Dihedral Group

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3. $H = \{(2,4)\}$

$D_4 = \{(1\ 2\ 3\ 4), (1\ 3)(2\ 4),$
$(1\ 4\ 3\ 2), (1), (2\ 4),$
$(1\ 3), (1\ 2)(3\ 4),$
$(1\ 4)(2\ 3)\}$

(a) The left cosets of H

$(1\ 2\ 3\ 4)H = (1\ 2\ 3\ 4)(2\ 4) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} = (1\ 2)(3\ 4)$ ✓

$(1\ 3)(2\ 4)H = (1\ 3)(2\ 4)(2\ 4) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix} = (1\ 3)(2)(4)$ ✓

$(1\ 4\ 3\ 2)H = (1\ 4\ 3\ 2)(2\ 4) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} = (1\ 4)(2\ 3)$ ✓

$(1)H = (1)(2\ 4) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix} = (2\ 4)(1)(3)$ ✓

$(2\ 4)H = (2\ 4)(2\ 4) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} = (1)(2)(3)(4)$ ✓

$(1\ 3)H = (1\ 3)(2\ 4) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} = (1\ 3)(2\ 4)$ ✓

$(1\ 2)(3\ 4)H = (1\ 2)(3\ 4)(2\ 4) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} = (1\ 2\ 3\ 4)$ ✓

$(1\ 4)(2\ 3)H = (1\ 4)(2\ 3)(2\ 4) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} = (1\ 4\ 3\ 2)$ ✓

(b) The right cosets of H:

$H(1\ 2\ 3\ 4) = (2\ 4)(1\ 2\ 3\ 4) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} = (1\ 4)(2\ 3)$ ✓

$H(1\ 3)(2\ 4) = (2\ 4)(1\ 3)(2\ 4) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix} = (1\ 3)(2)(4)$ ✓

$H(1\ 4\ 3\ 2) = (2\ 4)(1\ 4\ 3\ 2) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} = (1\ 2)(3\ 4)$ ✓

$H(1) = (2\ 4)(1) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix} = (2\ 4)(1)(3)$ ✓

$H(2\ 4) = (2\ 4)(2\ 4) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} = (1)(2)(3)(4)$ ✓

$H(1\ 3) = (2\ 4)(1\ 3) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} = (1\ 3)(2\ 4)$ ✓

$H(1\ 2)(3\ 4) = (2\ 4)(1\ 2)(3\ 4) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} = (1\ 4\ 3\ 2)$ ✓

$H(1\ 4)(2\ 3) = (2\ 4)(1\ 4)(2\ 3) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} = (1\ 2\ 3\ 4)$ ✓

(c) The left cosets of H = The right cosets of H

$\therefore$ H is a normal subgroup of $D_4$

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Example

The group of rigid motion of a square, $D_4$, consists of eight elements. With the vertices numbered 1, 2, 3, 4, the rotations are

$r = (1\ 2\ 3\ 4)$

$r^2 = (1\ 3)(2\ 4)$

$r^3 = (1\ 4\ 3\ 2)$

$r^4 = (1)$

and the reflections are

$s_1 = (2\ 4)$

$s_2 = (1\ 3)$

The order of $D_4$ is 8. The remaining two elements are

$rs_1 = (1\ 2)(3\ 4)$

$r^3 s_1 = (1\ 4)(2\ 3)$

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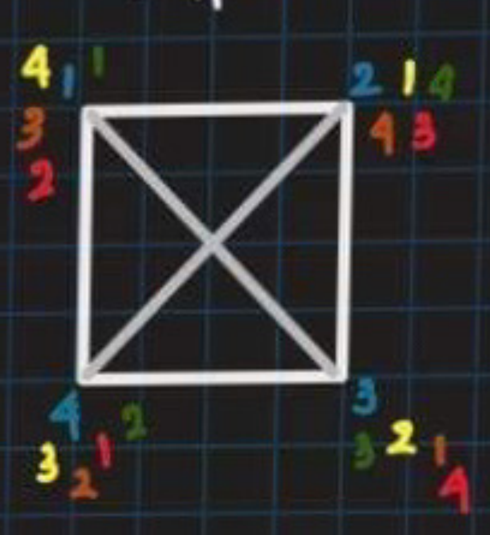
$S_4 = \{1, 2, 3, 4\}$ , $D_4 = ?$

$r^k = k\,\dfrac{360^\circ}{n}$



$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} = (1234)$

$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} = (13)(24)$

$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} = (1432)$

$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix} = (13)(24)$

Rotation

$r^1 = 90^\circ = (1234)$

$r^2 = 180^\circ = (13)(24)$

$r^3 = 270^\circ = (1432)$

$r^4 = (1)$

Reflection

$s_1 = (24)$

$s_2 = (13)$

Remaining two elements

$rs_1 = (1234)(24) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} = (12)(34)$

$r^3 s_1 = (1432)(24) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} = (14)(23)$

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