Inverse Problems 1: convolution and deconvolution

Lesson 12: Compressed Sensing and Dictionary Learning

Luca Ratti

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University of Helsinki

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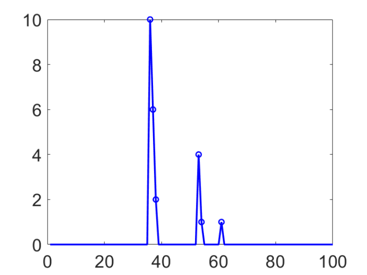
What is Compressed Sensing?

The notion of sparsity

A vector $f \in \mathbb{R}^n$ is said to be **sparse** if most of its components are vanishing.



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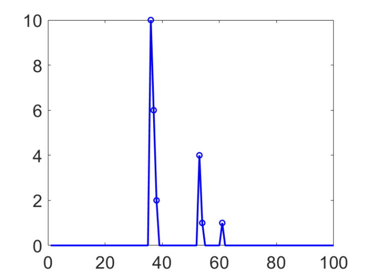

Example: a 1D signal recording the seismical activity in a certain region.

How to describe rigorously the notion of sparsity?

L^0 "norm" and sparsity

$\|f\|_0 = \#\{j : f_j \neq 0\}$. **Sparse** vector: $f \in \mathbb{R}^n$ s.t. $\|f\|_0 \ll n$.

Warning: $\|\cdot\|_0$ is not actually a norm, and is a delicate tool. Its name comes from the property $\|f\|_0 = \lim_{p\to 0} \|f\|_p^p$, but this does not imply the properties of the norm (e.g. the homogeneity).

Sparsity with respect to a dictionary

Canonical basis in $\mathbb{R}^n$: $\{e_i\}_{i=1}^n$ such that each $e_i \in \mathbb{R}^n$ has all components equal to 0, except the $i-$th one, which is equal to 1; briefly, $(e_i)_j = \delta_{ij}$.

It holds $f \in \mathbb{R}^n \Rightarrow f = \sum_{i=1}^{n} f_i \mathrm{e_i}$.

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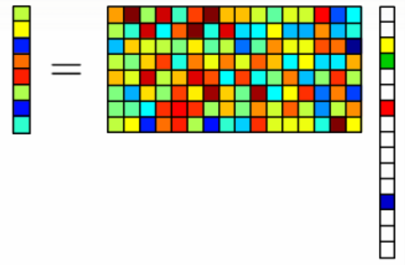
Sparsity w.r.t. a dictionary

Consider a matrix $D \in \mathbb{R}^{n \times m}$, and take $f \in \mathbb{R}^n$, $w \in \mathbb{R}^m$ s.t. $f = Dw$. This is equivalent to $f = \sum_{i=1}^{m} w_i \mathrm{d_i}$, being $\mathrm{d_i}$ the columns of D.
A vector $f \in range(D)$ is said to be sparse with respect to D, or to admit a sparse representation in D if $\|w\|_0 \ll n$.



Compressed Sensing: alternative formulations

Original definition of compressed sensing:

$$\text{find } w^* = \arg\min_{w \in \mathbb{R}^m} \left\{ \|w\|_0; \quad w \text{ s.t. } \|Dw - f\|_2^2 \leq \delta \right\}$$

Interpretation: find the sparsest representation of f with respect to D.

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This is the Morozov formulation of a **Variational Regularization problem** with discrepancy term $\mathcal{D}(w, f)$ and regularization term $\mathcal{R}(w)$:

$$\textit{(Morozov)} \quad w_M^* = \underset{w \in \mathbb{R}^m}{\arg\min} \{\mathcal{R}(w); \quad w \text{ s.t. } \mathcal{D}(w, f) \leq \delta\}$$

$$\textit{(Ivanov)} \quad w_I^* = \underset{w \in \mathbb{R}^m}{\arg\min} \{\mathcal{D}(w, f); \quad w \text{ s.t. } \mathcal{R}(w) \leq \epsilon\}$$

$$\textit{(Tikhonov)} \quad w_T^* = \underset{w \in \mathbb{R}^m}{\arg\min} \{\mathcal{D}(w, f) + \alpha \mathcal{R}(w)\}.$$

Under certain assumptions, the different formulations are equivalent or lead to similar results. In CS, $\mathcal{D}(w, f) = \|Dw - f\|_2^2$ and $\mathcal{R}(w) = \|w\|_0$.

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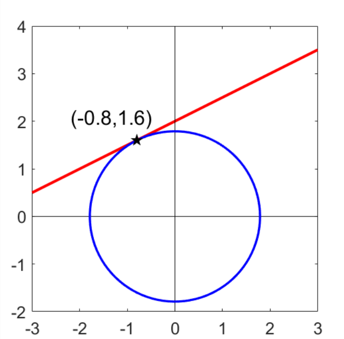
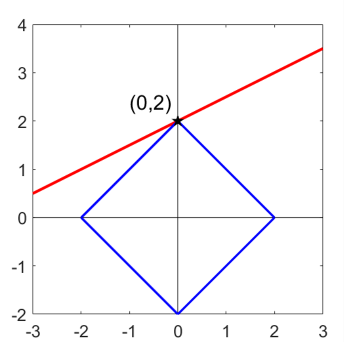
Under suitable conditions on $\mathcal{D}$, a sparsity promoting regularization is obtained by minimizing $\|w\|_1 = \sum_{i=1}^{m} |w_i|$.





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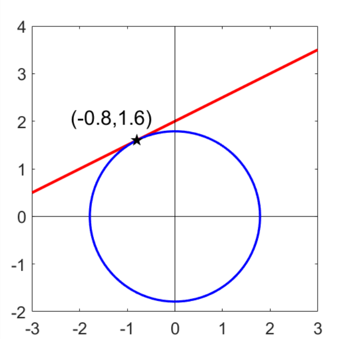

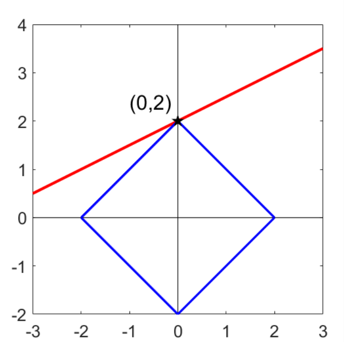

In CS, such conditions are connected to the **Restricted Isometry Property** and to the **incoherence** of the columns of D.

A particular case

Compressed Sensing: Tikhonov formulation with L^1 norm

$$w^* = \underset{w \in \mathbb{R}^n}{\arg\min} \left\{ \|Dw - f\|_2^2 + \alpha \|w\|_1 \right\}$$

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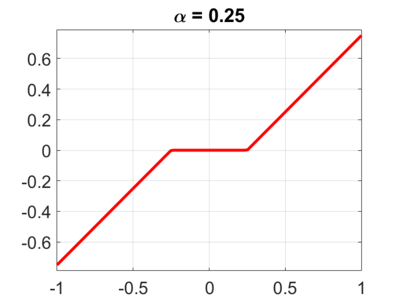
Since f is not differentiable, such problem cannot be solved by (just) computing a derivative. From basic convex analysis, we get

$$w_i = S_\alpha(y_i), \qquad S_\alpha\text{: Soft Thresholding operator}$$

The Soft Thresholding operator

The expression of $S_\alpha(x)$ is as follows:

$$S_\alpha(x) = \begin{cases} x + \alpha & \text{if } x \leq -\alpha \\ 0 & \text{if } |x| < \alpha \\ x - \alpha & \text{if } x \geq \alpha \end{cases}$$



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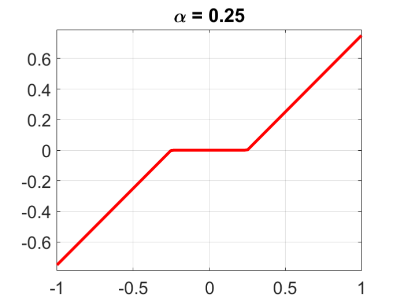

In conlcusion, the solution of

$$w^* = \arg\min_{w \in \mathbb{R}^n} \left\{ \|Dw - f\|_2^2 + \alpha \|w\|_1 \right\}$$

in case D is an orthogonal matrix is given by

$$w^* = \mathrm{S}_\alpha(D^T f),$$

where $(\mathrm{S}_\alpha(g))_i = S_\alpha(g_i)$ is the soft thresholding of each component of g.

Sparsity promoting regularization for the deconvolution problem

Sparsity in deconvolution

Consider the deconvolution problem, or any linear inverse problem:

$$\text{given } m_\delta = Af + \varepsilon, \text{ being } \|\varepsilon\| \leq \delta, \text{ determine } f.$$

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- **data fidelity** $\|Af - m_\delta\|_2$ is small;
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We are mainly focusing on two cases:

1. D is an **orthogonal basis**: $m = n$, $D^T D = DD^T = I$.
2. D is a **tight frame**: $m > n$, $DD^T = I$

Case I: D is an orthonormal base

Examples: $D = I$ (canonical basis), Haar wavelet basis, orhtogonal polinomials, Fourier goniometric polynomials, ...

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Two possible formulations:

Sparse deconvolution - synthesis formulation

$$w^* = \arg\min_{w \in \mathbb{R}^m} \{ \|ADw - m_\delta\|_2^2 + \alpha \|w\|_1 \}$$

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Result: if D is orthogonal, the sparsest solution f^* can be obtained by solving the analysis problem or, equivalently, by finding its weights w^* (solution of the synthesis) and computing $f^* = Dw^*$.

Case II: D is a tight frame

Description: $D \in \mathbb{R}^{n \times m}$, $m > n$, $DD^T = I$. This implies that $\forall f \in \mathbb{R}^n$ $\exists w \in \mathbb{R}^m$ s.t. $Dw = f$. Warning: this representation is in general not unique.

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In this case, synthesis and analysis do not coincide:

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It is possible to prove that, in general, $f_A^* \neq f_S^*$. What is the best one? It depends on the problem.

Iterative Soft Thresholding Algorithm

ISTA - introduction

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Main idea: if w^* is the solution of the synthesis problem,

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Then, w^* solves the fixed point equation:

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ISTA - Synthesis problem

$$\begin{cases} \text{select } w^{(0)} \in \mathbb{R}^m \\ \text{update } w^{(k+1)} = \mathsf{S}_\alpha(w^{(k)} + D^T A^T m_\delta - D^T A^T ADw^{(k)}) \end{cases}$$

ISTA - continuation

Result: under suitable conditions on A and D, for any choice of $w^{(0)}$, the sequence $w^{(k)}$ converges to w^*. ([D,D,D, 2004]).

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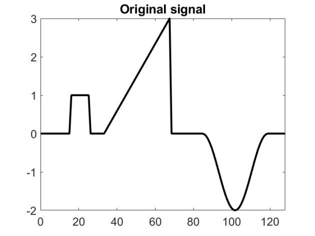
It is an easy exercise to check that, if D is orthogonal, the iterations of ISTA for the analysis and synthesis problems coincide: if $f^{(0)} = Dw^{(0)}$, then $f^{(k)} = Dw^{(k)}$ for every k. As a consequence, $f_A^* = Dw^* = f_S^*$, as we already knew.

Implementation

Example 1

1. Consider the same signal f as in Lecture 7. Let $n = 128$. Create the PSF by triple autoconvolution of [111]. Add a Gaussian noise, $\delta = 0.2$.



2. Use ISTA (1000 iterations) to find $f_1 = \arg\min\{\|Af - m\|_2^2 + \alpha\|f\|_1\}$
3. Use ISTA (1000 iterations) to find $f_2 = \arg\min\{\|Af - m\|_2^2 + \alpha\|H^T f\|_1\}$, being H the matrix of the Haar wavelet transform (an orthogonal matrix that can be obtained by the provided function `haarmtx(n,levels)`)
4. Combine the canonical basis and H to form a tight frame D. Compute f_S^* and f_A^* performing 1000 iterations of the two formulations of ISTA.

An example from a recent paper

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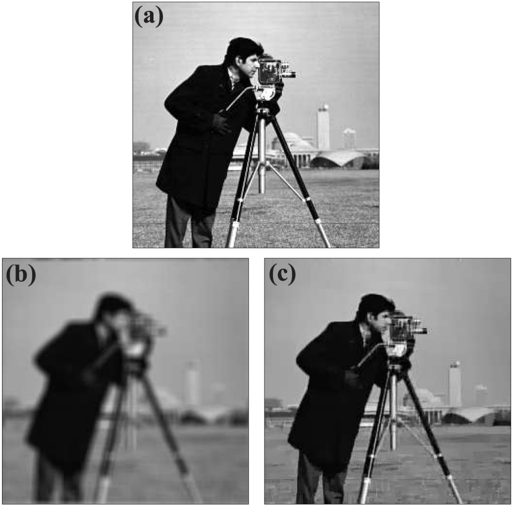
(a) original picture

(b) blurred version (no noise!)

(c) deblurring by ISTA with a tight frame (redundant Haar wavelets)



Dictionary Learning

A more challenging problem

Compressed Sensing (warning: no longer deconvolution!)

Unknown: $w \in \mathbb{R}^m$; Datum: $f = Dw + \varepsilon$;

A priori knowledge: $\|w\|_1 \ll n$, i.e., f is sparse with respect to a dictionary D, which we know.

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Dictionary learning - Morozov formulation

Data: $\{f_j\}_{j=1}^N \subset \mathbb{R}^n$;

Unknowns: $\{w_j\}_{j=1}^N \subset \mathbb{R}^m$, $D \in \mathbb{R}^{n \times m}$

$$\min_{D, w_1, \dots, w_N} \left\{ \sum_{j=1}^N \|w_j\|_1 \quad s.t. \quad \|Dw_j - f_j\|_2^2 \leq \delta \quad j = 1, \dots, N \right\}$$

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Dictionary learning - (Tikhonov) matricial formulation

$$\min_{(D,W)\in\mathbb{R}^{n\times m}\times\mathbb{R}^{m\times N}} \{\|DW - F\|_F^2 + \alpha\|W\|_{1,1}\}$$

where we used the following matricial norms:

- the Frobenius norm $\|\cdot\|_F$: $\|A\|_F^2 = \sum_{i=1}^n \sum_{j=1}^m A_{i,j}^2$
- the column-wise 1 norm $\|\cdot\|_{1,1}$: $\|A\|_{1,1} = \sum_{j=1}^m \|A_j\|_1$.

Alternative formulation

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Such function of D and W is convex with respect to each of the two variables, but not jointly convex.

The K-SVD algorithm

Input: training data F, initial guess $D^{(0)}$ for the dictionary, $i = 1$.
Iteration: while stopping criterion not satisfied do

1. update the weights W according to $D^{(i-1)}$, i.e.:
$$\text{find } W^{(i)} = \underset{W \in \mathbb{R}^{m \times N}}{\arg\min} \{ \|F - D^{(i-1)} W\|_F^2 + \alpha \|W\|_{1,1} \};$$
this problem can be decoupled in N compressed sensing problems.
2. update D (and again W), preserving the sparsity of W.
This is done each column at a time. The contribution of each column D_j to the term $\|DW - F\|_F^2$ is represented by a rank one matrix $D_j W^j$, where W^j is the $j-$th row of W. Such contribution can be optimized by means of a singular value decomposition, updating D_j and W^j. In order to preserve sparsity, we look for an update restricted to the actual support of W^j.

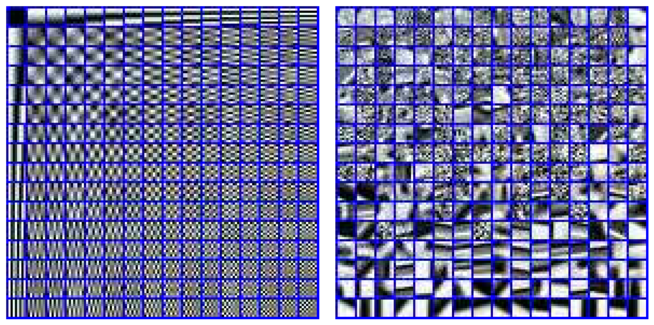
All the details can be found in [Chen, Needell, 2014].

Two examples from recent papers

A very clever idea to apply DL in image denoising is the local patch processing algorithm:

- divide the noisy picture in several smaller patches (e.g.: original picture = 256x256, divide it in 256 patches of size 8x8
- use the 8x8 patches as a training set to learn a dictionary which will allow for the sparse reconstructions of (8x8) pictures
- use K-SVD for the combined learning of the dictionary and of the sparse representation of each patch
- collect all the final patches into a denoised version of the picture

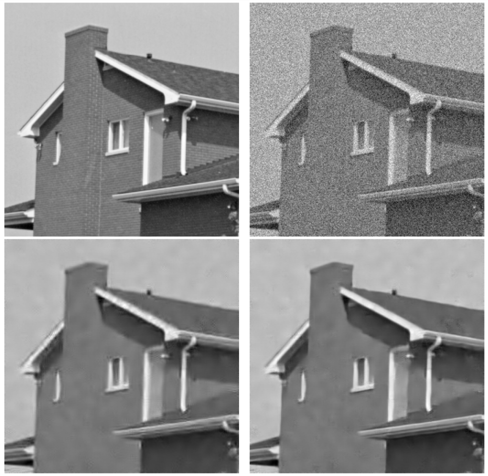
Two examples from recent papers



The initial guess $D^{(0)}$ for the dictionary - The learn dictionary at the last iteration

(pictures from [Elad, Figueiredo, Ma, 2010])

Two examples from recent papers



Original picture - Noisy version
Deblurring with initial dictionary - Deblurring with learnt dictionary
(pictures from [Elad, Figueiredo, Ma, 2010])

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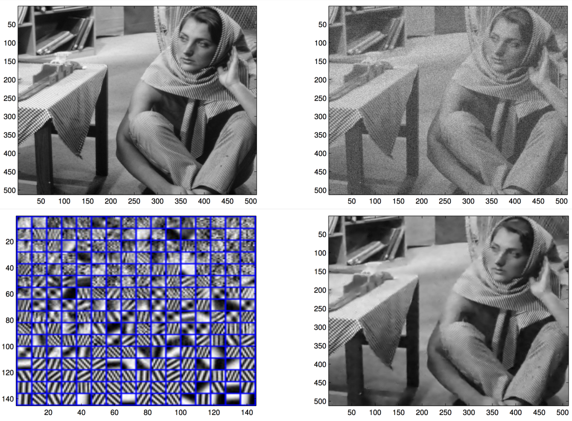

Original picture - Noisy version
Learnt dictionary - Deblurring with learnt dictionary
(pictures from [Chen, Needell, 2014])

What is the future of Compressed Sensing?

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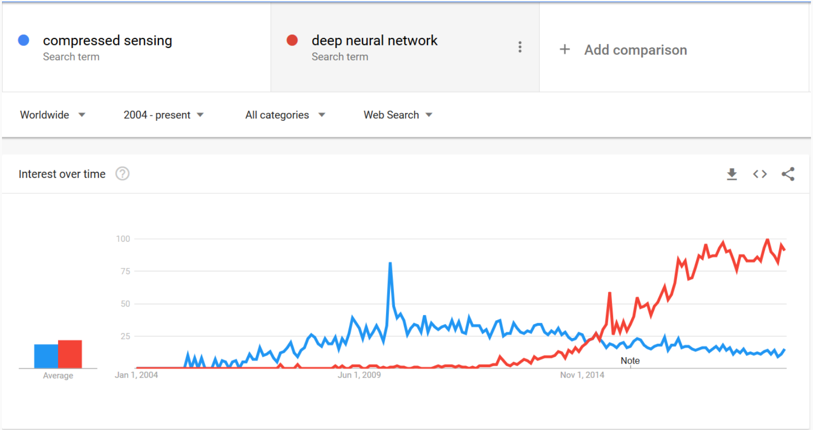

`https://trends.google.com/trends/explore?date=all&q=compressed%20sensing,deep%20neural%20network`