

Abstraction of Graph Theory

Wael Ali

2: Definitions

- A **simple graph** G consists of a non-empty finite set $V(G)$ of elements called vertices (or nodes), and a finite set $E(G)$ of distinct unordered pairs of distinct elements of $V(G)$ called edges.
- A **graph** G consists of a non-empty finite set $V(G)$ of elements called vertices, and a finite family $E(G)$ of unordered pairs of (not necessarily distinct) elements of $V(G)$ called edges.
- Two graphs G_1 and G_2 are **isomorphic** if there is a one-one correspondence between the vertices of G_1 and those of G_2 such that the number of edges joining any two vertices of G_1 is equal to the number of edges joining the corresponding vertices of G_2 .

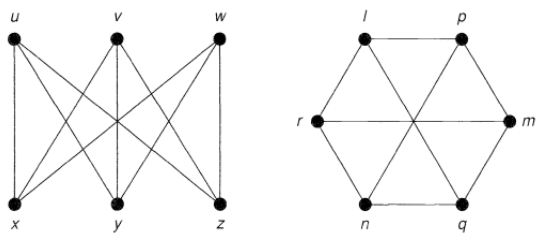


Figure 1: Two simple graphs that are isomorphic to each other

- A graph is **connected** if it cannot be expressed as the union of two graphs, and **disconnected**.
- Two vertices v and w of a graph G are **adjacent** if there is an edge vw joining them, and the vertices v and w are then **incident** with such an edge. Similarly, two distinct edges e and f are adjacent if they have a vertex in common.

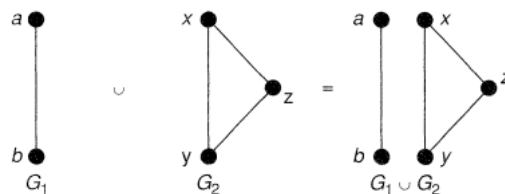
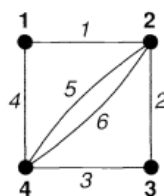


Figure 2: Each of G_1 and G_2 is a component of $G_1 \cup G_2$

- The **degree** of a vertex v of G is the number of edges incident with v , and is written $deg(v)$. Loop-edge increase node-degree by 2. vertex of degree 0 is an **isolated vertex** and a vertex of degree 1 is an **end-vertex**.
- **Handshaking lemma**; in any graph the sum of all the vertex-degrees is an even number - in fact, twice the number of edges, since each edge contributes exactly 2 to the sum.
- A **subgraph** of a graph G is a graph, each of whose vertices belongs to $V(G)$ and each of whose edges belongs to $E(G)$.
- **Matrix representation** one way to represent graph is by its adjacency matrix A , and its incidence matrix M as follows;

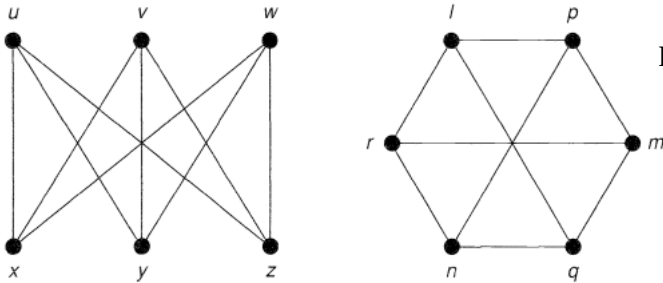


$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{bmatrix}$$

$$M = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Exercise 2

2a Write down the vertex-set and edge-set of each graph in Fig 2.5



The first graph G_1 is $(V(G_1), E(G_1))$

$$\begin{aligned} V(G_1) &= \{x, y, z, u, v, w\} \\ E(G_1) &= \{\{x, u\}, \{x, v\}, \{x, w\}, \\ &\quad \{y, u\}, \{y, v\}, \{y, w\}, \\ &\quad \{z, u\}, \{z, v\}, \{z, w\}\} \end{aligned}$$

The second graph G_2 is $(V(G_2), E(G_2))$

$$\begin{aligned} V(G_2) &= \{n, m, q, r, l, p\} \\ E(G_2) &= \{\{n, r\}, \{n, p\}, \{n, q\}, \\ &\quad \{m, r\}, \{m, p\}, \{m, q\}, \\ &\quad \{l, r\}, \{l, p\}, \{l, q\}\} \end{aligned}$$

2b Draw;

- (i) a simple graph.
- (ii) a non-simple graph with no loops.
- (iii) a non-simple graph with no multiple edges, each having 5 vertices each having 5 vertices and 8 edges.

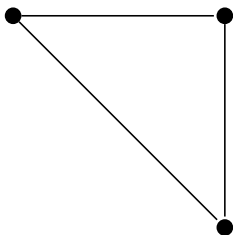


Figure 3: (i) a simple graph

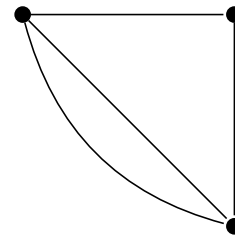


Figure 4: (ii) a non-simple graph with no loops

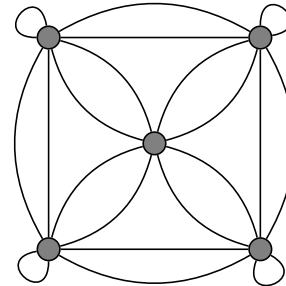


Figure 5: (iii) a 5-vertices and 8-degrees each

2c Draw;

- (i) Draw a graph on six vertices whose degrees are 5,5,5,5,3,3; does there exist a simple graph with these degrees?
- (ii) How does the answer to part (i) changed if the degrees are 5, 5, 4, 3, 3,2?

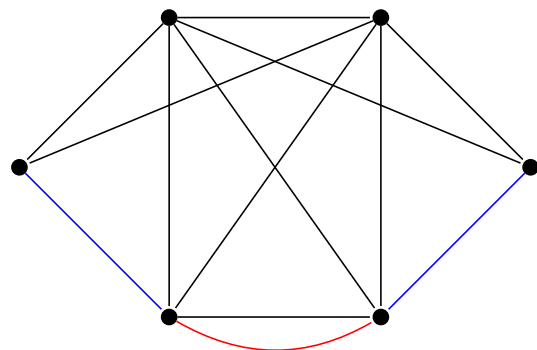


Figure 6: (i) a non-simple graph with 6-vertices with degrees [5,5,5,5,3,3]

There isn't a simple graph with last mentioned degrees for a 6-vertices graph. But it

we just remove the red-arc and one of the blue-arcs, we then get a simple graph as shown in the next figure.

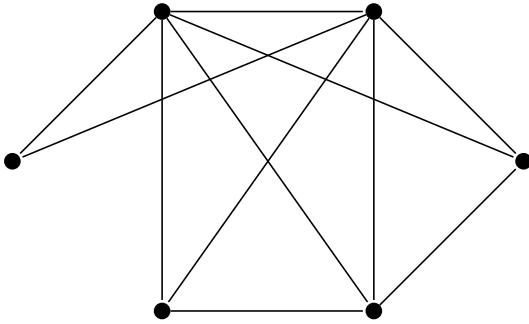
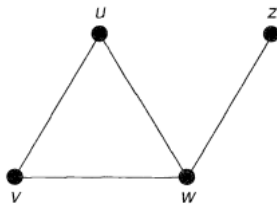


Figure 7: (i) a simple graph with 6-vertices with degrees $[5,5,4,3,3,2]$

(2d) Verify that handshaking lemma is hold for figure 2.1

As we can see in the last figure, the sum



of all vertex-degrees is $2 + 2 + 3 + 1 = 8$ which is an even number.

- (2f) (i) By suitably lettering the vertices, show that the two graphs in Fig. 2.20 are isomorphic.
(ii) Explain why two graphs in Fig. 2.21 are not isomorphic.

- (i) As shown in the following figure 8 the two graphs are labeled with the same letters in a way to emphasize that they are both isomorphic to each other.
(ii) As shown in the following figure 9, we cannot find the red part of the first graph as a subset in the second graph.

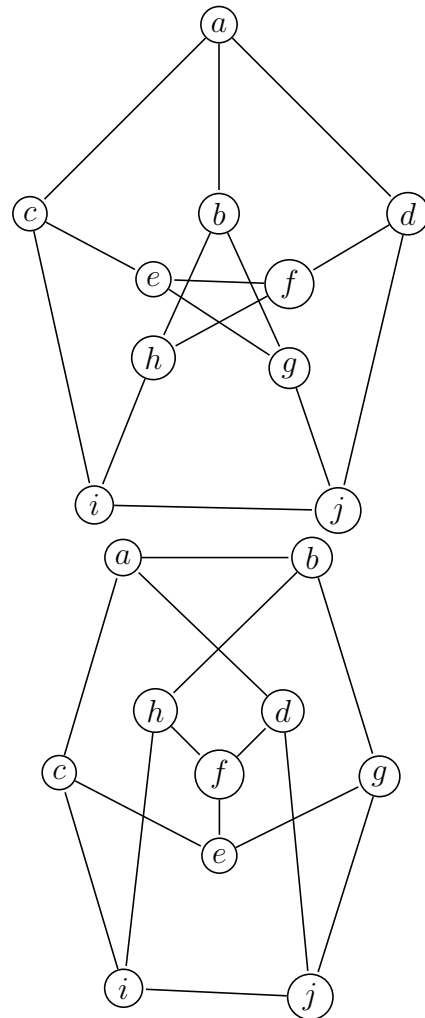


Figure 8: (i)

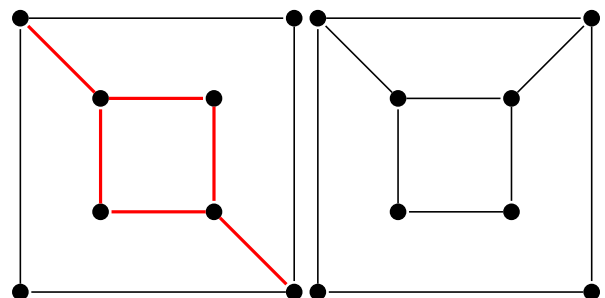
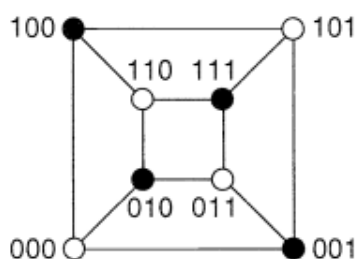


Figure 9: (ii) not isomorphic graphs

3: Examples of graphs

- **Null graphs**, a graph whose edge-set is empty. **Complete graph**, a simple graph in which each pair of distinct vertices are adjacent, in this case k -vertex must have $n(n-1)/2$ degree as n the total number of vertices.
- **Regular graph**, a graph in which each vertex has the same degree. Platonic graphs, formed from the vertices and edges of the five regular (Platonic) solids - the tetrahedron, octahedron, cube.
- **Bipartite graph**, If the vertex set of a graph G can be split into two disjoint sets A and B so that each edge of G joins a vertex of A and a vertex of B . A complete bipartite graph is a bipartite graph in which each vertex in A is joined to each vertex in B by just one edge.
- **Cubes**, the k -cube Q_k is the graph whose vertices correspond to the sequences (a_1, a_2, \dots, a_k) , where each $a_i = 0$ or 1 , and whose edges join those sequences that differ in just one place. You should check that Q_k has 2^k vertices and $k2^{k-1}$ edges, and is regular of degree k .



- **Complement of a simple graph**, if G is a simple graph with vertex set $V(G)$, its complement \overline{G} is the simple graph with vertex set $V(G)$ in which two vertices are adjacent if and only if they are not adjacent in G .

Exercise 3

(3a) Draw the following graphs:

- (i) the null graph N_5 .

(ii) the complete graph K_6 .

(iii) the complete bipartite graph $K_{2,4}$.

(iv) the union of $K_{1,3}$ and W_4 .

(v) the complement of the cycle graph C_4 .

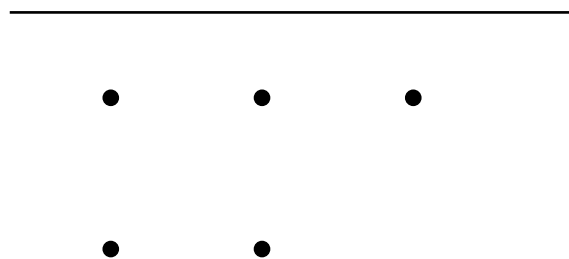


Figure 10: (i) Null graph N_5

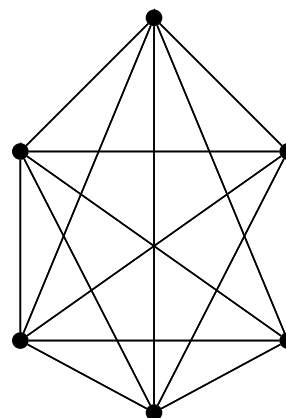


Figure 11: (ii) Complete graph K_6

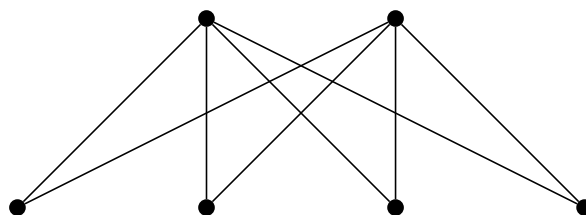


Figure 12: (iii) Complete bipartite graph $K_{2,4}$

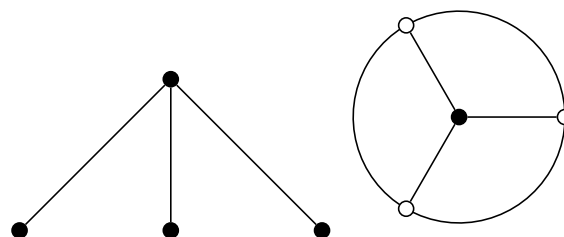


Figure 13: (iv) Union of $K_{1,3}$ and W_4

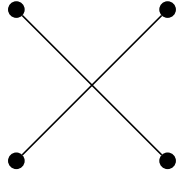


Figure 14: (iv) Complement of cycle graph C_4

- (3c) Draw the graphs $K_{2,2,2}$, and $K_{3,3,2}$, and write down the number of edges of $K_{3,4,5}$.

The graphs $K_{2,2,2}$ and $K_{3,3,2}$ are shown in the following figures, respectively. For the graph $K_{3,4,5}$, there is 47 edges.

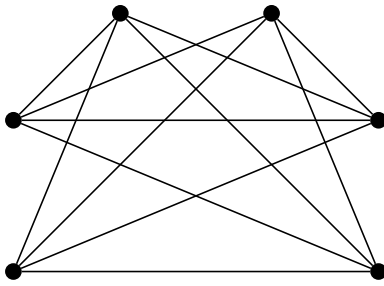


Figure 15: $K_{2,2,2}$

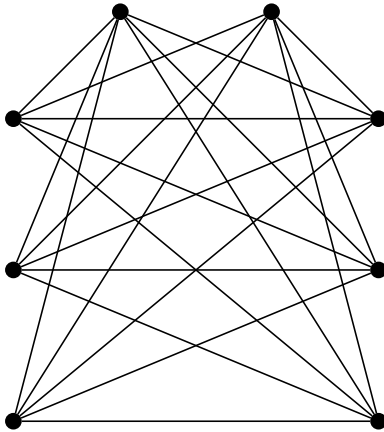


Figure 16: $K_{3,3,2}$

- (3g) A simple graph that is isomorphic to its complement is self-complementary.
- Prove that, if G is self-complementary, then G has $4k$ or $4k+1$ vertices, where k is an integer,
 - Find all self-complementary graphs with 4 and 5 vertices,
 - Find a self-complementary graph with 8 vertices.

(i) Proof

If G is a self-complementary with n vertices, and

$$G \cup \overline{G} = K_n.$$

But we know that, the total number of edge in the complete graph K_n i.e. $|E(K_n)|$ is $n(n-1)/2$, that is,

$$|E(G)| = |E(\overline{G})| = \frac{n(n-1)}{4}.$$

In other words, n or $n-1$ must be divisible by 4, that is, when n is $4k$ or $4k+1$.

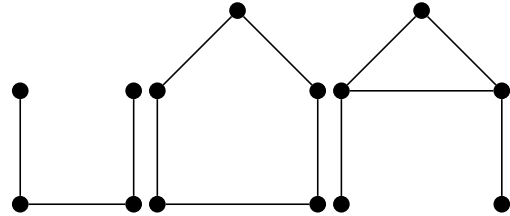


Figure 17: (ii) 4 and 5 vertices self-complementary graphs

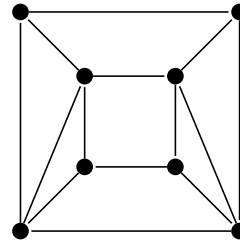


Figure 18: (iii) a self-complementary graph with 8 vertices

4: Embeddings of graphs

- Jordan curve** is a continuous curve which doesn't intersect itself.
- Graph embedding**: A graph G can be embedded (or has an embedding) in a given space if it is isomorphic to a graph drawn in the space with points representing vertices of G and Jordan curves representing edges in such a way that there are no crossings.

- **Theorem 4A:** Every graph can be embedded in Euclidean 3-space.
- **Planer graph:** a graph that can be embedded in a plane.
- **Theorem 4B:** A graph is planer if and only if it can be embedded on the surface of a sphere.
- If G is connected, its **edge connectivity** $\lambda(G)$ is the size of the smallest cutset in G . Thus $\lambda(G)$ is the minimum number of edges that we need to delete in order to disconnect G .
- If G is connected and not a complete graph, its **vertex connectivity** $\mathcal{K}(G)$ is the size of the smallest separating set in G . Thus $\mathcal{K}(G)$ is the minimum number of vertices that we need to delete in order to disconnect G .

5: More definitions

- A **walk** in G is a finite sequence of edges of the form $v_0v_1, v_1v_2, \dots, v_{m-1}v_m$, where v_0 is the initial vertex and v_m is the final vertex of the walk, also the number of edges is called length.
- **Trial** is a walk in which all the edges are distinct. **path** is a trial with all vertices are distinct also (except, possibly $v_0 = v_m$ where then we call the trial or the path **closed**). A closed path containing at least one edge is a **cycle**¹.
- A graph is **connected** if and only if there is a path between each pair of vertices.
- **Theorem 5.1** If G is a bipartite graph, then each of G has even length.
- **Theorem 5.2** Let G be a simple graph on n vertices. If G has k components, then the number m of edges of G satisfies

$$n - k \leq m \leq (n - k)(n - k + 1)/2. \quad (1)$$

- **Corollary 5.3** Any simple graph with n vertices and more than $(n - 1)(n - 2)/2$ edges is connected.
- A **disconnecting set** in a connected graph G is a set of edges whose removal disconnects G and increases the number of components of G .
- A **cutset** is defined to be a disconnecting set, no proper subset of which is a disconnecting set. If a cutset has only one edge e , we call e a **bridge**.

¹called circuit in 3rd edition of the textbook

Exercise 5

(5a) In the Petersen graph, find

- (i) a trail of length 5;
- (ii) a path of length 9;
- (iii) cycles of lengths 5, 6, 8 and 9;
- (iv) cutsets with 3, 4 and 5 edges.

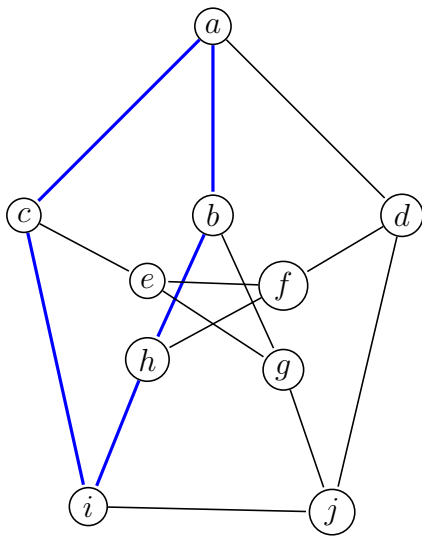


Figure 19: (i) a Petersen graph with trail of length 5 highlighted $\{ab, bh, hi, ic, ca\}$

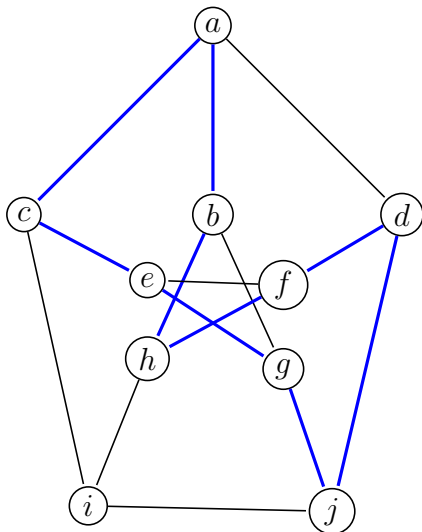


Figure 20: (ii) a Petersen graph with path of length 9 highlighted $\{ab, bh, hf, fd, dj, ig, ge, ec, ca\}$

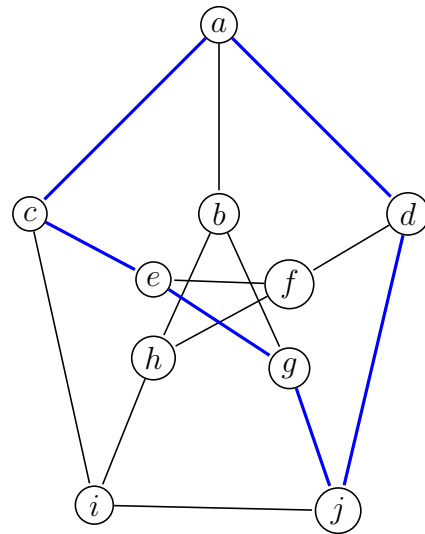


Figure 21: While (i) also represents a cycle of 5 length, this represents a cycle of 6 length highlighted $\{ac, ce, eg, gj, jd, da\}$

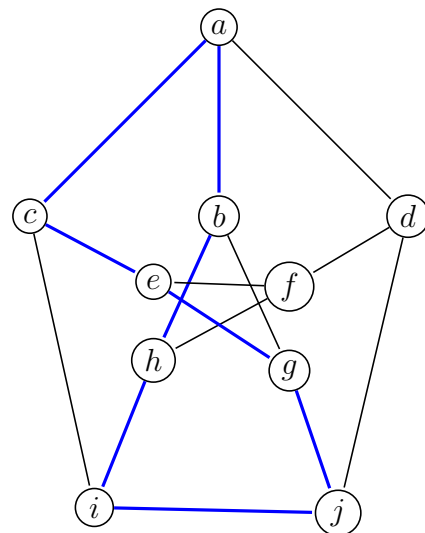


Figure 22: A cycle of 8 length highlighted $\{ac, ce, eg, gj, ji, ih, hb, ba\}$

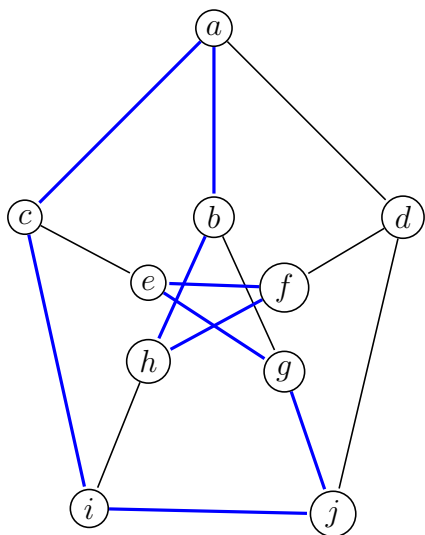


Figure 23: A cycle of 9 length highlighted $\{ac, ci, ij, jg, ge, ef, fh, hb, ba\}$

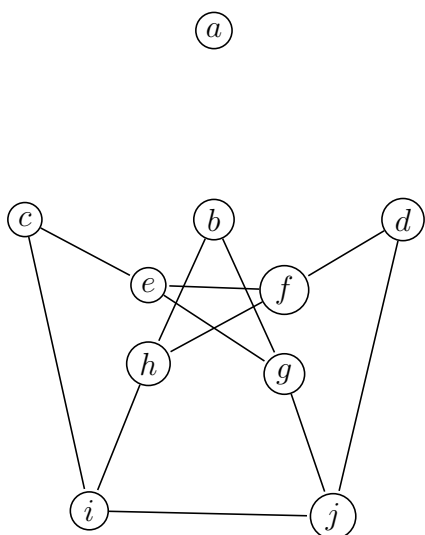


Figure 24: A cutset of 3 edges $\{ac, ab, bc\}$

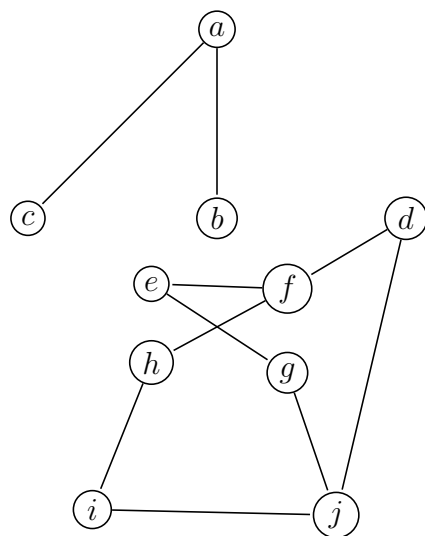


Figure 26: A cutset of 5 edges $\{ad, bh, bg, ce, ci\}$

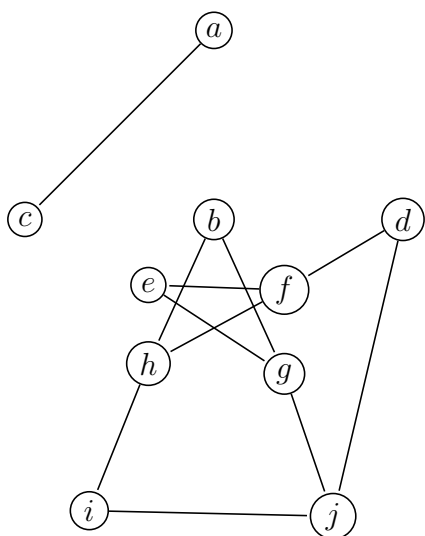


Figure 25: A cutset of 4 edges $\{ab, ac, ce, ci\}$

- (5c) Find $\mathcal{K}(G)$ and $\lambda(G)$ for each of the following graphs (i) C_6 , (ii) W_6 , (iii) $K_{4,7}$, (iv) Q_4
-

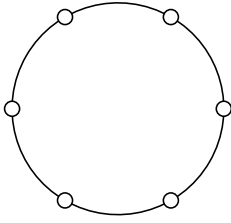


Figure 27: (i) $\mathcal{K}(C_6) = 2$ and $\lambda(C_6) = 2$

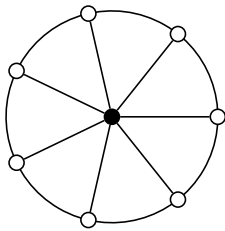


Figure 28: (ii) $\mathcal{K}(W_8) = 4$ and $\lambda(W_8) = 3$

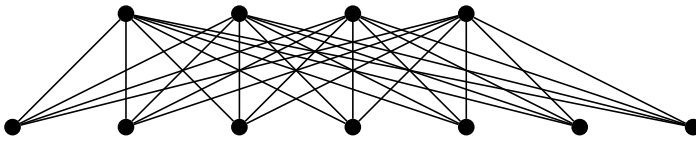


Figure 29: (iii) $\mathcal{K}(K_{4,7}) = 4$ and $\lambda(W_8) = 4$

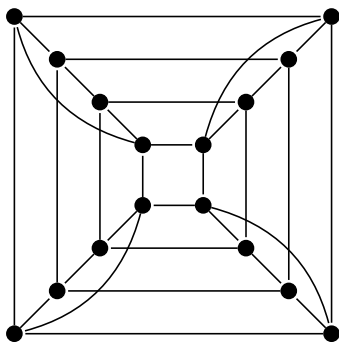


Figure 30: (iv) $\mathcal{K}(Q_4) = 3$ and $\lambda(Q_4) = 8$

- (5f) Prove that if G is a simple graph, then G and \overline{G} cannot both be disconnected
-

To prove this we need to prove that the complement of a disconnected graph G is connected.

That is without loss of generality, assume G is disconnected. Now consider two vertices x and y in \overline{G} . If x and y are not adjacent in G , then they will be adjacent in \overline{G} and we can find a trivial $x - y$ path. If x and y are adjacent in G then they must have been in the same component. This means that the edges xz and yz were not in G . This implies that they both must be edges in \overline{G} . This gives us the path $x \rightarrow y \rightarrow z$. Therefore, in \overline{G} we have that there exists a path between any two vertices and hence it is connected.

6: Eulerian graphs

- A connected graph G is **Eulerian** if there exists a closed trail containing every edge of G . Such trail is an **Eulerian trial**. A non-Eulerian graph is **semi-Eulerian** if there exists a trail containing every edge of G .

- Lemma 6a:** If G is a graph in which the degree of each vertex is at least 2, then G contains a cycle.

- Theorem 6b:** A connected graph G is Eulerian if and only if the degree of each vertex of G is even.

- Corollary 6c:** A connected graph is Eulerian if and only its set of edges can be split up into disjoint cycles.

- Corollary 6d:** A connected graph is semi-Eulerian if and only if it has exactly two vertices of odd degree.

- Theorem 6e:** Let G be an Eulerian graph. Then the following construction is always possible, and produces an Eulerian trial of G .

Start at any vertex u and traverse the edges in an arbitrary manner, subject only to the following rules:

- erase the edges as they traversed, and if any isolated vertices result, erase them too;
- at each stage, use a bridge only if there is no alternative.

Exercise 6

(6a) Which of the following graphs are Eulerian? semi-Eulerian?

- the complete graph K_5 .
- the complete bipartite graph $K_{4,3}$.
- the graph of cube.
- the graph of octahedron.
- the Petersen graph.

• **Theorem 7A** If G is a simple graph with $n(\geq 3)$ vertices, and if $\rho(v) + \rho(w) \geq n$ for each of non-adjacent vertices v and w , then G is Hamiltonian.²

• **Dirac 1952** If G is a simple graph with $n(\geq 3)$ vertices, and if $\rho(v) \geq \frac{1}{2}n$ for every vertex v then G is Hamiltonian.

Exercise 7

(6b) – For which values of n is K_n Eulerian?
 – Which complete bipartite graphs are Eulerian?
 – Which Platonic graphs are Eulerian?
 – For which values of n is the wheel W_n Eulerian?
 – For which values of k the k -cube Q_n Eulerian?

(7a) Which of the following graphs are Hamiltonian? semi-Hamiltonian?

- i the complete graph K_5 .
- ii the complete bipartite graph $K_{5,3}$.
- iii the graph of the octahedron.
- iv the wheel W_6 .
- v the 4-cube Q_4 .

(6c) Let G a connected graph with $k(> 0)$ vertices of odd degree.

- (i) Show that the minimum number of trials, which have no edges in common add which together include every edge of G , is $\frac{1}{2}k$.
- (ii) How many continuous pen-strokes are needed to draw the diagram in Figure without repeating any line?

(7b) i For which values of n is K_n Hamiltonian?

- ii Which complete bipartite graphs are Hamiltonian?
- iii Which Platonic graphs are Hamiltonian?
- iv For which values of n is the wheel W_n Hamiltonian?
- v For which values of k if the k -cube Q_n Hamiltonian?

7: Hamiltonian graphs

- a closed trail passing exactly once through each vertex of G . Such a trail must be a cycle, except when G is the graph N_1 . Such a cycle is a **Hamiltonian cycle** and G is a **Hamiltonian graph**. A **non-Hamiltonian graph** G is semi-Hamiltonian if there exists a path passing through every vertex.

Some applications

- The short path problem.
- The Chinese postman problem.
- The travelling salesman problem.

² ρ is referred as "deg" in 4th edition

9: Elementary properties of trees

- A forest is defined to be a graph which contains no cycles and a connected forest is called a tree.

- Theorem 9A** Let T be a graph with n vertices. Then the following statements are equivalent:

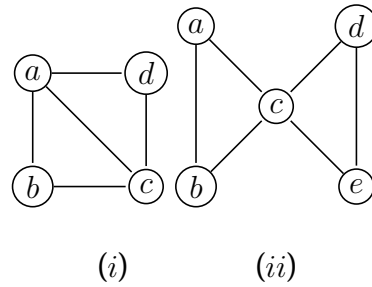
- T is a tree.
- T contains no cycles, and has $n - 1$ edges.
- T is connected, and has $n - 1$ edges.
- T is connected, and every edge is a bridge.
- any two vertices of T are connected by exactly one path.
- T contains no cycles, but the addition of any new edge creates exactly one cycle.

- Corollary 9B** If G be a forest with n vertices and k components, then G has $n - k$ edges.

- Given any connected graph G , we can choose a cycle and remove any one of its edges, and the resulting graph remains connected. If we repeat this procedure until there are no cycles left. The graph that remains is a tree that connects all the vertices of G . It is called a **spanning tree** of G . Generally, if G is an arbitrary graph with n vertices, m edges and k components, then we can carry out this procedure on each component of G . The result is called a **spanning forest**, and the total number of edges removed in this process is the cycle rank of G , denoted by $\gamma(G)$. the cutset rank of G to be the number of edges in a spanning forest, denoted by $\xi(G) = n - k$

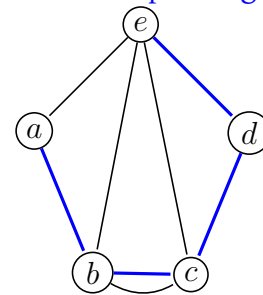
- Theorem 9C** If T is any spanning forest of a graph G , then

- every cutset of G has an edge in common with T .
- every cutset of G has an edge in common with the complement of T .



- Draw all the spanning trees in the graph of the following figure.

- Find the fundamental sets of cycles and cutsets of the graph if the following figure, associated with the spanning tree shown



10: The enumeration of trees

- Labelling** of a graph G on n vertices to be one-one mapping from the vertex-set of G onto the set $1, \dots, n$, a **labelled graph** is then a pair (G, ϕ) , where ϕ is a labelling of G .

- Theorem 10A** (Cayley) There are n^{n-2} distinct labelled trees on n vertices.

- Corollary 10B** The number of spanning trees of K_n is n^{n-2} .

- Theorem 10C** Let G be a connected simple graph with vertex set $\{v_1, \dots, v_n\}$, and let $M = (m_{ij})$ be $n \times n$ matrix in which $m_{ii} = \rho(v_i)$, $m_{ij} = -1$ if v_i and v_j are adjacent, and $m_{ij} = 0$ otherwise. Then the number of spanning trees of G is equal to the cofactor of any element of M .

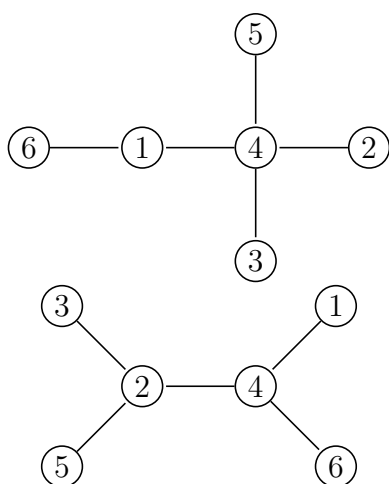
Exercise 9

- Prove that every tree is a bipartite graph.
- Which trees are complete bipartite graphs?

Exercise 10

- Verify directly that there are exactly 125 labelled trees on five vertices.

- (i) Show that there are exactly $2^{n(n-1)/2}$ labelled simple graphs on n vertices.
- (ii) How many of these have exactly m edges?
- In the first proof of Cayley's theorem, find:
 - (i) the labelled trees corresponding to the sequences (1,2,3,4) and (3,3,3,3).
 - (ii) the sequences corresponding to the labelled tree in the following figure



- How many spanning trees has $K_{2,s}$?

More Applications

- The minimum connector problem.
- Enumeration of chemical molecules.
- Electrical networks.

12: Planar graphs

- A **plane graph** is a graph drawn in the plane in such a way that no two edges intersect geometrically except at a vertex to which they are both incident. a planar graph is one which is isomorphic to a plane graph.
- **Theorem 12A:** K_5 , and $K_{3,3}$ are non-planar
- Two graphs are **homomorphic** if they can both be obtained from the same graph by inserting new vertices of degree two into its edges.

- **Theorem 12B:** A graph is **planar** iff it contains **no subgraph homeomorphic to K_5 , and $K_{3,3}$**
- **Theorem 12C:** A graph is **planar** iff it contains **no subgraph which contractible to K_5 , and $K_{3,3}$**

Exercise 12

- (12c) Prove that the Petersen graph is non-planar
 - (i) by using Theorem 12B,
 - (ii) by using Theorem 12C.
- (12d) Give an example of
 - (i) a non-planar graph which is not homeomorphic to K_5 or $K_{3,3}$,
 - (ii) a non-planar graph which is not contractible to K_5 or $K_{3,3}$.

Why does the existence of these graphs not contradict Theorems 12B and 12C?

13: Euler's formula for plane graphs

- If x is a point of the plane disjoint from G , the **face** of (G) containing x is the set of all points of the plane which can be reached from x by a Jordan curve all of whose points are disjoint from G .
- **Theorem 13A (Euler):** Let G be a connected plane graph, and let n, m and f denote respectively the number of vertices, edges and faces of G , then **$n - m + f = 2$** .
- **Corollary 13B:** let G be a **polyhedral** graph, then with above notation **$n - m + f = 2$** .
- **Corollary 13C:** let G be a **plane** graph with n vertices, m edges, f faces and k components, then **$n - m + f = k + 1$** .
- **Corollary 13D:** (i) If G is a connected simple planar graph with $n (\geq 3)$ vertices and m edges, then $m \leq 3n - 6$
(ii) If, in addition, G has no triangles, then $m \leq 2n - 4$.
- **Corollary 13E:** K_5 and $K_{3,3}$ are non-planar.

- **Corollary 13F:** Every simple planar graph contains a vertex whose degree is at most five.
- The **thickness** of a graph $t(G)$ is the smallest number of planar graphs which can be superimposed to form G .
- **Theorem 13G:** Let G be a simple graph with $n(\geq 3)$ vertices and m edges, then the thickness $t(G)$ of G satisfies the following inequalities:

$$t(G) \geq \left\lceil \frac{m}{3n-6} \right\rceil; \quad t(G) \geq \left\lfloor \frac{m+3n-7}{3n-6} \right\rfloor$$

Exercise 13

- (13c) (i) Use Euler's formula to prove that if G is a connected plane graph of girth 5 then, with the above notation, $m \leq \frac{5}{3}(n-2)$.
- (ii) Deduce the Petersen graph is non-planar.
- (iii) Obtain an inequality, generalizing that in part (i), for connected plane graphs of girth r .
- (13h) Find the thickness of
- (i) the Petersen graph.
- (ii) the 4-cube Q_4 .

14: Graphs on other surfaces

Exercise 14

15: Dual graphs

- **Geometric-dual G^*** of graph G constructed as follows:
 - inside each face F_i of G we choose a point v_i^* — these points are the vertices of G^* ,
 - corresponding to each edge e of G we draw a line e^* which crosses e (but no other edge of G), and joins the vertices v_i^* which lie in the faces F_i adjoining e — these lines are the edges of G^* .

- **Lemma 15A:** Let G be a plane connected graph with n vertices, m edges and f faces, and let its geometric-dual G^* have n^* vertices, m^* edges and f^* faces. Then $n^* = f$, $m^* = m$, and $f^* = n$.
- **Theorem 15B:** Let G be a plane connected graph. Then G^{**} is isomorphic to G .
- **Theorem 15C:** Let G be a planar and G^* be a geometric-dual of G . Then a set of edges in G forms a cycle in G iff the corresponding set of edges of G^* forms a cutset in G^* .
- **Corollary 15D:** A set of edges of G forms a cutset in G iff the corresponding set of edges of G^* forms a cycle in G^* .
- The **abstract-dual** G^* of a graph G if there is a one-one correspondence between the edges of G and those of G^* with the property that a set of edges of G forms a cycle in G iff the corresponding set of edges of G^* forms a cutset in G^* .
- **Theorem 15E:** If G^* is an abstract-dual of G , then G is an abstract-dual of G^* .
- **Theorem 15F:** A graph is planar iff it has an abstract-dual.

Exercise 15

16: Infinite graphs

Exercise 16