# Inverse Problems 1: Convolution and Deconvolution

## Lesson 6: Regularization and TSVD

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# Outline

Review

Regularized Inversion

Regularisation parameter

Truncated SVD

Examples

# Last Class

- Let $\boldsymbol{A}$ be a $k \times n$ matrix. Let the SVD of matrix $\boldsymbol{A}$ be denoted by $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{D}\boldsymbol{V}^T$ Then the pseudoinverse (or Moore-Penrose inverse) of $\boldsymbol{A}$ is denoted by $\boldsymbol{A}^+$ is defined by

$$\boldsymbol{A}^+\boldsymbol{m} = \boldsymbol{V}\boldsymbol{D}^+\boldsymbol{U}^T\boldsymbol{m}$$

  where $\boldsymbol{D}^+ = \mathrm{diag}\left(\frac{1}{d_1}, \frac{1}{d_2}, \ldots, \frac{1}{d_r}, 0, 0, \ldots, 0\right)$

- A vector $\mathcal{L}(\boldsymbol{m}) \in \mathbb{R}^n$ is called a least squares solution of the equation $\boldsymbol{A}\boldsymbol{f} = \boldsymbol{m}$ if

$$\|\boldsymbol{A}\mathcal{L}(\boldsymbol{m}) - \boldsymbol{m}\| = \min_{\boldsymbol{z} \in \mathbb{R}^n} \|(\boldsymbol{A}\boldsymbol{z} - \boldsymbol{m})\|$$

- $\mathcal{L}(\boldsymbol{m})$ is called the mimimum norm solution if

$$\|\mathcal{L}(\boldsymbol{m})\| = \inf\{\|\boldsymbol{z}\| : \boldsymbol{z} \quad \text{is a least-squares solution of} \quad \boldsymbol{A}\boldsymbol{f} = \boldsymbol{m}\}$$

# Outline

# Two kinds of Inverse Problems

- Let $m = \mathcal{A}f + \epsilon$. Given $m$ and $\delta > 0$ with $\|m - \mathcal{A}f\|_Y \leq \delta$, recover $f$ approximately.
  - We need to design a computational method that would map $m = \mathcal{A}f + \epsilon$ in a neighbourhood of $\mathcal{A}f$ of radius $\delta$ to some point in $X$ near $f$.
  - The naïve approach works for well-posed inverse problems.
  - But for ill-posed problems, even if $\mathcal{A}$ is bijective, the inverse $A^{-1}$ may not be continuous.
- Consider the restricted Inverse problem.
- Let $m = \mathcal{A}f + \epsilon$. Given $m$ and $\delta > 0$ with $\|m - \mathcal{A}f\|_Y \leq \delta$, extract any information about $f$
  - To look for the location of the inclusion in the known background material.

# Outline

# Regularization parameter

- Assume $\mathrm{Ker}(\mathcal{A}) = 0$.

**Definition**

Let $X$ and $Y$ be Hilbert spaces.Let $\mathcal{A} : X \to Y$ be an injective bounded linear operator. Consider the measurement $m = \mathcal{A} f + \epsilon$. A family of linear maps $\mathcal{R}_\alpha : Y \to X$ parametrized by $0 < \alpha < \infty$ is called a regularization strategy if

$$\lim_{\alpha \to 0} \mathcal{R}_\alpha \mathcal{A} f = f$$

for every $f \in X$

# Admissible choice of regularization of parameter

**Definition**

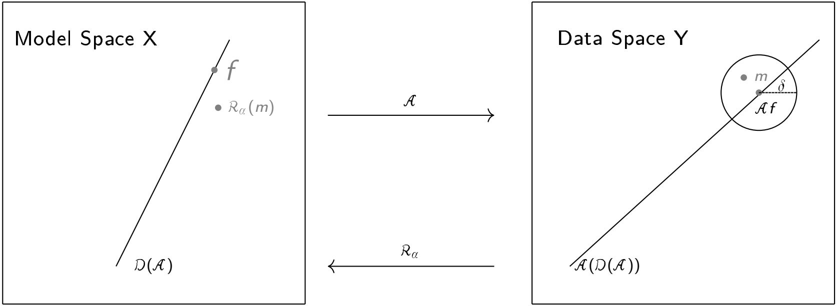
Assume we are given noise level $\delta > 0$ so that $\|m - \mathcal{A}f\|_Y \leq \delta$. A choice of regularization parameter $\alpha = \alpha(\delta)$ as a function of $\delta$ is called admissible if

$$\alpha(\delta) \to 0 \quad \text{as} \quad \delta \to 0,$$

and

$$\sup_m \{\|\mathcal{R}_{\alpha(\delta)} m - f\| : \|\mathcal{A}f - m\| \leq \delta\} \to 0 \quad \text{as} \quad \delta \to 0 \quad \text{for every} \quad f \in X$$

# Schematic representation of the regularization parameter



# Outline

# Basic Regularization Technique

- The first simplest regularization technique is Truncated Singular Value Decomposition (TSVD)

## Definition

For any $\alpha > 0$ define the truncated SVD (TSVD) by $\boldsymbol{A}_\alpha^+ = \boldsymbol{V}\boldsymbol{D}_\alpha^+\boldsymbol{U}^T$, where,

$$\boldsymbol{D}_\alpha^+ = \begin{bmatrix} \frac{1}{d_1} & 0 & \dots & 0 & & \dots & 0 \\ 0 & \frac{1}{d_2} & & & & & \vdots \\ \vdots & & \ddots & & & & \\ & & & \frac{1}{d_{r_\alpha}} & & & \\ & & & & 0 & & \\ \vdots & & & & & \ddots & \vdots \\ 0 & \dots & & & & \dots & 0 \end{bmatrix}$$

and $r_\alpha = \min\{r, max\{j : 1 \le j \le \min(k, n), d_j > \alpha\}$

# Reconstruction

- The reconstruction function $\mathcal{L}_\alpha$ is defined by the formula

$$\mathcal{L}_\alpha(m) = \boldsymbol{V}\boldsymbol{D}_\alpha^+\boldsymbol{U}^T\boldsymbol{m}$$

- The Hadamard's conditions are all satisfied.
  - $\mathcal{L}_\alpha : \mathbb{R}^k \to \mathbb{R}^n$ is well-defined and is single valued.
  - The norm of the mapping $\mathcal{L}_\alpha$ is given by

$$\begin{aligned} \|\mathcal{L}_\alpha\| &= \|\boldsymbol{V}\boldsymbol{D}_\alpha^+\boldsymbol{U}^T\| \\ &\leq \|\boldsymbol{V}\|\|\boldsymbol{D}_\alpha^+\|\|\boldsymbol{U}^T\| \\ &= \|\boldsymbol{D}_\alpha^+\| = \frac{1}{d_{r_\alpha}} \end{aligned}$$

  - This implies continuity.

1. In the strict mathematical sense

$$\begin{aligned}\mathcal{L}_\alpha(\boldsymbol{m}) &= \boldsymbol{V}\boldsymbol{D}_\alpha^+\boldsymbol{U}^T(\boldsymbol{A}\boldsymbol{f}+\epsilon)\\ &= \boldsymbol{V}\boldsymbol{D}_\alpha^+\boldsymbol{D}\boldsymbol{V}^T\boldsymbol{f}+\boldsymbol{V}\boldsymbol{D}_\alpha^+\boldsymbol{U}^T\boldsymbol{\epsilon}\end{aligned}$$

2. The vector $\boldsymbol{V}\boldsymbol{D}_\alpha^+\boldsymbol{D}\boldsymbol{V}^T\boldsymbol{f}$ is an approximation to $\boldsymbol{f}$.
3. Error term estimation is given by

$$\|\boldsymbol{V}\boldsymbol{D}_\alpha^+\boldsymbol{U}^T\boldsymbol{\epsilon}\| \leq \|\boldsymbol{V}\boldsymbol{D}_\alpha^+\boldsymbol{U}^T\|\|\boldsymbol{\epsilon}\| \leq \|\boldsymbol{D}_\alpha^+\|\|\boldsymbol{\epsilon}\| = d_{r_\alpha}^{-1}\|\boldsymbol{\epsilon}\|$$

4. By the ordering of singular values

$$d_1^{-1} \leq d_2^{-1} \leq \ldots \leq d_r^{-1}$$

5. If fewer singular values are kept in the reconstruction, then the noise in the inversion is amplified less and less.

Denoting $\boldsymbol{a} := \boldsymbol{D}_\alpha^+ \boldsymbol{U}^T \boldsymbol{m}$ in the inversion formula of $\mathcal{L}_\alpha(\boldsymbol{m}) = \boldsymbol{V}\boldsymbol{D}_\alpha^+ \boldsymbol{U}^T \boldsymbol{m}$, we can see that the reconstruction is a linear combination of the matrix

$$\boldsymbol{V} = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \dots & \boldsymbol{v}_n \end{bmatrix}$$

That is

$$\mathcal{L}_\alpha(\boldsymbol{m}) = \boldsymbol{V}\boldsymbol{a} = a_1\boldsymbol{v}_1 + \ldots + a_n\boldsymbol{v}_n$$

The columns $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ are called the singular vectors.

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## Previous Example

- Consider the example of the signal

$$f = 1 \quad \text{for} \quad n = [25 : 75] \qquad f = 0 \quad \text{otherwise}$$

  (Here $n = [1 : 100]$ ).
- Consider the point spread function given by $PSF = [1, 1, 1, 1, 1]^T$ and normalise the point spread function.
- Create data for the inverse problem that involves inverse crime and with added noise.
- Now find $\boldsymbol{f}$ using naive inversion and with pseudo inverse and plot them on a graph
- Use TSVD to compute the reconstruction

# Example 1

- Construct a smooth signal $f(x) = (2\pi - x)x^3(1 - cos(x))$ for $x \in [0, 2\pi]$
- Construct a Gaussian Point Spread Function. This can be done by taking $PSF = [1, 1, 1]$; and convolving with itself a few (say 3 times). Normalize the point spread function.
- Construct the convolution matrix.
- Compute the SVD of matrix $\boldsymbol{A}$.
- Take a look at the matrix and semi-log plots of the singular value.
- Construct a convolved signal that includes noise.
- Do a naive inversion and plot the signal and inversion.
- Do inversion with Pseudoinverse.
- Truncate the singular values.
- Reconstruct using TSVD.