Math 371 Homework 3

1. In 1685 John Wallis published a book called *Algebra* in which he described a method devised by Newton for solving equations. In a slightly modified form, this method was also published by Joseph Raphson in 1690. This form is the one commonly called Newton's method or the Newton-Raphson method. Newton himself discussed the method in 1669 and illustrated it with the equation $x^3 - 2x - 5 = 0$. Wallis used the same example. Find a root of this equation using Newton's method (either by hand or by coding), thus continuing the tradition that every numerical analysis student should solve this venerable equation.

2. Consdier $f(x) = x^3 - 2$.
   (a) Show that $f(x) = 0$ has a solution in $[1, 2]$ (use a theorem from class).
   (b) Compute by hand (feel free to use a calculator) the first three iterations of the following method (pick the appropriate initial value as needed).
      i. Bisection method.
      ii. Newton's method
      iii. Secant method
      iv. Linear interpolation
   (c) Write a Matlab code for each of the method mentioned in 1(b). Use your code to find the approximation of the solution of $f(x) = 0$. Create an error table for each method.

3. Use your code to find the first ten positive solutions of equation $\tan x = x$.

4. Investigate the behavior of the secant method on the function $f(x) = \sqrt{|x - a|}$. Explain your finding and discuss the cause.

5. Each of the functions

$$\begin{aligned}
f_1(x) &= \sin(x) - x - 1 \\
f_2(x) &= x(1 - \cos(x)) \\
f_3(x) &= e^x - x^2 + 3x - 2
\end{aligned}$$

   have a root in the interval $[-2, 1]$. Use all four of the above rootfinding methods to approximate the roots within absolute error tolerance 10^{-6} for each function. Limit the number of iterations to 500. For Newton's Method, use starting value $x_0 = 1$; for the Secant Method use $x_0 = 1$ and $x_1 = 0.9$. Summarize the results of the analysis for each method in table form.

Function	Number of Iterations	Approximate Root
$f_1(x)$		
$f_2(x)$		
$f_3(x)$		

   (a) Why did the Bisection Method require approximately the same number of iterations to converge to the approximate root for all three test problems?
   (b) Newton's method should have experienced difficulty approximating the root of one of the test functions. Identify which function presented a problem and explain why the difficulty occurred.

(c) Above, the Bisection Method was used to find the root of the function $f_1(x) = \sin(x) - x - 1$. Consider the function $g_1(x) = (\sin(x) - x - 1)^2$. Clearly f_1 and g_1 have the same root in $[-2, 1]$. Could the Bisection Method be used to approximate the root in g_1? Why or why not?

6. Let $f(x) = \sin(x)$ and $g(x) = \sin^2(x)$ and note that both have $x = \pi$ as a root.

   (a) Will Newton's method converge to root π for each of these functions? Why? What rate of convergence do you expect?

   (b) Use Newton's method to approximate π as roots of f and g accurate to machine error. Use Matlab to check the rate of convergence at each step. List your results.

   (c) Plot the number of iterations for Newton's method in part (b) against the absolute error at each step. Plot the results for f and g in the same graph. What does each curve resemble?

7. If the root of $f(x) = 0$ is a double root (a root with multiplicity 2), then Newton's method can be accelerated by using

$$x_{n+1} = x_n - 2\frac{f(x_n)}{f'(x_n)}$$

   Create two examples by your own and compare the convergence of this scheme with regular Newton's method. Explain why this algorithm will work.

8. Prove the convergence result of the secant method using a similar argument as in Newton's method.