Inverse Problems 1: Convolution and Deconvolution

Lesson 9: Total Variation regularization

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Outline

Regularization

- Singular Value decomposition

Singular Value Decomposition

$A = UDV^T$ U, V orthogonal matrices
$D = \text{diag}\{d_1, d_2, ...d_n\}, \quad d_1 \geq d_2 \geq d_3 \ldots \geq d_n$

- If $d_n > 0$, the the problem is well posed.
- If on the other hand $d_r > 0$, but $d_{r+1} = d_{r+2} = \ldots = d_n = 0$, then existence, uniqueness and stability fails.
- We need regularization techniques.

Two Techniques so far

- Truncated Singular Value Decomposition (TSVD)

$$T_r(m) = \sum_{i=0}^{r} \frac{m^T \mathrm{u}_i}{d_i} \mathrm{v}_i$$

  where d_i are the singular values and u_i and v_i are singular vectors.

- Tikhonov regularization

$$\begin{aligned} T_\alpha(m) &= \underset{f}{argmin} \left\{ \frac{1}{2} \|Af - m\|_2^2 + \alpha \|f\|_2^2 \right\} \\ &= \sum_{i=1}^{k} \frac{d_i (m^T \mathrm{u}_i)}{d_i^2 + \alpha} \mathrm{v}_i \end{aligned}$$

  where α is the regularization parameter.

Generalized Tikhnov Regularization

- Generalized Tikhnov regularization

$$T_\alpha(m) = \underset{f}{\operatorname{argmin}} \left\{ \frac{1}{2} \|Af - m\|_2^2 + \alpha \|Lf\|_2^2 \right\}$$

where L for instance can be the discrete differentiation matrix:

$$L = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & -1 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & -1 & 1 & 0 & \dots & 0 \\ \vdots & & & \ddots & & \vdots & \\ \vdots & & & \ddots & & \vdots & \\ 0 & & \dots & 0 & -1 & 1 & 0 \\ 0 & & \dots & 0 & 0 & -1 & 1 \\ 1 & & \dots & 0 & 0 & 0 & -1 \end{pmatrix}$$

Choice of Regularization parameter

- Morozov's Discrepancy Principle: Consider the residual

$$r(\alpha) = \|Af_{\alpha,\delta} - m_\delta\|, \quad f_{\alpha,\delta} = T_\alpha m_\delta$$

.

Morozov's Principle

Select α such that $r(\alpha) = \delta$

- Alternative formulation of Morozov:

$$r(\alpha) = \tau\delta$$

  where $\tau = 1.2$

- L-curve method

Outline

Total Variation

Definition

Let f be a real valued function defined on the interval $[a, b]$. The total variation of f, denoted by $TV(f)$, is defined to be

$$TV(f) = \sup \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})|$$

where the supremum is over all partitions $a = x_0 < x_1 < \ldots < x_k = b$ of $[a, b]$

- If f is piecewise continuous function with a finite number of jump discontinuities , then the total variation is the sum of the magnitude of the jumps.

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Total Variation

- If f is differentiable function, then let $\triangle x_i = x_i - x_{i-1}$ Consider

$$TV(f) = \sup \sum_{i=1}^{k} \frac{|f(x_i) - f(x_{i-1})|}{|x_i - x_{i-1}|} |x_i - x_{i-1}|$$

  Taking the limit $\triangle x_i \to 0$, we get in one dimension,

$$TV(f) = \int_a^b |f'(x)| dx$$

- In higher dimensions, that is $\Omega = [a, b]^n$ The formula generalizes to

$$TV(f) = \int_\Omega |\nabla f(x)| dx$$

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Variational Forms

Variational problems are of the form:

$$T_\alpha(m) = \underset{f}{\operatorname{argmin}} \left\{ \frac{1}{2} \|Af - m\|_2^2 + \alpha \mathcal{R}(f) \right\}$$

where $\mathcal{R}(f)$ includes *a priori* information on the unknown f.

- Tikhnov regularization: $\mathcal{R}(f) = \|f\|_2^2$
- Generalized Tikhnov regularization: $\mathcal{R}(f) = \|Lf\|_2^2$
- Total variation regularization: $\mathcal{R}(f) = \|Lf\|_1$

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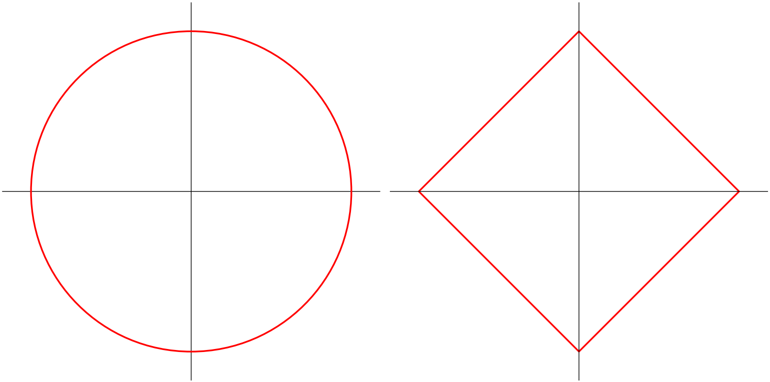
A brief history of l_p norms

Let $f \in \mathbb{R}^n$. The l_p norms for $1 \leq p < \infty$ is defined by

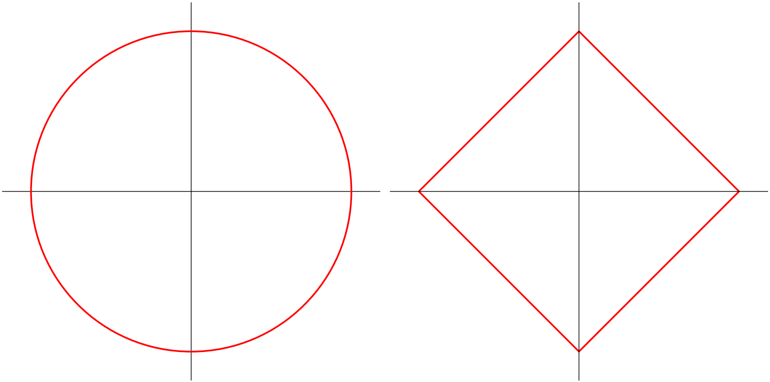
$$\|f\|_p = \left(\sum_{j=1}^{n} |f_j|^p \right)^{1/p}$$

In particular,

$$\|f\|_2^2 = \sum_{j=1}^{n} |f_j|^2, \qquad \|f\|_1 = \sum_{j=1}^{n} |f_j|$$



- By constraining our least squares minimization problem to minimize the total variation of the solution, we expect the solution to have a fewer oscillations.
- We expect the solution to be more of a block as there is no minimizing the function amplitude or derivative directly.



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What does Total Variation do?

Total Variation regularization can be understood as a balance between two requirements

- $T_\alpha(m)$ should give a small residual $AT_\alpha(m) - m$.
- $LT_\alpha(m)$should be small in l_1 norm.

TV regularization

Total Variation (TV) regularization promotes sparsity in the derivative favouring piecewise constant functions.

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Example1

- Consider a piecewise continuous function

$$h(x) = \begin{cases} 0 & 0 \le x \le 1/2 \\ 2 & 1/2 < x \le 1 \end{cases}$$

- Choose $n = 10$ and discretize the function at the points $x_1 = 0, x_2 = \frac{1}{10}, \ldots, x_{11} = \frac{9}{10}$.
- Define the derivative matrix L
- Compute the l_1 and l_2 norm of $\|Lh\|$

Example 2

- Consider a piecewise continuous function

$$f(x) = 2x$$

- Choose $n = 10$ and discretize the function at the points $x_1 = 0, x_2 = \frac{1}{10}, \ldots, x_{11} = \frac{9}{10}$.
- Define the derivative matrix L
- Compute the l_1 and l_2 norm of $\|Lf\|$

Example 3

- Consider a piecewise continuous function

$$f(x) = sin(2\pi x)$$

- Choose $n = 10$ and discretize the function at the points $x_1 = 0, x_2 = \frac{1}{10}, \ldots, x_{11} = \frac{9}{10}$.
- Define the derivative matrix L
- Compute the l_1 and l_2 norm of $\|Lf\|$
- Repeat the experiment by choosing different n values such as $n = 21, 51, 101$

Example 4

- Consider a piecewise continuous function

$$h(x) = \begin{cases} 0 & x = 0 \\ 1 & 0 < x < 0.5 \\ 0 & x = 0.5 \\ -1 & 0.5 < x < 1 \\ 0 & x = 1 \end{cases}$$

- Choose $n = 10$ and discretize the function at the points $x_1 = 0, x_2 = \frac{1}{10}, \ldots, x_{11} = \frac{9}{10}$.
- Define the derivative matrix L
- Compute the l_1 and l_2 norm of $\|Lh\|$
- Repeat the experiment by choosing different n values such as $n = 21, 51, 101$