Inverse Problems 1: convolution and deconvolution

Lesson 7: Tikhonov regularization, part I

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Table of contents

Review

Singular Value Decomposition

Linear inverse problems

Deconvolution can be seen as a linear inverse problem:

$$\text{given } m \in \mathbb{R}^n, \text{ find } f \in \mathbb{R}^n \text{ such that } Af = m, \text{ being } A \in \mathbb{R}^{n \times n}$$

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- if $d_n > 0$, the problem is well posed: $\forall m \in \mathbb{R}^n \quad \exists! f \in \mathbb{R}^n$ s.t. $Af = m$. Moreover, $f = A^{-1}m$ and $\|f - f_\delta\| \leq \|A^{-1}\|\|m - m_\delta\|$.
- if $d_r > 0$ and $d_{r+1} = \dots = d_n = 0$, then existence, uniqueness and stability fail.

What about deconvolution?

Main issue

The convolution matrix is an approximation in $\mathbb{R}^{n\times n}$ of a continuous operator $\mathcal{A}$ operating on real-valued function. As $n \to \infty$, A converges to $\mathcal{A}$ (in a suitable sense).

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For large n, due to finite precision, $d_n = 0$. $\Rightarrow$ ill-posedness.

Moore-Penrose pseudoinverse

The cure of ill-posedness is represented by the pseudoinverse:

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from now on, we will not distinguish between f and f^+

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   Existence fails: $range(A) \neq \mathbb{R}^n \Rightarrow \exists m \in \mathbb{R}^n : \quad \nexists f \in \mathbb{R}^n : Af = m$
   (you might never notice it if you do inverse crime). What to do?
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2. **A priori knowledge**
   Uniqueness fails: $ker(A) \neq \{0\} \Rightarrow \exists k \in \mathbb{R}^n : \quad Af = A(f + k)$. What to do? Use any a priori information at disposal on the solution E.g. the solution is smooth, has a known average, has small norm.

$$f^+ = \arg\min_{g \in \mathbb{R}^n} \{\|g\|, g \in \arg\min_{h \in \mathbb{R}^n} \{\|Ah - m\|\}\}$$

Regularization Theory

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Remark: a parameter choice $\alpha = \alpha(\delta)$ is defined *a priori*, since it holds for any m and any perturbation m_δ. We will focus more on *a posteriori* (heuristic) rules $\alpha = \alpha(\delta, m_\delta)$.

Graphical interpretation

Compute $f_{\alpha,\delta} = R_\alpha m_\delta$ instead of $f_\delta = A^+ m_\delta$. Error bound for the approximation of $f = f^+ = A^+ m$ (remember $\|m - m_\delta\| \leq \delta$):

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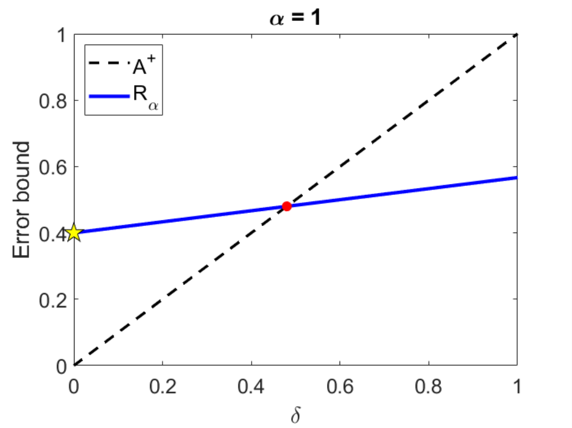

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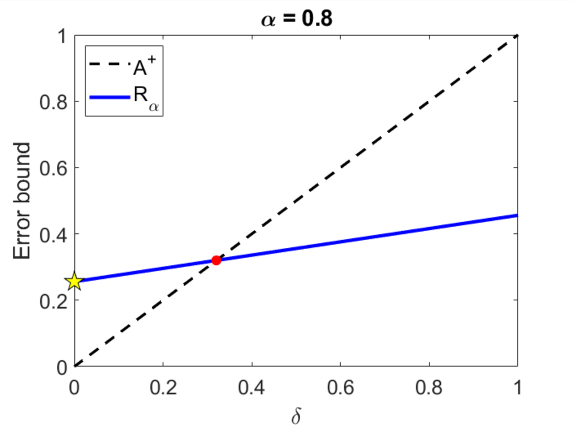

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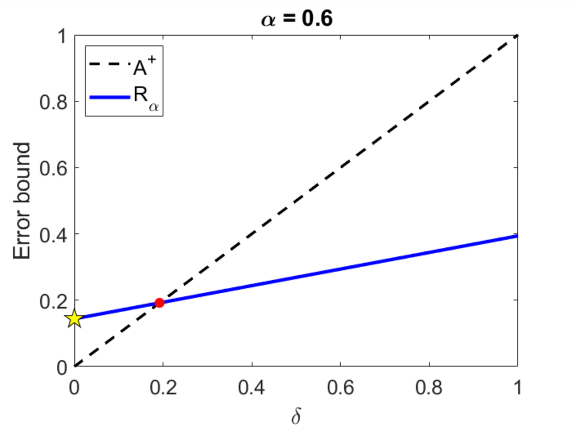

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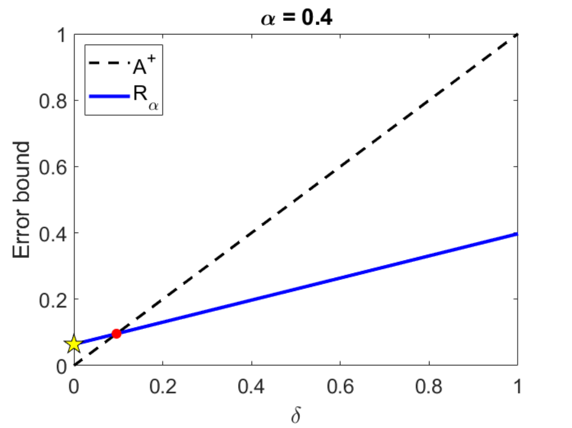

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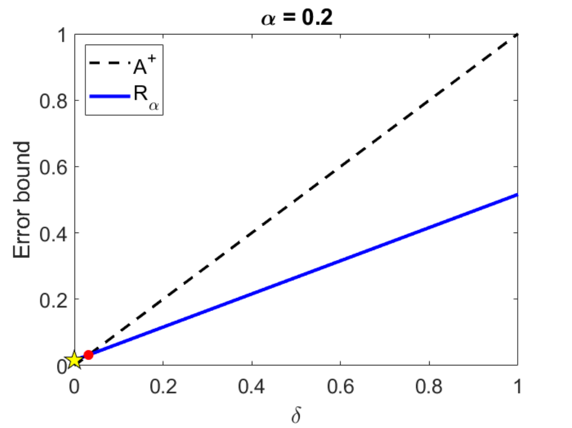

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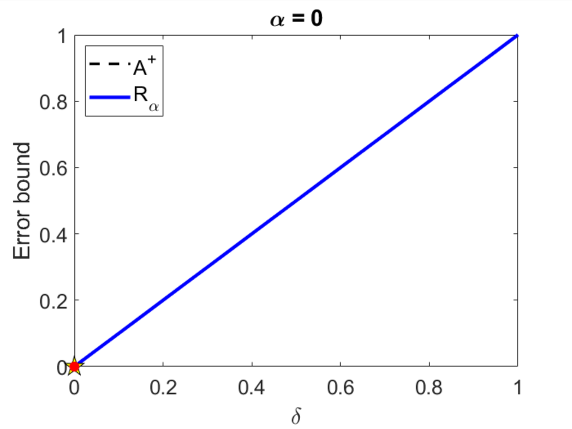

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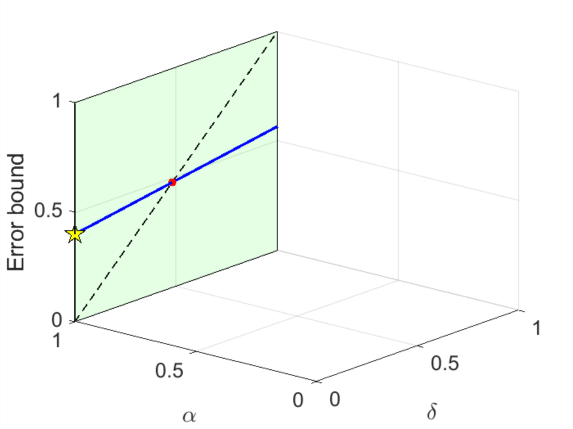

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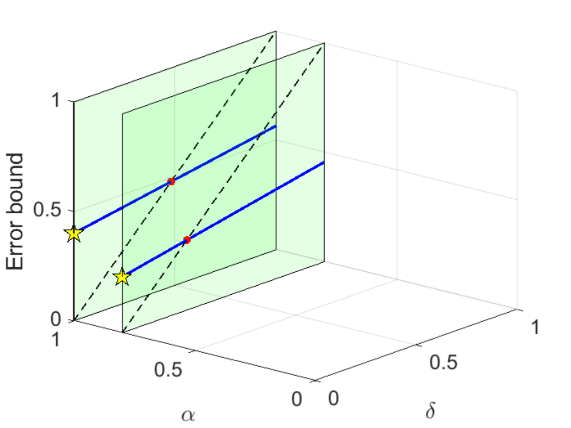

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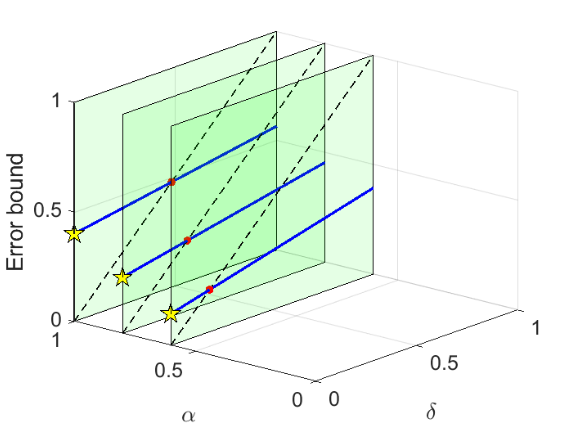

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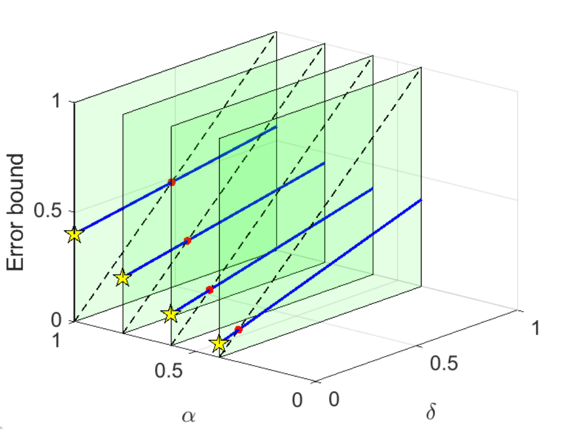

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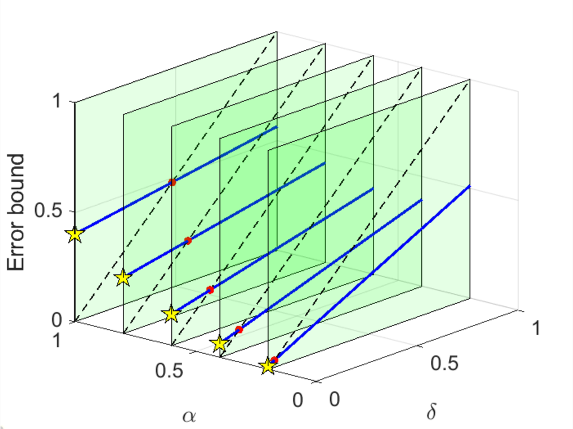

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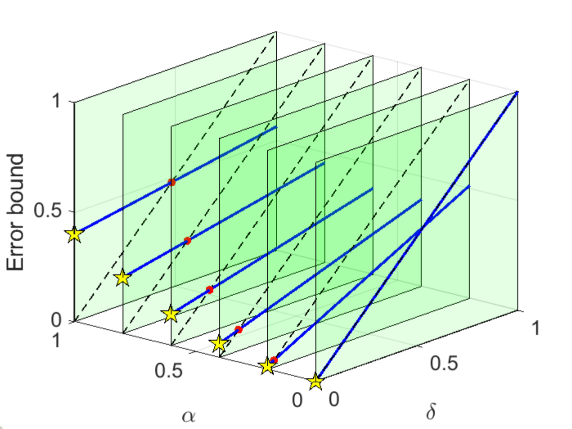

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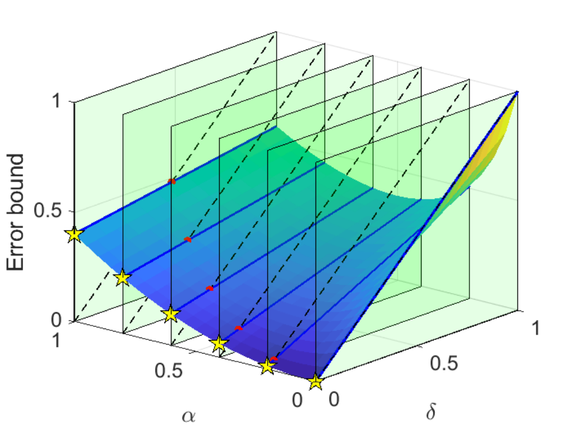

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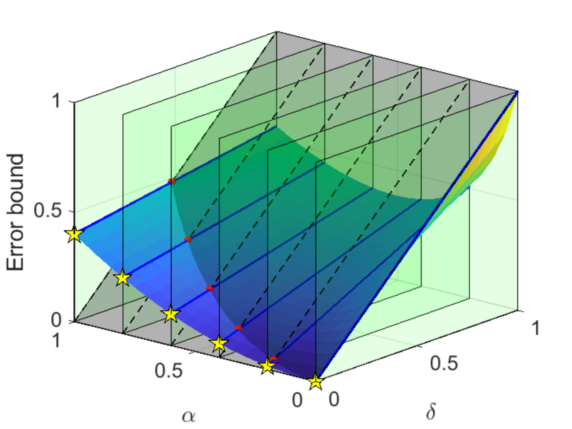

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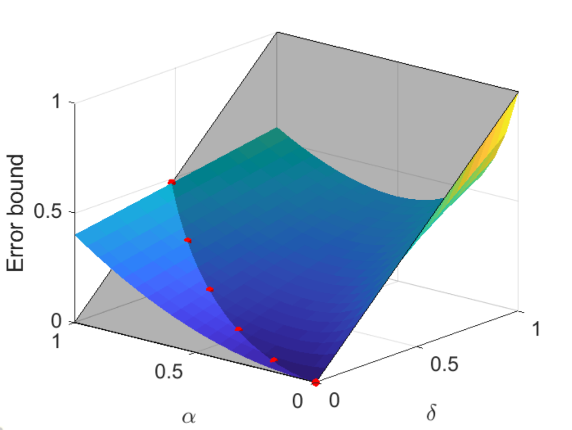

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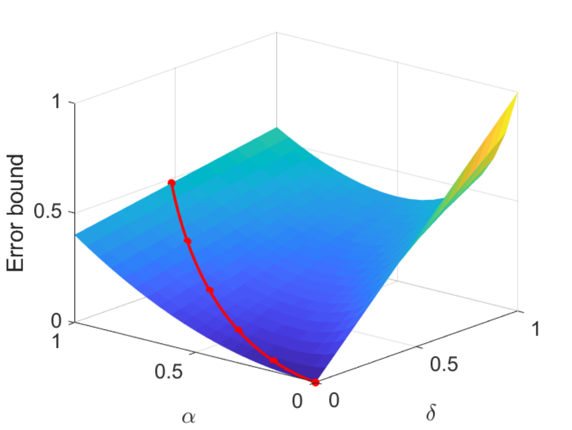

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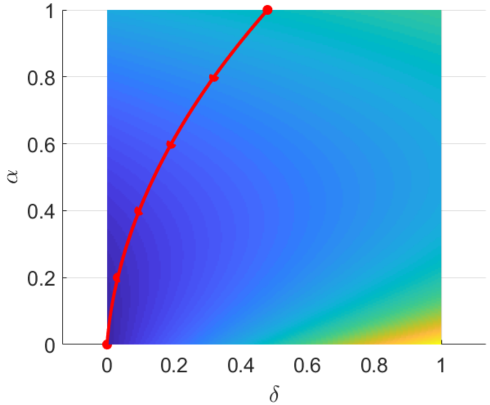

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Our goals:

- find an explicit expression for the matrix representing T_α;
- check that this choice is a regularization strategy;
- implement and see the benefits of regularization.

A first explicit expression

Theorem

Let $A = UDV^T$, being U, V orthogonal matrices and $D = diag(d_1, \ldots, d_n)$, such that $d_1 \geq \ldots \geq d_n$. Suppose that $\exists r \leq n$: $d_r > 0$ and $d_{r+1} = \ldots = d_n = 0$. Let

$$\mathcal{D}_\alpha^+ = diag\left(\frac{d_1}{d_1^2 + \alpha}, \ldots, \frac{d_n}{d_n^2 + \alpha}\right).$$

Then, $T_\alpha = V\mathcal{D}_\alpha^+ U^T$

A first explicit expression

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Let $A = UDV^T$, being U, V orthogonal matrices and $D = diag(d_1, \dots, d_n)$, such that $d_1 \geq \dots \geq d_n$. Suppose that $\exists r \leq n$: $d_r > 0$ and $d_{r+1} = \dots = d_n = 0$. Let

$$\mathcal{D}_\alpha^+ = diag\left(\frac{d_1}{d_1^2 + \alpha}, \dots, \frac{d_n}{d_n^2 + \alpha}\right).$$

Then, $T_\alpha = V\mathcal{D}_\alpha^+ U^T$

Preliminary remarks:

- in case $d_k = 0$, then $\dfrac{d_k}{d_k^2 + \alpha} = 0$;
- if $\alpha \to 0$ and $d_k \neq 0$, $\dfrac{d_k}{d_k^2 + \alpha} \to \dfrac{1}{d_k}$;

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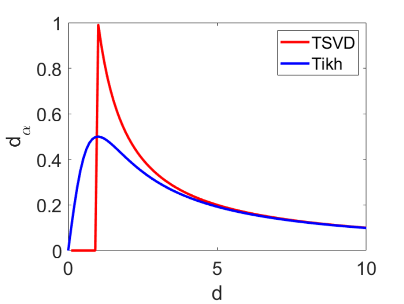
Comparison with TSVD:
we use a different manipulation on the singular values:
$d_\alpha^{Tikh}(d) = \frac{d}{d^2+\alpha}$,
whereas in TSVD we have:
$d_\alpha^{TSVD}(d) = \frac{1}{d}\chi_{(\alpha,\infty)}(d)$



Explicit expression - proof (part I)

- Since the matrix V is orthogonal, its columns consist in an orthonormal basis of $\mathbb{R}^n$. Hence, every vector $f \in \mathbb{R}^n$ can be expressed as $f = Va$, being $a \in \mathbb{R}^n$.

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- Let $T_\alpha m = Va^*$. Then, we can reformulate Tikhonov regularization as follows: find

$$a^* = \underset{a \in \mathbb{R}^n}{\arg\min}\{\|AVa - m\|^2 + \alpha\|Va\|^2\}$$

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- By the orthogonality of U and V, we notice that

$$\begin{aligned}
\|AVa - m\|^2 + \alpha\|Va\|^2 &= \|UDV^T Va - m\|^2 + \alpha a^T V^T Va \\
&= \|UDa - UU^T m\|^2 + \alpha\|a\|^2 = \|Da - U^T m\|^2 + \alpha\|a\|^2 \\
&= \|Da - m'\|^2 + \alpha\|a\|^2,
\end{aligned}$$

where we have defined $m' = U^T m$.

Explicit expression - proof (part II)

- D is a diagonal matrix; suppose $d_{r+i} = \ldots = d_n = 0$. By the definition of the norm of a vector:

$$\|Da - m'\|^2 + \alpha\|a\|^2 = \sum_{j=1}^{n}(d_j a_j - m'_j)^2 + \alpha \sum_{j=i}^{n} a_j^2$$

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$$= \sum_{j=1}^{r}(d_j^2 + \alpha)\left(a_j - \frac{d_j m'_j}{d_j^2 + \alpha}\right)^2 - \sum_{j=1}^{r} \frac{d_j^2 m'^2_j}{(d_j^2 + \alpha)^2} + \alpha \sum_{j=r+1}^{n} a_j^2 + \sum_{j=1}^{n} m'^2_j$$

Explicit expression - proof (part III)

- We are looking for a vector $a^* \in \mathbb{R}^n$ minimizing

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- The solution can be computed by hand selecting

$$a_j = \frac{d_j m_j'}{d_j^2+\alpha} \text{ for } j = 1, \dots, r; \qquad a_j = 0 \text{ for } j = r+1, \dots, n;$$

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  the first formula can be used in both cases ($d_{r+1} = \dots = d_n = 0$).
- We conclude $a^* = \mathcal{D}_\alpha^+ m' = \mathcal{D}_\alpha^+ U^T m$, hence $T_\alpha m = V\mathcal{D}_\alpha^+ U^T m$.

Theoretical results

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  hence, for any $m \in \mathbb{R}^n$, $T_\alpha \to A^+$ as $\alpha \to 0$.
- an a priori parameter choice is available for α, but we are not interested on it (theoretically demanding, sub-optimal with respect to the a posteriori rules we will learn)

Tikhonov regularization - implementation

Implementation

Example

1. Set $n = 200$. Define in Matlab the signal $f \in \mathbb{R}^n$ such that:

$$f = \begin{cases} 1 & \text{if } n_1 \leq n \leq n_2, \\ 3\dfrac{n - n_3}{n_4 - n_3} & \text{if } n_3 \leq n \leq n_4, \\ -1 - cos\left(\dfrac{2\pi n}{n_6 - n_5}\right) & \text{if } n_5 \leq n \leq n_6, \end{cases}$$

being
$n_1 = 2\lfloor \frac{n}{15} \rfloor$, $n_2 = 3\lfloor \frac{n}{15} \rfloor$,
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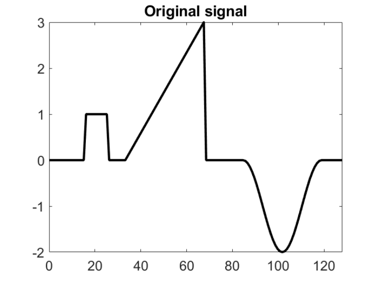

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2. Define the point spread function by normalizing the vector $[1, 4, 8, 16, 19, 15, 10, 7, 1]$. Create the convolution matrix associated to the zero-padding case.
3. Generate the noisy measurement $m_\delta = Af + \delta r$, being r a Gaussian random vector of n dimensions, $\|r\| = 1$. Consider $\delta = 10^{-5}$.
4. Compute the pseudoinverse A^+ of A and find $f_\delta = A^+ m_\delta$.
5. Select $\alpha = 0.1$. Compute T_α and find $f_{\alpha,\delta} = T_\alpha m_\delta$.
6. Compare $f_{\alpha,\delta}$ and f_δ by graphical inspection and by computing their normalized error with respect to $f^+ = f$.
7. Repeat the previous experiment with the following couples (δ, α):
   $(10^{-5}, 10)$, $(10^{-5}, 10^{-6})$, $(10^{-5}, 10^{-14})$; $(10^{-1}, 0.1)$, $(10^{-3}, 0.1)$, $(10^{-8}, 0.1)$.

Results interpretation