Abstraction of Graph Theory

Wael Ali

Section (2)

- A **simple graph** G consists of a non-empty finite set $V(G)$ of elements called vertices (or nodes), and a finite set $E(G)$ of distinct unordered pairs of distinct elements of $V(G)$ called edges.

- A **graph** G consists of a non-empty finite set $V(G)$ of elements called vertices, and a finite family $E(G)$ of unordered pairs of (not necessarily distinct) elements of $V(G)$ called edges.

- Two graphs $G1$ and $G2$ are **isomorphic** if there is a one-one correspondence between the vertices of G_x and those of $G2$ such that the number of edges joining any two vertices of G_x is equal to the number of edges joining the corresponding vertices of $G2$.

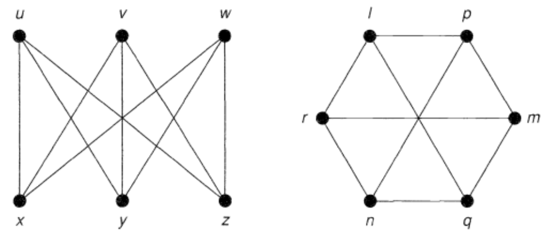


Figure 1: Two simple graphs that are isomorphic to each other

- A graph is **connected** if it cannot be expressed as the union of two graphs, and **disconnected**.

- Two vertices v and w of a graph G are **adjacent** if there is an edge vw joining them, and the vertices v and w are then **incident** with such an edge. Similarly, two distinct edges e and/are adjacent if they have a vertex in common.

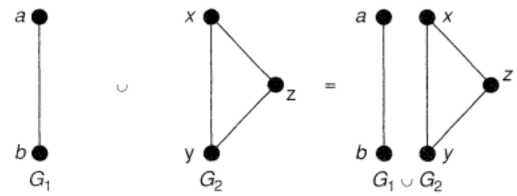


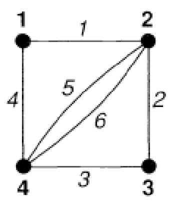
Figure 2: Each of G_1 and G_2 is a component of $G_1 \cup G_2$

- The **degree** of a vertex v of G is the number of edges incident with v, and is written $deg(v)$. *Loop*-edge increase node-degree by 2. vertex of degree 0 is an **isolated vertex** and a vertex of degree 1 is an **end-vertex**.

- **Handshaking lemma**; in any graph the sum of all the vertex-degrees is an even number - in fact, twice the number of edges, since each edge contributes exactly 2 to the sum.

- A **subgraph** of a graph G is a graph, each of whose vertices belongs to $V(G)$ and each of whose edges belongs to $E(G)$.

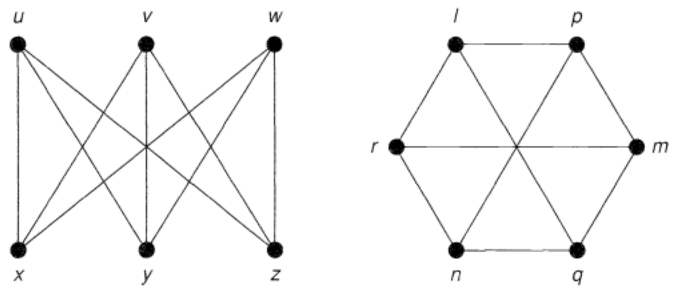
- **Matrix representation** one way to represent graph is by its **adjacency matrix** A, and its **incidence matrix** M as follows;



$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{pmatrix} \qquad \mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Exercise 2

2a Write down the vertex-set and edge-set of each graph in Fig 2.5



The first graph G_1 is $(V(G_1), E(G_1))$

$$\begin{aligned} V(G_1) &= \{x, y, z, u, v, w\} \\ E(G_1) &= \{\{x, u\}, \{x, v\}, \{x, w\}, \\ &\quad \{y, u\}, \{y, v\}, \{y, w\}, \\ &\quad \{z, u\}, \{z, v\}, \{z, w\}\} \end{aligned}$$

The second graph G_2 is $(V(G_2), E(G_2))$

$$\begin{aligned} V(G_2) &= \{n, m, q, r, l, p\} \\ E(G_2) &= \{\{n, r\}, \{n, p\}, \{n, q\}, \\ &\quad \{m, r\}, \{m, p\}, \{m, q\}, \\ &\quad \{l, r\}, \{l, p\}, \{l, q\}\} \end{aligned}$$

2b Draw;

(i) a simple graph.

(ii) a non-simple graph with no loops.

(iii) a non-simple graph with no multiple edges, each having 5 vertices each having 5 vertices and 8 edges.

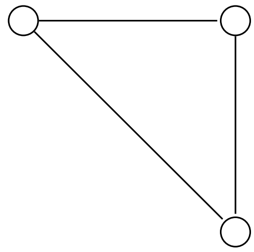


Figure 3: (i) a simple graph

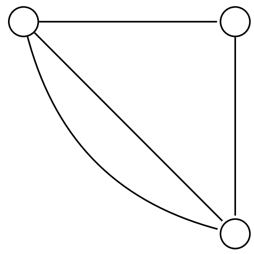


Figure 4: (ii) a non-simple graph with no loops

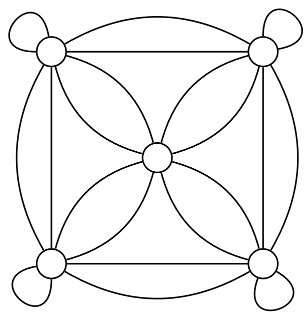


Figure 5: (iii) a 5-vertices and 8-degress each

2c Draw;

(i) Draw a graph on six vertices whose degrees are 5,5,5,5,3,3; does there exist a simple graph with these degrees?

(ii) How does the answer to part (i) changed if the degrees are 5, 5, 4, 3, 3,2?

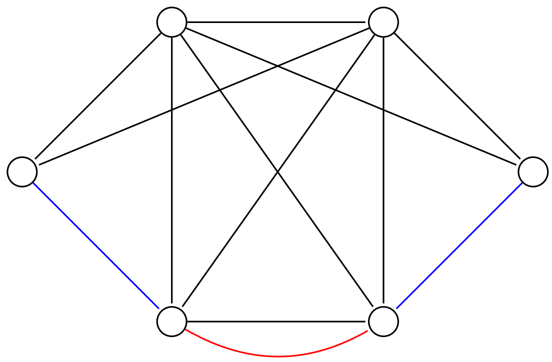


Figure 6: (i) a non-simple graph with with 6-vertices with degrees [5,5,5,5,3,3]

There isn't a simple graph with last mentioned degrees for a 6-vertices graph. But it we just remove the red-arc and one of the blue-arcs, we then get a simple graph as shown in the next figure.

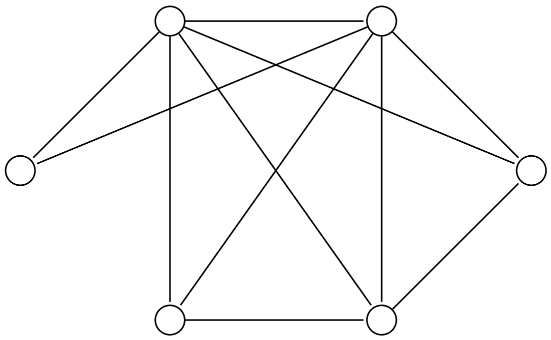
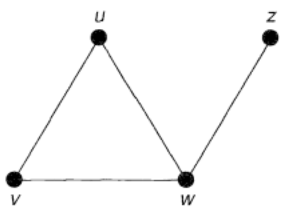


Figure 7: (i) a simple graph with with 6-vertices with degrees [5,5,4,3,3,2]

(2d) Verify that handshaking lemma is hold for figure 2.1

As we can see in the last figure, the sum



of all vertex-degrees is $2 + 2 + 3 + 1 = 8$ which is an even number.

(2f) (i) By suitably lettering the vertices, show that the two graphs in Fig. 2.20 are isomorphic.

(ii) Explain why two graphs in Fig. 2.21 are not isomorphic.

(i) As shown in the following figure 8 the two graphs are labeled with the same letters in a way to emphasize that they are both isomorphic to each other.
(ii) As shown in the following figure 9, we cannot find the red part of the first graph as a subset in the second graph.

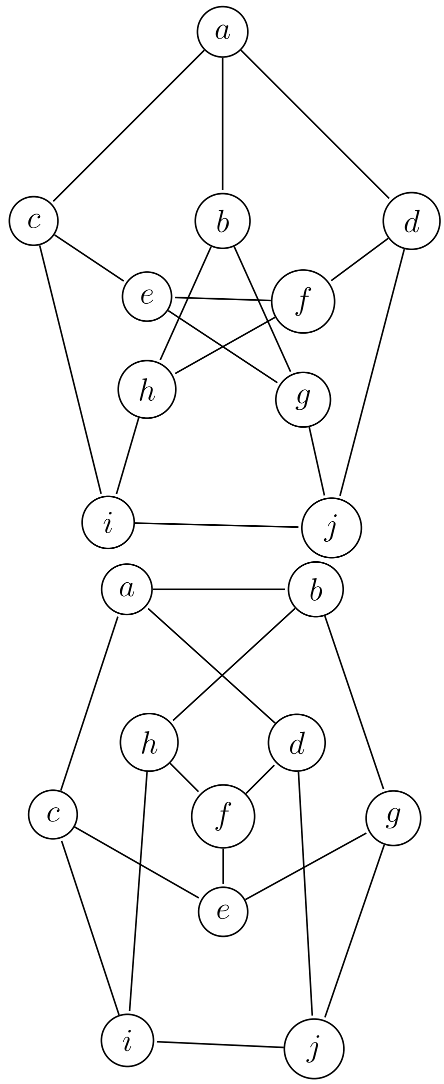


Figure 8: (i)

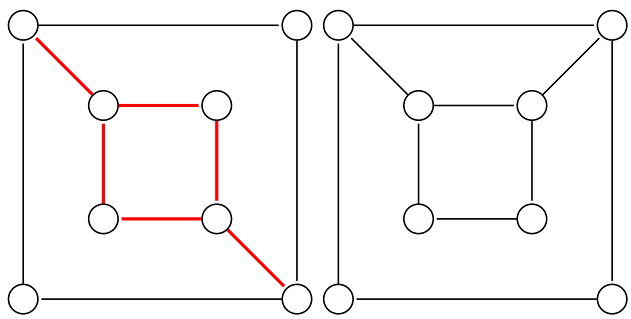


Figure 9: (ii) not isomorphic graphs

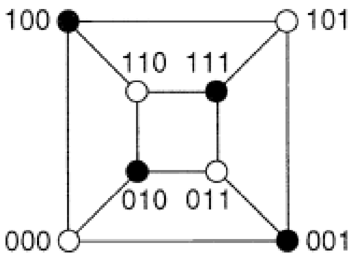
Section (3)

- **Null graphs**, a graph whose edge-set is empty. **Complete graph**, a simple graph in which each pair of distinct vertices are adjacent, in this case k-vertex must has $n(n-1)/2$ degree as n the total number of vertices.

- **Regular graph**, a graph in which each vertex has the same degree. **Platonic graphs**, formed from the vertices and edges of the five regular (Platonic) solids - the tetrahedron, octahedron, cube.

- **Bipartite graph**, If the vertex set of a graph G can be split into two disjoint sets A and B so that each edge of G joins a vertex of A and a vertex of B. A **complete bipartite graph** is a bipartite graph in which each vertex in A is joined to each vertex in B by just one edge.

- **Cubes**, the k-cube Q_k is the graph whose vertices correspond to the sequences $(a_1, a_2, \cdots, a_k)$, where each $a_i = 0$ or 1, and whose edges join those sequences that differ in just one place. You should check that Q_k has 2^k vertices and $k2^{k-1}$ edges, and is regular of degree k.



- **Complement of a simple graph**, if G is a simple graph with vertex set $V(G)$, its complement $\overline{G}$ is the simple graph with vertex set $V(G)$ in which two vertices are adjacent if and only if they are not adjacent in G.

Exercise 3

(3a) Draw the following graphs:

(i) the null graph N_5.

(ii) the complete graph K_6.

(iii) the complete bipartite graph $K_{2,4}$.

(iv) the union of $K_{1,3}$ and W_4.

(v) the complement of the cycle graph C_4.

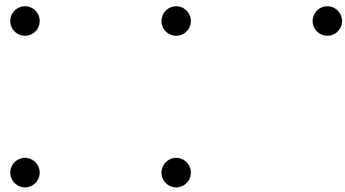


Figure 10: (i) Null graph N_5

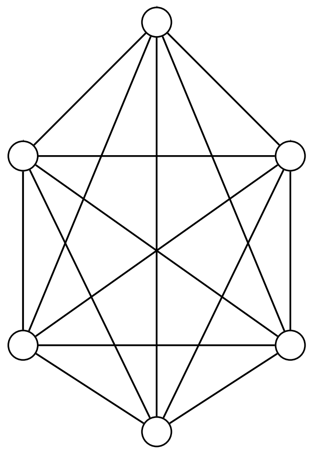


Figure 11: (ii) Complete graph K_6

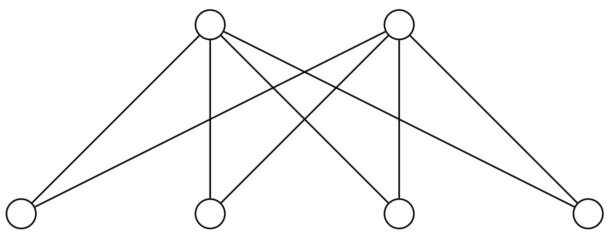


Figure 12: (iii) Complete bipartite graph $K_{2,4}$

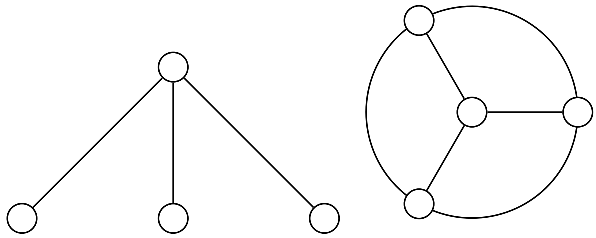


Figure 13: (iv) Union of $K_{1,3}$ and W_4

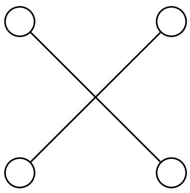


Figure 14: (iv) Complement of cycle graph C_4

(3c) Draw the graphs $K_{2,2,2}$, and $K_{3,3,2}$, and write down the number of edges of $K_{3,4,5}$.

The graphs $K_{2,2,2}$ and $K_{3,3,2}$ are shown in the following figures, respectively.
For the graph $K_{3,4,5}$, there is 47 edges.

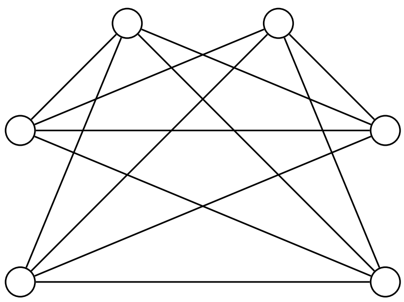


Figure 15: $K_{2,2,2}$

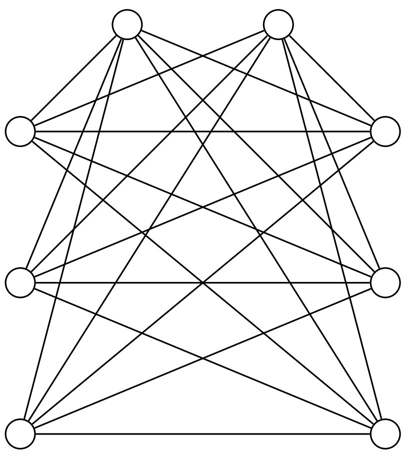


Figure 16: $K_{3,3,2}$

(3g) A simple graph that is isomorphic to its complement is self-complementary.

(i) Prove that, if G is self-complementary, then G has $4k$ or $4k+1$ vertices, where k is an integer,

(ii) Find all self-complementary graphs with 4 and 5 vertices,

(iii) Find a self-complementary graph with 8 vertices.

(i) Proof

If G is a self-complementary with n vertices, and

$$G \cup \overline{G} = K_n.$$

But we know that, the total number of edge in the complete graph K_n i.e. $|E(K_n)|$is $n(n-1)/2$, that is,

$$|E(G)| = |E(\overline{G})| = \frac{n(n-1)}{4}.$$

In other words, n or $n-1$ must be divisible by 4, that is, when n is $4k$ or $4k+1$.

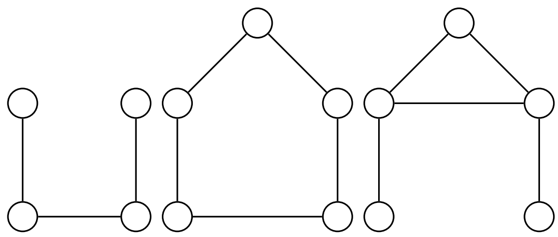


Figure 17: (ii) 4 and 5 vertices self-complementary graphs

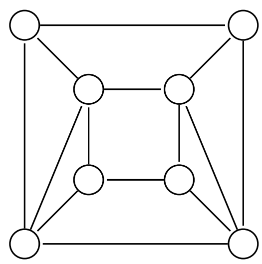


Figure 18: (iii) a self-complementary graph with 8 vertices