

Inverse Problems 1: Convolution and Deconvolution

Lesson 3: Ill-posedness in Inverse problems

Rashmi Murthy

University of Helsinki

September 10, 2019

Outline

- Introduction
 - What is a neural network?
 - Why use neural networks?
 - Types of neural networks
- Feedforward neural networks
 - Architecture
 - Training
 - Applications
- Convolutional neural networks
 - Architecture
 - Training
 - Applications
- Recurrent neural networks
 - Architecture
 - Training
 - Applications
- Deep learning
 - Overview
 - Challenges
 - Future research

Convolution and Deconvolution

Continuous Model

$$f : \mathbb{R} \rightarrow \mathbb{R},$$

$$p : [-\gamma, \gamma] \rightarrow \mathbb{R},$$

$$\int_{-\gamma}^{\gamma} p(x) dx = 1; p * f : \mathbb{R} \rightarrow \mathbb{R},$$

$$(p * f)(x) = \int_{-\gamma}^{\gamma} p(y) f(x - y) dy$$

Discrete Model

$$f = (f_1, \dots, f_n) \in \mathbb{R}^n,$$

$$p = (p_\nu, \dots, p_{-\nu}) \in \mathbb{R}^{2\nu+1},$$

$$\sum_{\ell=-\nu}^{\nu} p_\ell = 1; p * f \in \mathbb{R}^n,$$

$$(p * f)_i = \sum_{\ell=-\nu}^{\nu} p_\ell f_{i-\ell}$$

Outline

Forward Map

- The core of an inverse problem is the forward map $\mathcal{A} : \mathcal{D}(\mathcal{A}) \rightarrow Y$.
- $\mathcal{D}(\mathcal{A}) \subset X$ and Y are suitable Hilbert spaces called model space and data space.
- $\mathcal{D}(\mathcal{A}) \subset X$ is the domain of definition of the bounded linear operator \mathcal{A}
- Constructing the model space and data space and the forward map is not a trivial task.
- Physical process, Technical properties of the measurement device, geometry of the measurement and possible limitations in the data set must be considered.

Hadamard's condition for a well-posed problem

A problem is well-posed if the following conditions hold:

- The solution exists
- The solution is unique
- The solution depends continuously on the data.

Ill-posed inverse problem:

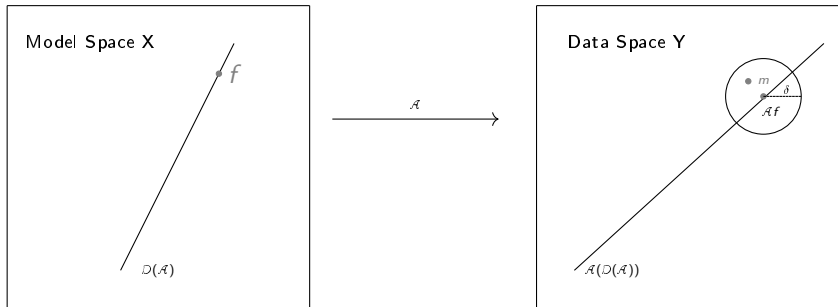
Given any noisy data $m = Af + \epsilon$,
recover f



Well posed and Ill-posed problems

1. If the forward map $\mathcal{A} : X \rightarrow Y$ is bijective and allows a continuous inverse \mathcal{A}^{-1} , then Naive inversion satisfies all the Hadamard's condition and we are dealing with a well-posed problem.
2. Ill-posed inverse problem will have one of the Hadamard conditions not satisfied.
 - If the noisy data does not belong to the range of the forward map, then the first Hadamard condition is violated.
 - If two quantities in model space $f, g \in \mathcal{X}$ give the same measurement, that is $Af = Ag$, then Hadamard's second condition is not satisfied.
 - If the forward map does not allow for a continuous inverse, then Hadamard's third condition is violated.
 - The solution does not depend continuously on the data.

Illustration of the linear forward map



Outline

- 1. Introduction
 - 1.1. Overview of the course
 - 1.2. Learning objectives
 - 1.3. Course structure
- 2. Fundamentals of Quantum Mechanics
 - 2.1. Wave-particle duality
 - 2.2. The Schrödinger equation
 - 2.3. Quantum states and wave functions
 - 2.4. Uncertainty principle
 - 2.5. Quantum tunneling
- 3. Quantum Mechanics in One Dimension
 - 3.1. Free particle
 - 3.2. Particle in a box
 - 3.3. Harmonic oscillator
 - 3.4. Potential barriers and wells
- 4. Angular Momentum and Spin
 - 4.1. Orbital angular momentum
 - 4.2. Spin angular momentum
 - 4.3. Addition of angular momenta
- 5. Three-Dimensional Quantum Mechanics
 - 5.1. Free particle in 3D
 - 5.2. Particle in a 3D box
 - 5.3. Central potentials
- 6. Perturbation Theory
 - 6.1. Time-independent perturbation theory
 - 6.2. Time-dependent perturbation theory
- 7. Scattering Theory
 - 7.1. Scattering cross-section
 - 7.2. Partial wave analysis
- 8. Quantum Mechanics and Relativity
 - 8.1. Dirac equation
 - 8.2. Relativistic quantum mechanics
- 9. Quantum Entanglement and Nonlocality
 - 9.1. Bell's theorem
 - 9.2. Quantum entanglement
- 10. Quantum Information and Quantum Computing
 - 10.1. Quantum bits (qubits)
 - 10.2. Quantum gates and circuits
 - 10.3. Quantum entanglement in quantum computing
- 11. Quantum Optics and Quantum Electrodynamics
 - 11.1. Coherent states
 - 11.2. Quantum electrodynamics (QED)
- 12. Quantum Mechanics in Condensed Matter
 - 12.1. Band structure
 - 12.2. Semiconductors
 - 12.3. Superconductivity
- 13. Quantum Mechanics in Atomic and Molecular Physics
 - 13.1. Atomic spectra
 - 13.2. Molecular spectra
- 14. Quantum Mechanics in Nuclear Physics
 - 14.1. Nuclear structure
 - 14.2. Nuclear reactions
- 15. Quantum Mechanics in Particle Physics
 - 15.1. Elementary particles
 - 15.2. Quantum field theory
- 16. Quantum Mechanics in Astrophysics and Cosmology
 - 16.1. Black holes
 - 16.2. Quantum cosmology
- 17. Quantum Mechanics in Quantum Gravity
 - 17.1. General relativity
 - 17.2. Quantum gravity
- 18. Quantum Mechanics in Quantum Gravity
 - 18.1. String theory
 - 18.2. Loop quantum gravity
- 19. Quantum Mechanics in Quantum Gravity
 - 19.1. Quantum gravity
 - 19.2. Quantum gravity
- 20. Quantum Mechanics in Quantum Gravity
 - 20.1. Quantum gravity
 - 20.2. Quantum gravity
- 21. Quantum Mechanics in Quantum Gravity
 - 21.1. Quantum gravity
 - 21.2. Quantum gravity
- 22. Quantum Mechanics in Quantum Gravity
 - 22.1. Quantum gravity
 - 22.2. Quantum gravity
- 23. Quantum Mechanics in Quantum Gravity
 - 23.1. Quantum gravity
 - 23.2. Quantum gravity
- 24. Quantum Mechanics in Quantum Gravity
 - 24.1. Quantum gravity
 - 24.2. Quantum gravity
- 25. Quantum Mechanics in Quantum Gravity
 - 25.1. Quantum gravity
 - 25.2. Quantum gravity
- 26. Quantum Mechanics in Quantum Gravity
 - 26.1. Quantum gravity
 - 26.2. Quantum gravity
- 27. Quantum Mechanics in Quantum Gravity
 - 27.1. Quantum gravity
 - 27.2. Quantum gravity
- 28. Quantum Mechanics in Quantum Gravity
 - 28.1. Quantum gravity
 - 28.2. Quantum gravity
- 29. Quantum Mechanics in Quantum Gravity
 - 29.1. Quantum gravity
 - 29.2. Quantum gravity
- 30. Quantum Mechanics in Quantum Gravity
 - 30.1. Quantum gravity
 - 30.2. Quantum gravity
- 31. Quantum Mechanics in Quantum Gravity
 - 31.1. Quantum gravity
 - 31.2. Quantum gravity
- 32. Quantum Mechanics in Quantum Gravity
 - 32.1. Quantum gravity
 - 32.2. Quantum gravity
- 33. Quantum Mechanics in Quantum Gravity
 - 33.1. Quantum gravity
 - 33.2. Quantum gravity
- 34. Quantum Mechanics in Quantum Gravity
 - 34.1. Quantum gravity
 - 34.2. Quantum gravity
- 35. Quantum Mechanics in Quantum Gravity
 - 35.1. Quantum gravity
 - 35.2. Quantum gravity
- 36. Quantum Mechanics in Quantum Gravity
 - 36.1. Quantum gravity
 - 36.2. Quantum gravity
- 37. Quantum Mechanics in Quantum Gravity
 - 37.1. Quantum gravity
 - 37.2. Quantum gravity
- 38. Quantum Mechanics in Quantum Gravity
 - 38.1. Quantum gravity
 - 38.2. Quantum gravity
- 39. Quantum Mechanics in Quantum Gravity
 - 39.1. Quantum gravity
 - 39.2. Quantum gravity
- 40. Quantum Mechanics in Quantum Gravity
 - 40.1. Quantum gravity
 - 40.2. Quantum gravity
- 41. Quantum Mechanics in Quantum Gravity
 - 41.1. Quantum gravity
 - 41.2. Quantum gravity
- 42. Quantum Mechanics in Quantum Gravity
 - 42.1. Quantum gravity
 - 42.2. Quantum gravity
- 43. Quantum Mechanics in Quantum Gravity
 - 43.1. Quantum gravity
 - 43.2. Quantum gravity
- 44. Quantum Mechanics in Quantum Gravity
 - 44.1. Quantum gravity
 - 44.2. Quantum gravity
- 45. Quantum Mechanics in Quantum Gravity
 - 45.1. Quantum gravity
 - 45.2. Quantum gravity
- 46. Quantum Mechanics in Quantum Gravity
 - 46.1. Quantum gravity
 - 46.2. Quantum gravity
- 47. Quantum Mechanics in Quantum Gravity
 - 47.1. Quantum gravity
 - 47.2. Quantum gravity
- 48. Quantum Mechanics in Quantum Gravity
 - 48.1. Quantum gravity
 - 48.2. Quantum gravity
- 49. Quantum Mechanics in Quantum Gravity
 - 49.1. Quantum gravity
 - 49.2. Quantum gravity
- 50. Quantum Mechanics in Quantum Gravity
 - 50.1. Quantum gravity
 - 50.2. Quantum gravity
- 51. Quantum Mechanics in Quantum Gravity
 - 51.1. Quantum gravity
 - 51.2. Quantum gravity
- 52. Quantum Mechanics in Quantum Gravity
 - 52.1. Quantum gravity
 - 52.2. Quantum gravity
- 53. Quantum Mechanics in Quantum Gravity
 - 53.1. Quantum gravity
 - 53.2. Quantum gravity
- 54. Quantum Mechanics in Quantum Gravity
 - 54.1. Quantum gravity
 - 54.2. Quantum gravity
- 55. Quantum Mechanics in Quantum Gravity
 - 55.1. Quantum gravity
 - 55.2. Quantum gravity
- 56. Quantum Mechanics in Quantum Gravity
 - 56.1. Quantum gravity
 - 56.2. Quantum gravity
- 57. Quantum Mechanics in Quantum Gravity
 - 57.1. Quantum gravity
 - 57.2. Quantum gravity
- 58. Quantum Mechanics in Quantum Gravity
 - 58.1. Quantum gravity
 - 58.2. Quantum gravity
- 59. Quantum Mechanics in Quantum Gravity
 - 59.1. Quantum gravity
 - 59.2. Quantum gravity
- 60. Quantum Mechanics in Quantum Gravity
 - 60.1. Quantum gravity
 - 60.2. Quantum gravity
- 61. Quantum Mechanics in Quantum Gravity
 - 61.1. Quantum gravity
 - 61.2. Quantum gravity
- 62. Quantum Mechanics in Quantum Gravity
 - 62.1. Quantum gravity
 - 62.2. Quantum gravity
- 63. Quantum Mechanics in Quantum Gravity
 - 63.1. Quantum gravity
 - 63.2. Quantum gravity
- 64. Quantum Mechanics in Quantum Gravity
 - 64.1. Quantum gravity
 - 64.2. Quantum gravity
- 65. Quantum Mechanics in Quantum Gravity
 - 65.1. Quantum gravity
 - 65.2. Quantum gravity
- 66. Quantum Mechanics in Quantum Gravity
 - 66.1. Quantum gravity
 - 66.2. Quantum gravity
- 67. Quantum Mechanics in Quantum Gravity
 - 67.1. Quantum gravity
 - 67.2. Quantum gravity
- 68. Quantum Mechanics in Quantum Gravity
 - 68.1. Quantum gravity
 - 68.2. Quantum gravity
- 69. Quantum Mechanics in Quantum Gravity
 - 69.1. Quantum gravity
 - 69.2. Quantum gravity
- 70. Quantum Mechanics in Quantum Gravity
 - 70.1. Quantum gravity
 - 70.2. Quantum gravity
- 71. Quantum Mechanics in Quantum Gravity
 - 71.1. Quantum gravity
 - 71.2. Quantum gravity
- 72. Quantum Mechanics in Quantum Gravity
 - 72.1. Quantum gravity
 - 72.2. Quantum gravity
- 73. Quantum Mechanics in Quantum Gravity
 - 73.1. Quantum gravity
 - 73.2. Quantum gravity
- 74. Quantum Mechanics in Quantum Gravity
 - 74.1. Quantum gravity
 - 74.2. Quantum gravity
- 75. Quantum Mechanics in Quantum Gravity
 - 75.1. Quantum gravity
 - 75.2. Quantum gravity
- 76. Quantum Mechanics in Quantum Gravity
 - 76.1. Quantum gravity
 - 76.2. Quantum gravity
- 77. Quantum Mechanics in Quantum Gravity
 - 77.1. Quantum gravity
 - 77.2. Quantum gravity
- 78. Quantum Mechanics in Quantum Gravity
 - 78.1. Quantum gravity
 - 78.2. Quantum gravity
- 79. Quantum Mechanics in Quantum Gravity
 - 79.1. Quantum gravity
 - 79.2. Quantum gravity
- 80. Quantum Mechanics in Quantum Gravity
 - 80.1. Quantum gravity
 - 80.2. Quantum gravity
- 81. Quantum Mechanics in Quantum Gravity
 - 81.1. Quantum gravity
 - 81.2. Quantum gravity
- 82. Quantum Mechanics in Quantum Gravity
 - 82.1. Quantum gravity
 - 82.2. Quantum gravity
- 83. Quantum Mechanics in Quantum Gravity
 - 83.1. Quantum gravity
 - 83.2. Quantum gravity
- 84. Quantum Mechanics in Quantum Gravity
 - 84.1. Quantum gravity
 - 84.2. Quantum gravity
- 85. Quantum Mechanics in Quantum Gravity
 - 85.1. Quantum gravity
 - 85.2. Quantum gravity
- 86. Quantum Mechanics in Quantum Gravity
 - 86.1. Quantum gravity
 - 86.2. Quantum gravity
- 87. Quantum Mechanics in Quantum Gravity
 - 87.1. Quantum gravity
 - 87.2. Quantum gravity
- 88. Quantum Mechanics in Quantum Gravity
 - 88.1. Quantum gravity
 - 88.2. Quantum gravity
- 89. Quantum Mechanics in Quantum Gravity
 - 89.1. Quantum gravity
 - 89.2. Quantum gravity
- 90. Quantum Mechanics in Quantum Gravity
 - 90.1. Quantum gravity
 - 90.2. Quantum gravity
- 91. Quantum Mechanics in Quantum Gravity
 - 91.1. Quantum gravity
 - 91.2. Quantum gravity
- 92. Quantum Mechanics in Quantum Gravity
 - 92.1. Quantum gravity
 - 92.2. Quantum gravity
- 93. Quantum Mechanics in Quantum Gravity
 - 93.1. Quantum gravity
 - 93.2. Quantum gravity
- 94. Quantum Mechanics in Quantum Gravity
 - 94.1. Quantum gravity
 - 94.2. Quantum gravity
- 95. Quantum Mechanics in Quantum Gravity
 - 95.1. Quantum gravity
 - 95.2. Quantum gravity
- 96. Quantum Mechanics in Quantum Gravity
 - 96.1. Quantum gravity
 - 96.2. Quantum gravity
- 97. Quantum Mechanics in Quantum Gravity
 - 97.1. Quantum gravity
 - 97.2. Quantum gravity
- 98. Quantum Mechanics in Quantum Gravity
 - 98.1. Quantum gravity
 - 98.2. Quantum gravity
- 99. Quantum Mechanics in Quantum Gravity
 - 99.1. Quantum gravity
 - 99.2. Quantum gravity
- 100. Quantum Mechanics in Quantum Gravity
 - 100.1. Quantum gravity
 - 100.2. Quantum gravity

Continuous Case

- Let $L(X, Y)$ denote the space of bounded linear mappings from a normed linear space X to normed linear space Y
- For a linear operator $\mathcal{A} : U \subset X \rightarrow V \subset Y$, the conditions of well-posedness imply:
 - ▶ \mathcal{A} is surjective (onto),
 - ▶ \mathcal{A} is injective (one-to-one)
 - ▶ \mathcal{A}^{-1} is continuous

A few theorems for the continuous case

Theorem

Let X and Y be Banach spaces. If $\mathcal{A} \in L(X, Y)$ is bijective, then $\mathcal{A}^{-1} \in L(Y, X)$

So then when will a continuous case of inverse problem does not have the inverse?

Theorem

Let X and Y be Banach spaces. Suppose $\mathcal{A} : U \subset X \rightarrow Y$ is a compact linear operator, and $\dim U$ is not finite. Then the problem $\mathcal{A}f = m$ is ill-posed.

Theorem

Let $\mathcal{A} : H \rightarrow H$ be a linear operator and $K_n \in \mathcal{L}(H, H)$ a sequence of compact operators. If $K_n \rightarrow \mathcal{A}$ in the operator norm, then \mathcal{A} is a compact operator.

Theorem

*Let $(\mathcal{A}f)(x) = \int_{\Omega} K(x, y)f(y)dy$
with Kernel $K \in L^2(\Omega \times \Omega)$. Then $\mathcal{A} \in \mathcal{L}(L^2(\Omega), L^2(\Omega))$ is compact.*

Outline

- 1. Introduction
 - 1.1. The Role of the Teacher
 - 1.2. The Role of the Student
 - 1.3. The Role of the School
- 2. Theoretical Framework
 - 2.1. Theoretical Foundations
 - 2.2. Theoretical Models
 - 2.3. Theoretical Perspectives
- 3. Research Methodology
 - 3.1. Research Design
 - 3.2. Research Instruments
 - 3.3. Research Procedures
- 4. Data Collection and Analysis
 - 4.1. Data Collection
 - 4.2. Data Analysis
 - 4.3. Data Interpretation
- 5. Results and Discussion
 - 5.1. Results
 - 5.2. Discussion
 - 5.3. Conclusion
- 6. References
- 7. Appendix

The singular value expansion

Consider Fredholm integral equation of the first kind, where we wish to find $f(t)$

$$\int_0^1 K(s, t)f(t)dt = g(s), \quad 0 \leq s \leq 1, \quad \|K\|_{L^2([0,1] \times [0,1])}^2 \leq C$$

The Singular Value Expansion (SVE) theorem states that any Kernel, K with $\|K\|_{L^2} < \infty$ can be written as

$$K(s, t) = \sum_{i=1}^{\infty} \mu_i u_i(s) v_i(t)$$

Here u_i and v_i are the singular functions of K and μ_i are the singular values of K .

Properties of the Singular values and Singular Functions

- Singular functions are orthonormal.
That is $\langle u_i, u_j \rangle = \int_0^1 u_i(s)u_j(s)ds = 1$ if $i = j$
- The singular values satisfy:

$$\mu_1 \geq \mu_2 \geq \dots \geq 0; \quad \sum_{i=1}^{\infty} \mu_i^2 = \|K\|_{L^2}$$

- And

$$\int_0^1 K(s, t)v_i(t)dt = \mu_i u_i(s)$$

- Algebra shows that

$$\langle v_i, f \rangle v_i(t) = \frac{1}{\mu_i} \langle u_i, g \rangle v_i(t)$$

- The solution $f(t)$ to the Fredholm integral equation is given by

$$\begin{aligned} f(t) &= \sum_{i=1}^{\infty} \langle v_i, f \rangle v_i(t) \\ &= \sum_{i=1}^{\infty} \frac{1}{\mu_i} \langle u_i, g \rangle v_i(t) \end{aligned}$$

A few things to note about the singular values and singular functions.

- Behaviour of the singular values and the singular functions depends on the Kernel, $K(s, t)$
- Smoother the Kernel $K(s, t)$, faster the singular values μ_i decay.
- The smaller the μ_i , more oscillatory the functions u_i and v_i will be.
- The factor $\frac{1}{\mu_i}$ amplifies highly oscillatory contributions in g .

Ill-posedness in terms of singular values

If there exists a real number $\alpha > 0$ such that the singular values satisfy $\mu_n = \mathcal{O}(n^{-\alpha})$, then α is the degree of ill-posedness.

1. If $0 < \alpha \leq 1$, the problem is mildly ill-posed.
2. If $\alpha > 1$, the problem is moderately ill posed.
3. If $\mu_n = \mathcal{O}(e^{-\alpha n})$, then the problem is severely ill-posed.

For the Fredholm integral equation to have a solution, g must satisfy

$$\sum_{i=1}^{\infty} \left(\frac{\langle u_i, g \rangle}{\mu_i} \right)^2 < \infty$$