Inverse Problems 1: Convolution and Deconvolution

Lesson 3: Ill-posedness in Inverse problems

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Outline

Convolution and Deconvolution

Continuous Model

$$f : \mathbb{R} \to \mathbb{R},$$

$$p : [-\gamma, \gamma] \to \mathbb{R},$$

$$\int_{-\gamma}^{\gamma} p(x)dx = 1; p * f : \mathbb{R} \to \mathbb{R},$$

$$(p * f)(x) = \int_{-\gamma}^{\gamma} p(y)f(x-y)dy$$

Discrete Model

$$f = (f_1, \dots f_n) \in \mathbb{R}^n,$$

$$p = (p_\nu, \dots\dots p_{-\nu}) \in \mathbb{R}^{2\nu+1},$$

$$\sum_{\ell=-\nu}^{\nu} p_\ell = 1; p * f \in \mathbb{R}^n,$$

$$(p * f)_i = \sum_{\ell=-\nu}^{\nu} p_\ell f_{i-\ell}$$

Outline

Forward Map

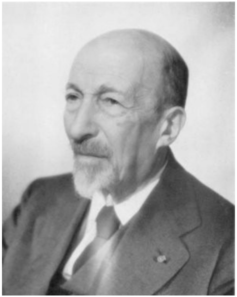
- The core of an inverse problem is the forward map $\mathcal{A} : \mathcal{D}(\mathcal{A}) \to Y$.
- $\mathcal{D}(\mathcal{A}) \subset X$ and Y are suitable Hilbert spaces called model space and data space.
- $\mathcal{D}(\mathcal{A}) \subset X$ is the domain of definition of the bounded linear operator $\mathcal{A}$
- Constructing the model space and data space and the forward map is not a trivial task.
- Physical process, Technical properties of the measurement device, geometry of the measurement and possible limitations in the data set must be considered.

Hadamard's condition for a well-posed problem

A problem is well-posed if the following conditions hold:

- The solution exists
- The solution is unique
- The solution depends continuously on the data.

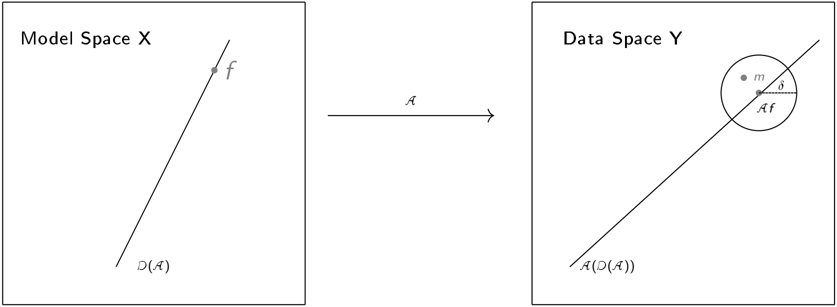
Ill-posed inverse problem:
Given any noisy data $m = Af + \epsilon$, recover f



Well posed and Ill-posed problems

1. If the forward map $\mathcal{A} : X \to Y$ is bijective and allows a continuous inverse $\mathcal{A}^{-1}$, then Naive inversion satisfies all the Hadamard's condition and we are dealing with a well-posed problem.
2. Ill-posed inverse problem will have one of the Hadamard conditions not satisfied.
   - If the nosiy data does not belong to the range of the forward map, then the first Hadamard condition is violated.
   - If two quantities in model space $f, g \in \mathcal{A}$ give the same measurement, that is $Af = Ag$, then Hadamard's second condition is not satisfied.
   - If the forward map does not allow for a continious inverse, then Hadamard's third condition is violated.
   - The solution does not depend continuously on the data.

Illustration of the linear forward map



Outline

Continuous Case

- Let $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ denote the space of bounded linear mappings from a normed linear space X to normed linear space Y
- For a linear operator $\mathcal{A} : U \subset X \to V \subset Y$, the conditions of well-posedness imply:
  - $\mathcal{A}$ is surjective (onto),
  - $\mathcal{A}$ is injective (one-to-one)
  - $\mathcal{A}^{-1}$ is continuous

A few theorems for the continuous case

Theorem

Let X and Y be Banach spaces. If $\mathcal{A} \in \mathbb{L}(\mathcal{X}, \mathcal{Y})$ is bijective, then $\mathcal{A}^{-1} \subset \mathbb{L}(\mathcal{X}, \mathcal{Y})$

So then when will a continuous case of inverse problem does not have the inverse?

Theorem

Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces. Suppose $\mathcal{A} : \mathcal{U} \subset \mathcal{X} \to \mathcal{Y}$ is a compact linear operator, and dim U is not finite. Then the problem $\mathcal{A}f = m$ is ill-posed.

Theorem

Let $\mathcal{A} : H \to H$ be a linear operator and $K_n \in \mathbb{L}(H, H)$ a sequence of compact operators. If $K_n \to \mathcal{A}$ in the operator norm, then $\mathcal{A}$ is a compact operator.

Theorem

Let $(\mathcal{A} f)(x) = \int_\Omega K(x, y) f(y) dy$
with Kernel $K \in L^2(\Omega \times \Omega)$. Then $\mathcal{A} \in \mathbb{L}(L^2(\Omega), L^2(\Omega))$ is compact.

Outline

The singular value expansion

Consider Fredholm integral equation of the first kind, where we wish to find $f(t)$

$$\int_0^1 K(s,t)f(t)dt = g(s), \quad 0 \le s \le 1, \quad \|K\|^2_{L^2([0,1]\times[0,1])} \le C$$

The Singular Value Expansion (SVE) theorem states that any Kernel, K with $\|K\|_{L^2} < \infty$ can be written as

$$K(s,t) = \sum_{i=1}^{\infty} \mu_i u_i(s) v_i(t)$$

Here u_i and v_i are the singular functions of K and μ_i are the singular values of K.

Properties of the Singular values and Singular Functions

- Singular functions are orthonormal.
  That is $\langle u_i, u_j \rangle = \int_0^1 u_i(s) u_j(s) ds = 1 \quad \text{if} \quad i = j$
- The singular values satisfy:

$$\mu_1 \geq \mu_2 \geq \ldots \geq 0; \quad \sum_{i=1}^{\infty} \mu_i^2 = \|K\|_{L^2}$$

- And

$$\int_0^1 K(s,t) v_i(t) dt = \mu_i u_i(s)$$

- Algebra shows that

$$\langle v_i, f \rangle v_i(t) = \frac{1}{\mu_i} \langle u_i, g \rangle v_i(t)$$

- The solution $f(t)$ to the Fredholm integral equation is given by

$$\begin{aligned} f(t) &= \sum_{i=1}^{\infty} \langle v_i, f \rangle v_i(t) \\ &= \sum_{i=1}^{\infty} \frac{1}{\mu_i} \langle u_i, g \rangle v_i(t) \end{aligned}$$

A few things to note about the singular values and singular functions.

- Behaviour of the singular values and the singular functions depends on the Kernel, $K(s, t)$
- Smoother the Kernel $K(s, t)$, faster the singualr values μ_i decay.
- The smaller the μ_i, more oscillatory the functions u_i and v_i will be.
- The factor $\frac{1}{\mu_i}$ amplifies highly oscillatory contributions in g.

Ill-posedness in terms of singular values

If there exists a real number $\alpha > 0$ such that the singular values satisfy $\mu_n = \mathcal{O}(n^{-\alpha})$, then α is the degree of ill-posedness.

1. If $0 < \alpha \leq 1$, the problem is mildly ill-posed.
2. If $\alpha > 1$,the problem is moderately ill posed.
3. If $\mu_n = \mathcal{O}(e^{-\alpha n})$, then the problem is severely ill-posed.

For the Fredholm integral equation to have a solution, g must satisfy

$$\sum_{i=1}^{\infty} \left(\frac{\langle u_i, g \rangle}{\mu_i} \right)^2 < \infty$$