Inverse Problems 1: Convolution and Deconvolution

Lesson 4: Inversion of the ill-posed problem

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Outline

Discretization

Singular Value Decomposition

Condition Number

Discretization

- Consider Fredholm integral equation of the first kind, where we wish to find $f(t)$

$$\int_0^1 K(s,t)f(t)dt = g(s), \quad 0 \leq s \leq 1, \quad \|K\|^2_{L^2([0,1]\times[0,1])} \leq C$$

Quadrature Method

- Consider M discrete points $0 \leq s_1 < s_2 < s_3 < ... < s_M \leq 1$
- Consider the quadrature method on

$$\int_0^1 K(s,t)f(t)dt = \sum_{j=1}^{n} w_j K(s_i,t_j)f(t_j)$$

- Then the integral equation reduces to M linear equations

$$\sum_{j=1}^{n} w_j K(s_i,t_j)f(t_j) = g(s_i)$$

Linear System

- Define a matrix $\boldsymbol{A}$ with entries $a_{ij} \equiv w_j K(s_i, t_j)$
- Define a vector $\boldsymbol{g}$ with entries $g_i \equiv g(s_i)$
- Define a vector $\boldsymbol{f}$ with entries $f_j \equiv f(t_j)$
- The system of equations becomes $\boldsymbol{Af} = \boldsymbol{g}$ with M equations in n unknowns.

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Singular Value Decomposition

- Consider an approximated discrete model of the form $\boldsymbol{m} = \boldsymbol{A}\boldsymbol{f} + \epsilon$ where $\boldsymbol{A}$ is a matrix, $\boldsymbol{f} \in \mathbb{R}^n$, $\boldsymbol{m} \in \mathbb{R}^k$
- Singular Value Decomposition (SVD) is a tool that allows for the explicit analysis of the Hadamard's condition.
- Matrix Algebra tells us that any matrix $\boldsymbol{A} \in \mathbb{R}^{k \times n}$can be written in the form of

$$\boldsymbol{A} = \boldsymbol{U}\boldsymbol{D}\boldsymbol{V}^T$$

- The right hand side of the above equation is called Singular Value Decomposition (SVD) of a matrix $\boldsymbol{A}$
- Here $\boldsymbol{U} \in \mathbb{R}^{k \times k}$ and $\boldsymbol{V} \in \mathbb{R}^{n \times n}$ are orthogonal matrices.
- $\boldsymbol{D} \in \mathbb{R}^{k \times n}$ is a diagonal matrix ; The diagonal elements d_j are the singular values of $\boldsymbol{A}$.

Orthogonality of U and V

- The matrices $\boldsymbol{U}$ and $\boldsymbol{V}$ are orthogonal means that

$$\boldsymbol{U}^T\boldsymbol{U} = \boldsymbol{U}\boldsymbol{U}^T = \boldsymbol{I}, \qquad \boldsymbol{V}^T\boldsymbol{V} = \boldsymbol{V}\boldsymbol{V}^T = \boldsymbol{I}$$

- The columns of $\boldsymbol{U}$ form an orthonormal basis for $\mathbb{R}^k$.
- The columns of $\boldsymbol{V}$ form an orthonormal basis for $\mathbb{R}^n$.
- $\boldsymbol{U}^{-1} = \boldsymbol{U}^T$

Three possibilities for Diagonal matrix

- If $k = n$, The matrix $\boldsymbol{D}$ is square.
- If $k > n$, then

$$\boldsymbol{D} = \begin{bmatrix} \text{diag}(d_1, & \ldots & , d_n)_{k \times n} \\ (0, & \ldots & , 0)_{(k-n \times n)} \end{bmatrix}$$

- If $k < n$, then

$$\boldsymbol{D} = \begin{bmatrix} \text{diag}(d_1, & \ldots & , d_k), (0, & \ldots & , 0)_{k \times (n-k)} \end{bmatrix}$$

- The diagonal elements d_j are non-negative and in decreasing order:

$$d_1 \geq d_2 \geq \ldots \geq d_{\min(k,n)} \geq 0$$

- Note that some or all of d_j can be zeros.

Some basic Linear Algebra Definitions

Linear Subspaces related to $\boldsymbol{A}$

- $\mathrm{Ker}(\boldsymbol{A}) = \{\boldsymbol{f} \in \mathbb{R}^n : \boldsymbol{A}\boldsymbol{f} = 0\}$
- $\mathrm{Range}(\boldsymbol{A}) = \{\boldsymbol{m} \in \mathbb{R}^k : \text{there exists } \boldsymbol{f} \in \mathbb{R}^n \text{such that} \boldsymbol{A}\boldsymbol{f} = \boldsymbol{m}\}$
- $\mathrm{Coker}(\boldsymbol{A}) = (\mathrm{Range}(\boldsymbol{A}))^{\perp} \subset \mathbb{R}^k$

Hadamard's condition and the matrix D

- if $n < k$, then by rank-nullity theorem, $\dim(\text{Range}(\boldsymbol{A})) < k$. Thus we can find a non-zero $\boldsymbol{m}_0 \in \text{Coker}(\boldsymbol{A})$ for which there is no $\boldsymbol{f}$. Thus existence condition is not satisfied.
- If $n > k$, then $\dim(\text{Ker}(\boldsymbol{A})) > 0$, that is there is a non-zero $\boldsymbol{f}_0 \in \text{Ker}(\boldsymbol{A})$ such that $\boldsymbol{A}(\boldsymbol{A}^{-1}\boldsymbol{m}) = \boldsymbol{m} = \boldsymbol{A}(\boldsymbol{A}^{-1}\boldsymbol{m} + \boldsymbol{f}_0)$.Thus the uniqueness is not satisfied.
- Consider $k = n$ and $\boldsymbol{A}$ is invertible.Now we have,

$$\boldsymbol{A}^{-1}\boldsymbol{m} = \boldsymbol{A}^{-1}(\boldsymbol{A}\boldsymbol{f} + \epsilon) = \boldsymbol{f} + \boldsymbol{A}^{-1}\epsilon$$

- The error $\boldsymbol{A}^{-1}\epsilon$ can be bounded by

$$\|\boldsymbol{A}^{-1}\epsilon\| \leq \|\boldsymbol{A}^{-1}\|\|\epsilon\|$$

Example1 of class 2

1. Define a vector $\boldsymbol{f} \in \mathbb{R}^{100}$ such that $f_i = 1$ if $25 \leq i \leq 75, \quad f_i = 0$. Plot the signal.
2. Define a point spread function $\boldsymbol{p} \in \mathbb{R}^5$ obtained by normalizing the vector $[1, 2, 3, 2, 1]'$.
3. Find the convolution matrix
4. Do the naive inversion. Plot the results
5. Let us analyse the failure of naive inversion with the help of SVD
6. Compute $\boldsymbol{A}^{-1}$
7. Compute the norm of $\boldsymbol{A}^{-1}$ using norm command
8. Find SVD of $\boldsymbol{A}$ using svd command

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Condition number

- Question is what is the problem when we can find the inverse of a matrix?
- Note that inverse of a matrix is

$$\boldsymbol{A}^{-1} = \boldsymbol{V}\boldsymbol{D}^{-1}\boldsymbol{U}^T, \qquad \boldsymbol{D}^{-1} = \mathrm{diag}\left(\frac{1}{d_1}, \frac{1}{d_2}, \dots, \frac{1}{d_k}\right)$$

.

- The problem for inversion arises from *Condition Number*

$$Cond(\boldsymbol{A}) := \frac{d_1}{d_k}$$

Example2 of Class2

1. Consider the same function as in the previous example.
2. Now define a different point spread function as $\boldsymbol{p} = [1, 1, 1, 1, 1]$, convolved with itself and normalized.
3. Find the Convolution Matrix
4. Add noise and do the naive inversion.
5. Analyse the results with norm of $\boldsymbol{A}^{-1}$,SVD, Condition number of $\boldsymbol{A}$
6. Compute$\boldsymbol{A}^{-1}$
7. Compute the norm of $\boldsymbol{A}^{-1}$ using norm command
8. Find SVD of $\boldsymbol{A}$ using svd command
9. Find Condition number of $\boldsymbol{A}$