Higher Algebra Midterm 3 - Winter 2021

December 16, 2021

Name:

This examination contains 8 pages and 7 problems.
You can write on the blank page after the problem if the space is not enough.
Please do NOT tear out the pages.

Read all the questions carefully before starting the exam.

Use of Calculators and Books are NOT permitted!

Show all your work!!!

Good Luck!!!!

Problem	Score
1	/20
2	/15
3	/15
4	/15
5	/15
6	/10
7	/10
Total	/100

1. True or False.

(a) ___F___ Let H be a subgroup of G. If H is abelian, then H is a normal subgroup of G.

① $G = S_3$, $H = \langle (1,2) \rangle$

$H = \{e, (1,2)\}$

$(1,3)H = \{(1,3), \underline{(1,2,3)}\}$

$(2,3)H = \{(2,3), (1,3,2)\}$

$H = \{e, (1,2)\}$

$H(1,3) = \{(1,3), \underline{(1,3,2)}\}$

$(1,3)H \neq H(1,3) \Rightarrow H$ is not normal

(b) ___T___ Let G be a group and $g \in G$. Suppose that $|G| = 45$, $g^9 \neq e$, and $g^{15} \neq e$. Then G is cyclic.

By Lagrange's Theorem,

$|g| \mid 45 = 3^2 \times 5$

$\Rightarrow |g| = \cancel{1}$ or $\cancel{3}$ or $\cancel{5}$ or $\cancel{9}$ or $\cancel{15}$ or 45

① $|g| = 1 \Rightarrow g = e$ ✗

② $|g| = 3$, $g^3 = e$

$\Rightarrow (g^3)^3 = e$ ✗

③ $|g| = 5$, $g^5 = e$

$\Rightarrow (g^5)^3 = e$ ✗

④ $|g| = 9$, $g^9 = e$ ✗

⑤ $|g| = 15$ ✗

⑥ $|g| = 45 \Rightarrow G$ is cyclic □

(c) ___T___ Let G be a group and H be a subgroup of G. If $[G:H] = 2$, then H is normal.

(d) ___F___ $(1,3,4,8,2)A_8 = (1,3,4,2)A_8$

$(1,3,4,8,2) = (1,2)(1,8)(1,4)(1,3)$ even

$(1,3,4,2) = (1,2)(1,4)(1,3)$ odd

(e) ___T___ Let G_1 and G_2 be groups and $h : G_1 \to G_2$ be a homomorphism. If h is one-to-one and G_2 is abelian, then G_1 is abelian.

Let $a, b \in G_1$

$$h(ab) \underset{h:\text{homomorphism}}{=} h(a)h(b) \underset{G_2:\text{abelian}}{=} h(b)h(a) \underset{h:\text{homomorphism}}{=} h(ba)$$

Since h is 1-1, we get $ab = ba$ □

2. (a) Please state the definition of index.

Let G be a group and H be a subgroup of G.

The index of H is the number of left (or right) H-cosets.

$[G:H]$

(b) Please state the Lagrange's theorem.

Let G be a group and H be a subgroup of G.

If $|G| < \infty$, then

$$|G| = [G:H]\,|H|$$

In other words,

$$|H| \mid |G| .$$

(c) Suppose that G is a group with $|G| = 18$. What are the orders of possible subgroups of G?

Let H be a subgroup of G.

Lagrange's theorem: $|H| \mid 18 = 2 \times 3^2$

$\Rightarrow |H| = 1$ or 2 or 3 or 6 or 9 or 18

3. Let $H = \langle(1,2,3)\rangle$. Consider H as a subgroup of A_4.

(a) Please list all left cosets of H. (including the elements of the cosets)

$H = \{ e, (1,2,3), (1,3,2) \}$

$\boxed{(1,2)(3,4)H = \{ (1,2)(3,4), (2,4,3), (1,4,3) \}}$

$(1,3)(2,4)H = \{ (1,3)(2,4), (1,4,2), (2,3,4) \}$

$(1,4)(2,3)H = \{ (1,4)(2,3), (1,2,4), (1,3,4) \}$

$A_4 = \{ \cancel{e},$
$(1,\cancel{2},3), (1,\cancel{3},2), (1,2,4), (1,\cancel{4},2),$
$(1,3,4), (1,\cancel{4},3), (2,\cancel{3},4), (2,\cancel{4},3),$
$(1,2)\cancel{(3,4)}, (1,3)\cancel{(2,4)}, (1,4)(2,3) \}$

(b) Please list all right cosets of H. (including the elements of the cosets)

$H = \{ e, (1,2,3), (1,3,2) \}$

$\boxed{H(1,2)(3,4) = \{ (1,2)(3,4), (1,3,4), (2,3,4) \}}$

$H(1,3)(2,4) = \{ (1,3)(2,4), (2,4,3), (1,2,4) \}$

$H(1,4)(2,3) = \{ (1,4)(2,3), (1,4,3), (1,4,2) \}$

$A_4 = \{ \cancel{e},$
$(1,\cancel{2},3), (1,\cancel{3},2), (1,\cancel{2},4), (1,4,2),$
$(1,\cancel{3},4), (1,4,3), (2,\cancel{3},4), (2,\cancel{4},3),$
$(1,2)\cancel{(3,4)}, (1,3)\cancel{(2,4)}, (1,4)\cancel{(2,3)} \}$

(c) Is H a normal subgroup of A_4? Ans: No

$(1,2)(3,4)H \neq H(1,2)(3,4)$

4. Consider the group D_4 and $H = \{e, (13)(24)\}$. H is a normal subgroup of D_4. The quotient group D_4/H contains 4 elements

$$H, rH, sH, srH \tag{1}$$

where $r = (1,2,3,4)$ and $s = (2,4)$.

(a) Please complete the Cayley (multiplication) table of D_4/H in terms of the expressions in (1). (6 points)

$*$	H	rH	sH	srH
H	H	rH	sH	srH
rH	rH	H	srH	sH
sH	sH	srH	H	rH
srH	srH	sH	rH	H

$(sH)(sH) = s^2H = eH = H$

(b) Is D_4/H cyclic? Ans: No (4 points)

H : order $= 1$

$(rH)^2 = H$: order $= 2$

$(sH)^2 = H$: order $= 2$

$(srH)^2 = H$: order $= 2$

order $\neq 4$

(c) Simplify $(2,4)H * [(4,3,2,1)H]^{-1} * [(1,4)(2,3)H]$. (5 points)
Please express your answer in terms of the expressions in (1).

$(2,4)H\,[(4,3,2,1)H]^{-1}(1,4)(2,3)H$

$= (2,4)H\,(4,3,2,1)^{-1}H\;(1,4)(2,3)H$

$= (2,4)(4,3,2,1)^{-1}(1,4)(2,3)H$

$= (2,4)(1,2,3,4)(1,4)(2,3)H$

$= eH$

$= H$

$1 \to 1$

$2 \to 2$

$3 \to 3$

$4 \to 4$

5. Let $\phi : \mathbb{R} \to GL_2(\mathbb{R})$ be defined by

$$\phi(x) = \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}.$$

(a) Prove that ϕ is a homomorphism. (10 points)
Hint: $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$, $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

Let $x, y \in \mathbb{R}$. Then

$$\phi(x)\phi(y) = \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}\begin{bmatrix} \cos y & -\sin y \\ \sin y & \cos y \end{bmatrix}$$

$$= \begin{bmatrix} \cos x\cos y - \sin x \sin y & -\cos x \sin y - \sin x \cos y \\ \sin x \cos y + \cos x \sin y & -\sin x \sin y + \cos x \cos y \end{bmatrix}$$

$$= \begin{bmatrix} \cos(x+y) & -\sin(x+y) \\ \sin(x+y) & \cos(x+y) \end{bmatrix}$$

$$= \phi(x+y)$$

Therefore, ϕ is a homomorphism #

(b) Please find the kernel of ϕ. (5 points)

$$\phi(x) = \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix} = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{cases} \cos x = 1 & \Rightarrow x = 2\pi n, \; n \in \mathbb{Z} \\ \sin x = 0 & \Rightarrow x = \pi n, \; n \in \mathbb{Z} \end{cases}$$

$\ker\phi = \{ 2\pi n ; n \in \mathbb{Z} \}$

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6. (a) Let G be a group and $a,b \in G$ with $a \neq b$. Suppose that $\phi : G \to S_3$ is a homomorphism such that $\phi(a) = (1,2)$ and $\phi(b) = (1,2,3)$. Evaluate $\phi(ab^{-1}a)$.

$$\phi(ab^{-1}a) = \phi(a)\phi(b^{-1})\phi(a)$$
$$= \phi(a)[\phi(b)]^{-1}\phi(a)$$
$$= (1,2)(1,2,3)^{-1}(1,2)$$
$$= (1,2)(1,3,2)(1,2)$$
$$= \boxed{(1,2,3)}$$

$1 \to 2 \to 1 \to 2$
$2 \to 1 \to 3 \to 3$
$3 \to 3 \to 2 \to 1$

(b) Let G be a group and $a,b \in G$ with $a \neq b$. Suppose that $\phi : G \to \mathbb{R}^2$: abelian is a homomorphism such that $\phi(a) = (1,2)$ and $\phi(b) = (2,1)$. Find a solution of x such that $\phi(x) = (4,-1)$.

Let $x = a^m b^n$, $m, n \in \mathbb{Z}$.

$$\phi(x) = \phi(a^m b^n) = m\phi(a) + n\phi(b)$$
$$= m(1,2) + n(2,1)$$
$$= (m+2n, 2m+n)$$
$$\overset{\text{want}}{=} (4,-1)$$

$$\Rightarrow \begin{cases} m + 2n = 4 & -③ \\ 2m + n = -1 \end{cases} \Rightarrow \begin{cases} 2m + 4n = 8 & -① \\ 2m + n = -1 & -② \end{cases}$$

① − ② $\Rightarrow 3n = 9 \Rightarrow \boxed{n = 3}$

Plug $n=3$ into ③

$\Rightarrow m + 2 \cdot 3 = 4$

$\Rightarrow m + 6 = 4$

$\Rightarrow \boxed{m = -2}$

$$\boxed{x = a^{-2} b^3}$$

7. Let G be an abelian group with $|G| = 15$. Prove that G has a subgroup of order 3 and a subgroup of order 5. Please use only the materials that were taught so far.

Let $g \in G$ s.t. $g \neq e$.

By the Lagrange's theorem $|g| \mid 15 \Rightarrow |g| = \cancel{1}$ or 3 or 5 or 15

① If $|g| = 15$

$\langle g^5 \rangle = \{e, g^3, g^6, g^9, g^{12}\}$

(subgroup of order 5)

$\langle g^5 \rangle = \{e, g^5, g^{10}\}$

(subgroup of order 3)

② $\boxed{\text{If } |g| = 3}$

$H = \langle g \rangle = \{e, g, g^2\}$

(subgroup of order 3)

Let $\boxed{a \in G \setminus H}$

Lagrange's theorem

$|a| = \cancel{1}$ or $\cancel{3}$ or 5 or 15

$(a \neq e)$

Since G is abelian, H is normal

If $|a| = 3$

$(aH)^3 = a^3 H = eH = H$

$\Rightarrow |aH| \mid 3 \Rightarrow |aH| = 1$ or 3

$|G/H| = [G:H] = |G|/|H| = 5$

By Lagrange's theorem

$|aH| = 1$ or 3

$\Rightarrow |aH| = 1 \Rightarrow aH = H \Rightarrow a \in H$

$\Rightarrow$ This is not possible

If $|a| = 5$

$\langle a \rangle$ is a subgroup of order 5

If $|a| = 15$

$\langle a^3 \rangle = \{e, a^3, a^6, a^9, a^{12}\}$

$\langle a^3 \rangle$ is a subgroup of order 5

③ If $|g| = 5$

Let $H = \langle g \rangle = \{e, g, g^2, g^3, g^4\}$

(subgroup of order 5)

Let $a \in G \setminus H$

Lagrange's theorem

$\Rightarrow |a| = \cancel{1}$ or $\boxed{3}$ or $\cancel{5}$ or $\boxed{15}$

H is a normal subgroup of G.

If $|a| = 5$

$(aH)^5 = a^5 H = eH = H$.

$\Rightarrow |aH| = 1$ or 5

$|G/H| = |G|/|H| = 3$

Lagrange's theorem

$|aH| = 1$ or 3

$\Rightarrow |aH| = 1 \Rightarrow aH = H \Rightarrow a \in H$.

$\Rightarrow$ This is not possible.

If $|a| = 3$, then $\langle a \rangle$ is a subgroup of order 3

If $|a| = 15$, then $\langle a^5 \rangle$ is a subgroup of order 3.

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