Inverse Problems 1: convolution and deconvolution

Lesson 1: introduction to convolution

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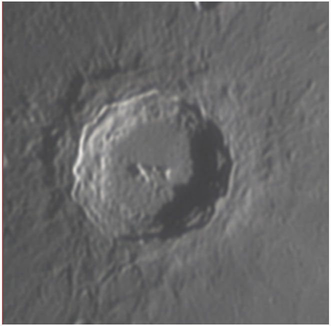
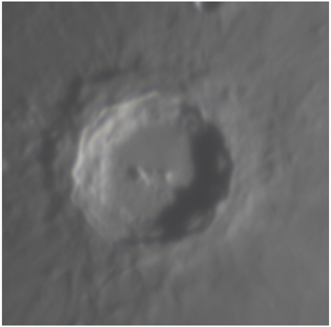
Motivation

Examples of convolution

Convolution is a mathematical model describing physical phenomena like image blurring and averaging of signals





Defocus aberration

Credits: `https://en.wikipedia.org/wiki/Deconvolution`

Examples of convolution

Convolution is a mathematical model describing physical phenomena like image blurring and averaging of signals

Motion blur

Examples of convolution

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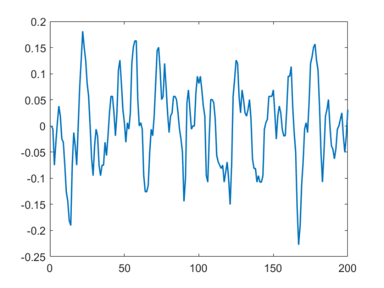

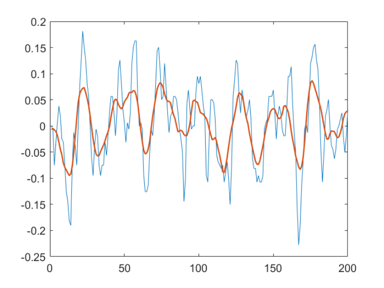

Blurred audio signal

Deconvolution

The task of deconvolution is to restore the original signal from the blurred one.

Original

Deconvolution

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Original

Blurred

Deconvolution

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Original Blurred Reconstructed

What is convolution?

Main idea

Convolution occurs when the value of a signal in one point is influenced by the values of the points close by

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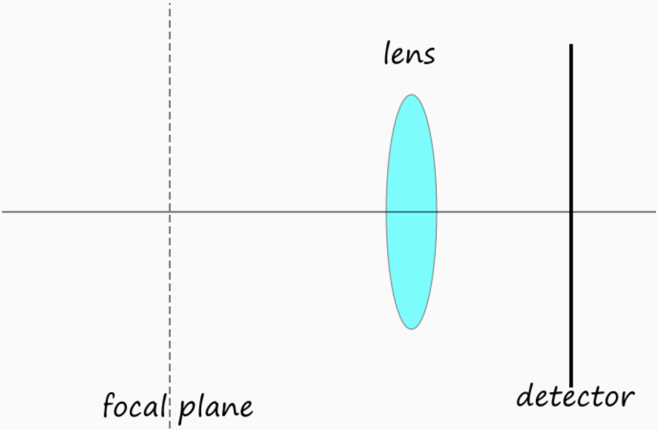

An example from optics

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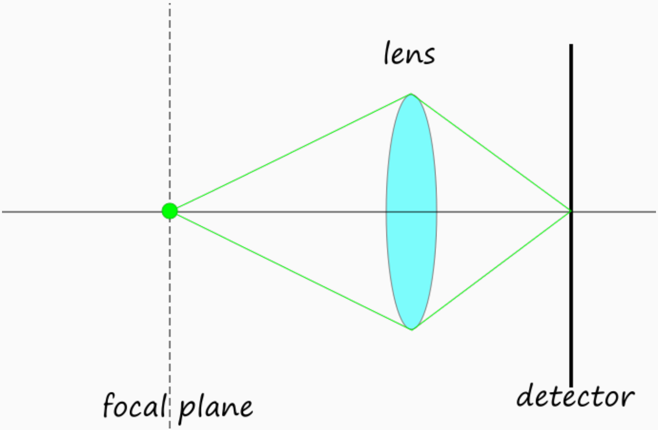

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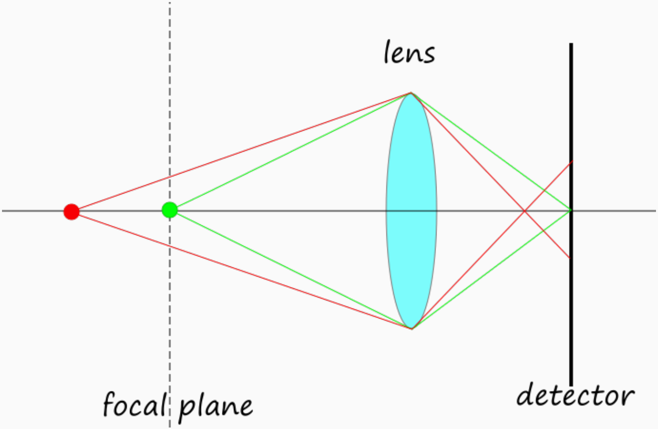

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Convolution in 1D: mathematical description

A first example

Replace each element of a vector with the average of the one before, itself and the one after.

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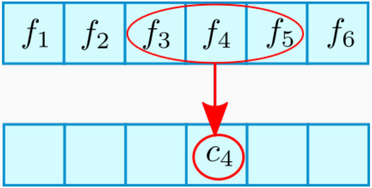

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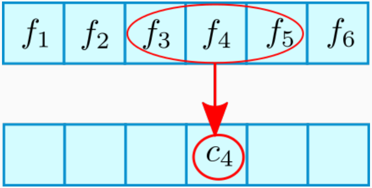

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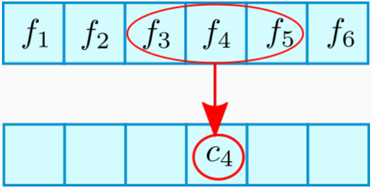

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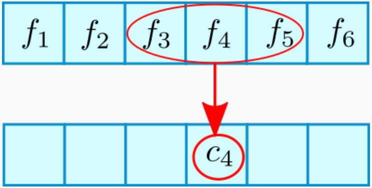

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$$c_i = \frac{1}{3}(f_{i-1} + f_i + f_{i+1})$$

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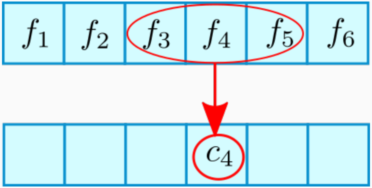

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$$c_i = \frac{1}{3} f_{i-1} + \frac{1}{3} f_i + \frac{1}{3} f_{i+1}$$

Point Spread Function

The effect of the proximal points is described by a vector defined as Point Spread Function:

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- average over 5 elements, asymmetric: $p = \left[\frac{1}{10}, \frac{1}{10}, \frac{3}{10}, \frac{3}{10}, \frac{2}{10}\right]$

Mathematical formulation

1D-signal: consider a vector $f \in \mathbb{R}^n$.

Point Spread Function: take $p \in \mathbb{R}^m$, $m = 2\nu + 1$.

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The **convolution** between f and p is a vector $p * f \in \mathbb{R}^n$ such that:

Convolution formula

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\begin{aligned}
(p * f)_j &= \sum_{\ell=-\nu}^{\nu} p_\ell f_{j-\ell} \\
&= p_{-\nu} f_{j+\nu} + \ldots + p_0 f_j + \ldots + p_\nu f_{j-\nu}
\end{aligned}
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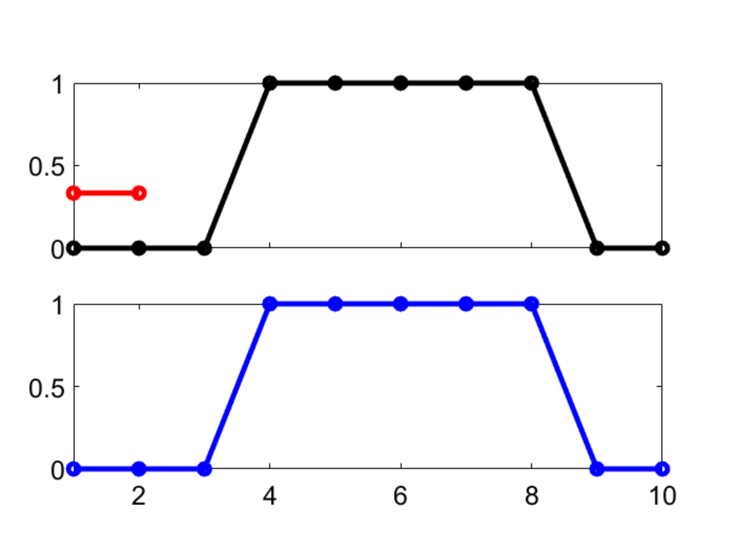
Problem: the formula requires the knowledge of the values $f_{-\nu+1}, f_{-\nu+2}, \ldots, f_0$ and also $f_{n+1}, \ldots, f_{n+\nu}$. How to define them? There is not an unique convention (next lesson).

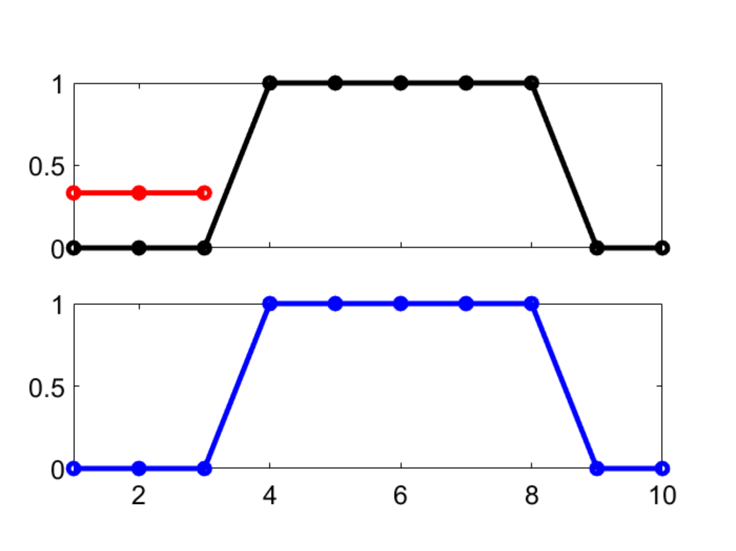
Convolution in 1D: examples

Example 1



Convolution with a symmetric filter (zero extension of f)

Example 1



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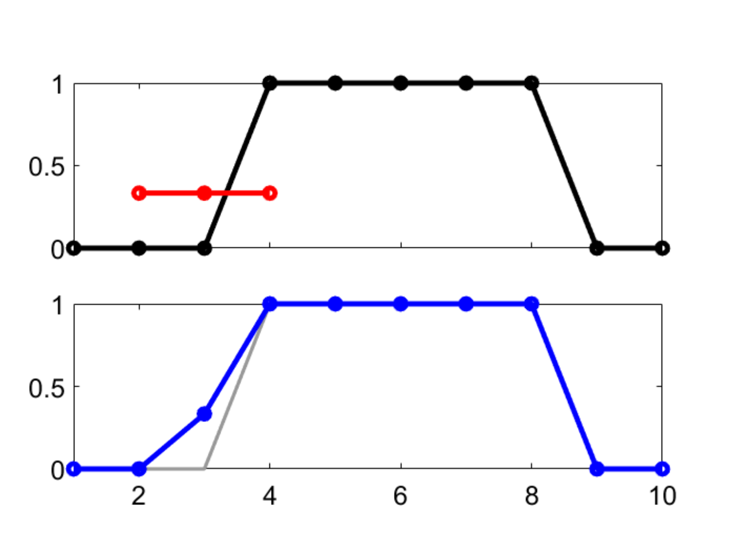

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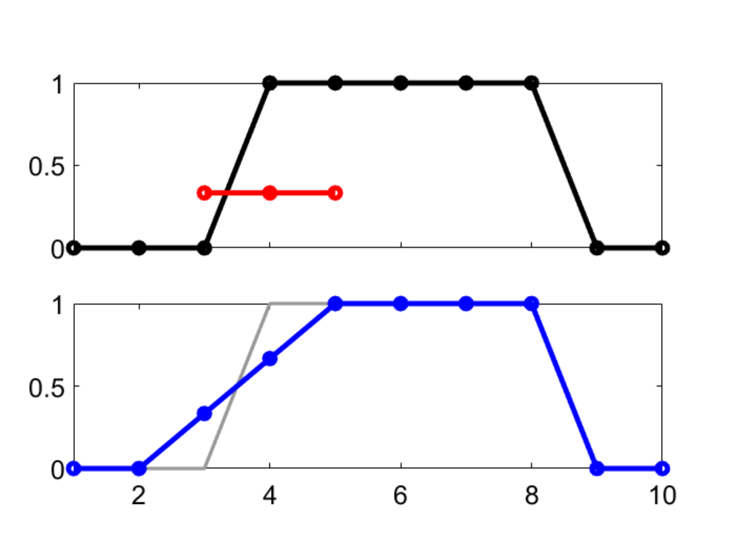

Convolution with a symmetric filter (zero extension of *f*)

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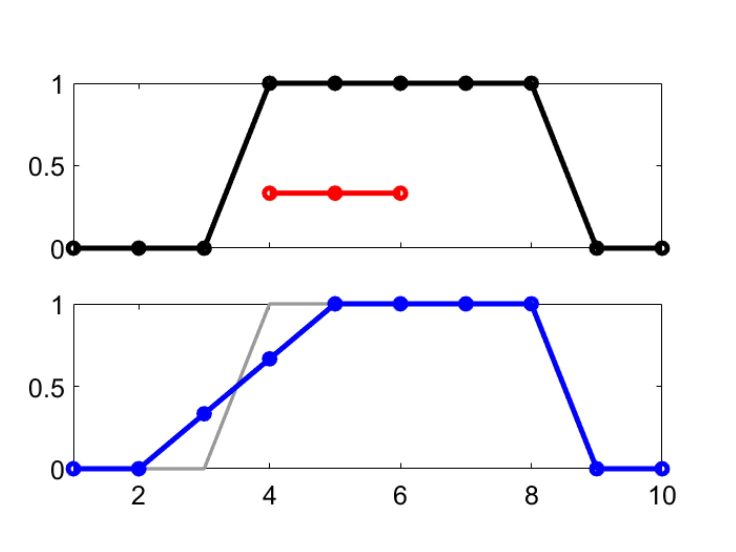

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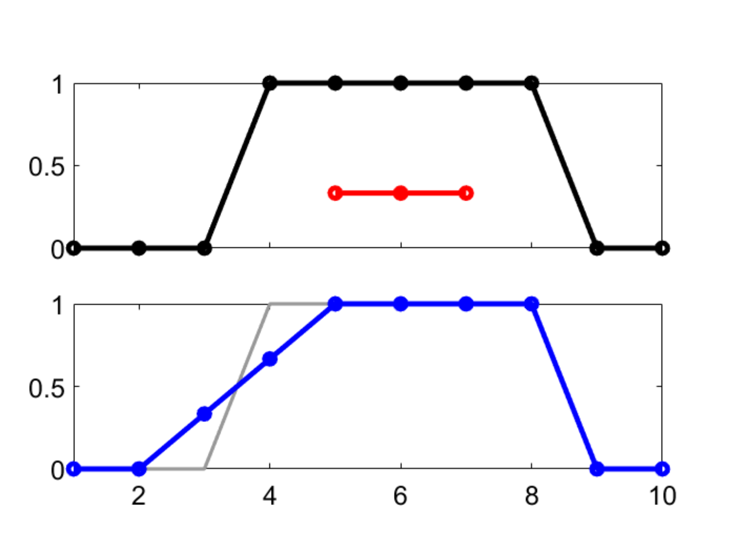

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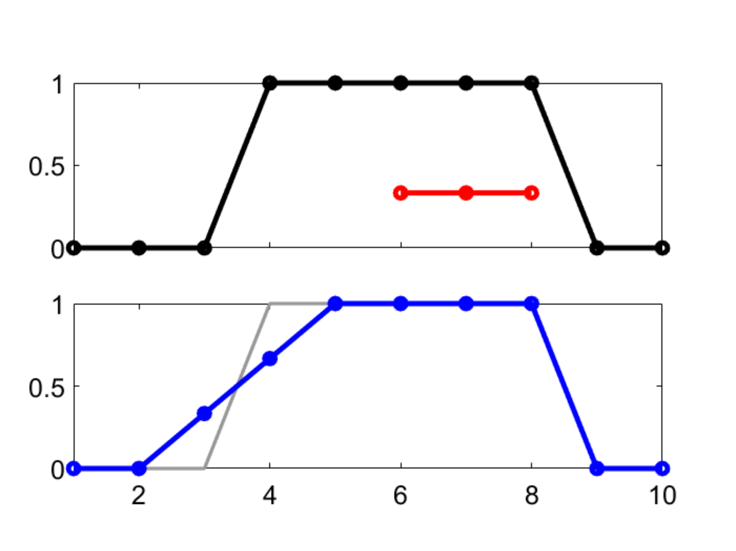

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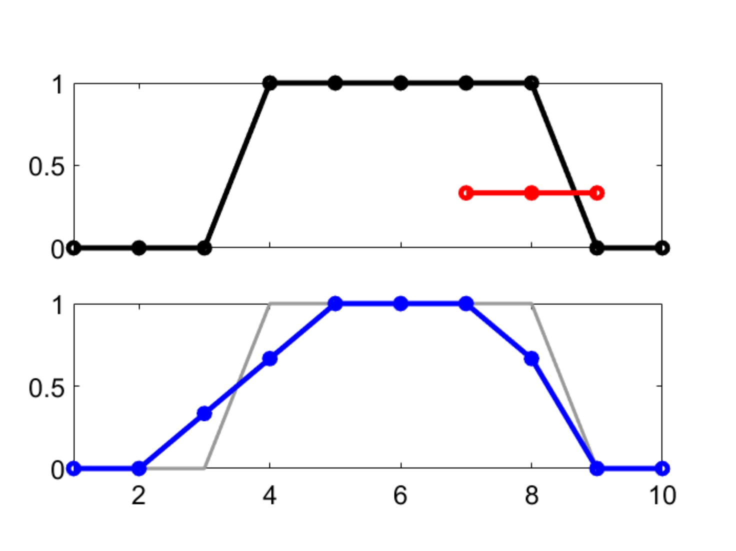

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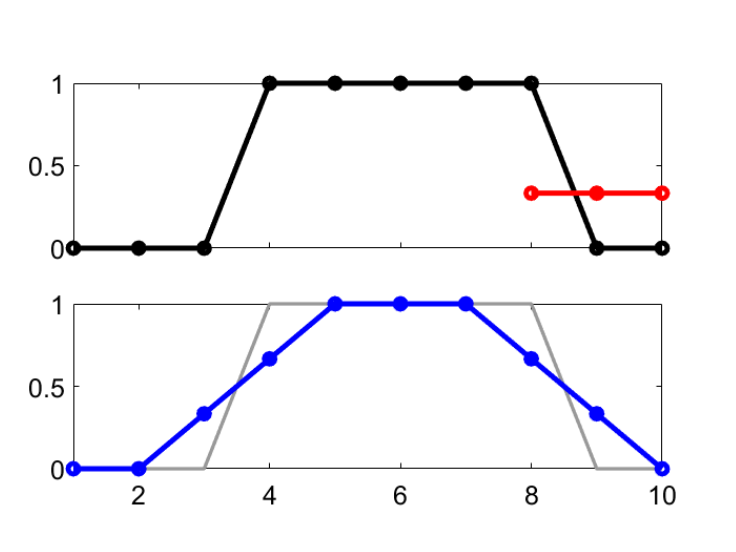

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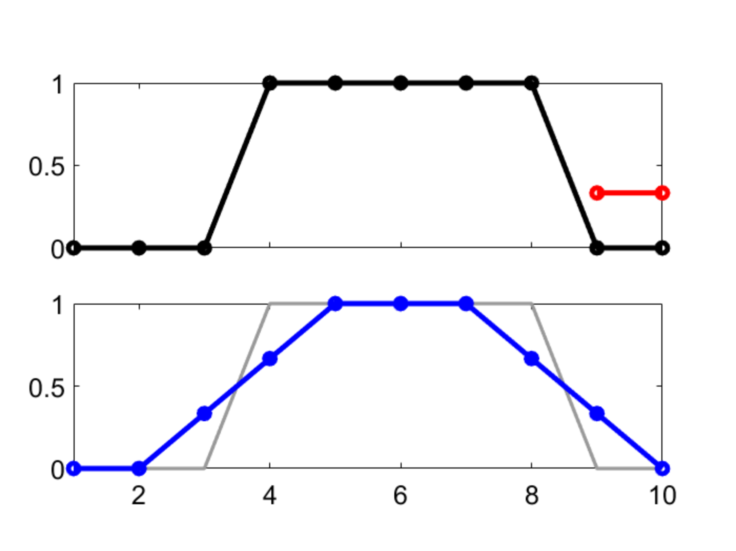

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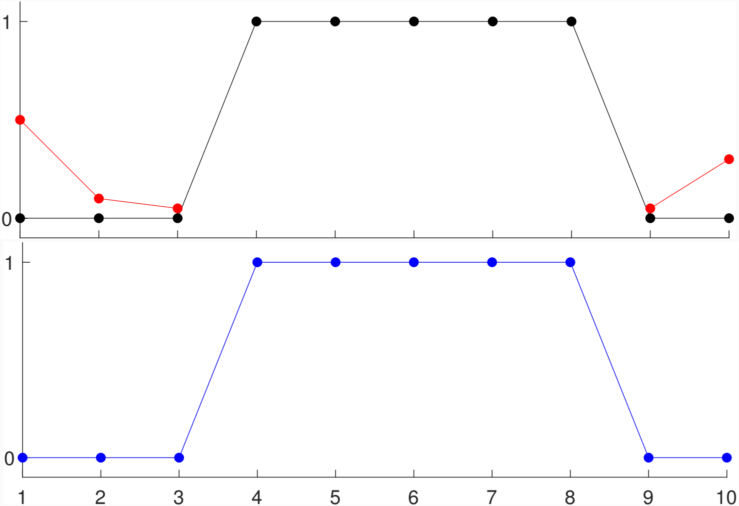

Convolution with an asymmetric filter (periodic extension of *f*)

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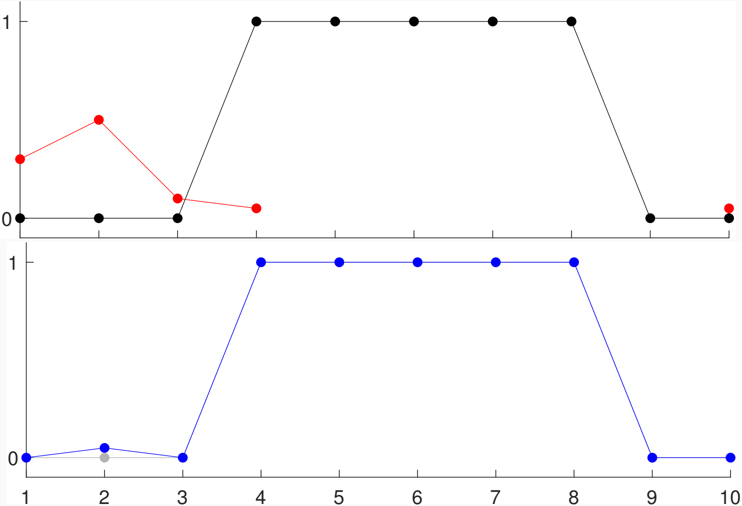

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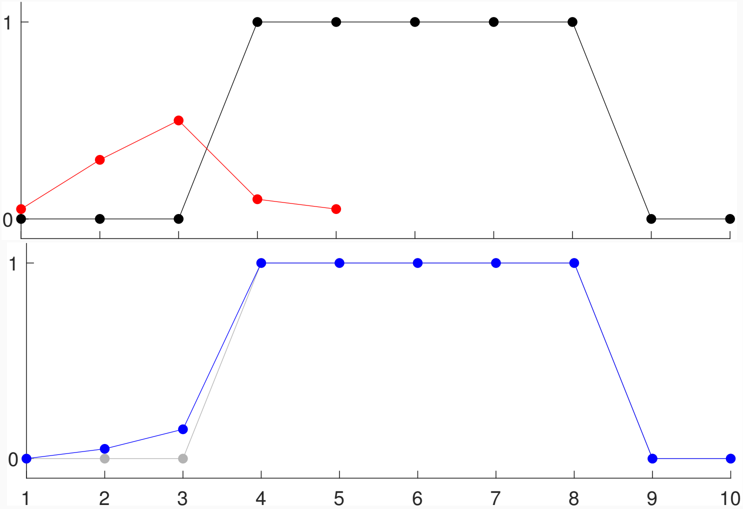

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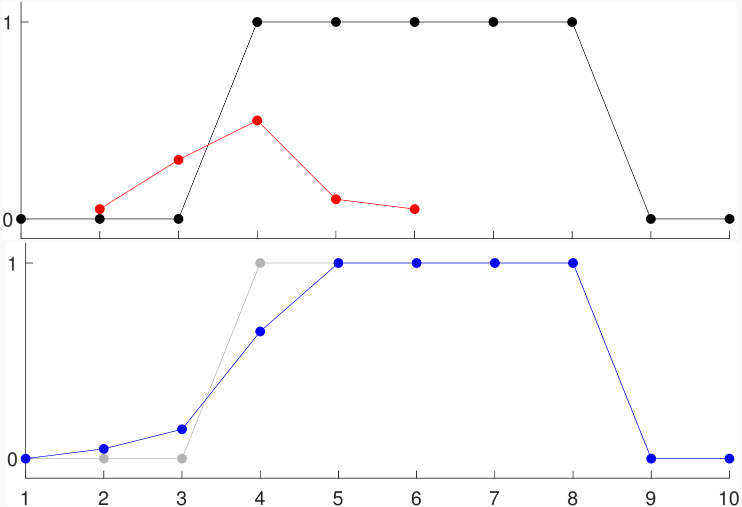

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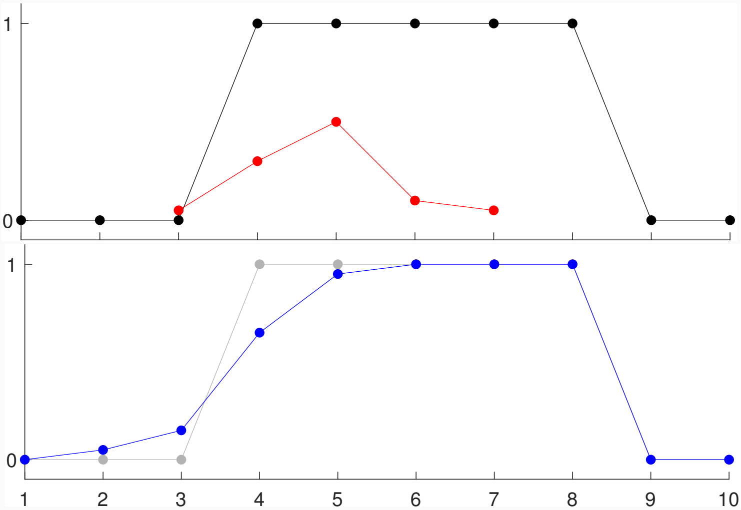

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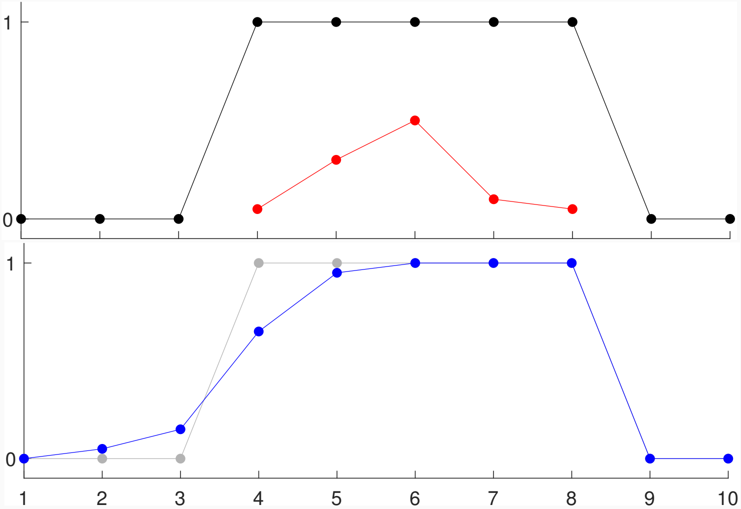

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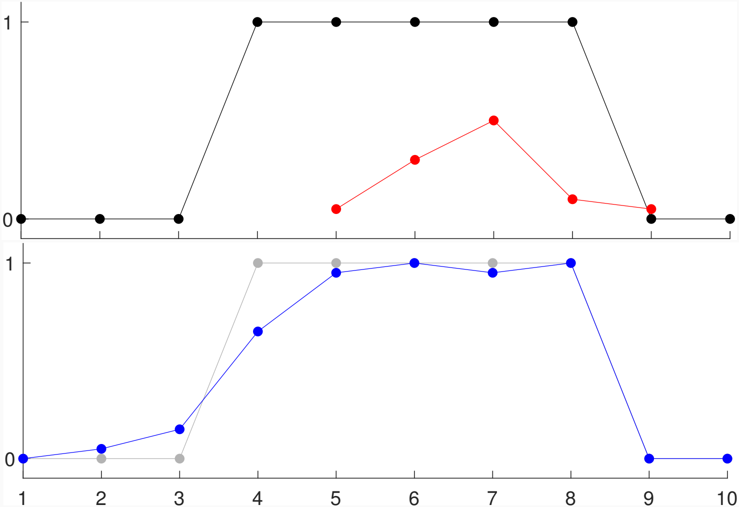

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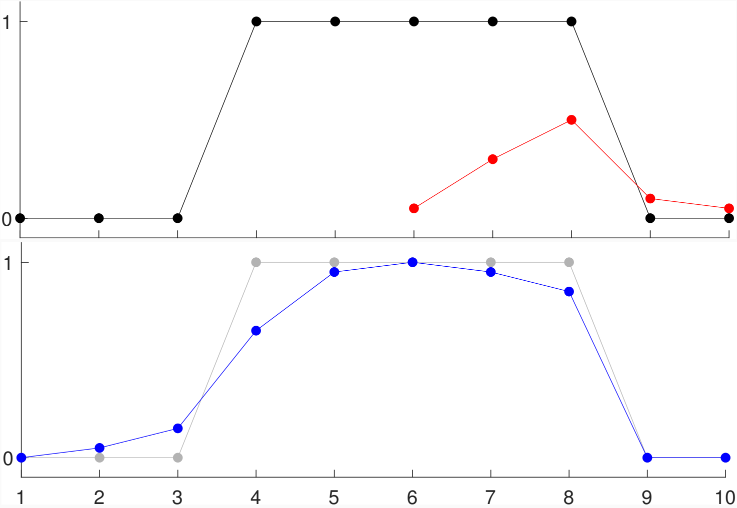

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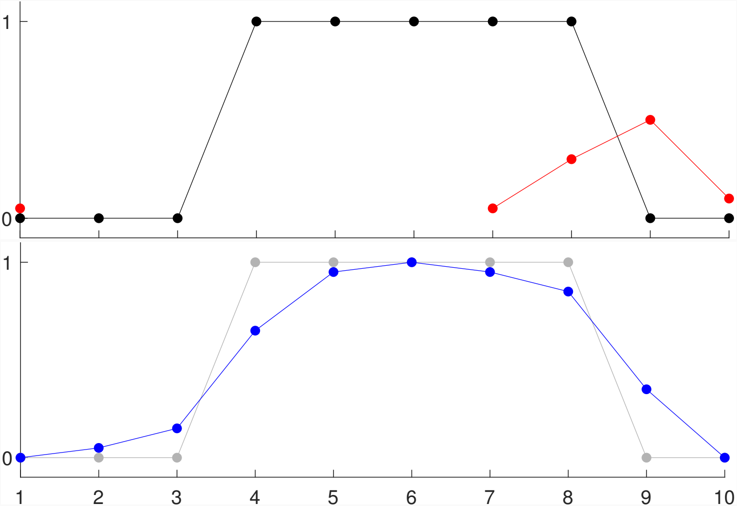

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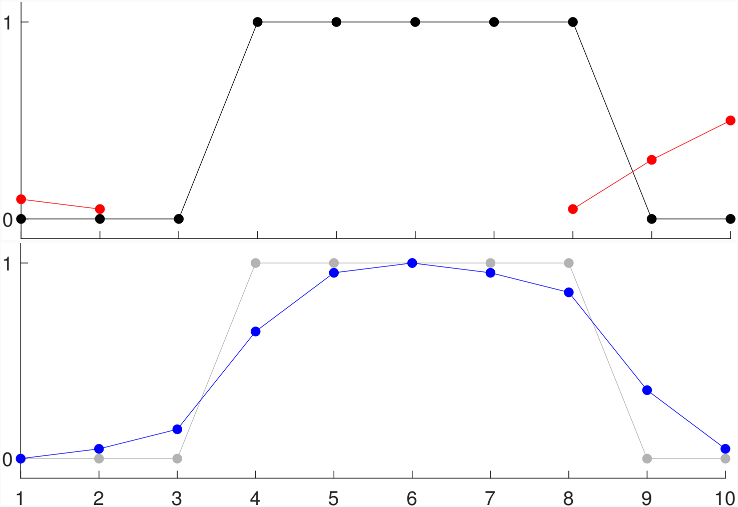
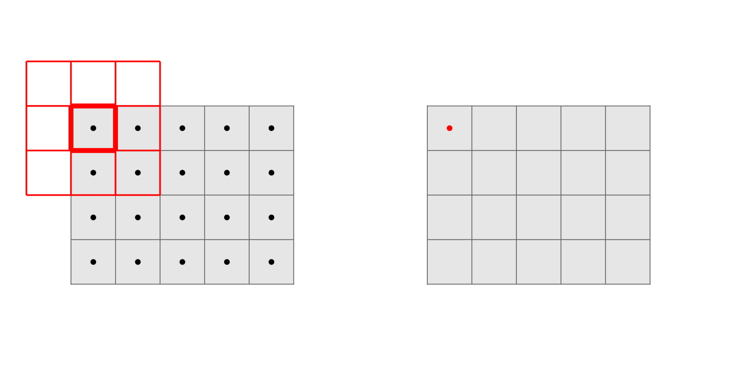


Convolution with an asymmetric filter (periodic extension of *f*)

Convolution in 2D: a brief excursion

Example 2

The same idea holds for 2D convolution, as explained by the illustration: in this case the signal is a matrix and the point spread function is a smaller square matrix.



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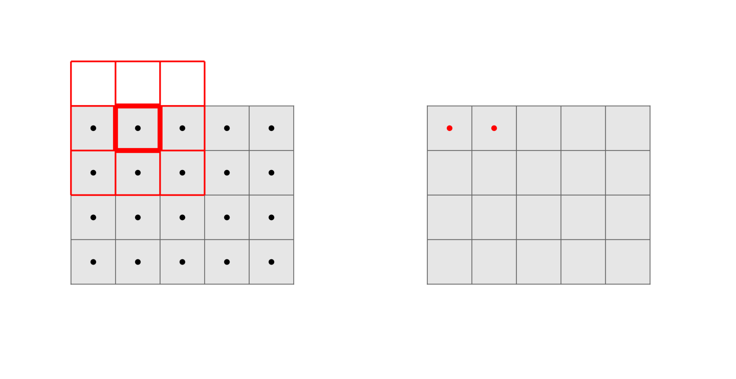

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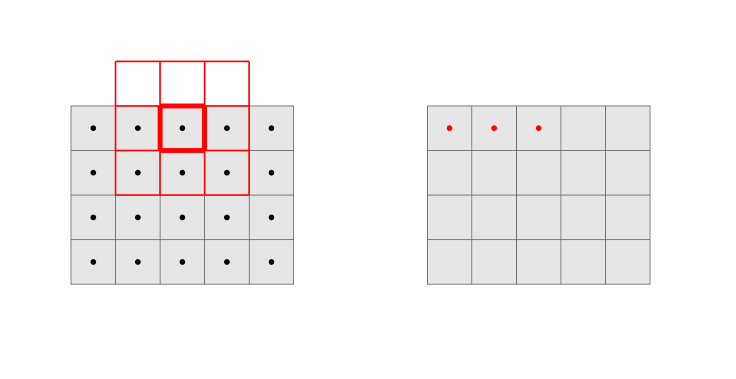

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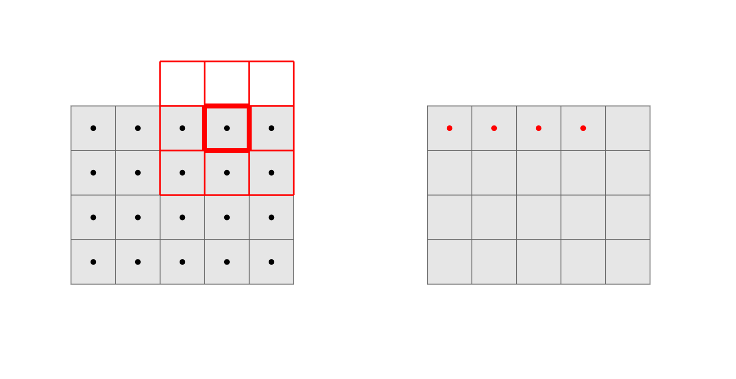

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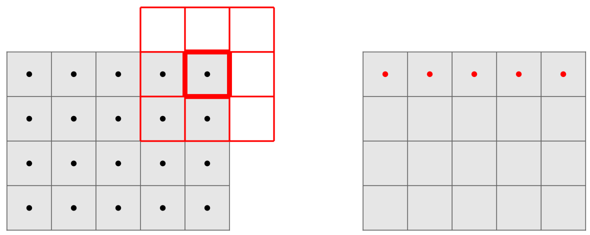

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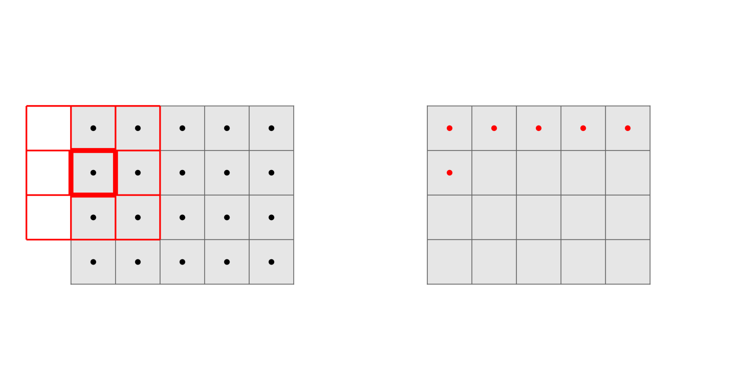

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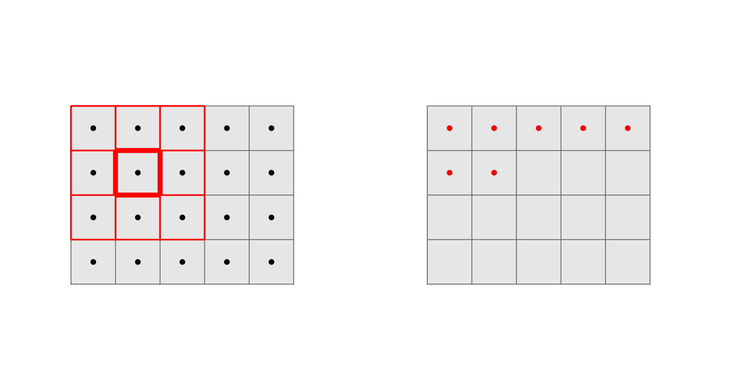

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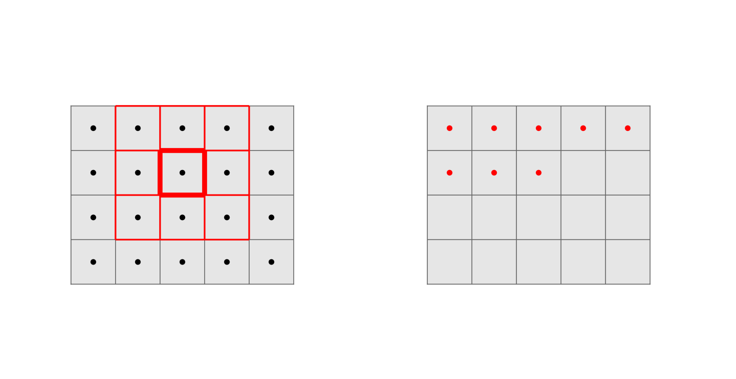

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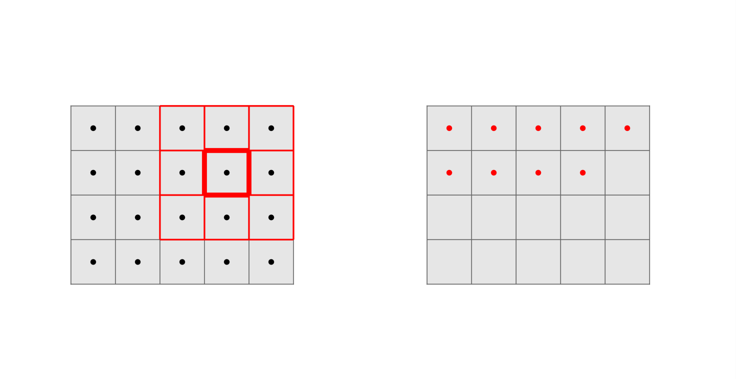

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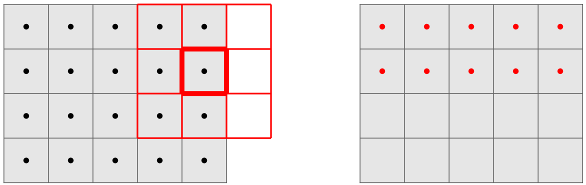

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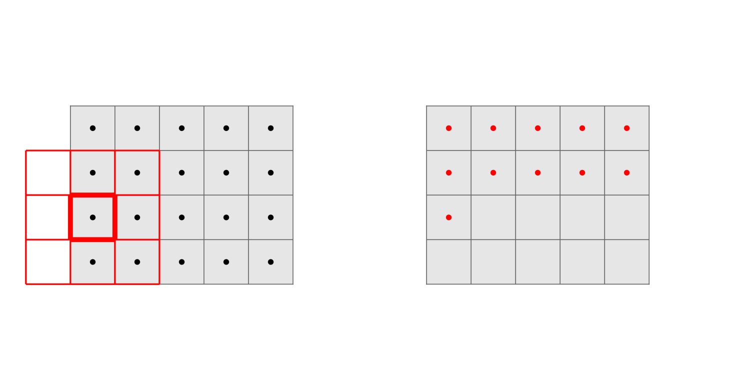

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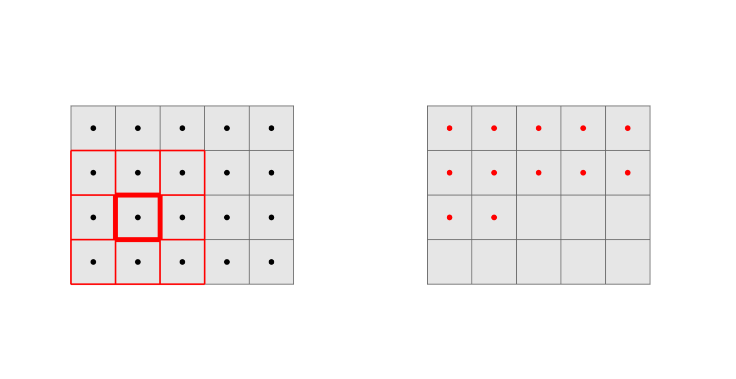

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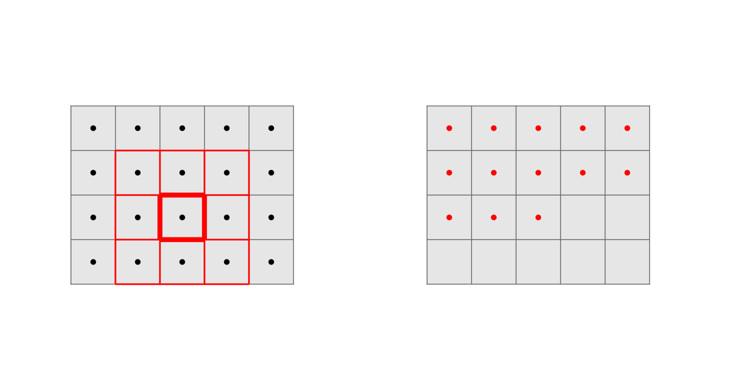

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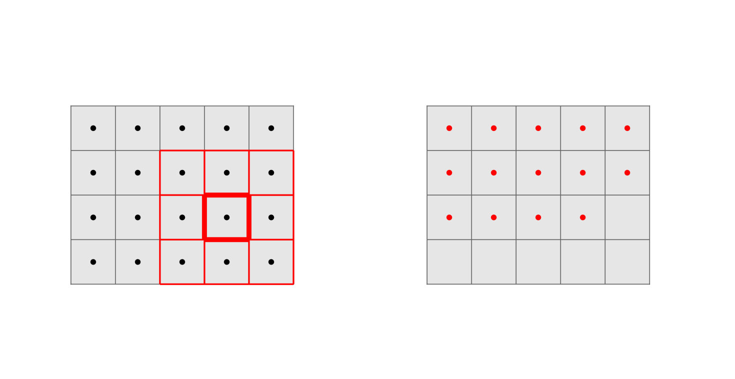

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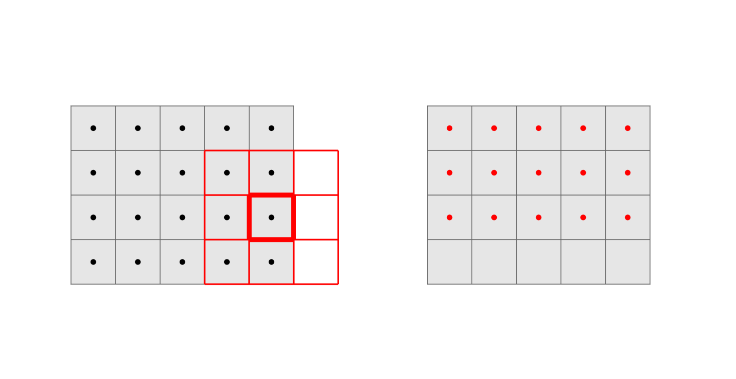

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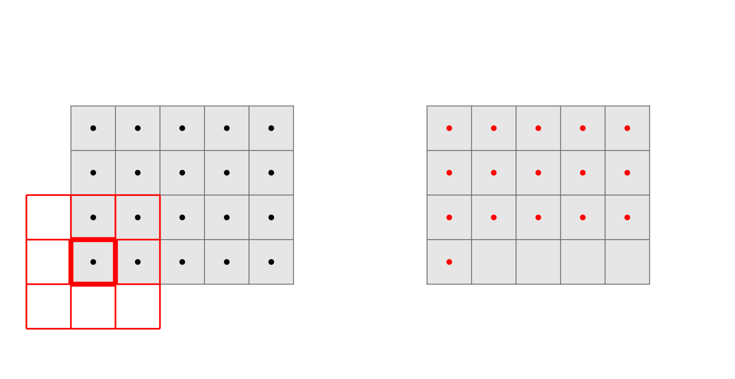

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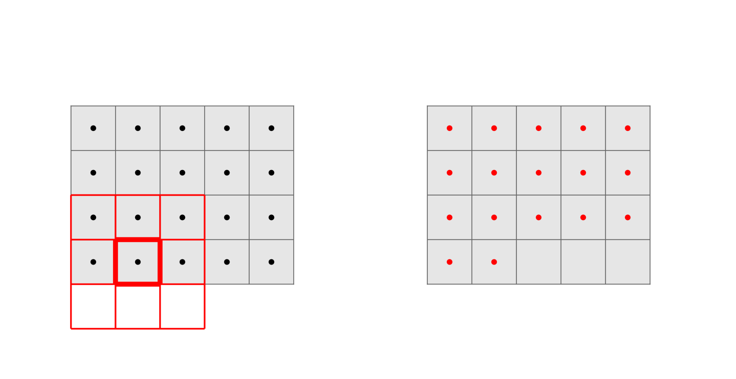

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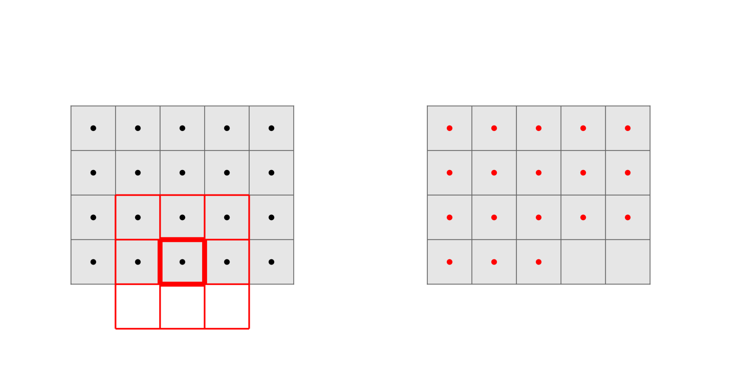

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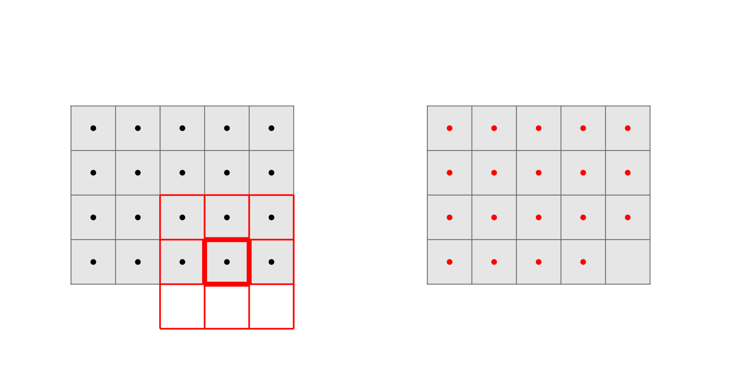

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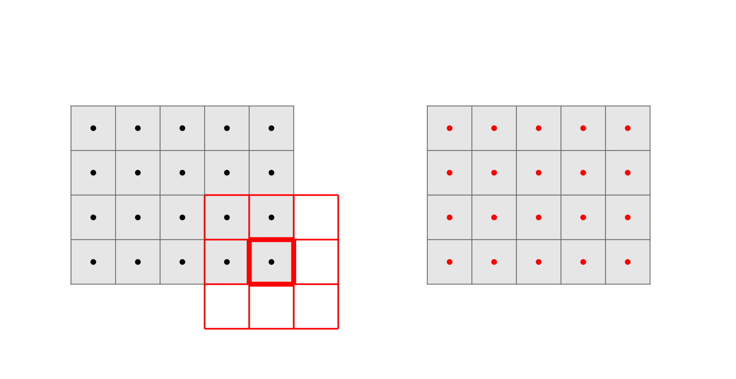

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About the course

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4. (optional, but recommended) actively explore the field of deconvolution, by a small project, and learn how to report scientific findings.

Practical information

- Lectures: Tue 12:15 - 14, Wed 14:15 - 16;
- Matlab exercises: online (MOOC) + exercise session: Thu 12:15-14;
- completion: home exam
- project: (optional) related to the topics of Inverse Problems 1 or Inverse Problems 2;
- material and additional information: `https://courses.helsinki.fi/en/mast31401/130419888`;
- contact: heli.virtanen@helsinki.fi