Module - 4

Chapter 7

Principles of Counting - II

In this chapter, we continue the studies on Principles of Counting. We first consider the *Principle of inclusion-exclusion* with appropriate details and illustrations. Then we deal with its applications to *derangements* as well as arrangements with forbidden positions. The so-called *rook polynomials* are used as a tool in the latter case.

7.1 The Principle of Inclusion-Exclusion

Recall that if S is a finite set, then the number of elements in S is called the *order* (or the *size*, or the *cardinality*) of S and is denoted by $|S|$. If A and B are *subsets* of S, then the order of $A \cup B$ is given by the formula

$$|A \cup B| = |A| + |B| - |A \cap B| \tag{1}$$

Thus, for determining the number of elements that are in $A \cup B$, we *include* all elements in A and B, but *exclude* all elements common to A and B.

If $\overline{A}$ is the *complement* of A (in S) and $\overline{B}$ is the complement of B, we recall that $\overline{A} \cap \overline{B} = \overline{(A \cup B)}$ and $|\overline{(A \cup B)}| = |S| - |A \cup B|$. Accordingly, we find by using formula (1), that

$$|\overline{A} \cap \overline{B}| = |\overline{(A \cup B)}| = |S| - |A \cup B| = |S| - |A| - |B| + |A \cap B| \tag{2}$$

The formulas (1) and (2) are equivalent to one another, and either of these is referred to as the *Addition Principle* or the *Principle of inclusion-exclusion*, for two sets.

Below we prove a generalization of this principle to n sets.

Principle of Inclusion-Exclusion for n sets

Let S be a finite set and $A_1, A_2, \ldots, A_n$ be subsets of S. Then the *Principle of inclusion-exclusion* for $A_1, A_2, \ldots, A_n$ states that

$$|A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n| = \Sigma|A_i| - \Sigma|A_i \cap A_j| + \Sigma|A_i \cap A_j \cap A_k| + \cdots + (-1)^{n-1}|A_1 \cap A_2 \cap \cdots \cap A_n| \tag{3}^*$$

*In expression (3) and in what follows, the following *notation* is used (See Foot Note in the next page):

Proof: Take any $x \in A_1 \cup A_2 \cup \ldots \cup A_n$. Then x is in m of the sets $A_1, A_2, \ldots, A_n$ where $1 \le m \le n$. Without loss of generality, let us assume that $x \in A_i$ for $1 \le i \le m$ and $x \notin A_i$ for $i > m$. Then x will be counted once in each of the terms $|A_i|, i = 1, 2, \ldots, m$. Thus, x will be counted m times in $\sum |A_i|$.

We note that there are $C(m, 2)$ pairs of sets A_i, A_j where x is in both A_i and A_j. As such, x will be counted $C(m, 2)$ times in $\sum |A_i \cap A_j|$.

Similarly, x will be counted $C(m, 3)$ times in $\sum |A_i \cap A_j \cap A_k|$, and so on.

Continuing in this way, we see that, in the right hand side of expression (3), x is counted

$$m - C(m, 2) + C(m, 3) + \cdots + (-1)^{m-1} C(m, m)$$

number of times. (Bear in mind that $C(m, n) = 0$ for $n > m$).

We note that

$$\begin{aligned} m - C(m, 2) + C(m, 3) + \cdots + (-1)^{m-1} C(m, m) \\ &= 1 - \{1 - m + C(m, 2) - C(m, 3) + \cdots + (-1)^m C(m, m)\} \\ &= 1 - (1 + (-1))^m, \text{ by binomial theorem} \\ &= 1. \end{aligned}$$

Thus, in the right hand side of expression (3) every element x of $A_1 \cup A_2 \cup \ldots \cup A_n$ is counted exactly once. This means that the number of elements in $A_1 \cup A_2 \cup \ldots \cup A_n$ is equal to the right hand side of expression (3). This completes the proof of expression (3). •

Corollary: By virtue of the De'Morgan law in set Theory, we have

$$\overline{(A_1 \cup A_2 \cup A_3 \ldots \cup A_n)} = \overline{A_1} \cap \overline{A_2} \cap \ldots \cap \overline{A_n}.$$

Since $|\overline{A}| = |S| - |A|$ for any subset A of S, this yields

$$\begin{aligned} |\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \ldots \cap \overline{A_n}| &= |\overline{(A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n)}| \\ &= |S| - |(A_1 \cup A_2 \cup \ldots \cup A_n)| \end{aligned}$$

*$\sum |A_i|$ = Sum of the orders of sets, $A_1, A_2, \ldots, A_n$,
$\sum |A_i \cap A_j|$ = Sum of the orders of intersections of $A_1, A_2, \ldots A_n$, taken 2 at a time,
$\sum |A_i \cap A_j \cap A_k|$ = Sum of the orders of intersections of $A_1, A_2, \ldots, A_n$, taken 3 at a time, and so on.

We note that:
the sum $\sum |A_i|$ contains n terms,
the sum $\sum |A_i \cap A_j|$ contains $C(n, 2) \equiv \binom{n}{2}$ terms,
the sum $\sum |A_i \cap A_j \cap A_k|$ contains $C(n, 3) \equiv \binom{n}{3}$ terms, and so on.

Using expression (3), this becomes

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \ldots \cap \overline{A_n}| = |S| - \Sigma|A_i| + \Sigma|A_i \cap A_j| - \Sigma|A_i \cap A_j \cap A_k| + \cdots$$
$$+ (-1)^n |A_1 \cap A_2 \cap \ldots \cap A_n| \tag{4}$$

This expression is an equivalent version of the Principle of inclusion-exclusion, given by (3). Note that, for $n = 2$, equations (3) and (4) reduce to expressions (1) and (2), respectively.

Alternative Versions

We may rewrite expressions (3) and (4) in other forms as well.

Suppose A_1 represents the set of all those elements of S which satisfy a certain condition c_1, A_2 represents the set of all those elements of S which satisfy a certain condition c_2, and so on.

Let us put

$$N = |S|, \quad N(c_i) = |A_i|, \quad N(\overline{c_i}) = |\overline{A_i}|,$$
$$N(c_i c_j) = |A_i \cap A_j|, \qquad N(\overline{c}_i \overline{c}_j) = |\overline{A_i} \cap \overline{A_j}|,$$
$$N(c_i c_j c_k) = |A_i \cap A_j \cap A_k|, \qquad N(\overline{c}_i \overline{c}_j \overline{c}_k) = |\overline{A_i} \cap \overline{A_j} \cap \overline{A_k}|,$$
$$\cdots\cdots$$
$$\cdots\cdots$$
$$N(c_1 c_2 c_3 \ldots c_n) = |\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \ldots \cap \overline{A_n}|,$$

and
$$N(c_i \text{ or } c_j) = |A_i \cup A_j|, \quad N(c_i \text{ or } c_j \text{ or } c_k) = |A_i \cup A_j \cup A_k|,$$
$$\cdots\cdots$$
$$N(c_1 \text{ or } c_2 \text{ or } c_3 \text{ or } \ldots \text{ or } c_n) = |A_1 \cup A_2 \cup \ldots \cup A_n|.$$

Then, expression (3) reads

$$N(c_1 \text{ or } c_2 \text{ or } c_3 \text{ or } \cdots \text{ or } c_n) = \Sigma N(c_i) - \Sigma N(c_i c_j) + \Sigma N(c_i c_j c_k) - \cdots$$
$$+ (-1)^{n-1} N(c_1 c_2 c_3 \cdots c_n) \tag{5}$$

and expression (4) reads

$$\overline{N} = N - \Sigma N(c_i) + \Sigma N(c_i c_j) - \Sigma N(c_i c_j c_k) + \cdots$$
$$+ (-1)^n N(c_1 c_2 c_3 \cdots c_n) \tag{6}$$

Using expression (6), expression (5) can be written as

$$N(c_1 \text{ or } c_2 \text{ or } c_3 \text{ or } \cdots \text{ or } c_n) = N - \overline{N} \tag{7}$$

Putting

$$S_0 = N = |S|, \quad S_1 = \Sigma N(c_i) = \Sigma|A_i|, \quad S_2 = \Sigma N(c_i c_j) = \Sigma|A_i \cap A_j|,$$
$$S_3 = \Sigma N(c_i c_j c_k) = \Sigma|A_i \cap A_j \cap A_k|, \ldots \tag{8}$$

and so on, expressions (5) and (6) can be rewritten respectively as follows :

$$N(c_1 \text{ or } c_2 \text{ or } c_3 \text{ or } \cdots \text{ or } c_n) = S_1 - S_2 + S_3 - \cdots + (-1)^{n-1} S_n \tag{9}$$

$$\overline{N} = S_0 - S_1 + S_2 - S_3 + \cdots + (-1)^n S_n \tag{10}$$

Generalization

The principle of inclusion-exclusion as given by expression (10) gives the number of elements in S that satisfy none of the conditions $c_1, c_2, \ldots, c_n$. The following expression determines *the number of elements in S that satisfy exactly m of the n conditions* $(0 \leq m \leq n)$:

$$E_m = S_m - \binom{m+1}{1} S_{m+1} + \binom{m+2}{2} S_{m+2} - \cdots + (-1)^{n-m} \binom{n}{n-m} S_n \tag{11}$$

For $m = 0$, this expression reduces to expression (10).

Further, the following expression determines *the number of elements in S that satisfy at least m of the n conditions* $(1 \leq m \leq n)$:

$$L_m = S_m - \binom{m}{m-1} S_{m+1} + \binom{m+1}{m-1} S_{m+2} - \cdots + (-1)^{n-m} \binom{n-1}{m-1} S_n \tag{12}$$

For $m = 1$, this expression reduces to expression (9).

The proofs of expressions (11) and (12) are omitted.

Example 1 *Among the students in a hostel,* 12 *students study Mathematics (A),* 20 *study Physics (B),* 20 *study Chemistry (C), and* 8 *study Biology (D). There are* 5 *students for A and B,* 7 *students for A and C,* 4 *students for A and D,* 16 *students for B and C,* 4 *students for B and D, and* 3 *students for C and D. There are* 3 *students for A, B and C,* 2 *for A, B, and D,* 2 *for B, C and D,* 3 *for A, C and D. Finally, there are* 2 *who study all of these subjects. Furthermore, there are* 71 *students who do not study any of these subjects. Find the total number of students in the hostel.*

► From, what is given, we have

$$|A| = 12, \quad |B| = 20, \quad |C| = 20, \quad |D| = 8,$$
$$|A \cap B| = 5, \quad |A \cap C| = 7, \quad |A \cap D| = 4,$$
$$|B \cap C| = 16, \quad |B \cap D| = 4, \quad |C \cap D| = 3,$$

$$|A \cap B \cap C| = 3, \quad |A \cap B \cap D| = 2, \quad |B \cap C \cap D| = 2,$$
$$|A \cap C \cap D| = 3, \quad |A \cap B \cap C \cap D| = 2, \quad |\overline{A} \cap \overline{B} \cap \overline{C} \cap \overline{D}| = 71,$$

in the notation which is obvious.

We are required to find $|S|$ where S is the set of all students in the hostel.

The Principle of inclusion-exclusion (as given by equation (4)) applied to the above data gives

$$71 = |S| - (12 + 20 + 20 + 8) + (5 + 7 + 4 + 16 + 4 + 3) - (3 + 2 + 2 + 3) + 2 = |S| - 29$$

This gives
$$|S| = 71 + 29 = 100.$$

Thus, the total number of students in the hostel is 100. ■

Example 2 *Out of 30 students in a hostel, 15 study History, 8 study Economics, and 6 study Geography. It is known that 3 students study all these subjects. Show that 7 or more students study none of these subjects.*

► Let S denote the set of all students in the hostel, and A_1, A_2, A_3 denote the sets of students who study History, Economics and Geography, respectively. Then, from what is given, we have

$$S_1 = \Sigma|A_i| = 15 + 8 + 6 = 29, \quad \text{and} \quad S_3 = |A_1 \cap A_2 \cap A_3| = 3.$$

The number of students who do not study any of the three subjects is $|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}|$. This is given by (see expression (4))

$$\begin{aligned} |\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}| &= |S| - \Sigma|A_i| + \Sigma|A_i \cap A_j| - \Sigma|A_1 \cap A_2 \cap A_3| \\ &= |S| - S_1 + S_2 - S_3 \\ &= 30 - 29 + S_2 - 3 = S_2 - 2 \end{aligned} \qquad ((i))$$

where $S_2 = \Sigma|A_i \cap A_j|$.

We note that $(A_1 \cap A_2 \cap A_3)$ is a subset of $(A_i \cap A_j)$ for $i, j = 1, 2, 3$. Therefore, *each* of $|A_i \cap A_j|$, which are 3 in number, is greater than or equal to $|A_1 \cap A_2 \cap A_3|$. Hence

$$S_2 = \Sigma|A_i \cap A_j| \geq 3|A_1 \cap A_2 \cap A_3| = 9.$$

Using this in (i), we find that

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}| \geq 9 - 2 = 7.$$

This proves the required result ■

Example 3 *Determine the number of positive integers n such that $1 \le n \le 100$ and n is not divisible by 2, 3, or 5.*

▶ Let $S = \{1, 2, 3, \ldots, 100\}$. Then $|S| = 100$. Let A_1, A_2, A_3 be the subsets of S whose elements are divisible by 2, 3 and 5 respectively. Then, we have to find $|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}|$.

We note that*

$$\begin{aligned}
|A_1| &= \text{No. of elements in } S \text{ that are divisible by 2} \\
&= \lfloor 100/2 \rfloor = \lfloor 50 \rfloor = 50, \\
|A_2| &= \text{No. of elements in } S \text{ that are divisible by 3} \\
&= \lfloor 100/3 \rfloor = \lfloor 33.333 \rfloor = 33, \\
|A_3| &= \text{No. of elements in } S \text{ that are divisible by 5} \\
&= \lfloor 100/5 \rfloor = \lfloor 20 \rfloor = 20, \\
|A_1 \cap A_2| &= \text{No. of elements in } S \text{ that are divisible by 2 and 3} \\
&= \lfloor 100/6 \rfloor = \lfloor 16.666 \rfloor = 16, \\
|A_1 \cap A_3| &= \text{No. of elements in } S \text{ that are divisible by 2 and 5} \\
&= \lfloor 100/10 \rfloor = \lfloor 10 \rfloor = 10, \\
|A_2 \cap A_3| &= \text{No. of elements in } S \text{ that are divisible by 3 and 5} \\
&= \lfloor 100/15 \rfloor = \lfloor 6.666 \rfloor = 6, \\
|A_1 \cap A_2 \cap A_3| &= \text{No. of elements in } S \text{ that are divisible by 2, 3 and 5} \\
&= \lfloor 100/30 \rfloor = \lfloor 3.333 \rfloor = 3.
\end{aligned}$$

Now, the Principle of inclusion-exclusion gives (see expression (4))

$$\begin{aligned}
|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}| &= |S| - \{|A_1| + |A_2| + |A_3|\} + \{|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|\} - |A_1 \cap A_2 \cap A_3| \\
&= 100 - (50 + 33 + 20) + (16 + 10 + 6) - 3 = 26.
\end{aligned}$$

Thus, the required number is 26. ■

Example 4 *How many integers between 1 and 300 (inclusive) are*

(i) *divisible by at least one of 5, 6, 8?*

(ii) *divisible by none of 5, 6, 8?*

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*The following is a standard result in Number Theory:

For positive integers a and b, the number of positive integers less than or equal to a and divisible by b is $\lfloor a/b \rfloor$, the greatest integer less than or equal to a/b.

► Let $S = \{1, 2, \ldots, 300\}$ so that $|S| = 300$. Also, let A_1, A_2, A_3 be subsets of S whose elements are divisible by 5, 6, 8 respectively. Then:

(i) The number of elements of S that are divisible by at least one of 5, 6, 8 is $|A_1 \cup A_2 \cup A_3|$. This is given by (see expression (3))

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - \{|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|\} + |A_1 \cap A_2 \cap A_3| \quad \text{(i)}$$

We note that

$$|A_1| = \lfloor 300/5 \rfloor = 60, \quad |A_2| = \lfloor 300/6 \rfloor = 50, \quad |A_3| = \lfloor 300/8 \rfloor = 37,$$
$$|A_1 \cap A_2| = \lfloor 300/30 \rfloor = 10, \quad |A_1 \cap A_3| = \lfloor 300/40 \rfloor = 7,$$
$$|A_2 \cap A_3| = \lfloor 300/24 \rfloor = 12 \quad \text{(Note that the l.c.m. of 6 and 8 is 24)},$$
$$|A_1 \cap A_2 \cap A_3| = \lfloor 300/120 \rfloor = 2 \text{ (Note that the l.c.m. of 5, 6, 8 is 120)}.$$

Using these in (i), we get

$$|A_1 \cup A_2 \cup A_3| = (60 + 50 + 37) - (10 + 7 + 12) + 2 = 120.$$

Thus, 120 elements of S are divisible by at least one of 5, 6, 8.

(ii) The number of elements of S that are divisible by none of 5, 6, 8 is

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}| = |S| - |A_1 \cup A_2 \cup A_3| = 300 - 120 = 180.$$

■

Example 5 *Find the number of nonnegative integer solutions of the equation*

$$x_1 + x_2 + x_3 + x_4 = 18$$

X

under the condition $x_i \leq 7$, *for* $i = 1, 2, 3, 4$.

► Let S denote the set of all nonnegative integer solutions of the given equation. The number of such solutions is* $C(4 + 18 - 1, 18) = C(21, 18)$, so that

$$|S| = C(21, 18).$$

Let A_1 be the subset of S that contains the nonnegative integer solutions of the given equation under the conditions $x_1 > 7$, $x_2 \geq 0$, $x_3 \geq 0$, $x_4 \geq 0$. That is,

$$A_1 = \{(x_1, x_2, x_3, x_4) \in S \,|\, x_1 > 7\}$$

*See Section 4.5.

Similarly, let

$$A_2 = \{(x_1, x_2, x_3, x_4) \in S \,|\, x_2 > 7\},$$
$$A_3 = \{(x_1, x_2, x_3, x_4) \in S \,|\, x_3 > 7\},$$
$$A_4 = \{(x_1, x_2, x_3, x_4) \in S \,|\, x_4 > 7\}.$$

Then the required number of solutions would be $|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4}|$.

Let us set $y_1 = x_1 - 8$. Then $x_1 > 7$ (i.e., $x_1 \geq 8$) corresponds to $y_1 \geq 0$. When written in terms of y_1, the given equation reads

$$y_1 + x_2 + x_3 + x_4 = 10.$$

The number of nonnegative integer solutions of this equation is $C(4 + 10 - 1, 10) = C(13, 10)$. This is precisely $|A_1|$. Thus, $|A_1| = C(13, 10)$.

Similarly, by symmetry,

$$|A_2| = |A_3| = |A_4| = C(13, 10).$$

Let us take $y_1 = x_1 - 8$, $y_2 = x_2 - 8$. Then $x_1 > 7$ and $x_2 > 7$ correspond to $y_1 \geq 0$ and $y_2 \geq 0$. When written in terms of y_1 and y_2, the given equation reads

$$y_1 + y_2 + x_3 + x_4 = 2$$

The number of nonnegative integer solutions of this equation is $C(4 + 2 - 1, 2) = C(5, 2)$. This is precisely $|A_1 \cap A_2|$. Thus, $|A_1 \cap A_2| = C(5, 2)$.

Similarly, by symmetry,

$$|A_1 \cap A_3| = |A_1 \cap A_4| = |A_2 \cap A_3| = |A_2 \cap A_4| = |A_3 \cap A_4| = C(5, 2).$$

In the given equation, more than two x_i's cannot be greater than 7 simultaneously. Hence

$$|A_1 \cap A_2 \cap A_3| = |A_1 \cap A_2 \cap A_4| = |A_1 \cap A_3 \cap A_4| = |A_2 \cap A_3 \cap A_4| = 0,$$

$$\text{and} \qquad |A_1 \cap A_2 \cap A_3 \cap A_4| = 0.$$

Accordingly, we find that (using the Principle of inclusion-exclusion as given by equation (4))

$$\begin{aligned}
|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4}| &= |S| - \Sigma|A_i| + \Sigma|A_i \cap A_j| - \Sigma|A_i \cap A_j \cap A_k| + |A_1 \cap A_2 \cap A_3 \cap A_4| \\
&= C(21, 18) - \binom{4}{1} \times C(13, 10) + \binom{4}{2} \times C(5, 2) - 0 + 0 \\
&= 1330 - (4 \times 286) + (6 \times 30) = 366.
\end{aligned}$$

This is the required number of solutions. ■

Example 6 *Find the number of integer solutions of the equation*

$$x_1 + x_2 + x_3 = 20$$

such that $2 \le x_1 \le 5,\ \ 4 \le x_2 \le 7,\ \ -2 \le x_3 \le 9$.

► Let $y_1 = x_1 - 2,\ y_2 = x_2 - 4,\ y_3 = x_3 + 2$. Since $x_1 \ge 2,\ x_2 \ge 4,\ x_3 \ge -2$, we have $y_1 \ge 0$, $y_2 \ge 0,\ y_3 \ge 0$. When written in terms of y_i the given equation reads

$$y_1 + y_2 + y_3 = 16 \tag{i}$$

The number of nonnegative integer solutions of this equation is $C(3 + 16 - 1, 16) = C(18, 16)$. This is precisely the number of integer solutions of the given equation for which $x_1 \ge 2,\ x_2 \ge 4,\ x_3 \ge -2$. If S is this set of solutions, we have $|S| = C(18, 16)$.

When $x_1 \le 5$ we have $y_1 \le 3$, when $x_2 \le 7$ we have $y_2 \le 3$, and when $x_3 \le 9$ we have $y_3 \le 11$. Now, let

$$A_1 = \{(y_1, y_2, y_3) \in S \mid y_1 > 3\}$$
$$A_2 = \{(y_1, y_2, y_3) \in S \mid y_2 > 3\}$$
$$A_3 = \{(y_1, y_2, y_3) \in S \mid y_3 > 11\}$$

We have to find $|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}|$.

Let us set $z_1 = y_1 - 4$. Then $y_1 > 3$ (i.e. $y_1 \ge 4$) corresponds to $z_1 \ge 0$. When written in terms of z_1, equation (i) reads

$$z_1 + y_2 + y_3 = 12.$$

The number of nonnegative integer solutions of this equation is $C(3 + 12 - 1, 12) = C(14, 12)$. This is the number of solutions for which $y_1 > 3$. Thus, $|A_1| = C(14, 12)$.

Similarly, we find that

$$|A_2| = C(3 + 12 - 1, 12) = C(14, 12),$$
$$|A_3| = C(3 + 4 - 1, 4) = C(6, 4).$$

Next, let us set $z_1 = y_1 - 4$ and $w_2 = y_2 - 4$. Then, $z_1 \ge 0$ and $w_2 \ge 0$ if $y_1 > 3$ and $y_2 > 3$. When written in terms of z_1 and w_2, equation (i) reads

$$z_1 + w_2 + y_3 = 8.$$

The number of nonnegative integer solutions of this equation is $C(3 + 8 - 1, 8) = C(10, 8)$. This means that $|A_1 \cap A_2| = C(10, 8)$.

Similarly, we find that

$$|A_1 \cap A_3| = C(3+0-1,0) = C(2,0),$$
$$|A_2 \cap A_3| = C(3+0-1,0) = C(2,0).$$

In any solution of equation (i) we cannot have $y_1 > 3$, $y_2 > 3$ and $y_3 > 11$. Therefore, $|A_1 \cap A_2 \cap A_3| = 0$.

Now, we find that (by using the Principle of inclusion-exclusion)

$$\begin{aligned}|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}| &= |S| - \{|A_1| + |A_2| + |A_3|\} + \{|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|\} \\ &\quad - |A_1 \cap A_2 \cap A_3| \\ &= C(18,16) - \{C(14,12) + C(14,12) + C(6,4)\} \\ &\quad + \{C(10,8) + C(2,0) + C(2,0)\} - 0 \\ &= 153 - \{(2 \times 91) + 15\} + (45+2) = 3.\end{aligned}$$

Thus, under the given conditions, the given equation has exactly three integer solutions. ■

Example 7 *Determine the number of positive integers x where $x \le 9{,}999{,}999$ and the sum of the digits in x equals* 31.

► Here, x is of the form

$$x = x_1x_2x_3x_4x_5x_6x_7$$

where $0 \le x_i \le 9$ for $i = 1, 2, 3, \ldots, 7$, and

$$x_1 + x_2 + x_3 + \cdots + x_7 = 31. \qquad \text{(i)}$$

Thus, we have to find the number of nonnegative integer solutions of equation (i) for which $x_i \le 9$, $i = 1, 2, 3, \ldots 7$.

Let S denote the set of all nonnegative integer solutions of equation (i). Then

$$|S| = C(7+31-1, 31) = C(37, 31).$$

Let A_k denote the set of those solutions in S for which $x_k > 9$ and other x's are non-negative. Taking $y_k = x_k - 10$, we find that $y_k \ge 0$. Substituting for x_k in (i), we get

$$y_k + \text{ sum of } x_i'\text{s excluding } x_k = 21 \qquad \text{(ii)}$$

The number of nonnegative integer solutions of this equation is $C(7+21-1, 21) = C(27, 21)$. This number is precisely $|A_k|$; that is

$$|A_k| = C(27, 21).$$

This holds for all A_k, $k = 1, 2, \ldots 7$.

Let A_p denote the set of those solutions in S for which $x_p > 9$ and other x's are nonnegative. Taking $z_p = x_p - 10$, we find that $z_p \geq 0$. In terms of z_p, equation (ii) reads

$$y_k + z_p + (\text{sum of } x_i'\text{s excluding } x_k \text{ and } x_p) = 11 \tag{iii}$$

The number of nonnegative integer solutions of this equation is

$$|A_k \cap A_p| = C(7 + 11 - 1, 11) = C(17, 11).$$

This holds for $k, p = 1, 2, 3, \ldots 7, k \neq p$. Further, there are $C(7, 2)$ number of intersections $A_k \cap A_p$.

Similarly, we find that

$$|A_k \cap A_p \cap A_q| = C(7 + 1 - 1, 1) = C(7, 1).$$

This holds for all $k,\ p,\ q = 1, 2, \ldots, 7,\ k \neq p,\quad k \neq q,\quad p \neq q$. Further, there are $C(7, 3)$ number of intersections $A_k \cap A_p \cap A_q$, $k \neq p \neq q$.

In equation (i), more than three x_i cannot be greater than 9 simultaneously. Hence the order of the intersection of more than A_i's is zero.

Accordingly, the number of nonnegative integer solutions of equation (i) for which $x_i \leq 9$, $i = 1, 2, \ldots 7$, is given by

$$\begin{aligned} |\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \ldots \cap \overline{A_7}| &= |S| - \Sigma|A_k| + \Sigma|A_k \cap A_p| + \Sigma|A_k \cap A_p \cap A_q| + 0 \\ &= C(37, 31) - 7 \times C(27, 21) + C(7, 2) \times C(17, 11) - C(7, 3) \times C(7, 1) \end{aligned}$$

This is the required number. ■

Example 8 *In how many ways 5 number of a's, 4 number of b's and 3 number of c's can be arranged so that all the identical letters are not in a single block?*

► The given letters are $5 + 4 + 3 = 12$ in number, of which 5 are a's 4 are b's and 3 are c's. If S is the set of all permutations (arrangements) of these letters, we have

$$|S| = \frac{12!}{5!\,4!\,3!}.$$

Let A_1 be the set of arrangements of the letters where the 5 a's are in a single block. The number of such arrangements is

$$|A_1| = \frac{8!}{4!\,3!}.$$

(Because in such an arrangement all the a's taken together can be regarded as a single letter and the remaining letters consist of 4 b's and 3 c's).

Similarly, if A_2 is the set of arrangements where the 4 b's are in a single block, and A_3 is the set of arrangements where the 3 c's are in a single block, we have

$$|A_2| = \frac{9!}{5!\,3!} \quad \text{and} \quad |A_3| = \frac{10!}{5!\,4!}.$$

Likewise,

$$|A_1 \cap A_2| = \frac{5!}{3!}, \quad |A_1 \cap A_2| = \frac{6!}{4!}, \quad |A_2 \cap A_3| = \frac{7!}{5!}, \quad |A_1 \cap A_2 \cap A_3| = 3!.$$

Accordingly, the required number of arrangements is

$$\begin{aligned}
|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}| &= |S| - \{|A_1| + |A_2| + |A_3|\} + \{|A_1 \cap A_2| + |A_1 \cap A_3| \\
&\qquad + |A_2 \cap A_3|\} - |A_1 \cap A_2 \cap A_3| \\
&= \frac{12!}{5!\,4!\,3!} - \left\{\frac{8!}{4!\,3!} + \frac{9!}{5!\,3!} + \frac{10!}{5!\,4!}\right\} + \left\{\frac{5!}{3!} + \frac{6!}{4!} + \frac{7!}{5!}\right\} - 3! \\
&= 27720 - (280 + 504 + 1260) + (20 + 30 + 42) - 6 \\
&= 25762.
\end{aligned}$$

■

Example 9 *Find the number of permutations of the digits* 1 *through* 9 *in which*

(a) *the blocks* 23, 57, 468 *do not appear.*

(b) *the blocks* 36, 78, 672 *do not appear.*

► Let S denote the set of all permutations of the digits 1 thorugh 9 (without repetetion). Then $|S| = 9!$.

(a) Let A_1 be the subset of S which contains the block 23. Thus, A_1 consists of all permutations which contain the block 23 as a single object and the seven remaining objects 1, 4, 5, 6, 7, 8, 9. Thus,

$$|A_1| = (1 + 7)! = 8!$$

Similarly, if A_2 and A_3 are the subsets of S which respectively contain the blocks 57 and 468, we find that

$$|A_2| = (1 + 7)! = 8!, \quad |A_3| = (1 + 6)! = 7!$$

Further, we find that:

$$|A_1 \cap A_2| = \text{No. of permutations of distinct objects consisting of the two blocks 23 and 57, and the (five) digits not present in these blocks.}$$
$$= (2+5)! = 7!$$

Likewise,

$$|A_1 \cap A_3| = (2+4)! = 6!, \quad |A_2 \cap A_3| = (2+4)! = 6!,$$

$$|A_1 \cap A_2 \cap A_3| = \text{No. of permutations of distinct objects consisting of the three blocks 23, 57 and 468, and the (two) digits not present in these blocks}$$
$$= (3+2)! = 5!$$

Accordingly, the required number of permutations is

$$\begin{aligned}|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}| &= |S| - \{|A_1| + |A_2| + |A_3|\} \\ &\quad + \{|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|\} - |A_1 \cap A_2 \cap A_3| \\ &= 9! - 2 \times (8! + 8! + 7!) + (7! + 6! + 6!) - 5! \\ &= 9! - 2 \times (8!) + 2 \times (6!) - 5! \\ &= 2,83,560.\end{aligned}$$

(b) Let A_1, A_2, A_3 be the subsets of S which respectively contain the blocks 36, 78, 672. Like in case (a) above, we find

$$|A_1| = 8!, \quad |A_2| = 8!, \quad |A_3| = 7!$$

$|A_1 \cap A_2|$ = No. of permutations of distinct objects consisting of the two blocks 36 and 78 and the (five) digits not present in these blocks.
= $(2+5)! = 7!$

$|A_1 \cap A_3|$ = No. of permutations of distinct objects consisting of the blocks 36 and 672, and the digits not present in these blocks.
= No. of permutations of distinct objects consisting of the single block 3672 and the (five) digits not present in this block*
= $(1+5)! = 6!$

*Note that the block 3672 includes in it both of the blocks 36 and 672.

$|A_2 \cap A_3| = 0$. (Because, no permutation of distinct objects can contain both of blocks 78 and 672.)[†]

Likewise, $|A_1 \cap A_2 \cap A_3| = 0$.

Accordingly, the required number of permutations is

$$\begin{aligned}
|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}| &= |S| - \{|A_1| + |A_2| + |A_3|\} \\
&\quad + \{|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap |A_3|\} - |A_1 \cap A_2 \cap A_3| \\
&= 9! - (8! + 8! + 7!) + (7! + 6! + 0) - 0 \\
&= 9! - 2 \times (8!) + 6! \\
&= 2,82,960.
\end{aligned}$$

■

Example 10 *Find the number of permutations of the letters a, b, c, . . . , x, y, z in which none of the patterns* spin, game, path *or* net *occurs.*

▶ The specified letters are 26 in number. Let S denote the set of all permutations of these letters without repetition. Then $|S| = 26!$.

Let A_1 be the subset of S which contains the pattern *spin*. Thus, A_1 consists of all permutations (of the above-mentioned 26 letters) which contain the 4-letter pattern *spin* as a single object and the remaining 22 letters as 22 objects. Therefore,

$$\begin{aligned}
|A_1| &= \text{number of permutations of distinct objects consisting of the single pattern } spin \text{ and the 22 letters not present in this pattern.} \\
&= (1 + 22)! = 23!
\end{aligned}$$

Similarly, if A_2, A_3, A_4 are the subsets of S which respectively contain the patterns *game, path, net*, we find that

$$|A_2| = 23!, \quad |A_3| = 23!, \quad |A_4| = (1 + 23)! = 24!$$

Likewise, we find that:

$$\begin{aligned}
|A_1 \cap A_2| &= \text{number of permutations of distinct objects consisting of the two patterns } spin \text{ and } game\text{, and the 18 letters not present in these two patterns} \\
&= (2 + 18)! = 20!
\end{aligned}$$

$|A_1 \cap A_3| = 0$. (Because, no permutation of distinct objects can contain both of the patterns *spin* and *path*).[**]

[†]If there is a permutation containing the blocks 78 and 672, this permutation contains a repeated digit, 7.

[**]If there is a permutation containing both of the patterns *spin* and *path*, this permutation contains a repeated letter, *p*.

Likewise,

$$|A_2 \cap A_3| = 0, \quad |A_2 \cap A_4| = 0, \quad |A_3 \cap A_4| = 0,$$
$$|A_1 \cap A_2 \cap A_3| = 0, \quad |A_1 \cap A_2 \cap A_4| = 0,$$
$$|A_1 \cap A_3 \cap A_4| = 0, \quad |A_2 \cap A_3 \cap A_4| = 0,$$
$$|A_1 \cap A_2 \cap A_3 \cap A_4| = 0.$$

$|A_1 \cap A_4|$ = No. of permutations of distinct objects consisting of the patterns *spin* and *net*, and the letters not present in these patterns.

= No. of permutations of distinct objects consisting of the single pattern *spinet*, and the (20) letters not present in this pattern[†]

= $(1 + 20)! = 21!$

Accordingly, the required number of permutations is given by

$$\begin{aligned}
|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4}| &= |S| - \Sigma|A_i| + \Sigma|A_i \cap A_j| \\
&\quad - \Sigma|A_i \cap A_j \cap A_k| + |A_1 \cap A_2 \cap A_3 \cap A_4| \\
&= 26! - (23! + 23! + 23! + 24!) \\
&\quad + (20! + 0 + 0 + 0 + 0 + 21!) - (0 + 0 + 0 + 0) + 0 \\
&= 26! - \{3 \times (23!) + 24!\} + (20! + 21!).
\end{aligned}$$

■

Example 11 *In how many ways can the* 26 *letters of the English alphabet be permuted so that none of the patterns* CAR, DOG, PUN *or* BYTE *occurs?*

► Let S denote the set of all permutations of the 26 letters. Then $|S| = 26!$.

Let A_1 be the set of all permutations in which CAR appears. This word, CAR, consists of three letters which form a single block. The set A_1 therefore consists of all permutations which contain this single block and the 23 remaining letters. Therefore, $|A_1| = 24!$

Similarly, if A_2, A_3, A_4 are the sets of all permutations which contain DOG, PUN and BYTE respectively, we have

$$|A_2| = 24!, \quad |A_3| = 24!, \quad |A_4| = 23!.$$

Likewise, we find that*

$$|A_1 \cap A_2| = |A_1 \cap A_3| = |A_2 \cap A_3| = (26 - 6 + 2)! = 22!,$$
$$|A_1 \cap A_4| = |A_2 \cap A_4| = |A_3 \cap A_4| = (26 - 7 + 2)! = 21!,$$

[†]Note that the pattern *spinet* includes in it both of the patterns *spin* and *net*.

*Note that no letter is repeated in the given patterns.

$$|A_1 \cap A_2 \cap A_3| = (26 - 9 + 3)! = 20!,$$
$$|A_1 \cap A_2 \cap A_4| = |A_1 \cap A_3 \cap A_4| = |A_2 \cap A_3 \cap A_4| = (26 - 10 + 3)! = 19!,$$
$$|A_1 \cap A_2 \cap A_3 \cap A_4| = (26 - 13 + 4)! = 17!.$$

Therefore, the required number of permutations is given by

$$\begin{aligned}|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4}| &= |S| - \Sigma|A_i| + \Sigma|A_i \cap A_j| - \Sigma|A_i \cap A_j \cap A_k| + |A_1 \cap A_2 \cap A_3 \cap A_4| \\ &= 26! - (3 \times 24! + 23!) + (3 \times 22! + 3 \times 21!) - (20! + 3 \times 19!) + 17!\end{aligned}$$ ■

Example 12 *In a certain area of the country side, there are five villages* A, B, C, D, E. *An Engineer is to devise a system of roads so that, after the system is completed, no village will be isolated. In how many ways can he do this?*

► There are $C(5, 2) = 10$ possible roads between any two of the five villages. Each of these roads can be either included or excluded from the system. Therefore, the number of ways of divising the system of roads is $N = |S| = 2^{10}$, where S is the set of all roads.

For each $1 \leq i \leq 5$, let c_i be the condition that a system of these roads isolates the villages A, B, C, D, E respectively. We have to find $\overline{N} = N(\overline{c_1 c_2 c_3 c_4 c_5})$.

The possible roads in the system that isolate the village A are BC, BD, BE, CD, CE, DE, which are 6 in number. Each of these roads can be included or excluded from the system. Therefore, $N(c_1) = 2^6$. Similarly, by symmetry, $N(c_i) = 2^6$ for $i = 2, 3, 4, 5$ also.

The possible roads in the system that isolate the villages A and B are CD, DE, CE, which are 3 in number. Each of these roads can be included or excluded. Therefore, $N(c_1 c_2) = 2^3$. In view of this, we note, because of symmetry, that $N(c_i c_j) = 2^3, \quad i \neq j,$

Likewise, we find that

$$N(c_i c_j c_k) = 2^1, \quad i \neq j \neq k,$$
$$N(c_i c_j c_k c_p) = 2^0, \quad i \neq j \neq k \neq p, \quad N(c_1 c_2 c_3 c_4 c_5) = 2^0.$$

Accordingly, the Principle of inclusion-exclusion (as given by expression (6)) gives

$$\overline{N} = 2^{10} - \binom{5}{1} \times 2^6 + \binom{5}{2} 2^3 - \binom{5}{3} 2^1 + \binom{5}{4} 2^0 - \binom{5}{5} 2^0 = 768.$$

This is the required answer. ■

Example 13 *Six married couple are to be seated at a circular table. Find in how many ways can they arrange themselves so that no wife sits next to her husband, given that two seating arrangements are considered the same if one is a rotation of the other.*

► First, we note that the total number of arrangements of 6 couples (= 12 persons) around a circular table with no restrictions is $N = (12 - 1)! = 11!$

For $1 \leq i \leq 6$, let c_i denote the condition where a seating arrangement has the husband and wife in a couple i seated next to each other.

To determine $N(c_1)$, we consider arranging 11 distinct objects - namely, couple 1 (cons. ered as one object) and the other 10 people. Eleven distinct objects can be arranged around a circular table in $(11-1)! = 10!$ ways. Further, the wife in couple 1 can be seated to the left or right of her husband. Therefore we have $N(c_1) = 2 \times (10!)$.

Similarly,

$$N(c_2) = N(c_3) = N(c_4) = N(c_5) = N(c_6) = 2 \times 10!$$

Next, let us compute $N(c_i c_j)$ for $1 \leq i \leq 6$, $1 \leq j \leq 6$, $i \neq j$. Here we are arranging 10 distinct objects (couple i considered as one object and couple j as another object) and the other eight people. Ten objects can be arranged around a circular table in $(10-1)! = 9!$ ways. Further, there are two ways for the wife in couple i to be seated next to her husband, and two ways for the wife in couple j to be seated next to her husband. Therefore, $N(c_i c_j) = 2^2 \times (9!)$.

Proceeding like this, we find (for $1 \leq i, j, k, p, q \leq 6$ with $i \neq j \neq k \neq p \neq q$)

$$\begin{aligned}
&N(c_i c_j c_k) = 2^3 \times (8!),\\
&N(c_i c_j c_k c_p) = 2^4 \times (7!),\\
&N(c_i c_j c_k c_p c_q) = 2^5 \times (6!),\\
\text{and}\quad &N(c_1 c_2 c_3 c_4 c_5 c_6) = 2^6 \times (5!).
\end{aligned}$$

We note that $N(c_i)$ are 6 in number, $N(c_i c_j)$ are $C(6,2) \equiv \binom{6}{2}$ in number, $N(c_i c_j c_k)$ are $\binom{6}{3}$ in number, and so on.

Accordingly, the Principle of inclusion-exclusion (as given by expression (6)) gives

$$\begin{aligned}
\overline{N} = 11! &- \binom{6}{1} \times \{2 \times (10!)\} + \binom{6}{2} \times \{2 \times (9!)\}\\
&- \binom{6}{3} \times \{2^3 \times (8!)\} + \binom{6}{4} \times \{2^4 \times (7!)\}\\
&- \binom{6}{5} \times \{2^5 \times (6!)\} + \binom{6}{6} \times \{2^6 \times (5!)\}
\end{aligned}$$

■

Example 14 *Determine the number of integers between* 1 *and* 300 *(inclusive) which are* (i) *divisible by exactly two of* 5, 6, 8, *and* (ii) *divisible by at least two of* 5, 6, 8.

► Let the sets S, A_1, A_2, A_3 be as defined in the Example 4. Using the computations made in that Example, we find that

$$S_0 = |S| = 300, \quad S_1 = \Sigma|A_i| = 60 + 50 + 37 = 147,$$
$$S_2 = \Sigma|A_i \cap A_j| = 10 + 7 + 12 = 29, \quad S_3 = |A_1 \cap A_2 \cap A_3| = 2.$$

Therefore:

(i) the number of integers between 1 and 300 which are divisible by exactly two of 5, 6, 8 is (see expression (11), page 312)

$$E_2 = S_2 - \binom{3}{1} S_3 = 29 - \binom{3}{1} \times 2 = 29 - (3 \times 2) = 23,$$

(ii) the number of integers between 1 and 300 which are divisible by at least two of 2, 3, 5 is (see expression (12), page 312)

$$L_2 = S_2 - \binom{2}{1} S_3 = 29 - (2 \times 2) = 25.$$ ■

Example 15 *Find the number of permutations of the English letters which contain*
(i) *exactly two,* (ii) *at least two,* (iii) *exactly three, and* (iv) *at least three,*
of the patterns CAR, DOG, PUN *and* BYTE.

► Let the sets S, A_1, A_2, A_3, A_4 be defined as in Example 11. Then we note that (by using the computations made in that Example)

$$\begin{aligned} S_0 &= |S| = 26! \\ S_1 &= \Sigma|A_i| = (3 \times 24!) + 23!, \\ S_2 &= \Sigma|A_i \cap A_j| = (3 \times 22!) + (3 \times 21!), \\ S_3 &= \Sigma|A_i \cap A_j \cap A_k| = 20! + (3 \times 19!), \\ S_4 &= |A_1 \cap A_2 \cap A_3 \cap A_4| = 17!. \end{aligned}$$

Therefore (on using expressions (11) and (12), page 312) we find that the number of permutations in the required four cases are as given below:

(i) $E_2 = S_2 - \binom{3}{1} S_3 + \binom{4}{2} S_4 = 3 \times (22! + 21!) - 3 \times \{20! + (3 \times 19!)\} + 6 \times 17!.$

(ii) $L_2 = S_2 - \binom{2}{1} S_3 + \binom{3}{1} S_4 = 3 \times (22! + 21!) - 2 \times \{20! + (3 \times 19!)\} + 3 \times 17!.$

(iii) $E_3 = S_3 - \binom{4}{1} S_4 = \{20! + (3 \times 19!)\} - 4 \times 17!.$

(iv) $L_3 = S_3 - \binom{3}{2} S_4 = \{20! + (3 \times 19!)\} - (3 \times 17!).$ ■

Example 16 *Consider the students referred to in Example 1. Find, among the subjects indicated, how many study* (i) *exactly* 1 *subject,* (ii) *exactly* 2 *subjects,* (iii) *exactly* 3 *subjects,* (iv) *at least* 1 *subject,* (v) *at least* 2 *subjects,* (vi) *at least* 3 *subjects.*

▶ From the details available in Example 1, we note the following:

$$S_0 = |S| = 100, \quad S_1 = 60, \quad S_2 = 39, \quad S_3 = 10, \quad S_4 = 2.$$

The required numbers are, respectively,

(i) $E_1 = S_1 - \binom{2}{1}S_2 + \binom{3}{2}S_3 - \binom{4}{3}S_4 = 60 - (2\times 39) + (3\times 10) - (4\times 2) = 4,$

(ii) $E_2 = S_2 - \binom{3}{1}S_3 + \binom{4}{2}S_4 = 39 - (3\times 10) + (6\times 2) = 21,$

(iii) $E_3 = S_3 - \binom{4}{1}S_4 = 10 - (4\times 2) = 2,$

(iv) $L_1 = S_1 - \binom{1}{0}S_2 + \binom{2}{0}S_3 - \binom{3}{0}S_4 = 60 - 39 + 10 - 2 = 29,$

(v) $L_2 = S_2 - \binom{2}{1}S_3 + \binom{3}{1}S_4 = 39 - (2\times 10) + (3\times 2) = 25,$

(vi) $L_3 = S_3 - \binom{3}{2}S_4 = 10 - (3\times 2) = 4.$ ■

Example 17 *In how many ways can one arrange the letters in the word* CORRESPONDENTS *so that*

(i) *there is no pair of consecutive identical letters?*

(ii) *there are exactly two pairs of consecutive identical letters?*

(iii) *there are at least three pairs of consecutive identical letters?*

▶ In the word CORRESPONDENTS, there occur one each of C, P, D and T, and two each of O, R, E, S, N. If S is the set of all permutations of these 14 letters, we have

$$|S| = \frac{14!}{(2!)^5}.$$

Let A_1, A_2, A_3, A_4, A_5 be the sets of permutations in which O's, R's, E's, S's, N's appear in pairs, respectively. Then

$$|A_i| = \frac{13!}{(2!)^4} \quad \text{for } i = 1,2,3,4,5.$$

Also,

$$|A_i \cap A_j| = \frac{12!}{(2!)^3}, \qquad |A_i \cap A_j \cap A_k| = \frac{11!}{(2!)^2},$$

$$|A_i \cap A_j \cap A_k \cap A_p| = \frac{10!}{(2!)}, \qquad |A_1 \cap A_2 \cap \cdots \cap A_5| = 9!.$$

From these, we get

$$S_0 = N = |S| = \frac{14!}{(2!)^5}, \quad S_1 = C(5,1) \times \frac{13!}{(2!)^4}, \quad S_2 = C(5,2) \times \frac{12!}{(2!)^3},$$

$$S_3 = C(5,3) \times \frac{11!}{(2!)^2}, \quad S_4 = C(5,4) \times \frac{10!}{2!}, \quad S_5 = C(5,5) \times 9!.$$

Accordingly, the number of permutations where there is no pair of consecutive identical letters is

$$\begin{aligned} E_0 &= S_0 - \binom{1}{1}S_1 + \binom{2}{2}S_2 - \binom{3}{3}S_3 + \binom{4}{4}S_4 - \binom{5}{5}S_5 \\ &= \frac{14!}{(2!)^5} - \binom{5}{1} \times \frac{13!}{(2!)^4} + \binom{5}{2} \times \frac{12!}{(2!)^3} - \binom{5}{3} \times \frac{11!}{(2!)^2} + \binom{5}{4} \times \frac{10!}{2!} - \binom{5}{5} \times 9!. \end{aligned}$$

Next, the number of permutations where there are exactly two pairs of consecutive identical letters is

$$\begin{aligned} E_2 &= S_2 - \binom{3}{1}S_3 + \binom{4}{2}S_4 - \binom{5}{3}S_5 \\ &= \binom{5}{2} \times \frac{12!}{(2!)^3} - \binom{3}{1}\binom{5}{3} \times \frac{11!}{(2!)^2} + \binom{4}{2}\binom{5}{4} \times \frac{10!}{2!} - \binom{5}{3}\binom{5}{5} \times 9!. \end{aligned}$$

Lastly, the number of permutations where there are at least three pairs of consecutive identical letters is

$$\begin{aligned} L_3 &= S_3 - \binom{3}{2}S_4 + \binom{4}{2}S_5 \\ &= \binom{5}{3} \times \frac{11!}{(2!)^2} - \binom{3}{2}\binom{5}{4} \times \frac{10!}{2!} + \binom{4}{2}\binom{5}{5} \times 9! \end{aligned}$$

■

Exercises

1. The sets A_1, A_2, A_3, A_4 are subsets of a set S with $|S| = 200$. It is known that

$$|A_1| = 125, \quad |A_2| = 83, \quad |A_3| = 50, \quad |A_4| = 35,$$
$$|A_1 \cap A_2| = 41, \quad |A_1 \cap A_3| = 25, \quad |A_1 \cap A_4| = 17,$$
$$|A_2 \cap A_3| = 16, \quad |A_2 \cap A_4| = 11, \quad |A_3 \cap A_4| = 7,$$
$$|A_1 \cap A_2 \cap A_3| = 8, \quad |A_1 \cap A_2 \cap A_4| = 5, \quad |A_1 \cap A_3 \cap A_4| = 3,$$
$$|A_2 \cap A_3 \cap A_4| = 2, \quad |A_1 \cap A_2 \cap A_3 \cap A_4| = 1.$$

Show that $|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4}| = 7$ and $|A_1 \cup A_2 \cup A_3 \cup A_4| = 193$.

2. Determine the number of positive integers n such that $1 \leq n \leq 300$ and n is divisible by at least one of 3, 5, 7.

3. Determine the number of integers between 1 and 100 (inclusive) which are (i) divisible by exactly two of 2, 3, 5, and (ii) divisible by at least two of 2, 3, 5.

4. Find the number of integers between 1 and 10,000 (inclusive) that are divisible by none of 5, 6, and 8.

5. Find the number of integers between 1 and 1000 that are not divisible by 2, 3, 5 or 7.

6. Determine the number of positive integers n, $1 \leq n \leq 2000$ that are (i) not divisible by 2, 3, or 5, (ii) not divisible by 2, 3, 5, or 7.

7. Find the number of integer solutions of the equation $x_1 + x_2 + x_3 + x_4 = 17$, where $1 \leq x_1 \leq 3$, $2 \leq x_2 \leq 4$, $3 \leq x_3 \leq 5$, and $4 \leq x_4 \leq 6$.

8. Find the number of integer solutions of the equation $x_1 + x_2 + x_3 + x_4 = 18$ such that $1 \leq x_1 \leq 5$, $-2 \leq x_2 \leq 4$, $0 \leq x_3 \leq 5$, $3 \leq x_4 \leq 9$.

9. Find the number of integer solutions of the equation $x_1 + x_2 + x_3 + x_4 = 20$ such that $1 \leq x_1 \leq 6$, $1 \leq x_2 \leq 7$, $1 \leq x_3 \leq 8$ and $1 \leq x_4 \leq 9$.

10. In how many ways 4 a's, 2 b's and 2 c's be arranged so that all identical letters are not in a single block?

11. In how many ways can one distribute 10 distinct prizes among 4 students with exactly 2 students getting nothing? How many ways have at least 2 students getting nothing?

12. For the set considered in Exercise 1, evaluate E_1, E_2, E_3 and L_1, L_2, L_3.

13. For the situation in Worked Example 12, find the number of systems of roads so that (i) exactly two, and (ii) atleast two of the villages remain isolated.

14. In how many ways can one arrange all of the letters in the word INFORMATION so that no pair of consecutive letters occurs more than once?

15. Find the number of permutations of the digits 1 through 9 in which (i) none of the blocks 12, 34 and 567 appears, (ii) none of the blocks 415, 12 and 23 appears.

16. Find the number of permutations of the English letters in which none of the patterns ABC, EFG, PQRS or XYZ occurs.

17. In how many ways can three x's, three y's and three z's be arranged so that no consecutive triple of the same letter appears?

18. In how many ways 5 a's, 4 b's and 3 c's can be arranged so that (i) there are exactly two blocks of identical letters? (ii) there are at least two blocks of identical letters?

19. Fifteen different plants are to be arranged on five shelves. In how many ways the arrangement can be made so that each shelf has at least one but no more than four plants?

20. Determine in how many ways can the letters in the word ARRANGEMENT be arranged so that (i) there are exactly two pairs of consecutive identical letters. (ii) at least two pairs of consecutive identical letters.

Answers

2. 162 **3.** (i) 23 (ii) 26 **4.** 6000 **5.** 228

6. (i) 534 (ii) 458 **7.** 4 **8.** 55 **9.** 217 **10.** 871

11. $E_2 = 6132,\ L_2 = 6136$ **12.** $E_1 = 9,\ E_2 = 69,\ E_3 = 14,\ L_1 = 93,\ L_2 = 84, L_3 = 15.$

13. (i) 40, (ii) 51 **14.** 4,460,400

15. (i) $9! - (8! + 8! + 7!) + (7! + 6! + 6!) - 5!$ (ii) $9! - (8! + 8! + 7!) + (7! + 0 + 6!) - 0$

16. (i) $26! - \{(3 \times 24!) + 23!\} + \{(3 \times 22!) + (3 \times 21!)\} - \{20! + (3 \times 19!)\} + 17!$

17. $\dfrac{9!}{(3!)^3} - 3 \times \dfrac{7!}{(3!)^2} + 3 \times \dfrac{5!}{3!} - 3!$

18. (i) $\left(\dfrac{5!}{3!} + \dfrac{6!}{4!} + \dfrac{7!}{5!}\right) - 3 \times 3!$ (ii) $\left(\dfrac{5!}{3!} + \dfrac{6!}{4!} + \dfrac{7!}{5!} - 2 \times 3!\right)$

19. $(15!) \times \left\{\binom{14}{10} - \binom{5}{1}\binom{10}{6} + \binom{5}{2}\binom{6}{2}\right\}$

20. (i) 3,32,640 (ii) 3,98,160.

7.2 Derangements

A permutation of n distinct objects in which *none* of the objects is in its natural (original) place is called a ***derangement***. For example, a permutation of the integers $1, 2, 3, 4, \ldots n$, in which 1 is not in the first place, 2 is not in the second place, 3 is not in the third place, and so on, and n is not in the nth place is a derangement.

The number of possible derangements of n distinct objects $1, 2, 3, \ldots, n$ is denoted by d_n. If there is only one object, it continues to be in its original place in every arrangement; therefore $d_1 = 0$. If there are two objects, a derangement can be done in only one way – by interchanging their places; therefore $d_2 = 1$. For three objects 1, 2, 3, the possible derangements are 231 and 312; therefore $d_3 = 2$.

Formula for d_n

The following is the formula for d_n for $n \geq 1$:

$$d_n = n!\left\{1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right\}$$

$$= n! \times \sum_{k=0}^{n} \frac{(-1)^k}{k!} \tag{1}$$

Proof: Let S be the set of all (possible) permutations of $1, 2, 3, \ldots, n$. Then $|S| = n!$. Let A_1 be the set of all permutations of $1, 2, 3, \ldots, n$ where 1 is in its natural place, A_2 be the set of all permutations where 2 is in its natural place, and so on. Then the set of all derangements of $1, 2, 3, \ldots, n$ is $\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}$. Consequently,

$$d_n = |\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \ldots \cap \overline{A_n}|.$$

Using the Principle of inclusion-exclusion, we may rewrite this as

$$d_n = |S| - \Sigma|A_i| + \Sigma|A_i \cap A_j| - \Sigma|A_i \cap A_j \cap A_k| + \cdots + (-1)^n|A_1 \cap A_2 \cap \ldots A_n|$$

$$= (n!) - S_1 + S_2 - S_3 + \cdots + (-1)^n S_n \tag{2}$$

We note that the permutations in A_1 are all of the form $1\ b_2b_3 \ldots b_n$, where $b_2b_3 \ldots b_n$ is a permutation of $2, 3, \ldots n$. Thus A_1 consists of all permutations of $2, 3, \ldots n$. As such $|A_1| = (n-1)!$. Similarly, $|A_2| = |A_3| = \cdots = |A_n| = (n-1)!$. Consequently,

$$S_1 = \Sigma|A_i| = n \times (n-1)! = C(n, 1) \times (n-1)!.$$

Next, the permutations in $A_1 \cap A_2$ are all of the form $12b_3b_4 \ldots b_n$ where $b_3b_4 \ldots b_n$ is a permutation of $3, 4, \ldots n$. Thus, $A_1 \cap A_2$ consists of all permutations of $3, 4, \ldots n$. As such, $|A_1 \cap A_2| = (n-2)!$. Similarly, the order of the intersection of every two of the sets $A_1, A_2, \ldots A_n$ is $(n-2)!$. The number of such intersections is $C(n, 2)$.

Accordingly,

$$S_2 = \Sigma|A_i \cap A_j| = C(n, 2) \times (n-2)!.$$

Likewise, we find

$$S_3 = \Sigma|A_i \cap A_j \cap A_k| = C(n, 3) \times (n-3)!$$

$$\cdots\cdots$$

$$S_n = |A_1 \cap A_2 \cap \ldots \cap A_n| = C(n, n) \times (n-n)! = C(n, n).$$

Using these in expression (2), we get

$$d_n = (n!) - C(n,1)\times(n-1)! + C(n,2)\times(n-2)! - C(n,3)\times(n-3)! + \cdots + (-1)^n C(n,n)$$
$$= n! - \frac{n!}{1!} + \frac{n!}{2!} - \frac{n!}{3!} + \cdots + (-1)^n\frac{n!}{n!}$$
$$= n!\left\{1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right\} = n! \times \sum_{k=0}^{n}\frac{(-1)^k}{k!}.$$

This completes the proof.* •

Remark: Recall the exponential expansion

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty}\frac{x^k}{k!}.$$

From this, we get

$$e^{-1} = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}.$$

To five places of decimals, e^{-1} is known to be equal to 0.36788. Thus,

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{k!} = 0.36788 \quad \text{to five places.}$$

On the other hand, we find that

$$\sum_{k=0}^{7}\frac{(-1)^k}{k!} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots - \frac{1}{7!} \approx 0.36786.$$

Thus, for $n \geq 7$, we may take

$$\sum_{k=0}^{n}\frac{(-1)^k}{k!} \approx e^{-1} \approx 0.3679 \quad \text{to four places.}$$

Consequently, for $n \geq 7$, the formula (1) for d_n may be rewritten as

$$d_n \approx \lfloor (n!) \times e^{-1} \rfloor \approx \lfloor (0.3679) \times n! \rfloor \tag{3}$$

*For another proof, see Example 19, Section 7.3.

Example 1 *Find the number of derangements of* $1, 2, 3, 4$.

► Here, there are 4 objects. Therefore, the number of derangements is

$$d_4 = 4! \times \left\{1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right\}$$

$$= 24 \times \left\{1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24}\right\}$$

$$= 12 - 4 + 1 = 9.$$

We can check that the nine derangements of 1, 2, 3, 4 are:

2143	2341	2413
3142	3412	3421
4123	4312	4321

■

Example 2 *Evaluate* d_5, d_6, d_7, d_8.

► We have

$$d_5 = (5!)\left\{1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right\}$$

$$= (120)\left(\frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120}\right) = 44,$$

$$d_6 = (6!)\left\{1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!}\right\}$$

$$= (720)\left(\frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720}\right) = 265,$$

$$d_7 \approx \lfloor (7!) \times e^{-1} \rfloor \approx \lfloor 5040 \times 0.3679 \rfloor \approx 1854,$$

$$d_8 \approx \lfloor (8!) \times e^{-1} \rfloor \approx \lfloor 40320 \times 0.3679 \rfloor \approx 14833.$$

■

Example 3 *While at the race track, a person bets on each of the nine horses in a race to come in accordance to how they are favoured. In how many ways can they reach the finish line so that he loses all his bets?*

► Here, we have to find the number of ways of arranging the horses $1, 2, 3, \ldots, 9$ so that 1 is not in its favoured place, 2 is not in its favoured place,..., and 9 is not in its favoured place. Thus, the required number of ways is the number of derangements of 9 objects, namely,

$$d_9 = e^{-1} \times 9! \approx 0.3679 \times 9!$$

$$= 133504$$

■

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Example 4 *In how many ways can we arrange the numbers* 1, 2, 3, ..., 10 *so that* 1 *is not in the first place,* 2 *is not in the second place, and so on, and* 10 *is not in the* 10^{th} *place?*

▶ The required number of ways is

$$d_{10} \approx (10!)(e^{-1}) \approx 10! \times 0.3679$$
$$\approx 13,35,036. \qquad \blacksquare$$

Example 5 *From the set of all permutations of n distinct objects, one permutation is chosen at random. What is the probability that it is not a derangement?*

▶ The number of permutations of n distinct objects is $n!$. The number of derangements of these objects is d_n. Therefore, the probability that a permutation chosen is *not* a derangement is

$$p = 1 - \frac{d_n}{n!} = 1 - \left\{1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right\}$$
$$= 1 - \frac{1}{2!} + \frac{1}{3!} - + \cdots - \frac{(-1)^n}{n!}. \qquad \blacksquare$$

Example 6 *There are n pairs of children's gloves in a box. Each pair is of a different colour. Suppose the right gloves are distributed at random to n children, and thereafter the left gloves are also distributed to them at random. Find the probability that* (i) *no child gets a matching pair,* (ii) *every child gets a matching pair,* (iii) *exactly one child gets a matching pair, and* (iv) *at least* 2 *children get matching pairs.*

▶ Any one distribution of n right gloves to n children determines a set of n places for the n pairs of gloves. Let us take these as the natural places for the pairs of gloves. The left gloves can be distributed to n children in $n!$ ways.

(i) The event of no child getting a matching pair occurs if the distribution of the left gloves is a derangement. The number of derangements is d_n. Therefore, the required probability, in this case, is

$$p_1 = \frac{d_n}{n!} = \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right)$$

(ii) The event of every child getting a matching pair occurs in only one distribution of the left gloves. Therefore, the required probability, in this case, is $p_2 = \dfrac{1}{n!}$.

(iii) The event of exactly one child getting a matching pair occurs when only one left glove is in the natural place, and all others are in wrong places. The number of such distributions is d_{n-1}. The required probability, in this case, is

$$p_3 = \frac{d_{n-1}}{n!} = \frac{1}{n}\left\{1 - \frac{1}{1!} + \frac{1}{2!} + \cdots + (-1)^{n-1}\frac{1}{(n-1)!}\right\}.$$

(iv) The event of at least 2 children getting a matching pair occurs if the event of no child or one child getting a matching pair *does not* occur. The probability, in this case, is $p_4 = 1 - (p_1 + p_3)$. ■

Example 7 *There are eight letters to eight different people to be placed in eight different addressed envelopes. Find the number of ways of doing this so that at least one letter gets to the right person.*

► The number of ways of placing 8 letters in 8 envelopes is 8!.

The number of ways of placing 8 letters in 8 envelopes such that no letter is in the right envelope is d_8.

Therefore, the number of ways of placing 8 letters in 8 envelopes such that at least one letter is in the right envelope is

$(8!) - \lfloor(8!) \times e^{-1}\rfloor$

$$\begin{aligned} 8! - d_8 &\approx (8!) - \lfloor(8!) \times e^{-1}\rfloor \\ &\approx \lfloor(8!)(1 - 0.3679)\rfloor \\ &= \lfloor 40320 \times 0.6321 \rfloor = 25486. \end{aligned}$$ ■

$8!, d_8$

$8!$

$8![1 - 0.3679$

40320×0.6321

Example 8 *Each of the n students is given a book. The books are to be returned and redistributed to the same students. In how many ways can the two distributions be made so that no student will get the same book in both the distributions.*

► At the first instance, the books can be distributed in $n!$ ways. Each such distribution fixes the positions of the n books. The redistribution of the books such that no student will get the same book again is a derangement of these positions. This can be done in d_n ways. Therefore, by the product rule, the answer is

$$(n!)d_n = (n!)^2 \times \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$ ■

Example 9 *Find the number of derangements of the integers from* 1 *to n (inclusive) such that, in each derangement,*

(i) *the elements in the first k places are* $1, 2, 3, \ldots k$ *in some order.*

(ii) *the elements in the first* $n - k$ *places are* $k + 1$, $k + 2, \ldots n$ *in some order.*

(Here $0 < k < n$*).*

► (i) The number of derangements of the integers $1, 2, 3, \ldots, k$ in the first k places is d_k. Consequently, the number of derangements of the remaining $n - k$ integers (which are to be in the last $n - k$ places) integers is d_{n-k}. Hence the answer (in this case) is $d_k \times d_{n-k}$.

$|S|-S_1+S_2-S_3+S_4-S_5$

(ii) Any arrangement of the $n-k$ integers $k+1, k+2, \ldots, n$ in the first $n-k$ places is a dearrangement; the number of such arrangements is $(n-k)!$. Likewise, any arrangement of the k integers $1, 2, 3, \ldots, k$ in the last k places is a dearrangement and the number of such arrangements is $k!$. Hence the answer (in this case) is $(n-k)! \times k!$. ■

Example 10 *For the positive integers $1, 2, 3, \ldots n$, there are 11660 derangements where $1, 2, 3, 4, 5$ appear in the first five positions. What is the value of n?*

► The integers 1, 2, 3, 4 and 5 can be deranged in the first five places in d_5 ways; the last $n-5$ integers in d_{n-5} ways. Hence, the number of derangements (being considered) is $d_5 \times d_{n-5}$. This is given as 11660. Thus, we have $d_5 \times d_{n-5} = 11660$, so that

$$d_{n-5} = \frac{11660}{d_5} = \frac{11660}{44} = 265.$$

But $265 = d_6$. Thus, $n-5=6$ so that $n=11$. ■

Example 11 *In how many ways can the integers $1, 2, 3, \ldots, 10$ be arranged in a line so that no even integer is in its natural place.*

► Let A_1 be the set of all permutations of the given integers where 2 is in its natural place, A_2 be the set of all permutations in which 4 is in its natural place, and so on. Then, the number of permutations where no even integer is in its natural place is $|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4} \cap \overline{A_5}|$. This is given by *

$$|\overline{A_1} \cap \overline{A_2} \cap \ldots \cap \overline{A_5}| = |S| - S_1 + S_2 - S_3 + S_4 - S_5 \tag{i}$$

We note that $|S| = 10!$.

Now, the permutations in A_1 are all of the form $b_1 b_3 b_4 \ldots b_{10}$, where $b_1\ b_3\ b_4 \ldots b_{10}$ is a permutation of 1,3,4,5...10. As such, $|A_1| = 9!$.

Similarly,

$$|A_2| = |A_3| = |A_4| = |A_5| = 9!,$$

so that

$$S_1 = \Sigma |A_i| = 5 \times 9! = C(5,1) \times 9!.$$

The permutations in $A_1 \cap A_2$ are all of the form $b_1 2 b_3 4 b_5 b_6 \ldots b_{10}$, where $b_1 b_3 b_5 b_6 \ldots b_{10}$ is a permutation of $1, 3, 5, 6, \ldots 10$. As such, $|A_1 \cap A_2| = 8!$. Similarly, each of $|A_i \cap A_j| = 8!$, and there are $C(10,2)$ such terms. Hence

$$S_2 = \Sigma |A_i \cap A_j| = C(5,2) \times 8!.$$

Likewise, we find

$$S_3 = C(5,3) \times 7!, \quad S_4 = C(5,4) \times 6!, \quad S_5 = C(5,5) \times 5!.$$

*Recall expression (4) of section 7.1.

Accordingly, expression (i) gives the required number as

$$|\overline{A_1} \cap \overline{A_2} \cap \ldots \cap \overline{A_5}| = 10! - C(5,1) \times 9! + C(5,2) \times 8! - C(5,3) \times 7!$$
$$+ C(5,4) \times 6! - C(5,5) \times 5!$$
$$= 2170680.$$ ■

Example 12 *Prove that, for any positive integer n,*

$$n! = \sum_{k=0}^{n} \binom{n}{k} d_k$$

► For any positive integer n, the total number of permutations of $1, 2, 3, \ldots, n$, is $n!$. In each such permutation there exist k (where $0 \leq k \leq n$) elements which are in their natural (original) positions called fixed elements, and $n-k$ elements which are not in their original positions. The k elements can be chosen in $\binom{n}{k}$ ways, and the remaining $n-k$ elements can then be chosen in d_{n-k} ways. Hence there are $\binom{n}{k} d_{n-k}$ permutations of $1, 2, 3, \ldots, n$, with k fixed elements and $n-k$ deranged elements. As k varies from 0 to n, we count all of the $n!$ permutations of $1, 2, 3, \ldots, n$. Thus,

$$n! = \sum_{k=0}^{n} \binom{n}{k} d_{n-k}$$
$$= \binom{n}{0} d_n + \binom{n}{1} d_{n-1} + \binom{n}{2} d_{n-2} + \cdots + \binom{n}{n} d_0$$
$$= \sum_{k=0}^{n} \binom{n}{n-k} d_k = \sum_{k=0}^{n} \binom{n}{k} d_k$$ ■

Exercises

1. How many permutations of 1, 2, 3, 4, 5, 6, 7 are not derangements?
2. Find the number of derangements of 1, 2, 3, 4, 5, 6 where the first three numbers are 1, 2, 3 in some order.
3. How many derangements of 1, 2, 3, 4, 5, 6, 7, 8 start with 5, 6, 7, or 8 in some order?
4. Thirty students take a quiz. Then for the purpose of grading, the teacher asks the students to exchange papers so that no one is grading his own paper. In how many ways can this be done?
5. A simple code is made by permuting the letters of the English alphabet with every letter being replaced by a different letter. How many distinct codes can be made in this way?

6. In how many ways can each of 10 people select a left glove and a right glove out of a total of 10 pairs of gloves so that no person selects a matching pair of gloves.

7. At a restaurant, 10 men hand over their umbrellas to the receptionist. In how many ways can their umbrellas be returned so that (i) no man receives his own umbrella? (ii) at least one of the men receives his own umbrella? (iii) at least two of the men receive their own umbrellas?

8. Find the number of permutations of the integers 1 to 10 (inclusive)

   (i) such that exactly 4 of the integers are in their natural positions.

   (ii) that do not begin with 1 and do not end with 10.

   (iii) such that 6 or more of the integers are deranged.

9. Let $A = \{1, 2, 3, \ldots, n\}$. How many one-to-one functions $f : A \to A$ have at least one fixed point? (An element $x \in A$ is called a fixed point of f if $f(x) = x$).

10. Prove the following:

    (i) $d_n = (n-1)(d_{n-1} + d_{n-2})$ for $n \geq 3$.

    (ii) $d_n = nd_{n-1} + (-1)^{n-1}$ for $n \geq 2$.

Answers

1. 3186 **2.** 4 **3.** 576 **4.** d_{30} **5.** d_{26}

6. $(10!) \times d_{10}$ **7.** (i) d_{10} (ii) $(10!) - d_{10}$ (iii) $10! - d_{10} - 10d_9$

8. (i) $C(10, 6) \times d_6$ (ii) $10! - 2 \times 9! + 8!$

(iii) $C(10, 6) \times d_6 + C(10, 7) \times d_7 + C(10, 8) \times d_8 + C(10, 9) \times d_9 + C(10, 10) \times d_{10}$.

9. $n! - d_n$.

7.3 Rook Polynomials

Consider a board that resembles a full chess board or a part of a chess board. Let n be the number of squares present in the board. Pawns are placed in the squares of the board such that not more than one pawn occupies a square. Then, according to the *Pigeonhole Principle**, not more than n pawns can be used. Two pawns placed on a board having 2 or more squares are said to *capture* (or *take*) each other if they (pawns) are in the same row or in the same column of the board. For $2 \leq k \leq n$, let r_k denote the number of ways in which k pawns can be placed on a board such that no two pawns capture each other – that is, no two pawns are in the same row or in the same column of the board. Then the polynomial

$$1 + nx + r_2x^2 + \cdots + r_nx^n$$

*See: Section 5.4.

is called the *rook polynomial*[†] for the board considered. If the board is given the name C, then the polynomial is denoted by $r(C, x)$. Thus, by definition,

$$r(C, x) = 1 + nx + r_2 x^2 + \cdots + r_n x^n. \tag{1}$$

While defining this polynomial, it has been assumed that $n \geq 2$. In the trivial case where $n = 1$ (that is, in the case where a board contains only one square), $r_2, r_3 \ldots$ are identically zero and the rook polynomial $r(C, x)$ is defined by

$$r(C, x) = 1 + x. \tag{2}$$

The expressions (1) and (2) can be put in the following combined form which holds for a board C with $n \geq 1$ squares:

$$r(C, x) = 1 + r_1 x + r_2 x^2 + \cdots + r_n x^n \tag{3}$$

Here, $\quad r_1 = n =$ number of squares in the board.

Example 1 . Consider the board shown in Figure 7.1, which contains 4 squares.

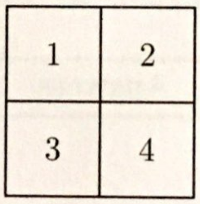


Figure 7.1

For this board, $r_1 = 4$. The number of ways in which two rooks can be placed on this board such that no two of them capture each other is 2; the two possible positions are (1,4) and (2,3). Thus, $r_2 = 2$. Three rooks cannot be placed on the board such that no two pawns capture each other. Thus, $r_3 = 0$. Similarly, $r_4 = 0$.

Accordingly, the rook polynomial for the board is

$$r(C, x) = 1 + r_1 x + r_2 x^2 = 1 + 4x + 2x^2.$$

Example 2 . Consider the board containing 5 squares (marked 1 to 5) shown in Figure 7.2.

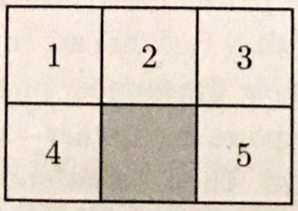


Figure 7.2

[†] *Rook* is the term used for a pawn placed on a chess board. It is also called *castle*.

For this board, $r_1 = 5$. We note that 2 non-capturing rooks can be placed on the board in the following positions: (1,5), (2,4), (2,5), (3,4). Thus, $r_2 = 4$. We check that the board has no positions for more than two mutually non-capturing rooks. That is, $r_3 = 0$, $r_4 = 0$, $r_5 = 0$. Thus, for this board, the rook polynomial is

$$r(C, x) = 1 + 5x + 4x^2.$$

Example 3 . Consider the board containing 6 squares, shown in Figure 7.3.

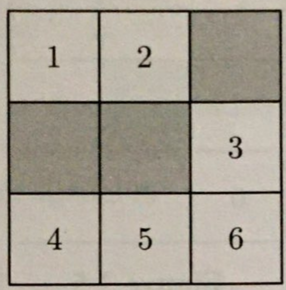


Figure 7.3

For this board $r_1 = 6$. We observe that 2 non-capturing rooks can have the following positions: (1,3), (1,5), (1,6), (2,3), (2,4), (2,6), (3,4), (3,5). These positions are 8 in number; therefore $r_2 = 8$.

Next, 3 mutually non-capturing rooks can be placed only in the following two positions: (1,3,5), (2,3,4). Thus, $r_3 = 2$. We find that four or more mutually noncapturing rooks cannot be placed on the board. Thus $r_4 = 0$, $r_5 = 0$, $r_6 = 0$. Accordingly, for this board, the rook polynomial is

$$r(C, x) = 1 + 6x + 8x^2 + 2x^3.$$

Example 4 . Consider the board with squares marked from 1 to 7 shown in Figure 7.4.

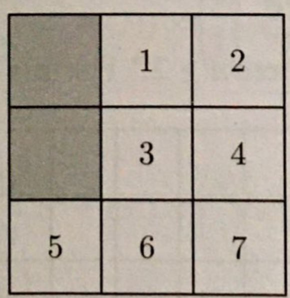


Figure 7.4

For this board we have $r_1 = 7$.

The positions for 2 non-capturing rooks are: (1,4), (1,5), (1,7), (2,3), (2,5), (2,6), (3,5), (3,7), (4,5), (4,6). These are 10 in number; therefore $r_2 = 10$. The positions of 3 mutually non-capturing rooks are: (1,4,5), (2,3,5). Thus $r_3 = 2$.

The board has no positions for four or more mutually non-capturing rooks. Hence, $r_4 = r_5 = r_6 = r_7 = 0$.

Thus, for this board, the rook polynomial is

$$r(C, x) = 1 + 7x + 10x^2 + 2x^3.$$

Example 5 . Consider the board containing 8 squares (marked 1 to 8) as shown in Figure 7.5.

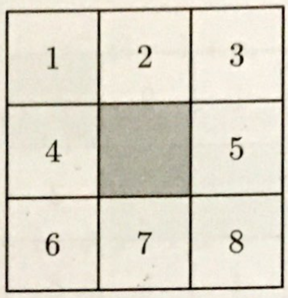


Figure 7.5

For this board, $r_1 = 8$.

In this board, the positions of 2 non-capturing rooks are: (1,5), (1,7), (1,8), (2,4), (2,5), (2,6), (2,8), (3,4), (3,6), (3,7), (4,7), (4,8), (5,6), (5,7). These are 14 in number; therefore $r_2 = 14$. The positions of 3 mutually non-capturing rooks are: (1,5,7), (2,4,8), (2,5,6), (3,4,7). These are 4 in number, therefore $r_3 = 4$.

We check that the board has no positions for more than 3 mutually non-capturing rooks. Hence $r_4 = r_5 = r_6 = r_7 = r_8 = 0$.

Thus, for this board, the rook polynomial is

$$r(C, x) = 1 + 8x + 14x^2 + 4x^3.$$

Rook Polynomial for $n \times n$ board

Consider an $n \times n$ board $C_{n \times n}$, where $n \geq 2$.* For this board, $r_1 = n^2$.

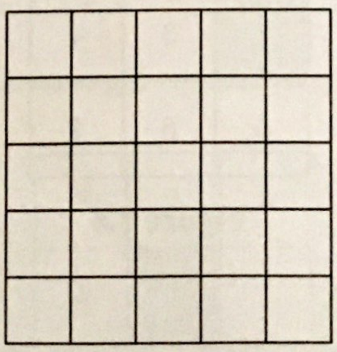


Figure 7.6

*For $n = 5$, the board is as shown in Figure 7.6.

In this board, one can choose k (with $2 \le k \le n$) rows out of the n rows in $C(n,k)$ ways. After that, k rooks can be placed in these k rows so that there is exactly 1 rook in each row and no 2 rooks are in the same column, in $P(n,k)$ ways. Thus, there are $C(n,k) \times P(n,k)$ ways of placing the k rooks such that no two rooks are in the same row or in the same column. This means that, for $2 \le k \le n$,

$$\begin{aligned} r_k &= C(n,k) \times P(n,k) \\ &= \frac{n!}{(n-k)!\,k!} \times \frac{n!}{(n-k)!} = k! \times \{C(n,k)\}^2 \end{aligned}$$

Further, $r_k = 0$ for $k > n$. Because it is not possible to place more than n rooks such that no 2 rooks are in the same row or in the same column.

Hence, for this board, the rook polynomial is

$$\begin{aligned} r(C_{n\times n}, x) &= 1 + n^2 x + + \sum_{k=2}^{n} k! \times \{C(n,k)\}^2 x^k \\ &= 1 + \binom{n}{1}^2 x + (2!) \times \binom{n}{2}^2 x^2 + (3!) \times \binom{n}{3}^2 x^3 + \cdots + (n!) \times \binom{n}{n}^2 x^n \end{aligned} \tag{4}$$

Taking $n = 2, 3, 4, \ldots$ in this polynomial, we get the rook polynomials for the boards $C_{2\times2}$, $C_{3\times3}$, $C_{4\times4}$, ... Thus,

$$\begin{aligned} r(C_{2\times2}, x) &= 1 + \binom{2}{1}^2 x + (2!) \times \binom{2}{2}^2 x^2 \\ &= 1 + 4x + 2x^2 \quad \text{(This agrees with that obtained in Example 1).} \end{aligned}$$

$$\begin{aligned} r(C_{3\times3}, x) &= 1 + \binom{3}{1}^2 x + (2!) \times \binom{3}{2}^2 x^2 + (3!) \times \binom{3}{3}^2 x^3 \\ &= 1 + 9x + 18x^2 + 6x^3, \end{aligned}$$

$$\begin{aligned} r(C_{4\times4}, x) &= 1 + \binom{4}{1}^2 x + (2!) \times \binom{4}{2}^2 x^2 + (3!) \times \binom{4}{3}^2 x^3 + (4!) \times \binom{4}{4}^2 x^4 \\ &= 1 + 16x + 72x^2 + 96x^3 + 24x^4, \end{aligned}$$

$$\begin{aligned} r(C_{5\times5}, x) &= 1 + \binom{5}{1}^2 x + (2!) \times \binom{5}{2}^2 x^2 + (3!) \times \binom{5}{3}^2 x^3 + (4!) \times \binom{5}{4}^2 x^4 + (5!) \times \binom{5}{5}^2 x^5 \\ &= 1 + 25x + 200x^2 + 600x^3 + 600x^4 + 120x^5, \end{aligned}$$

and so on.

Expansion Formula

The rook polynomials have many interesting properties. Below we indicate two of these.

In a given board C, suppose we choose a particular square and mark it as $\circledast$. Let D be the board obtained from C by deleting the row and the column containing the square $\circledast$, and let E be the board obtained from C by deleting only the square $\circledast$. Then, the rook polynomial $r(C,x)$ for the board C has the following property:

$$r(C,x) = x\,r(D,x) + r(E,x) \tag{5}$$

This is known as the ***expansion formula*** for $r(C,x)$.

Example 6 *Find the rook polynomial for the 2×2 board by using the expansion formula.*

► The 2×2 board is shown in Figure 7.1. Let us mark the square numbered 1 as $\circledast$. Then the boards D and E appear as shown below.

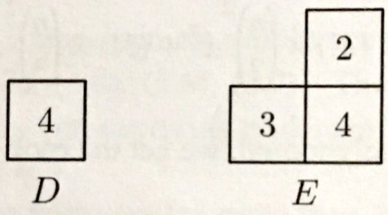


Figure 7.7

The board D contains only one square. Therefore,

$$r(D,x) = 1 + x.$$

For the board E, we find that $r_1 = 3$, $r_2 = 1$ and $r_3 = 0$. Therefore,

$$r(E,x) = 1 + 3x + x^2.$$

Now, the expansion formula yields

$$r(C_{2\times2},x) = x\,r(D,x) + r(E,x) = x(1+x) + (1+3x+x^2) = 1 + 4x + 2x^2.$$

(This result agrees with the one got in Example 1). ∎

Example 7 *Find the rook polynomial for the 3×3 board by using the expansion formula.*

▶ The 3×3 board is as shown below.

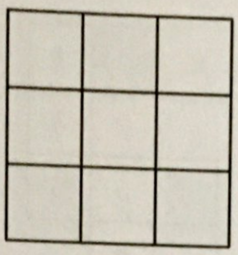
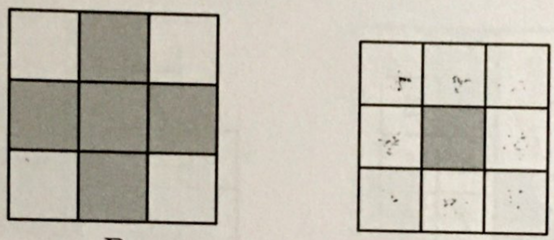


Figure 7.8

Let us mark the square which is at the centre of the board as ⊛. Then the boards D and E appear as shown below (the shaded parts are the deleted parts):



D E

Figure 7.9

For the board D, we find that $r_1 = 4, r_2 = 2, r_3 = r_4 = 0$. Hence

$$r(D, x) = 1 + 4x + 2x^2.$$

The board E is the same as the one considered in Example 5 (See Figure 6.5). As such (for this board)

$$r(E, x) = 1 + 8x + 14x^2 + 4x^3.$$

Now, the expansion formula gives

$$\begin{aligned} r(C_{3\times3}, x) &= x\,rD(x) + r(E, x) \\ &= x(1 + 4x + 2x^2) + (1 + 8x + 14x^2 + 4x^3) \\ &= 1 + 9x + 18x^2 + 6x^3. \end{aligned}$$

(This result agrees with that noted earlier). ■

Example 8 *By using the expansion formula, obtain the rook polynomial for the board C shown below.*

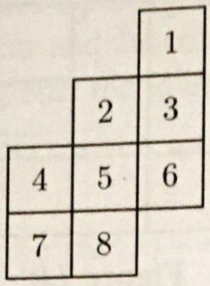


Figure 7.10

► Let us mark the topmost square 1 in the given board as ⊛. Then the boards D and E appear as shown below.

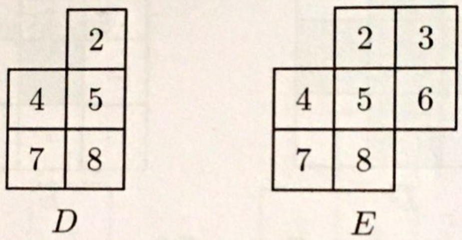


Figure 7.11

For the board D, we have $r_1 = 5$, $r_2 = 4$ (the positions of two non-capturing rooks being (2,4), (2,7), (4,8), (5,7)), and $r_3 = r_4 = r_5 = 0$. Thus,

$$r(D, x) = 1 + 5x + 4x^2.$$

For the board E, we have $r_1 = 7$, $r_2 = 11$ (the positions of two non-capturing rooks being (2,4), (2,6), (2,7), (3,4), (3,5), (3,7), (3,8), (4,8), (5,7), (6,7), (6,8)), $r_3 = 3$ (the positions of three mutually non-capturing rooks being (2,6,7), (3,4,8) (3,5,7)), and $r_k = 0$ for $k = 4, 5, 6, 7$. Thus,

$$r(E, x) = 1 + 7x + 11x^2 + 3x^3.$$

By using the expansion formula, we find that

$$\begin{aligned} r(C, x) = x\,r(D, x) + r(E, x) &= x(1 + 5x + 4x^2) + (1 + 7x + 11x^2 + 3x^3) \\ &= 1 + 8x + 16x^2 + 7x^3. \end{aligned}$$

∎

Example 9 *By using the expansion formula, find the rook polynomial for the board C shown below (madeup of unshaded parts):*

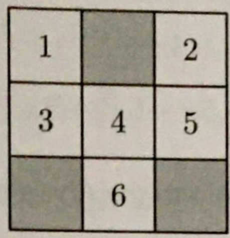
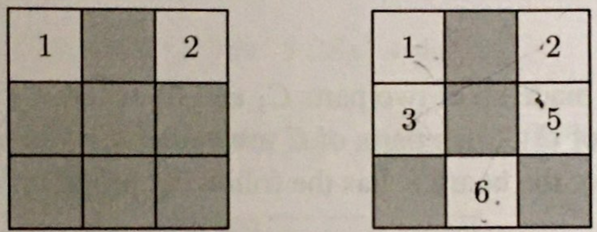


Figure 7.12

► Marking the square numbered 4 as ⊛, we get the boards D and E shown below (unshaded parts):



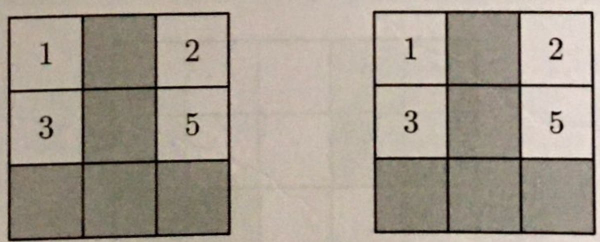
D E

Figure 7.13

For the board D, we have $r_1 = 2$, $r_2 = 0$. Hence

$$r(D, x) = 1 + 2x \tag{i}$$

In the board E, let us mark the square numbered 6 as ⊛. Then, consider the boards D' and E' shown below (unshaded parts):



D' E'

Figure 7.14

We observe that D' and E' are identical. For these boards, $r_1 = 4$, $r_2 = 2$, $r_3 = r_4 = 0$. Hence

$$r(D', x) = 1 + 4x + 2x^2, \quad r(E', x) = 1 + 4x + 2x^2$$

Accordingly, the expansion formula applied to the board E gives

$$
\begin{aligned}
r(E, x) &= x\,r(D', x) + r(E', x) \\
&= x(1 + 4x + 2x^2) + (1 + 4x + 2x^2) \\
&= 1 + 5x + 6x^2 + 2x^3 \qquad \text{(ii)}
\end{aligned}
$$

Now, applying the expansion formula to the given board C, and using (i) and (ii), we get

$$
\begin{aligned}
r(C, x) &= x\,r(D, x) + r(E, x) \\
&= x(1 + 2x) + (1 + 5x + 6x^2 + 2x^3) \\
&= 1 + 6x + 8x^2 + 2x^3.
\end{aligned}
$$
∎

Product formula

Suppose a board C is made up of two parts C_1 and C_2 where C_1 and C_2 have no squares in the same row or column of C – such parts of C are called *disjoint sub-boards of* C. Then the rook polynomial $r(C, x)$ for the board C has the following property:

$$r(C, x) = r(C_1, x) \times r(C_2, x) \qquad (6)$$

This is known as the ***product formula*** for $r(C, x)$.

The following is a natural generalization of this formula:

If a board C is made up of *pairwise disjoint* sub-boards $C_1, C_2, C_3, \ldots, C_n$, then

$$r(C, x) = r(C_1, x) \times r(C_2, x) \times r(C_3, x) \times \cdots \times r(C_n, x) \qquad (7)$$

Example 10 *Find the rook polynomial for the board shown below (shaded part).*

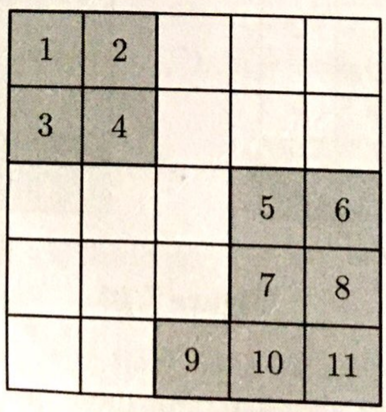


Figure 7.15

► We note that the given board C is made up of two *disjoint* sub-boards C_1 and C_2, where C_1 is the 2×2 board with squares numbered 1 to 4 and C_2 is the board with squares numbered 5 to 11.

Since C_1 is the 2×2 board, we have

$$r(C_1, x) = 1 + 4x + 2x^2 \qquad \text{(see Example 1).}$$

We note that C_2 is the same as the board considered in Example 4. For this board, we have

$$r(C_2, x) = 1 + 7x + 10x^2 + 2x^3.$$

Therefore, the product formula yields the rook polynomial for the given board as

$$\begin{aligned} r(C, x) &= r(C_1, x) \times r(C_2, x) = (1 + 4x + 2x^2)(1 + 7x + 10x^2 + 2x^3). \\ &= 1 + 11x + 40x^2 + 56x^3 + 28x^4 + 4x^5 \end{aligned}$$ ■

Example 11 *A board consists of the shaded part of Figure* 7.16. *Find its rook polynomial.*

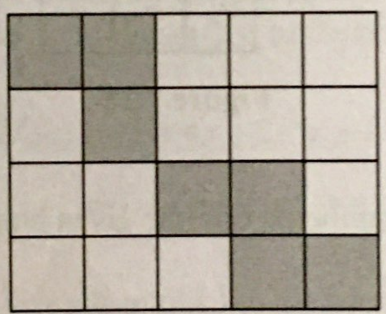
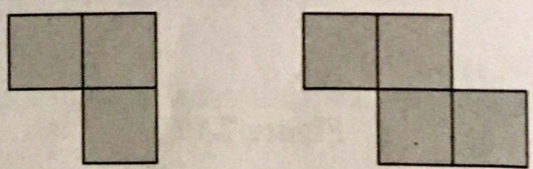


Figure 7.16

► We observe that the given board, C, is made up of two *disjoint* sub-boards C_1 and C_2 shown below.



C_1 C_2

Figure 7.17

We note that, for C_1, $r_1 = 3, r_2 = 1, r_3 = 0$, so that

$$r(C_1, x) = 1 + 3x + x^2.$$

For the sub-board C_2, we find that $r_1 = 4, r_2 = 3,\ r_3 = r_4 = 0$ so that

$$r(C_2, x) = 1 + 4x + 3x^2.$$

Therefore, by the product formula, the rook polynomial for the given board is

$$r(C, x) = r(C_1, x) \times r(C_2, x)$$
$$= (1 + 3x + x^2)(1 + 4x + 3x^2) = 1 + 7x + 16x^2 + 13x^3 + 3x^4 \quad ■$$

Example 12 *Find the rook polynomial for the board shown below:*

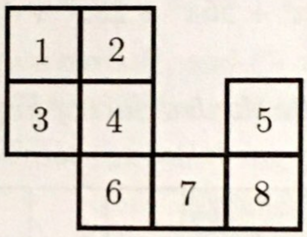
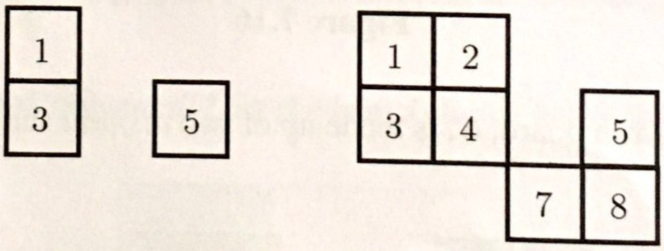


Figure 7.18

► First, let us mark the square numbered 6 in the given board, C, as ⊛ and obtain the boards D and E shown below.



D E

Figure 7.19

We observe that for the board D, $r_1 = 3, r_2 = 1, r_3 = 0$. Therefore,

$$r(D, x) = 1 + 3x + x^2 \tag{i}$$

In the board E, let us mark the square numbered 5 as ⊛ and obtain the boards D' and E' shown below.

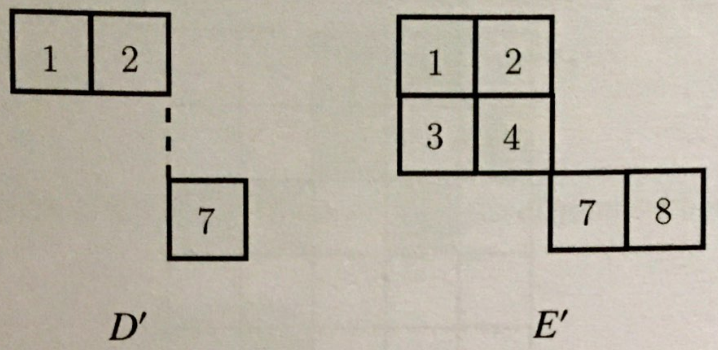


Figure 7.20

We observe that for the board D', $r_1 = 3$, $r_2 = 2$, $r_3 = 0$. Therefore,

$$r(D', x) = 1 + 3x + 2x^2 \tag{ii}$$

The board E' is made up of two *disjoint* subboards one of which is a 2×2 board and the other is a 2×1 board. The product formula applied to the board E' yields

$$\begin{aligned} r(E', x) &= (1 + 4x + 2x^2)(1 + 2x) \\ &= 1 + 6x + 10x^2 + 4x^3 \end{aligned} \tag{iii}$$

Consequently, the expansion formula applied to the board E yields (using (ii) and (iii))

$$\begin{aligned} r(E, x) &= x\, r(D', x) + r(E', x) \\ &= x(1 + 3x + 2x^2) + (1 + 6x + 10x^2 + 4x^3) \\ &= 1 + 7x + 13x^2 + 6x^3 \end{aligned} \tag{iv}$$

Finally, the expansion formula applied to the given board C yields (using (i) and (iv))

$$\begin{aligned} r(c, x) &= x\, r(D, x) + r(E, x) \\ &= x(1 + 3x + x^2) + (1 + 7x + 13x^2 + 6x^3) \\ &= 1 + 8x + 16x^2 + 7x^3. \end{aligned}$$

This is the rook polynomial for the given board. ∎

Example 13 *Find the rook polynomial for the board made up of the shaded squares in Figure 7.21 below.*

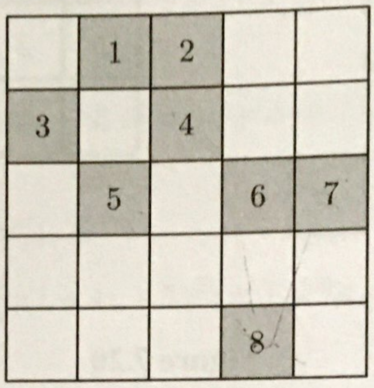


Figure 7.21

► First, let us mark the square numbered 6 in the given board, C, as ⊛ and obtain the boards D and E shown below.

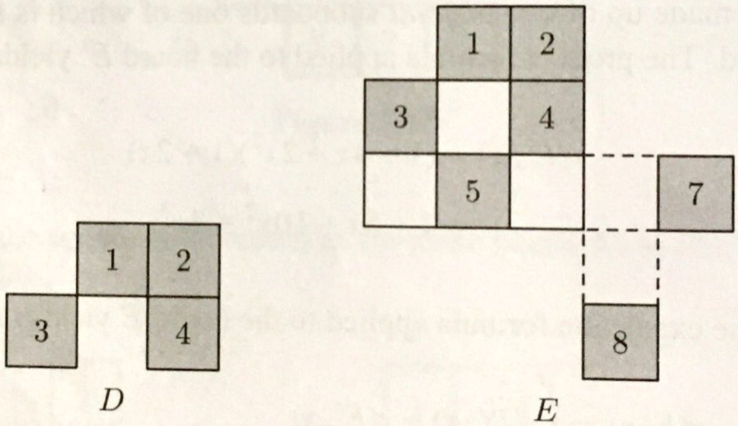


Figure 7.22

In the board D let us mark the square numbered 2 as ⊛ and obtain the boards D' and E' shown below:

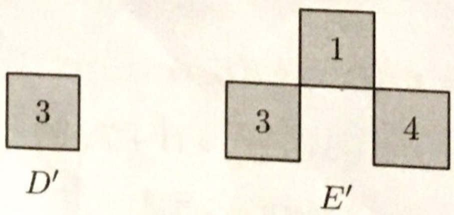


Figure 7.23

Then, we have (by the expansion formula applied to the board D)

$$\begin{aligned} r(D, x) &= x\,r(D', x) + r(E', x) \\ &= x(1 + x) + (1 + 3x + 2x^2) \\ &= 1 + 4x + 3x^2. \end{aligned} \tag{i}$$

Now, let us look at the board E. This is made up of the *disjoint* subboards E_1 and E_2 shown below:

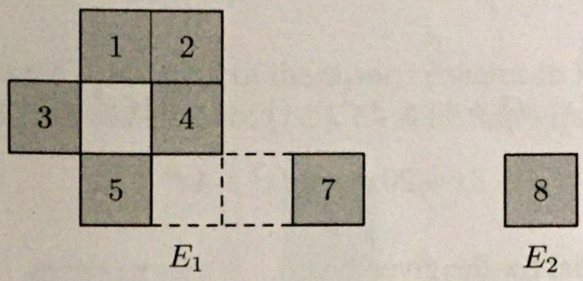


Figure 7.24

Then, we have, using the product formula (applied to the board E),

$$r(E, x) = r(E_1, x) \times r(E_2, x) = r(E_1, x) \times (1 + x) \tag{ii}$$

In the board E_1, let us mark the square numbered 5 as ⊛, and obtain the boards D'' and E'' shown below.

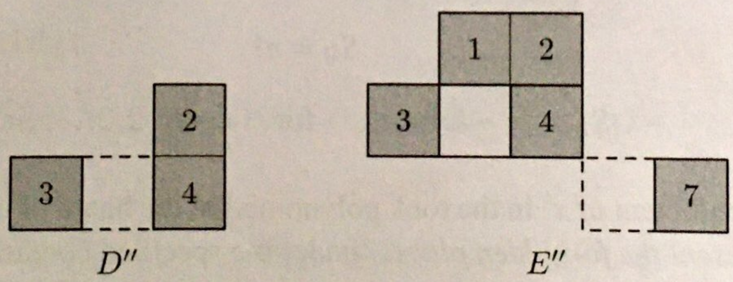


Figure 7.25

The expansion formula (applied to the board E_1) gives

$$r(E_1, x) = x\,r(D'', x) + r(E'', x) = x(1 + 3x + x^2) + r(E'', x) \tag{iii}$$

We observe that E'' is made up of two *disjoint* boards (— one board made up of squares numbered from 1 to 4 and the other board with a single square numbered 7). Using the product formula, we find that

$$r(E'', x) = (1 + 4x + 3x^2) \times (1 + x) = 1 + 5x + 7x^2 + 3x^3 \tag{iv}$$

Using this in (iii), we get

$$r(E_1, x) = x(1 + 3x + x^2) + (1 + 5x + 7x^2 + 3x^3) = 1 + 6x + 10x^2 + 4x^3 \tag{v}$$

Putting this into (ii), we get

$$r(E, x) = (1 + x)(1 + 6x + 10x^2 + 4x^3) = 1 + 7x + 16x^2 + 14x^3 + 4x^4 \tag{vi}$$

Finally, employing the expansion formula to the given board C, we obtain, on using (i) and (vi),

$$\begin{aligned} r(C, x) &= x(1 + 4x + 3x^2) + (1 + 7x + 16x^2 + 14x^3 + 4x^4) \\ &= 1 + 8x + 20x^2 + 17x^3 + 4x^4 \end{aligned}$$

This is the rook polynomial for the given board. ∎

Arrangements with forbidden positions

Suppose m objects are to be arranged in n places, where $n \geq m$. Suppose there are constraints under which some objects cannot occupy certain places – such places are called the *forbidden positions* for the said objects. The number of ways of carrying out this task is given by the following rule:

$$\overline{N} = S_0 - S_1 + S_2 - S_3 + \cdots + (-1)^n S_n \tag{8}$$

where
$$S_0 = n! \tag{9}$$

and
$$S_k = (n-k)! \times r_k, \quad \text{for} \quad k = 1, 2, 3, \ldots, n. \tag{10}$$

Here, r_k is the coefficient of x^k in the rook polynomial of the board of m rows and n columns *whose squares represent the forbidden places* (under the specified conditions).

The formula (8) is a consequence of and analogous to the formula (9) of Section 7.1.

Example 14 *An apple, a banana, a mango and an orange are to be distributed to four boys B_1, B_2, B_3, B_4. The boys B_1 and B_2 do not wish to have apple, the boy B_3 does not want banana or mango, and B_4 refuses orange. In how many ways the distribution can be made so that no boy is displeased?*

► The situation can be described by the board shown in Figure 7.26 in which the rows respectively represent apple, banana, mango and orange, and the columns represent the boys B_1, B_2, B_3, B_4, respectively. Also, the shaded squares together represent the forbidden places in the distribution.

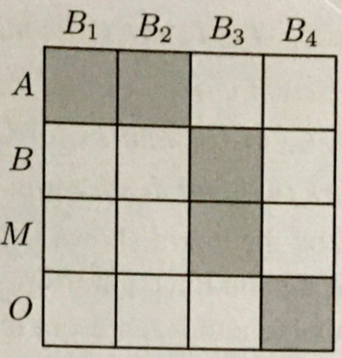


Figure 7.26

Let us consider the board C consisting of the shaded squares in Figure 7.26. We note that C is formed by the mutually disjoint boards C_1, C_2, C_3 shown in Figure 7.27.

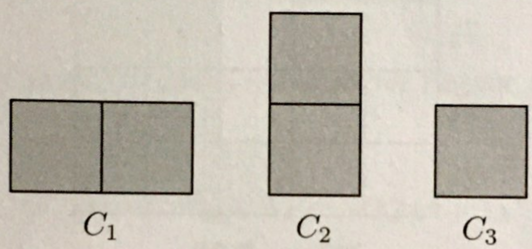


Figure 7.27

As such, the rook polynomial for C is (by the product formula)

$$r(C, x) = r(C_1, x) \times r(C_2, x) \times r(C_3, x).$$

By inspection, we find that

$$r(C_1, x) = 1 + 2x, \quad r(C_2, x) = 1 + 2x, \quad r(C_3, x) = 1 + x.$$

Accordingly, we have

$$r(C, x) = (1 + 2x)^2(1 + x) = 1 + 5x + 8x^2 + 4x^3.$$

Thus, for C, $r_1 = 5, r_2 = 8, r_3 = 4$.

Consequently (by expressions (9) and (10))

$$S_0 = 4! = 24, \quad S_1 = (4 - 1)! \times r_1 = 30,$$

$$S_2 = (4 - 2)! \times r_2 = 16, \quad S_3 = (4 - 3)! \times r_3 = 4.$$

Therefore (by expression (8)),

$$\overline{N} = S_0 - S_1 + S_2 - S_3 = 24 - 30 + 16 - 4 = 6.$$

This is the number of ways of distributing the fruits under the given constraints. ■

Example 15 *Five teachers T_1, T_2, T_3, T_4, T_5 are to be made class teachers for five classes, C_1, C_2, C_3, C_4, C_5, one teacher for each class. T_1 and T_2 do not wish to become the class teachers for C_1 or C_2, T_3 and T_4 for C_4 or C_5, and T_5 for C_3 or C_4 or C_5. In how many ways can the teachers be assigned the work (without displeasing any teacher)?*

► The situation can be represented by the board shown below in which the rows respectively represent the teachers T_1, T_2, T_3, T_4, T_5 and the columns respectively represent the classes C_1, C_2, C_3, C_4, C_5, and the shaded squares together represent the forbidden places involved.

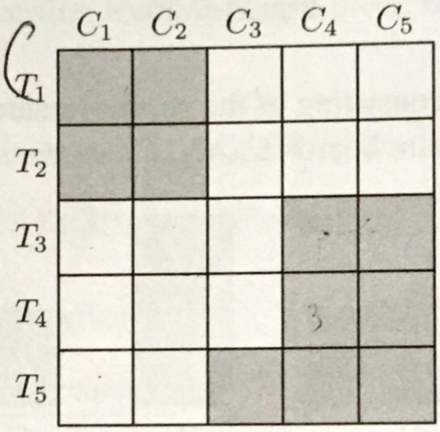


Figure 7.28

For the board C made up of the shaded squares in the above Figure, the rook polynomial is given by (see Example 10)

$$r(C, x) = 1 + 11x + 40x^2 + 56x^3 + 28x^4 + 4x^5.$$

Thus, here, $r_1 = 11$, $r_2 = 40$, $r_3 = 56$, $r_4 = 28$, $r_5 = 4$.

Consequently,

$$S_0 = 5! = 120, \quad S_1 = (5-1)! \times r_1 = 264,$$
$$S_2 = (5-2)! \times r_2 = 240, \quad S_3 = (5-3)! \times r_3 = 112,$$
$$S_4 = (5-4)! \times r_4 = 28, \quad S_5 = (5-5)! \times r_5 = 4.$$

Accordingly, the number of ways in which the work can be assigned is

$$S_0 - S_1 + S_2 - S_3 + S_4 - S_5 = 120 - 264 + 240 - 112 + 28 - 4 = 8.$$ ■

Example 16 *Four persons P_1, P_2, P_3, P_4 who arrive late for a dinner party find that only one chair at each of five tables T_1, T_2, T_3, T_4 and T_5 is vacant. P_1 will not sit at T_1 or T_2, P_2 will not sit at T_2, P_3 will not sit at T_3 or T_4, and P_4 will not sit at T_4 or T_5. Find the number of ways they can occupy the vacant chairs.*

► Consider the board shown below, representing the situation. The shaded squares in the first row indicate that tables T_1 and T_2 are forbidden for P_1, and so on.

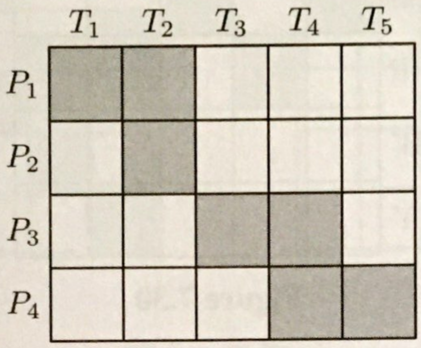


Figure 7.29

For the board made up of shaded squares in the above Figure, the rook polynomial is given by (see Example 11)

$$r(C, x) = 1 + 7x + 16x^2 + 13x^3 + 3x^4.$$

Thus, here, $r_1 = 7$, $r_2 = 16$, $r_3 = 13$, $r_4 = 3$.

Hence,

$$S_0 = 5! = 120, \quad S_1 = (5-1)! \times r_1 = 168, \quad S_2 = (5-2)! \times r_2 = 96,$$
$$S_3 = (5-3)! \times r_3 = 26, \quad S_4 = (5-4)! \times r_4 = 3.$$

Consequently, the number of ways in which the four persons can occupy the chairs is

$$S_0 - S_1 + S_2 - S_3 + S_4 = 120 - 168 + 96 - 26 + 3 = 25. \qquad \blacksquare$$

Example 17 *A girl student has sarees of 5 different colours: blue, green, red, white and yellow. On Mondays she does not wear green; on Tuesdays blue or red; on Wednesdays blue or green; on Thursdays red or yellow; on Fridays red. In how many ways can she dress without repeating a colour during a week (from Monday to Friday)?*

► The situation here can be represented by the board shown below in which the rows respectively represent the colours B, G, R, W, Y and the columns respectively represent Mondays through Fridays, and the shaded squares together represent the constraints on the colours worn.

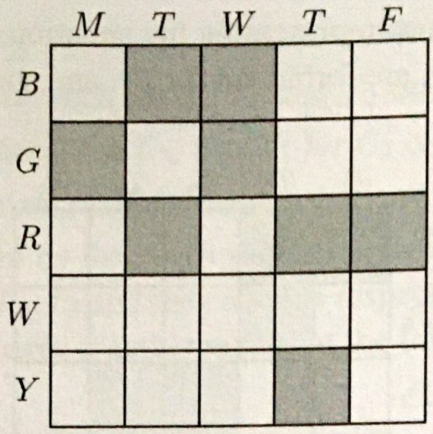


Figure 7.30

For the board C made up of the shaded squares in the above Figure, the rook polynomial is (see Example 13)

$$r(C, x) = 1 + 8x + 20x^2 + 17x^3 + 4x^4.$$

Thus, here, $r_1 = 8$, $r_2 = 20$, $r_3 = 17$, $r_4 = 4$.

Consequently,

$$S_0 = 5! = 120, \quad S_1 = (5-1)! \times r_1 = 192, \quad S_2 = (5-2)! \times r_2 = 120$$

$$S_3 = (5-3)! \times r_3 = 34, \quad S_4 = (5-4)! \times r_4 = 4.$$

Therefore, the number of ways of dressing is

$$S_0 - S_1 + S_2 - S_3 + S_4 = 120 - 192 + 120 - 34 + 4 = 18. \quad ■$$

Example 18 *A pair of dice, one red and the other green is rolled six times. Find the probability that we obtain all six values on both the red die and the green die under the restriction that the ordered pairs* $(1,1)$, $(1,5)$, $(2,4)$, $(3,6)$, $(4,2)$, $(4,4)$, $(5,1)$ *and* $(5,5)$ *do not occur. [Here an ordered pair* (a, b) *indicates a on the red die and b on the green].*

▶ The situation being considered here can be represented by the board shown in Figure 7.31(a) in which the rows represent the values appearing on the red die and the columns represent the values appearing on the green die. The shaded squares represent the value-pairs that do not occur.

For the purpose of writing down the rook polynomial, let us redraw the board by relabeling the rows and columns as shown in Figure 7.31(b).

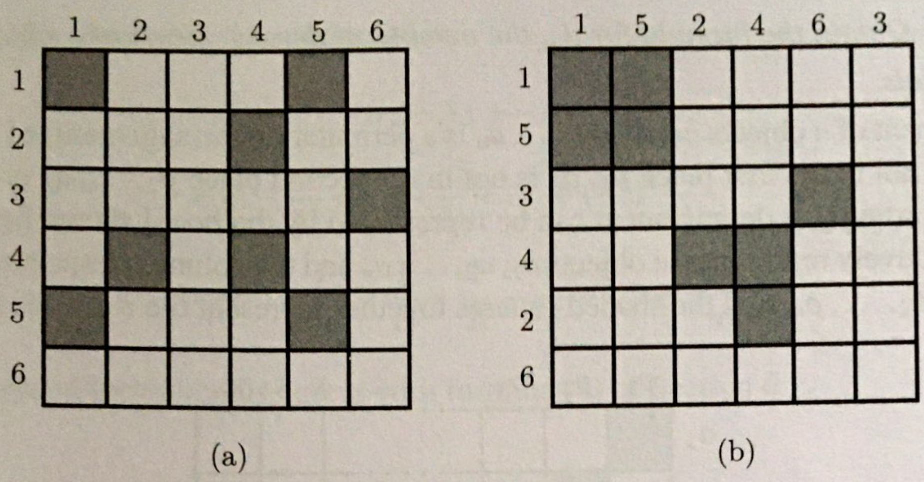


Figure 7.31

For the shaded part of the board in Figure 7.31(b), the rook polynomial is (using the Product formula)

$$\begin{aligned} r(C,x) &= (1+4x+2x^2)(1+3x+x^2)(1+x) \\ &= 1+8x+22x^2+25x^3+12x^4+2x^5 \end{aligned}$$

This gives

$$S_0 = 6! = 720, \quad S_1 = (5!)\times 8 = 960,$$

$$S_2 = (4!)\times 22 = 528, \quad S_3 = (3!)\times 25 = 150,$$

$$S_4 = (2!)\times 12 = 24, \quad S_5 = (1!)\times 2 = 2.$$

Therefore,

$$\begin{aligned} \overline{N} &= S_0 - S_1 + S_2 - S_3 + S_4 - S_5 \\ &= 720 - 960 + 528 - 150 + 24 - 2 = 160 \end{aligned}$$

Consequently, the number of ordered sequences of the six rolls of the dice for the event we are interested in is

$$(6!)\times \overline{N} = 720 \times 160.$$

Since the sample space consists of all sequences of six ordered pairs selected with repetitions from the 28 unshaded squares of the board, the probability of this event is

$$P = \frac{720 \times 160}{(28)^6} \qquad \blacksquare$$

Example 19 *Obtain the formula for d_n, the number of derangements of n objects, by using rook polynomials.*

► A derangement of n objects $a_1, a_2, a_3, \ldots, a_n$ is a permutation (arrangement) of $a_1, a_2, \ldots, a_n$ such that a_1 is not in the first place p_1, a_2 is not in the second place $p_2, \ldots, a_n$ is not in the n^{th} place p_n. Accordingly, a derangement can be represented by the board shown below in which the rows respectively represent the objects $a_1, a_2, \ldots, a_n$ and the columns respectively represent the places $p_1, p_2, \ldots, p_n$, and the shaded squares together represent the constraints involved.

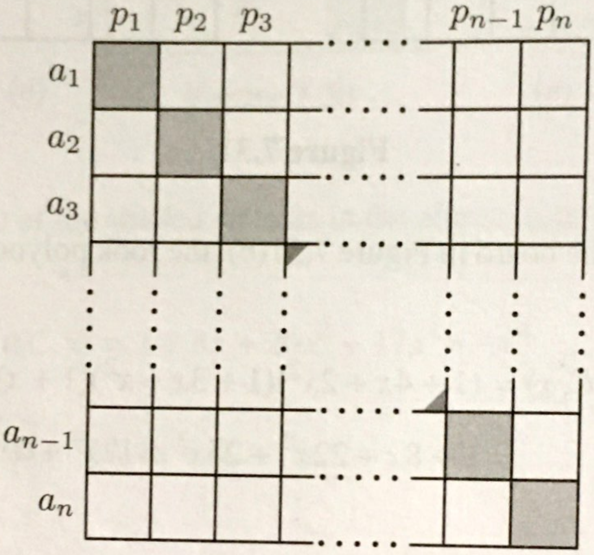


Figure 7.32

The board C made up of the shaded squares consists of n mutually disjoint squares, one square in each row/column. For a board with single square, the rook polynomial is $(1 + x)$. Therefore, by the product formula, the rook polynomial for C is

$$\begin{aligned} r(C, x) &= (1 + x) \times (1 + x) \times \cdots \times (1 + x) \ (n \text{ factors}) \\ &= (1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k, \text{ by the binomial theorem.} \end{aligned}$$

Thus, here,

$$r_k = \binom{n}{k} = \frac{n!}{k!(n-k)!}, \text{ for } k = 1, 2, \ldots n.$$

Consequently, $S_0 = n!$, and

$$S_k = (n - k)! \times r_k = \frac{n!}{k!}, \quad \text{for} \quad k = 1, 2, 3, \ldots, n.$$

Therefore, the number of derangements is

$$\begin{aligned}
d_n &= S_0 - S_1 + S_2 - S_3 + \cdots + (-1)^n S_n \\
&= n! - \frac{n!}{1!} + \frac{n!}{2!} - \frac{n!}{3!} + \cdots + (-1)^n \frac{n!}{n!} \\
&= (n!) \times \left\{ 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!} \right\}
\end{aligned}$$

This is the required formula; this agrees with formula (1) of Section 6.2. ■

Exercises

1. Find the rook polynomials for the boards shown below.

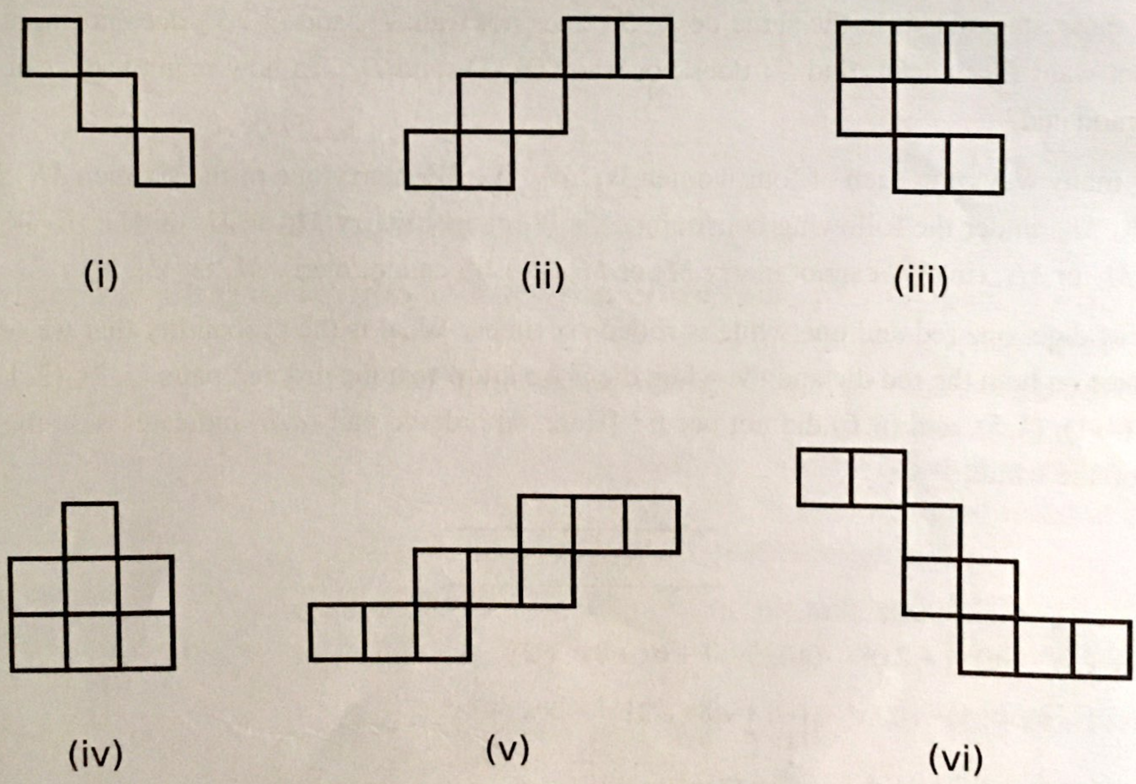


Figure 7.33

2. Find the rook polynomials for the shaded parts of the boards shown below.

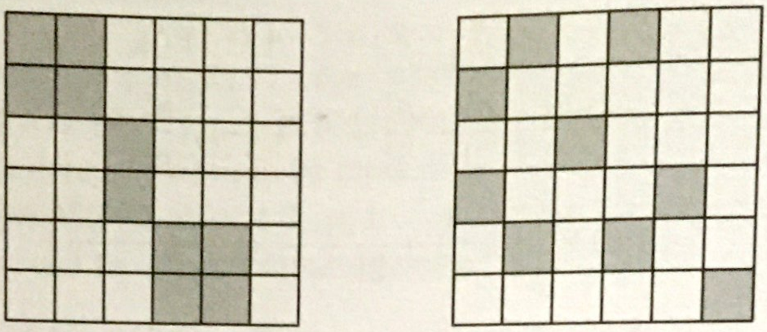


Figure 7.34

3. Five teachers T_1, T_2, T_3, T_4, T_5 are required to examine five classes C_1, C_2, C_3, C_4, C_5, one for each class. T_1 and T_2 both dislike C_3; T_3 wants to avoid C_2 and C_5; T_4 rejects C_1 and C_2, and T_5 refuses C_3 and C_4. In how many ways can they do their work without being offended?

4. Students S_1, S_2, S_3, S_4 are to be accommodated in desks $D_1, D_2, D_3, D_4, D_5, D_6$ such that no two of these students sit in the same desk. S_1 does not want D_1 and D_2; S_2 does not want D_3; S_3 does not want D_3 and D_4, and S_4 does not want D_4, D_5 and D_6. In how many ways can they be accommodated?

5. In how many ways can each of four women W_1, W_2, W_3, W_4 marry one of the six men $M_1, M_2, M_3, M_4, M_5, M_6$, under the following constraints: (i) W_1 cannot marry M_1 or M_3 or M_6, (ii) W_2 cannot marry M_2 or M_4, (iii) W_3 cannot marry M_3 or M_6, (iv) W_4 cannot marry M_4 or M_5.

6. A pair of dice, one red and one white is rolled six times. What is the probability that we obtain all six values on both the red die and the white die if we know that the ordered pairs $(1,2)$, $(2,1)$, $(2,5)$, $(3,4)$, $(4,1)$, $(4,5)$, and $(6,6)$ did not occur? [Here, an ordered pair (a,b) indicates a on the red die and b on the white].

Answers

1. (i) $(1+x)^3$ (ii) $(1+2x)^3$ (iii), (iv) $1+6x+8x^2+2x^3$
   (v) $1+9x+25x^2+21x^3$ (vi) $1+8x+21x^2+20x^3+4x^4$

2. (i) $1+10x+36x^2+56x^3+36x^4+8x^5$
   (ii) $1+9x+30x^2+47x^3+37x^4+14x^5+2x^6$

3. 20 4. 76 5. 63 6. $(192\times 6!)/(29)^6$.