\frac{3}{4}\right)\cos n\pi x\right]_{1/2}^{1}+\frac{1}{n^2\pi^2}[\sin n\pi x]_{1/2}^{1}\right\}$$

$$= 2\left\{\frac{-1}{n\pi}\left(-\frac{1}{4}\cos\frac{n\pi}{2}-\frac{1}{4}\right)-\frac{1}{n^2\pi^2}\left(\sin\frac{n\pi}{2}\right)\right.$$

$$\left.-\frac{1}{n\pi}\left(\frac{1}{4}\cos n\pi+\frac{1}{4}\cos\frac{n\pi}{2}\right)+\frac{1}{n^2\pi^2}\left(0-\sin\frac{n\pi}{2}\right)\right\}$$

$$= 2\left\{\frac{1}{4n\pi}\left(\cos\frac{n\pi}{2}+1-\cos n\pi-\cos\frac{n\pi}{2}\right)-\frac{2}{n^2\pi^2}\sin\frac{n\pi}{2}\right\}$$

$$= 2\left\{\frac{1}{4n\pi}(1-\cos n\pi)-\frac{2}{n^2\pi^2}\sin\frac{n\pi}{2}\right\}$$

$$b_n = \frac{1}{2n\pi}\{1-(-1)^n\}-\frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}$$

Thus the sine half range series is given by

$$\boxed{f(x) = \sum_{n=1}^{\infty}\left[\frac{1}{2n\pi}\{1-(-1)^n\}-\frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}\right]\sin n\pi x}$$

[41] *Obtain the sine half range Fourier series for the function*

$$f(x) = \begin{cases}\dfrac{2kx}{l} & in\ \ 0\le x\le\dfrac{l}{2}\\[2mm] \dfrac{2k}{l}(l-x) & in\ \ \dfrac{l}{2}\le x\le l\end{cases}$$

☞ $f(x)$ is defined in $(0, l)$ and the sine half range series is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \quad \text{where } b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{2}{l}\left\{\int_0^{l/2} \frac{2kx}{l}\sin\frac{n\pi x}{l}\,dx + \int_{l/2}^{l} \frac{2k}{l}(l-x)\sin\frac{n\pi x}{l}\,dx\right\}$$

$$= \frac{4k}{l^2}\left\{\int_0^{l/2} x\sin\frac{n\pi x}{l}\,dx + \int_{l/2}^{l} \frac{2k}{l}(l-x)\sin\frac{n\pi x}{l}\,dx\right\}$$

On integration [*similar to problem* - [39] (i)] we obtain $b_n = \dfrac{8k}{n^2\pi^2}\sin\dfrac{n\pi}{2}$

Thus the required sine half range series is given by

$$\boxed{f(x) = \frac{8k}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{l}}$$

[42] *Obtain the half range cosine series for the function*

$$f(x) = \begin{cases} \dfrac{\pi}{3}, & 0 < x < \dfrac{\pi}{3} \\ 0, & \dfrac{\pi}{3} < x < \dfrac{2\pi}{3} \\ \dfrac{-\pi}{3}, & \dfrac{2\pi}{3} < x < \pi. \end{cases}$$

☞ $f(x)$ is defined in $(0, \pi)$ and the cosine half range series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{where,}$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx; \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$a_0 = \frac{2}{\pi}\left\{\int_0^{\pi/3}\frac{\pi}{3}\,dx + \int_{\pi/3}^{2\pi/3} 0\,dx + \int_{2\pi/3}^{\pi}\frac{-\pi}{3}\,dx\right\}$$

$$= \frac{2}{\pi} \cdot \frac{\pi}{3} \left\{ \int_0^{\pi/3} 1\, dx - \int_{2\pi/3}^{\pi} 1\, dx \right\}$$

$$= \frac{2}{3} \left\{ [\, x \,]_0^{\pi/3} - [\, x \,]_{2\pi/3}^{\pi} \right\}$$

$$= \frac{2}{3} \left\{ \left(\frac{\pi}{3} - 0 \right) - \left(\pi - \frac{2\pi}{3} \right) \right\} = 0$$

$$\therefore \quad a_0/2 = 0$$

$$a_n = \frac{2}{\pi} \cdot \frac{\pi}{3} \left\{ \int_0^{\pi/3} \cos nx\, dx - \int_{2\pi/3}^{\pi} \cos nx\, dx \right\}$$

$$= \frac{2}{3} \left\{ \left[\frac{\sin nx}{n} \right]_0^{\pi/3} - \left[\frac{\sin nx}{n} \right]_{2\pi/3}^{\pi} \right\}$$

$$= \frac{2}{3n} \left\{ \left(\sin \frac{n\pi}{3} - 0 \right) - \left(0 - \sin \frac{2n\pi}{3} \right) \right\}$$

$$\therefore \quad a_n = \frac{2}{3n} \left(\sin \frac{n\pi}{3} + \sin \frac{2n\pi}{3} \right)$$

Thus the cosine half range series is given by

$$\boxed{f(x) = \frac{2}{3} \sum_{n=1}^{\infty} \frac{1}{n} \left(\sin \frac{n\pi}{3} + \sin \frac{2n\pi}{3} \right) \cos nx}$$

ASSIGNMENT

Obtain the cosine half range Fourier series for the following functions over the indicated interval.

1. $f(x) = \left(1 - \frac{x}{l}\right)^2$ in $(0, l)$

2. $f(x) = 2x - 1$ in $(0, 2)$

3. $f(x) = \cos ax$, a is not an integer in $(0, \pi)$

4. $f(x) = \begin{cases} \frac{1}{4} - x & \text{in } 0 < x < \frac{1}{2} \\ x - \frac{3}{2} & \text{in } \frac{1}{2} < x < 1 \end{cases}$ 5. $f(x) = \begin{cases} x & \text{in } 0 < x < 4 \\ 8 - x & \text{in } 4 < x < 8 \end{cases}$

Obtain the sine half range series for the following functions over the indicated interval.

6. $f(x) = 2x - 1$ in $0 < x < 1$ 7. $f(x) = x^2 - x$ in $0 < x < 1$

8. $f(x) = 1 - \left(\frac{x}{\pi}\right)$ in $0 < x < \pi$

9. $f(x) = \begin{cases} x & \text{in } 0 \le x < \frac{1}{2} \\ 1 - x & \text{in } \frac{1}{2} \le x \le 1 \end{cases}$ 10. $f(x) = \begin{cases} kx & \text{in } 0 \le x \le \frac{l}{2} \\ k(l - x) & \text{in } \frac{l}{2} \le x \le l \end{cases}$

11. Show that the half range Fourier sine series for $f(x) = k$ in $(0, \pi)$ is $\frac{4k}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right)$ and hence deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots$

12. Show that the cosine half range series for $f(x) = a \sin x$ in $(0, \pi)$ is

$$\frac{4a}{\pi}\left(\frac{1}{2} - \frac{\cos 2x}{1\cdot 3} - \frac{\cos 4x}{3\cdot 5} - \frac{\cos 6x}{5\cdot 7} \cdots\right)$$ and hence deduce that

$$\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \cdots = \frac{\pi - 2}{4}$$

ANSWERS

1. $\frac{1}{3} + \frac{4}{\pi^2}\sum_{1}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{l}$

2. $1 - \frac{8}{\pi^2}\{1 - (-1)^n\}\cos\frac{n\pi x}{2}$

3. $\frac{\sin a\pi}{a\pi} + \frac{2a}{\pi}\sum_{1}^{\infty}(-1)^{n+1}\frac{\sin a\pi \cos nx}{n^2 - a^2}$

4. $\frac{2}{\pi^2}\sum_1^\infty \frac{1}{n^2}\left\{1+\cos n\pi - 2\cos\frac{n\pi}{2}\right\}\cos n\pi x$

5. $\frac{-16}{\pi^2}\sum_1^\infty \frac{1}{n^2}\left\{1+\cos n\pi - 2\cos\frac{n\pi x}{8}\right\}$

6. $\frac{-4}{\pi}\sum_{n=2,4,6,\ldots}^\infty \frac{1}{n}\sin n\pi x$

7. $\frac{-8}{\pi^3}\sum_{n=1,3,5,\ldots}^\infty \frac{1}{n^3}\sin 2n\pi x$

8. $\frac{2}{\pi}\sum_1^\infty \frac{1}{n}\sin nx$

9. $\frac{4}{\pi^2}\sum_1^\infty \frac{1}{n^2}\sin\frac{n\pi}{2}\sin n\pi x$

10. $\frac{4kl}{\pi^2}\sum_1^\infty \frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{l}$

2.8 Practical Harmonic Analysis

So far, we have discussed the methods of obtaining the Fourier series of a known function $f(x)$ in a given interval. However, there will also be situations where in there will be no known functional expression for $f(x)$ but only the values at some equidistant points will be known.

Harmonic analysis is the process of finding the constant term and the first few cosine and sine terms numerically.

The Fourier series of period 2π of a function $y = f(x)$ will be of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^\infty a_n \cos nx + \sum_{n=1}^\infty b_n \sin nx,$$

where the Fourier coefficients are given by

$$a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\,dx,\ a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx,\ b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx.$$

$a_0/2$ is called the constant term and the groups of terms $(a_1 \cos x + b_1 \sin x)$, $(a_2 \cos 2x + b_2 \sin 2x)$ etc. are called the *first harmonics, second harmonics etc.*

To derive the relevant formulae, we need the following principle.

The mean value of a continuous function $y = f(x)$ in the range (a, b) is given by $\frac{1}{b-a}\int_a^b f(x)\,dx$

Suppose we have a set of N values of $y = f(x)$ having period 2π at equidistant points of x in the interval $c \le x < c + 2\pi$ or $c < x \le c + 2\pi$, [*if the values of y at* $x = c$ *and* $x = c + 2\pi$ *are given, we must omit one of them. Infact* $(y)_{x=c} = (y)_{x=c+2\pi}$ *by the periodic property* $f(x) = f(x + 2\pi)$] the Fourier coefficients a_0, a_n, b_n, assume the following form by the earlier stated principle.

$$a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\,dx = 2\left[\frac{1}{(c+2\pi)-c}\int_c^{c+2\pi} f(x)\,dx\right]$$

Here, $a = c$, $b = c + 2\pi$.

ie., $a_0 = 2$ [Mean value of $f(x) = y$ in $(c, c + 2\pi)$]

ie., $a_0 = 2\left[\frac{\sum y}{N}\right]$ or $a_0 = \frac{2}{N}\sum y$

$$a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx = 2\left[\frac{1}{(c+2\pi)-c}\int_c^{c+2\pi} f(x)\cos nx\,dx\right]$$

$a_n = 2$ [Mean value of $y \cos nx$ in $(c, c + 2\pi)$]

ie., $a_n = 2\left[\frac{\sum y\cos nx}{N}\right]$ or $a_n = \frac{2}{N}\sum y\cos nx$

Similarly, $b_n = 2\left[\frac{\sum y\sin nx}{N}\right]$ or $b_n = \frac{2}{N}\sum y\sin nx$

Suppose we have a set of N values of $y = f(x)$ having period $2l$ at equidistant points of x in the interval $c \le x < c + 2l$ or $c < x \le c + 2l$

$$a_n = 2\left[\frac{\sum y \cos(n\pi x/l)}{N}\right]$$

$$b_n = 2\left[\frac{\sum y \sin(n\pi x/l)}{N}\right]$$

Taking, $\theta = \pi x/l$ we have,

$$a_n = \frac{2}{N}\sum y \cos n\theta$$

$$b_n = \frac{2}{N}\sum y \sin n\theta.$$

Note : *All these formulae holds good for half range Fourier series also.*

Step by step working procedure for problems

Step - 1 We have to first write down the period of $y = f(x)$ from the given range of the values of x.

Step - 2 If the period is 2π, depending on the harmonics required we prepare the relevant table along with the summations $(\sum\ )$ of $y; y\cos x, y\cos 2x \ldots,\ y \sin x,\ y \sin 2x$. . . and the compute the harmonics using the formulae derived by taking $n = 1, 2, \ldots$

Step - 3 If the period is not 2π, we equate it with $2l$ to obtain the value of l.

Step - 4 The summations of y ; $y \cos\theta$, $y \cos 2\theta$, . . . ; $y \sin\theta$, $y \sin 2\theta$... where $\theta = \pi x/l$ will be required to compute the desired harmonics.

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WORKED PROBLEMS

[43] *Determine the constant term and the first cosine and sine terms of Fourier series expansion of y from the following data.* **[Dec 2018]**

x^0	0	45	90	135	180	225	270	315
y	2	3/2	1	1/2	0	1/2	1	3/2

☛ Here the interval of 0^0 to 360^0. That is $0 \le x < 2\pi$. We have to find a_0 ; a_1, b_1. This requires the summation of y, $y \cos x$, $y \sin x$.

x^0	y	$\cos x$	$y \cos x$	$\sin x$	$y \sin x$
0	2.0	1	2.0	0	0
45	1.5	0.7071	1.06065	0.7071	1.06065
90	1.0	0	0	1	1.0
135	0.5	–0.7071	–0.35355	0.7071	0.35355
180	0	–1	0	0	0
225	0.5	–0.7071	–0.35355	–0.7071	–0.35355
270	1.0	0	0	–1	–1
315	1.5	0.7071	1.06065	–0.7071	–1.06065
Totals	**8.0**		**3.4142**		**0**

$$a_0 = \frac{2}{N}\sum y,\ a_1 = \frac{2}{N}\sum y\cos x,\ b_1 = \frac{2}{N}\sum y\sin x,\ N = 8.$$

Also from the table,

$$\sum y = 8.0,\ \sum y\cos x = 3.4142,\ \sum y\sin x = 0.$$

$$a_0 = \frac{2}{8}(8.0) = 2 \quad \text{or} \quad \frac{a_0}{2} = 1\ ;\ a_1 = \frac{2}{8}(3.4142) = 0.85355\ b_1 = \frac{2}{8}(0) = 0$$

The Fourier series of y up to the first harmonic is given by

$$y = a_0/2 + a_1 \cos x + b_1 \sin x$$

Thus, $\boxed{y = 1 + 0.85355 \cos x}$

[44] *Obtain the Fourier series of y upto the second harmonics for the following values.*

x^0	45	90	135	180	225	270	315	360
y	4.0	3.8	2.4	2.0	–1.5	0	2.8	3.4

☞ The interval of x is $0<x\le 2\pi$ and period of $y = f(x)$ is 2π.

We have to compute a_0 ; a_1, b_1 ; a_2, b_2.

$$a_0 = \frac{2}{N}\sum y,\ a_1 = \frac{2}{N}\sum y\cos x,\ b_1 = \frac{2}{N}\sum y\sin x$$

$$a_2 = \frac{2}{N}\sum y\cos 2x,\ b_2 = \frac{2}{N}\sum y\sin 2x.\ \text{Here } N = 8;\frac{2}{N} = \frac{1}{4}$$

x^0	y	cos x	y cos x	sin x	y sin x	cos $2x$	y cos $2x$	sin $2x$	y sin $2x$
45	4.0	0.7071	2.8284	0.7071	2.8284	0	0	1	4.0
90	3.8	0	0	1	3.8	–1	–3.8	0	0
135	2.4	–0.7071	–1.69704	0.7071	1.69704	0	0	–1	–2.4
180	2.0	–1	–2.0	0	0	1	2.0	0	0
225	–1.5	–0.7071	1.06065	–0.7071	1.06065	0	0	1	–1.5
270	0	0	0	–1	0	–1	0	0	0
315	2.8	0.7071	1.97988	–07071	–1.97988	0	0	–1	–2.8
360	3.4	1	3.4	0	0	1	3.4	0	0
Totals	16.9		5.57189		7.40621		1.6		–2.7

From the table,

$$\sum y = 16.9,\ \sum y\cos x = 5.57189,\ \sum y\sin x = 7.40621,$$

$$\sum y\cos 2x = 1.6,\ \sum y\sin 2x = -2.7$$

$a_0 = 1/4\cdot(16.9) = 4.225,$ $\qquad a_0/2 = 2.1125$

$a_1 = 1/4\cdot(5.57189) = 1.393.$ $\qquad b_1 = 1/4\cdot(7.40621) = 1.8516$

$a_2 = 1/4\cdot(1.6) = 0.4$ $\qquad b_2 = 1/4\cdot(-2.7) = -0.675$

The Fourier series of y up to the second harmonic is given by

$$y = a_0/2 + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x)$$

Thus, $\boxed{y = 2.1125 + (1.393 \cos x + 1.8516 \sin x) + (0.4 \cos 2x - 0.675 \sin 2x)}$

[45] *Given the following table*

x^0	0	60^0	120^0	180^0	240^0	300^0
y	7.9	7.2	3.6	0.5	0.9	6.8

obtain the Fourier series neglecting terms higher than first harmonics.

☞ Here the interval of x is 0^0 to 360^0. That is $0 \le x < 2\pi$.

We are required to find a_0, a_1, b_1 only.

x^0	y	$\cos x$	$y \cos x$	$\sin x$	$y \sin x$
0	7.9	1	7.9	0	0
60	7.2	0.5	3.6	0.866	6.2352
120	3.6	–0.5	–1.8	0.866	3.1176
180	0.5	–1	–0.5	0	0
240	0.9	–0.5	–0.45	–0.866	–0.7794
300	6.8	0.5	3.4	–0.866	–5.8888
Totals	**26.9**		**12.15**		**2.6846**

Here, $N = 6$; $2/N = 1/3$

$$a_0 = \frac{2}{N}\sum y = \frac{1}{3}(26.9) = 8.9667 \; ; \; \frac{a_0}{2} = 4.48335$$

$$a_1 = \frac{2}{N}\sum y \cos x = \frac{1}{3}(12.15) = 4.05$$

$$b_1 = \frac{2}{N}\sum y \sin x = \frac{1}{3}(2.6846) = 0.8949$$

The Fourier series up to the first harmonic is given by

$$y = a_0/2 + (a_1 \cos x + b_1 \sin x)$$

Thus, $\boxed{y = 4.48335 + (4.05 \cos x + 0.8949 \sin x)}$

[46] *The turning moment T on the crank shaft of a steam engine for the crank angle θ is given as follows.*

θ^0	0	30	60	90	120	150	180	210	240	270	300	330
T	0	2.7	5.2	7	8.1	8.3	7.9	6.8	5.5	4.1	2.6	1.2

Expand T as a Fourier series upt to first harmonics.

☞ Here the interval of θ is $0 \le \theta < 2\pi$. Period of T is 2π. We are required to find a_0, a_1, b_1. The corresponding formulae are

$$a_0 = \frac{2}{N}\sum T,\ a_1 = \frac{2}{N}\sum T\cos\theta,\ b_1 = \frac{2}{N}\sum T\sin\theta,\ N = 12,\ \frac{2}{N} = \frac{1}{6}$$

θ^0	T	$\cos\theta$	$T\cos\theta$	$\sin\theta$	$T\sin\theta$
0	0	1	0	0	0
30	2.7	0.866	2.3382	0.5	1.35
60	5.2	0.5	2.6	0.866	4.5032
90	7.0	0	0	1	7.0
120	8.1	–0.5	–4.05	0.866	7.0146
150	8.3	–0.866	–7.1878	0.5	4.15
180	7.9	–1	–7.9	0	0
210	6.8	–0.866	–5.8888	–0.5	–3.4
240	5.5	–0.5	–2.75	–0.866	–4.763
270	4.1	0	0	–1	–4.1
300	2.6	0.5	1.3	–0.866	–2.2516
330	1.2	0.866	1.0392	–0.5	–0.6
Totals	59.4		–20.4992		8.9032

$$a_0 = \frac{1}{6}\sum T \qquad = \frac{1}{6}(59.4) = 9.9 \quad ; \quad \frac{a_0}{2} = 4.95$$

$$a_1 = \frac{1}{6}\sum T\cos\theta = \frac{1}{6}(-20.4922) = -3.4165$$

$$b_1 = \frac{1}{6}\sum T\sin\theta = \frac{1}{6}(8.9032) = 1.4839$$

The Fourier series upto the first harmonic is given by

$$T = f(\theta) = a_0/2 + (a_1\cos\theta + b_1\sin\theta)$$

Thus, $\boxed{T = 4.95 - 3.4165\cos\theta + 1.4839\sin\theta}$

[47] *Express y as a Fourier series upto the third harmonics given*

x	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
y	1.98	1.30	1.05	1.30	–0.88	–0.25	1.98

☞ Here the interval of x is $(0 \le x \le 2\pi)$ and the values of y at $x = 0$ and $x = 2\pi$ must be same by the periodic property $f(x + 2\pi) = f(x)$. In the given problem the values of y at $x = 0$ and 2π both are given and we must omit one of them. Let us omit the last value. The values of x in degrees are 0, 60, 120, 180, 240, 300 and $N = 6$.

The relevant table is formulated by considering the values of $\cos x \sin x$; $\cos 2x$, $\sin 2x$; $\cos 3x$, $\sin 3x$ rounded off to three places of decimals and then multiplied by the values of y.

x^0	y	$y\cos x$	$y\cos 2x$	$y\cos 3x$	$y\sin x$	$y\sin 2x$	$y\sin 3x$
0	1.98	1.98	1.98	1.98	0	0	0
60	1.3	0.65	–0.65	–1.3	1.1258	1.1258	0
120	1.05	–0.525	–0.525	1.05	0.9093	–0.9093	0
180	1.3	–1.3	1.3	–1.3	0	0	0
240	–0.88	0.44	0.44	–0.88	0.76208	–0.76208	0
300	–0.25	–0.125	0.125	0.25	0.2165	0.2165	0
Totals	**4.5**	**1.12**	**2.67**	**–0.2**	**3.01368**	**–0.32908**	**0**

$$a_0 = \frac{2}{N}\sum y \quad = \frac{1}{3}(4.5) = 1.5 \;;\; \frac{a_0}{2} = 0.75$$

$$a_1 = \frac{2}{N}\sum y\cos x \quad = \frac{1}{3}(1.12) = 0.3733$$

$$a_2 = \frac{2}{N}\sum y\cos 2x = \frac{1}{3}(2.67) = 0.89$$

$$a_3 = \frac{2}{N}\sum y\cos 3x = \frac{1}{3}(-0.2) = -0.0667$$

$$b_1 = \frac{2}{N}\sum y\sin x \quad = \frac{1}{3}(3.01368) = 1.00456$$

$$b_2 = \frac{2}{N}\sum y\sin 2x = \frac{1}{3}(-0.32908) = -0.1097$$

$$b_3 = \frac{2}{N}\sum y\sin 3x = \frac{1}{3}(0) = 0.$$

Fourier series up to third harmonics is given by

$$y = a_0/2 + (a_1\cos x + b_1\sin x) + (a_2\cos 2x + b_2\sin 2x) + (a_3\cos 3x + b_3\sin 3x)$$

Thus, $$\boxed{\begin{aligned} y &= 0.75 + (0.3733\cos x + 1.00456\sin x) \\ &\quad + (0.89\cos 2x - 0.1097\sin 2x) + (-0.0667\cos 3x) \end{aligned}}$$

Note : *In the earlier problems also directly columns of y cos x, y sin x . . . can be presented along with their totals.*

[48] *Find the Fourier series to represent y (x) upto the second harmonic from the following data :*

x^0	30	60	90	120	150	180	210	240	270	300	330	360
y	2.34	3.01	3.68	4.15	3.69	2.20	0.83	0.51	0.88	1.09	1.19	1.64

☛ The period of $y(x)$ is $2\pi = 360^0$ and the Fourier series upto the second harmonic is given by

$$y(x) = a_0/2 + a_1\cos x + a_2\cos 2x + b_1\sin x + b_2\sin 2x$$

where, $a_0 = \frac{2}{N}\sum y,\; a_1 = \frac{2}{N}\sum y\cos x,\; a_2 = \frac{2}{N}\sum y\cos 2x$

$$b_1 = \frac{2}{N}\sum y\sin x,\; b_2 = \frac{2}{N}\sum y\sin 2x \;;$$

x^0	y	$\cos x$	$\cos 2x$	$\sin x$	$\sin 2x$	$y \cos x$	$y \cos 2x$	$y \sin x$	$y \sin 2x$
30	2.34	0.87	0.5	0.5	0.87	2.0358	1.17	1.17	2.0358
60	3.01	0.5	−0.5	0.87	0.87	1.505	−1.505	2.6187	2.6187
90	3.68	0	−1	1	0	0	−3.68	3.68	0
120	4.15	−0.5	−0.5	0.87	−0.87	−2.075	−2.075	3.6105	−3.6105
150	3.69	−0.87	0.5	0.5	−0.87	−3.2103	1.845	1.845	−3.2103
180	2.20	−1	1	0	0	−2.2	2.2	0	0
210	0.83	−0.87	0.5	−0.5	0.87	−0.7221	0.415	−0.415	0.7221
240	0.51	−0.5	−0.5	−0.87	0.87	−0.255	−0.255	−0.4437	0.4437
270	0.88	0	−1	−1	0	0	−0.88	−0.88	0
300	1.09	0.5	−0.5	−0.87	−0.87	0.545	−0.545	−0.9483	−0.9483
330	1.19	0.87	0.5	−0.5	−0.87	1.0353	0.595	−0.595	−1.0353
360	1.64	1	1	0	0	1.64	1.64	0	0
Totals	**25.21**					**−1.7013**	**−1.075**	**9.6422**	**−2.9841**

$$a_0 = \frac{25.21}{6} = 4.2017 \; ; \; \frac{a_0}{2} = 2.1$$

$$a_1 = \frac{-1.7013}{6} = -0.28, \; a_2 = \frac{-1.075}{6} = -0.18$$

$$b_1 = \frac{9.6422}{6} = 1.61, \qquad b_2 = \frac{-2.9841}{6} = -0.5$$

Thus the required Fourier series upto the second harmonic is given by

$$\boxed{y(x) = 2.1 + (-0.28\cos x + 1.61\sin x) + (-0.18\cos 2x - 0.5\sin 2x)}$$

[49] *Compute the first two harmonics of the Fourier series of $f(x)$ given the following table.*

x	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
$f(x)$	1.0	1.4	1.9	1.7	1.5	1.2	1.0

☞ We have values of $f(x) = y$ in the interval $0 \le x \le 2\pi$ and hence we omit the last value $f(2\pi) = 1.0$ which is same as $f(0)$. The relevant table for computing the first two harmonics is as follows.

x^0	y	cos x	cos $2x$	sin x	sin $2x$	y cos x	y cos $2x$	y sin x	y sin $2x$
0	1	1	1	0	0	1	1	0	0
60	1.4	0.5	−0.5	0.866	0.866	0.7	−0.7	1.2124	1.2124
120	1.9	−0.5	−0.5	0.866	−0.866	−0.95	−0.95	1.6454	−1.6454
180	1.7	−1	1	0	0	−1.7	1.7	0	0
240	1.5	−0.5	−0.5	−0.866	0.866	−0.75	−0.75	−1.299	1.299
300	1.2	0.5	−0.5	−0.866	−0.866	0.6	−0.6	−1.0392	−1.0392
Totals						**−1.1**	**−0.3**	**0.5196**	**−0.1732**

$$a_1 = \frac{2}{N}\sum y\cos x = \frac{2}{6}(-1.1) = -0.367$$

$$a_2 = \frac{2}{N}\sum y\cos 2x = \frac{2}{6}(-0.3) = -0.1$$

$$b_1 = \frac{2}{N}\sum y\sin x = \frac{2}{6}(0.5196) = 0.1732$$

$$b_2 = \frac{2}{N}\sum y\sin 2x = \frac{2}{6}(-0.1732) = -0.0577$$

The first two harmonics are

$(a_1\cos x + b_1\sin x)$ and $(a_2\cos 2x + b_2\sin 2x)$. Thus they are

$$\boxed{(-0.367\cos x + 0.1732\sin x) \text{ and } (-0.1\cos 2x - 0.0577\sin 2x)}$$

[50] *Obtain the constant term and the coefficients of the first cosine and sine terms in the Fourier expansion of y from the table.* **[June 2017]**

x	0	1	2	3	4	5
y	9	18	24	28	26	20

☞ The values at 0, 1, 2, 3, 4, 5 are given ($N = 6$) and hence the interval of x should be $0 \le x < 6$.

$\therefore$ length of the interval is $6 - 0 = 6$. Comparing with $2l$ we have $2l = 6$. or $l = 3$. The Fourier series of period $2l$ is given by

$$y = f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

Since $l = 3$, the series containing the first harmonics is

$$y = f(x) = \frac{a_0}{2} + a_1 \cos\frac{\pi x}{3} + b_1 \sin\frac{\pi x}{3}$$

Writing $\frac{\pi x}{3} = \theta$, $y = \frac{a_0}{2} + a_1 \cos\theta + b_1 \sin\theta$; $N = 6$ and $\frac{2}{N} = \frac{1}{3}$

x	$\theta = \pi x/3$	y	$\cos\theta$	$y\cos\theta$	$\sin\theta$	$y\sin\theta$
0	0	9	1	9	0	0
1	60^0	18	0.5	9	0.866	15.588
2	120^0	24	–0.5	–12	0.866	20.784
3	180^0	28	–1	–28	0	0
4	240^0	26	–0.5	–13	–0.866	–22.516
5	300^0	20	0.5	10	–0.866	–17.32
Total		**125**		**–25**		**–3.464**

$$a_0 = \frac{2}{N}\sum y = \frac{1}{3}(125) \approx 41.67 \; ; \quad \frac{a_0}{2} = 20.835$$

$$a_1 = \frac{2}{N}\sum y\cos\theta = \frac{1}{3}(-25) \approx -8.333$$

$$b_1 = \frac{2}{N}\sum y\sin\theta = \frac{1}{3}(-3.464) \approx -1.155$$

Thus we have,

Constant term = $a_0/2$ = 20.835	
Coefficient of the first cosine term	= a_1 = –8.333
Coefficient of the first sine term	= b_1 = –1.155

[51] *(a) Express y as a Fourier series up to the third harmonics given the following values.*

x	0	1	2	3	4	5
y	4	8	15	7	6	2

(b) Find the Fourier series upto the first harmonic. **[Dec. 2016, June 18]**

☞ As in the previous problem the interval of x is $0 \le x < 6$

$\therefore \quad 2l = 6$ or $l = 3, N = 6$; $2/N = 1/3$

Fourier series upto the third harmonics is given by

$$y = \frac{a_0}{2} + \left(a_1 \cos\frac{\pi x}{l} + b_1 \sin\frac{\pi x}{l}\right) + \left(a_2 \cos\frac{2\pi x}{l} + b_2 \sin\frac{2\pi x}{l}\right)$$

$$+\left(a_3 \cos\frac{3\pi x}{l} + b_3 \sin\frac{3\pi x}{l}\right), \text{ where } l = 3.$$

$$\therefore \quad y = \frac{a_0}{2} + \left(a_1 \cos\frac{\pi x}{3} + b_1 \sin\frac{\pi x}{3}\right) + \left(a_2 \cos\frac{2\pi x}{3} + b_2 \sin\frac{2\pi x}{3}\right)$$

$$+\left(a_3 \cos\frac{3\pi x}{3} + b_3 \sin\frac{3\pi x}{3}\right)$$

Putting $\pi x/3 = \theta$

$$y = a_0/2 + (a_1 \cos\theta + b_1 \sin\theta) + (a_2 \cos 2\theta + b_2 \sin 2\theta) + (a_3 \cos 3\theta + b_3 \sin 3\theta)$$

x	$\theta = \pi x/3$	y	$y \cos\theta$	$y \cos 2\theta$	$y \cos 3\theta$	$y \sin\theta$	$y \cos 2\theta$	$y \sin 3\theta$
0	0	4	4	4	4	0	0	0
1	60°	8	4	−4	−8	6.928	6.928	0
2	120°	15	−7.5	−7.5	15	12.99	−12.99	0
3	180°	7	−7	7	−7	0	0	0
4	240°	6	−3	−3	6	−5.196	5.196	0
5	300°	2	1	−1	−2	−1.732	−1.732	0
Total		42	−8.5	−4.5	8	12.99	−2.598	0

$$a_0 = \frac{2}{N}\sum y = \frac{1}{3}(42) = 14 \ ; \ \frac{a_0}{2} = 7$$

$$a_1 = \frac{2}{N}\sum y\cos\theta = \frac{1}{3}(-8.5) = -2.833\ ;\quad b_1 = \frac{2}{N}\sum y\sin\theta = \frac{1}{3}(12.99) = 4.33$$

$$a_2 = \frac{2}{N}\sum y\cos 2\theta = \frac{1}{3}(-4.5) = -1.5;\quad b_2 = \frac{2}{N}\sum y\sin 2\theta = \frac{1}{3}(-2.598) = -0.866$$

$$a_3 = \frac{2}{N}\sum y\cos 3\theta = \frac{1}{3}(8) \approx 2{,}667 \quad b_3 = \frac{2}{N}\sum y\sin 3\theta = \frac{1}{3}(0) = 0$$

Thus the required Fourier series upto the third harmonics is given by

$$\boxed{y = 7 - 2.833\cos\frac{\pi x}{3} + 4.33\sin\frac{\pi x}{3} - 1.5\cos\frac{2\pi x}{3} - 0.866\sin\frac{2\pi x}{3} + 2.667\cos\pi x}$$

[52] *Obtain the constant term and the first three coefficients in the Fourier cosine series for y using the following table*

x	0	1	2	3	4	5
y	4	8	15	7	6	2

☞ Here the interval of x is $0 \le x < 6$ and since the coefficients of the Fourier cosine series are to be found we have to conclude that it should be the cosine half range Fourier series of $y = f(x)$ in $(0, 6)$. Comparing with half the range $(0, l)$ we get $l = 6$ and the Fourier cosine series is of the form

$$y = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}.$$

ie., $$y = \frac{a_0}{2} + a_1\cos\left(\frac{\pi x}{6}\right) + a_2\cos\left(\frac{2\pi x}{6}\right) + a_3\cos\left(\frac{3\pi x}{6}\right) + \cdots$$

Putting $\pi x/6 = \theta$ the Fourier series upto third harmonics assumes the form

$$y = a_0/2 + a_1\cos\theta + a_2\cos 2\theta + a_3\cos 3\theta \qquad \ldots (1)$$

We have to compute $a_0/2$ (*constant term*) a_1, a_2, a_3 by the formulae

$$a_0 = \frac{2}{N}\sum y,\quad a_1 = \frac{2}{N}\sum y\cos\theta,\quad a_2 = \frac{2}{N}\sum y\cos 2\theta,\quad a_3 = \frac{2}{N}\sum y\cos 3\theta$$

Here, $N = 6$; $\frac{2}{N} = \frac{1}{3}$

The relevant table is as follows :

x	$\theta = \pi x/6$	y	$\cos\theta$	$\cos 2\theta$	$\cos 3\theta$	$y\cos\theta$	$y\cos 2\theta$	$y\cos 3\theta$
0	0	4	1	1	1	4	4	4
1	30^0	8	0.866	0.5	0	6.928	4	0
2	60^0	15	0.5	–0.5	–1	7.5	–7.5	–15
3	90^0	7	0	–1	0	0	–7	0
4	120^0	6	–0.5	–0.5	1	–3	–3	6
5	150^0	2	–0.866	0.5	0	–1.732	1	0
Total		**42**				**13.696**	**–8.5**	**–5**

Here, $\sum y = 42,\ \sum y\cos\theta = 13.696,\ \sum y\cos 2\theta = -8.5,\ \sum y\cos 3\theta = -5$

$$a_0 = \frac{1}{3}(42) = 14 \ \therefore\ \frac{a_0}{2} = 7\ ;\ a_1 = \frac{1}{3}(13.696) \approx 4.565$$

$$a_2 = \frac{1}{3}(-8.5) \approx -2.833\ ;\ a_3 = \frac{1}{3}(-5) \approx -1.667$$

Thus the required values $a_0/2,\ a_1,\ a_2,\ a_3$ are respectively

$$\boxed{7,\ 4.565 - 2.833 \ \text{and}\ -1.667}$$

[53] *Obtain the constant term and the coefficients of* $\sin\theta$ *and* $\sin 2\theta$ *in the Fourier expansion of* y *given the following data.*

θ^0	0	60	120	180	240	300	360
y	0	9.2	14.4	17.8	17.3	11.7	0

☞ Here the interval of θ is $(0, 360^0)$. That is $0 \le \theta \le 2\pi$ and the value of y at $\theta = 0$ and $\theta = 2\pi$ must be the same by the periodic property $f(\theta + 2\pi) = f(\theta)$. When the values are given both at $\theta = 0$ and $\theta = 2\pi$, we must omit one of them. We need to compute the coefficient of : $\sin\theta$ being b_1 and $\sin 2\theta$ being b_2 in the Fourier expansion of y using the formulae,

$b_1 = \frac{2}{N}\sum y\sin\theta,\ b_2 = \frac{2}{N}\sum y\sin 2\theta$ where $N = 6$ by omitting the last value.

The relevant table is as follows.

θ^0	y	$\sin\theta$	$\sin 2\theta$	$y\sin\theta$	$y\sin 2\theta$
0	0	0	0	0	0
60	9.2	0.866	0.866	7.9672	7.9672
120	14.4	0.866	−0.866	12.4704	−12.4704
180	17.8	0	0	0	0
240	17.3	−0.866	0.866	−14.9818	14.9818
300	11.7	−0.866	−0.866	−10.1322	−10.1322
Total				**−4.6764**	**0.3464**

Thus we have,

$$\boxed{b_1 = \frac{2}{6}(-4.6764) = -1.5588 \quad b_2 = \frac{2}{6}(0.3464) = 0.1155}$$

[54] *The following values of y and x are given. Find the Fourier series of y upto second harmonics.*

x	0	2	4	6	8	10	12
y	9.0	18.2	24.4	27.8	27.5	22.0	9.0

☞ The values of y at $x = 0$ and $x = 12$ are same. Hence the interval of x is (0, 12). That is $0 \le x \le 12$ and we shall omit the value of y for $x = 12$ in the process of calculation.

The Fourier series of period $2l$ is given by

$$y = f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

Putting $l = 6$, the Fourier series upto the second harmonics is given by

$$y = f(x) = \frac{a_0}{2} + \left(a_1 \cos\frac{\pi x}{6} + b_1 \sin\frac{\pi x}{6}\right) + \left(a_2 \cos\frac{2\pi x}{6} + b_2 \sin\frac{2\pi x}{6}\right)$$

Putting $\theta = \pi x/6$ we have,

$$y = a_0/2 + (a_1 \cos\theta + b_1 \sin\theta) + (a_2 \cos 2\theta + b_2 \sin 2\theta)$$

The relevant table is as follows.

x	y	$\theta=\pi x/6$	$\cos\theta$	$y\cos\theta$	$\cos 2\theta$	$y\cos 2\theta$	$\sin\theta$	$y\sin\theta$	$\sin 2\theta$	$y\sin 2\theta$
0	9.0	0	1	9	1	9.0	0	0	0	0
2	18.2	60	0.5	9.1	−0.5	−9.1	0.866	15.7612	0.866	15.7612
4	24.4	120	−0.5	−12.2	−0.5	−12.2	0.866	21.1304	−0.866	−21.1304
6	27.8	180	−1	−27.8	1	27.8	0	0	0	0
8	27.5	240	−0.5	−13.75	−0.5	−13.75	−0.866	−23.815	0.866	23.815
10	22.0	300	0.5	11.0	−0.5	−11.0	−0.866	−19.052	−0.866	−19.052
Total	128.9			−24.65		−9.25		−5.9754		−0.6062

$$a_0 = \frac{2}{N}\sum y \qquad = \frac{2}{6}(128.9) \approx 42.967 \quad \therefore \quad \frac{a_0}{2} \approx 21.4835$$

$$a_1 = \frac{2}{N}\sum y\cos\theta \qquad = \frac{2}{6}(-24.65) \approx -8.217$$

$$a_2 = \frac{2}{N}\sum y\cos 2\theta \qquad = \frac{2}{6}(-9.25) \approx -3.083$$

$$b_1 = \frac{2}{N}\sum y\sin\theta \qquad = \frac{2}{6}(-5.9754) \approx -1.9918$$

$$b_2 = \frac{2}{N}\sum y\sin 2\theta \qquad = \frac{2}{6}(-0.6062) \approx -0.202$$

Thus the required Fourier series upto the second harmonics is given by

$$\boxed{y = f(x) = 21.4835 + \left(-8.217\cos\frac{\pi x}{6} - 1.9918\sin\frac{\pi x}{6}\right) + \left(-3.083\cos\frac{\pi x}{3} - 0.202\sin\frac{\pi x}{3}\right)}$$

[55] *The following data gives the variations of a periodic current over a period.*

t secs	0	$T/6$	$T/3$	$T/2$	$2T/3$	$5T/6$	T
A amps	9.0	18.2	24.4	27.8	27.5	22.0	9.0

Find numerically the direct current part of the variable current and obtain the amplitudes upto the second harmonic.

☞ We observe that the values of A at $t = 0$ and $t = T$ are the same. Hence we shall omit the last value. We convert $A = f(t)$ to the period 2π by putting $\theta = 2\pi(t/T)$ so that we have $\theta = 0$ when $t = 0$ and $\theta = 2\pi$ when $t = T$. The corresponding values of θ are respectively 0^0, 60^0, 120^0, 180^0, 240^0, 300^0, 360^0

The Fourier series upto the second harmonics is represented by

$$A = a_0/2 + (a_1 \cos\theta + b_1 \sin\theta) + (a_2 \cos 2\theta + b_2 \sin 2\theta)$$

We prepare the relevant table considering the values of A and θ in $0 \le \theta < 2\pi$.

Since the data is same as in Problem - [54] The Fourier series is given by

$$A = f(\theta) = 21.4835 + (-8.217\cos\theta - 1.9918\sin\theta) + (-3.083\cos 2\theta - 0.202\sin 2\theta)$$

The *direct current part* of the variable current is the constant term in the Fourier series being $\boxed{21.4835}$

Amplitude of the first harmonic = $\sqrt{a_1^2 + b_1^2} = \boxed{8.455}$

Amplitude of the second harmonic = $\sqrt{a_2^2 + b_2^2} = \boxed{3.09}$

Note : Similar Problem. *The following table gives the variations of a periodic current A over a period T.* **[June, Dec 2017, Dec 18]**

t (sec)	0	T/6	T/3	T/2	2T/3	5T/6	T
A(Amp)	1.98	1.30	1.05	1.03	−0.88	− 0.25	1.98

Find numerically the direct current part of the variable current and the amplitude of the first harmonic.

Refer [47] for computations. Direct current part is $a_0/2 = 0.75$ and the amplitude of the first harmonic is $\sqrt{a_1^2 + b_1^2} = 1.07168$

ASSIGNMENT

1. Analayse the following data harmonically upto the second harmonic.

x^0	0	30	60	90	120	150	180
y	134	444	326	106	94	144	66
x^0	210	240	270	300	330		
y	−44	−126	−306	−494	−344		

2. Determine first two harmonics in the Fourier series of y given data :

x^0	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
y	0.8	0.6	0.4	0.7	0.9	1.1	0.8

3. Express y as a Fourier series upto the second harmonic given

x^0	0	60	120	180	240	300	360
y	4	3	2	4	5	6	4

4. In a machine the displacement y of a given point is given for certain angles of θ.

θ^0	0	30	60	90	120	150	180
y	7.9	8	7.2	5.6	3.6	1.7	0.5
θ^0	210	240	270	300	330		
y	0.2	0.9	2.5	4.7	6.8		

Find the coefficients of $\sin\theta$ and $\sin 2\theta$ in the Fourier expansion of y.

5. Determine the first harmonic of the Fourier series for the values.

x^0	0	30	60	90	120	150	180
y	−3	2.07	5.29	9.57	10.2	4.68	3
x^0	210	240	270	300	330	360	
y	6.12	6.3	2.25	−3.79	−6.87	−3	

6. The values of y, a periodic function of x, are given below for twelve equidistant values of x covering the whole period. Express y in a Fourier series as far as the third harmonics if the first value is for $x = 30^0$.

1.8, 1.1, 0.3, 0.16, 0.5, 1.5, 2.16, 1.88, 1.25, 1.3, 1.76, 2.

(*Values of the Fourier coefficients be given correct to two decimals*)

7. The turning moment T on the crank shaft of a steam engine for the crank angle θ is given

θ^0	0	15	30	45	60	75	90
T	0	2.7	5.2	7	8.1	8.3	7.9
θ^0	105	120	135	150	165		
T	6.8	5.5	4.1	2.6	1.2		

Expand T as a series of sine upto the second harmonics.

8. Obtain the constant term and coefficients of $\cos\theta$ and $\cos 2\theta$ in the Fourier expansion of y given the data :

θ^0	0	45	90	135	180	225	270	315	360
y	5	4	2	1	–5	–4	–2	–1	5

9. Analyse the following data upto the first harmonic.

x	0	1	2	3	4	5	6	7	8	9	10	11
y	7	9	11	13	14	8	–7	–9	–11	–13	–14	–8

10. Obtain the constant term and the first three coefficients in the Fourier cosine series of y using the following table.

x	0	1	2	3	4	5
y	8	6	4	7	9	11

ANSWERS

1. $(300\sin x)+(100\cos 2x+173\sin 2x)$
2. $(0.1\cos x-0.289\sin x)$ and $(0\cdot\cos 2x+0\cdot\sin 2x)$
3. $4+(0.33\cos x-1.732\sin x)$ [second harmonic is zero]
4. $b_1 = 1.492;\ b_2 = -0.072$
5. $-4.502\cos x+3.1718\sin x$
6. $1.31+(-0.07\cos x-0.62\sin x)+(0.64\cos 2x-0.18\sin 2x)$
   $+(-0.11\cos 3x-0.02\sin 3x)$
7. $7.837\sin\theta+1.484\sin 2\theta$
8. Const. term = 0, Coeff. of $\cos\theta$ = 3.56, Coeff. of $\cos 2\theta$ = 0
9. $0+[2.122\cos(\pi x/6)+14.384\sin(\pi x/6)]$
10. 7.5, 0.39, 1, 4.333

TRANSFORM CALCULUS, FOURIER SERIES & NUMERICAL TECHNIQUES

MODULE - 3

Many engineering problems lead to ordinary or partial differential equations which have to be solved under various types of conditions formulated from the problem. We are already familiar with the solution of higher order ordinary differential equations with initial conditions (initial value problems) using Laplace transforms. Solution of some partial differential equations with boundary conditions (boundary value problems) can be obtained with the help of Fourier transforms.

Further, we proceed to discuss ***Z-Transforms*** *which operates on the sequences of functions of a single variable defined for non negative integral values of the variable.* ***Z-Transforms*** *serves as a tool to solve* ***Difference equations.***