On the Euclidean Distance of Images

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Abstract

We present a new Euclidean distance for images, which we call IMage Euclidean Distance (IMED). Unlike the traditional Euclidean distance, IMED takes into account the spatial relationships of pixels. Therefore it is robust to small perturbation of images. We argue that IMED is the only intuitively reasonable Euclidean distance for images. IMED is then applied to image recognition. The key advantage of this distance measure is that it can be embedded in most image classification techniques such as SVM, LDA and PCA. The embedding is rather efficient by involving a transformation referred to as Standardizing Transform (ST). We show that ST is a transform domain smoothing. Using the Face Recognition Technology (FERET) database and two state-of-the-art face identification algorithms, we demonstrate a consistent performance improvement of the algorithms embedded with the new metric over their original versions.

1   Introduction

A central problem in image recognition and computer vision is determining the distance between images. Considerable efforts have been made to define image distances that provide intuitively reasonable results [4, 16, 1, 10]. Among others, two representative measures are the tangent distance [16] and the generalized Hausdorff distance [4]. Tangent distance is locally invariant with respect to some chosen transformations, and has been widely used in handwritten digit recognition. The generalized Hausdorff distance is not only robust to noise but also allows portions of one image to be compared with another, and has become a standard tool for comparing shapes.

However, from the image recognition point of view, many existing image distances suffer from some of the following disadvantages:

1) It is difficult to combine the metric with those powerful image recognition techniques such as SVM, LDA, PCA, etc

2) The computation of the measure is complicated.

3) The distance does not obey the triangle inequality, so sometimes two highly dissimilar images can be both similar to an unknown object.

Among all the image metrics, Euclidean distance is the most commonly used due to its simplicity. Let x, y be two M by N images, $x = (x^1, x^2, \cdots, x^{MN})$, $y = (y^1, y^2, \cdots, y^{MN})$, where x^{kN+l}, y^{kN+l} are the gray levels at location (k, l). The Euclidean distance $d_E(x, y)$ is given by

$$d_E^2(x, y) = \sum_{k=1}^{MN} \left(x^k - y^k\right)^2 \tag{1}$$
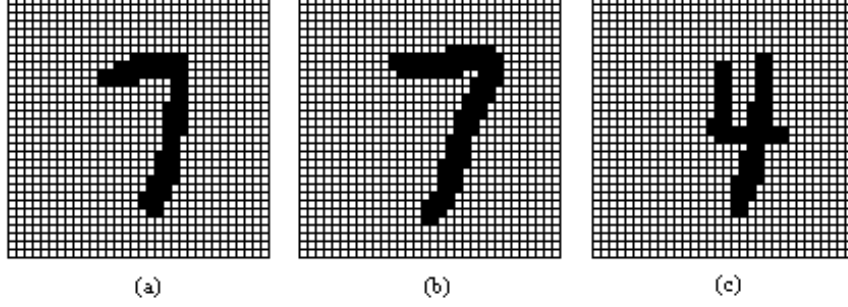

Figure 1: Similar and dissimilar images

However, this distance measure suffers from a high sensitivity even to small deformation. Fig. 1 shows three 32 by 32 images. A reasonable image metric should present smaller distance between (a), (b) than that of (a), (c). But the Euclidean distance gives counter intuitive result. For simplicity, let the gray levels be one at the black pixels and zero elsewhere. Computing the Euclidean distances yields $d_E(a, b) = 54$ and $d_E(a, c) = 49$. The pair with more similarity has a larger Euclidean distance! This phenomenon is caused by the fact that the Euclidean distance defined in (1) does not take into account that x, y are **images**, x^k, y^k are gray levels on **pixels**. For images, there are spatial relationships between pixels. The traditional Euclidean distance is only a summation of the pixel-wise intensity differences, and consequently small deformation may result in a large Euclidean distance.

This paper proposes a new Euclidean distance, which we call IMage Euclidean Distance (IMED). Unlike the traditional one, IMED takes into consideration the spatial relationships of pixels. Based on three properties that (arguably) any intuitively reasonable image metric should satisfy, we show that IMED is the only Euclidean distance possessing these properties.

IMED is then applied to image recognition. The key advantages of this metric are:

1) Relative insensitivity to small perturbation (deformation);

2) Simplicity of computation;

3) It can be efficiently embedded in most of the powerful image recognition techniques.

In order to simplify the computation of IMED, we introduce an image transformation referred to as Standardizing Transform (ST). We show that ST is a transform domain smoothing. This result directly relates image Euclidean distance to smoothing. It indicates that smoothing noiseless images can still increase the recognition rate.

The rest of this paper is organized as follows. Section 2 presents the image Euclidean distance. In Section 3, we provide a study on embedding IMED in image recognition algorithms. We also introduce the Standardizing transform (ST) and analyze its properties. Section 4 presents the experimental results. Finally, a conclusion is given in Section 5.

2 Image Euclidean Distance

All the M by N images are easily discussed in an MN dimensional Euclidean space, called image space. It is natural to adopt the base $e_1, e_2, \cdots, e_{MN}$ to form a coordinate system of the image space, where e_{kN+l} corresponds to an ideal point source with unit intensity at location (k, l). Thus an image $x = (x^1, x^2, \cdots, x^{MN})$, where x^{kN+l} is the gray level at the (k, l)th pixel, is represented as a point in the image space, and x^{kN+l} is the coordinate with respect to e_{kN+l}. The origin of the image space is an image whose gray levels are zero everywhere.

Although the algebra of the image space can be easily formulated as above, the Euclidean distance of images (i.e. the distance between their corresponding points in the image space) could not be determined until the metric coefficients [14, Lecture 14] of the basis are given. The metric coefficients g_{ij} $i, j = 1, 2, \cdots, MN$, are defined as

$$g_{ij} = <e_i, e_j> = \sqrt{<e_i, e_i>}\sqrt{<e_j, e_j>} \cdot \cos\theta_{ij} \tag{2}$$

where $<,>$ is the scalar product, and θ_{ij} is the angle between e_i and e_j. Note that, if $<e_i, e_i> = <e_j, e_j> = \cdots$, i.e. all the base vectors have the same length, then g_{ij} depends completely on the angle θ_{ij}. Given the metric coefficients, the Euclidean distance of two images x, y is written by

$$d_E^2(x, y) = \sum_{i,j=1}^{MN} g_{ij}(x^i - y^i)(x^j - y^j) = (x - y)^T G(x - y) \tag{3}$$

where the symmetric matrix $G = (g_{ij})_{MN \times MN}$ will be referred to as metric matrix.

For images of fixed size M by N, every MNth order symmetric and positive definite matrix G induces a Euclidean distance. But most of them are not appropriate for measuring image distances. For example, suppose any two base vectors e_i, e_j $(i \neq j)$, no matter which pixels they correspond to, are mutually perpendicular, the basis then forms a Cartesian coordinate system. Accordingly, G is the identity matrix, and it induces the traditional Euclidean distance given by (1). (If the base vectors have different lengths, G is a diagonal matrix. It induces the weighted Euclidean distance.) We have illustrated in the previous section (see Fig.1) this traditional metrics sensitivity to deformation, which is caused by the regardless of the fact that the two objects being compared are images. Geometrically, this defect is due to the orthogonality of the base vectors $e_1, e_2, \cdots, e_{MN}$, which correspond to pixels. Clearly, the information about the spatial relationship, i.e. the distances between the pixels, cannot be reflected by all mutually perpendicular base vectors. Such information, however, often appears in intuitive image distance as in the following statement: A slightly deformed image is very similar to the original one. Here, slightly deformed means that pixels in the deformed image are close to the corresponding pixels in the original image. This implies that a good Euclidean distance for images should contain the information of pixel distances. Accordingly, the metric coefficients, which define the Euclidean distance, have to be related to the pixel distances.

We next illustrate that if the metric coefficients depend properly on the pixel distances, the obtained Euclidean distance is insensitive to small deformation. The reader should be aware that two distances are being discussed here, one is the image distance measured in the high dimensional image space, the other is the pixel distance. Let $P_i, P_j, i, j = 1, 2, \cdots, MN$, be pixels. The pixel distance, written as $|P_i - P_j|$, is the distance between P_i and P_j on the image lattice. For example, if P_i is at location (k, l), and P_j is at (k', l'), $|P_i - P_j|$ may be $\sqrt{(k - k')^2 + (l - l')^2}$. Consider a simplification of the images (a), (b) in Fig. 1: Let the two digits seven, denoted by x, y respectively, are different on only two pixels P_i and P_j, as shown in Fig. 2 (a) and (b). Their Euclidean distance is completely determined by g_{ij}, or θ_{ij} if $g_{ii} = g_{jj} = 1$, i.e. e_i and e_j have the unit length (see (2) and (3)). It is a small deformation because P_i and P_j are close to each other. In Fig. 2 (c), like the traditional Euclidean distance, we set $\theta_{ij} = \pi/2$. Then $g_{ij} = \cos\theta_{ij} = 0$, hence

$$d_E(x,y) = \sqrt{g_{ii}(x^i - y^i)^2 + g_{jj}(x^j - y^j)^2 + 2g_{ij}(x^i - y^i)(x^j - y^j)} = \sqrt{g_{ii} + g_{jj}} = \sqrt{2}$$

While in Fig. 2 (d), we let , for the two nearby pixels, small, or equivalently thus obtained Euclidean distance is:

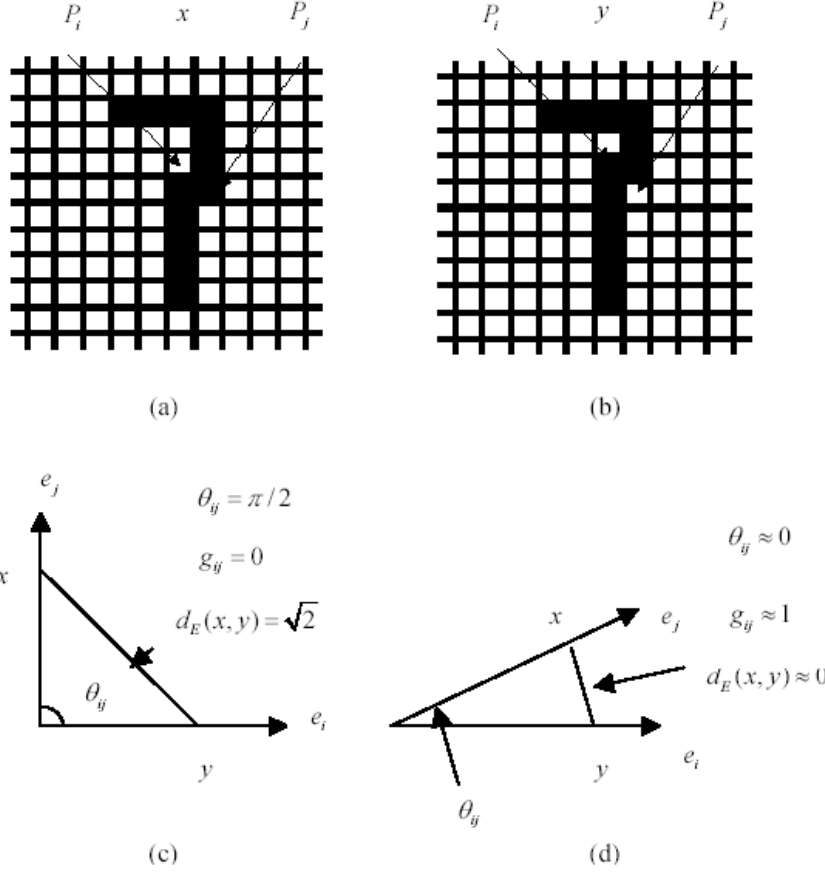$$d_E(x,y) = \sqrt{g_{ii} + g_{jj} - 2g_{ij}} \approx 0$$

Figure 2: Depending properly on pixel distance, metric coefficients induce a Euclidean distance that is robust to small deformation

This result indeed reflects the similarity between the two images. (It suffices to show x, y on the $P_i P_j$ plane in Fig. 2 (c) and (d), since they are identical on all the other pixels.) We see that giving metric coefficient g_{ij} a larger value for nearby pixels P_i, P_j makes the Euclidean distance robust to small perturbation.

Generally, we call a Euclidean distance IMage Euclidean Distance (IMED) if the metric satisfies three conditions which lead to appealing properties.

Definition 1. *A Euclidean distance $d(x,y) = \left[(x-y)^T G(x-y)\right]^{1/2}$, $G = (g_{ij})_{MN \times MN}$ is said to be an IMED, if the following conditions C1-C3 are satisfied.*

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(C1) The metric coefficient g_{ij} depends on the distance between pixels P_i and P_j. Let f represent this dependency.

$$g_{ij} = f\left(|P_i - P_j|\right), \qquad i, j = 1, 2, \cdots, MN. \tag{4}$$

(C2) f is continuous, and g_{ij} decreases monotonically as $|P_i - P_j|$ increases.

(C3) The functional dependency f is a universal function. That is, it is not for images of a particular size or resolution.

C1 means that the information about pixel distance must be considered in the metric. Depending only on $|P_i - P_j|$ makes g_{ij} (and hence the induced Euclidean distance) invariant to linear transformation of images. It also implies that all the base vectors have the same length, and therefore g_{ij} is proportional to $\cos\theta_{ij}$. C2 says how to merge the pixel distance into the metric coefficients so that the induced distance is intuitively reasonable. The continuity of f is a general necessity. The request that g_{ij} decreases as $|P_i - P_j|$ increases means that the distance depends on the extent of the deformation. Finally, C3 guarantees the universal validity of this distance measure.

More precisely, conditions C1-C3 imply that IMED is characterized by the following properties:

1) Small deformation yields small image distance. The stronger the deformation, the larger the distance. And the distance is continuous to the extent of deformation;

2) The distance between two images remains invariant if we perform the same translation, rotation and reflection to the images;

3) The metric applies to images of any size and resolution.

We argue that any intuitively reasonable image distance should have these properties. Thus IMED is the only reasonable Euclidean distance for images. (But there may exist infinitely many satisfactory non-Euclidean image metrics.) As a counterexample, the traditional Euclidean distance is not an IMED because f is discontinuous

$$g_{ij} = f\left(|P_i - P_j|\right) = \begin{cases} 1 & if \quad i = j \\ 0 & otherwise \end{cases}.$$

In order that the conditions C1-C3 are satisfied, f has to belong to a very special functional class. Note that the metric matrix G is positive definite for images of any size and resolution, so f must be the so-called positive definite function [15, p.815], [2, p.92]. A positive definite function is one that for arbitrary n, and arbitrary $P_1, P_2, \cdots, P_n$, the matrix $\left(f(|P_i - P_j|)\right)_{n \times n}$ is positive definite. The most important positive definite function, which we make use of to construct the metric coefficients, is the Gaussian function, written as

$$g_{ij} = f\left(|P_i - P_j|\right) = \frac{1}{2\pi\sigma^2} \exp\left\{-|P_i - P_j|^2 / 2\sigma^2\right\} \tag{5}$$

where σ is the width parameter and will be set to be 1 in the rest of this paper for simplicity. We denote the induced IMED as d_{IME}. Let two images be $x = (x^1, x^2, \cdots, x^{MN})$, $y = (y^1, y^2, \cdots, y^{MN})$, then $d_{IME}(x, y)$ is given by

$$d_{IME}^2(x, y) = \frac{1}{2\pi} \sum_{i,j=1}^{MN} \exp\left\{-|P_i - P_j|^2 / 2\right\} \left(x^i - y^i\right) \left(x^j - y^j\right) \tag{6}$$

Recomputing the image distances in Fig. 1 by (6) yields $d_{IME}(a, b) = 23.3$, $d_{IME}(a, c) = 27.3$. The new metric does provide intuitively reasonable results.

3 Embedding IMED in Image Recognition Algorithms

Although image metrics may be utilized for image recognition in a direct matching (nearest neighbor) algorithm, a key issue of an image distance is whether it can be embedded in other powerful image recognition techniques. Because experimental results have demonstrated that the state-of-the-art recognition techniques outperform direct matching in various applications.

The major significance of IMED is that one can easily embed it in most of the existing recognition algorithms, because these algorithms are based on Euclidean distance. Some of them are

Radial Basis Function Support Vector Machines (RBF SVMs) [18]: The discriminant function RBF SVMs depends only on the Euclidean distance.

$$h(x) = \sum_{i=1}^{n} \alpha_i \exp\left\{-\gamma \|x - x_i\|^2\right\}$$

Principal Component Analysis (PCA) [7]: Also called Eigenfaces method in human face recognition literatures [9], [17]. It is well known that computing the eigenspace spanned by the leading eigenvectors is a least square problem, which involves only Euclidean distance.

Bayesian similarity[12]: A powerful method for face recognition. It essentially performs PCA for two kinds of image differences (intra-class and extra-class) respectively.

Embedding IMED in these techniques is straightforward, just use IMED instead of the traditional Euclidean distance in the algorithms. For example, after the embedding, the discriminant function of SVMs with RBF kernel becomes

$$h(x) = \sum_{i=1}^{n} \alpha_i \exp\left\{-\gamma (x - x_i)^T G(x - x_i)\right\}$$

In these algorithms, one often needs to compute IMED, i.e. $(x_i - x_j)^T G(x_i - x_j)$, for all pairs of images. Thus, for large database this evaluation is expensive. However, these computations can be greatly simplified by introducing a linear transformation. Consider a decomposition of matrix G, $G = A^T A$. If we transform all images $x, y, \cdots$ by A, and denote $u = Ax$, $v = Ay, \cdots$, then IMED between x, y is equal to the traditional Euclidean distance between u, v:

$$(x - y)^T G(x - y) = (x - y)^T A^T A(x - y) = (u - v)^T (u - v)$$

Thus one avoids unnecessary and repeated computation of IMED by utilizing the transformed images $u, v, \cdots$ as inputs to the image recognition algorithms.

Although there are infinitely many decompositions of the form $G = A^T A$, we adopt a very special one. It reveals the nature of IMED, which will be evident later. The decomposition is written by

$$G = G^{1/2} G^{1/2} \tag{7}$$

where the symmetric matrix $G^{1/2}$ is uniquely defined as

$$G^{1/2} = \Gamma \Lambda^{1/2} \Gamma^T \tag{8}$$

Here, Λ is a diagonal matrix whose elements are eigenvalues of G (Remember that G is positive definite, so the diagonal entries of $\Lambda^{1/2}$ are positive real numbers), and Γ is an orthogonal matrix whose column vectors are eigenvectors of G. Since $G = \Gamma \Lambda \Gamma^T$, it is obvious that (7) is valid. Thus applying the transformation $G^{1/2}$ to the images x, y

$$u = G^{1/2}x, v = G^{1/2}y \tag{9}$$

reduces IMED between x and y to the traditional Euclidean distance between u, v:

$$(x - y)^T G(x - y) = (x - y)^T G^{1/2}G^{1/2}(x - y) = (u - v)^T(u - v) \tag{10}$$

We call the transformation $G^{1/2}(\cdot)$ Standardizing transform (ST). Hence, feeding the transformed images to a recognition algorithm automatically embeds IMED in it.

An interesting result is that ST is a transform domain smoothing. Note that ST is a composition of three operations $G^{1/2} = \Gamma\Lambda^{1/2}\Gamma^T$: the orthogonal transform $\Gamma^T(\cdot)$, followed by multiplying the square roots of the eigenvalues of G, and finally the inverse transform of $\Gamma^T(\cdot)$ ($\Gamma = (\Gamma^T)^{-1}$ since it is an orthogonal matrix). We show in Fig. 3 the column vectors of Γ (as images), which are usually called basis images. Each basis image is of size 8×8. The low frequency basis images are eigenvectors of G corresponding to larger eigenvalues, while high frequency ones to the smaller eigenvalues. ST is to compress the high frequency components of an image and is therefore a smoothing. Some facial images (the top row) and those after transformation (the second row) are shown in Fig. 4. The smoothing result is perceptible.
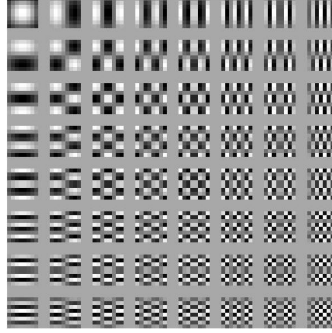

Figure 3: Basis images of the (two-dimensional) orthogonal transform Γ^T
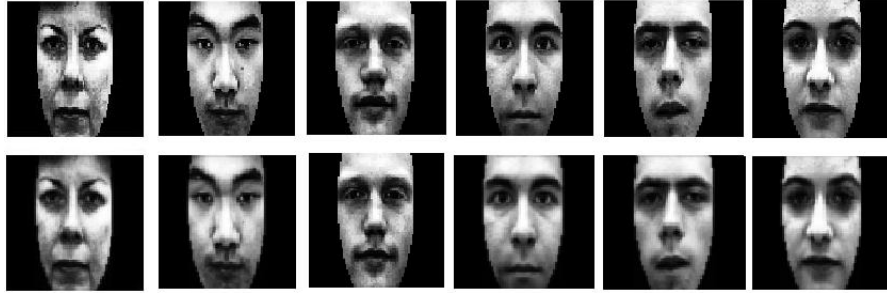

Figure 4: Smoothing effects of the Standardizing transform (ST) on facial images

There exist fast algorithms for ST, i.e. $u = G^{1/2}x = \Gamma\Lambda^{1/2}\Gamma^T x$. Note that G is separable [6, p.36], [5]:

$$g_{iN+j,i'N+j'} = \exp\left\{-\left[(i - i')^2 + (j - j')^2\right]/2\right\} = \exp\left\{-(i - i')^2/2\right\} \cdot \exp\left\{-(j - j')^2/2\right\}$$

The MN by MN matrix G can be written as the Kronecker product of an M by M and an N by N matrices. Consequently, the eigenvectors of G, i.e. Γ, is also separable [6, p.165]. The transformation $\Gamma^T x$ can be realized by a succession of two one-dimensional transforms on x. Moreover, most entries of $G^{1/2}$ are nearly zero, so ST may be well approximated in spatial domain by a 5×5 mask for a practical use.

Finally, we pointed out that IMED must NOT be understood as a Mahalanobis distance [11], given by

$$d_M^2(\xi, \eta) = (\xi - \eta)^T \Sigma^{-1}(\xi - \eta) \tag{11}$$

That is, the matrix G in IMED (see (3)) must NOT be viewed as some inverse covariance matrix Σ^{-1}, although both G and Σ^{-1} are symmetric and positive definite matrices.

On the surface, IMED and Mahalanobis distance have similar expressions. However, the two distance measures are essentially different for the following two reasons:

1) In the derivation of IMED, no random variables or statistics were involved.

2) More importantly, the Mahalanobis distance has a completely opposite behavior to IMED. That is, Mahalanobis distance is even more sensitive to small deformation than the traditional Euclidean distance. In fact, the covariance matrix Σ of images (random fields) have been studied in KLT based image compression [6, chapter 2, 5], [6] for a long time. One of the commonly used covariance model $\Sigma = (\sigma_{ij})_{MN \times MN}$ is given by

$$\sigma_{ij} = \exp\{-|P_i - P_j|\} \tag{12}$$

Comparing it to (5), it is Σ but not Σ^{-1} that is similar to G. Accordingly, using this Mahalanobis distance for image recognition leads to an opposite effect to IMED.

4 Experimental Results

We have conducted two sets of experiments. The goal of the first set was to compare IMED with several other image metrics in terms of recognition rate using a nearest neighbor classifier. The second set of experiments was to test whether embedding IMED in an image recognition technique can improve that algorithms accuracy.

4.1 A Comparison of Image Metrics on Handwritten Digit Recognition

We first conducted experiments that compare IMED with the following image metrics. For more details about these distances, please see [4], [16], and [10]. Here only present short descriptions.
Traditional Euclidean distance: The most commonly used metric, given in (1).
Tangent distance [16]: Developed particularly for handwritten digit recognition. This distance is locally invariant to seven chosen transformations: line thickening and thinning, and translation, rotation, scaling, and two hyperbolic transforms.
Generalized Hausdorff distance [4]: Very useful in comparing shapes. One limitation is that it is for binary images. To evaluate it in our experiment, we transfer the images to binary images using an edge detector [3] before recognition. The parameters of the generalized Hausdorff distance are tuned by cross validation on the training set.
Fuzzy Image Metric (FIM) [10]: Defined by fuzzy integral. Primarily used in image quality assessment for image compression.

The recognition methodology is the direct matching (nearest neighbor classifier) in terms of each metric respectively. With a certain image distance, when a test pattern is to be classified, the nearest

prototype in the training set is found, and the pattern is given the class as that prototype. We test them on handwritten digit recognition, which is a ten-class problem, using the US postal service (USPS) database. It consists of 16 by 16 pixel size-normalized images of handwritten digits, divided into a training set of 7,291 prototypes and a test set of 2,007 pattern. Several examples are shown in Fig. 5 [8].

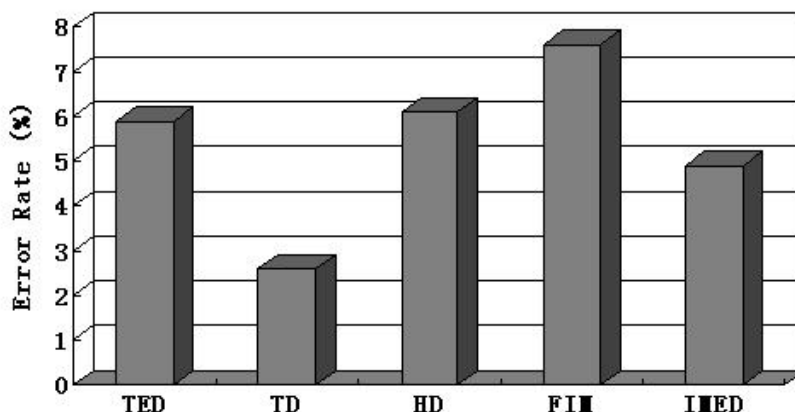Figure 5: Some examples of the handwritten digits in the USPS database.

Figure 6: Comparison of image metrics on USPS database. TED: traditional Euclidean distance. TD: tangent distance. HD: generalized Hausdorff distance. FIM: fuzzy image metric. IMED: Image Euclidean distance.

The recognition results are plotted in Fig. 6. IMED outperforms all the metrics except for the tangent distance, which is robust to common variations in handwritten digits, and is particularly appropriate for this task.

4.2   Embedding IMED in Recognition Techniques for Face Identification

The primary goal of the experiments in this section is to test a more important ability of IMED. That is, whether embedding IMED in an image recognition algorithm can improve that algorithms performance. Note that it is difficult to combine those intelligent metrics that are good at direct matching with the recognition techniques described below.

Embedding IMED in an algorithm is simple: first transform all images by ST (see (9)), and then run the algorithm with the transformed images as inputs.

The recognition task is human face identification, which has become an active area of research over the last decade. The facial images are all from the FERET database [13]. FERET database contains four probe categories, named FB, duplicate I, duplicate II, and fc respectively, and they all share a common gallery. A probe set consists of images of unknown faces, while the gallery contains images of known individuals. For details about the FERET database and the terminologies, please see [13]. We choose a training set of 1068 frontal images from 554 classes to train the models.

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We embedded IMED in the following two methods. For more details the reader is referred to [17], [9], [12]

Eigenfaces (PCA) [17], [9]: A ground-breaking work in face recognition. The algorithm used here was presented as the baseline in the FERET competition. Faces were represented by their projection onto the first 200 eigenvectors and were identified by a nearest neighbor classifier using the L_1 metric.

Bayesian similarity [12]: The top performing system in the FERET competition. It defines a probabilistic measure of similarity based on a Bayesian (MAP) analysis of image differences. We use in the experiments a simplified measure called ML similarity, which was reported to result in only a minor deficit in recognition error rate while cutting the computation cost by a factor of 2.

The front end to the experiments consists of a preprocessing procedure, where all the faces were 1) translated, rotated, and scaled so that the centers of the eyes were placed on specific pixel, 2) faces were masked to remove background and hair, and 3) the non-masked pixels were processed by a histogram equalization. Several examples of the preprocessed facial images (of the same individual) are shown in Fig. 7.

Figure 7: Facial images of one individual from the FERET database.

The results of the two pairs of algorithms *Eigenfaces* and *IMED + Eigenfaces*, *Bayesian similarity* (Shortened as *Bayes*) and *IMED + Bayesian similarity* (Shortened as *IMED + Bayes*) on the four probe sets were plotted in Fig. 8. The performance statistics is cumulative match score. In all the experiments, the algorithms embedded with IMED outperform the original ones respectively.

5   Conclusion

It is desirable to define an image metric that can be efficiently embedded in the existing image recognition methods. Euclidean distance is consequently a candidate because, representing images as points in a high dimensional Euclidean space, the so-called image space, is a common starting point of most recognition algorithms. Although there are infinitely many Euclidean distances for images (for every symmetric and positive definite matrix defines a Euclidean distance, see Section 2), they often provide counter intuitive results. For example, the traditional Euclidean distance is sensitive to deformation and translation due to the lack of consideration of the spatial relationship of pixels. IMED, to a certain extent, overcomes this defect. Experiments on FERET datasets demonstrated a consistent performance improvement of two state-of-the-art algorithms when embedded with IMED.

By an analysis on the Standardizing Transform, we relate smoothing to image Euclidean distance. This theoretical result indicates that smoothing noiseless images can still increase the recognition rate.

One limitation of IMED is that it does not always provide the best recognition result comparing with some other intelligent metrics under the nearest neighbor rule. We think there are two future directions:

1) Find an efficient way (if possible) to embed, for instance the tangent distance, in those image recognition algorithms.
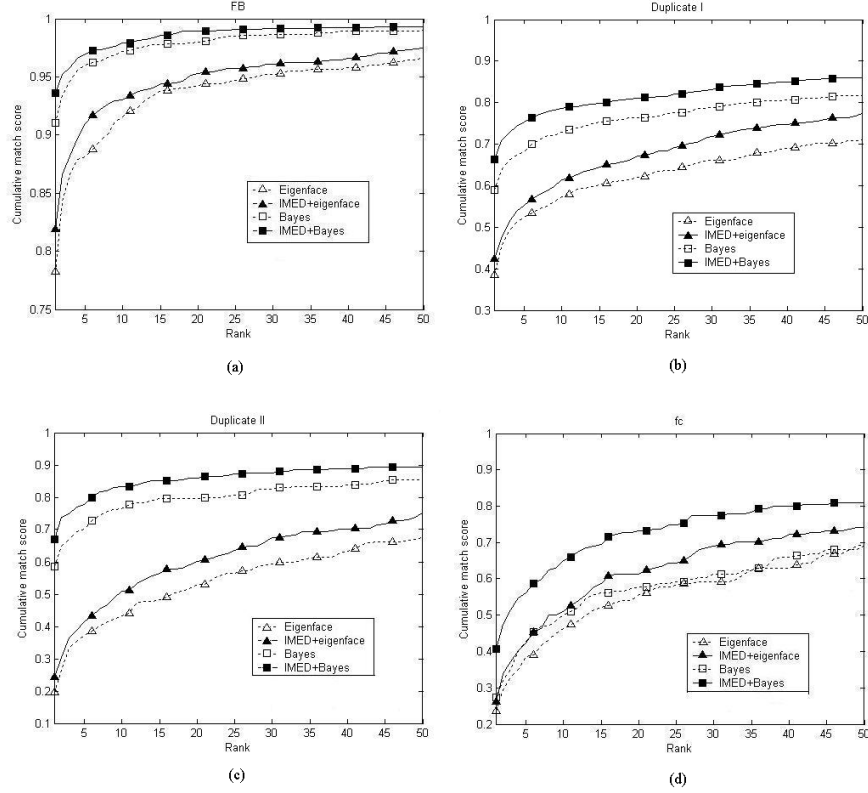
Figure 8: Identification performance on FERET database. (a) FB probes. (b) Duplicate I probes. (c) Duplicate II probes. (d) fc probes.

2) Looking for new image metrics that are good at direct matching as well as embedding ability.

References

[1] R. Bajcsy and S. Kovačič. Multiresolution elastic matching. *Computer vision, graphics, and image processing*, 46(1):1–21, 1989.

[2] S. Bochner. *Lectures on Fourier integrals*. Number 42. Princeton University Press, 1959.

[3] J. Canny. A computational approach to edge detection. *Pattern Analysis and Machine Intelligence, IEEE Transactions on*, (6):679–698, 1986.

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[4] D.P. Huttenlocher, G.A. Klanderman, and W.J. Rucklidge. Comparing images using the hausdorff distance. *Pattern Analysis and Machine Intelligence, IEEE Transactions on*, 15(9):850–863, 1993.

[5] A.K. Jain. Advances in mathematical models for image processing. *Proceedings of the IEEE*, 69(5):502–528, 1981.

[6] A.K. Jain. *Fundamentals of digital image processing*, volume 3. prentice-Hall Englewood Cliffs, 1989.

[7] I.T. Jolliffe. *Principal component analysis*. Springer-Verlag, New York, 1986.

[8] D. Keysers, R. Paredes, H. Ney, and E. Vidal. Combination of tangent vectors and local representations for handwritten digit recognition. In *Structural, Syntactic, and Statistical Pattern Recognition*, pages 538–547. Springer, 2002.

[9] M. Kirby and L. Sirovich. Application of the karhunen-loeve procedure for the characterization of human faces. *Pattern Analysis and Machine Intelligence, IEEE Transactions on*, 12(1):103–108, 1990.

[10] J. Li, G. Chen, and Z. Chi. A fuzzy image metric with application to fractal coding. *Image Processing, IEEE Transactions on*, 11(6):636–643, 2002.

[11] K.V. Mardia, J.T. Kent, and J.M. Bibby. *Multivariate analysis*. Academic press, 1979.

[12] B. Moghaddam, T. Jebara, and A. Pentland. Bayesian face recognition. *Pattern Recognition*, 33(11):1771–1782, 2000.

[13] P.J. Phillips, H. Moon, S.A. Rizvi, and P.J. Rauss. The feret evaluation methodology for face-recognition algorithms. *Pattern Analysis and Machine Intelligence, IEEE Transactions on*, 22(10):1090–1104, 2000.

[14] M. Postnikov. *Lectures in Geometry: Semester I: Analytic Geometry*. Translated by V. Shokurov, MirPublishers, Moscow, 1982.

[15] I.J. Schoenberg. Metric spaces and completely monotone functions. *Annals of Mathematics*, pages 811–841, 1938.

[16] P. Simard, Y.L. LeCun, and J. Denker. Efficient pattern recognition using a new transformation distance. In *Advances in neural information processing systems*, pages 50–58, 1993.

[17] M.A. Turk and A.P. Pentland. Face recognition using eigenfaces. In *Proc. IEEE Conference on Computer Vision and Pattern Recognition*, pages 586–591, 1991.

[18] V. Vapnik. *Statistical learning theory*. John Wiley, New York, 1998.