Math 371 Homework 4

1. Show that
$$x = \frac{1}{2}\cos x \tag{1}$$
has a solution in $[0, 0.5]$.
Hint: use the Brouwer's fixed point theorem to show that $y = \frac{1}{2}\cos x$ has a fixed point in $[0, 0.5]$.

2. This problem is a writing project from Calculus II. The goal is to show that the iteration
$$x_{n+1} = \frac{1}{2}\cos(x_n)$$
will converge to the real solution thus can be used as a numerical algorithm to solve (1). This problem also captures the idea of the proof of the Brouwer's fixed point theorem in general.

   Construct the following sequence
$$x_{n+1} = \frac{1}{2}\cos x_n, \quad a_1 = 0.5 \quad \text{for} \quad n = 1, 2, 3, \ldots$$

   (a) Show that if the sequence x_n converges, the limit $x = \lim_{n\to\infty} x_n$ will be the solution of (1).

   (b) Now we only need to show that sequence x_n converges. Below are the step by step instructions. Warp them up and write a complete proof. Make you proof precise and good looking with all necessary details included.

      i. A function $f(x)$ is called a **CONTRACTION** if there is a positive constant $C < 1$ such that
$$|f(a) - f(b)| \leq C|a - b| \quad \text{for any } a, b \quad \text{in} \quad \mathbb{R}.$$
      Use the **Mean Value Theorem** to show that $f(x) = \frac{1}{2}\cos x$ is a contraction with $C = 1/2$.

      ii. Define a series $y_n = x_{n+1} - x_n$. Use i to show that
$$|y_n| \leq \left(\frac{1}{2}\right)^{n-1} |y_1| \quad \text{for any positive integer } n \geq 2.$$

      iii. Show that $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} |y_1|$ is convergent. Then by the comparison test $\sum_{n=1}^{\infty} |y_n|$ is convergent.

      iv. Use iii to show that $\sum_{n=1}^{\infty} y_n$ is convergent.

      v. The conclusion in iv implies that $\lim_{n\to\infty}(a_n - 1)$ is convergent. Therefore $\lim_{n\to\infty} a_n$ is convergent.

3. (Challenge) Based on problem 2 prove the following variation of the fixed point theorem.

   Fixed theorem: let $f(x) : [a, b] \to [a, b]$. Suppose there is a constant $C < 1$ such that $|f'(x)| \leq C$, for any x in $[a, b]$. Then

(a) $f(x)$ has exactly one fixed point x in $[a, b]$

(b) Using any x_0 in $[a, b]$, the scheme $x_{n+1} = f(x_n)$ converges to x at least linearly.

4. Consider solving $f(x) = x^3 + 6x^2 - 8 = 0$

   (a) Use the Intermediate Value Theorem to show f has a root on $[1, 2]$.

   (b) Consider the fixed point iteration defined by

      i. Function $g_1(x) = x^3 + 6x^2 + x - 8$

      ii. Function $g_2(x) = \sqrt{\dfrac{8}{x+6}}$

      iii. Function $g_3(x) = \sqrt{\dfrac{8-x^3}{6}}$

      For each scheme, show that if it converges, it will converge to the zero of f.

   (c) Try each scheme with several random initial x_0 in Matlab. What do you see?

   (d) Use the theorem in problem 3 to explain what you saw in (c).

5. Fixed point iteration: Consider the iteration $x_n = \frac{1}{2}\left(x_{n-1} + \frac{c}{x_{n-1}}\right)$ for constant $c > 0$.

   (a) Prove this iteration converges at least quadratically.

   (b) What will the iteration converge to? Explain.

   (c) Verify your results of parts (a) and (b) in Matlab.

   (d) Relate this iteration to Newton's Method.

6. Find a solution p to $x^4 + 2x^2 - x - 3 = 0$ using the following methods.

   - A fixed point iteration for $g_1(x) = (3 + x - 2x^2)^{1/4}$ with initial guess $x_0 = 0.5$.
   - A fixed point iteration for $g_2(x) = \dfrac{3x^4 + 2x^2 + 3}{4x^3 + 4x - 1}$ with a initial guess $x_0 = 0.5$.
   - The Secant Method with initial guesses $x_0 = 0.5, x_1 = 2$.
   - Newton's Method with initial guess $x_0 = 0.5$.

   For each, compute until you reach absolute error less than 10^{-14}.

   (a) For each method, print the approximation, absolute error, and computed rate of convergence. Use a table format and label the columns.

   (b) How many steps did each method take to reach the required accuracy?

   (c) For each method, how does the convergence rate compare to the number of correct digits gained at each step?

   (d) (Challenge) Prove the rate of convergence for scheme defined by $g_1(x)$ and $g_2(x)$. Use it to support your computed rate of convergence.

7. Read the first two part of the article at `http://en.wikipedia.org/wiki/Fixed_point_(mathematics)` (everything up to but not including **Applications**). Wrap up the idea of using fixed point theorem to solve equations.

   (a) Write a step by step instruction.

   (b) Give three distinct examples that works fine.

   (c) Give three distinct examples that will fail.