Inverse Problems 1: Convolution and Deconvolution

Lesson 5: Minimum norm solutions and TSVD

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September 17, 2019

Outline

- Ill-posedness of the Inverse Problem.
- Singular Value Decomposition (SVD) $A = UDV^T$
- Hadamard's conditions with respect to SVD matrix D
- $$A^{-1} = VD^{-1}U^T; \qquad D^{-1} = \mathrm{diag}\left(\frac{1}{d_1}, \frac{1}{d_2}, \ldots, \frac{1}{d_k}\right)$$
- Condition Number $\mathrm{Cond}(A) := \frac{d_1}{d_k}$

A theorem

Theorem

Suppose that a continuum measurement is modelled by a sequence of matrices A_k *of the size* $k \times k$ *for* $k = k_0, k_0 + 1, k_0 + 2, \ldots$ *such that approximation to the forward problem becomes better as k grows. Then, Hadamard's third condition fails if*

$$\lim_{k \to \infty} Cond(A_k) = \infty$$

Outline

Review

Pseudoinverse or Moore-Penrose Inverse

Minimum norm solution

Examples

- Let A be a $k \times n$ matrix. Let the SVD of matrix A be denoted by $A = UDV^T$ Then the pseudoinverse (or Moore-Penrose inverse) of A is denoted by A^+ is defined by

$$A^+ \mathbf{m} = VD^+ U^T \mathbf{m}$$

where

$$D^+ = \begin{bmatrix} \frac{1}{d_1} & 0 & \dots & 0 & & \dots & 0 \\ 0 & \frac{1}{d_2} & \ddots & & & & \vdots \\ \vdots & & & \frac{1}{d_r} & & & \\ & & & & 0 & & \\ \vdots & & & & & \ddots & \vdots \\ 0 & \dots & & & & \dots & 0 \end{bmatrix} \in \mathbb{R}^{n \times k}.$$

Outline

Review

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Examples

Minimum Norm solution

- A vector $\mathcal{L}(\mathbf{m}) \in \mathbb{R}^n$ is called a least squares solution of the equation $A\mathbf{f} = \mathbf{m}$ if

$$\|A\mathcal{L}(\mathbf{m}) - \mathbf{m}\| = \min_{\mathbf{z} \in \mathbb{R}^n} \|(A\mathbf{z} - \mathbf{m})\|$$

- $\mathcal{L}(\mathbf{m})$ is called the mimimum norm solution if

$$\|\mathcal{L}(\mathbf{m})\| = \inf\{\|\mathbf{z}\| : \mathbf{z} \quad \text{is a least-squares solution of} \quad A\mathbf{f} = \mathbf{m}\}$$

Theorem

Let A be a $k \times n$ matrix. Denote the SVD of the matrix A by $A = UDV^T$. The minimum norm solution of the equation $A\mathbf{f} = \mathbf{m}$ is given by $A^+\mathbf{m}$, where

$$A^+\mathbf{m} = VD^+U^T\mathbf{m}$$

Outline

A system of equations

- Consider the system:

$$
\begin{aligned}
y &= x; \\
y &= 2x + 3
\end{aligned}
$$

  - Find the solution of the system by matrix inversion.
  - Find the SVD of A
  - Find the condition number of A
  - Plot the system and the solution

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Previous Example

- Consider the example of the signal

$$f = 1 \quad \text{for} \quad n = [25:75] \qquad f = 0 \quad \text{otherwise}$$

  (Here $n = [1:100]$).
- Consider the point spread function given by $PSF = [1, 1, 1, 1, 1]^T$ and normalise the point spread function.
- Create data for the inverse problem that involves inverse crime and with added noise.
- Now find $\mathbf{f}$ using naive inversion and with pseudo inverse and plot them on a graph