Please submit your solutions via Ilias. The submission is not a formal requirement for passing the exam but doing the exercises will be very helpful to do so. Submissions should be a single PDF document (note that Jupyter notebooks can and should also be downloaded as PDFs, and not only submitted as .ipynb files).

Sampling

1. **Short Theory Questions — *Monte-Carlo* sampling**

   a) Provide the formula for Monte-Carlo estimation of integrals of the form $\int p(x)f(x)dx$.

   b) The entropy of a random variable is defined as $H[X] = \int -p(x)\log p(x)dx$. Provide the formula for a Monte-Carlo estimate of the entropy.

   c) At what rate does the average error of Monte-Carlo samples decay as more samples are used?

   d) Why would one use Monte-Carlo estimation to estimate integrals instead of computing the integrals numerically (e.g., with the trapezoidal rule for integration)

2. **Theory 1 — *Rejection sampling* an unfair coin toss**

   Suppose we want to simulate an unfair coin toss. The result of this random variable X should be 0 with probability $p \leq 0.5$ and 1 with probability $1 - p$. A friend provides us with a fair coin, i.e. a coin that comes out heads (0) with probability 50% and tails (1) with probability 50%.

   Our goal is to simulate the unfair coin toss with rejection sampling.

   a) Provide the probability mass function of the fair coin $q(x)$.

   b) Provide the probability mass function of the unfair coin $p(x)$.

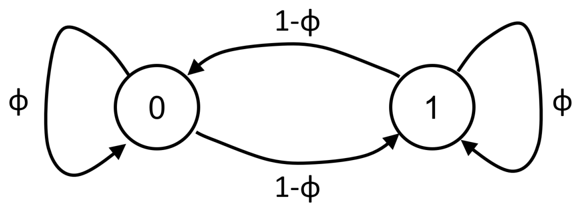
   c) Compute the smallest value c for which $c \cdot q(x) \geq p(x)$ for all x.

   d) Compute the rejection rate of the rejection sampler for the optimal choice of c. Evaluate this term for $p = 0.5$ and $p = 0.0$.

3. **Theory 2 — *MCMC sampling* an unfair coin toss**

   We are interested in the same setup as in the previous question. We now aim to simulate the unfair coin toss with MCMC.

   To do so, we are setting up an MCMC sampler with two states: Heads (0) and tails (1). We are using a transition kernel which remains in its current state with probability ϕ and proposes to change the state with probability $1 - \phi$.



   a) Assume that the chain is currently in state 0. What is the probability that the chain will be in state 1 after the next step?

   b) Assume that the chain is currently in state 1. What is the probability that the chain will be in state 0 after the next step?

c) Using your results from a) and b), provide the transition matrix

$$T = \begin{bmatrix} p(x_t = 0|x_{t-1} = 0) & p(x_t = 1|x_{t-1} = 0) \\ p(x_t = 0|x_{t-1} = 1) & p(x_t = 1|x_{t-1} = 1) \end{bmatrix}$$

of a single step.

d) For $\phi = 0.1$ and $p = 0.4$, the transition matrix T is

$$T = \begin{bmatrix} p(x_t = 0|x_{t-1} = 0) & p(x_t = 1|x_{t-1} = 0) \\ p(x_t = 0|x_{t-1} = 1) & p(x_t = 1|x_{t-1} = 1) \end{bmatrix} = \begin{bmatrix} 0.1 & 0.9 \\ 0.6 & 0.4 \end{bmatrix}$$

Assume that, initially, the states follow the distribution $x^* = [1.0, 0.0]$. Compute the distribution of the states after 2, 5, 10, and 20 steps. Interpret the result with respect to the unfair coin toss.

4. **Practical Question — *Exercise_07.ipynb*.** In the lecture you learned about different methods that allow sampling from complex distributions. In this programming exercise, you will explore importance sampling in order to refine laplace approximations and implement a Gibbs-sampler for an Ising model.