BEFORE MIDTERM II (18. 01. 2022)

Zoltán

WHAT WILL BE ON TEST?

- LINEAR NON-HOMOGENOUS RECURRENCE RELATIONS,
- GRAPH THEORY,
- EULERIAN AND HAMILTONIAL GRAPHS.

1) LINEAR NON-HOMOGENOUS RECURRENCE RELATIONS:

FORM:

$$\underbrace{F_n = AF_{n-1} + BF_{n-2}}_{\text{HOMOGENOUS RELATION}} + \underset{\substack{\uparrow \\ \text{PARTICULAR SOLUTION}}}{f(n)} \quad \text{WHERE } f(n) \neq 0$$

THEREFORE:

$a_n = a_h + a_t$ WHERE:

- a_h – HOMOGENOUS RECURENCE RELATION
- a_t – PARTICULAR SOLUTION ($f(n)$).

TO FIND THE PARTICULAR SOLUTION, WE FIND APPROPRIATE TRIAL SOLUTION.

LET $f(n) = cx^n$; LET $x^2 = Ax + B$ BE THE CHARACTERISTIC EQUATION OF THE ASSOCIATED HOMOGENOUS RECURRENCE RELATION AND LET x_1 AND x_2 BE ITS ROOTS.

- IF $x \neq x_1$ AND $x \neq x_2$, THEN $a_t = Ax^n$
- IF $x = x_1$, $x \neq x_2$, THEN $a_t = Anx^n$
- IF $x = x_1 = x_2$, THEN $a_t = An^2x^n$

EXAMPLE:

LET A NON-HOMOGENOUS RELATION BE $F_n = AF_{n-1} + BF_{n-2} + f(n)$ WITH CHARACTERISTIC ROOTS $x_1 = 2$ AND $x_2 = 5$. TRIAL SOLUTIONS FOR DIFFERENT POSSIBLE VALUES OF $f(n)$ ARE:

$f(n)$	TRIAL SOLUTIONS
4	A
$5 \cdot 2^n$	$An2^n$
$8 \cdot 5^n$	$An5^n$
4^n	$A4^n$
$2n^2+3n+1$	An^2+Bn+C

$f(n)$	a_n^t
C	A
n	$A_1n + A_0$
n^2	$A_2n^2 + A_1n + A_0$
r^n	Ar^n

PROBLEM:

SOLVE THE RECURRENCE RELATION $F_n = 3F_{n-1} + 10F_{n-2} + 7 \cdot 5^n$ WHERE $F_0 = 4$ AND $F_1 = 3$

SOLUTION:

THIS IS A LINEAR NON-HOMOGENEOUS RELATION, WHERE THE ASSOCIATED HOMOGENOUS EQUATION IS: $F_n = 3F_{n-1} + 10F_{n-2}$ AND $f(n) = 7 \cdot 5^n$

THE CHARACTERISTIC EQUATION OF ITS ASSOCIATED HOMOGENOUS RELATION IS:

$$x^2 - 3x - 10 = 0$$

$$\Rightarrow (x-5)(x+2) = 0$$

$$\Rightarrow x_1 = 5 \text{ AND } x_2 = -2$$

HENCE: $a_n^h = a5^n + b(-2)^n$, WHERE a AND b ARE CONSTANTS.

SINCE $f(n) = 7 \cdot 5^n$, I.E OF THE FORM cx^n, A RESONABLE TRIAL SOLUTION OF a_t WILL BE Anx^n:

$$a_n^p = Anx^n = An5^n$$

AFTER PUTTING THE SOLUTION IN THE RECURRENCE RELATION, WE GET:

$$An5^n = 3A(n-1)5^{n-1} + 10A(n-2)5^{n-2} + 7 \cdot 5^n$$

DIVIDING BOTH SIDES BY 5^{n-2}, WE GET:

$$An5^2 = 3A(n-1)5 + 10A(n-2)5^0 + 7 \cdot 5^2$$

$$\Rightarrow 25An = 15An - 15A + 10An - 20A + 175$$

$$\Rightarrow 35A = 175$$

$$\Rightarrow A = 5$$

SO:

$$a_n^p \cancel{a_t} = An5^n = 5n5^n = n5^{n+1}$$

THE SOLUTION OF THE RECRRENCE RELATION CAN BE WRITTEN AS:

$$F_n = a_h + a_t \Rightarrow a5^n + b(-2)^n + n5^{n+1}$$

PUTTING VALUES OF $F_0 = 4$ AND $F_1 = 3$, IN THE ABOVE EQUATION, WE GET $a = -2$ AND $b = 6$

SO THE SOLUTION IS: $\underline{\underline{F_n = n5^{n+1} + 6(-2)^n - 2 \cdot 5^n}}$

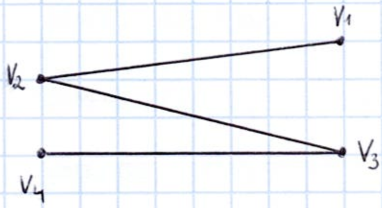
2) GRAPH THEORY

DEFINITION:

A GRAPH $G=(V,E)$ CONSISTS OF TWO FINE SETS V AND E. THE ELEMENTS OF V ARE CALLED THE VERTICES AND ELEMENTS OF E THE EDGES OF G. EACH EDGE IS A TWO-ELEMENT SUBSET OF V WRITTEN AS $\{x,y\}$ OR xy.

$|V|$ - ORDER OF GRAPH $(|G|)$
$|E|$ - SIZE OF GRAPH $(\|G\|)$

ORDER IS 4 $\to V=\{V_1; V_2; V_3; V_4\}$
SIZE IS 5 $\to E=\{V_1V_2; V_2V_4; V_4V_3; V_3V_1; V_3V_2\}$



ORDER IS 4 $\to V=\{V_1; V_2; V_3; V_4\}$

SIZE IS 3 $\to E=\{V_1V_2; V_2V_3; V_3V_4\}$

SPECIAL GRAPHS:

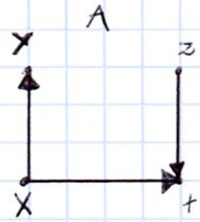
- A <u>DIRECTED GRAPH</u> OR <u>DIGRAPH</u> - BY REPLACING THE SET E WITH A SET OF ORDERED PAIRS OF VERTICES $((x,y)\neq(y,x))$

Ex.

$E=\{\{xy\},\{z,t\},\{x,t\}\}$

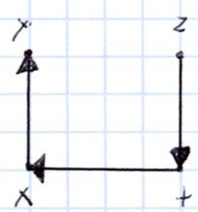
$V=\{x;y;z;t\}$

A



$\to A=\{(x,y),(z,t),(x,t)\}$

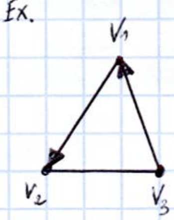
$A\neq A'$



$\to A'=\{(x,y),(z,t),(t,x)\}$

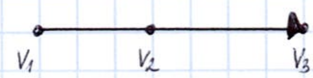
- A <u>MIXED GRAPH</u> - IF E CONTAINS BOTH DIRECTED AND UNDIRECTED EDGES.

Ex.



V_1V_2 AND V_3V_1 - DIRECTED
V_2V_3 - UNDIRECTED

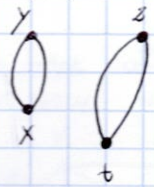
Ex.



V_2V_3 - DIRECTED
V_1V_2 - UNDIRECTED

- A <u>MULTIGRAPH</u> - IF E IS A MULTISET

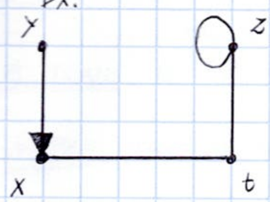
Ex.



$\rightarrow \quad E = \{\{x,y\}; \{y,x\}, \{z,t\}, \{t,z\}\}$

- A <u>PSEUDOGRAPH</u> - IF E HAS EDGES THAT CONNECT A VERTEX TO ITSELF (LOOP)

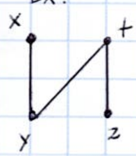
Ex.



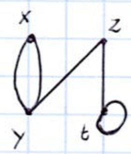
$E = \{\{y,x\}; \{x,t\}; \{t,z\}; \{z\}\}$

- A SIMPLE GRAPH - FINITE GRAPH WITHOUT LOOPS OR MULTIPLE EDGES.

Ex.



← SIMPLE GRAPH



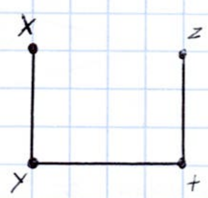
← NOT A SIMPLE GRAPH

RELATIONS BETWEEN EDGES AND VERTICES:

LET $G = (V, E)$ BE A GRAPH:

- $v, w \in V(G)$ - <u>ADJACENT</u> IF THERE IS AN EDGE BETWEEN TWO VERTICES (NEIGHBOR
- $v, w \in V(G)$ - <u>INCIDENT</u> (ENDPOINT)
- THE <u>NEIGHBORHOOD</u> OF A VERTEX $v \in V(G)$, IS A SET OF ALL VERTICES ADJACENT TO v

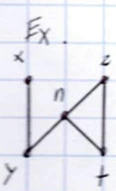
Ex.



$N(x) = \{y\}$
$N(y) = \{x, t\}$
$N(t) = \{y, z\}$
$N(z) = \{t\}$

- THE <u>DEGREE</u> OF A VERTEX $v \in V(G)$ $(deg(v))$ IS A NUMBER OF EDGES INCIDENT WITH v.

Ex.



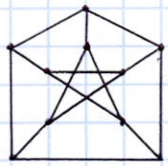
$deg(x) = 1$
$deg(y) = 2$
$deg(n) = 3$
$deg(z) = 2$
$deg(t) = 2$

IF $deg(x) = 0$ THEN x IS ISOLATED

- THE <u>MAX DEGREE</u> OF A GRAPH G: $\Delta(G) = \max\{d(v) : v \in V\}$
- THE <u>MIN DEGREE</u> OF A GRAPH G: $\delta(G) = \min\{d(v) : v \in V\}$

• **R-REGULAR** GRAPH - IF ALL VERTICES HAVE THE SAME NUMBER OF EDGES

Ex. 3-REGULAR GRAPH



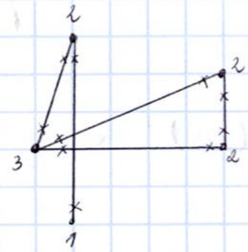
← PETERSEN GRAPH

HANDSHAKE LEMMA:

$$\sum_{v \in V(G)} \deg(v) = 2|E(G)|$$

(2 • NUMBER OF EDGES)

Ex.



$3+2+2+2+1 = 10$

$2 \cdot 5 = 1$

↑ NUMBER OF EDGES

COROLLARY (NASTĘPSTWO):

IN ANY GRAPH THE NUMBER OF VERTICES HAVING THE ODD DEGREE IS EVEN.

A DEGREE SEQUENCE:

DEF.

A DEGREE SEQUENCE OF A SIMPLE GRAPH G IS A SET OF DEGREES OF ALL VERTICES IN V WRITTEN IN NON-INCREASING ORDER.

ANY GRAPH HAS EXACLY ONE DEGREE SEQUENCE.

Ex.

→ (2, 2, 1, 1) ← DEGREE SEQUENCE

A GRAPHIC SEQUENCE:

DEF.

A GRAPHIC SEQUENCE IS A SEQUENCE OF NUMBERS WHICH CAN BE THE DEGREE SEQUENCE OF SOME GRAPH.

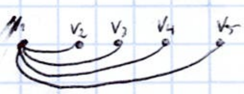
Ex.

$\begin{array}{ccccc} V_1 & V_2 & V_3 & V_4 & V_5 \end{array}$

IS $(4, 3, 2, 2, 1)$ GRAFIC?

COVER 4 AND SUBTRACT

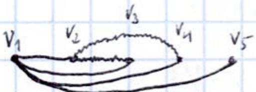
→ $4+3+2+2+1 = 12$ → EVEN - WE DON'T KNOW



$V_2\ V_3\ V_4\ V_5$

$(2, 1, 1, 0)$

COVER 2 AND SUBTRACT



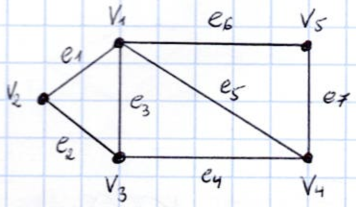
$V_3\ V_4\ V_5$

→ $(0, 0, 0)$

WALK:

DEF.

A WALK CONSISTS OF ALTERNATING SEQUENCE OF VERTICES AND EDGES CONSETUCIVE ELEMENTS OF WITCH ARE INCIDENT, THAT BEGINS AND ENDS WITH A VERTEX.



Ex.

$$(V_1, e_1, V_2, e_2\, V_3, e_4, V_4, e_7, V_5)$$

A TRAIL IS A WALK WITHOUT REPETED EDGES.

A PATH IS A WALK WITHOUT REPETED VERTICES.

THE LENGHT OF A WALK = THE NUMBER OF EDGES IN THE WALK.

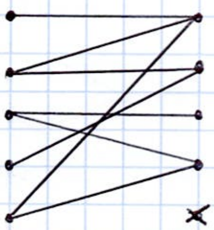
P_n - A PATH OF A LENGHT n.

A WALK IS CLOSED IF $V_0 = V_k$, OPEN WHEN $V_0 \neq V_k$ (V_0 - START, V_k - END)

BIPART GRAPHS:

A GRAPH G IS BIPART IF THE VERTICES OF G CAN BE PARTITIONED INTO TWO SUBSETS V_1 AND V_2 IN SUCH A WAY NO TWO VERTICES IN THE SAME SUBSET ARE ADJACENT (NO NEIGHBOURS).

Ex.



A COMPLETE BIPARTITE GRAPH $G = (V_1, V_2; E)$ IS A BIPARTITE GRAPH SUCH THAT FOR ANY TWO VERTICES $V_1 \in V_2$ AND $V_2 \in V_2$, $V_1, V_2 \in E$. THE COMPLETE BIPARTITE GRAPH WITH PARTITIONS OF SIZE $|V_1| = m$ AND $|V_2| = n$ IS DENOTED BY $K_{m,n}$.

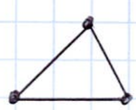
Ex.

← A COMPLETE BIPART GRAPH WITH $m=5$ AND $n=3$ $(K_{5,3})$

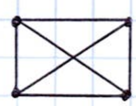
COMPLETE GRAPH:

THE COMPLETE GRAPH IS A GRAPH OF ORDER n THAT IS AN $(n-1)$ - REGULAR. IT IS DENOTED BY K_n

Ex. K_3



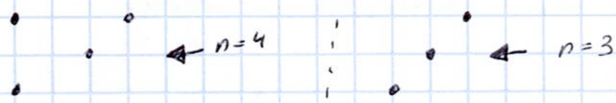
K_4



THE EMPTY GRAPH:

A EMPTY GRAPH $E_n = (V, \emptyset)$. E_n IS 0-REGULAR GRAPH.

Ex.

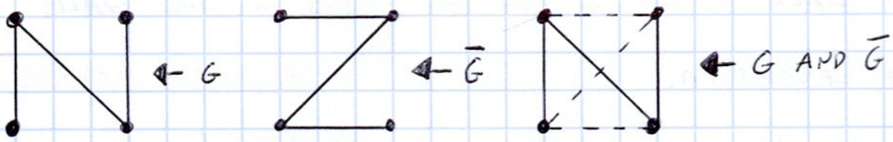


THE COMPLEMENT OF A GRAPH:

THE COMPLEMENT OF A GRAPH G IS A GRAPH $\overline{G}$ WITH VERTEX SET $V(\overline{G}) = V(G)$ AND EDGE SET $E(\overline{G}) = \{xy : x, y \in V(G),\ xy \notin E(G)\}$. $\overline{K_n} = E_n$

NOTE: $\overline{\overline{G}} = G$

Ex

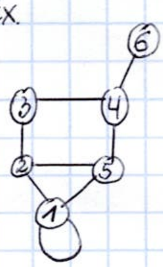


ADJACENCY MATRIX:

THE ADJACENCY MATRIX OF A GRAPH G WITH THE VERTEX SET $V = \{v_1, v_2, \dots v_n\}$ IS A BINARY SQUARE $n \times n$ MATRIX $A(G) = (a_{ij})_{n \times n}$ SUCH THAT

$$a_{ij} = \begin{cases} 1 & v_i v_j \in E(G) \\ 0 & v_i v_j \notin E(G) \end{cases}$$

Ex.



	1	2	3	4	5	6
1	2	1	0	0	1	0
2	1	0	1	0	1	0
3	0	1	0	1	0	0
4	0	0	1	0	1	1
5	1	1	0	1	0	0
6	0	0	0	1	0	0

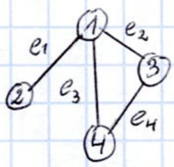
$$\rightarrow \begin{pmatrix} 2 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

INCIDENCE MATRIX:

FOR A SIMPLE GRAPH $G = (V, E)$ THE INCIDENCE MATRIX A $n \times m$ MATRIX $B(G) = (b_{ij})_{n \times m}$ WHERE n AND m ARE THE NUMBERS OF VERTICES AND EDGES RESPECTIVELY ($|V| = n$ AND $|E| = m$) SUCH THAT:

$$b_{ij} = \begin{cases} 1 & v_i \in e_j \\ 0 & v_i \notin e_j \end{cases}$$

Ex.



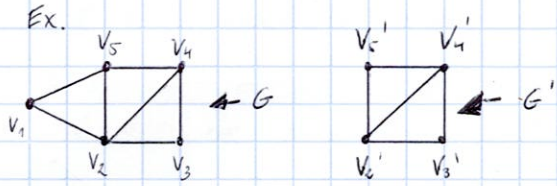
	e_1	e_2	e_3	e_4
1	1	1	1	0
2	1	0	0	0
3	0	1	0	1
4	0	0	1	1

$$\rightarrow \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

SUBGRAPH:

A GRAPH $G' = (V', E')$ IS A SUBGRAPH OF ANOTHER GRAPH $G = (V, E)$
IFF $V \subset V'$ AND $E' \subset E$

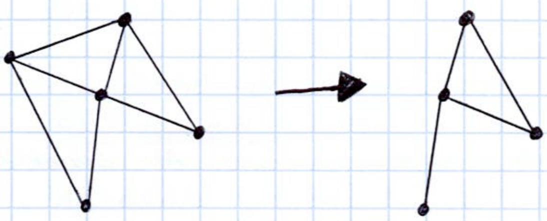
Ex.



INDUCED SUBGRAPH:

LET $G = (V, E)$ BE ANY GRAPH, AND LET $S \subset V$ BE ANY SUBSET OF VERTICES OF G. THEN THE INDUCED SUBGRAPH $G[S]$ IS THE GRAPH WHOSE VERTEX SET IS S AND WHOSE EDGE SET CONSISTS OF ALL OF THE EDGES IN E THAT HAVE BOTH ENDPOINTS IN

Ex.



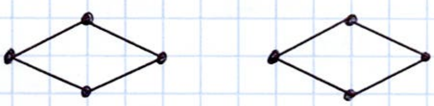
AND DISCONNECTED

CONNECTED GRAPH:

A GRAPH IS CONNECTED IF ANY TWO VERTICES ARE JOINT BY A PATH. IF A GRAPH IS NOT CONNECTED THEN IT IS DISCONNECTED. FOR A DISCONNECTED GRAPH G EVERY MAXIMAL CONNECTED SUBGRAPH IS CALLED A <u>CONNECTED COMPONENT</u> OF G

Ex. (CONNECTED)

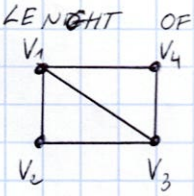
Ex. (DISCONNECTED)



DISTANCE ($dist(x, y)$):

THE DISTANCE BETWEEN TWO VERTICES X AND Y, DENOTED BY $dist(x, y)$ IS THE LENGHT OF A SHORTEST PATH JOINING THEM

Ex.



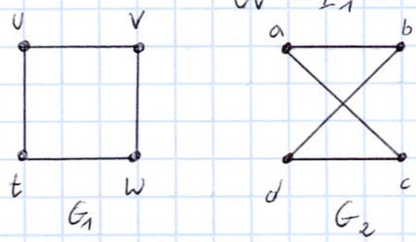
$dist(V_3, V_1) = 1$ $\quad$ $dist(V_2, V_4) = 2$

ISOMORPHISM OF GRAPHS:

TWO GRAPHS $G_1 = (V_1, E_1)$ AND $G_2 = (V_2, E_2)$ ARE ISOMORPHIC IF THERE IS A BIJECTION $f: V_1 \to V_2$ THAT PRESERVES ADJACENCY.

Ex.

$$uv \in E_1 \iff f(u)f(v) \in E_2$$



$G_1 \cong G_2$

$f(u) = a$, $f(v) = b$, $f(w) = d$, $f(t) = c$

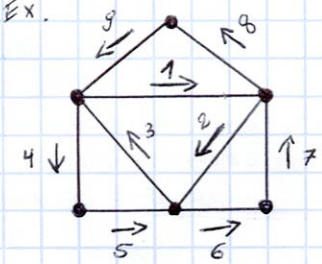
EULERIAN CYCLE:

AN EULERIAN CYCLE IN A GRAPH G IS A CLOSED WALK THAT CONTAINS EACH EDGE EXACTLY ONCE (CLOSED TRAIL).

EULERIAN GRAPH:

AN EULERIAN GRAPH IS A CONNECTED GRAPH THAT CONTAINS AN EULERIAN WALK.

Ex.

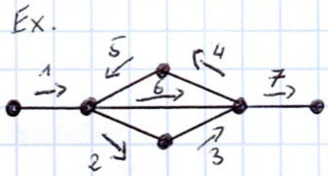


A CONNECTED GRAPH G POSSESES AN EULERIAN CYCLE IFF THE DEGREE OF EVERY VERTEX IS **EVEN**

EULERIAN WALK:

AN EULERIAN WALK IS A GRAPH G THAT USES EACH EDGE EXACLLY ONCE. IF SUCH A PATH EXISTS, THEN THE GRAPH G IS CALLED <u>SEMIEULERIAN GRAPH</u>.

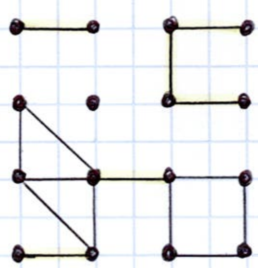
Ex.



A BRIDGE (CUT-EDGE):

A BRIDGE (CUT-EDGE) IS AN EDGE OF A GRAPH WHOSE DELETION INCREASES ITS NUMBER OF CONNECTED COMPONENTS.

Ex.



← 16 VERTICES AND 6 BRIDGES.

FLEURY'S ALGORITHM:

1. MAKE SURE THAT GRAPH HAS EITHER 0 OR 2 ODD VERTICES.

2. IF THERE ARE 0 ODD VERTICES, START ANYWHERE. IF THERE ARE 2 ODD VERTICES, START AT ONE OF THEM.

3. FOLLOW EDGES ONE AT A TIME. IF YOU HAVE CHOICE BETWEEN A BRIDGE OR NON-BRIDGE, ALWAYS CHOOSE A NON-BRIDGE

4. STOP WHEN YOU RUN OUT OF EDGES.

HAMILTONIAN GRAPH:

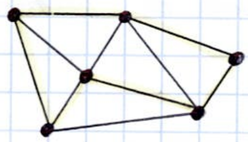
A HAMILTONIAN CYCLE IN A GRAPH G IS A CYCLE WITCH CONTAINS ALL THE VERTICES OF G.

A HAMILTONIAN PATH IN A GRAPH G IS A PATH WITCH CONTAINS ALL THE VERTICES OF G.

A GRAPH G WITCH POSSESSES A HAMILTONIAN CYCLE IS CALLED THE HAMILTONIAN GRAPH.

A GRAPH WHICH POSSESSES A HAMILTONIAN PATH (BUT NOT CYCLE) IS CALLED THE SEMIHAMILTONIAN GRAPH.

Ex.



← HAMILTONIAN PATH/CYCLE

← SEMIHAMILTONIAN GRAPH

DIRAC'S THEOREM:

A SIMPLE GRAPH WITH n VERTICES $(n \geq 3)$ IS HAMILTONIAN IF, EVERY VERTEX HAS DEGREE $\frac{n}{2}$ OR ~~GREATER~~ GREATER

ORE'S THEOREM:

A SIMPLE GRAPH WITH n VERTICES $(n \geq 3)$ IS HAMILTONIAN IF, FOR EVERY PAIR OF NON-ADJACENT VERTICES, THE SUM OF THEIR DEGREES IS n OR GREATER.

TREE:

A TREE IS A CONNECTED GRAPH WITHOUT CYCLES.

EVERY TREE IS A BIPART GRAPH.

Ex.

IF $G=(V,E)$ IS A CONNECTED GRAPH, THEN THE FOLLOWING ARE EQUIVALENT:

- G IS A TREE
- G HAS NO CYCLES
- FOR EVERY PAIR OF DISTINCT VERTICES u AND v IN G, THERE IS EXACLY ONE PATH FROM u TO v
- $G-e$ IS DISCONNECTED FOR ANY EDGE $e \in E(G)$.
- $|E| = |V| - 1$

A TREE WITH n VERTICES HAS EXACTLY $n-1$ EDGES

FOREST:

A FOREST IS A DISJOINT UNION OF TREES.

Ex.

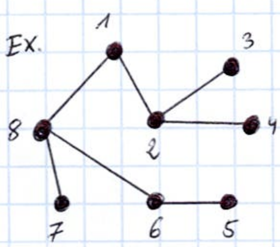
PRÜFER SEQUENCE:

A PRÜFER SEQUENCE OF LENGTH $n-2$, FOR $n \geq 2$, IS ANY SEQUENCE OF INTERGERS BETWEEN 1 AND n, WITH REPETITIONS ALLOWED.

A LABELED TREE GIVES A PRÜFER SEQUENCE BY:

- PICK THE LEAF WITH THE SMALLEST LABEL, CALL IT v.
- PUT THE LABEL OF v'S NEIGHBOOR IN THE OUTPUT SEQUENCE,
- REMOVE v FROM TREE

Ex.



$\longrightarrow (2, 2, 1, 8, 6, 8)$

PRÜFER DECODING:

A PRÜFER SEQUENCE DETERMINES A LABELED TREE BY:

$L \rightarrow$ THE ORDERED LIST OF NUMBERS $1, 2, \dots, n$.

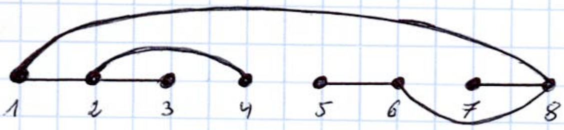
$P \rightarrow$ PRÜFER SEQUENCE.

START WITH n LABELED ISOLATED VERTICES.

REPEAT $n-2$ TIMES:

- LET K BE THE SMALLEST NUMBER IN L ~~[illegible]~~ WHICH IS NOT IN P
- LET j BE THE FIRST NUMBER IN P.
- ADD THE EDGE Kj TO THE GRAPH
- REMOVE K FROM L AND THE FIRST NUMBER IN P

WHEN THIS IS COMPLETED THERE WILL BE TWO NUMBERS LEFT IN L, ADD THE EDGE CORRESPONDING TO THESE TWO NUMBERS



$L = (1, 2, 3, 4, 5, 6, 7, 8)$

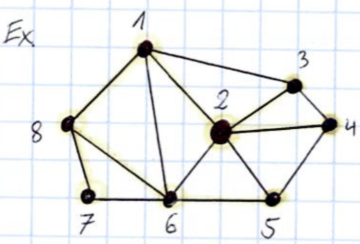
$P = (2, 2, 1, 8, 6, 8)$

PRÜFER CODING AND DECODING ARE INVERSE OPERATIONS, THAT MEANS THAT THERE IS ONE-TO-ONE CORRESPONDENCE BETWEEN LABELED TREES WITH n VERTICES AND PRÜFER SEQUENCES OF LENGTH n-2.

SPANNING TREE:

A SPANNING TREE T OF A CONNECTED, UNDIRECTED GRAPH G IS A TREE COMPOSED OF ALL THE VERTICES AND SOME (OR ALL) OF EDGES OF G.

Ex



A ROOTED TREE IS A TREE WITH VERTEX DESIGNATED AS ROOT (IT CAN BE ANY VERTEX). A ROOTED TRE INTRODUCES A PARENT-CHILD RELATIONSHIP BETWEEN THE VERTICES AND THE NOTION OF DEPTH IN THE TREE.

CALEY'S FORMULA:

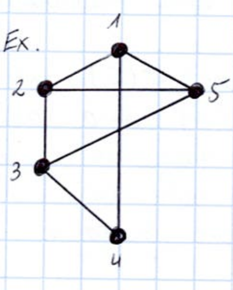
THE NUMBER OF SPANNING LABELED TREES OF K_n IS n^{n-2}, FOR $n \geq 2$.

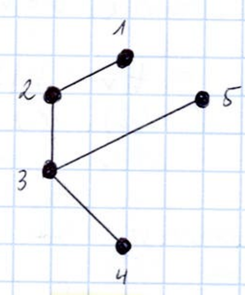
THE NUMBER OF SPANNING TREES OF K_n IS THE SAME AS NUMBER OF SEQUENCES WITH $(n-2)$ ELEMENTS WITH REPETITION FROM THE SET $\{1, 2, \ldots, n\}$. THUS EACH PRÜFER SEQUENCE CORRESPONDS TO EXACTLY ONE SPANNING TREE OF K_n

DEPTH-FIRST SEARCH: (DFS)

- CHOSE A ROOT r, LABEL IT.
- LET $v := r$
- LABEL A NON-LABELED NEIGHBOR OF V, $V := r$, OTHERWISE GO UP.

Ex.



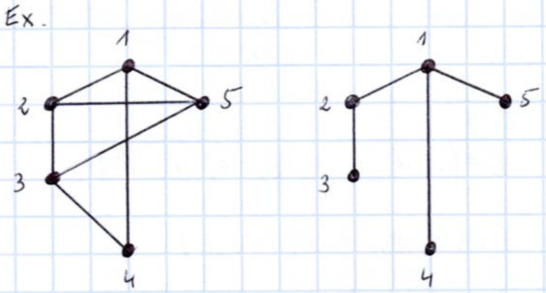


DFS

BREADTH - FIRST SEARCH (BFS):

- CHOSE A ROOT r, LABEL IT
- LET $v := r$
- LABEL ALL NON-LABELED NEIGHBORS w OF v, OTHERWISE GO UP.

Ex.



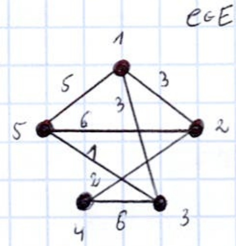
BFS

WEIGHTED GRAPH:

A WEIGHTED GRAPH IS A GRAPH, IN WITH EACH EDGE HAS A WEIGHT (SOME REAL NUMBER). FORMALLY $G=(V,E,w)$. THE WEIGHT OF A GRAPH IS SUM OF WEIGHTS OF ALL EDGES:

$$w(G) = \sum_{e \in E} w(e)$$

Ex.



$$\longrightarrow \quad w(G) = \sum_{e \in E} w(e) = 5+3+3+6+1+2+6 = \underline{\underline{26}}$$

DIJKSTRA'S ALGORITHM:

PROBLEM: FIND THE SHORTEST TRAIL IN THE WEIGHTED GRAPH FROM THE ROOT S.

SOLUTION:

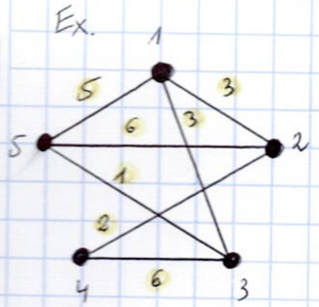
$l(v)$ - DISTANCES (THE WEIGHT OF A PATH FROM s TO v)

$p(v)$ - A NEIGHBOR OF v ON THE PATH

- LET $l(s)=0$, $l(v)=\infty$ FOR $v \neq s$ AND $p(v)=0$ FOR $v \in V$
- LET $S = \emptyset$
  UNLESS $S \neq V$ DO:
  - PICK $u \in V \setminus S$ WITH MINIMAL $l(u)$ AND ADD IT TO S.
  - $\forall v \in V \setminus S$: $uv \in E(G)$ AND $l(v) > l(u) + w(vu)$ SET $l(v) = l(u) + w(uv)$ AND $p(v) = u$ IF IT SMALLER THAN THE PREVIOUS ONE, THEN PUT IT IN THE TABLE

THE SHORTEST PATH FROM s TO v IS. $v, p(v), p(p(v)), \dots, s$.

Ex.



← WEIGHT

	1	2	3	4	5	S
1	0	∞	∞	∞	∞	1
2	X	3	3	∞	5	2
3	X	X	3	5	5	3
4	X	X	X	5	4	5
5	X	X	X	5	X	4

	1	2	3	4	5
1	0	0	0	0	0
2	X	1	1	0	1
3	X	X	1	2	1
4	X	X	X	2	3
5	X	X	X	2	X

MINIMUM SPANNING TREE:

A MINIMUM SPANNING TREE IN UNDIRECTED CONNECTED WEIGHTED GRAPH IS A SPANNING TREE OF MINIMUM WEIGHT (AMONG ALL SPANNING TREES.).

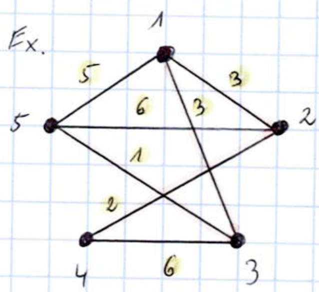
REMARK: THE MINIMUM SPANNING TREE MAY NOT BE UNIQUE. HOEWER, IF THE WEIGHTS OF ALL EDGES ARE PAIRWISE DISTINCT, IT IS UNIQUE.

KRUSCAL ALGORITHM:

LET $T=(V,E)$ BE A TREE OF SIZE $|E|=m$:

- $T=(V,E_1)$, $E_1 \neq \emptyset$, (T - FOREST OF ISOLATED VERTICES)
- LET $e_1, \dots, e_m$: $w(e_1) \leq w(e_2) \leq \dots w(e_m)$
- FOR $i=1,\dots,m$ IF $e_i = uv$ FOR u AND v IN DIFFERENT COMPONENTS OF T, THEN $e_i \in E_1$

Ex.



← WEIGHT

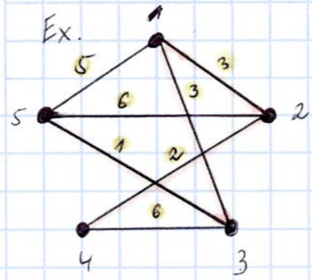
PRIM ALGORITHM:

PRIM ALGORITHM SIMILAR TO DIJKSTRA (WE COUNT DISTANCES TO THE TREE NOT A VERTEX):

$K(v)$ - A DISTANCE FROM A TREE
$p(v)$ - A NEIGHBOR OF v IN THE TREE

- TAKE ANY $s \in V$ AND LET $l(s)=0$, $K(v)=\infty$ FOR $v \neq s$ AND $p(v)=0$ FOR $v \in V$
- LET $S=\emptyset$
  UNLESS $S \neq V$ do
  - PICK $u \in V \setminus S$ WITH MINIMAL $K(u)$ AND ADD IT TO S
  - $\forall v \in V \setminus S$: $uv \in E(G)$ AND $K(v) > (vu)$ SET $k(v) = w(vu)$ AND $p(v)=u$
    MOREOVER $E_1 = \{vp(v) : v \in S\}$

Ex.



← WEIGHT

	1	2	3	4	5	S
1	0	∞	∞	∞	∞	1
2	x	3	3	∞	5	2
3	x	x	3	2	5	4
4	x	x	3	x	5	3
5	x	x	x	x	1	5

	1	2	3	4	5
1	0	0	0	0	0
2	x	1	1	0	1
3	x	x	1	2	1
4	x	x	1	x	1
5	x	x	x	x	3

$E_1 = (V_3V_5,\ V_2V_4,\ V_1V_3,\ V_1V_2)$