Abstraction of Graph Theory

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Section (2)

- A **simple graph** G consists of a non-empty finite set $V(G)$ of elements called vertices (or nodes), and a finite set $E(G)$ of distinct unordered pairs of distinct elements of $V(G)$ called edges.

- A **graph** G consists of a non-empty finite set $V(G)$ of elements called vertices, and a finite family $E(G)$ of unordered pairs of (not necessarily distinct) elements of $V(G)$ called edges.

- Two graphs $G1$ and $G2$ are **isomorphic** if there is a one-one correspondence between the vertices of G_x and those of $G2$ such that the number of edges joining any two vertices of G_x is equal to the number of edges joining the corresponding vertices of $G2$.

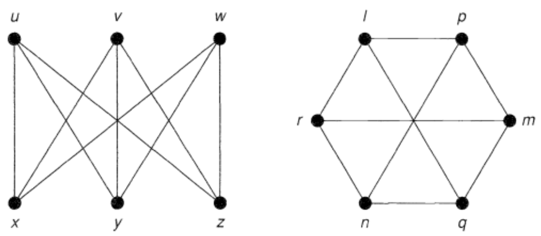


Figure 1: Two simple graphs that are isomorphic to each other

- A graph is **connected** if it cannot be expressed as the union of two graphs, and **disconnected**.

- Two vertices v and w of a graph G are **adjacent** if there is an edge vw joining them, and the vertices v and w are then **incident** with such an edge. Similarly, two distinct edges e and/are adjacent if they have a vertex in common.

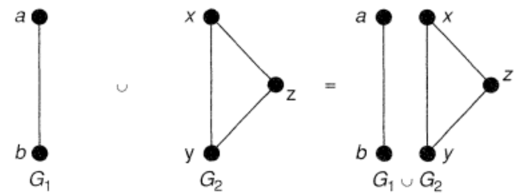


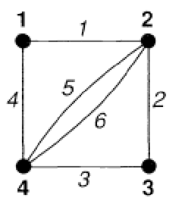
Figure 2: Each of G_1 and G_2 is a component of $G_1 \cup G_2$

- The **degree** of a vertex v of G is the number of edges incident with v, and is written $deg(v)$. *Loop*-edge increase node-degree by 2. vertex of degree 0 is an **isolated vertex** and a vertex of degree 1 is an **end-vertex**.

- **Handshaking lemma**; in any graph the sum of all the vertex-degrees is an even number - in fact, twice the number of edges, since each edge contributes exactly 2 to the sum.

- A **subgraph** of a graph G is a graph, each of whose vertices belongs to $V(G)$ and each of whose edges belongs to $E(G)$.

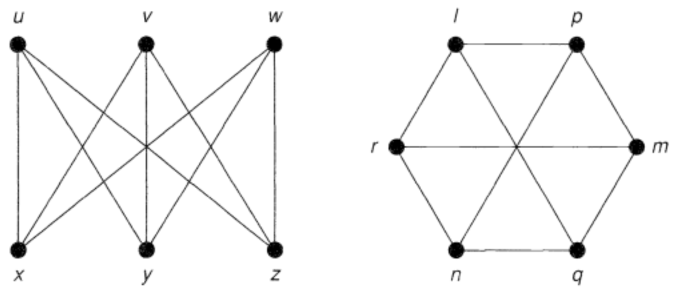
- **Matrix representation** one way to represent graph is by its **adjacency matrix** A, and its **incidence matrix** M as follows;



$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{pmatrix} \qquad \mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Exercise 2

2a Write down the vertex-set and edge-set of each graph in Fig 2.5



The first graph G_1 is $(V(G_1), E(G_1))$

$$\begin{aligned} V(G_1) &= \{x, y, z, u, v, w\} \\ E(G_1) &= \{\{x, u\}, \{x, v\}, \{x, w\}, \\ &\{y, u\}, \{y, v\}, \{y, w\}, \\ &\{z, u\}, \{z, v\}, \{z, w\}\} \end{aligned}$$

The second graph G_2 is $(V(G_2), E(G_2))$

$$\begin{aligned} V(G_2) &= \{n, m, q, r, l, p\} \\ E(G_2) &= \{\{n, r\}, \{n, p\}, \{n, q\}, \\ &\{m, r\}, \{m, p\}, \{m, q\}, \\ &\{l, r\}, \{l, p\}, \{l, q\}\} \end{aligned}$$

2b Draw;

(i) a simple graph.

(ii) a non-simple graph with no loops.

(iii) a non-simple graph with no multiple edges, each having 5 vertices each having 5 vertices and 8 edges.

2c Draw;

(i) Draw a graph on six vertices whose degrees are 5,5,5,5,3,3; does there exist a simple graph with these degrees?

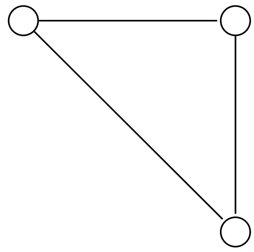


Figure 3: (i) a simple graph

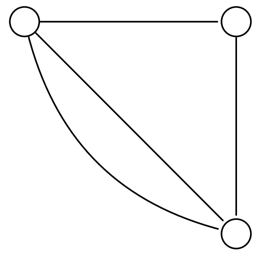


Figure 4: (ii) a non-simple graph with no loops

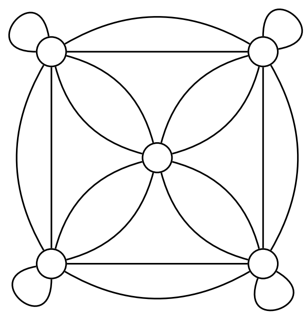


Figure 5: (iii) a 5-vertices and 8-degress each

(ii) How does the answer to part (i) changed if the degrees are 5, 5, 4, 3, 3,2?

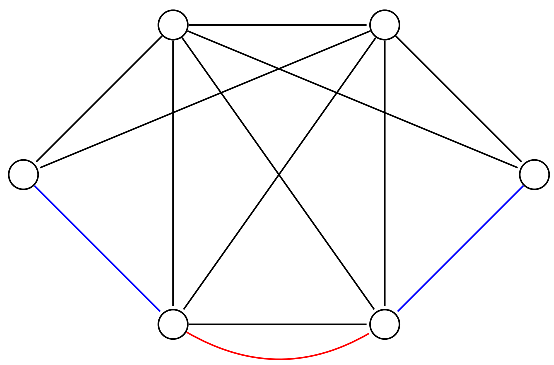


Figure 6: (i) a non-simple graph with with 6-vertices with degrees [5,5,5,5,3,3]

There isn't a simple graph with last mentioned degrees for a 6-vertices graph. But it we just remove the red-arc and one of the blue-arcs, we then get a simple graph as shown in the next figure.

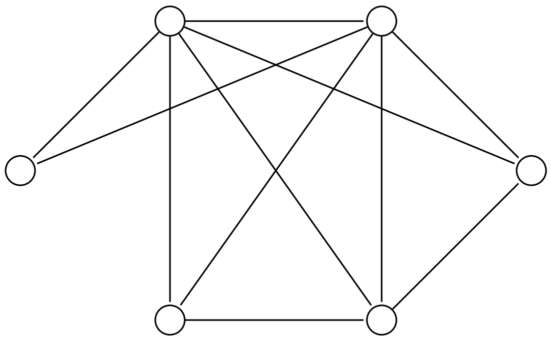
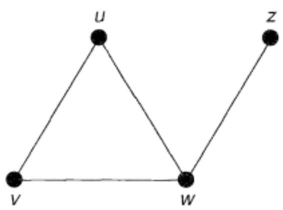


Figure 7: (i) a simple graph with with 6-vertices with degrees [5,5,4,3,3,2]

(2d) Verify that handshaking lemma is hold for figure 2.1

As we can see in the last figure, the sum



of all vertex-degrees is $2+2+3+1=8$ which is an even number.

(2f) (i) By suitably lettering the vertices, show that the two graphs in Fig. 2.20 are isomorphic.

(ii) Explain why two graphs in Fig. 2.21 are not isomorphic.

(i) As shown in the following figure 8 the two graphs are labeled with the same letters in a way to emphasize that they are both isomorphic to each other.
(ii) As shown in the following figure 9, we cannot find the red part of the first graph as a subset in the second graph.

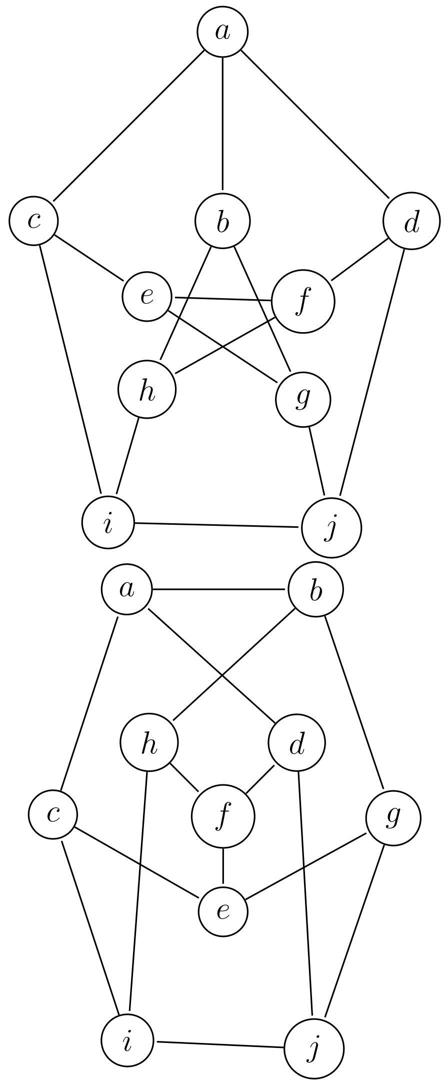


Figure 8: (i)

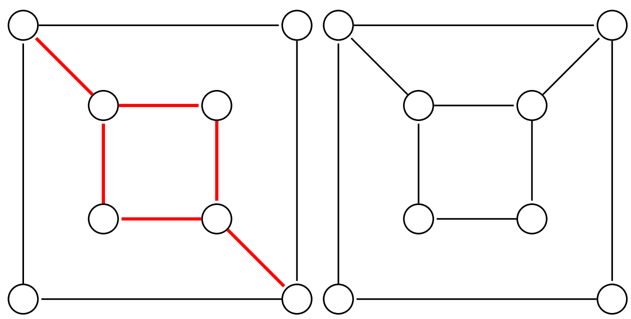


Figure 9: (ii) not isomorphic graphs

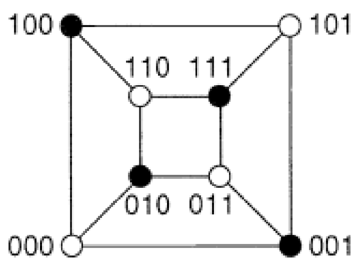
Section (3)

- **Null graphs**, a graph whose edge-set is empty. **Complete graph**, a simple graph in which each pair of distinct vertices are adjacent, in this case k-vertex must has $n(n-1)/2$ degree as n the total number of vertices.

- **Regular graph**, a graph in which each vertex has the same degree. **Platonic graphs**, formed from the vertices and edges of the five regular (Platonic) solids - the tetrahedron, octahedron, cube.

- **Bipartite graph**, If the vertex set of a graph G can be split into two disjoint sets A and B so that each edge of G joins a vertex of A and a vertex of B. A **complete bipartite graph** is a bipartite graph in which each vertex in A is joined to each vertex in B by just one edge.

- **Cubes**, the k-cube Q_k is the graph whose vertices correspond to the sequences $(a_1, a_2, \cdots, a_k)$, where each $a_i = 0$ or 1, and whose edges join those sequences that differ in just one place. You should check that Q_k has 2^k vertices and $k2^{k-1}$ edges, and is regular of degree k.



- **Complement of a simple graph**, if G is a simple graph with vertex set $V(G)$, its complement $\overline{G}$ is the simple graph with vertex set $V(G)$ in which two vertices are adjacent if and only if they are not adjacent in G.

Exercise 3

(3a) Draw the following graphs:

(i) the null graph N_5.

(ii) the complete graph K_6.

(iii) the complete bipartite graph $K_{2,4}$.

(iv) the union of $K_{1,3}$ and W_4.

(v) the complement of the cycle graph C_4.

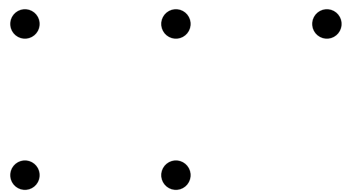


Figure 10: (i) Null graph N_5

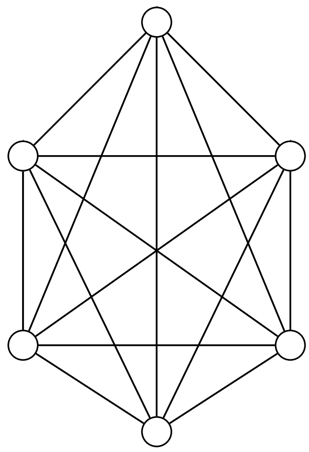


Figure 11: (ii) Complete graph K_6

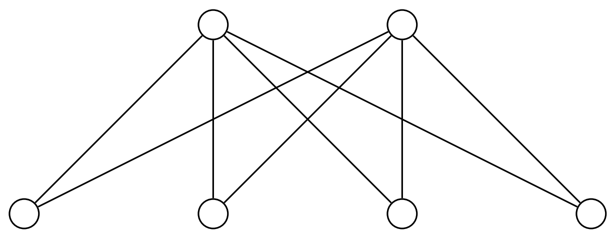


Figure 12: (iii) Complete bipartite graph $K_{2,4}$

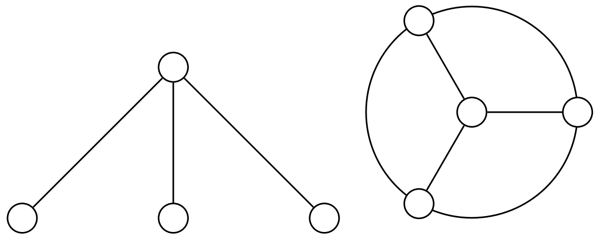


Figure 13: (iv) Union of $K_{1,3}$ and W_4

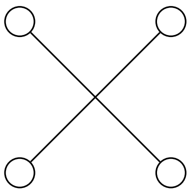


Figure 14: (iv) Complement of cycle graph C_4

(3c) Draw the graphs $K_{2,2,2}$, and $K_{3,3,2}$, and write down the number of edges of $K_{3,4,5}$.

The graphs $K_{2,2,2}$ and $K_{3,3,2}$ are shown in the following figures, respectively.
For the graph $K_{3,4,5}$, there is 47 edges.

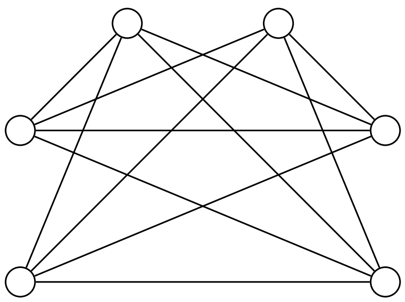


Figure 15: $K_{2,2,2}$

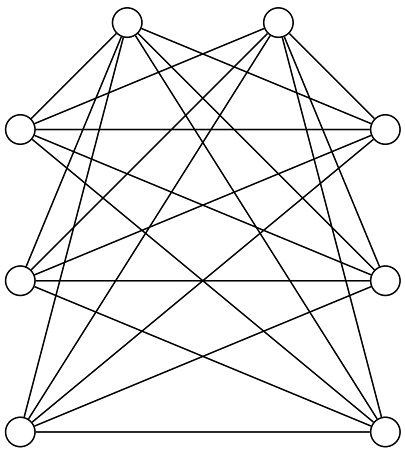


Figure 16: $K_{3,3,2}$

(3g) A simple graph that is isomorphic to its complement is self-complementary.

(i) Prove that, if G is self-complementary, then G has $4k$ or $4k + 1$ vertices, where k is an integer,

(ii) Find all self-complementary graphs with 4 and 5 vertices,

(iii) Find a self-complementary graph with 8 vertices.

(i) Proof
If G is a self-complementary with n vertices, and

$$G \cup \overline{G} = K_n.$$

But we know that, the total number of edge in the complete graph K_n i.e. $|E(K_n)|$is $n(n-1)/2$, that is,

$$|E(G)| = |E(\overline{G})| = \frac{n(n-1)}{4}.$$

In other words, n or $n-1$ must be divisible by 4, that is, when n is $4k$ or $4k+1$.

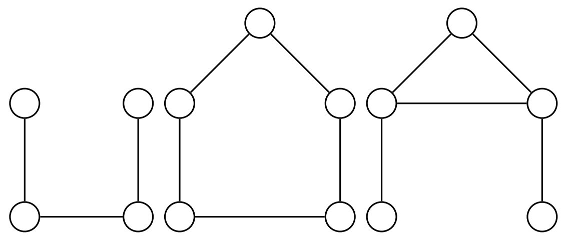


Figure 17: (ii) 4 and 5 vertices self-complementary graphs

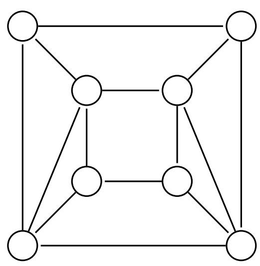


Figure 18: (iii) a self-complementary graph with 8 vertices

Section (4)

- **Jordan curve** is a continuous curve which doesn't intersect itself.
- **Graph embedding**: A graph G can be embedded (or has an embedding) in a given space if it is isomorphic to a graph drawn in the space with points representing vertices of G and Jordan curves representing edges in such a way that there are no crossings.
- **Theorem 4A**: *Every graph can be embedded in Euclidean 3-space.*
- **Planer graph**: a graph that can be embedded in a plane.
- **Theorem 4B**: *A graph is planer if and only if it can be embedded on the surface of a sphere.*

Section (5)

- A **walk** in G is a finite sequence of edges of the form $v_0v_1, v_1v_2, \cdots, v_{m-1}v_m$, where v_0 is the **initial vertex** and v_m is the **finial vertex** of the walk, also the number of edges is called **length**.
- **Trial** is a walk in which all the edges are distinct. **path** is a trial with all vertices are distinct also (expect, possibly $v_0 = v_m$ where then we call the trial or the path **closed**). A closed path containing at least one edge is a **cycle**[1].
- A graph is **connected** if and only if there is a path between each pair of vertices.
- **Theorem 5.1** *If G is a bipartite graph, then each of G has even length.*
- **Theorem 5.2** *Let G be a simple graph on n vertices. If G has k components, then the number m of edges of G satisfies*

$$n - k \leq m \leq (n-k)(n-k+1)/2. \quad (1)$$

- **Corollary 5.3** *Any simple graph with n vertices and more than $(n-1)(n-2)/2$ edges is connected.*
- A **disconnecting set** in a connected graph G is a set of edges whose removal disconnects G and increases the number of components of G.
- A **cutset** is defined to be a disconnecting set, no proper subset of which is a disconnecting set. If a cutset has only one edge e, we call e a **bridge**.
- If G is connected, its **edge connectivity** $\lambda(G)$ is the size of the smallest cutset in G. Thus $\lambda(G)$ is the minimum number of edges that we need to delete in order to disconnect G.
- If G is connected and not a complete graph, its **vertex connectivity** $\mathcal{K}(G)$ is the size of the smallest separating set in G. Thus $\mathcal{K}(G)$ is the minimum number of vertices that we need to delete in order to disconnect G.

Exercise 5

(5a) In the Petersen graph, find

(i) a trail of length 5;

(ii) a path of length 9;

(iii) cycles of lengths 5, 6, 8 and 9;

(iv) cutsets with 3, 4 and 5 edges.

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[1] called *circuit* in 3rd edition of the textbook

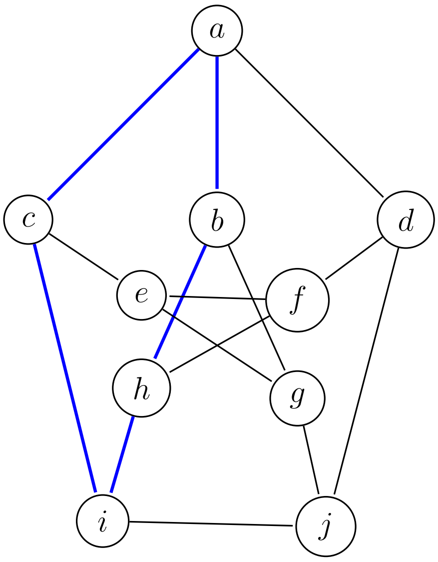

Figure 19: (i) a Petersen graph with trail of length 5 highlighted $\{ab, bh, hi, ic, ca\}$

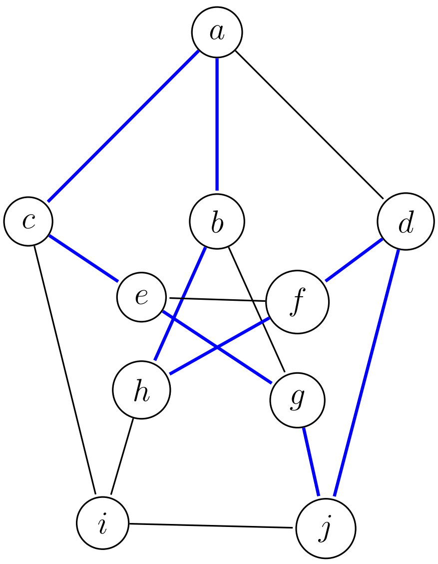

Figure 20: (ii) a Petersen graph with path of length 9 highlighted $\{ab, bh, hf, fd, dj, ig, ge, ec, ca\}$

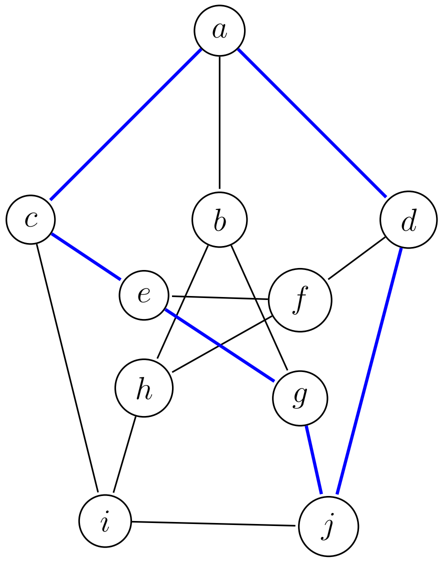

Figure 21: While (i) also represents a cycle of 5 length, this represents a cycle of 6 length highlighted $\{ac, ce, eg, gj, jd, da\}$

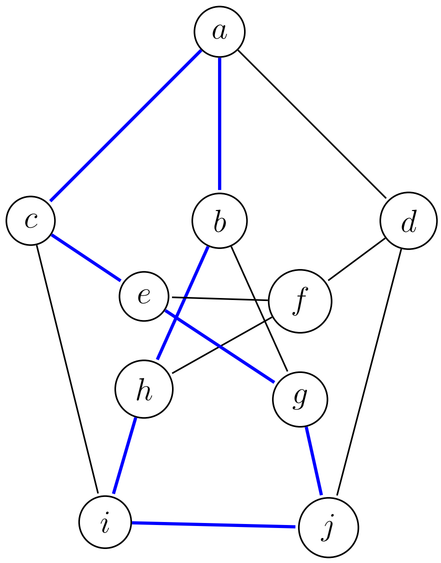

Figure 22: A cycle of 8 length highlighted $\{ac, ce, eg, gj, ji, ih, hb, ba\}$

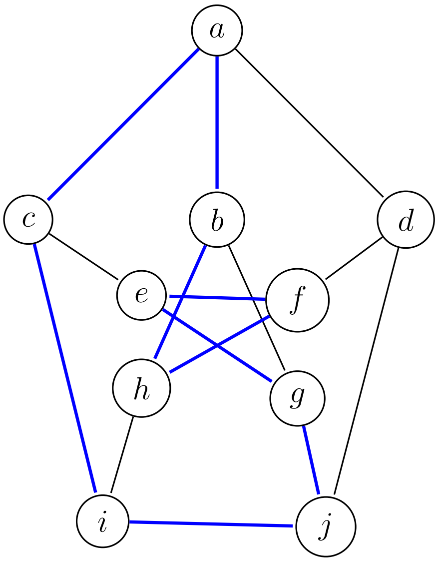

Figure 23: A cycle of 9 length highlighted $\{ac, ci, ij, jg, ge, ef, fh, hb, ba\}$

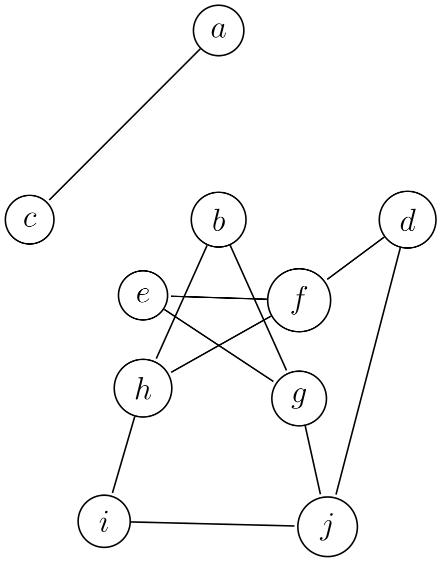

Figure 25: A cutset of 4 edges $\{ab, ac, ce, ci\}$

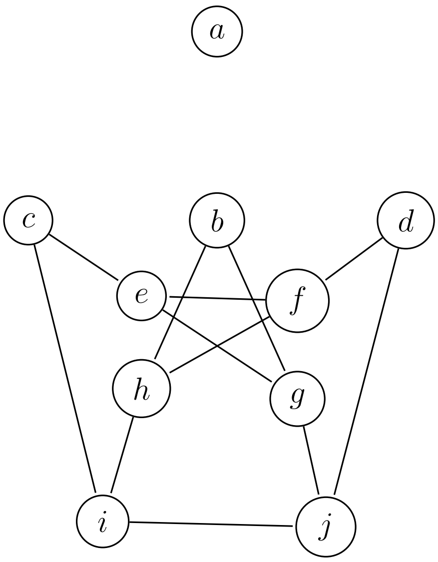

Figure 24: A cutset of 3 edges $\{ac, ab, ac\}$

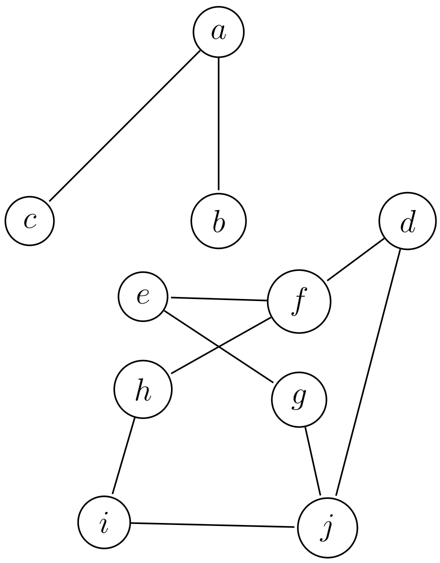

Figure 26: A cutset of 5 edges $\{ad, bh, bg, ce, ci\}$

(5c) Find $\mathcal{K}(G)$ and $\lambda(G)$ for each of the following graphs (i) C_6, (ii) W_6, (iii) $K_{4,7}$, (iv) Q_4

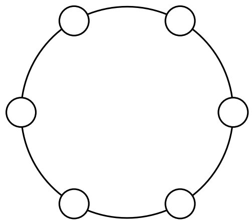

Figure 27: (i) $\mathcal{K}(C_6) = 2$ and $\lambda(C_6) = 2$

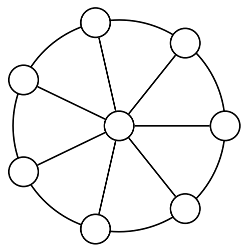

Figure 28: (ii) $\mathcal{K}(W_8) = 4$ and $\lambda(W_8) = 3$

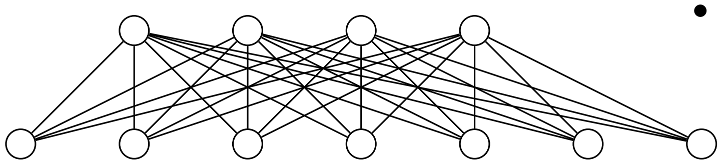

Figure 29: (iii) $\mathcal{K}(K_{4,7}) = 4$ and $\lambda(W_8) = 4$

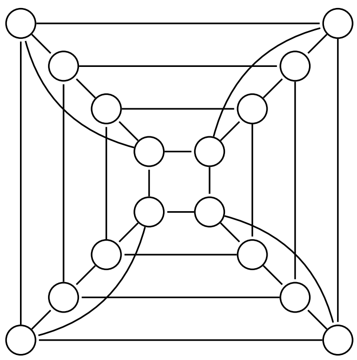

Figure 30: (iv) $\mathcal{K}(Q_4) = 3$ and $\lambda(Q_4) = 8$

(5f) Prove that if G is a simple graph, then G and $\overline{G}$ cannot both be disconnected

To prove this we need to prove that the complement of a disconnected graph G is connected.
That is without loss of generality, assume G is disconnected. Now consider two vertices x and y in $\overline{G}$. If x and y are not adjacent in G, then they will be adjacent in $\overline{G}$ and we can find a trivial $x-y$ path. If x and y are adjacent in G then they must have been in the same component. This means that the edges xz and yz were not in G. This implies that they both must be edges in $\overline{G}$. This gives us the path $x \to y \to z$. Therefore, in $\overline{G}$ we have that there exists a path between any two vertices and hence it is connected.

Section (6)

- A connected graph G is **Eulerian** if there exists a closed trial containing every edge of G. Such trial is an **Eulerian trial**. A non-Eulerian graph is **semi-Eulerian** if there exists a trail containing every edge of G.

- **Lemma 6a**: *If G is a graph in which the degree of each vertex is at least 2, then G contains a cycle.*

- **Theorem 6b**: *A connected graph G is Eulerian if and only if the degree of each vertex of G is even.*

- **Corollary 6c**: *A connected graph is Eulerian if and only its set of edges can be split up into disjoint cycles.*

- **Corollary 6d**: *A connected graph is semi-Eulerian if and only if it has eactly two vertices of odd degree.*

- **Theorem 6e**: *Let G be an Eulerian graph. Then the following construction is always possible, and produces an Eulerian trial of G.*
  Start at any vertex u and traverse the edges in an arbitrary manner, subject only to the following rules:

  (i) erase the edges as they traversed, and if any isolated vertices result, erase them too;

(ii) at each stage, use a bridge only if there is no alternative.