Inverse Problems 1: convolution and deconvolution

Lesson 11: Convolution and deconvolution in 2D

Luca Ratti

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University of Helsinki

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Definition and examples

Signals in 2D, images and matrices

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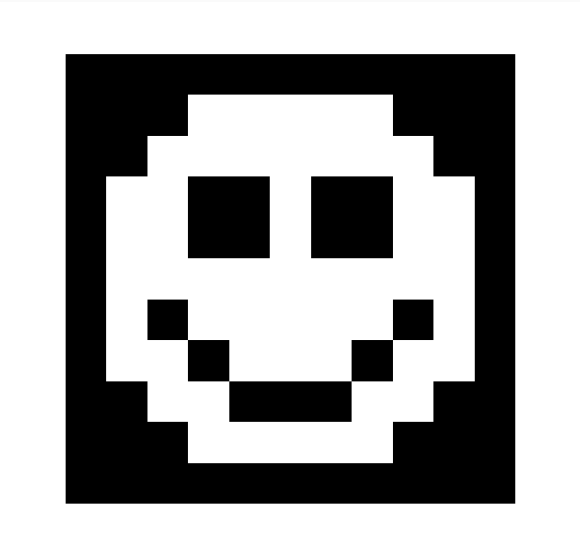

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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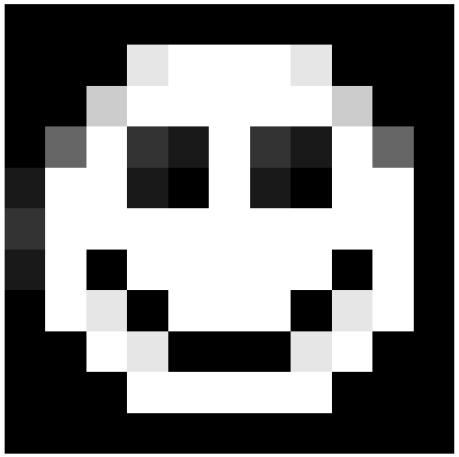

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.9 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0.8 & 1 & 1 & 1 & 1 & 1 & 0.8 & 0 & 0 \\ 0 & 0.4 & 1 & 0.2 & 0.1 & 1 & 0.2 & 0.1 & 1 & 0.4 & 0 \\ 0.1 & 1 & 1 & 0.1 & 0 & 1 & 0.1 & 0 & 1 & 1 & 0 \\ 0.2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0.1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0.9 & 0 & 1 & 1 & 1 & 0 & 0.9 & 1 & 0 \\ 0 & 0 & 1 & 0.9 & 0 & 0 & 0 & 0.9 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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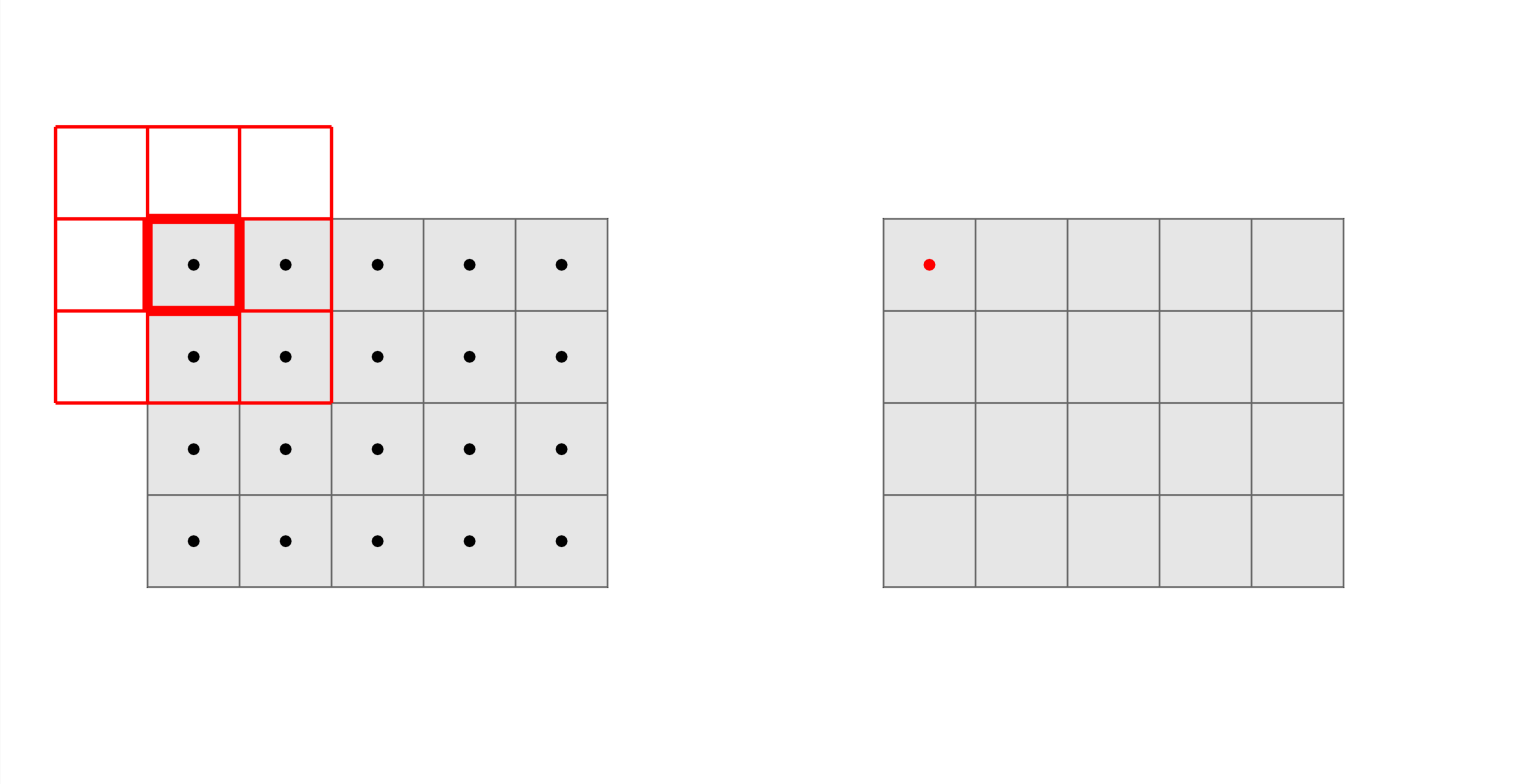
Convolution - graphical intuition

Convolution: the value of the blurred picture in a pixel is influenced by the respective one in the original picture and by the points close to it.



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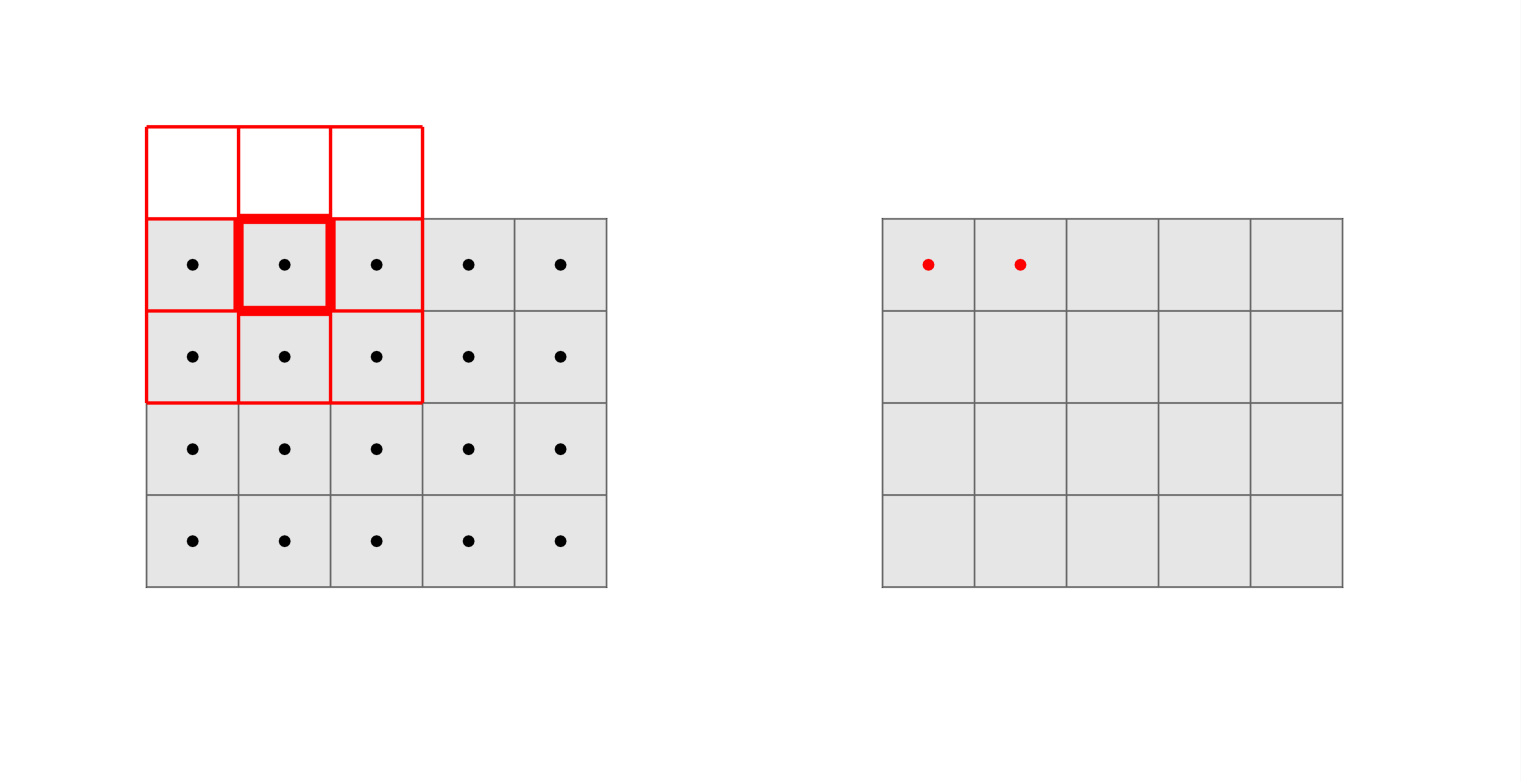

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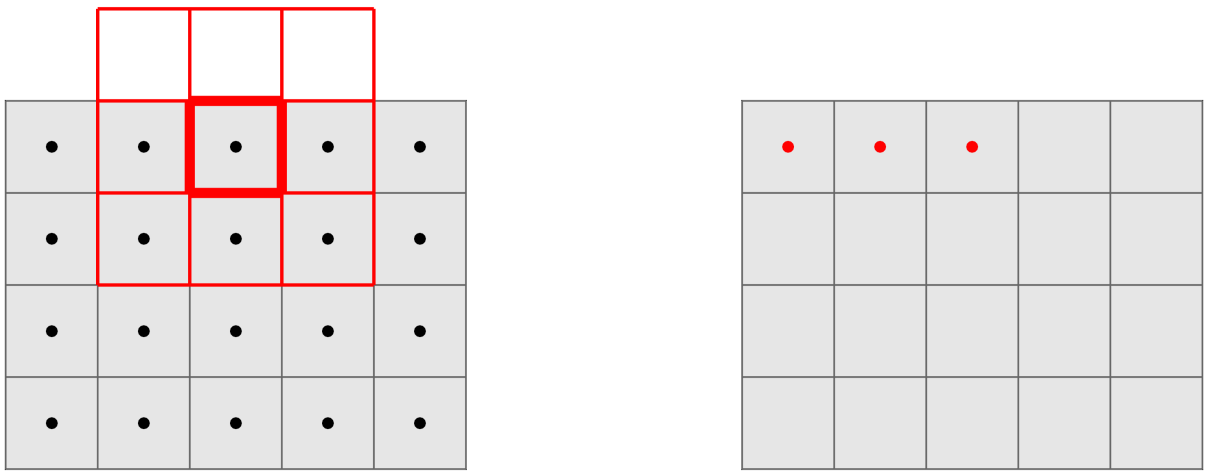

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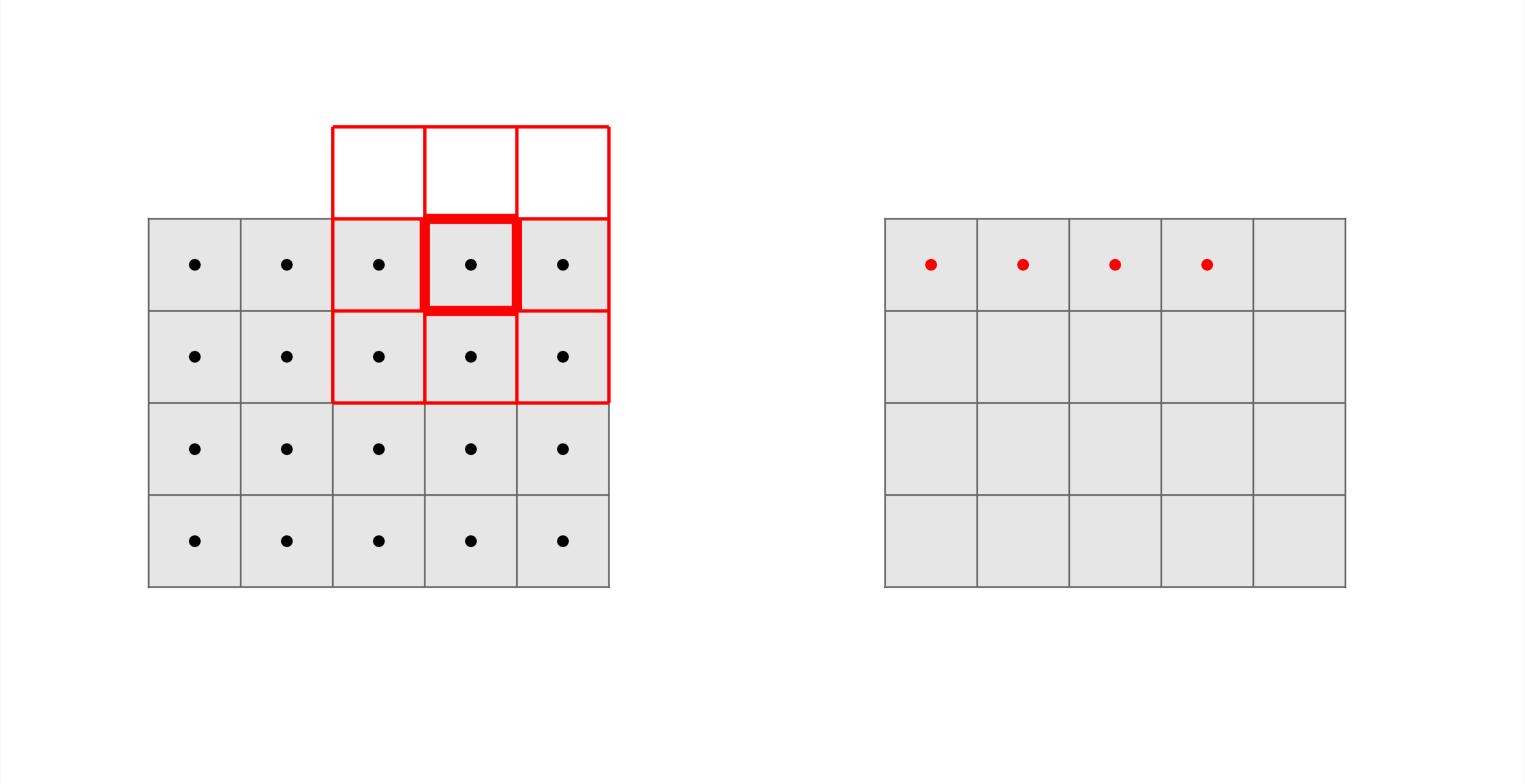

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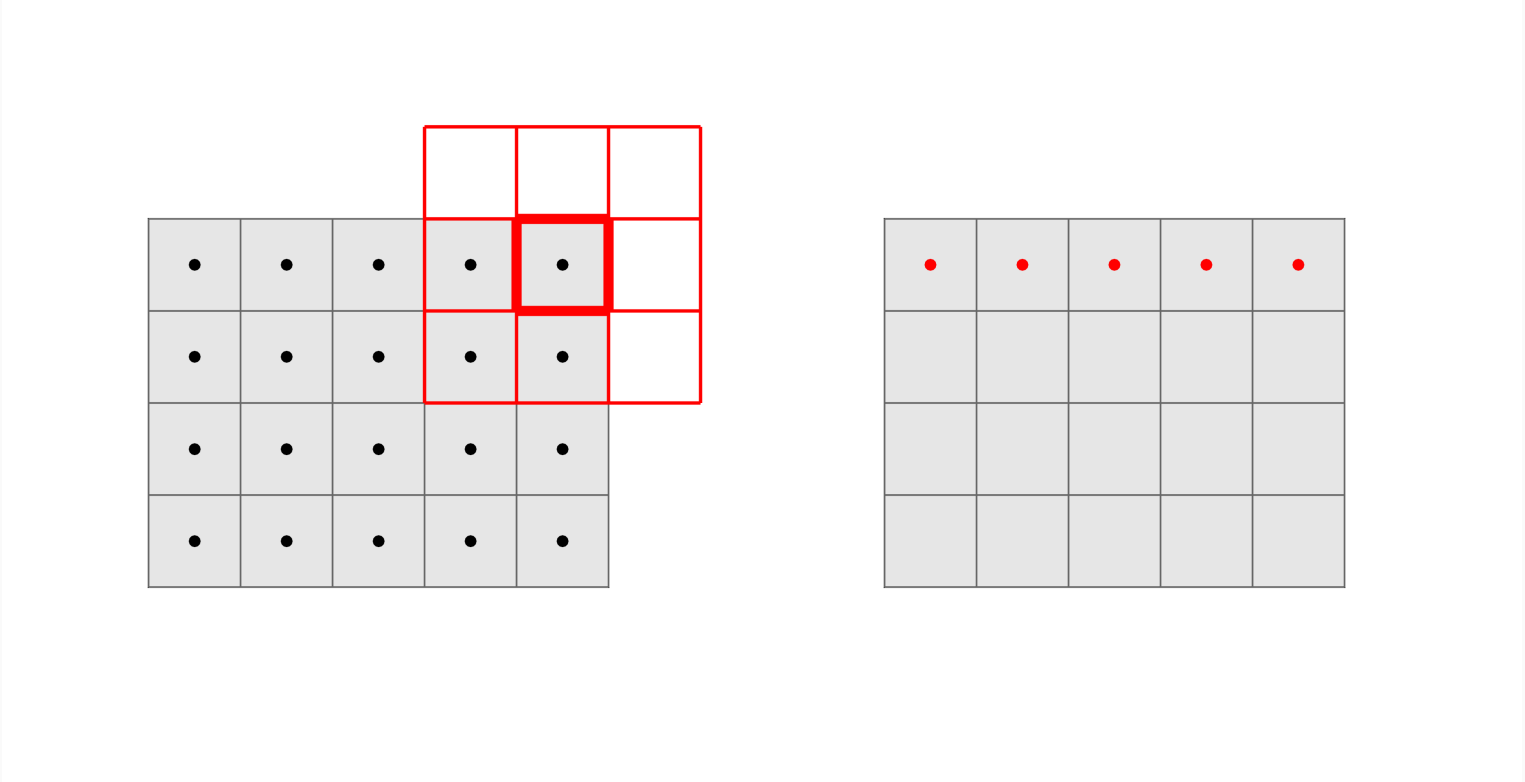

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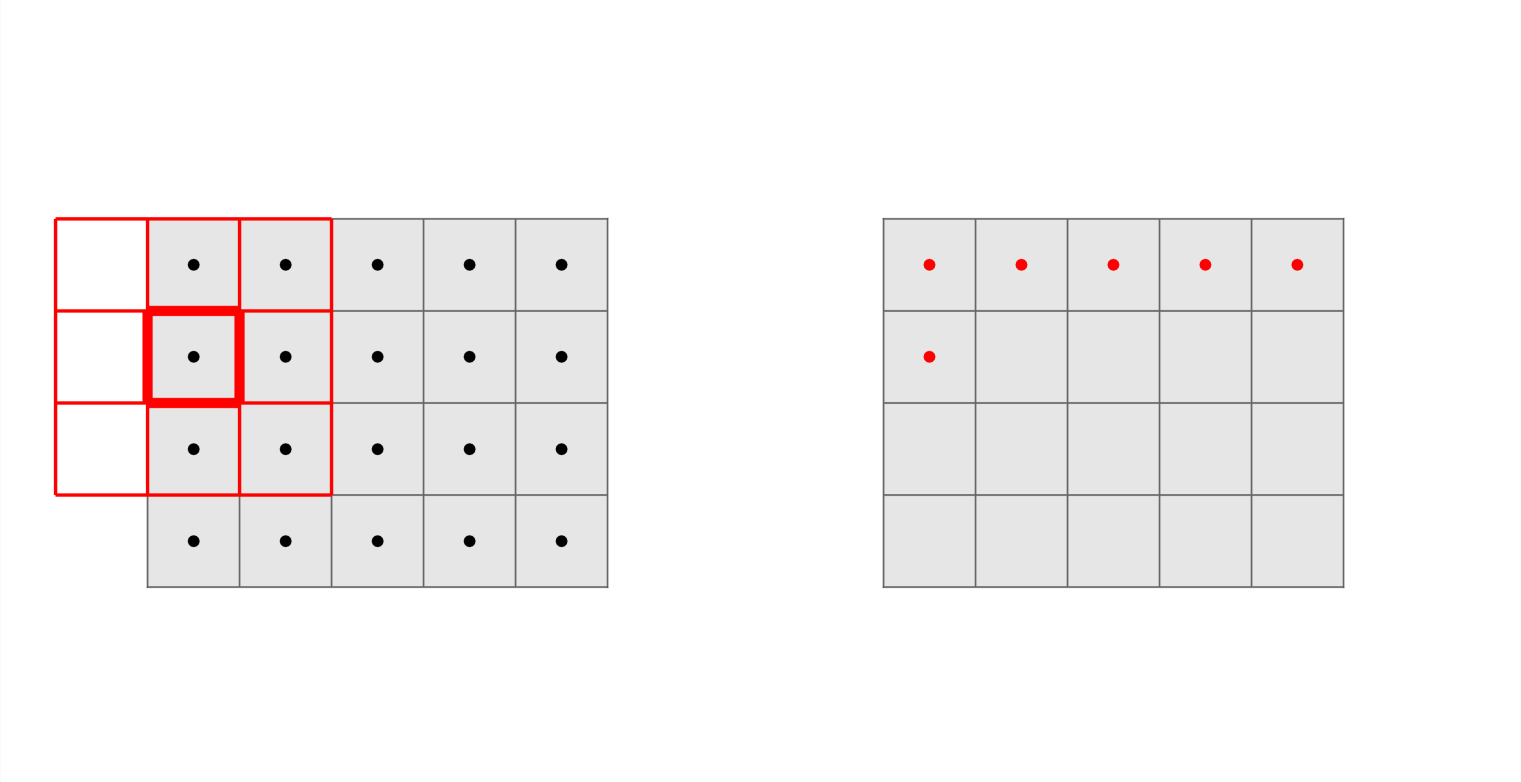

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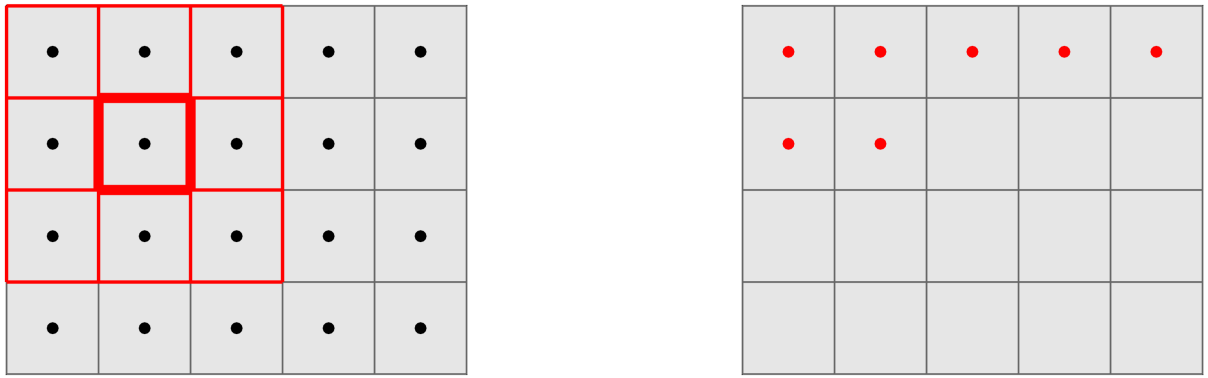

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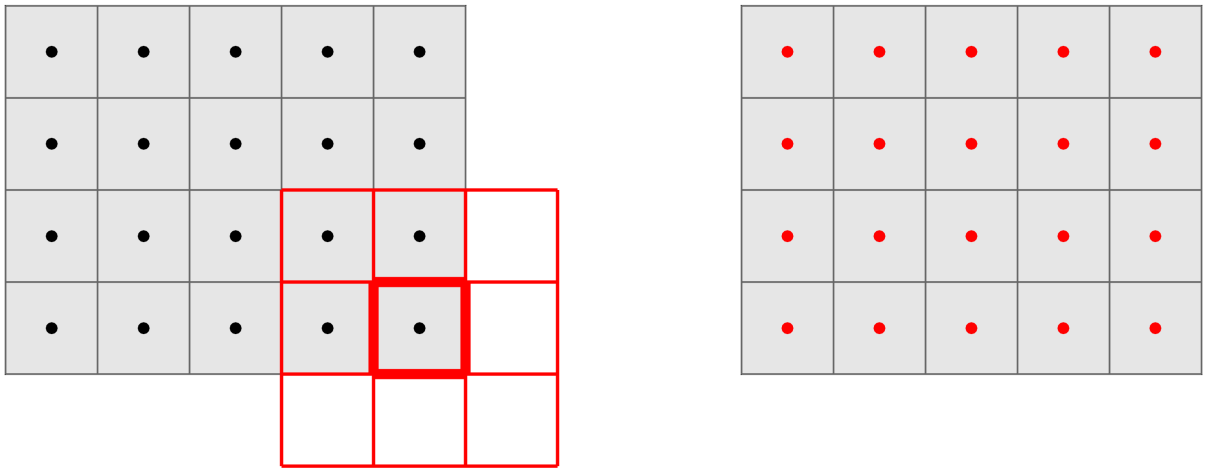

2D convolution

Point Spread Function

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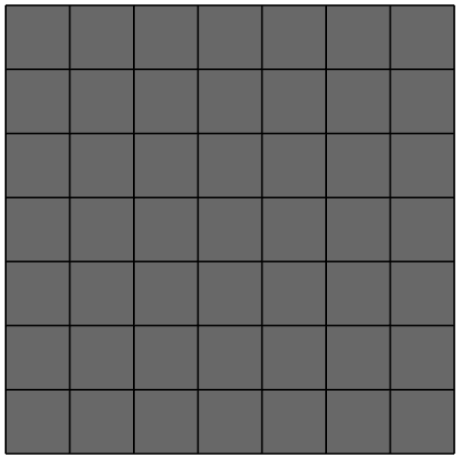
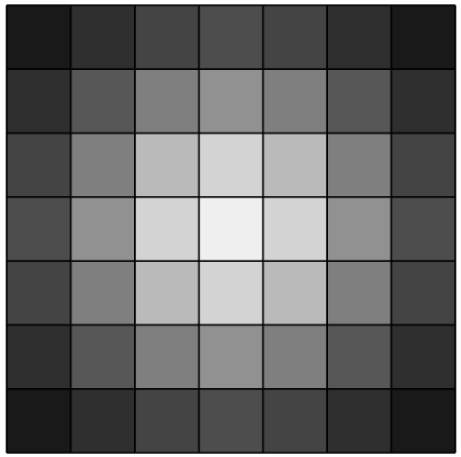
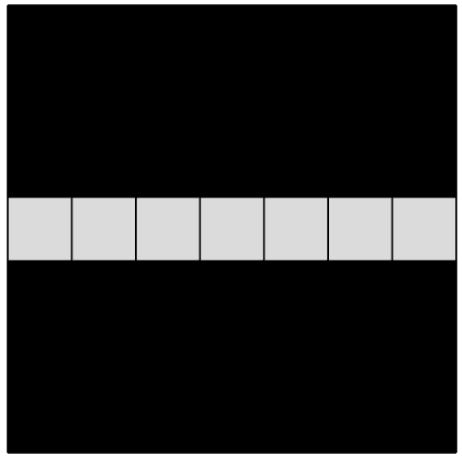
Some of the most important examples (please note: non-matching grayscales!)



Average PSF



Gaussian PSF



Motion PSF

A complicated formula

Let $F \in \mathbb{R}^{n \times m}$ be a 2D signal. Let $P \in \mathbb{R}^{q \times r}$ be the PSF.

Normalization: suppose $\sum_{i=1}^{q} \sum_{j=1}^{r} P_{i,j} = 1$

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For simplicity, let q and r be odd numbers: $q = 2\nu + 1$, $r = 2\mu + 1$. Rename the indices of P according to the following order:

$$\begin{bmatrix} P_{-\nu,-\mu} & .. & P_{-\nu,0} & .. & P_{-\nu,\mu} \\ .. & & .. & & .. \\ P_{0,-\mu} & .. & P_{0,0} & .. & P_{0,\mu} \\ .. & & .. & & .. \\ P_{\nu,-\mu} & .. & P_{\nu,0} & .. & P_{\nu,\mu} \end{bmatrix}$$

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Convolution formula

The convolved signal $C = P * F$ is a matrix in $\mathbb{R}^{n\times m}$ s.t.

$$C_{i,j} = \sum_{\ell=-\nu}^{\nu}\sum_{k=-\mu}^{\mu} P_{\ell,k} F_{i-\ell,j-k}.$$

We always consider zero padding for the missing values of F.

Examples

Here are some examples of convolution kernels acting on a picture:

Original picture

Convolution with 9x9 Gaussian filter

Examples

Here are some examples of convolution kernels acting on a picture:

Original picture

Convolution with 9x9 average filter

Examples

Here are some examples of convolution kernels acting on a picture:

Original picture

Convolution with 9x9 motion filter

Implementation

Example 1

1. Import the picture `mri.tif`.
   Matlab image add on contains several pictures. The command to import one of those, or a picture from the current folder, is `imread`. For our purposes, it is better to deal with picture with values between 0 and 1 (sometimes the default is from -255 to 255): use the command `im2double`. Show the picture with `imshow`
2. Create 7x7 gaussian, average and motion PSFs.
   Use the command `fspecial`. For the Gaussian filter, impose variance equal to 2.
3. Show the result of convolution with those filters
   The command in Matlab is `conv2`. The default option is `'full'`, but we are interested in the option `'same'`
4. Add some Gaussian random noise to the picture with mean 0 and variance 0.005. Use the command `imnoise`.

Matrix formulation

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How to do it in Matlab:

- given a matrix $A \in \mathbb{R}^{n \times m}$, `a = A(:)` stores A in a nm dimensional vector;
- given a matrix $a \in \mathbb{R}^{nm}$, the command `A = reshape(A,[n,m])` creates the associated matrix with dimensions n and m.

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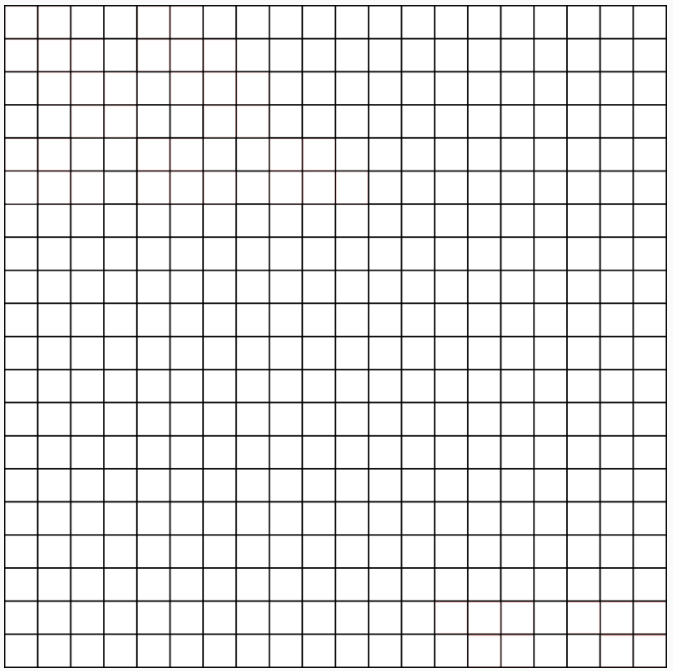
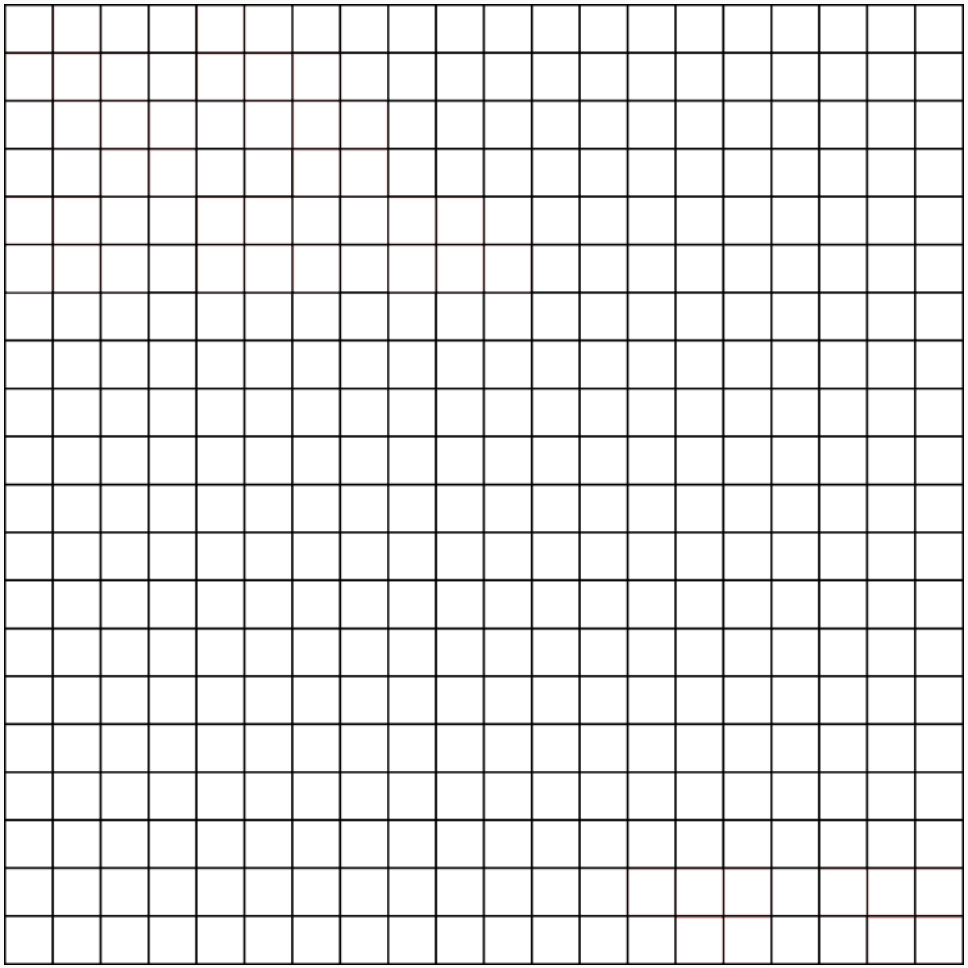


Matrix representation - graphical intuition

What elements of the matrix A are different from 0?



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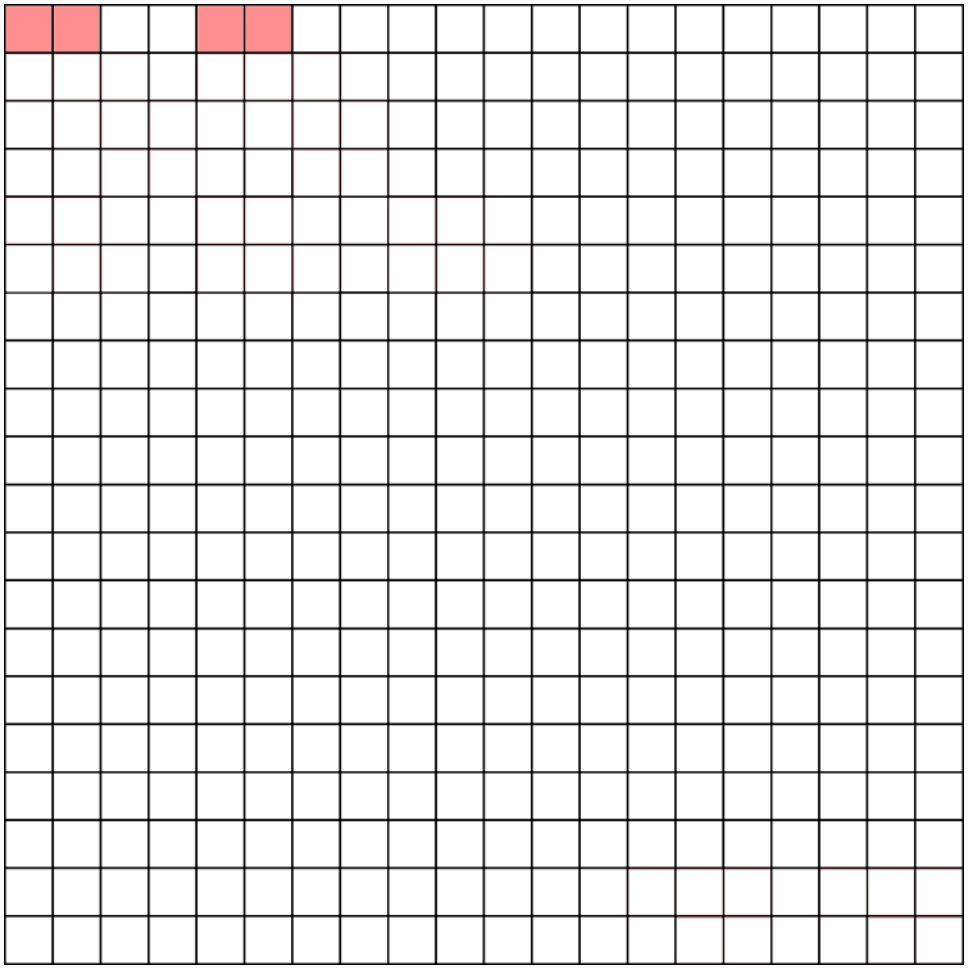

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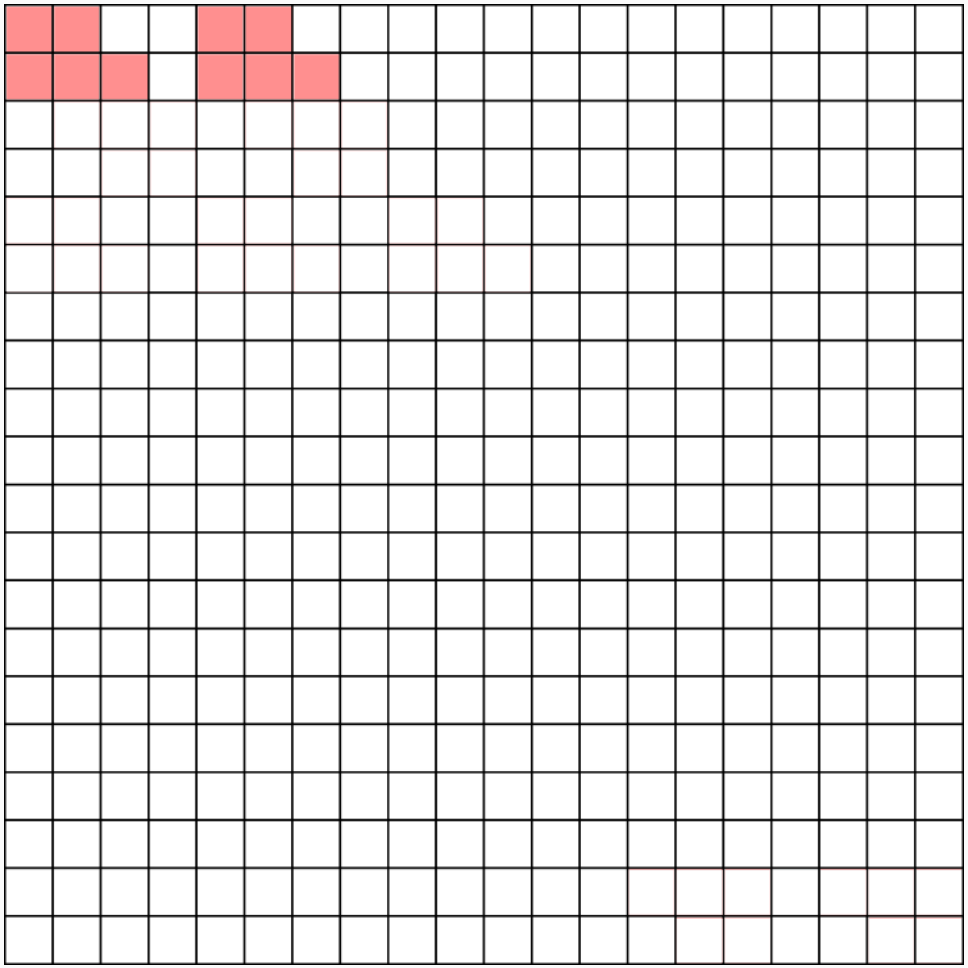

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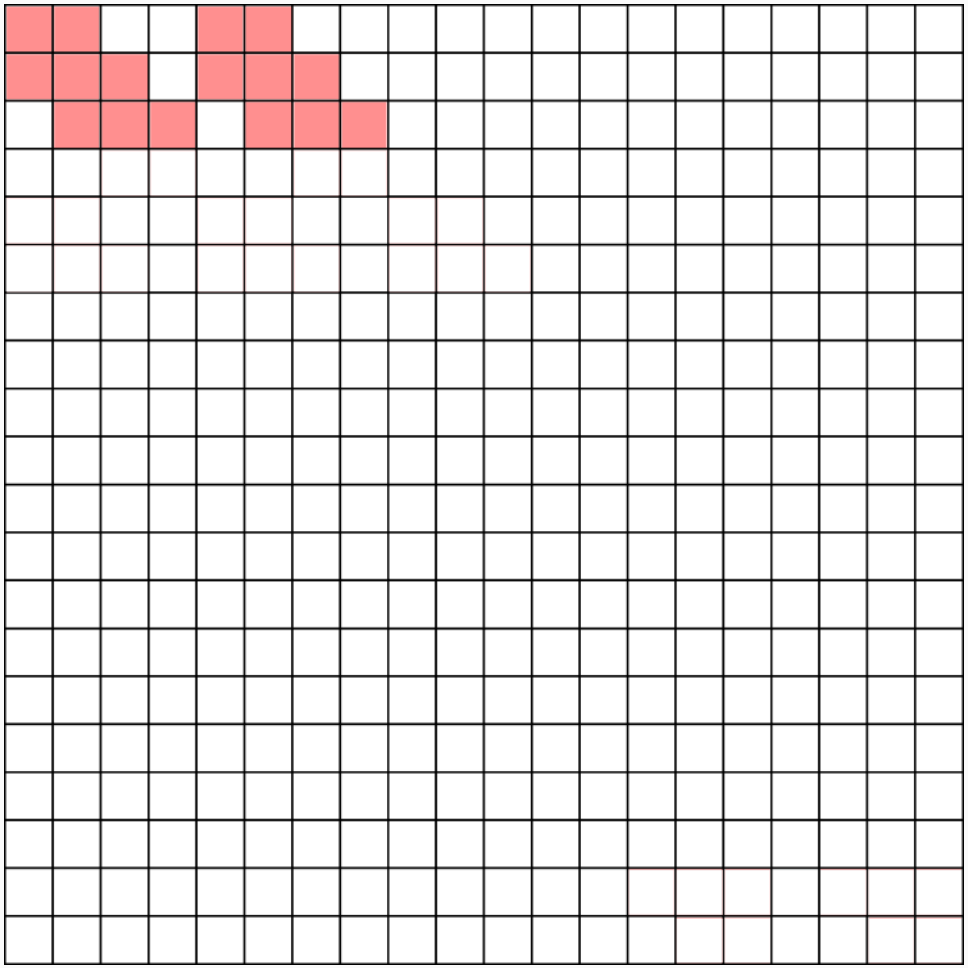

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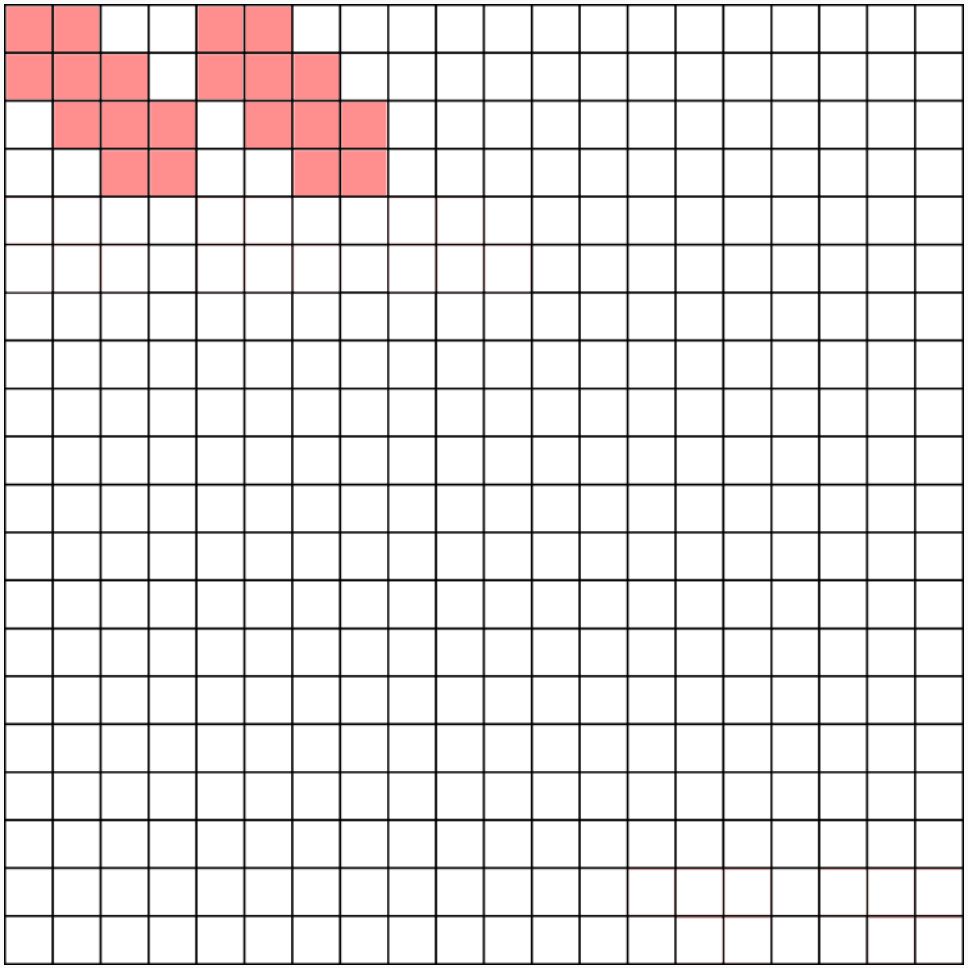

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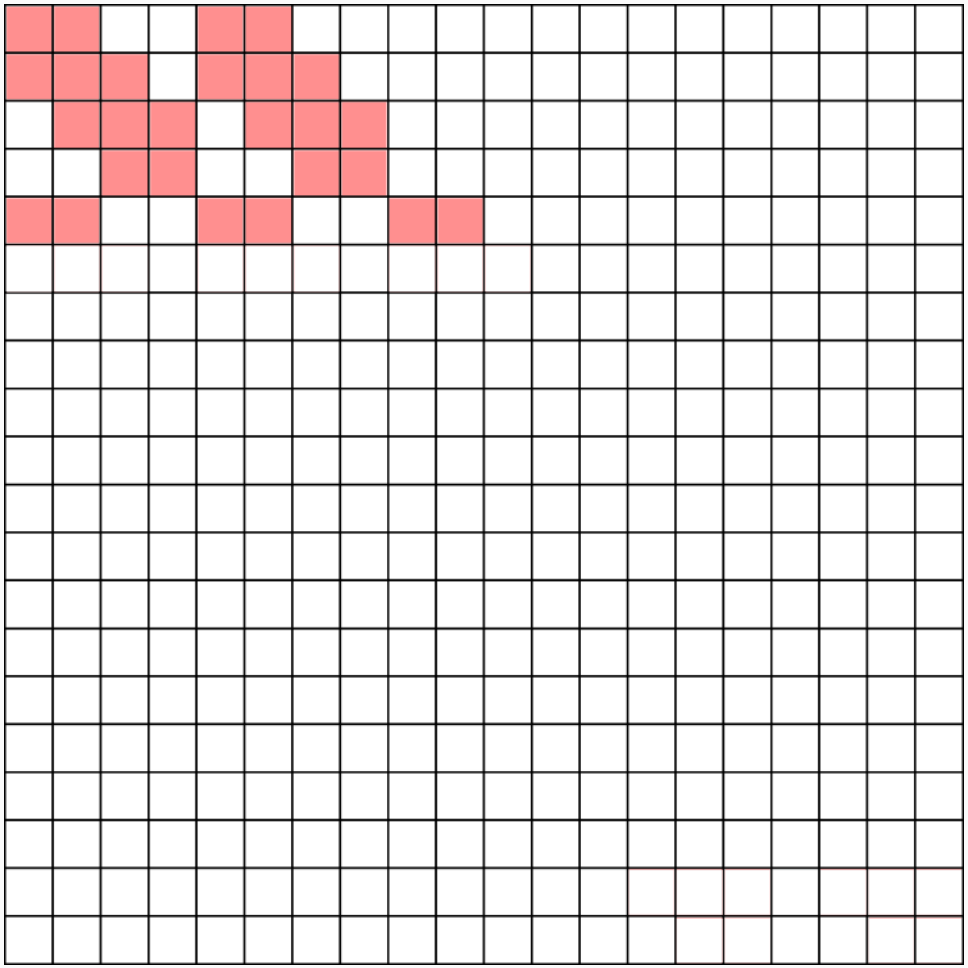

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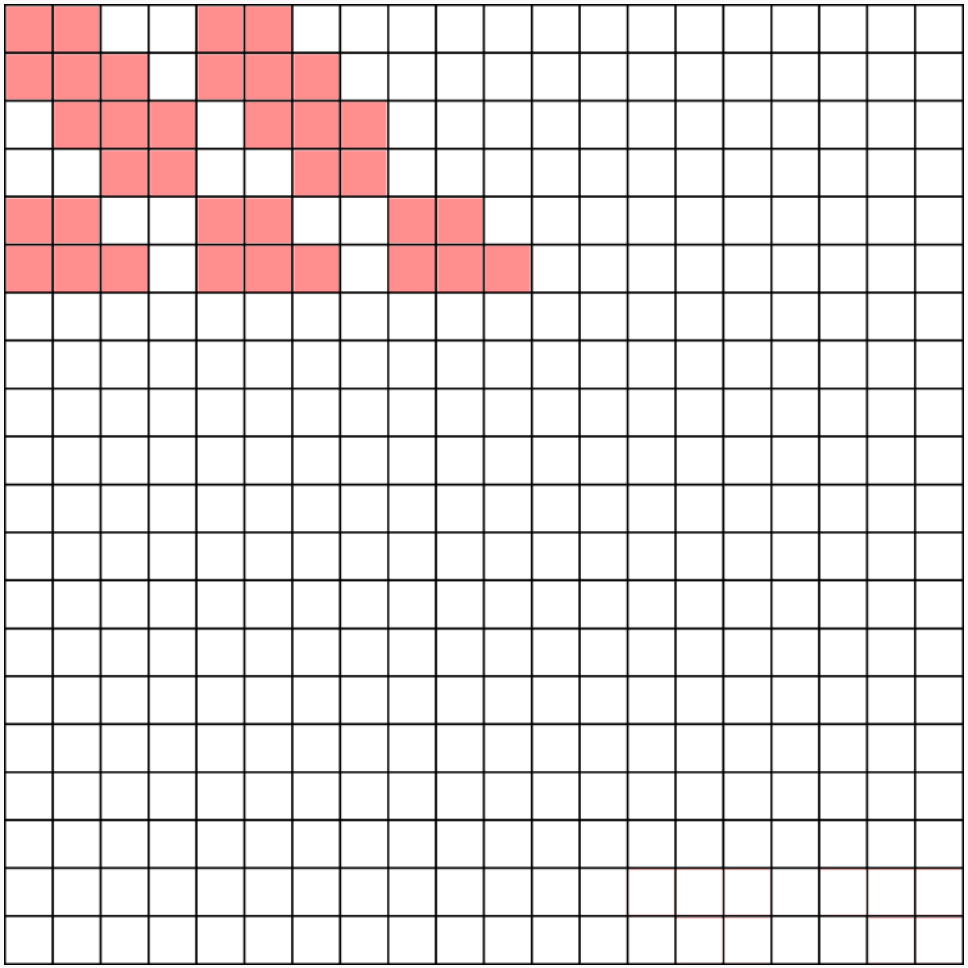

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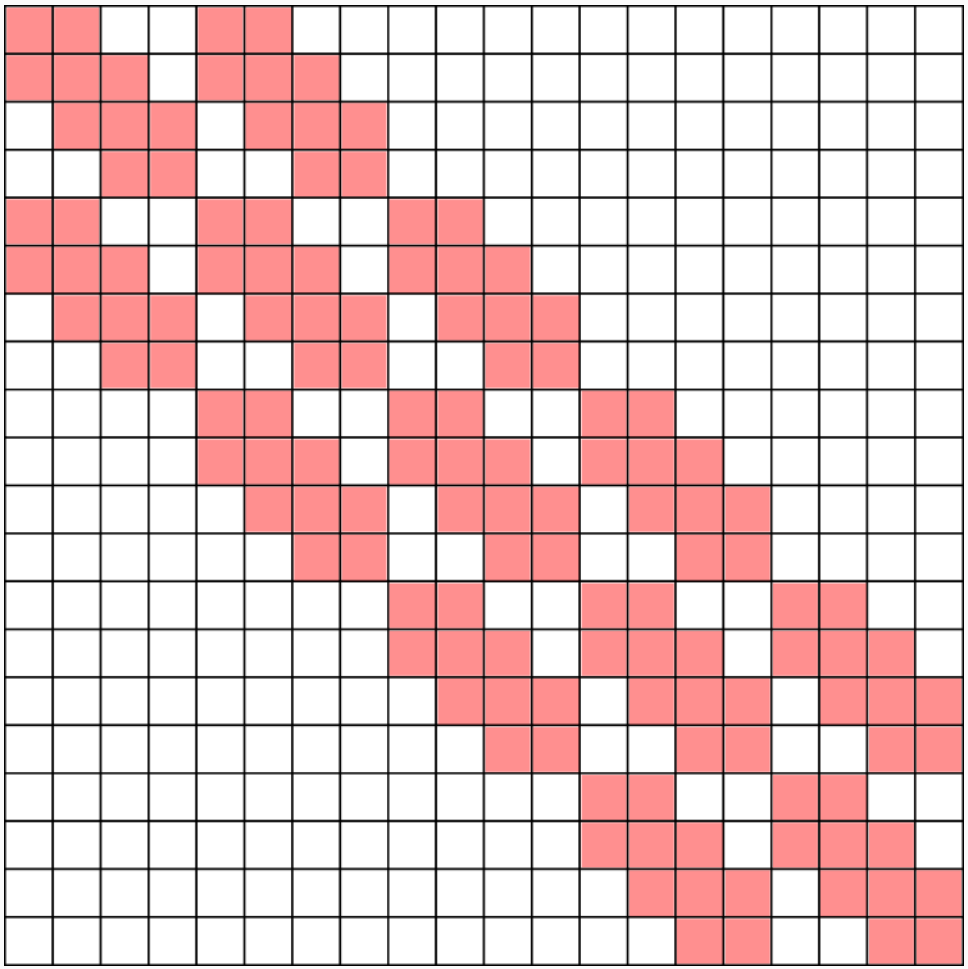
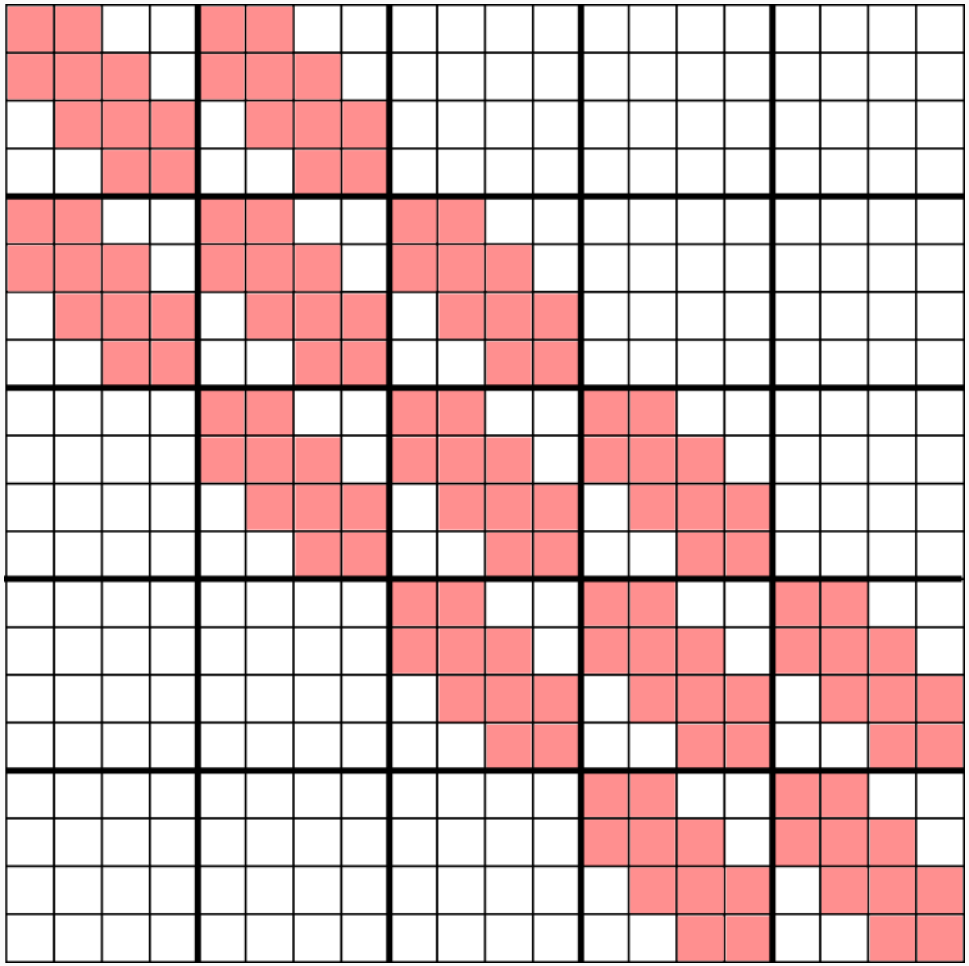


$$\begin{bmatrix} f_1 & f_5 & f_9 & f_{13} & f_{17} \\ f_2 & f_6 & f_{10} & f_{14} & f_{18} \\ f_3 & f_7 & f_{11} & f_{15} & f_{19} \\ f_4 & f_8 & f_{12} & f_{16} & f_{20} \end{bmatrix}$$

Matrix representation: some details



- A is a block matrix, consisting of $m \times m$ blocks of size $n \times n$;
- A is a block band matrix, with r (block) diagonals;
- every non null block of A is a band matrix with q diagonals;
- A is a **sparse** matrix, i.e., few elements are different from 0 (use the command `spy`).

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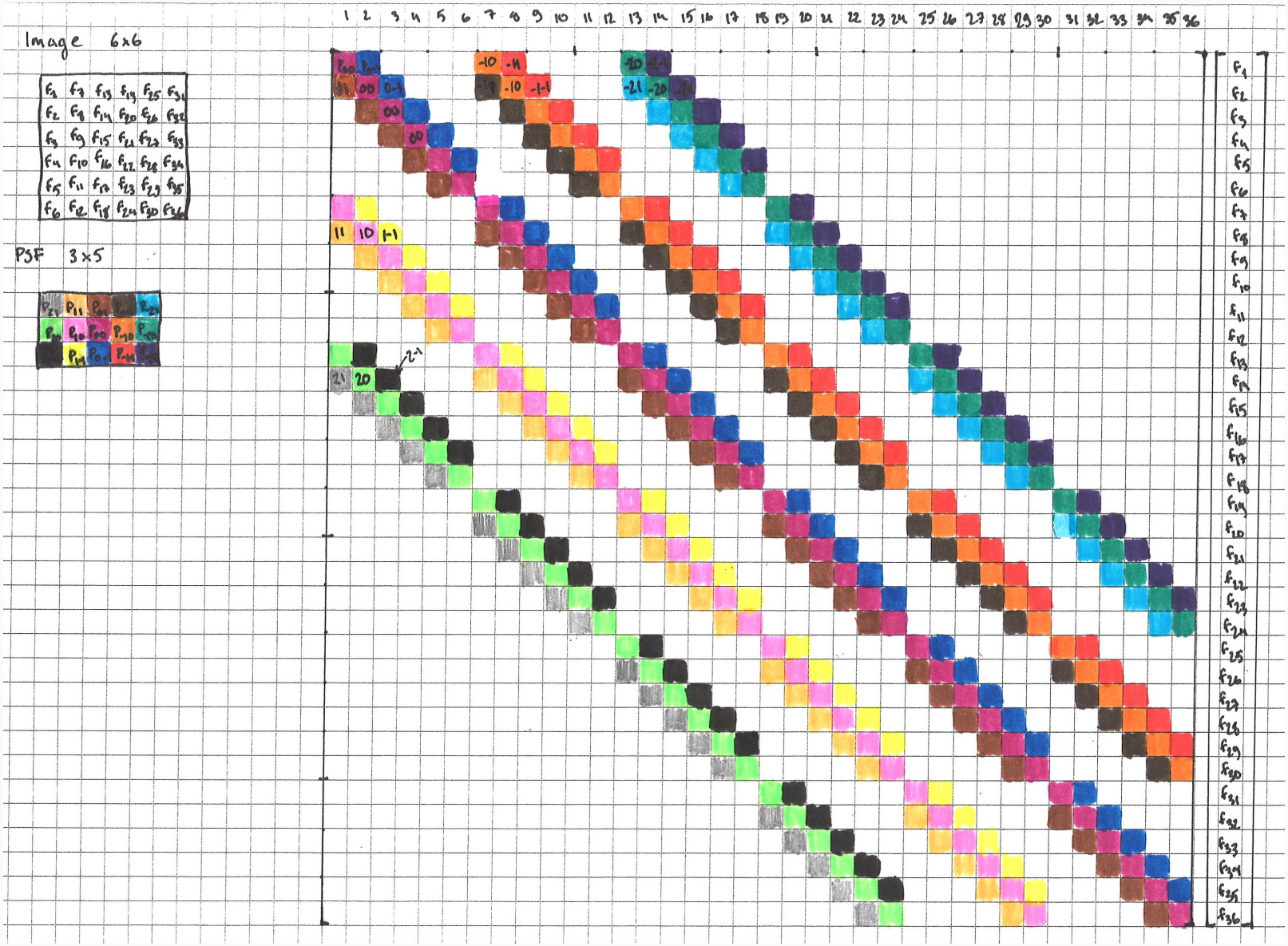

Credits: Heli Virtanen

The Matlab command `convmtx2`

The command `A = convmtx2(P,n,m)` creates the convolution matrix associated with P and with an original signal of s $n \times m$.

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Goal: remove all the rows of A associated to undesired elements in the final signal.

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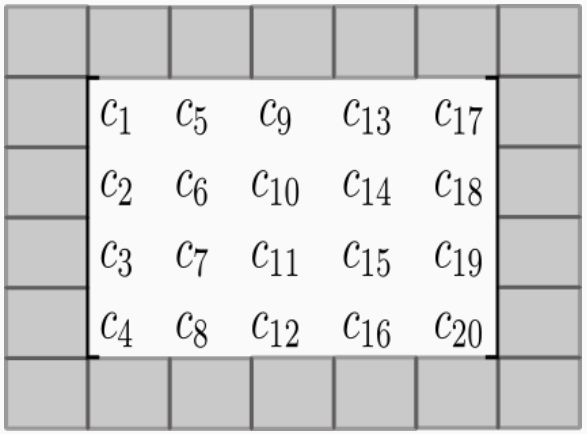

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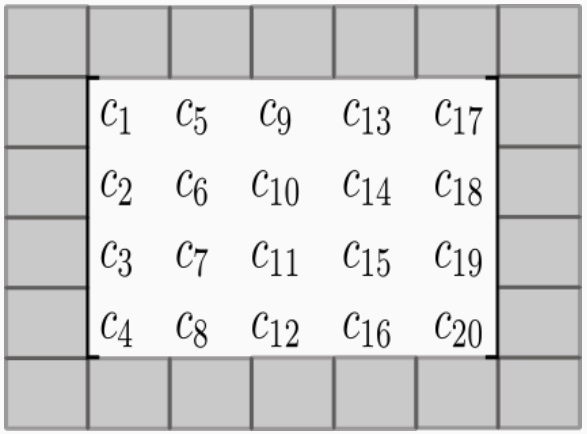

Proposed code:

```
A = convmtx2(P,n,m);
Nr = (n+2*nu)*(m+2*mu);
I = [1:(n+2*nu)*mu, (Nr-mu*(n+2*nu)+1):Nr];
for i = 1:nu
I = [I, i:(n+2*nu):Nr, (n+2*nu-i+1):(n+2*nu):Nr];
end
A(I,:)  = [];
```

Implementation

Example 2

1. Import the picture `cameraman.tif`.
2. Create a 5x5 average PSF with the command `fspecial`.
3. Compute the blurred picture `C1` by the command `conv2`.
4. Assemble the matrix A and remove the undesired lines.
5. Compute the blurred picture `C2` by matrix product and rescaling.
6. Compare the results.

Deconvolution: regularization

Naïve inversion clearly fails

Matrix formulation of the 2D deconvolution problem

Given $M \in \mathbb{R}^{n \times m}$ (unfolded in a vector $m \in \mathbb{R}^{nm}$), find $F \in \mathbb{R}^{n \times m}$ (i.e., $f \in \mathbb{R}^{nm}$) such that $Af = m$.

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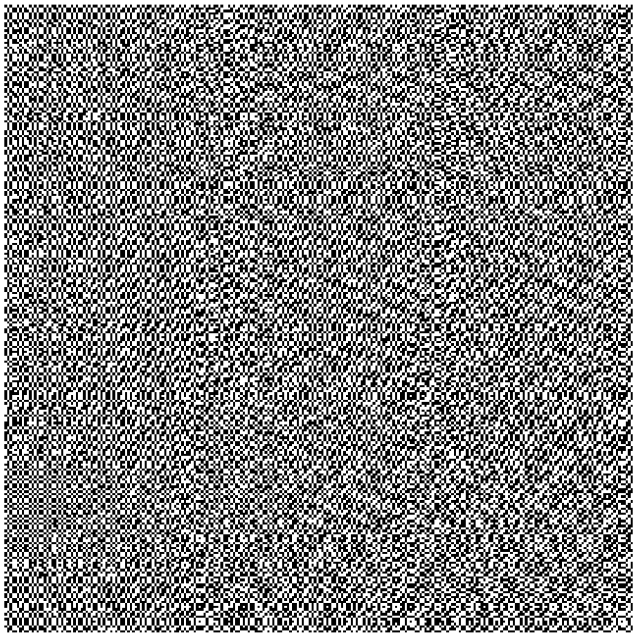
Blurred and noisy data

No regularization



The Matlab commands `deconvwnr` and `deconvreg`

Matlab has two built-in methods for image deconvolution: `deconvwnr` uses a technique related to the study of Wiener involving the Fourier transform: this is extremely effective in absence of noise. The method `deconvreg` is studied to guarantee stability with noisy measurements.

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The 2D convolution problem is a (ill-posed) linear inverse problem. All the technique we studied in 1D can be applied also here.

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The solution can be found via the **normal equations**:

$$f_{\alpha,\delta} = (A^T A + \alpha I)^{-1} A^T m$$

The other techniques are computationally impracticable (expecially computing the SVD).
Remark: the matrix $A^T A + \alpha I$ is sparse: use a sparse solver for the linear system!

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Tikhonov, α = 0.15

Generalized Tikhonov

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- small issue: how to build L_x and L_y?

Generalized Tikhonov: details

Approximation of the derivatives:

$$\frac{\partial F}{\partial x}(x_i, y_j) \approx \frac{F_{i,j} - F_{i,j-1}}{\Delta x} \qquad \frac{\partial F}{\partial y}(x_i, y_j) \approx \frac{F_{i,j} - F_{i-1,j}}{\Delta y}$$

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Some problems when unfolding the matrix ($\Delta x = \Delta y = 1$):

$$(L_x f)_i = f_i - f_i^{left} \qquad (L_y f)_i = f_i - f_i^{above},$$

where f_i^{left} is the component of f corresponding to the element that was on the left of the i-th component before unfolding, and f_i^{above} is the one above.

$$\begin{bmatrix} f_1 & f_5 & f_9 & f_{13} & f_{17} \\ f_2 & f_6 & f_{10} & f_{14} & f_{18} \\ f_3 & f_7 & f_{11} & f_{15} & f_{19} \\ f_4 & f_8 & f_{12} & f_{16} & f_{20} \end{bmatrix}$$

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Proposed code:

```
Lx = spdiags([ones(n*m,1) -ones(n*m,1)],[0,-n],n*m,n*m);
Ly = spdiags([ones(n*m,1) -ones(n*m,1)],[0,1],n*m,n*m);
Ly((m:m:n*m-1)+1,(m:m:n*m)) = 0;
```

Generalized Tikhonov - results

$$f_{\alpha,\delta} = (A^T A + \alpha L_x^T L_x + \alpha L_y^T L_y)^{-1} A^T m$$

Generalized Tikhonov, α = 0.15

Generalized Tikhonov, α = 1.75

The noise gets removed, but all the discontinuity in the signal (sharp edges) are lost.

Total Variation

A good strategy to enhance the sharpness of the edges would be the Total Variation regularization:

Total Variation

Set $\alpha > 0$. Find $f_\alpha = \underset{f \in \mathbb{R}^{nm}}{\arg\min} \left\{ \|Af - m\|^2 + \alpha \|Lf\|_1 \right\}$, where $L = [L_x, L_y]^T$ is the matrix approximating the partial derivatives.

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How to solve it? Interpretation as quadratic programming:

$$\text{minimize } \left\{ \|Af\|^2 - 2m^T Af + \alpha \mathbf{1}^T v_+ + \alpha \mathbf{1}^T v_- + \alpha \mathbf{1}^T w_+ + \alpha \mathbf{1}^T w_- \right\}$$

$$\text{subject to: } v_+ - v_- = L_x f;\ w_+ - w_- = L_x f;\ v_+, v_-, w_+, w_- \geq 0.$$

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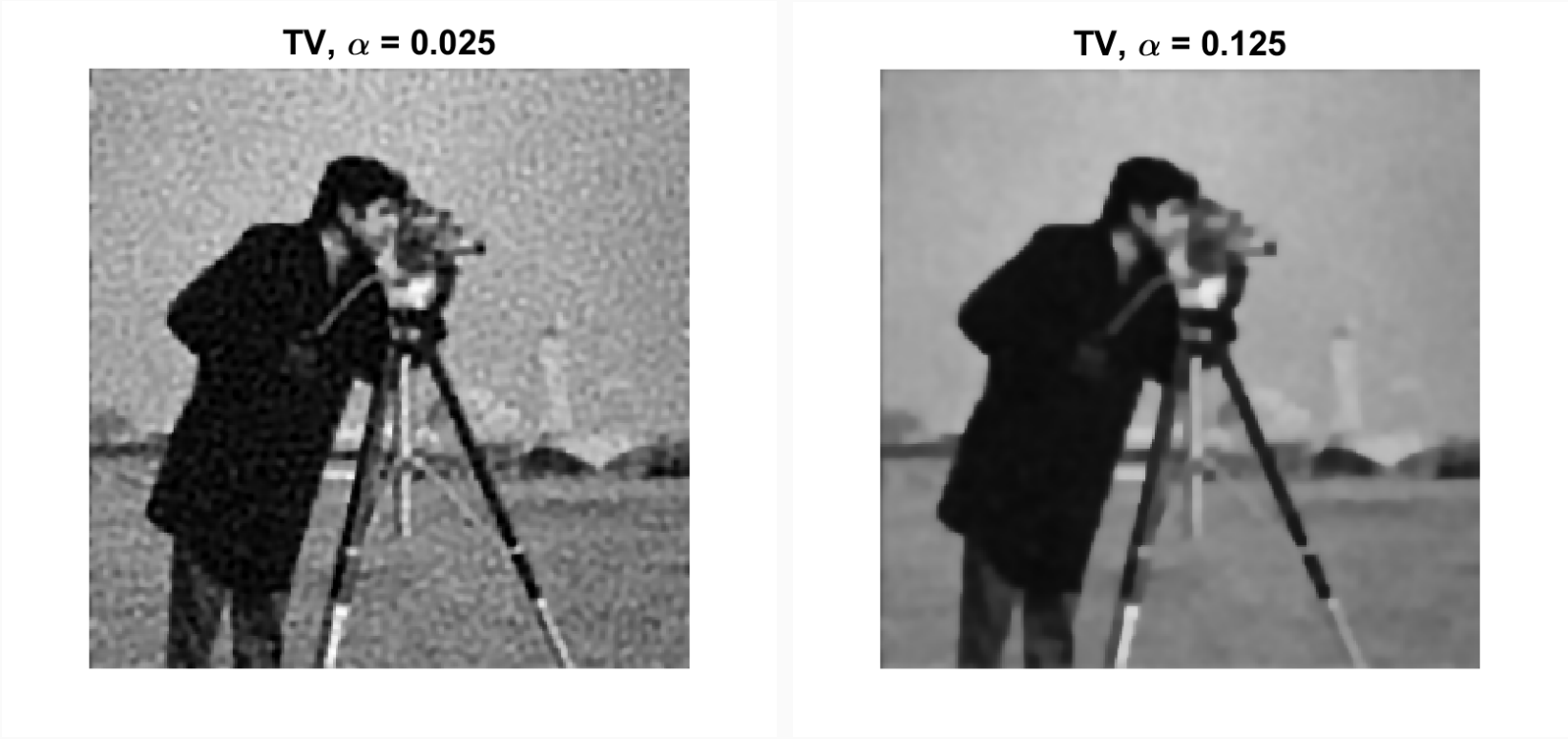
The implementation in Matlab via `quadprog` requires to define a matrix $H \in R^{5mn \times 5mn}$. Without sparse matrix solvers, this would be too demanding for most of the computers.

Total Variation: details

The time required by `quadprog` to process a 256x256 picture on a common laptop is around 1 minute.



The noise is removed the edges are clearly visible, but many small details are completely missing.

What regularization strategy would you pick? - I

Example 3

1. We are given a blurred and noise picture to deconvolve (filter: 5x5, Gaussian).
2. According to a priori knowledge, what regularization strategy would you recommend?

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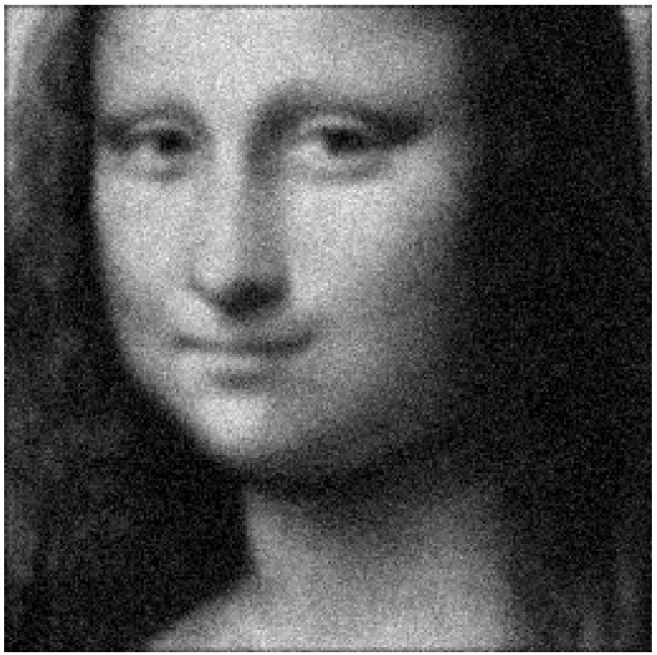

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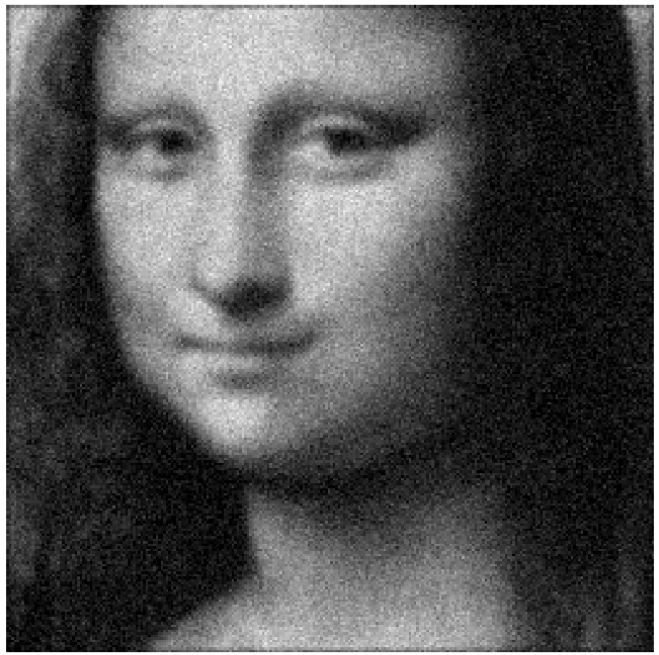
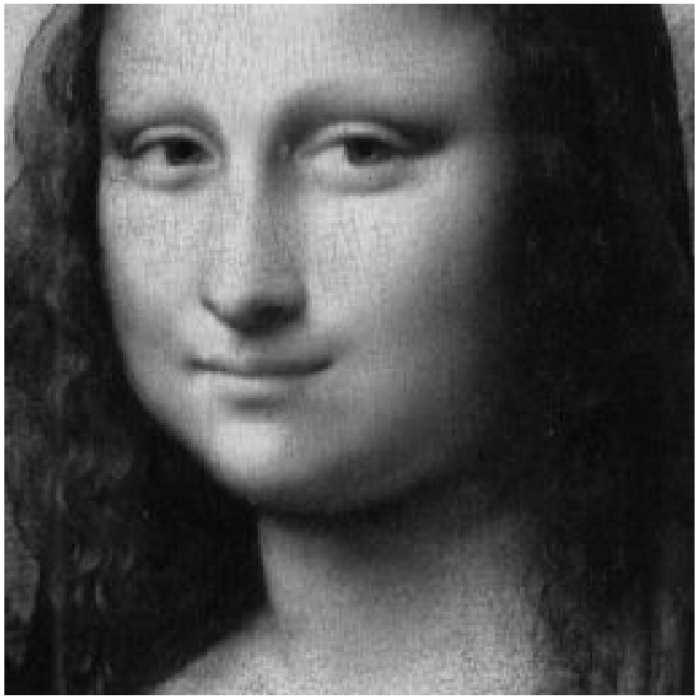


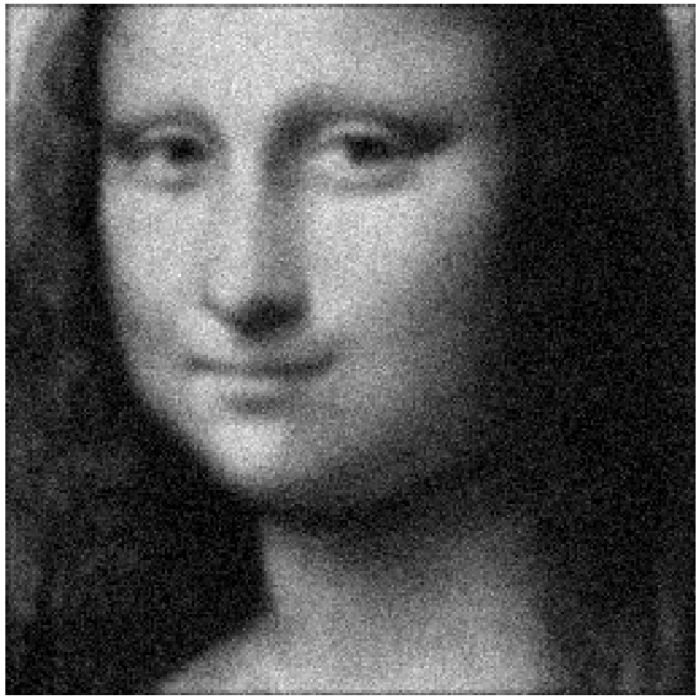
Hint: Leonardo Da Vinci was an expert of *sfumato*, a painting technique consisting in softening the transition between colours.

What regularization strategy would you pick? - I

Original Image



Blurred and noisy data



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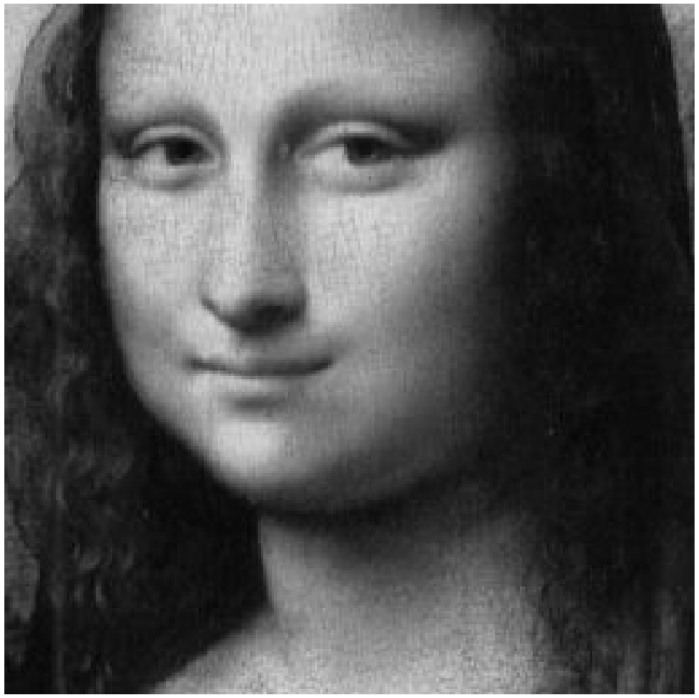
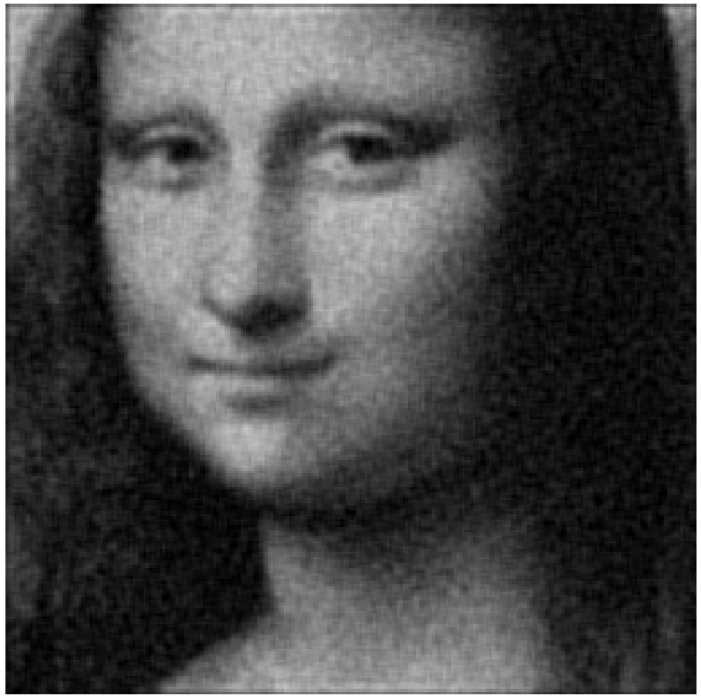


Tikhonov, α = 0.15



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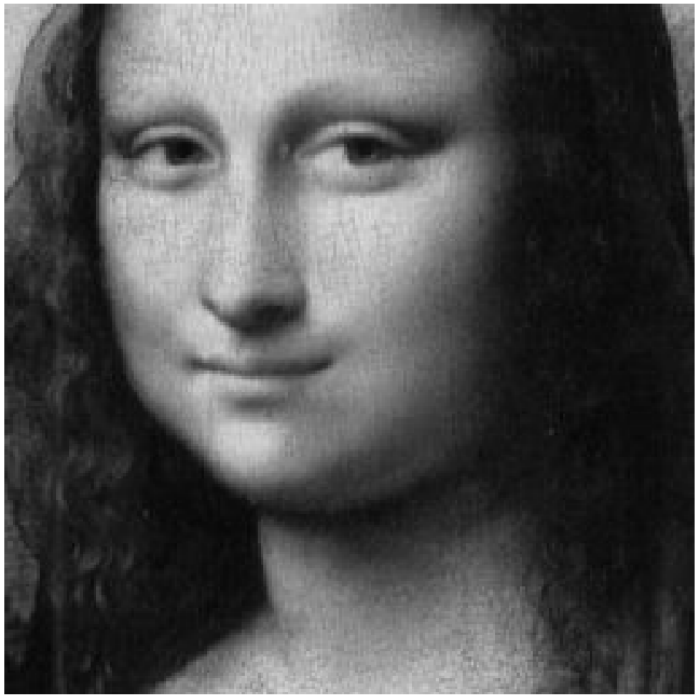
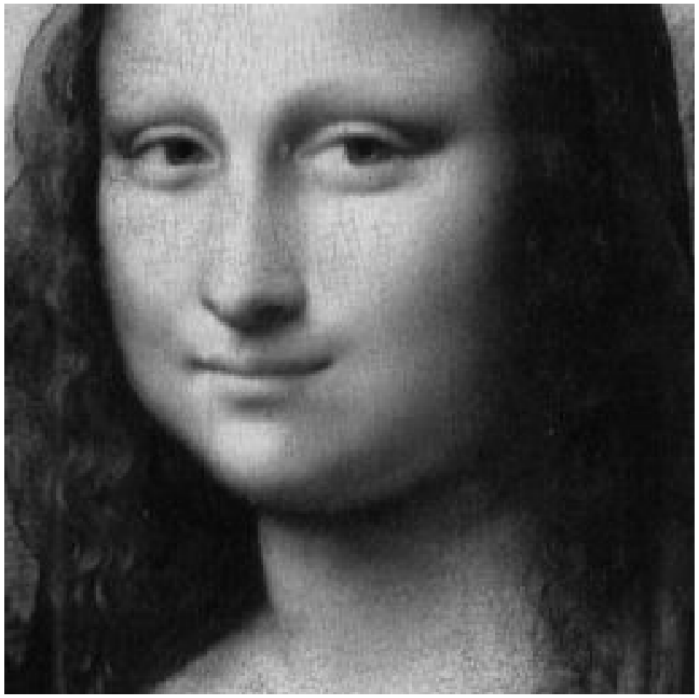


Generalized Tikhonov, α = 0.9

What regularization strategy would you pick? - I

Original Image



TV, α = 0.125

What regularization strategy would you pick? - I

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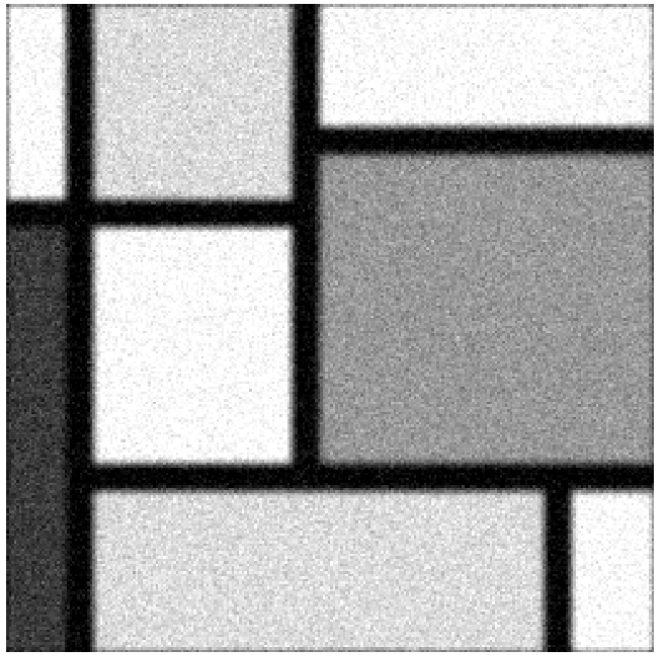

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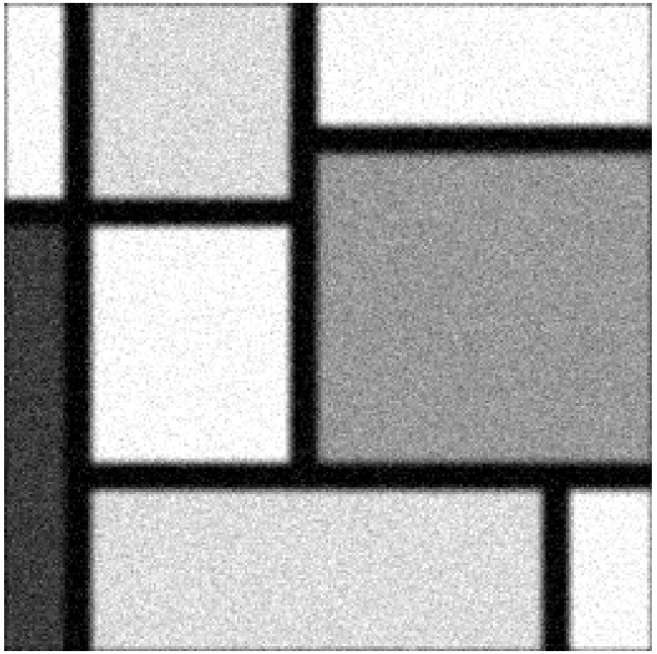
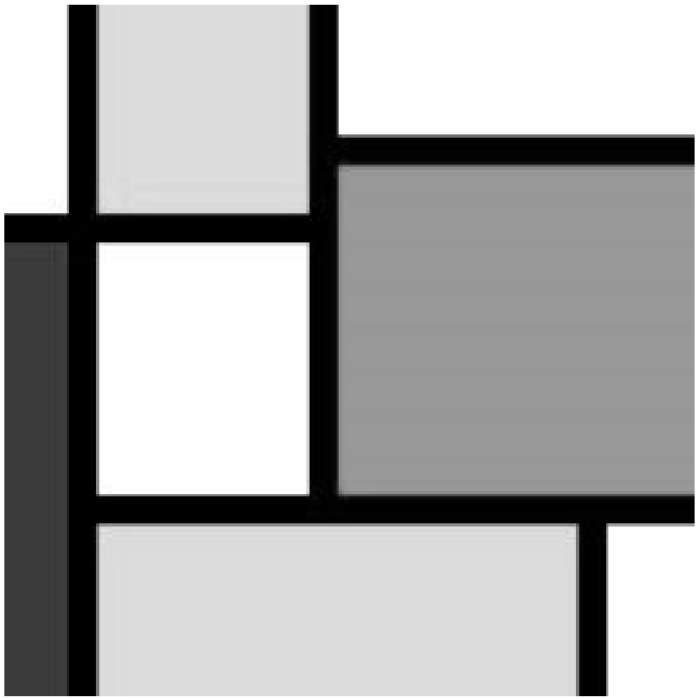
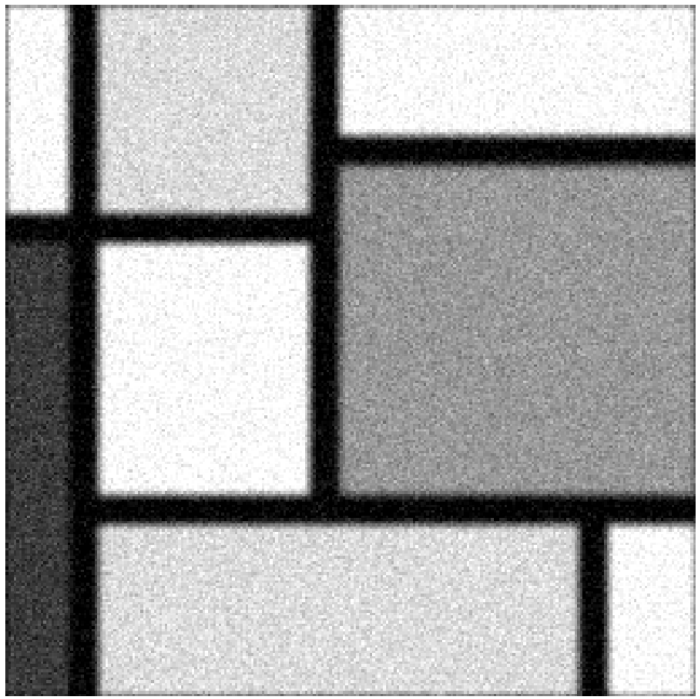


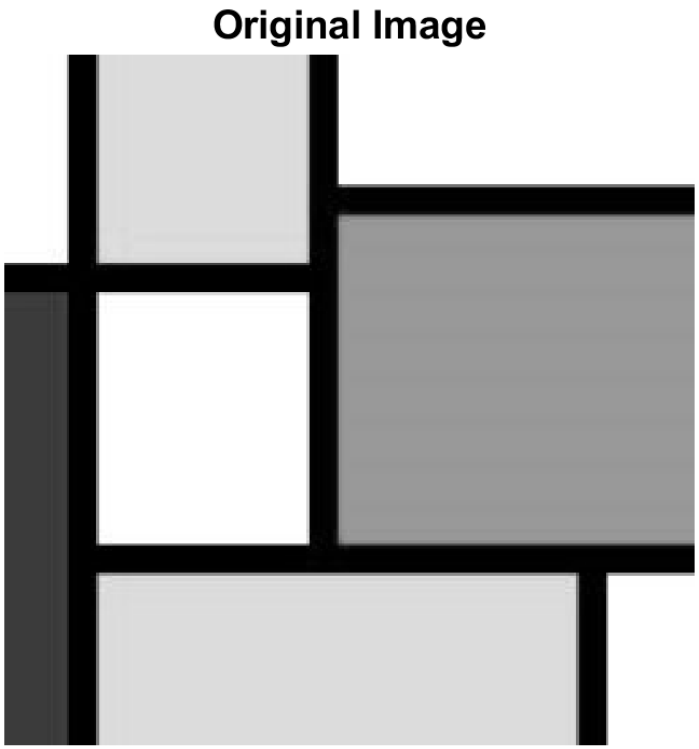
Hint: Piet Mondrian was the founder of Neoplasticism, whose main elements are the use of primary colors (red, blue and yellow), three primary values (black, white and gray) and two primary directions (horizontal and vertical).

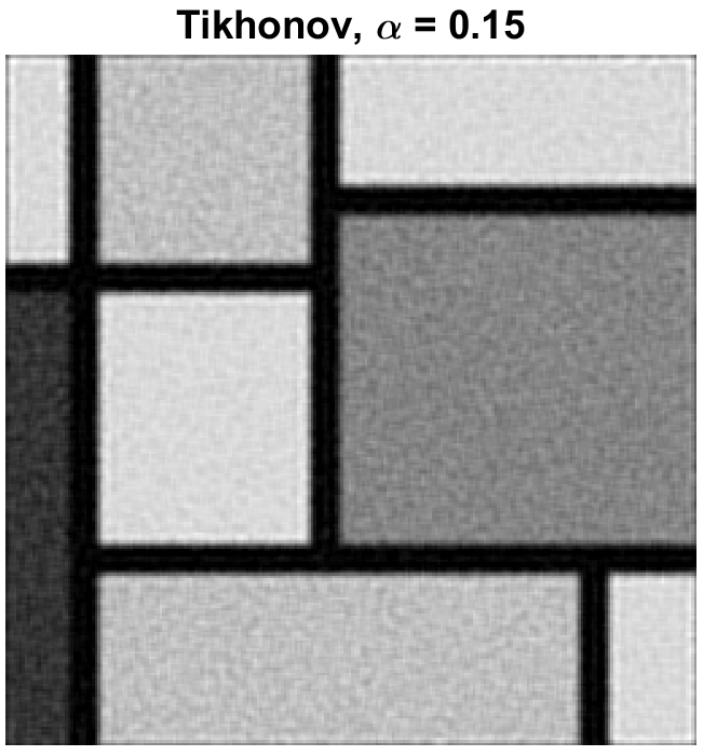
What regularization strategy would you pick? - I



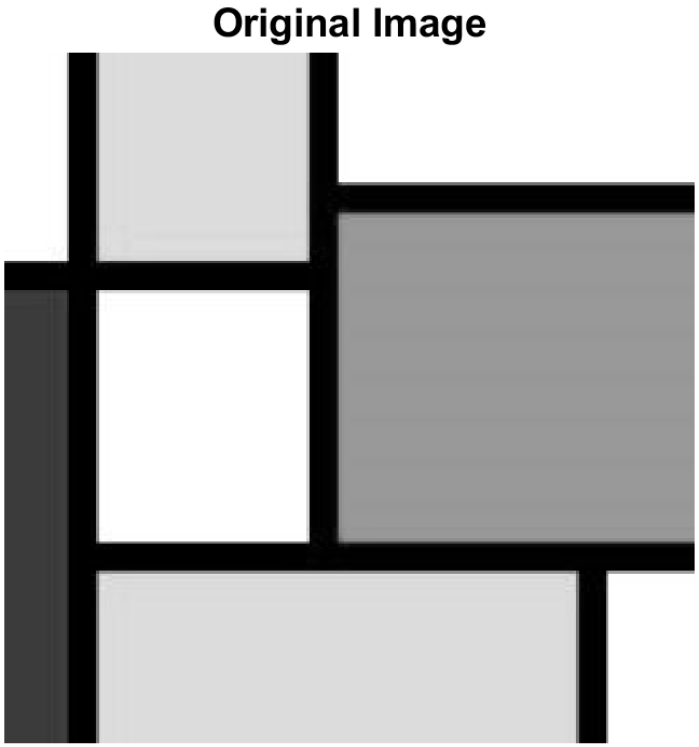


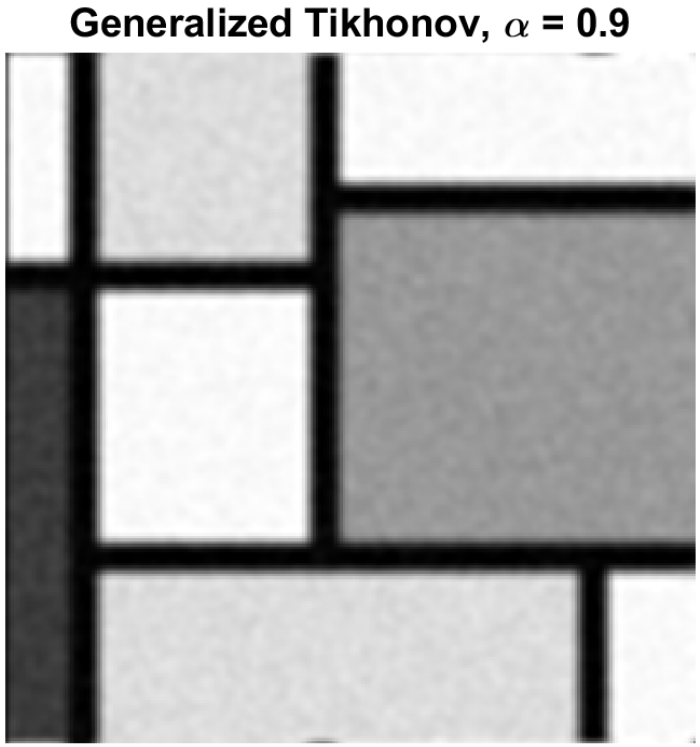
What regularization strategy would you pick? - I





What regularization strategy would you pick? - I





What regularization strategy would you pick? - I