Homework 1

1. Solve Kleinberg and Tardos, Chapter 1, Exercise 1. (5pts)

2. Solve Kleinberg and Tardos, Chapter 1, Exercise 2. (5pts)

3. Determine whether the following statement is true or false. If it is true, give an example. If it is false, give a short explanation. (5pts)

   For some $n \geq 2$, there exists a set of preferences for n men and n women such that in the stable matching returned by the G-S algorithm when men are proposing, every woman is matched with their most preferred man, even though that man does not prefer that woman the most.

4. A stable roommate problem with 4 students a, b, c, d is defined as follows. Each student ranks the other three in strict order of preference. A matching is defined as the partition of the students into two groups of two roommates. A matching is stable if no two separated students prefer each other to their current roommate.

   Does a stable matching always exist? If yes, give a proof. Otherwise, give an example of roommate preferences where no stable matching exists. (8pts)

5. Solve Kleinberg and Tardos, Chapter 1, Exercise 4. (15pts)

6. Solve Kleinberg and Tardos, Chapter 1, Exercise 8. (10pts)

7. Determine whether the following statement is true or false. If it is true, give a short explanation. If it is false, give a counterexample.

   For all $n \geq 2$, there exists a set of preferences for n men and n women such that in the stable matching returned by the G-S algorithm when men are proposing, every man is matched with their most preferred woman.

8. Consider a stable marriage problem where the set of men is given by $M = m_1, m_2, ..., m_N$ and the set of women is $W = w_1, w_2, ..., w_N$. Consider their preference lists to have the following properties:

   $$\forall w_i \in W : w_i \text{ prefers } m_i \text{ over } m_j \quad \forall j > i$$
   $$\forall m_i \in M : m_i \text{ prefers } w_i \text{ over } w_j \quad \forall j > i$$

   Prove that a unique stable matching exists for this problem. Note: the $\forall$ symbol means "for all". (12pts)

UNGRADED PRACTICE PROBLEMS

1. Determine whether the following statement is true or false. If it is true, give a short explanation. If it is false, give a counterexample.

   For all $n \geq 2$, there exists a set of preferences for n men and n women such that in the stable matching returned by the G-S algorithm when men are proposing, every woman is matched with their least preferred man.

2. Consider the Gale-Shapley algorithm operating on n men and n women, with women proposing.

   a) What is the maximum number of times a woman may be rejected, with respect to the problem size n? Give an example where this can happen.

   b) Consider the following modification to the G-S algorithm: at each iteration, we always pick the free woman with the highest average preference among men, i.e. the most "popular" remaining woman (when taking an average across all men's preference lists). Prove or disprove: this will help reduce the number of rejections for some women.

HW-1

1. False

In every instance of the Stable Matching problem, it is not always possible to have a pair (m, w) in a stable matching such that m is ranked first on the preference list of w & w is ranked first on the preference list of m.
The following example acts as a counter:

m_1	m_2	m_3	w_1	w_2	w_3
w_1	w_1	w_2	m_3	m_2	m_1
w_2	w_3	w_1	m_2	m_1	m_2
w_3	w_2	w_3	m_1	m_3	m_3

The resulting stable matching is (m_3, w_1), (m_1, w_2), (m_2, w_3)
∴ We can see that the is no (m, w) where m is ranked highest on the preference list of w & w is ranked highest on the preference list of m in this instance.

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2. True

If there exist a man m & a woman w who have each other as their first preference then their pair would always exist in S, i.e. $(m, w) \in S$.

Because when m gets his turn to propose, his first preference will be w. If at that moment, w is free she will get engaged and if she is already engaged, to say m', she will break the engagement with m' & get engaged to m as he is her first preference.
m & w will maintain their pair as w doesn't have any man higher than m in her preference list.

3. True

Example:

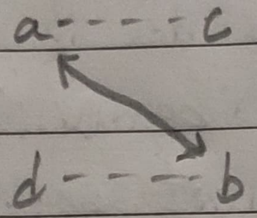
m_1	m_2	m_3	w_1	w_2	w_3
w_1	w_1	w_2	m_3	m_1	m_2
w_2	w_3	w_1	m_2	m_2	m_1
w_3	w_2	w_3	m_1	m_3	m_3

∴ Resulting ordered pairs are (m_3, w_1), (m_1, w_2) & (m_2, w_3)

∴ Every woman is matched with their most preferred significant other, but this is not the case for the men.

4. Yes, stable matching always exist.

Suppose an instability exists in the pairing S, between (a,c) & (b,d), such that a prefers b over c and b prefers a over d.



∴ Pair (a,b) is an instability.
This means a must have asked for b to be his roomate, prior to asking c and then got rejected for some other roommate a' which is higher in b's preference list. If b's last matching was with d, this means either a' = d or d is higher in preference than a'. Whatsoever, the reason, this means that d is higher in preference than a for b. This contradicts our initial assumption that b prefers a over d.

∴ The stable matching holds

5. In order to show that there is always a stable assignment of students to hospitals, let us assume that an instability exist:

Instability: (i) First

s is assigned to h & s' is free. h prefers s' to s.

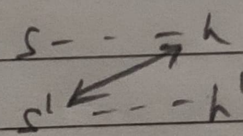
Did h ask s' prior to asking s? If h didn't then s ranks higher on h's preference list than s'. This will contradict our initial assumption.

If h did ask then s' must have rejected in preference of some other hospital h'' to whom he/she is paired with. However s' is free. This means s' never got asked by h''.

∴ This contradicts our assumption that h prefers s' to s.

(ii) Second

- s is assigned to h & s' is assigned to h'
- h prefers s' to s & s' prefers h to h'



- s was h's last pairing. Did h ask s' prior to asking s? If h didn't then s ranks higher on h's preference list than s'. This will contradict our initial assumption that h prefers s' to s.

If h did ask then s' must have rejected h in favour of h''. h'' is either h' or higher in preference than h'. However, this means h'

is higher in preference than h.
This contradicts our assumption that
s' prefers h over h'.

Algorithm

Initially $s \in S$ & $h \in H$ are free
While hospitals are not filled
choose hospital h which has unfilled position
let s be the highest ranked student in h's
preference list to whom h has not yet asked.
If s is free then
(h, s) pair up
Else s is currently paired with h'
If s prefers h' to h then
h remains unfilled
Else s prefers h to h'
(h, s) pair up
h' becomes unfilled
End If
End If
Endwhile
Return the set of ordered pairs & free students

6. For any set of preference lists, switching the order of a pair on the list cannot improve a woman's partner in the GS algorithm.

Suppose for a stable matching S, (m'', w) is the ordered pair where w prefers m to m'.

We run the algorithm again with w preferring m' to m. Now we get (m'', w) as a pair. Where m'' is either m' or a higher ranked man. If m'' = m' then w has not improved her chances of getting a better man.
If m'' is a higher ranked man than m' & m. Then m'' was going to propose to w before m or m', which means the order change between m' & m doesn't matter & doesn't improve w's chances of getting a better man.

7. True.

If all men have different women as their most preferred women then every man will be matched with their most prefered woman as the woman won't get a chance to reject any man. Every man will be paired in their first proposal as no other man will propose to the same woman later.