# CS275 Discrete Mathematics

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# 8.2 Solving Linear Recurrence Relations

# Find the solution to $a_n = 7a_{n-2} + 6a_{n-3}$ with $a_0 = 9$, $a_1 = 10$, and $a_2 = 32$.

- Step 1. Find the characteristic equation
  - $a_n = r^3$, $a_{n-1} = r^2$, $a_{n-2} = r$, $a_{n-3} = 1$
  - The characteristic equation:

    $r^3 = 0 \bullet r^2 + 7\,r + 6 \bullet 1$

    $r^3 - 7r - 6 = 0$
- Step 2. Solve the characteristic equation

  $r^3 - 7r - 6 = 0$

  $(r+1)(r^2-r-6) = 0$

  $(r+1)(r+2)(r-3) = 0$

  $r = -1, -2, 3$

# Find the solution to $a_n = 7a_{n-2} + 6a_{n-3}$ with $a_0 = 9$, $a_1 = 10$, and $a_2 = 32$.

- Step 3. Solve the recurrence relation $a_n = \alpha_1(-1)^n + \alpha_2(-2)^n + \alpha_3 \bullet 3^n$
  - Substitute n=0
    - $9 = \alpha_1 + \alpha_2 + \alpha_3$ (1)
  - Substitute n=1
    - $10 = \alpha_1(-1) + \alpha_2(-2) + \alpha_3 \bullet 3$
    - $10 = -\alpha_1 - 2\alpha_2 + 3\alpha_3$ (2)
  - Substitute n=2
    - $32 = \alpha_1(-1)^2 + \alpha_2(-2)^2 + \alpha_3 \bullet 3^2$
    - $32 = \alpha_1 + 4\alpha_2 + 9\alpha_3$ (3)
  - (1) + (2)
    - $-\alpha_2 + 4\alpha_3 = 19$ (4)
  - (2) + (3)
    - $2\alpha_2 + 12\alpha_3 = 42$ (5)

# Find the solution to $a_n = 7a_{n-2} + 6a_{n-3}$ with $a_0 = 9$, $a_1 = 10$, and $a_2 = 32$.

- Step 3. Solve the recurrence relation (continue)
  - (4)x2 + (5)
    - $20\alpha_3 = 80$
    - $\alpha_3 = 4$
  - Substitute $\alpha_3 = 4$ in (4)
    - $\alpha_2 = -3$
  - Substitute $\alpha_2 = -3$, $\alpha_3 = 4$ in (1)
    - $\alpha_1 = 8$
  - $\alpha_1 = 8$
  - $\alpha_1 = 8$, $\alpha_2 = -3$, $\alpha_3 = 4$ in $a_n = \alpha_1(-1)^n + \alpha_2(-2)^n + \alpha_3 \bullet 3^n$
    - $a_n = 8(-1)^n - 3(-2)^n + 4 \bullet 3^n$

# What is the general form of the solutions of a linear homogeneous recurrence relation if its characteristic equation has roots 1,1,1,1,−2,−2,−2,3,3,−4?

**Theorem 4:** Let $c_1, c_2, ..., c_k$ be real numbers. Suppose that the characteristic equation $r^k - c_1 r^{k-1} - c_2 r^{k-2} - ... - c_k = 0$ has $t$ **distinct roots** $r_1, r_2, ....r_t$ $(t \leq k)$ **with multiplicities** $m_1, m_2, ..., m_t$, respectively, so that $m_i \geq 1$ for $i = 1, 2, ..., t$ and $m_1 + m_2 + ... + m_t = k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k}$ if and only if $a_n = (\alpha_{1,0} + \alpha_{1,1} n + ... + \alpha_{1,m_1-1} n^{m_1-1}) r_1^n + (\alpha_{2,0} + \alpha_{2,1} n + ... + \alpha_{2,m_2-1} n^{m_2-1}) r_2^n + ... + (\alpha_{t,0} + \alpha_{t,1} n + ... + \alpha_{t,m_t-1} n^{m_t-1}) r_t^n$ for $n = 0, 1, 2, ....,$ where $\alpha_{i,j}$ are constants for $1 \leq i \leq t$ and $0 \leq j \leq m_i - 1$.

**Note:** Theorems 1, 2 and 3 are corollaries to Theorem 4

# What is the general form of the solutions of a linear homogeneous recurrence relation if its characteristic equation has roots $1,1,1,1,-2,-2,-2,3,3,-4$?

- The characteristic equation has roots: 1, -2, 3, -4
- The multiplies are: 4, 3, 2, 1
- The solution of the recurrence relation is:

$$
\begin{aligned}
a_n &= \left(\alpha_{1,0}+\alpha_{1,1}n+\alpha_{1,2}n^2+\alpha_{1,4-1}n^{4-1}\right)(1)^n+\left(\alpha_{2,0}+\alpha_{2,1}n+\alpha_{2,3-1}n^{3-1}\right)(-2)^n \\
&\quad +\left(\alpha_{3,0}+\alpha_{3,2-1}n^{2-1}\right)(3)^n+\left(\alpha_{4,1-1}n^{1-1}\right)(-4)^n \\
&= \left(\alpha_{1,0}+\alpha_{1,1}n+\alpha_{1,2}n^2+\alpha_{1,3}n^3\right)(1)+\left(\alpha_{2,0}+\alpha_{2,1}n+\alpha_{2,2}n^2\right)(-2)^n \\
&\quad +\left(\alpha_{3,0}+\alpha_{3,1}n\right)(3)^n+\left(\alpha_{4,0}n^0\right)(-4)^n \\
&= \boxed{\begin{aligned}&\left(\alpha_{1,0}+\alpha_{1,1}n+\alpha_{1,2}n^2+\alpha_{1,3}n^3\right)+\left(\alpha_{2,0}+\alpha_{2,1}n+\alpha_{2,2}n^2\right)(-2)^n \\ &+\left(\alpha_{3,0}+\alpha_{3,1}n\right)(3)^n+\alpha_{4,0}(-4)^n\end{aligned}}
\end{aligned}
$$

# Consider the nonhomogeneous linear recurrence relation $a_n=3a_{n-1}+2^n$.

- A) Show that $a_n = -2^{n+1}$ is a solution of this recurrence relation.
  - Since $a_n = -2^{n+1}$, $a_{n-1} = -2^n$
  - Substitute $a_{n-1} = -2^n$ in $a_n=3a_{n-1}+2^n$
    - $a_n = 3(-2^n) + 2^n$

$$
\begin{aligned}
&= -3 \bullet 2^n + 2^n \\
&= 2^n \bullet (-3+1) \\
&= 2^n \bullet (-2) \\
&= -2^{n+1}
\end{aligned}
$$

# Consider the nonhomogeneous linear recurrence relation $a_n = 3a_{n-1} + 2^n$.

- B) Use Theorem 5 to find all solutions of this recurrence relation

**Theorem 5:** If $\{a_n^{(p)}\}$ is a particular solution of the non-homogeneous linear recurrence relation with constant co-efficients $a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k} + F(n)$, then every solution is of the form $\{a_n^{(p)} + a_n^{(h)}\}$, where $\{a_n^{(h)}\}$ is a solution of the associated linear homogeneous recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k}$.

# Consider the nonhomogeneous linear recurrence relation $a_n=3a_{n-1}+2^n$.

- B) Use Theorem 5 to find all solutions of this recurrence relation
  - Step 1: Find the characteristic equation of the associated homogeneous recurrence relation, $a_n=3a_{n-1}$
    - $a_n= r$, $a_{n-1} = 1$
    - $r = 3$
    - the characteristic equation: $a_n = \alpha 3^n$

# Consider the nonhomogeneous linear recurrence relation $a_n=3a_{n-1}+2^n$.

- B) Use Theorem 5 to find all solutions of this recurrence relation
  - Step 2:
    - Assume $a_n^{(p)}=A2^n$ is a particular solution, $A$ is a constant
    - Substitute $a_n^{(p)}=A2^n$ in $a_n=3a_{n-1}+2^n$

      $A2^n=3(A2^{n-1})+2^n$

      $A2^n=(3/2)\,A2^n+2^n$

      $A = -2$
    - The particular solution $a_n^{(p)}=(-2)\bullet 2^n=-2^{n+1}$
  - The general solution of $a_n=3a_{n-1}+2^n$ is
    - $a_n = \alpha 3^n - 2^{n+1}$

# Consider the nonhomogeneous linear recurrence relation $a_n=3a_{n-1}+2^n$.

- C) Find the solution with $a_0 = 1$
  - Apply $a_0 = 1$ to $a_n = \alpha 3^n - 2^{n+1}$
    - $A_0 = \alpha 3^0 - 2^{0+1}$
    - $1 = \alpha - 2$
    - $\alpha = 3$
  - The solution with $a_0 = 1$ is $a_n = 3 \bullet 3^n - 2^{n+1} = 3^{n+1} - 2^{n+1}$

What is the general form of the particular solution guaranteed to exist by Theorem 6 of the linear nonhomogeneous recurrence relation $a_n = 8a_{n-2} - 16a_{n-4} + F(n)$ if $\mathbf{F(n) = n^3}$

**Theorem 6:** If $\{a_n\}$ satisfies the linear non-homogeneous recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + F(n)$, where $F(n) = (b_t n^t + b_{t-1} n^{t-1} + \ldots + b_1 n + b_0) s^n$. When $s$ **is not a root** of the characteristic equation of the associated linear homogeneous recurrence relation, there is a particular solution of the form $(p_t n^t + p_{t-1} n^{t-1} + \ldots + p_1 n + p_0) s^n$. When $s$ **is a root** of this characteristic equation of **multiplicity** $m$, there is a particular solution of the form $n^m (p_t n^t + p_{t-1} n^{t-1} + \ldots + p_1 n + p_0) s^n$.

# What is the general form of the particular solution guaranteed to exist by Theorem 6 of the linear nonhomogeneous recurrence relation $a_n = 8a_{n-2} - 16a_{n-4} + F(n)$ if $\mathbf{F(n) = n^3}$

- Step 1, solve the characteristic equation of the associated homogeneous recurrence relation
  - The associated homogeneous recurrence relation, $a_n = 8a_{n-2} - 16a_{n-4}$
  - The characteristic equation of $a_n = 8a_{n-2} - 16a_{n-4}$
    - $r^4 - 8r^2 + 16 = 0$
    - $r = 2, -2$ or multiplicity 2
- Step 2, find *s* in $F(n) = (b_t n^t + b_{t-1}n^{t-1} + \ldots + b_1 n^1 + b_0)s^n$
  - In our case, t=3, s=1
    - $F(n) = (b_3 n^3)1^n$
- Step 3, apply Theorem 6,
  - s=1, is NOT a root of the associated characteristic equation
  - According to Theorem 6
    - There is a particular solution of the form $(p_3n^3+p_2n^2+p_1n^1+p_0)x1^n$ = $p_3n^3+p_2n^2+p_1n^1+p_0$

# What is the general form of the particular solution guaranteed to exist by Theorem 6 of the linear nonhomogeneous recurrence relation $a_n = 8a_{n-2} - 16a_{n-4} + F(n)$ if **$F(n) = (-2)^n$**

- Step 1, solve the characteristic equation of the associated homogeneous recurrence relation
  - The associated homogeneous recurrence relation, $a_n = 8a_{n-2} - 16a_{n-4}$
  - The characteristic equation of $a_n = 8a_{n-2} - 16a_{n-4}$
    - $r^4 - 8r^2 + 16 = 0$
    - r = 2, -2 or multiplicity 2
- Step 2, find *s* in $F(n) = (b_t n^t + b_{t-1}n^{t-1} + \ldots + b_1 n^1 + b_0)s^n$
  - In our case, t=0, s=-2
    - $F(n) = (b_0 n^0)(-2)^n$
  - s=-2, is a root of the associated characteristic equation
  - According to Theorem 6
    - There is a particular solution of the form $n^2(p_0 n^0)x(-2)^n = n^2 p_0(-2)^n$

# 8.5 Inclusion-exclusion principle

There are 2504 computer science students at a school. Of these, 1876 have taken a course in Java, 999 have taken a course in Linux, and 345 have taken a course in C. Further, 876 have taken courses in both Java and Linux, 231 have taken courses in both Linux and C, and 290 have taken courses in both Java and C. If 189 of these students have taken courses in Linux, Java, and C, how many of these 2504 students have not taken a course in any of these three programming languages?

- Solution:
  - We know:
    - $|J| = 1876, |L| = 999, |C| = 345$
    - $|J \cap L| = 876, |J \cap C| = 290, |L \cap C| = 231,$
    - $|J \cap L \cap C| = 189$
  - According to Inclusion-exclusion principle
    - $|J \cup L \cup C| = |J|+|L|+|C| - |J \cap L| - |J \cap C| - |L \cap C| + |J \cap L \cap C| = 2012$
  - Thus, 2504-2012 = 492 students have not taken a course in any of the three.

How many elements are in the union of four sets if the sets have 50, 60, 70, and 80 elements, respectively, each pair of the sets has 5 elements in common, each triple of the sets has 1 common element, and no element is in all four sets?

- Solution:
  - We have four sets $A_1, A_2, A_3, A_4$,
    - $|A_1| = 50, |A_2| = 60, |A_3| = 70, |A_4| = 80$
    - Assume $1 \le i < j < k \le 4$
      - $|A_i \cap A_j| = 5, |A_i \cap A_j \cap A_k| = 1, |A_1 \cap A_2 \cap A_3 \cap A_4| = 0,$
  - According to Inclusion-exclusion principle

    $|A_1 \cup A_2 \cup A_3 \cup A_4|$

    $= \sum_{1 \le i \le 4} |A_i| - \sum_{1 \le i < j \le 4} |A_i \cap A_j| + \sum_{1 \le i < j < k \le 4} |A_i \cap A_j \cap A_k| - |A_1 \cap A_2 \cap A_3 \cap A_4|$

    $= 50+60+70+80 - 6(5)+4(1)-0 = 234$