Inverse Problems 1: convolution and deconvolution

Lesson 2: matrix representation and naïve inversion

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Mathematical description of 1D convolution

Convolution formula

1D-signal: consider a vector $f \in \mathbb{R}^n$.

Point Spread Function: take $p \in \mathbb{R}^m$, $m = 2\nu + 1$.

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Open problem: the formula requires the knowledge of the values $f_{-\nu+1}, f_{-\nu+2}, \ldots, f_0$ and also $f_{n+1}, \ldots, f_{n+\nu}$.

Continuous version and discretization

Discrete model (DM)

$f \in \mathbb{R}^n,\ f = (f_1, \ldots, f_n);$

$p \in \mathbb{R}^{2\nu+1},\ p = (p_\nu, \ldots, p_{-\nu}),$

$$\sum_{\ell=-\nu}^{\nu} p_\ell = 1; \quad p * f \in \mathbb{R}^n,$$

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Continuous model (CM)

$$f : \mathbb{R} \to \mathbb{R},\ f = f(x);$$
$$p : [-\gamma, \gamma] \to \mathbb{R},$$
$$\int_{-\gamma}^{\gamma} p(x)dx = 1; \quad p * f : \mathbb{R} \to \mathbb{R},$$
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- the real physical model is (CM): signals vary continuously in time or space, there is no 'minimum scale' for convolution;
- **discretization**, the passage from (CM) to (DM), is mandatory for applications: sampling of signals, finite algebra of computers;
- how to discretize: introduce a grid of points $x_1, \dots, x_n$ to represent (a segment of) the real line - represent f and p as vectors (interpolation, fitting, splines,...) - approximate integrals with sums (quadrature formulas).

Historical background

Many famous mathematicians contributed to the definition and the accurate description of convolution

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V. Volterra
introduced the notation with the symbol $*$

Extended points

In order to define the value of

$$(p * f)_1, \ldots (p * f)_\nu \text{ and } (p * f)_{n-\nu}, \ldots (p * f)_n$$

we need to prescribe the values of the extended signal points,

$$f_{-\nu+1}, \ldots f_0 \text{ and } f_{n+1}, \ldots f_{n+\nu}.$$

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$$f_{-\nu+1} = \ldots = f_0 = 0;$$

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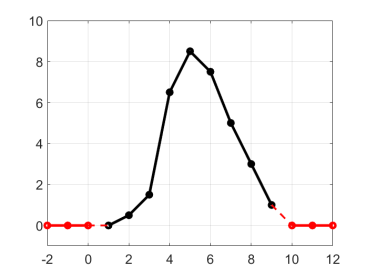

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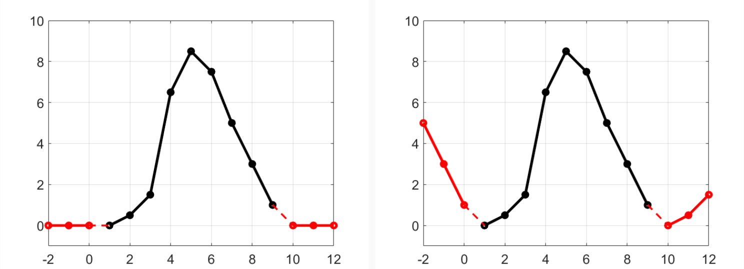
periodic extension:

$$f_{-\nu+1} = f_{n-\nu}, \dots, f_0 = f_n;$$

$$f_{n+1} = f_1, \dots, f_{n+\nu} = f_\nu;$$



Matrix representation

Matrix - vector product

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The convolution of the PSF p with a vector in $f \in \mathbb{R}^n$ can be represented as the product between a suitable matrix A (depending on p) and the vector f.

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Why?

Rigorous argument:

The convolution with p is a linear operator from $\mathbb{R}^n$ to $\mathbb{R}^n$ (indeed, $p * (f + g) = p * f + p * g$ and $p * (\lambda f) = \lambda p * f$). By the matrix representation theorem, there exists a matrix $A \in \mathbb{R}^{n \times n}$ such that $p * f = Af$ for all $f \in \mathbb{R}^n$.

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Intuitive argument:

$$\begin{bmatrix} A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} \\ A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} \\ A_{3,1} & A_{3,2} & A_{3,3} & A_{3,4} \\ A_{4,1} & A_{4,2} & A_{4,3} & A_{4,4} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}$$

Sizes of the problem

- $f \in \mathbb{R}^n$

f_1	f_2	f_3	f_4	f_5	f_6

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f_{-1}	f_0	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8

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- $c = p * f \in \mathbb{R}^n$

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f_{-1}	f_0	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8

- $c = p * f \in \mathbb{R}^n$

c_1	c_2	c_3	c_4	c_5	c_6

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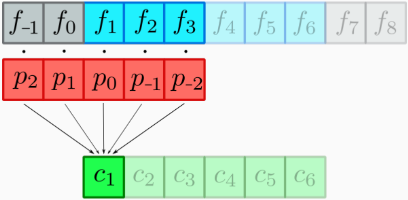
Remember the formulas:

$$(Af)_i = \sum_{j=1}^{n} A_{i,j} f_j; \qquad (p * f)_i = \sum_{\ell=-\nu}^{\nu} p_\ell f_{i-\ell}$$

Matrix: zero padding case



$$
\begin{aligned}
c_1 &= p_2 f_{-1} + p_1 f_0 + p_0 f_1 + p_{-1} f_2 + p_{-2} f_3 \\
&= p_2 0 + p_1 0 + p_0 f_1 + p_{-1} f_2 + p_{-2} f_3 \\
&= p_0 f_1 + p_{-1} f_2 + p_{-2} f_3
\end{aligned}
$$

$$
\begin{bmatrix} p_0 & p_{-1} & p_{-2} & 0 & 0 & 0 \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{bmatrix}
\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{bmatrix}
=
\begin{bmatrix} c_1 \\ \\ \\ \\ \\ \\ \end{bmatrix}
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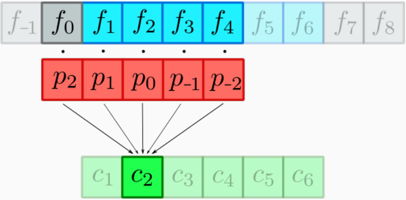

$$
\begin{aligned}
c_2 &= p_2 f_0 + p_1 f_1 + p_0 f_2 + p_{-1} f_3 + p_{-2} f_4 \\
&= p_2 0 + p_1 f_1 + p_0 f_2 + p_{-1} f_3 + p_{-2} f_4 \\
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$$
\begin{bmatrix}
p_0 & p_{-1} & p_{-2} & 0 & 0 & 0 \\
p_1 & p_0 & p_{-1} & p_{-2} & 0 & 0 \\
 & & & & & \\
 & & & & & \\
 & & & & & \\
 & & & & &
\end{bmatrix}
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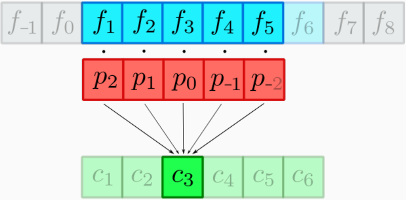

$$c_3 = p_2 f_1 + p_1 f_2 + p_0 f_3 + p_{-1} f_4 + p_{-2} f_5$$

$$\begin{bmatrix} p_0 & p_{-1} & p_{-2} & 0 & 0 & 0 \\ p_1 & p_0 & p_{-1} & p_{-2} & 0 & 0 \\ p_2 & p_1 & p_0 & p_{-1} & p_{-2} & 0 \\ & & & & & \\ & & & & & \\ & & & & & \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \\ \\ \end{bmatrix}$$

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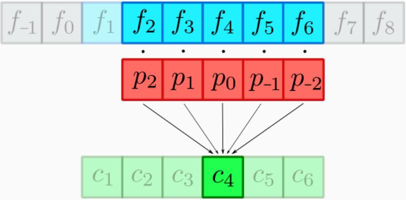
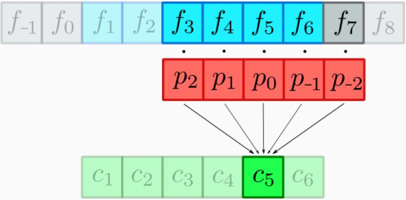


$$c_4 = p_2 f_2 + p_1 f_3 + p_0 f_4 + p_{-1} f_5 + p_{-2} f_6$$

$$\begin{bmatrix} p_0 & p_{-1} & p_{-2} & 0 & 0 & 0 \\ p_1 & p_0 & p_{-1} & p_{-2} & 0 & 0 \\ p_2 & p_1 & p_0 & p_{-1} & p_{-2} & 0 \\ 0 & p_2 & p_1 & p_0 & p_{-1} & p_{-2} \\ & & & & & \\ & & & & & \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ \\ \end{bmatrix}$$

Matrix: zero padding case



$$\begin{aligned} c_5 &= p_2 f_3 + p_1 f_4 + p_0 f_5 + p_{-1} f_6 + p_{-2} f_7 \\ &= p_2 f_3 + p_1 f_4 + p_0 f_5 + p_{-1} f_6 \end{aligned}$$

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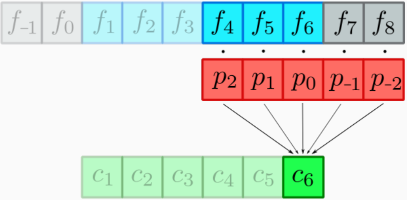
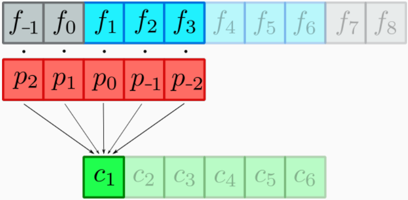


$$\begin{aligned} c_6 &= p_2 f_4 + p_1 f_5 + p_0 f_6 + p_{-1} f_7 + p_{-2} f_8 \\ &= p_2 f_4 + p_1 f_5 + p_0 f_6 \end{aligned}$$

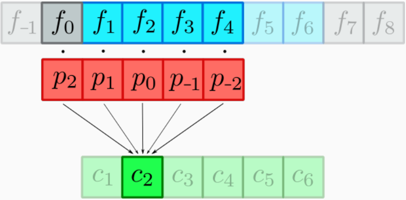
$$\begin{bmatrix} p_0 & p_{-1} & p_{-2} & 0 & 0 & 0 \\ p_1 & p_0 & p_{-1} & p_{-2} & 0 & 0 \\ p_2 & p_1 & p_0 & p_{-1} & p_{-2} & 0 \\ 0 & p_2 & p_1 & p_0 & p_{-1} & p_{-2} \\ 0 & 0 & p_2 & p_1 & p_0 & p_{-1} \\ 0 & 0 & 0 & p_2 & p_1 & p_0 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{bmatrix}$$



$$\begin{aligned} c_1 &= p_2 f_{-1} + p_1 f_0 + p_0 f_1 + p_{-1} f_2 + p_{-2} f_3 \\ &= p_2 f_5 + p_1 f_6 + p_0 f_1 + p_{-1} f_2 + p_{-2} f_3 \end{aligned}$$

$$\begin{bmatrix} p_0 & p_{-1} & p_{-2} & 0 & p_2 & p_1 \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{bmatrix} = \begin{bmatrix} c_1 \\ \\ \\ \\ \\ \\ \end{bmatrix}$$

Matrix: periodic extension



$$\begin{aligned} c_2 &= p_2 f_0 + p_1 f_1 + p_0 f_2 + p_{-1} f_3 + p_{-2} f_4 \\ &= p_2 f_6 + p_1 f_1 + p_0 f_2 + p_{-1} f_3 + p_{-2} f_4 \end{aligned}$$

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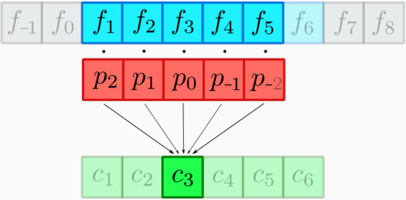

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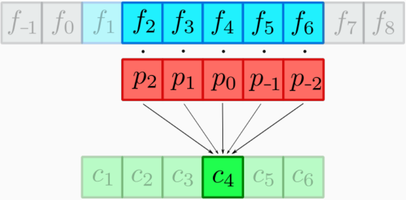

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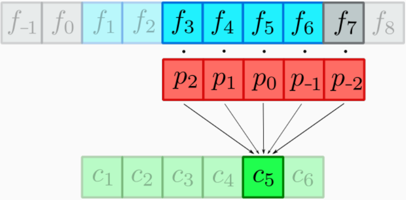
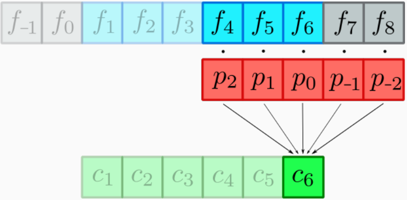


$$\begin{aligned} c_5 &= p_2 f_3 + p_1 f_4 + p_0 f_5 + p_{-1} f_6 + p_{-2} f_7 \\ &= p_2 f_3 + p_1 f_4 + p_0 f_5 + p_{-1} f_6 + p_{-2} f_1 \end{aligned}$$

$$\begin{bmatrix} p_0 & p_{-1} & p_{-2} & 0 & p_2 & p_1 \\ p_1 & p_0 & p_{-1} & p_{-2} & 0 & p_1 \\ p_2 & p_1 & p_0 & p_{-1} & p_{-2} & 0 \\ 0 & p_2 & p_1 & p_0 & p_{-1} & p_{-2} \\ p_{-2} & 0 & p_2 & p_1 & p_0 & p_{-1} \\ & & & & & \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ \\ \end{bmatrix}$$

Matrix: periodic extension



$$\begin{aligned} c_6 &= p_2 f_4 + p_1 f_5 + p_0 f_6 + p_{-1} f_7 + p_{-2} f_8 \\ &= p_2 f_4 + p_1 f_5 + p_0 f_6 + p_{-1} f_1 + p_{-2} f_2 \end{aligned}$$

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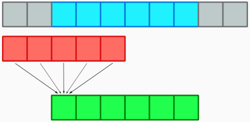

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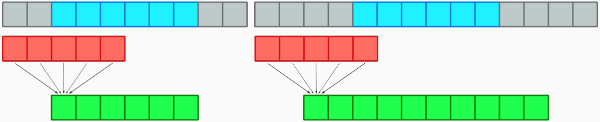

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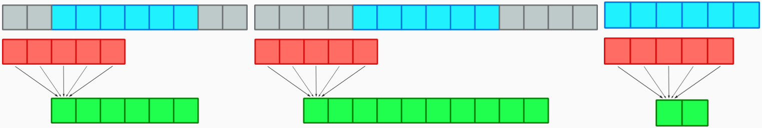

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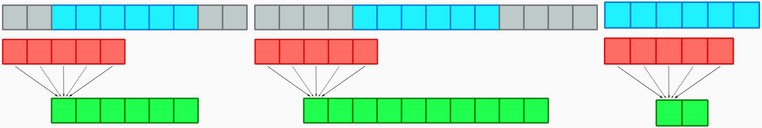

- convmtx: `A = convmtx(p,n)` returns a matrix of size $(n+m-1) \times 1$ such that `A*f = c2`, the full vector. To obtain `c1` one must extract `A = A(1+nu:n+nu,:)`.

Exercise

Example 1

1. define a vector $f \in \mathbb{R}^{100}$ such that $f_i = 1$ if $i \in \{25, \dots, 75\}$, $f_i = 0$ otherwise. Plot the signal f;
2. define a point spread function $p \in \mathbb{R}^5$ obtained by normalizing the vector $[1, 2, 3, 2, 1]'$;
3. compute $p * f$ with zero padding by the command `conv`;
4. compute $p * f$ with zero padding by the command `convmtx`;
5. compare the results of the previous two points with the original signal.
6. (Extra): create the convolution matrix by hand and compare it with the one obtained by `convmtx`. *(The following commands can be useful:* `diag` *and* `spy`*)*

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Since the mapping from f to m is linear, this is called a **linear inverse problem**. Main questions:

1. is this problem meaningful?
2. how can we solve it? (spoiler: A^{-1})

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The problem of approximating f from $m_\delta = Af + \varepsilon$ is usually referred to as **deconvolution and denoising**.

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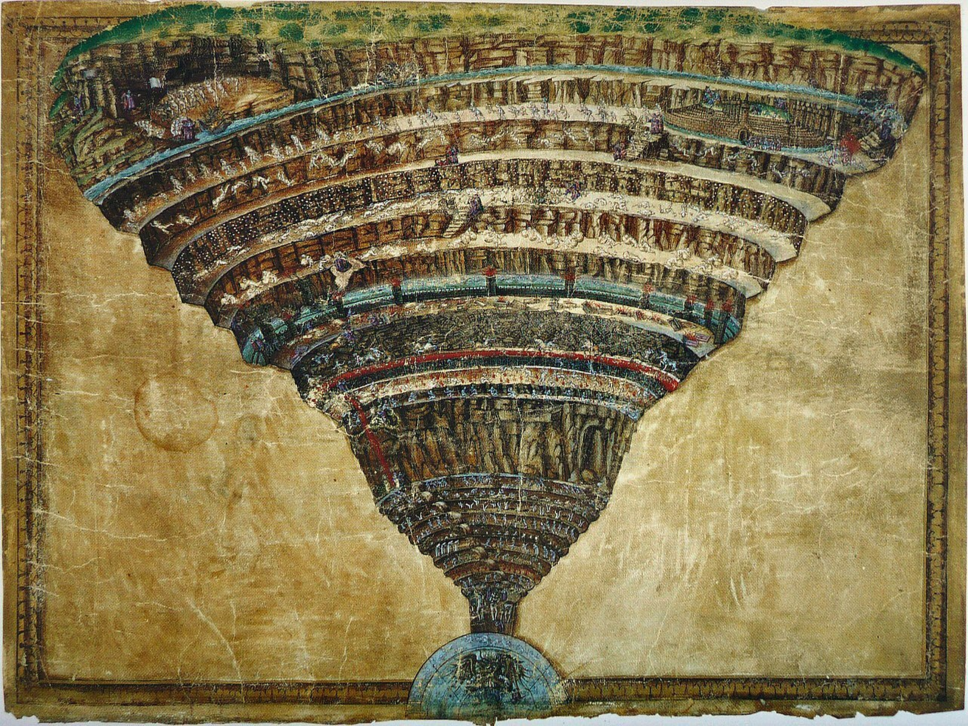
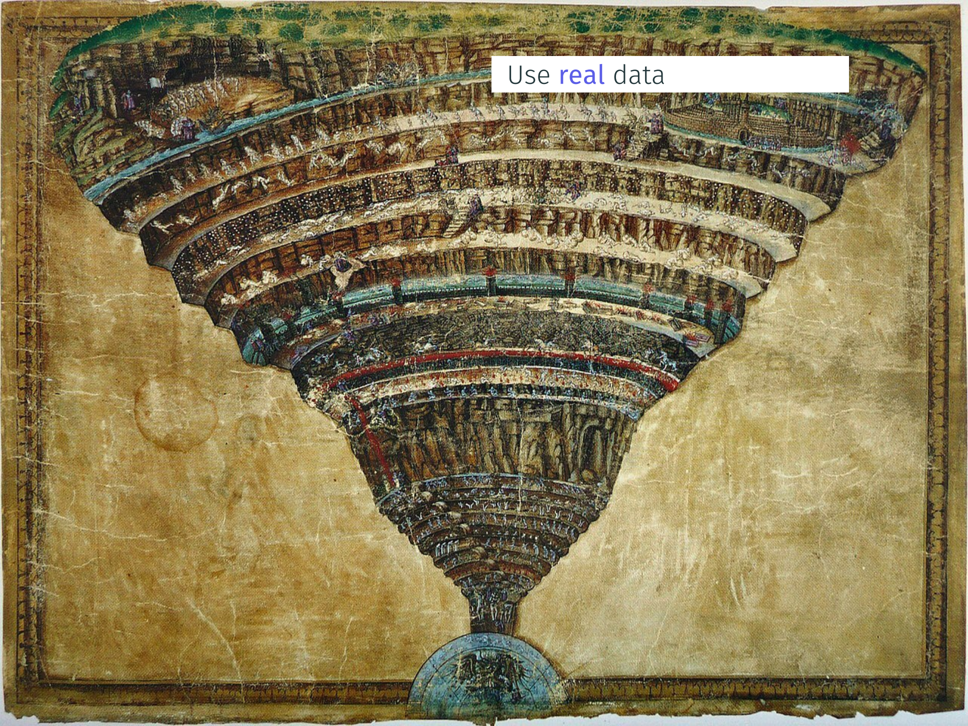
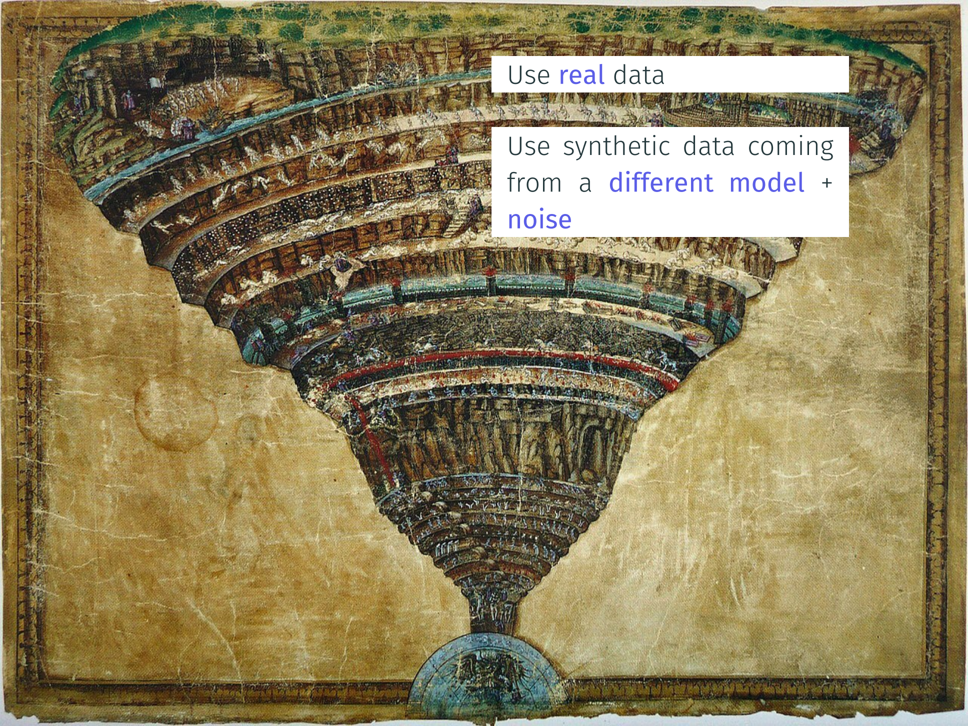
Inverse Crimes



Use real data





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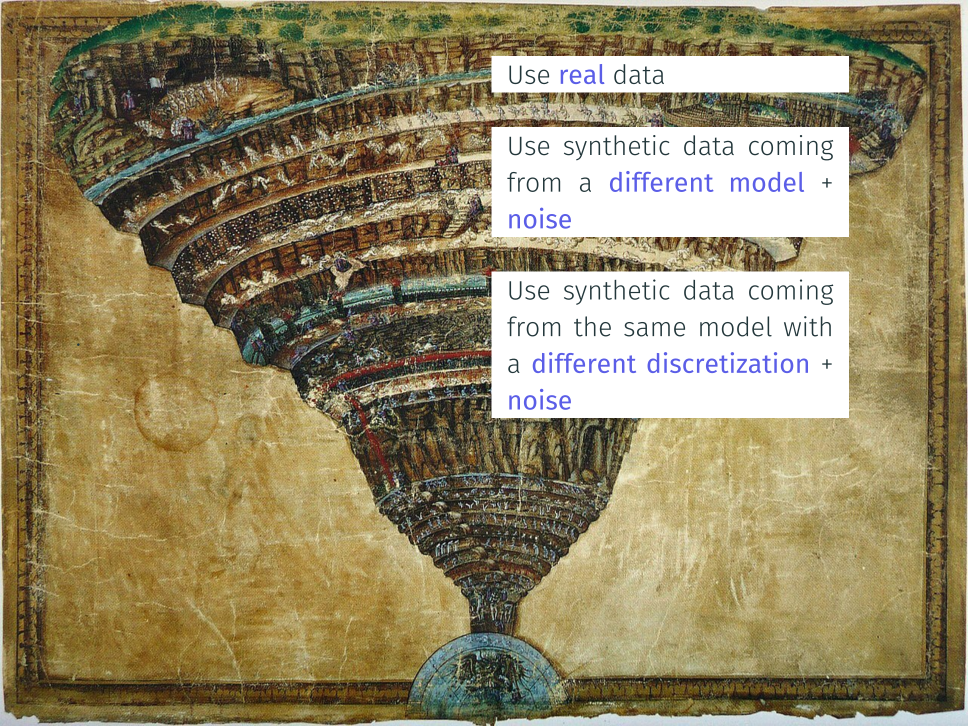
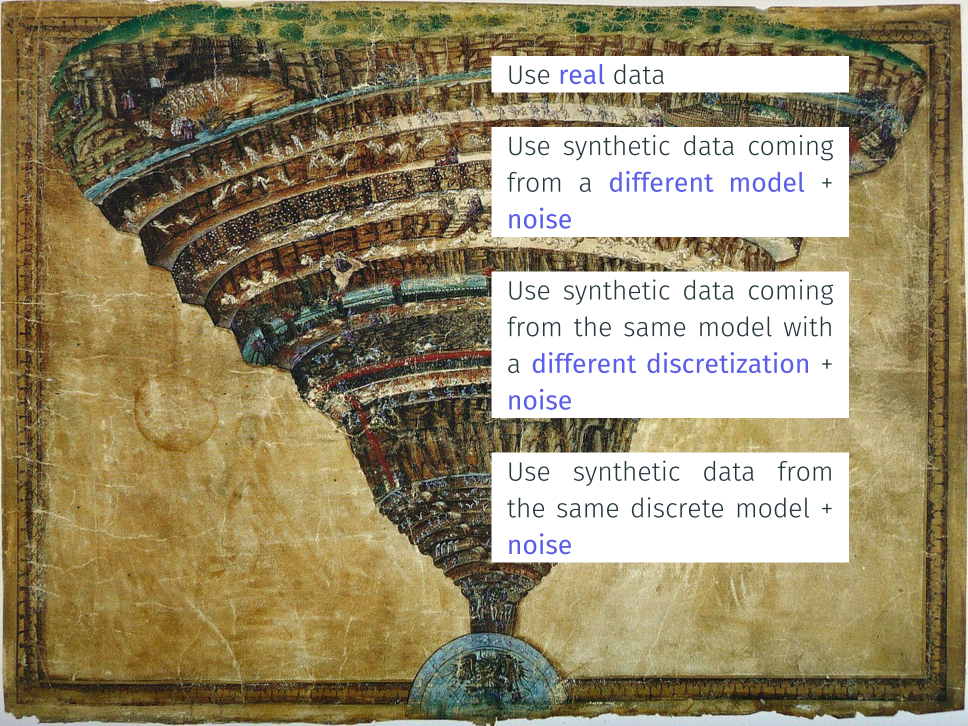


Use real data

Use synthetic data coming from a different model + noise

Use synthetic data coming from the same model with a different discretization + noise



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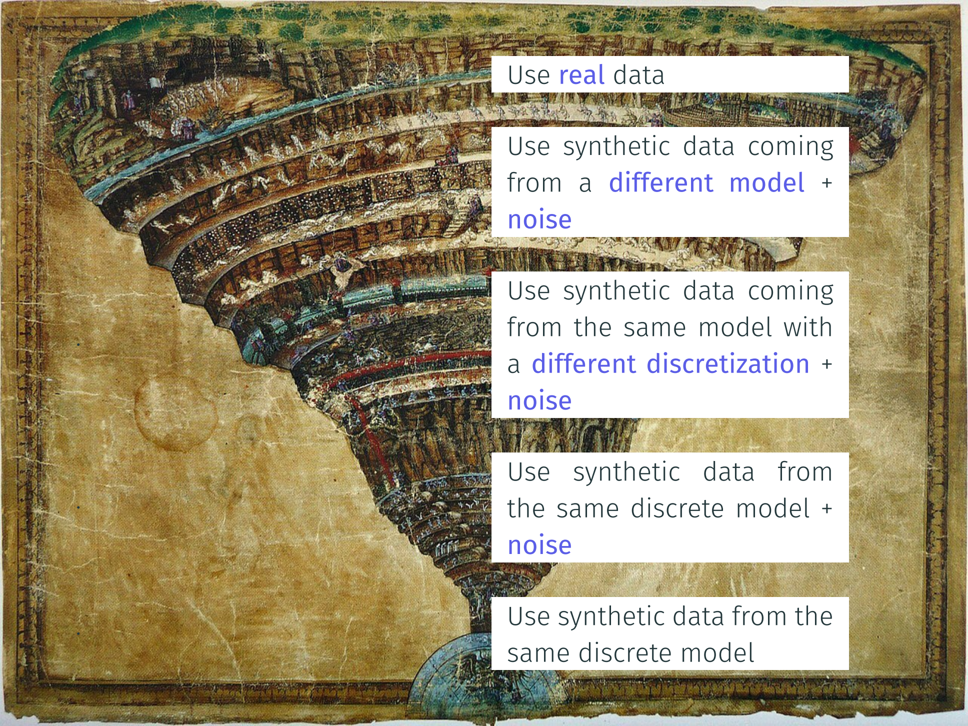
Use synthetic data from the same discrete model + noise



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Naïve inversion

How do we solve the Deconvolution problem?

Recall the deconvolution problem: given $m = Af$, find f.

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Problem 1: what if A is not invertible? How can we tell that?
Problem 2: is this solution stable and well conditioned? Remember: the real world application is a combination of **deconvolution and denoising**, i.e., recover a good approximation of f from $m_\delta = Af + \varepsilon$, $\|\varepsilon\| \leq \delta$.

Is $\widetilde{f_\delta} := A^{-1}m_\delta$ a good approximation of f?

Matlab example

Example 2

1. Consider a signal f as in Exercise 1, and a point spread function obtained from $p = [1, 1, 1, 1, 1]$, convolved with itself and normalized.
2. Compute the convolution matrix A.
3. Generate the synthetic datum $m = Af$. Compute $A^{-1}m$ and compare it to f by plotting both and by computing the norm of their difference. *(use the command \ and not* `inv`*)*
4. Generate the synthetic datum $m_\delta = Af + \varepsilon$, being ε a Gaussian vector of null mean and standard deviation $\delta = 10^{-4}$. Compute $\widetilde{f_\delta} = A^{-1}m_\delta$ and compare it to f as above.
5. Repeat the previous point with $\delta = 0.01$.
6. Repeat the previous point considering a different point spread function obtained from $p = [1, 1, 1, 1, 1]$, convolved with itself three times and normalized.

Why does naïve inversion fail?

A little computation shows that

$$\widetilde{f_\delta} = A^{-1}m_\delta = A^{-1}Af + A^{-1}\varepsilon = f + A^{-1}\varepsilon$$

This implies that

$$\|\widetilde{f_\delta} - f\| \leq \|A^{-1}\|\|\varepsilon\| \leq \|A^{-1}\|\delta.$$

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The harsh reality

The deconvolution problem is ill-posed, or at least (in this discrete context) ill-conditioned. Thus we cannot just naïvely invert A.