Abstraction of Graph Theory

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2: Definitions

- A *simple graph* G consists of a non-empty finite set $V(G)$ of elements called vertices (or nodes), and a finite set $E(G)$ of distinct unordered pairs of distinct elements of $V(G)$ called edges.

- A *graph* G consists of a non-empty finite set $V(G)$ of elements called vertices, and a finite family $E(G)$ of unordered pairs of (not necessarily distinct) elements of $V(G)$ called edges.

- Two graphs $G1$ and $G2$ are *isomorphic* if there is a one-one correspondence between the vertices of G_x and those of $G2$ such that the number of edges joining any two vertices of G_x is equal to the number of edges joining the corresponding vertices of $G2$.

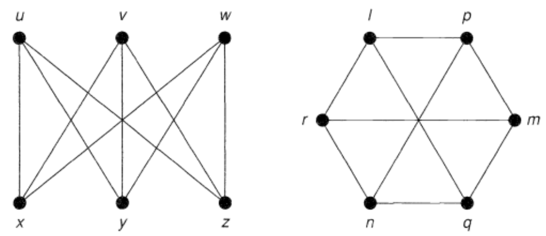


Figure 1: Two simple graphs that are isomorphic to each other

- A graph is *connected* if it cannot be expressed as the union of two graphs, and *disconnected*.

- Two vertices v and w of a graph G are *adjacent* if there is an edge vw joining them, and the vertices v and w are then *incident* with such an edge. Similarly, two distinct edges e and f are adjacent if they have a vertex in common.

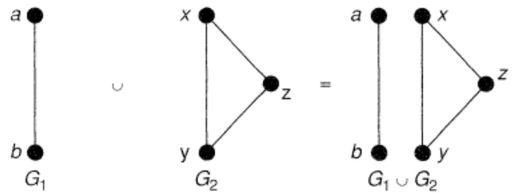


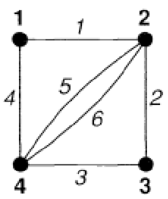
Figure 2: Each of G_1 and G_2 is a component of $G_1 \cup G_2$

- The *degree* of a vertex v of G is the number of edges incident with v, and is written $deg(v)$. *Loop*-edge increase node-degree by 2. vertex of degree 0 is an *isolated vertex* and a vertex of degree 1 is an *end-vertex*.

- *Handshaking lemma*; in any graph the sum of all the vertex-degrees is an even number - in fact, twice the number of edges, since each edge contributes exactly 2 to the sum.

- A *subgraph* of a graph G is a graph, each of whose vertices belongs to $V(G)$ and each of whose edges belongs to $E(G)$.

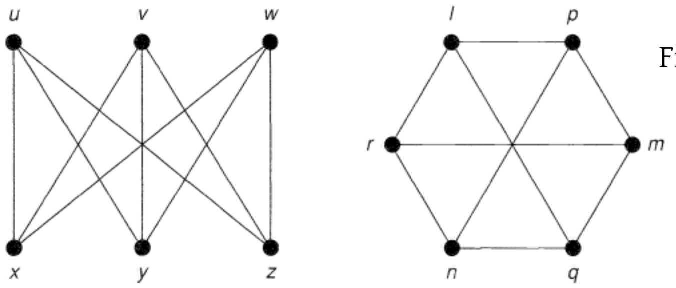
- *Matrix representation* one way to represent graph is by its adjacency matrix A, and its incidence matrix M as follows;



$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{pmatrix} \qquad \mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Exercise 2

2a Write down the vertex-set and edge-set of each graph in Fig 2.5



The first graph G_1 is $(V(G_1), E(G_1))$

$$\begin{aligned} V(G_1) &= \{x, y, z, u, v, w\} \\ E(G_1) &= \{\{x, u\}, \{x, v\}, \{x, w\}, \\ &\quad \{y, u\}, \{y, v\}, \{y, w\}, \\ &\quad \{z, u\}, \{z, v\}, \{z, w\}\} \end{aligned}$$

The second graph G_2 is $(V(G_2), E(G_2))$

$$\begin{aligned} V(G_2) &= \{n, m, q, r, l, p\} \\ E(G_2) &= \{\{n, r\}, \{n, p\}, \{n, q\}, \\ &\quad \{m, r\}, \{m, p\}, \{m, q\}, \\ &\quad \{l, r\}, \{l, p\}, \{l, q\}\} \end{aligned}$$

2b Draw;

(i) a simple graph.

(ii) a non-simple graph with no loops.

(iii) a non-simple graph with no multiple edges, each having 5 vertices each having 5 vertices and 8 edges.

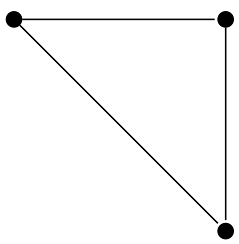


Figure 3: (i) a simple graph

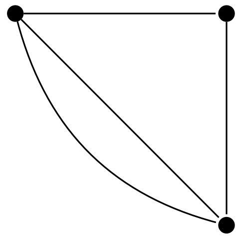


Figure 4: (ii) a non-simple graph with no loops

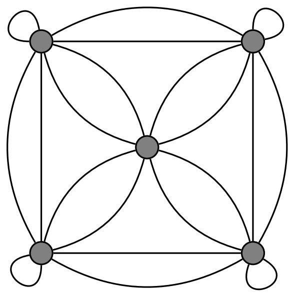


Figure 5: (iii) a 5-vertices and 8-degress each

2c Draw;

(i) Draw a graph on six vertices whose degrees are 5,5,5,5,3,3; does there exist a simple graph with these degrees?

(ii) How does the answer to part (i) changed if the degrees are 5, 5, 4, 3, 3,2?

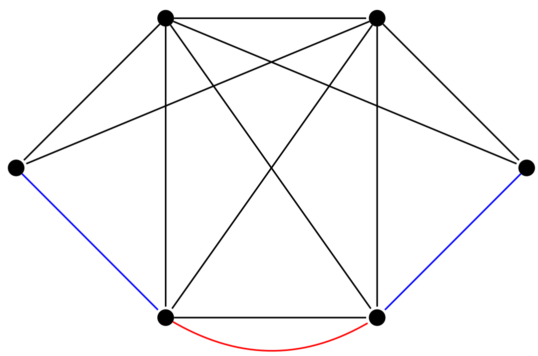


Figure 6: (i) a non-simple graph with with 6-vertices with degrees [5,5,5,5,3,3]

There isn't a simple graph with last mentioned degrees for a 6-vertices graph. But it we just remove the red-arc and one of the blue-arcs, we then get a simple graph as shown in the next figure.

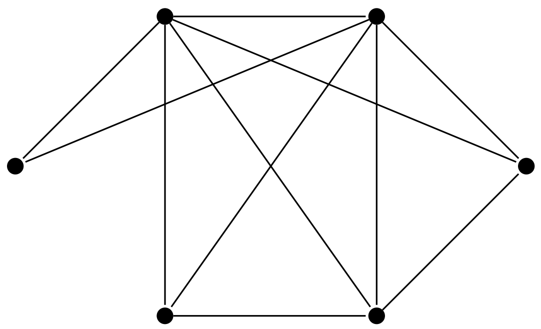
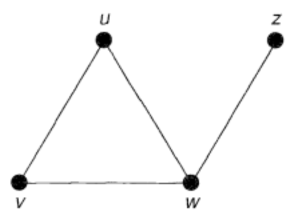


Figure 7: (i) a simple graph with with 6-vertices with degrees [5,5,4,3,3,2]

(2d) Verify that handshaking lemma is hold for figure 2.1

As we can see in the last figure, the sum



of all vertex-degrees is $2 + 2 + 3 + 1 = 8$ which is an even number.

(2f) (i) By suitably lettering the vertices, show that the two graphs in Fig. 2.20 are isomorphic.

(ii) Explain why two graphs in Fig. 2.21 are not isomorphic.

(i) As shown in the following figure 8 the two graphs are labeled with the same letters in a way to emphasize that they are both isomorphic to each other.
(ii) As shown in the following figure 9, we cannot find the red part of the first graph as a subset in the second graph.

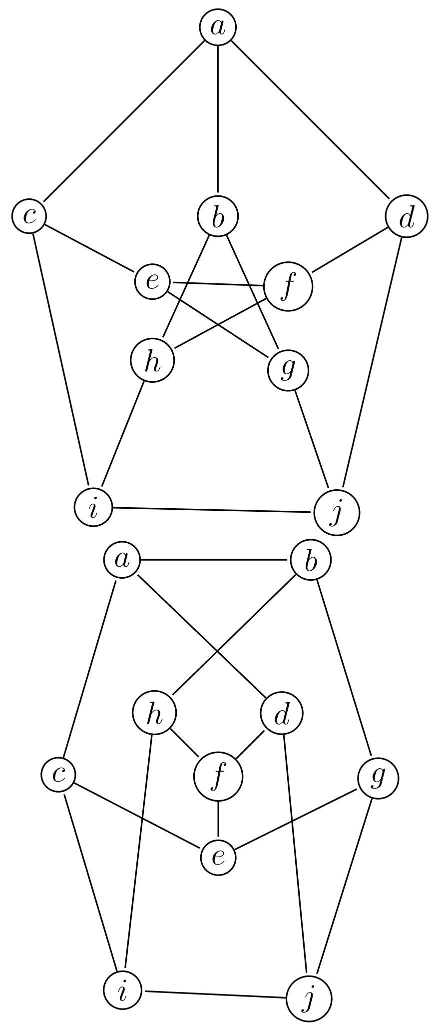


Figure 8: (i)

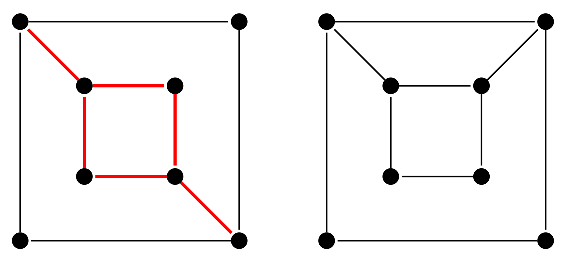


Figure 9: (ii) not isomorphic graphs

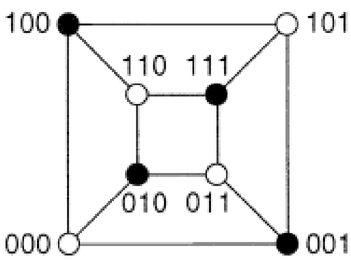
3: Examples of graphs

- *Null graphs*, a graph whose edge-set is empty. *Complete graph*, a simple graph in which each pair of distinct vertices are adjacent, in this case k-vertex must has $n(n-1)/2$ edges as n the total number of vertices.

- *Regular graph*, a graph in which each vertex has the same degree. Platonic graphs, formed from the vertices and edges of the five regular (Platonic) solids - the tetrahedron, octahedron, cube.

- *Bipartite graph*, If the vertex set of a graph G can be split into two disjoint sets A and B so that each edge of G joins a vertex of A and a vertex of B. A complete bipartite graph is a bipartite graph in which each vertex in A is joined to each vertex in B by just one edge.

- *Cubes*, the k-cube Q_k is the graph whose vertices correspond to the sequences $(a_1, a_2, \cdots, a_k)$, where each $a_i = 0$ or 1, and whose edges join those sequences that differ in just one place. You should check that Q_k has 2^k vertices and $k2^{k-1}$ edges, and is regular of degree k.



- *Complement of a simple graph*, if G is a simple graph with vertex set $V(G)$, its complement $\overline{G}$ is the simple graph with vertex set $V(G)$ in which two vertices are adjacent if and only if they are not adjacent in G.

Exercise 3

(3a) Draw the following graphs:

(i) the null graph N_5.
(ii) the complete graph K_6.
(iii) the complete bipartite graph $K_{2,4}$.
(iv) the union of $K_{1,3}$ and W_4.
(v) the complement of the cycle graph C_4.

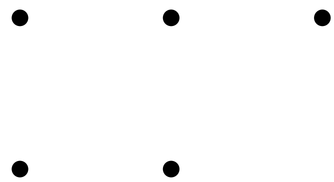


Figure 10: (i) Null graph N_5

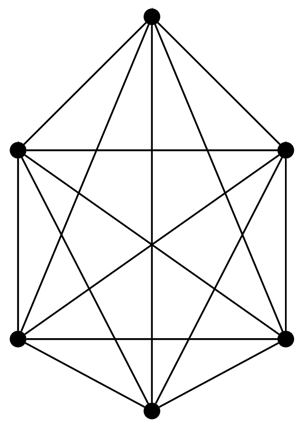


Figure 11: (ii) Complete graph K_6

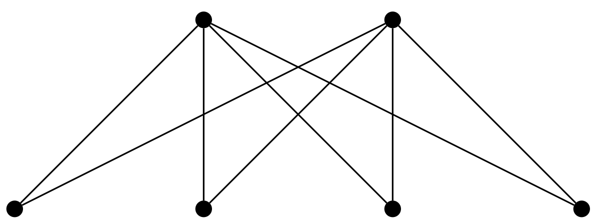


Figure 12: (iii) Complete bipartite graph $K_{2,4}$

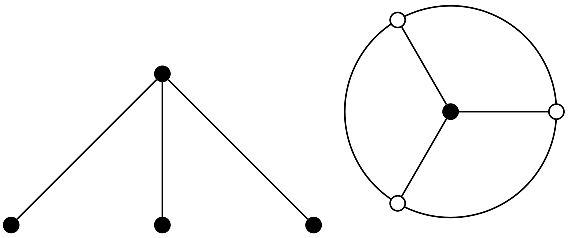


Figure 13: (iv) Union of $K_{1,3}$ and W_4

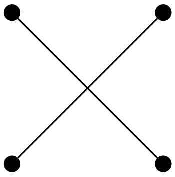


Figure 14: (iv) Complement of cycle graph C_4

(3c) Draw the graphs $K_{2,2,2}$, and $K_{3,3,2}$, and write down the number of edges of $K_{3,4,5}$.

The graphs $K_{2,2,2}$ and $K_{3,3,2}$ are shown in the following figures, respectively.
For the graph $K_{3,4,5}$, there is 47 edges.

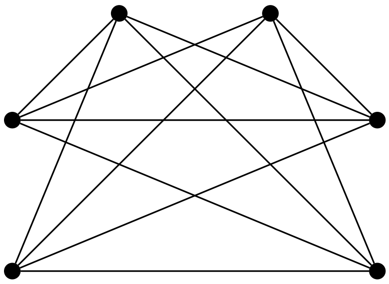


Figure 15: $K_{2,2,2}$

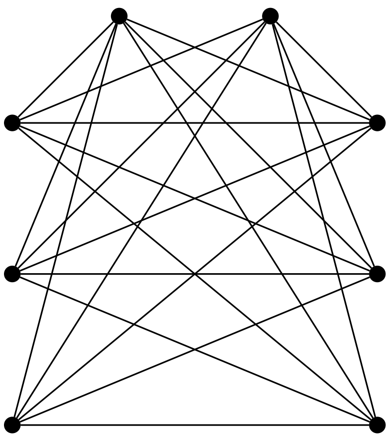


Figure 16: $K_{3,3,2}$

(3g) A simple graph that is isomorphic to its complement is self-complementary.

(i) Prove that, if G is self-complementary, then G has $4k$ or $4k+1$ vertices, where k is an integer,

(ii) Find all self-complementary graphs with 4 and 5 vertices,

(iii) Find a self-complementary graph with 8 vertices.

(i) <u>Proof</u>
If G is a self-complementary with n vertices, and

$$G \cup \overline{G} = K_n.$$

But we know that, the total number of edge in the complete graph K_n i.e. $|E(K_n)|$is $n(n-1)/2$, that is,

$$|E(G)| = |E(\overline{G})| = \frac{n(n-1)}{4}.$$

In other words, n or $n-1$ must be divisible by 4, that is, when n is $4k$ or $4k+1$.

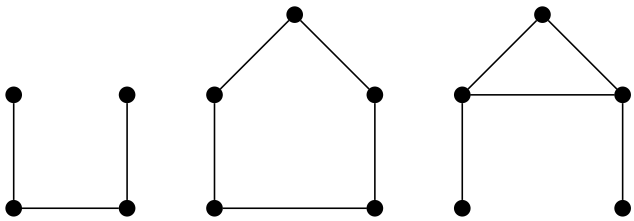


Figure 17: (ii) 4 and 5 vertices self-complementary graphs

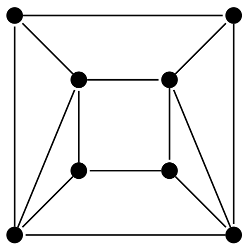


Figure 18: (iii) a self-complementary graph with 8 vertices

4: Embeddings of graphs

- *Jordan curve* is a continuous curve which doesn't intersect itself.
- *Graph embedding*: A graph G can be embedded (or has an embedding) in a given space if it is isomorphic to a graph drawn in the space with points representing vertices of G and Jordan curves representing edges in such a way that there are no crossings.
- *Theorem 4A*: *Every graph can be embedded in Euclidean 3-space*.
- *Planer graph*: a graph that can be embedded in a plane.
- *Theorem 4B*:*A graph is planer if and only if it can be embedded on the surface of a sphere*.

5: More definitions

- A *walk* in G is a finite sequence of edges of the form $v_0v_1, v_1v_2, \cdots, v_{m-1}v_m$, where v_0 is the initial vertex and v_m is the finial vertex of the walk, also the number of edges is called length.
- *Trial* is a walk in which all the edges are distinct. *path* is a trial with all vertices are distinct also (expect, possibly $v_0 = v_m$ where then we call the trial or the path *closed*). A closed path containing at least one edge is a *cycle*[1].
- A graph is *connected* if and only if there is a path between each pair of vertices.
- *Theorem 5.1* *If G is a bipartite graph, then each of G has even length*.
- *Theorem 5.2* *Let G be a simple graph on n vertices. If G has k components, then the number m of edges of G satisfies*

$$n - k \leq m \leq (n-k)(n-k+1)/2. \quad (1)$$

- *Corollary 5.3* *Any simple graph with n vertices and more than $(n-1)(n-2)/2$ edges is connected.*
- A *disconnecting set* in a connected graph G is a set of edges whose removal disconnects G and increases the number of components of G.
- A *cutset* is defined to be a disconnecting set, no proper subset of which is a disconnecting set. If a cutset has only one edge e, we call e a *bridge*.
- If G is connected, its *edge connectivity* $\lambda(G)$ is the size of the smallest cutset in G. Thus $\lambda(G)$ is the minimum number of edges that we need to delete in order to disconnect G.
- If G is connected and not a complete graph, its *vertex connectivity* $\mathcal{K}(G)$ is the size of the smallest separating set in G. Thus $\mathcal{K}(G)$ is the minimum number of vertices that we need to delete in order to disconnect G.

[1] called *circuit* in 3rd edition of the textbook

Exercise 5

(5a) In the Petersen graph, find

- (i) a trail of length 5;
- (ii) a path of length 9;
- (iii) cycles of lengths 5, 6, 8 and 9;
- (iv) cutsets with 3, 4 and 5 edges.

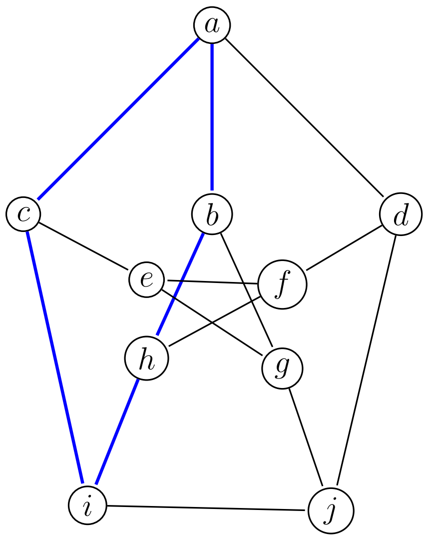


Figure 19: (i) a Petersen graph with trail of length 5 highlighted $\{ab, bh, hi, ic, ca\}$

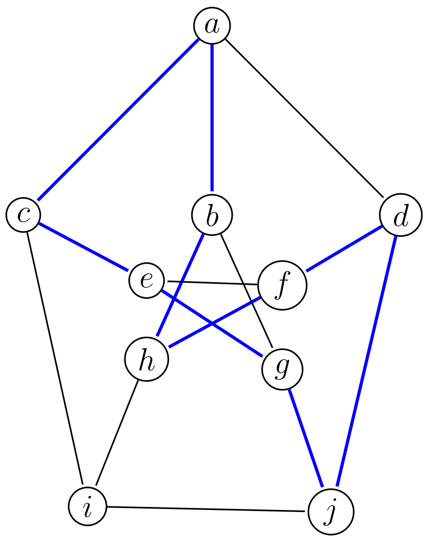


Figure 20: (ii) a Petersen graph with path of length 9 highlighted $\{ab, bh, hf, fd, dj, ig, ge, ec, ca\}$

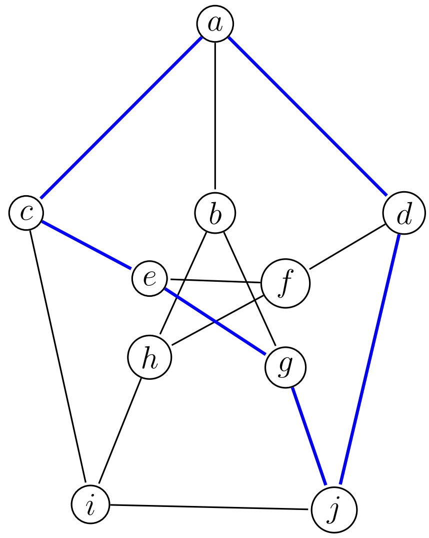


Figure 21: While (i) also represents a cycle of 5 length, this represents a cycle of 6 length highlighted $\{ac, ce, eg, gj, jd, da\}$

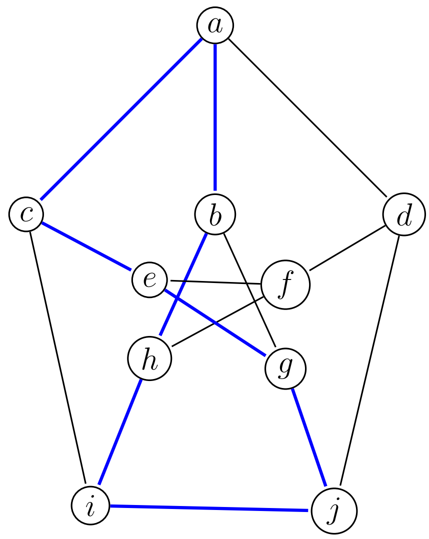


Figure 22: A cycle of 8 length highlighted $\{ac, ce, eg, gj, ji, ih, hb, ba\}$

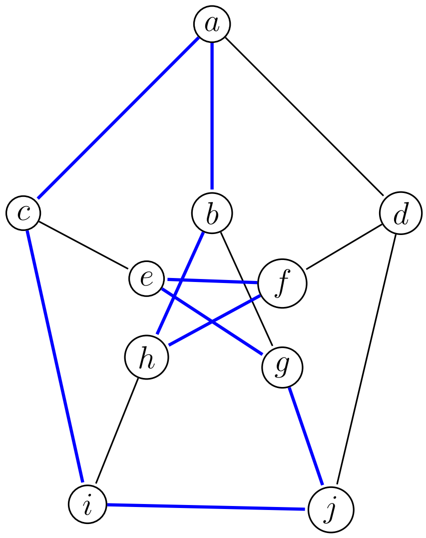


Figure 23: A cycle of 9 length highlighted $\{ac, ci, ij, jg, ge, ef, fh, hb, ba\}$

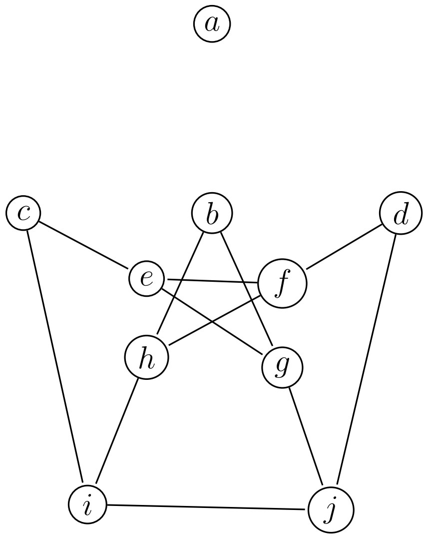


Figure 24: A cutset of 3 edges $\{ac, ab, ac\}$

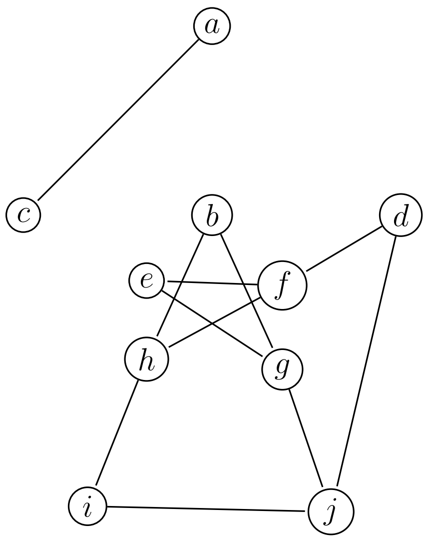


Figure 25: A cutset of 4 edges $\{ab, ac, ce, ci\}$

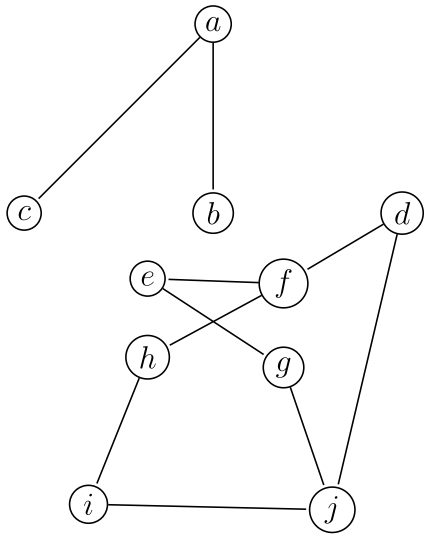


Figure 26: A cutset of 5 edges $\{ad, bh, bg, ce, ci\}$

(5c) Find $\mathcal{K}(G)$ and $\lambda(G)$ for each of the following graphs (i) C_6, (ii) W_6, (iii) $K_{4,7}$, (iv) Q_4

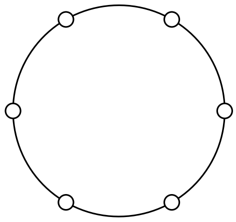


Figure 27: (i) $\mathcal{K}(C_6) = 2$ and $\lambda(C_6) = 2$

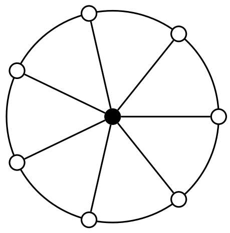


Figure 28: (ii) $\mathcal{K}(W_8) = 4$ and $\lambda(W_8) = 3$

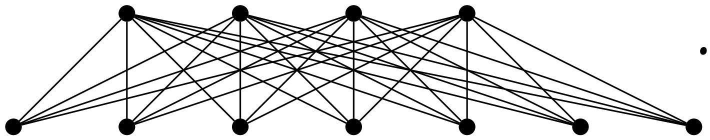


Figure 29: (iii) $\mathcal{K}(K_{4,7}) = 4$ and $\lambda(W_8) = 4$

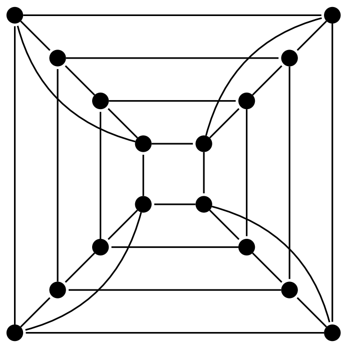


Figure 30: (iv) $\mathcal{K}(Q_4) = 3$ and $\lambda(Q_4) = 8$

(5f) Prove that if G is a simple graph, then G and $\overline{G}$ cannot both be disconnected

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To prove this we need to prove that the complement of a disconnected graph G is connected.

That is without loss of generality, assume G is disconnected. Now consider two vertices x and y in $\overline{G}$. If x and y are not adjacent in G, then they will be adjacent in $\overline{G}$ and we can find a trivial $x - y$ path. If x and y are adjacent in G then they must have been in the same component. This means that the edges xz and yz were not in G. This implies that they both must be edges in $\overline{G}$. This gives us the path $x \to y \to z$. Therefore, in $\overline{G}$ we have that there exists a path between any two vertices and hence it is connected.

6: Eulerian graphs

- A connected graph G is Eulerian if there exists a closed trial containing every edge of G. Such trial is an Eulerian trial. A non-Eulerian graph is semi-Eulerian if there exists a trail containing every edge of G.
- Lemma 6a: *If G is a graph in which the degree of each vertex is at least 2, then G contains a cycle.*
- Theorem 6b: *A connected graph G is Eulerian if and only if the degree of each vertex of G is even.*
- Corollary 6c: *A connected graph is Eulerian if and only its set of edges can be split up into disjoint cycles.*
- Corollary 6d: *A connected graph is semi-Eulerian if and only if it has eactly two vertices of odd degree.*
- Theorem 6e: *Let G be an Eulerian graph. Then the following construction is always possible, and produces an Eulerian trial of G.*
  Start at any vertex u and traverse the edges in an arbitrary manner, subject only to the following rules:
  - *(i) erase the edges as they traversed, and if any isolated vertices result, erase them too;*
  - *(ii) at each stage, use a bridge only if there is no alternative.*

Exercise 6

(6a) Which of the following graphs are Eulerian? semi-Eulerian?

- the complete graph K_5. Ans. *"Eulerian", since all nodes has even degree.*
- the complete bipartite graph $k_{4,3}$.
  Ans. *"Semi-Eulerian"*
- the graph of cube.
  Ans. *"Semi-Eulerian"*
- the graph of octahedron.
  Ans. *"Eulerian"*
- the Petersen graph.
  Ans. *"Semi-Eulerian"*

(6b)

- For which values of n is K_n Eulerian?
  Ans. *For n is odd, that always gives an even degree complete graph.*
- Which complete bipartite graphs are Eulerian?
  Ans. $\{K_{n,m} | n, m \text{ are even}\}$
- Which Platonic graphs are Eulerian?

Graph	*Hamiltonian*	*Eulerian*
tetrahedral	yes	no
octahedral	yes	yes
icosahedral	yes	no
cubical	yes	no
dodecahedral	yes	no

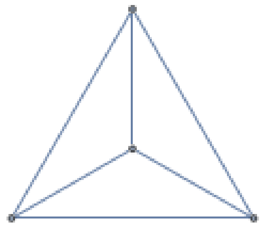
The next column shows how Mathemtica software helps with drawing all graphs mentioned in the last table.

- For which values of n is the wheel W_n Eulerian?
  Ans. *W_n is never be Eulerian.*
- For which values of k the $k-$cube Q_n Eulerian?
  Ans. $\{k | k \text{ is even}\}$

Platonic Graphs

```
GraphData["TetrahedralGraph"]
```

Out[1]=



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In[2]:= GraphData["OctahedralGraph"]
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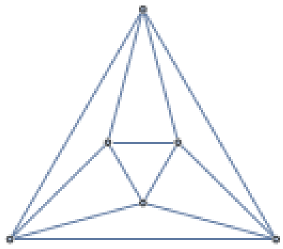
Out[2]=



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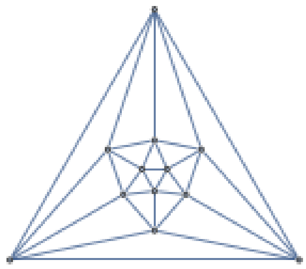
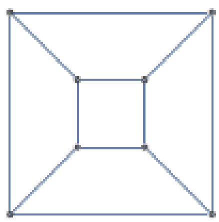
In[3]:= GraphData["IcosahedralGraph"]
```

Out[3]=



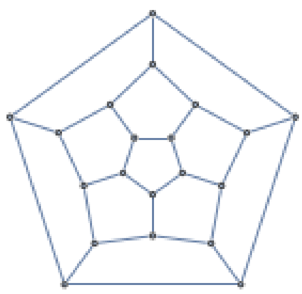
```
In[4]:= GraphData["CubicalGraph"]
```

Out[4]=



```
In[5]:= GraphData["DodecahedralGraph"]
```

Out[5]=



(6c) Let G a connected graph with $k(> 0)$ vertices of odd degree. Show that the minimum number of trials, which have no edges in common add which together include every edge of G, is $\frac{1}{2}k$.

7: Hamiltonian graphs

- a closed trail passing exactly once through each vertex of G. Such a trail must be a cycle, except when G is the graph N_1. Such a cycle is a Hamiltonian cycle and G is a Hamiltonian graph. A non-Hamiltonian graph G is semi-Hamiltonian if there exists a path passing through every verte$x$.
- Theorem 7A *If G is a simple graph with $n(\geq 3)$ vertices, and if $\rho(\nu)+\rho(w)\geq n$ for each of non-adjacent vertices ν and w, then G is Hamiltonian.* [2]
- Dirac 1952 *If G is a simple graph with $n(\geq 3)$ vertices, and if $\rho(\nu)\geq \frac{1}{2}n$ for every vertex ν then G is Hamiltonian.*

Exercise 7

(7a) Which of the following graphs are Hamilitonain? semi-Hamiltonian?

i the complete graph K_5.
Ans. "*Hamiltonian*" as $\rho(\nu)+\rho(w)=8>|V(K_5)|\ \forall\nu,w\in V(K_5)$

ii the complete bipartite graph $K_{5,3}$.
Ans. "*Semi-Hamiltonian*" as $\rho(\nu)+\rho(w)<|V(K_{5,3})|\ \forall\nu\in\{x_i\}\&w\in\{y_i\}$ as shown in the figure 31, but $\exists$ a trial that goes through all vertices.

iii the graph of the octahedron.
Ans. "*Hamiltonian*", in fact all Platonic graphs are Hamiltonian

iv the wheel W_6.
Ans. "*Hamiltonian*, as we can easily get a path that passing through all vertices exactly once."

[2] ρ is referred as "deg" in 4th edition

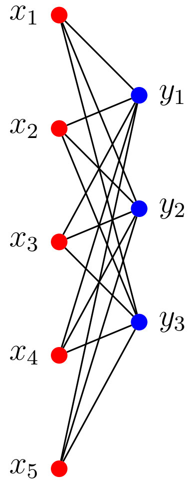


Figure 31: $K_{5,3}$

v the 4-cube Q_4.
Ans. "*Hamiltonian*, as shown in figure 30, we can see that $|V(Q_4)|=16$, while $\rho(\nu)=4,\forall\nu\in V(Q_4)$.

(7b) i For which values of n is K_n Hamiltonian?
Ans. If $n=2$.

ii Which complete bipartite graphs are Hamiltonian?
Ans. All $K_{n,n}$.

iii Which Platonic graphs are Hamiltonian?
Ans. All Platonic graphs

iv For which values of n is the wheel W_n Hamiltonian?
Ans. All Wheels.

v For which values of k if the $k-$cube Q_n Hamiltonian?

Some applications

- The short path problem.
- The Chinese postman problem.
- The travelling salesman problem.

9: Elementary properties of trees

- A forest is defined to be a graph which contains no cycles and a connected forest is called a tree.

- Theorem 9A *Let T be a graph with n vertices. Then the following statements are equivalent:*

  (i) T is a tree.

  (ii) T contains no cycles, and has $n-1$ edges.

  (iii) T is connected, and has $n-1$ edges.

  (iv) T is connected, and every edge is a bridge.

  (v) any two vertices of T are connected by exactly one path.

  (vi) T contains no cycles, but the addition of any new edge creates exactly one cycle.

- Corollary 9B *If G be a forest with n vertices and k components, then G has $n-k$ edges.*

- Given any connected graph G, we can choose a cycle and remove any one of its edges, and the resulting graph remains connected. If we repeat this procedure until there are no cycles left. The graph that remains is a tree that connects all the vertices of G. It is called a *spanning tree* of G. Generally, if G is an arbitrary graph with n vertices, m edges and k components, then we can carry out this procedure on each component of G. The result is called a *spanning forest*, and the total number of edges removed in this process is the *cycle rank* of G, denoted by $\gamma(G) = m - n + k$. the cutset rank of G to be the number of edges in a spanning forest, denoted by $\xi(G) = n - k$

- Theorem 9C *If T is any spanning forest of a graph G, then*

  (i) every cutset of G has an edge in common with T.

  (ii) every cutset of G has an edge in common with the complement of T.

- For T and G in last theorem, if we add to T any edge of G not contained in T, then by statment (vi) in theorem (9A) we get a unique cycle. The set of all cycles formed in this way is called *fundamental set of cycles* associated with T. Note, if T is spanning tree of G, then the *fundamental set of cycles* of T is equal to *cycle rank* of G.

Exercise 9

- Prove that every tree is a bipartie graph.
  Let's first investigate the following example: with the aid of the example demonstrated in

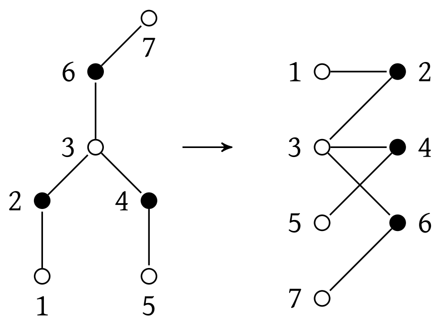


Figure 32: Example, shows how a tree can be treated as a bipartite graph

the last figure, we can formulate this property,

$$\begin{aligned} \forall u, v, w &\in V(G), \\ \text{if } \exists vw, uv &\in E(G), \text{then } vw \notin E(G), \end{aligned}$$

vthat is, for any tree, we can construct two sets/components x, y where there are only edges between vertices from x to vertices from y. (definition of bipartite graph)

- Which trees are complete bipartite graphs?
  It is easily to show that $K_{1,n}$, where $n \geq 2$ is a tree and a complete bipartite graph in the same time as shown in the next figure, but $K_{2,2}$ does not, as it has a cycle between its edges, as shown in the same figure,

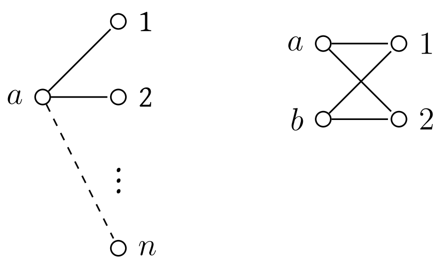
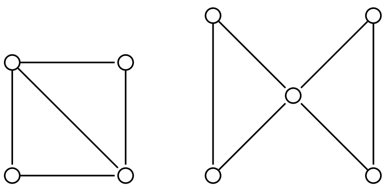


Figure 33: $K_{1,n}, n \geq 2$ is a tree, but not $K_{m,n}, m, n \geq 2$

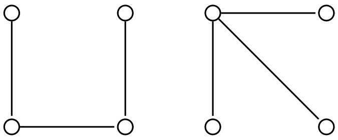
and in fact, we can embed $K_{2,2}$ in any $K_{m,n}, m, n \geq 2$. That is, our conculsion in the last figure caption.

- Draw all the spanning trees in the graph of the following figure.

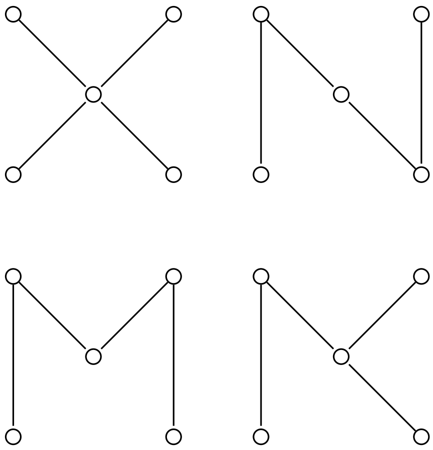


(i) (ii)

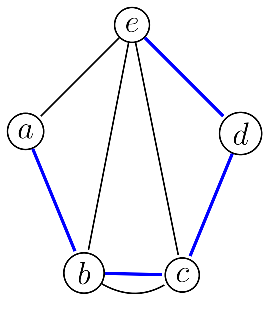
Ans. of (i)



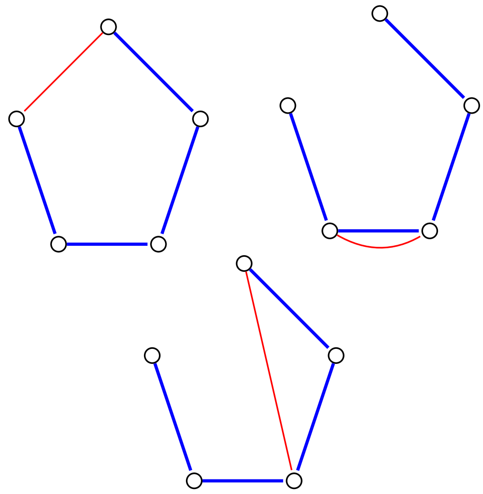
Ans. of (ii)



- Find the fundamental sets of cycles and cut-sets of the graph if the following figure, associated with the spanning tree shown



Ans.



10: The enumeration of trees

- *Labelling* of a graph G on n vertices to be one-one mapping from the vertex-set of G onto the set $1, \cdots, n$, a *labelled graph* is then a pair (G, ϕ), where ϕ is a labelling of G.
- *Theorem 10A* (Cayley) *There are n^{n-2} distinct labelled trees on n vertices*.
- *Corrollary 10B* *The number of spanning trees of K_n is n^{n-2}.*
- *Theorem 10C* *Let G be a connected simple graph with vertex set $\{v_1, \cdots, v_n\}$, and let $\mathrm{M} = (m_{ij})$ be $n \times n$ matrix in which $m_{ii} = \rho(v_i)$, $m_{ij} = -1$ if v_i and v_j are adjacent, and $m_{ij} = 0$ otherwise. Then the number of spanning trees of G is equal to the cofactor of any element of* M.

Exercise 10

- Verify directly that there are exactly 125 labelled trees on five vertices. Ans. for $n = 5$ vertices, there exists $5^{5-2} = 125$ labelled trees.
- (i) Show that there are exactly $2^{n(n-1)/2}$ labelled simple graphs on n vertices.

  proof 1:

  For an n-vertices simple graph, we have $\binom{n}{2}$ of all possible edges, any one of this edges may or may not be in the graph. That is, the total number of collections of edges is $2^{\binom{n}{2}}$. As each graph on n-vertices is uniquely

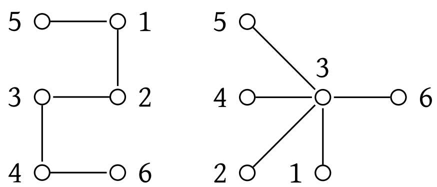
described by which edges are present in the graph, we get $2^{n(n-1)/2}$ different labelled graph.
Proof 2: by induction.

(ii) How many of these have exactly m edges? Ans.

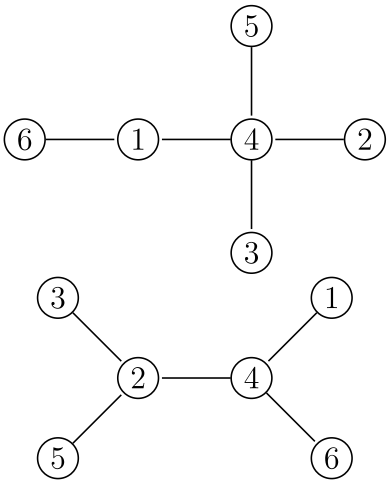
$$\binom{\binom{n}{2}}{m}$$

- In the first proof of Cayley's theorem, find:

  (i) the labelled trees corresponding to the sequences (1,2,3,4) and (3,3,3,3).



  (ii) the sequences corresponding to the labelled tree in the following figure



Ans. $(2,3,5,4)$, $(1,3,5,6)$ respectively.

More Applications

- The minimum connector problem.
- Enumeration of chemical molecules.
- Electrical networks.

12: Planar graphs

- A *plane graph* is a graph drawn in the plane in such a way that no two edges intersect geometrically except at a vertex to which they are both incident. a planar graph is one which is isomorphic to a plane graph.
- *Theorem 12A*: *K_5, and $K_{3,3}$ are non-planer*
- Two graphs are *homomorphic* if they can both be obtained from the same graph by inserting new vertices of degree two into its edges.
- *Theorem 12B*: *A graph is planar iff it contains no subgraph homeomorphic to K_5, and $K_{3,3}$*
- *Theorem 12C*: *A graph is planar iff it contains no subgraph which contractible to K_5, and $K_{3,3}$*

Exercise 12

(12c) Prove that the Petersen graph is non-planar

  (i) by using Theorem 12B,

  (ii) by using Theorem 12C.

(12d) Give an example of

  (i) a non-planar graph which is not homeomorphic to K_5 or $K_{3,3}$,

  (ii) a non-planar graph which is not contractible to K_5 or $K_{3,3}$.

  Why does the existence of these graphs not contradict Theorems 12B and 12C?

13: Euler's formula for plane graphs

- If x is a point of the plane disjoint from G, the *face* of (G) *colormagentacontaining* x is the set of all points of the plane which can be reached from x by a Jordan curve all of whose points are disjoint from G.
- *Theorem 13A (Euler)*: Let G be a connected plane graph, and let n, m and f denote respectively the number of vertices, edges and faces of G, then $n - m + f = 2$.

- Corollary 13B: *let G be a polyhedral graph, then with above notation $n - m + f = 2$.*

- Corollary 13C: *let G be a plane graph with n vertices, m edges, f faces and k components, then $n - m + f = k + 1$.*

- Corollary 13D: *(i) If G is a connected simple planar graph with $n(\geq 3)$ vertices and m edges, then $m \leq 3n - 6$*
(ii) If, in addition, G has no triangles, then $m \leq 2n - 4$.

- Corollary 13E: *K_5 and $K_{3,3}$ are non-planar.*

- Corollary 13F: *Every simple planar graph contains a vertex whose degree is at most five.*

- The thickness of a graph $t(G)$ is the smallest number of planar graphs which can be superimposed to form G.

- Theorem 13G: *Let G be a simple graph with $n(\geq 3)$ vertices and m edges, then the thickness $t(G)$ of G satisfies the following inequalities:*

$$t(G) \geq \lceil \frac{m}{3n-6} \rceil; \quad t(G) \geq \lfloor \frac{m+3n-7}{3n-6} \rfloor$$

Exercise 13

(13c) (i) Use Euler's formula to prove that if G is a connected plane graph of girth 5 then, with the above notation, $m \leq \frac{5}{3}(n-2)$.

(ii) Deduce the Petersen graph is non-planar.

(iii) Obtain an inequality, generalizing that in part (i), for connected plane graphs of girth r.

(13h) Find the thickness of

(i) the Petersen graph.

(ii) the 4-cube Q_4.

14: Graphs on other surfaces

Exercise 14

15: Dual graphs

- Geometric-dual G^* of graph G constructed as follows:

  (i) inside each face F_i of G we choose a point v_i^* — these points are the vertices of G^*,

  (ii) corresponding to each edge e of G we draw a line e^* which crosses e (but no other edge of G), and joins the vertices v_i^* which lie in the faces F_i adjoining e — these lines are the edges of G^*.

- Lemma 15A: *Let G be a plane connected graph with n vertices, m edges and f faces, and let its geometric-dual G^* have n^* vetices. m^* edges and f^* faces. Then $n^* = f, m^* = m$, and $f^* = n$.*

- Theorem 15B: *Let G be a plane connected graph. Then G^{**} is isomorphic to G.*

- Theorem 15C: *Let G be a planar and G^* be a geometric-dual of G. Then a set of edges in G forms a cycle in G iff the corresponding set of edges of G^* forms a cutset in G^*.*

- Corollary 15D: *A set of edges of G forms a cutset in G iff the corresponding set of edges of G^* forms a cycle in G^*.*

- *The abstract-dual G^* of a graph G if there is a one-one correspondance between the edges of G and those of G^* with the property that a set of edges of G forms a cycle in G iff the corresponding set of edges of G^* forms a cutset in G^*.*

- Theorem 15E: *If G^* is an abstact-dual of G, then G is an abstact-dual of G^*.*

- Theorem 15F: *A graph is planar iff it has an abstract-dual.*

Exercise 15

16: Infinite graphs

Exercise 16