Linear System Models

Lecture 3

Outline

- Laplace Transforms and Inverse Laplace Transforms
- Transfer Function Models
- Block Diagram representation of a System
- State Space Models
- Converting SS Model into TF Model
- Converting TF model into SS Model

Laplace Transforms

Let us define

$f(t)$ = a function of time t such that $f(t) = 0$ for $t < 0$

s = a complex variable

$\mathcal{L}$ = an operational symbol indicating that the quantity that it prefixes is to be transformed by the Laplace integral $\int_0^\infty e^{-st}\,dt$

$F(s)$ = Laplace transform of $f(t)$

Then the Laplace transform of $f(t)$ is given by

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty e^{-st}\,dt[f(t)] = \int_0^\infty f(t)e^{-st}\,dt$$

Examples

Consider the exponential function

$$\begin{aligned} f(t) &= 0, && \text{for } t < 0 \\ &= Ae^{-\alpha t}, && \text{for } t \geq 0 \end{aligned}$$

$$\mathscr{L}[Ae^{-\alpha t}] = \int_0^\infty Ae^{-\alpha t}e^{-st}\,dt = A\int_0^\infty e^{-(\alpha+s)t}\,dt = \frac{A}{s+\alpha}$$

Consider the step function

$$\begin{aligned} f(t) &= 0, && \text{for } t < 0 \\ &= A, && \text{for } t > 0 \end{aligned}$$

$$\mathscr{L}[A] = \int_0^\infty Ae^{-st}\,dt = \frac{A}{s}$$

	$f(t)$	$F(s)$
1	Unit impulse $\delta(t)$	1
2	Unit step $1(t)$	$\frac{1}{s}$
3	t	$\frac{1}{s^2}$
4	$\frac{t^{n-1}}{(n-1)!} \quad (n = 1, 2, 3, \dots)$	$\frac{1}{s^n}$
5	$t^n \quad (n = 1, 2, 3, \dots)$	$\frac{n!}{s^{n+1}}$
6	e^{-at}	$\frac{1}{s+a}$
7	te^{-at}	$\frac{1}{(s+a)^2}$
8	$\frac{1}{(n-1)!} t^{n-1}e^{-at} \quad (n = 1, 2, 3, \dots)$	$\frac{1}{(s+a)^n}$

9	$t^n e^{-at} \qquad (n = 1, 2, 3, \ldots)$	$\dfrac{n!}{(s+a)^{n+1}}$
10	$\sin \omega t$	$\dfrac{\omega}{s^2 + \omega^2}$
11	$\cos \omega t$	$\dfrac{s}{s^2 + \omega^2}$
12	$\sinh \omega t$	$\dfrac{\omega}{s^2 - \omega^2}$
13	$\cosh \omega t$	$\dfrac{s}{s^2 - \omega^2}$
14	$\dfrac{1}{a}(1 - e^{-at})$	$\dfrac{1}{s(s+a)}$
15	$\dfrac{1}{b-a}(e^{-at} - e^{-bt})$	$\dfrac{1}{(s+a)(s+b)}$
16	$\dfrac{1}{b-a}(be^{-bt} - ae^{-at})$	$\dfrac{s}{(s+a)(s+b)}$
17	$\dfrac{1}{ab}\left[1 + \dfrac{1}{a-b}(be^{-at} - ae^{-bt})\right]$	$\dfrac{1}{s(s+a)(s+b)}$

18	$\frac{1}{a^2}(1 - e^{-at} - ate^{-at})$	$\frac{1}{s(s+a)^2}$
19	$\frac{1}{a^2}(at - 1 + e^{-at})$	$\frac{1}{s^2(s+a)}$
20	$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2 + \omega^2}$
21	$e^{-at} \cos \omega t$	$\frac{s+a}{(s+a)^2 + \omega^2}$
22	$\frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin \omega_n \sqrt{1-\zeta^2}\, t$	$\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$
23	$-\frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_n \sqrt{1-\zeta^2}\, t - \phi)$ $\phi = \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta}$	$\frac{s}{s^2 + 2\zeta\omega_n s + \omega_n^2}$
24	$1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_n \sqrt{1-\zeta^2}\, t + \phi)$ $\phi = \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta}$	$\frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$
25	$1 - \cos \omega t$	$\frac{\omega^2}{s(s^2 + \omega^2)}$

26	$\omega t - \sin \omega t$	$\frac{\omega^3}{s^2(s^2 + \omega^2)}$
27	$\sin \omega t - \omega t \cos \omega t$	$\frac{2\omega^3}{(s^2 + \omega^2)^2}$
28	$\frac{1}{2\omega} t \sin \omega t$	$\frac{s}{(s^2 + \omega^2)^2}$
29	$t \cos \omega t$	$\frac{s^2 - \omega^2}{(s^2 + \omega^2)^2}$
30	$\frac{1}{\omega_2^2 - \omega_1^2} (\cos \omega_1 t - \cos \omega_2 t) \qquad (\omega_1^2 \neq \omega_2^2)$	$\frac{s}{(s^2 + \omega_1^2)(s^2 + \omega_2^2)}$
31	$\frac{1}{2\omega} (\sin \omega t + \omega t \cos \omega t)$	$\frac{s^2}{(s^2 + \omega^2)^2}$

Laplace Transform Properties

1	$\mathcal{L}[Af(t)] = AF(s)$
2	$\mathcal{L}[f_1(t) \pm f_2(t)] = F_1(s) \pm F_2(s)$
3	$\mathcal{L}_{\pm}\left[\frac{d}{dt} f(t)\right] = sF(s) - f(0\pm)$
4	$\mathcal{L}_{\pm}\left[\frac{d^2}{dt^2} f(t)\right] = s^2F(s) - sf(0\pm) - \dot{f}(0\pm)$
5	$\mathcal{L}_{\pm}\left[\frac{d^n}{dt^n} f(t)\right] = s^nF(s) - \sum_{k=1}^{n} s^{n-k} \overset{(k-1)}{f}(0\pm)$ where $\overset{(k-1)}{f}(t) = \frac{d^{k-1}}{dt^{k-1}} f(t)$
6	$\mathcal{L}_{\pm}\left[\int f(t)\, dt\right] = \frac{F(s)}{s} + \frac{1}{s}\left[\int f(t)\, dt\right]_{t=0\pm}$
7	$\mathcal{L}_{\pm}\left[\int \cdots \int f(t)(dt)^n\right] = \frac{F(s)}{s^n} + \sum_{k=1}^{n} \frac{1}{s^{n-k+1}}\left[\int \cdots \int f(t)(dt)^k\right]_{t=0\pm}$

8	$\mathscr{L}\left[\int_0^t f(t)\,dt\right] = \dfrac{F(s)}{s}$
9	$\int_0^\infty f(t)\,dt = \lim_{s\to 0} F(s) \qquad \text{if } \int_0^\infty f(t)\,dt \text{ exists}$
10	$\mathscr{L}[e^{-at} f(t)] = F(s + a)$
11	$\mathscr{L}[f(t - \alpha)1(t - \alpha)] = e^{-\alpha s} F(s) \qquad \alpha \geq 0$
12	$\mathscr{L}[tf(t)] = -\dfrac{dF(s)}{ds}$
13	$\mathscr{L}[t^2 f(t)] = \dfrac{d^2}{ds^2} F(s)$
14	$\mathscr{L}[t^n f(t)] = (-1)^n \dfrac{d^n}{ds^n} F(s) \qquad n = 1, 2, 3, \ldots$

15	$\mathcal{L}\left[\frac{1}{t} f(t)\right] = \int_s^\infty F(s)\, ds \qquad \text{if } \lim_{t\to 0} \frac{1}{t} f(t) \text{ exists}$
16	$\mathcal{L}\left[f\left(\frac{t}{a}\right)\right] = aF(as)$
17	$\mathcal{L}\left[\int_0^t f_1(t-\tau) f_2(\tau)\, d\tau\right] = F_1(s)F_2(s)$ Convolution
18	$\mathcal{L}[f(t)g(t)] = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F(p)G(s-p)\, dp$

Final Value Theorem

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

Initial Value Theorem

$$f(0+) = \lim_{s \to \infty} sF(s)$$

Inverse Laplace Transform

$$\mathcal{L}^{-1}[F(s)] = f(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F(s) e^{st}\, ds, \qquad \text{for } t > 0$$

Partial Fraction Expansion (PFE)

Case I: Distinct Poles

$$F(s) = \frac{B(s)}{A(s)} = \frac{K(s + z_1)(s + z_2)\cdots(s + z_m)}{(s + p_1)(s + p_2)\cdots(s + p_n)}, \qquad \text{for } m < n$$

$$F(s) = \frac{B(s)}{A(s)} = \frac{a_1}{s + p_1} + \frac{a_2}{s + p_2} + \cdots + \frac{a_n}{s + p_n}$$

$$a_k = \left[(s + p_k)\frac{B(s)}{A(s)}\right]_{s=-p_k} \qquad \mathcal{L}^{-1}\left[\frac{a_k}{s + p_k}\right] = a_k e^{-p_k t}$$

Example:

Find the inverse Laplace transform of

$$F(s) = \frac{s+3}{(s+1)(s+2)}$$

The partial-fraction expansion of $F(s)$ is

$$F(s) = \frac{s+3}{(s+1)(s+2)} = \frac{a_1}{s+1} + \frac{a_2}{s+2}$$

where a_1 and a_2 are found by using Equation (2–15):

$$a_1 = \left[(s+1)\frac{s+3}{(s+1)(s+2)}\right]_{s=-1} = \left[\frac{s+3}{s+2}\right]_{s=-1} = 2$$

$$a_2 = \left[(s+2)\frac{s+3}{(s+1)(s+2)}\right]_{s=-2} = \left[\frac{s+3}{s+1}\right]_{s=-2} = -1$$

Thus

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}[F(s)] \\ &= \mathcal{L}^{-1}\left[\frac{2}{s+1}\right] + \mathcal{L}^{-1}\left[\frac{-1}{s+2}\right] \\ &= 2e^{-t} - e^{-2t}, \qquad \text{for } t \geq 0 \end{aligned}$$

Case II: Repeated Poles

$$F(s) = \frac{s^2 + 2s + 3}{(s+1)^3}$$

$$F(s) = \frac{B(s)}{A(s)} = \frac{b_1}{s+1} + \frac{b_2}{(s+1)^2} + \frac{b_3}{(s+1)^3}$$

$$\left[(s+1)^3 \frac{B(s)}{A(s)}\right]_{s=-1} = b_3$$

$$\frac{d}{ds}\left[(s+1)^3 \frac{B(s)}{A(s)}\right]_{s=-1} = b_2$$

$$\frac{d^2}{ds^2}\left[(s+1)^3 \frac{B(s)}{A(s)}\right] = 2b_1$$

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}[F(s)] \\ &= \mathcal{L}^{-1}\left[\frac{1}{s+1}\right] + \mathcal{L}^{-1}\left[\frac{0}{(s+1)^2}\right] + \mathcal{L}^{-1}\left[\frac{2}{(s+1)^3}\right] \\ &= e^{-t} + 0 + t^2 e^{-t} \\ &= (1 + t^2)e^{-t}, \qquad \text{for } t \geq 0 \end{aligned}$$

[5]

```
import sympy as sym
from sympy.abc import s,t,x,y,z

f = 1/(s**2*(s**2+1))

print(f)

# PFE
sym.apart(f)

```

```
1/(s**2*(s**2 + 1))
-1/(s**2 + 1) + s**(-2)
```

[6]

```
import sympy as sym
from sympy.abc import s

F = (s+3)/((s+1)*(s+2))

# PFE
sym.apart(F)
```

```
-1/(s + 2) + 2/(s + 1)
```

Transfer Functions

Transfer function of a linear, time-invariant system is the ratio of Laplace transform of the output function to the Laplace transform of the input under the assumption of zero initial conditions.

$$\text{Transfer function} = G(s) = \left.\frac{\mathscr{L}[\text{output}]}{\mathscr{L}[\text{input}]}\right|_{\text{zero initial conditions}}$$

$$= \frac{Y(s)}{X(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

By using the concept of transfer function, it is possible represent the system dynamics by algebraic equations in s. Highest power of s in the denominator is called the order of the system.

$$G(s) = \frac{Y(s)}{X(s)} \qquad Y(s) = G(s)X(s)$$

For Impulse input: $x(t) = \delta(t) \Rightarrow \mathscr{L}[x(t)] = X(s) = 1$

This gives $Y(s) = G(s)$

$$\mathscr{L}^{-1}[G(s)] = g(t)$$

g(t) is called the impulse-response function of the linear system and its Laplace Transform G(s) gives the transfer function.

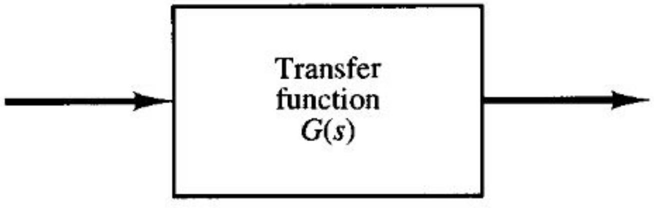
Impulse Response or Transfer functions contains the complete information about the dynamic characteristics of a linear time-invariant system.

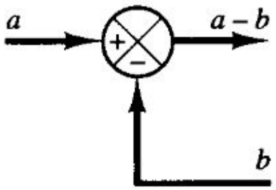
```python
from control import *
a = [[0,1],[-1,-1]]
b = [[0],[1]]
c = [1, 0]
d = 0
sys = ss(a,b,c,d)
print(sys)

g = tf(1, [1,1,1])
print(g)

# casting TF to SS
sys2 = ss(g)
print(sys2)

# casting SS to TF
g2 = tf(sys)
print(g2)
```

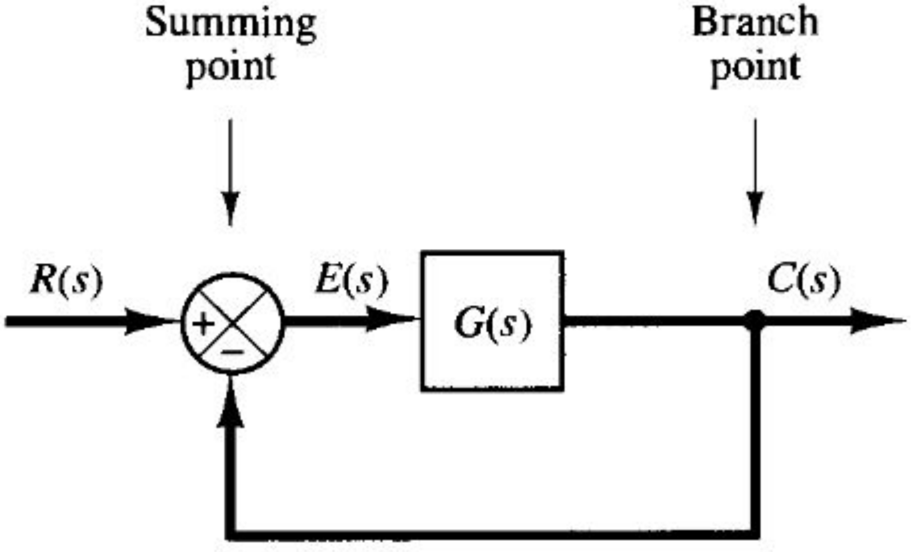
```
A = [[ 0.  1.]
 [-1. -1.]]

B = [[0.]
 [1.]]

C = [[1. 0.]]

D = [[0.]]


     1
-----------
s^2 + s + 1

A = [[-1. -1.]
 [ 1.  0.]]

B = [[1.]
 [0.]]

C = [[0. 1.]]

D = [[0.]]


     1
-----------
s^2 + s + 1
```

Block Diagrams

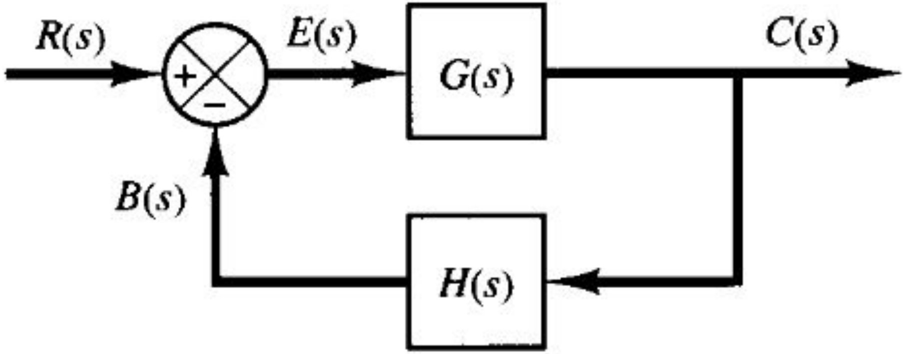
- It is a pictorial representation of the functions performed by various components of a system and the flow of signals between them.
- It is an easy to understand the interrelationship between various components of a system.
- A block diagram contains three kinds of elements
  - Functional block represented by transfer functions
  - Summing points where multiple signals are added or subtracted.
  - Branch point is a point from which signal goes out from a block to another blocks or summing joints.



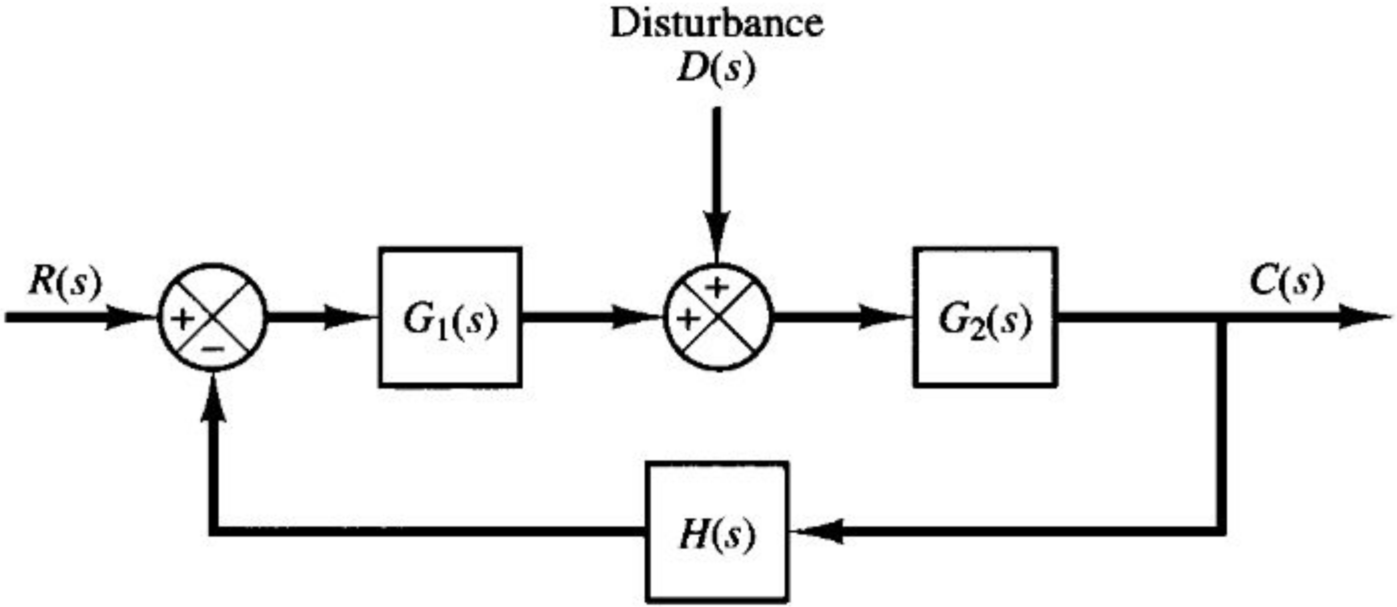




$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$



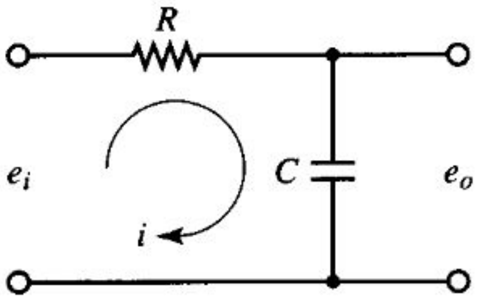
$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$



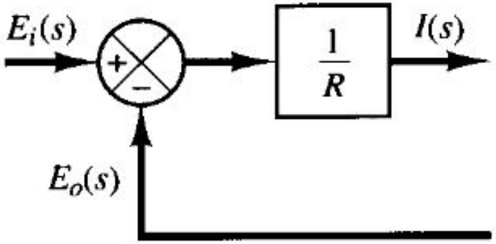
$$C(s) = C_R(s) + C_D(s)$$

$$= \frac{G_2(s)}{1 + G_1(s)G_2(s)H(s)}\left[G_1(s)R(s) + D(s)\right]$$

Example:



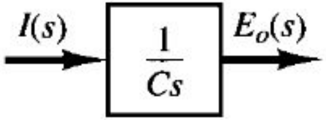
(a)



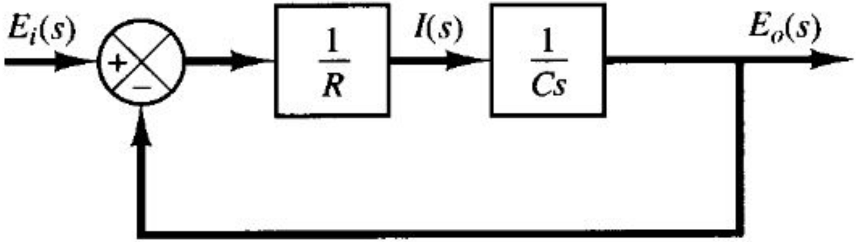
(b)

$$i = \frac{e_i - e_o}{R}$$

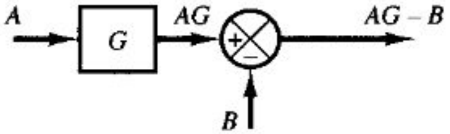
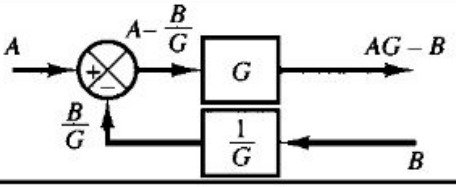
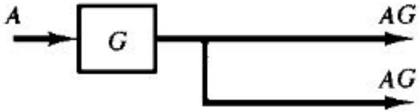
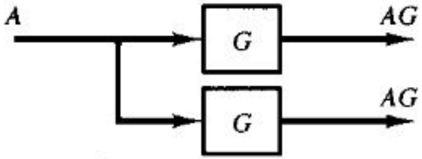
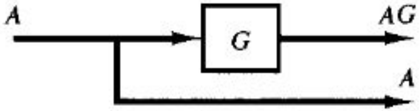
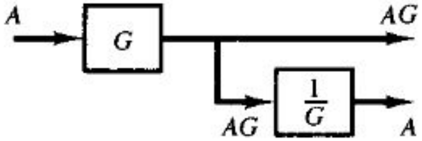
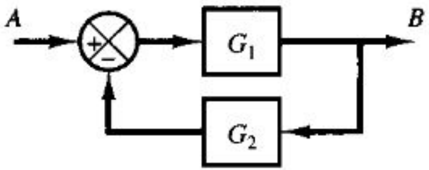
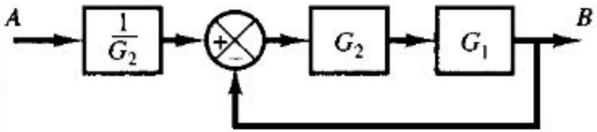
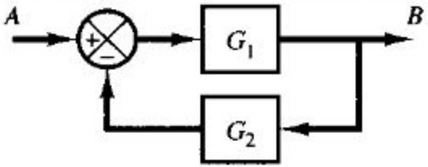
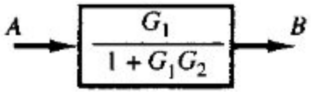
$$e_o = \frac{\int i\,dt}{C}$$



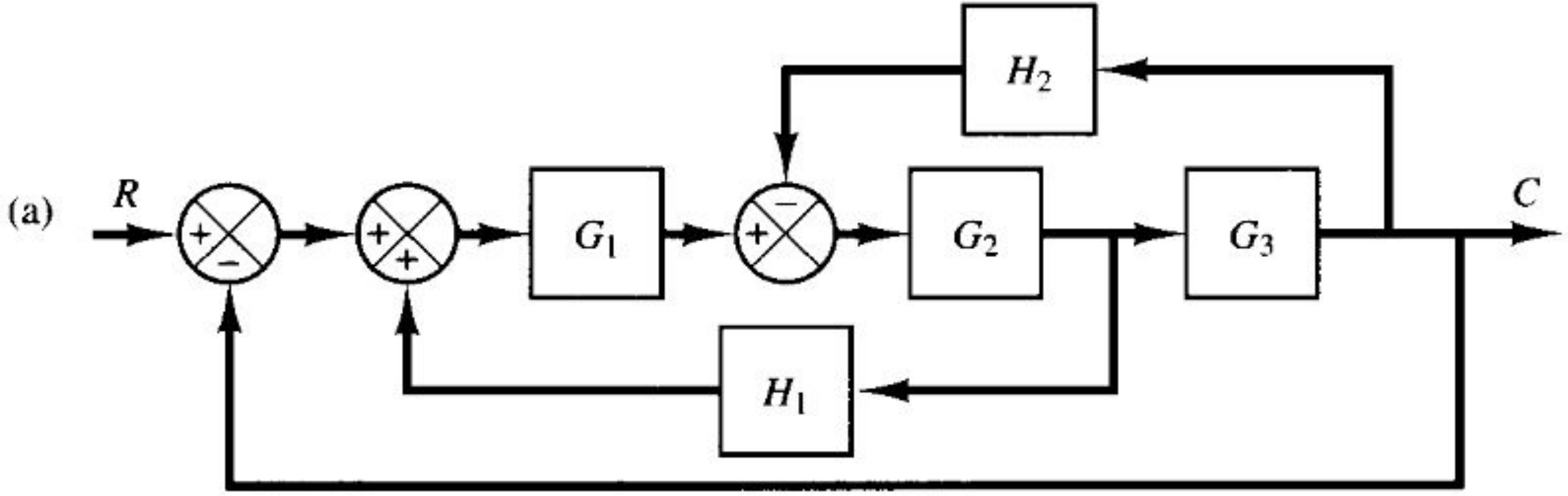
(c)



(d)

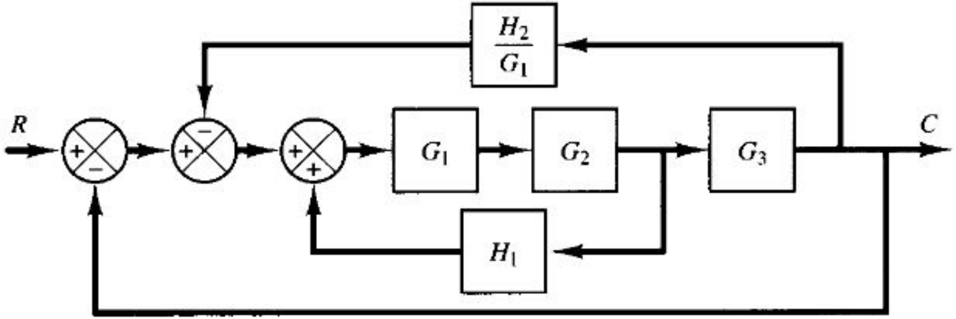
	Original Block Diagrams	Equivalent Block Diagrams
1		
2		
3		
4		
5		

Block Diagram Reduction



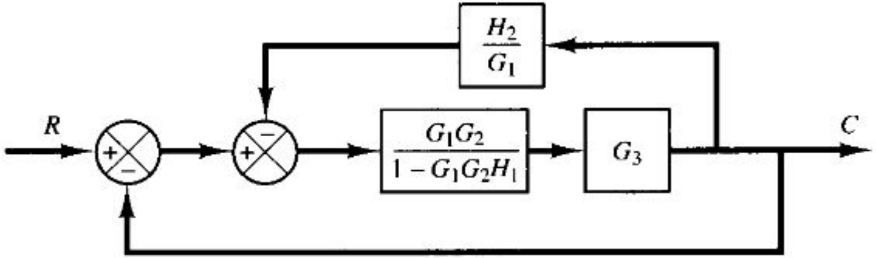
Find $\dfrac{C(s)}{R(s)}$

(b)



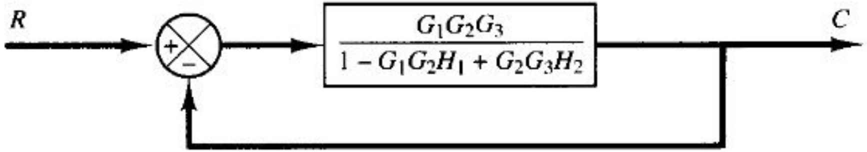
R
C
$\frac{H_2}{G_1}$
G_1
G_2
G_3
H_1

(c)



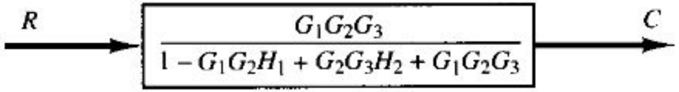
R
C
$\frac{H_2}{G_1}$
$\frac{G_1G_2}{1 - G_1G_2H_1}$
G_3

(d)



R
C
$\frac{G_1G_2G_3}{1 - G_1G_2H_1 + G_2G_3H_2}$

(e)



R
C
$\frac{G_1G_2G_3}{1 - G_1G_2H_1 + G_2G_3H_2 + G_1G_2G_3}$

```python
from control import *

g1 = tf(1,[1,1])
g2 = tf(1, [1,2])

# Parallel connection
g3 = parallel(g1,g2)
print(g3)

# series connection
g4 = series(g1,g2)
print(g4)

# Feedback connection
g5 = feedback(g1,g2,-1)
print(g5)
```

```
   2 s + 3
-------------
s^2 + 3 s + 2


      1
-------------
s^2 + 3 s + 2


    s + 2
-------------
s^2 + 3 s + 3
```

State Space Model

- State - The state of a dynamic system is the smallest set of variables (called state variables) such that the knowledge of these variables at t = t0, together with the knowledge of input u(t) for t >= t0, completely determines the behaviour of the system for any time t >= t0.
- State variables of a dynamic system are the smallest set of variables that determine the state of the system.
- State vector: n state variables of a system can be represented as a n-dimensional state vector.
- State space: The n-dimensional space spanned by the (basis) state vectors.
-

- State-space equations

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}, \mathbf{u}, t)$$
$$\mathbf{y}(t) = \mathbf{g}(\mathbf{x}, \mathbf{u}, t)$$

- Linearized SS equations:

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t)$$

- Linear Time-invariant model:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

Converting State Space Model into Transfer Function Models

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}U(s)$$
$$Y(s) = \mathbf{C}\mathbf{X}(s) + DU(s)$$

Assuming $x(0) = 0$

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{B}U(s)$$

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}U(s)$$

$$Y(s) = [\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D]U(s)$$

$$\frac{Y(s)}{U(s)} = G(s) = C(sI - A)^{-1}B + D$$

Example

$$m\ddot{y} + b\dot{y} + ky = u$$

$$x_1(t) = y(t)$$

$$x_2(t) = \dot{y}(t)$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{k}{m}x_1 - \frac{b}{m}x_2 + \frac{1}{m}u$$

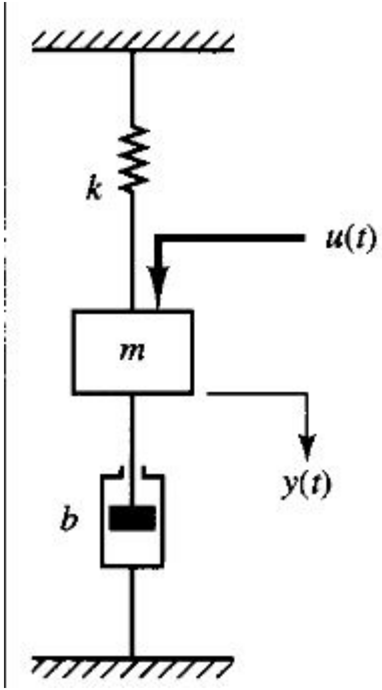
$$y = x_1$$

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$y = \mathbf{C}\mathbf{x} + Du$$

State-space model

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}, \quad \mathbf{C} = [1 \quad 0], \quad D = 0$$



$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \left\{ \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \right\}^{-1} \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} + 0$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ \frac{k}{m} & s + \frac{b}{m} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}$$

Transfer function model

$$\begin{bmatrix} s & -1 \\ \frac{k}{m} & s + \frac{b}{m} \end{bmatrix}^{-1} = \frac{1}{s^2 + \frac{b}{m}s + \frac{k}{m}} \begin{bmatrix} s + \frac{b}{m} & 1 \\ -\frac{k}{m} & s \end{bmatrix}$$

$$G(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{1}{s^2 + \frac{b}{m}s + \frac{k}{m}} \begin{bmatrix} s + \frac{b}{m} & 1 \\ -\frac{k}{m} & s \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}$$

$$= \frac{1}{ms^2 + bs + k}$$

Converting TF Models into SS Models

Case I: $$\frac{Y(s)}{U(s)} = \frac{1}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

$$\overset{(n)}{y} + a_1 \overset{(n-1)}{y} + \cdots + a_{n-1}\dot{y} + a_n y = u$$

$$\begin{aligned} x_1 &= y \\ x_2 &= \dot{y} \\ &\ \ \vdots \\ x_n &= \overset{(n-1)}{y} \end{aligned}$$

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\ \ \vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= -a_n x_1 - \cdots - a_1 x_n + u \end{aligned}$$

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

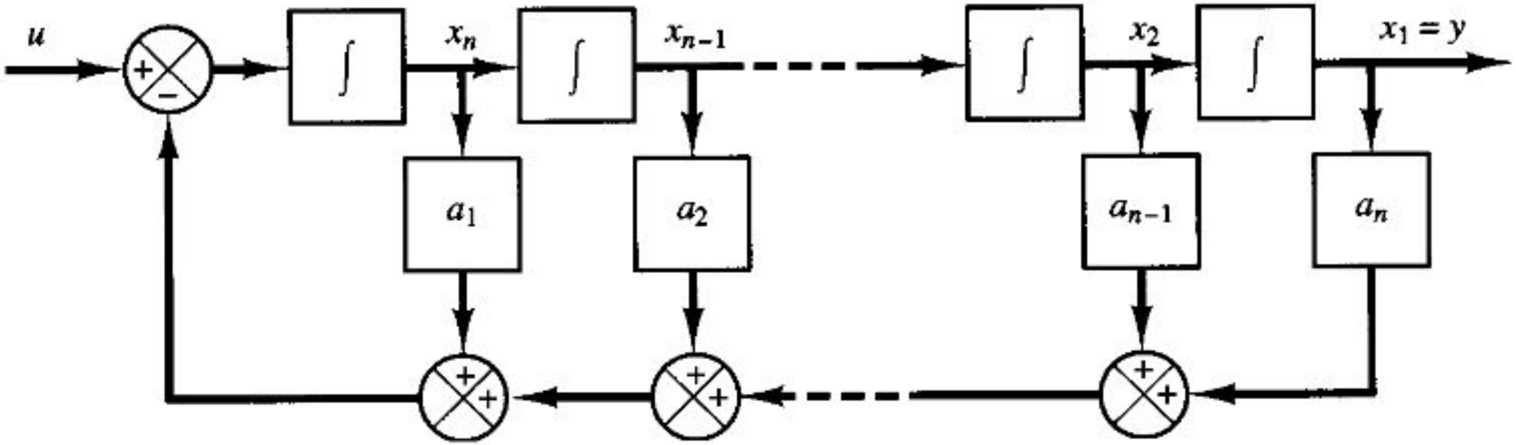
$$y = \mathbf{C}\mathbf{x}$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix}$$

The output can be given by

$$y = [1 \quad 0 \quad \cdots \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}$$



u
+
−
∫
x_n
∫
x_{n-1}
∫
x_2
∫
$x_1 = y$
a_1
a_2
a_{n-1}
a_n
+
+

Case II (A) :

$$G(s) = \frac{Y(s)}{U(s)} = \frac{\beta_0 s^m + \beta_1 s^{m-1} + \ldots + \beta_m}{s^n + \alpha_1 s^{n-1} + \ldots + \alpha_n} \quad m \leq n$$

$$\frac{Z(s)}{U(s)} = \frac{1}{s^n + \alpha_1 s^{n-1} + \ldots + \alpha_n}$$

$$p^n z(t) + \alpha_1 p^{n-1} z(t) + \ldots + \alpha_n z(t) = u(t)$$
or, $p^n z(t) = -\alpha_1 p^{n-1} z(t) - \ldots - \alpha_n z(t) + u(t)$

where $p = \frac{d}{dt}$

$$Y(s) = \left(\beta_0 s^n + \beta_1 s^{n-1} + \ldots + \beta_n\right) Z(s)$$

Assume $m = n$

$$y(t) = \beta_0 p^n z(t) + \beta_1 p^{n-1} z(t) + \ldots + \beta_n z(t)$$

Assuming $x_1 = z$

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\vdots &= \vdots \\
\dot{x}_n &= -\alpha_n x_1 - \alpha_{n-1} x_2 - \ldots - \alpha_1 x_n + u
\end{aligned}$$

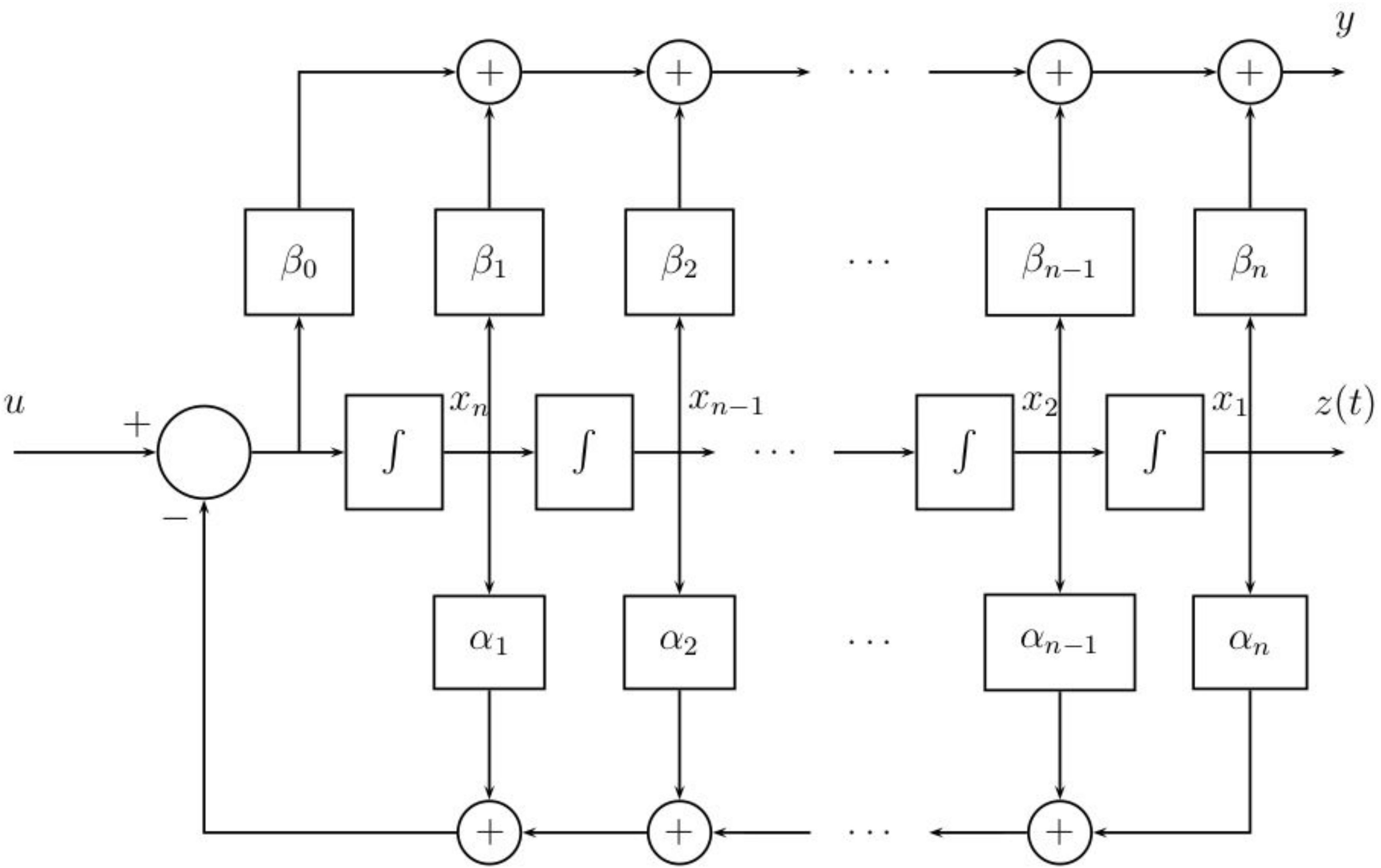
$$y(t) = (\beta_n - \alpha_n \beta_0) x_1 + (\beta_{n-1} - \alpha_{n-1}\beta_0) x_2 + \ldots + (\beta_1 - \alpha_1 \beta_0) x_n + \beta_0 u$$

$$\begin{aligned}
\dot{\mathbf{x}} &= A\mathbf{x} + Bu \\
y &= C\mathbf{x} + du
\end{aligned}$$

Final state-space model

$$A = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 1 \\ -\alpha_n & -\alpha_{n-1} & -\alpha_{n-2} & \ldots & -\alpha_1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$C = [\beta_n - \alpha_n \beta_0 \;\; \beta_{n-1} - \alpha_{n-1}\beta_0 \;\; \ldots \;\; \beta_1 - \alpha_1 \beta_0] \qquad d = \beta_0$$

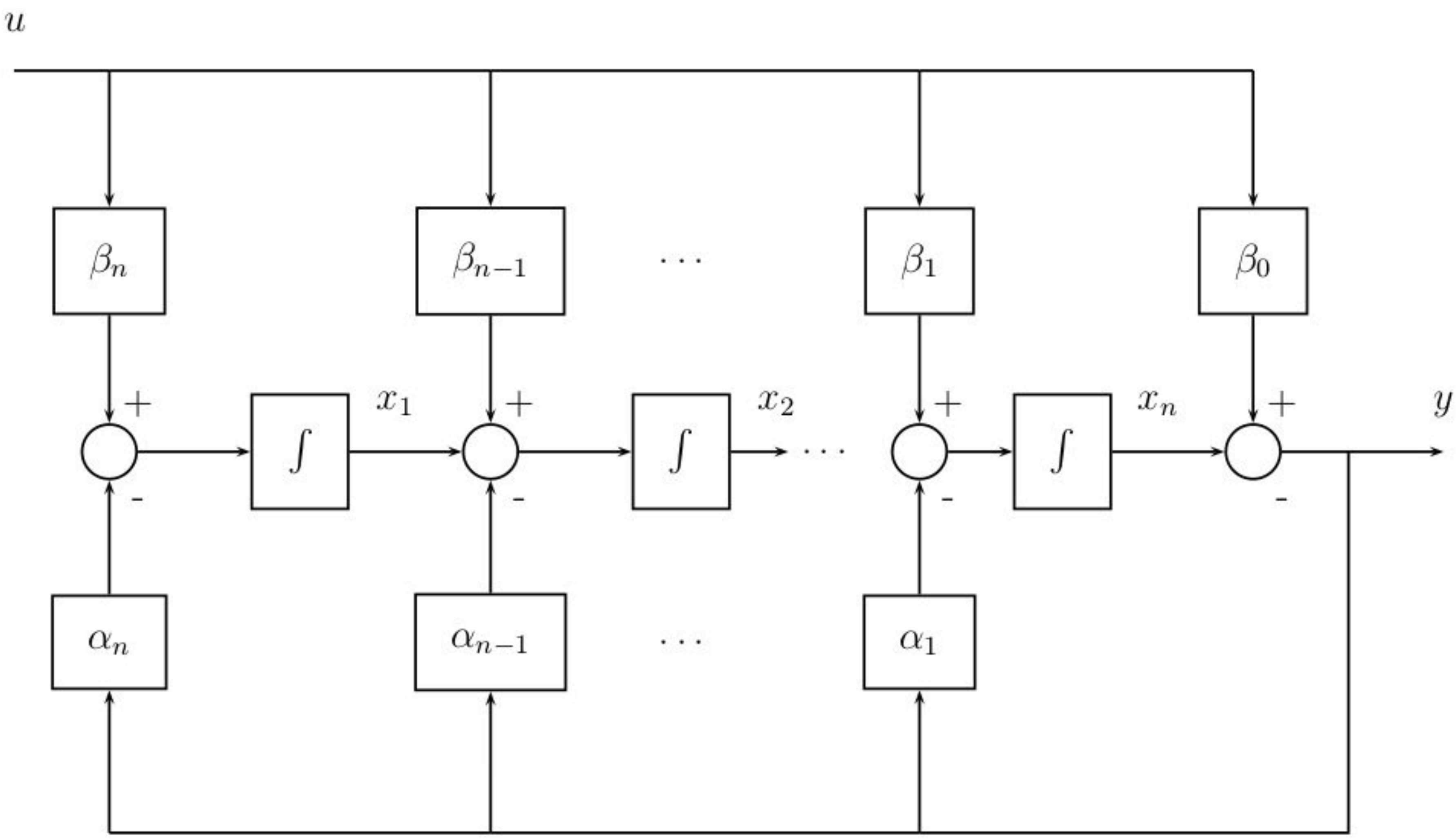


y
β_0
β_1
β_2
β_{n-1}
β_n
u
$+$
$-$
x_n
x_{n-1}
x_2
x_1
$z(t)$
$\int$
α_1
α_2
α_{n-1}
α_n

Case II(B): $$G(s) = \frac{Y(s)}{U(s)} = \frac{\beta_0 s^m + \beta_1 s^{m-1} + \ldots + \beta_m}{s^n + \alpha_1 s^{n-1} + \ldots + \alpha_n} \quad m \leq n$$

$$\left(s^n + \alpha_1 s^{n-1} + \ldots + \alpha_n\right) Y(s) = \left(\beta_0 s^n + \beta_1 s^{n-1} + \ldots + \beta_n\right) U(s)$$
$$s^n[Y(s) - \beta_0 U(s)] + s^{n-1}[\alpha_1 Y(s) - \beta_1 U(s)] + \ldots + [\alpha_n Y(s) - \beta_n U(s)] = 0$$

$$Y(s) = \beta_0 U(s) + \tfrac{1}{s}[\beta_1 U(s) - \alpha_1 Y(s)] + \ldots + \tfrac{1}{s^n}[\beta_n U(s) - \alpha_n Y(s)]$$


u
β_n
β_{n-1}
$\cdots$
β_1
β_0
$+$
x_1
$+$
x_2
$+$
x_n
$+$
y
$\int$
$\int$
$\cdots$
$\int$
-
-
-
-
α_n
α_{n-1}
$\cdots$
α_1

$$
\begin{aligned}
\dot{x}_n &= x_{n-1} - \alpha_1(x_n + \beta_0 u) + \beta_1 u \\
\dot{x}_{n-1} &= x_{n-2} - \alpha_2(x_n + \beta_0 u) + \beta_2 u \\
\vdots &= \vdots \\
\dot{x}_1 &= -\alpha_n(x_n + \beta_0 u) + \beta_n u \\
y &= x_n + \beta_0 u
\end{aligned}
$$

$$
A = \begin{bmatrix} 0 & 0 & 0 & \dots & -\alpha_n \\ 1 & 0 & 0 & \dots & -\alpha_{n-1} \\ 0 & 1 & 0 & \dots & -\alpha_{n-2} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \quad B = \begin{bmatrix} \beta_n - \alpha_n \beta_0 \\ \beta_{n-1} - \alpha_{n-1}\beta_0 \\ \vdots \\ \beta_1 - \alpha_1 \beta_0 \end{bmatrix} \quad C = [0 \ \ 0 \ \ \dots \ \ 1] \quad d = \beta_0
$$

Summary

We learn the following concepts in this lecture:

- Laplace Transforms & Inverse Laplace Transforms - Partial Fraction Expansion
- Transfer Function Models
- Block diagram representation of Systems
- State Space models
- Converting SS models into TF models and vice-versa.

Lab Session

- We will explore Python Control System Toolbox functions for creating Transfer Function and State-space Models

- Practice some examples for creating TF/SS models and apply block diagram reduction techniques.