• The rank-nullity theorem

- PROOF: LET B BE A BASIS FOR THE NULL SPACE OF T AND LET C BE A BASIS FOR THE RANGE OF T. SINCE B IS A BASIS FOR THE NULL SPACE, EVERY VECTOR IN V CAN BE WRITTEN AS A LINEAR COMBINATION OF VECTORS IN B AND A VECTOR IN THE RANGE OF T. THEREFORE, WE CAN EXPRESS EVERY VECTOR IN V AS A LINEAR COMBINATION OF VECTORS IN B AND A VECTOR IN C. THIS MEANS THAT B UNION C IS A BASIS FOR V.
- SINCE B AND C ARE BOTH BASES, THEY MUST BOTH BE FINITE SETS, AND HENCE B UNION C MUST ALSO BE A FINITE SET. THEREFORE, DIM(V) = |B UNION C| = |B| + |C| = NULLITY(T) + RANK(T).

1. The inverse matrix theorem:

- PROOF: WE NEED TO SHOW THAT T^(-1) IS A LINEAR TRANSFORMATION AND THAT ITS MATRIX REPRESENTATION WITH RESPECT TO SOME BASES OF W AND V IS GIVEN BY A^(-1).
- FIRST, LET'S SHOW THAT T^(-1) IS A LINEAR TRANSFORMATION. SUPPOSE THAT W1 AND W2 ARE IN W AND C1 AND C2 ARE SCALARS. WE NEED TO SHOW THAT T^(-1)(C1W1 + C2W2) = C1T^(-1)(W1) + C2T^(-1)(W2).
- SINCE A IS INVERTIBLE, WE CAN MULTIPLY BOTH SIDES OF THE EQUATION T(T^(-1)(W)) = W BY A^(-1) TO GET:
- A^(-1)*T(T^(-1)(W)) = A^(-1)*W
- THIS SIMPLIFIES TO:

- T^(-1)(W) = A^(-1)*W
- THEREFORE, T^(-1)(C1W1 + C2W2) = C1A^(-1)W1 + C2A^(-1)W2 = C1T^(-1)(W1) + C2T^(-1)(W2).
- THIS SHOWS THAT T^(-1) IS A LINEAR TRANSFORMATION.
- NOW, LET'S SHOW THAT THE MATRIX REPRESENTATION OF T^(-1) WITH RESPECT TO SOME BASES OF W AND V IS GIVEN BY A^(-1). LET W1, W2, ..., WN BE A BASIS FOR W AND LET V1, V2, ..., VM BE A BASIS FOR V. THE MATRIX REPRESENTATION OF T^(-1) WITH RESPECT TO THESE BASES IS GIVEN BY THE MATRIX B = [T^(-1)(W1) | T^(-1)(W2) | ... | T^(-1)(WN)].
- SINCE THE MATRIX REPRESENTATION OF T WITH RESPECT TO THESE BASES IS A = [T(V1) | T(V2) | ... | T(VM)], WE HAVE:
- B*A = [T^(-1)(W1) | T^(-1)(W2) | ... | T^(-1)(WN)] * [T(V1) | T(V2) | ... | T(VM)]
- = [T^(-1)(W1) * T(V1) + T^(-1)(W2) * T(V2) + ... + T^(-1)(WN) * T(VM)]
- = [I]
- WHERE I IS THE IDENTITY MATRIX. THIS MEANS THAT A*B = I, SO B = A^