Introduction to graph theory

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Akademia Górniczo-Hutnicza w Krakowie

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Each edge is a two-element subset of V. For notation convenience, instead of representing the edge $\{x, y\}$ we will denote it by xy.

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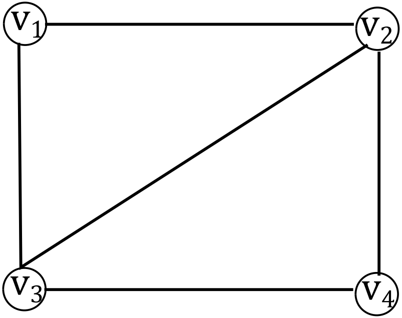

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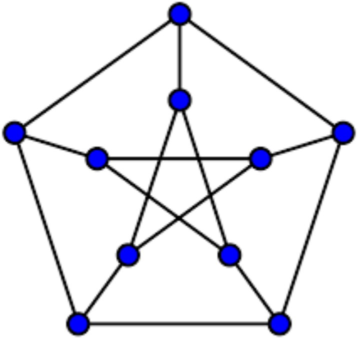
G – r-regular, if $\forall v \in V(G)\ d(v) = r$.

3-regular graph



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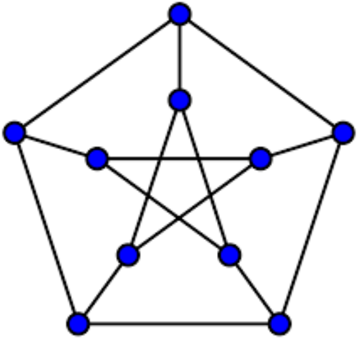

Petersen graph

Theorem: handshake lemma

The degree sum formula states that for a given graph $G = (V, E)$

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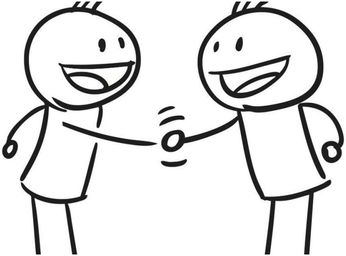

From Newsweek.

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A degree sequence

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It is clear that each simple graph has exactly one degree sequence, but that the converse need not hold.

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Is $(4, 1, 1)$ graphic?

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Is $(4, 3, 2, 2, 1)$ graphic?

A degree sequence

Theorem: Erdős, Gallai, 1960

A sequence $\{d_1, \ldots, d_n\}$ is graphic iff the sum of the elemnts is even and the sequence obeys the property

$$\sum_{i=1}^{r} d_i \leq r(r-1) + \sum_{i=r+1}^{n} \min\{r, d_i\}$$

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Theorem: Havel (1955) and Hakimi (1962)

A degree sequence with $n \geq 3$ and $d_1 \geq 1$ is graphical iff the sequence $\{d_{2-1}, d_{3-1}, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_p\}$ is graphical.

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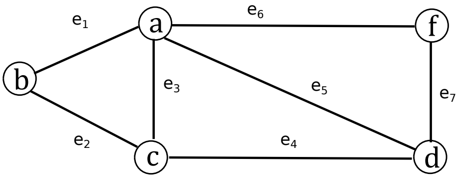

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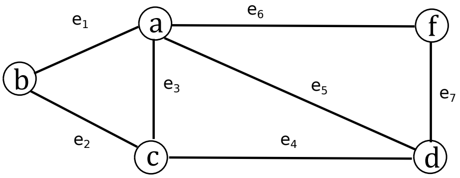

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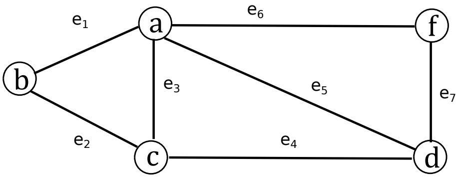

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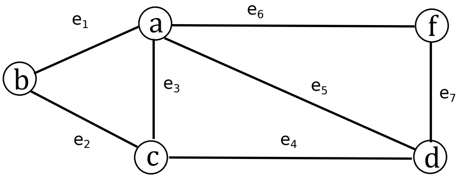

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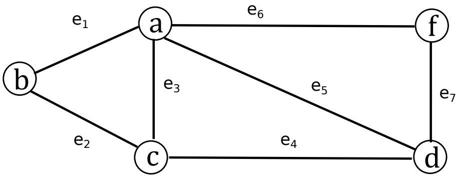

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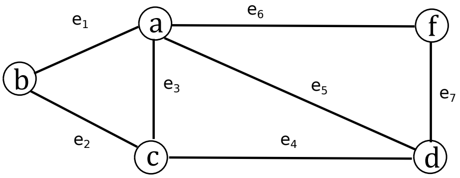
P_n – a path of length n.



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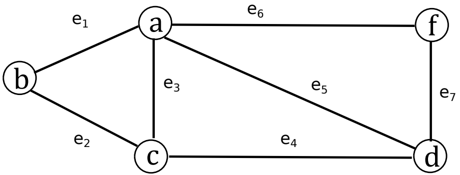

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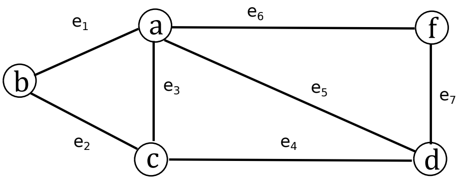
By C_n we denote a cycle with n edges (n vertices).



Bipartite graphs

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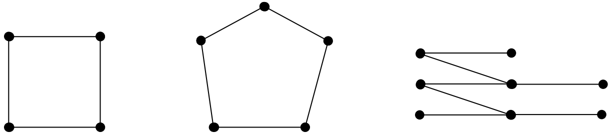

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Theorem:

A graph G is bipartite iff G has no cycle of odd size.

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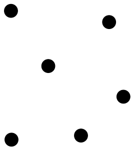
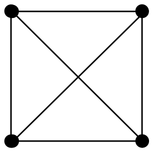


E_6



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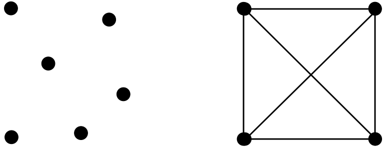

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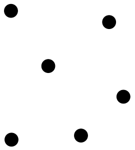

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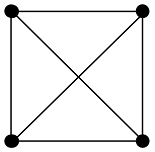

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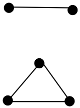

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More examples of graphs:

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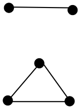

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Note: $\bar{\bar{G}} = G$.

Matrices associated with graphs

Definition:

The adjacency matrix of a graph G with the vertex set $V = \{v_1, v_2, \ldots, v_n\}$ is a binary square $n \times n$ matrix $A(G) = (a_{ij})_{n \times n}$ such that

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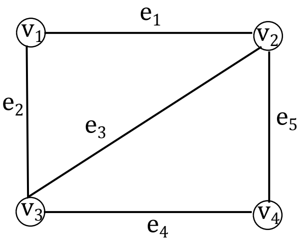

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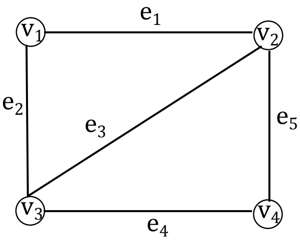

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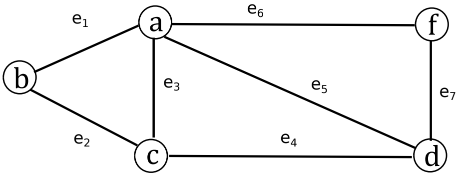

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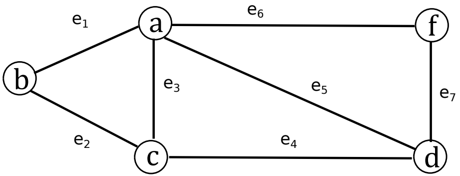

Graph G

Is $G' = (V', E')$ such that $V' = \{a, b, c\}$ and $E' = \{e_5\}$ a subgraph of G?

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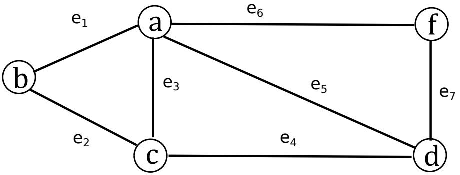

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Is $G' = (V', E')$ such that $V' = \{a, b, c\}$ and $E' = \{e_1\}$ a subgraph of G?

Yes

Subgraphs

Definition:

Let $G = (V, E)$ be any graph, and let $S \subset V$ be any subset of vertices of G. Then the induced subgraph $G[S]$ is the graph whose vertex set is S and whose edge set consists of all of the edges in E that have both endpoints in S.

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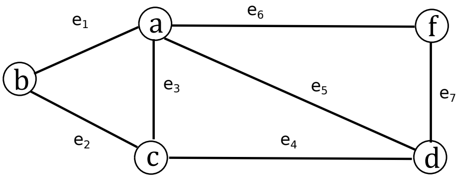

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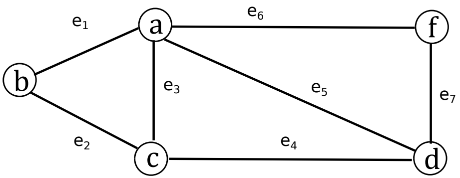

Graph G

Is $G^{'} = (V^{'}, E^{'})$ such that $V^{'} = \{a, b, c\}$ and $E^{'} = \{e_1\}$ an induced subgraph G (i.e. $G^{'} = G[V^{'}]$)?

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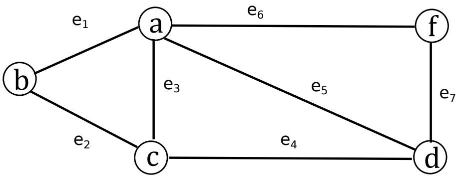

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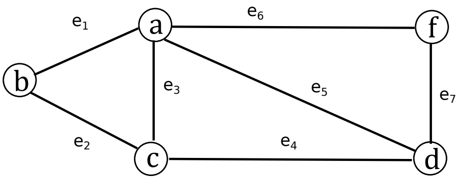

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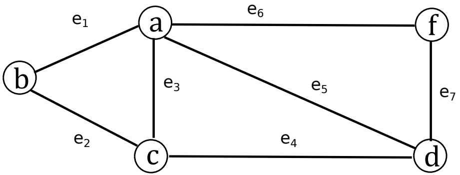

Graph G

Is G connected?

YES

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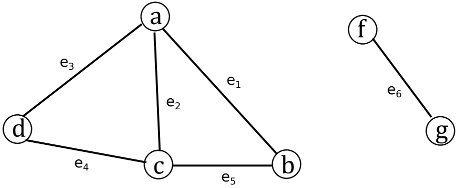

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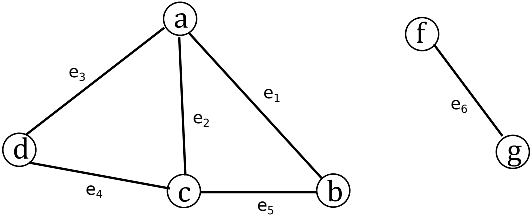

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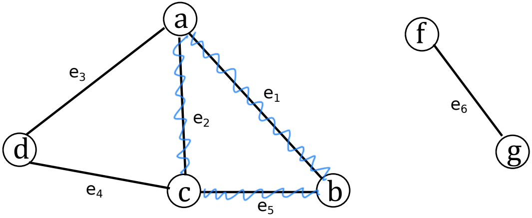

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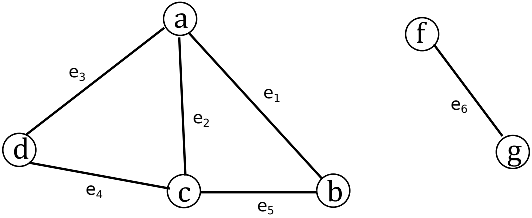

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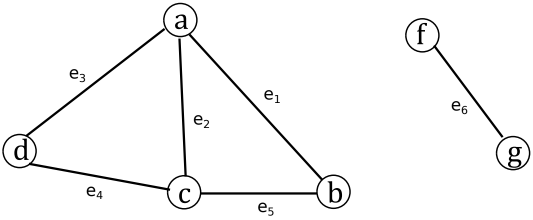

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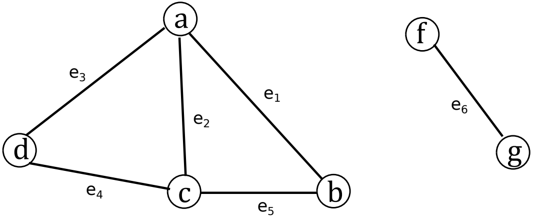

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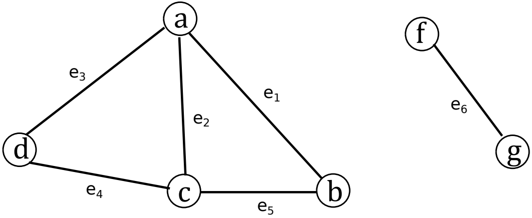

Graph G

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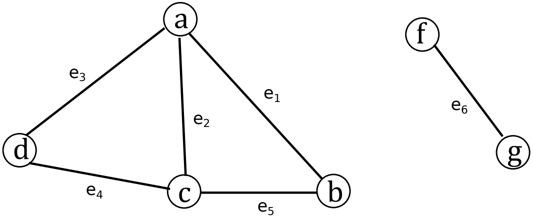

Graph G

$\operatorname{dist}(b, f)$ =? ∞

Isomorphism of graphs

Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are **isomorphic** if there is a bijection $f \colon V_1 \to V_2$ that preservers the adjacency

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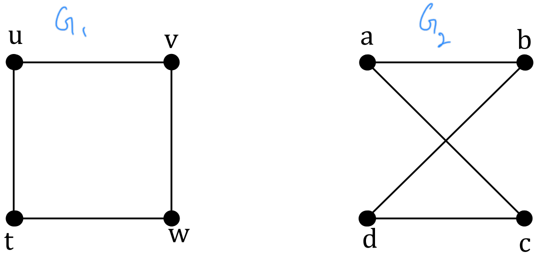

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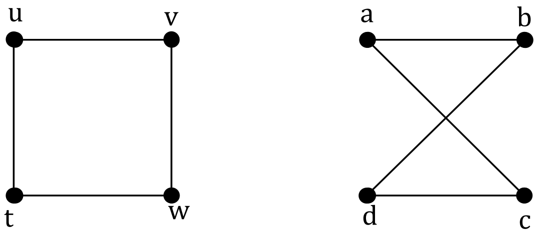

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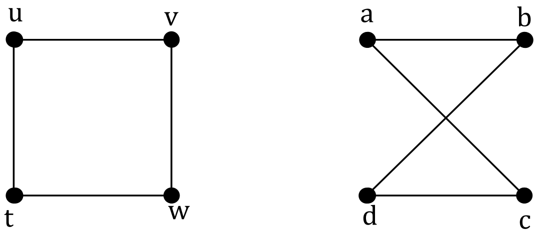

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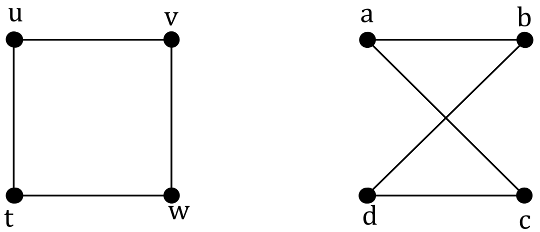

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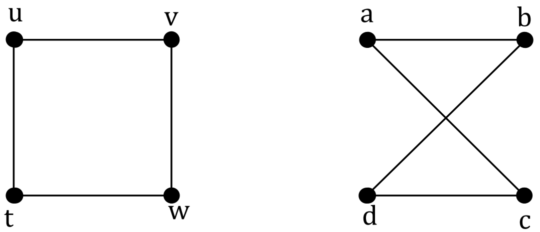

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