Practice Midterm 3

1. True or False.

   (a) Let H be a normal subgroup of G. If H and G/H are abelian, then G is abelian.

   (b) Let G be a group and $g \in G$. Suppose that $|G| = 60$, $g^{10} \neq e$, and $g^{15} \neq e$. Then G is cyclic.

   (c) Let G be a group and H be a subgroup of G. If $[G : H] = 3$, then H is normal.

   (d) $(1,3)D_4 = (2,4)D_4$.

   (e) Let G_1 and G_2 be groups and $h : G_1 \to G_2$ be a homomorphism. If G_1 is abelian, then G_2 is abelian.

2. Suppose that G is a group with $|G| = 50$. What are the orders of possible subgroups of G?

3. Let $H = \langle (2,4) \rangle$. Consider H as a subgroup of D_4.

   (a) Please list all left cosets of H. (including the elements of the cosets)

   (b) Please list all right cosets of H. (including the elements of the cosets)

   (c) Is H a normal subgroup of D_4?

4. Let $H = \{e, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$. H is a normal subgroup of A_4. The quotient group A_4/H contains 3 elements
$$H, (2,4,3)H, (1,3,2)H. \tag{1}$$

   (a) Please complete the Cayley (multiplication) table in terms of the elements in the list (1).

   (b) Is G/H cyclic?

   (c) Simplify $(1,3)(2,4)H * [(1,2,3)H]^{-1} * [(1,4)(2,3)H]$. Please express your answer in terms of the elements in the list (1).

5. Let $U = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$ and $V = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{bmatrix}$. Let $\phi : GL_2(\mathbb{R}) \to GL_2(\mathbb{R})$ be defined by $\phi(A) = UAV$.

   (a) Prove that ϕ is a homomorphism. (Hint: $U = V^{-1}$)

   (b) Please find the kernel of ϕ.

6. (a) Let G be a group and $a, b \in G$ with $a \neq b$. Suppose that $\phi : G \to S_4$ is a homomorphism such that $\phi(a) = (2,4)$ and $\phi(b) = (1,2,3)$. Evaluate $\phi(ab^{-1}a^2)$.

   (b) Let G be a group and $a, b \in G$ with $a \neq b$. Suppose that $\phi : G \to \mathbb{R}^2$ is a homomorphism such that $\phi(a) = (4,1)$ and $\phi(b) = (2,2)$. Find a solution of x such that $\phi(x) = (0, -6)$.