Inverse Problems 1: Convolution and Deconvolution

Lesson 10: Total Variation regularization

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Outline

Review

Quadratic Programming

Matlab's Quad Programming

Examples

Variational problems are of the form:

$$T_\alpha(m) = \operatorname*{argmin}_f \left\{ \frac{1}{2}\|Af - m\|_2^2 + \alpha \mathcal{R}(f) \right\}$$

where $\mathcal{R}(f)$ includes *a priori* information on the unknown f.

- Tikhnov regularization: $\mathcal{R}(f) = \|f\|_2^2$
- Generalized Tikhnov regularization: $\mathcal{R}(f) = \|Lf\|_2^2$
- Total variation regularization: $\mathcal{R}(f) = \|Lf\|_1$

Total Variation Regularization

$$\underset{f}{\operatorname{argmin}} \left\{ \frac{1}{2} \|Af - m\|_2^2 + \alpha \|Lf\|_1 \right\}$$

We can minimize this by

- Approximate the absolute value of the function by

$$|t_\beta| = \sqrt{t^2 + \beta}$$

- We can use the gradient-based minimization algorithms.
- Using algorithms for nonsmooth objective functions (primal-dual,thresholding,Bregman iteration etc)

Outline

Quadratic Programming

- We want to find a vector $f \in \mathbb{R}^n$ that solves

$$T_\alpha(m) = \underset{f}{\operatorname{argmin}} \left\{ \frac{1}{2} \|Af - m\|_2^2 + \alpha \|L(f)\|_1 \right\}$$

- We write the vector $Lf \in \mathbb{R}^n$ in the form

$$Lf = v_+ - v_-$$

  where $v_\pm$ are non-negative vectors. That is $(v_\pm) \in \mathbb{R}_+^n$
- The minimization problem now becomes:

$$\|Af\|_2^2 - 2m^T Af + \alpha 1^T v_+ + \alpha 1^T v_-$$

  where $1 = [1, 1, 1, \ldots, 1]^T \in \mathbb{R}^n$

Quadratic programming

- The minimization is over $y \in \mathbb{R}^{3n}$ defined by

$$y = \begin{bmatrix} \mathsf{f} \\ \mathsf{v}_+ \\ \mathsf{v}_- \end{bmatrix}$$

  where $\mathsf{f} \in \mathbb{R}^n \quad \mathsf{v}_+ \in \mathbb{R}^n_+ \quad \mathsf{v}_- \in \mathbb{R}^n_+$
- Note the identity $\|Af\|_2^2 = f^T A^T A f$, and let

$$H = \begin{bmatrix} 2A^T A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \mathsf{h} = \begin{bmatrix} -2A^T \mathsf{m} \\ \alpha \mathsf{1} \\ \alpha \mathsf{1} \end{bmatrix} \qquad \mathsf{y} = \begin{bmatrix} \mathsf{f} \\ \mathsf{v}_+ \\ \mathsf{v}_- \end{bmatrix}$$

- We have the quadratic optimization problem in the standard form as

$$\underset{y}{\operatorname{argmin}} \left[\frac{1}{2} \mathrm{y}^T H \mathrm{y} + \mathrm{h}^T \mathrm{y} \right]$$

- The constraints are given by

$$L \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} y_{n+1} \\ \vdots \\ y_{2n} \end{bmatrix} - \begin{bmatrix} y_{2n+1} \\ \vdots \\ y_{3n} \end{bmatrix}$$

and

$$y_j \geq 0 \qquad \text{for} \quad j = n+1, \ldots, 3n$$

Outline

quadprog

is a solver for quadratic objective functions with linear constraints:

quadprog

$$\operatorname*{argmin}_{y} \left\{ \frac{1}{2} \mathrm{y}^T H \mathrm{y} + \mathrm{h}^T \mathrm{y} \right\}$$

such that

$$\begin{aligned} B\mathrm{y} &\leq \mathrm{b}, && \text{inequality constraints} \\ \mathrm{c}\mathrm{y} &= \mathrm{c}, && \text{equality constraints} \\ \mathrm{lb} \leq \mathrm{y} &\leq \mathrm{ub}, && \text{box constraints} \end{aligned}$$

Here H, B and C are matrices, and $\mathrm{y}, \mathrm{h}, \mathrm{b}, \mathrm{c}, \mathrm{lb}$, and ub are vectors.

Outline

Example 1

- Set $n = 100$ and define a signal as $f(n) = 1$ for $25 \leq n \leq 75$.
- Generate a noisy measurement.
- Use Total variation to do the reconstruction.
- Repeat the experiment with several regularization parameters.

Example 2

- Define the matlab signal and point spread signal as we considered for the Tikhnov problem.
- Generate a noisy measurement.
- Use Total variation to do the reconstruction.
- Repeat the experiment with several regularization parameters.