Recurrence relations

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Definition:

A sequence is a function from a subset of the set of integers (usually either the set $\{0, 1, 2, \ldots\}$ or the set $\{1, 2, 3, \ldots\}$) to a set S. We use the notation a_n to denote the image of the integer n. We call a_n a **term** of the sequence.

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We use the notation $\{a_n\}_{n \in \mathbb{N}}$ to describe the infinite sequence. We describe sequences by listing the terms of the sequence in order of increasing subscripts.

Definition:

A recurrence relation for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer.

$a_n = n!$ $\quad n \in \mathbb{N} \quad a_0 = 0! = 1$

$a_n = (n-1)!\,n = n\,a_{n-1}$

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Many sequences can be a solution for the same recurrence relation.

Example:

Let $a_n = 2a_{n-1} - a_{n-2}$, for $n \geq 2$.

- $a_n = 3n$ for all $n \geqslant 0$ $\quad a_0 = 0,\ a_1 = 3.$
  - $a_{n-1} = 3(n-1)$
  - $a_{n-2} = 3(n-2)$

$$a_n = 2.3(n-1) - 3(n-2) = 6n - 6 - 3n + 6$$
$$= 3n$$

$a_n = 2a_{n-1} - a_{n-2} \quad n \geqslant 2$

$a_n = 5 \quad , n \in \mathbb{N} \qquad a_0 = a_1 = 5$

$a_{n-1} = 5 \quad , a_{n-2} = 5$

$a_n = 2.5 - 5 = 5$

Definition:

The initial conditions for a sequence specify the terms that precede the first term where the recurrence relation takes effect.

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The recurrence relations together with the initial conditions can uniquely determines the sequence.

Fibonacci Sequence

From Wikipedia

Fibonacci Sequence

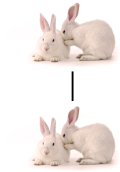
From www.tapeciarnia.pl

Example:

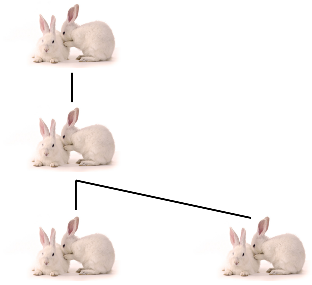
A young pair of rabbits (one of each gender) is placed on an island. A pair of rabbits does not breed until they are 2 months old. After they are 2 months old, each pair of rabbits produces another pair each month. Find a recurrence relation for the number of pairs of rabbits on the island after n months, assuming that rabbits never die.

Fibonacci Sequence

Fibonacci Sequence



Fibonacci Sequence



Fibonacci Sequence

f_n

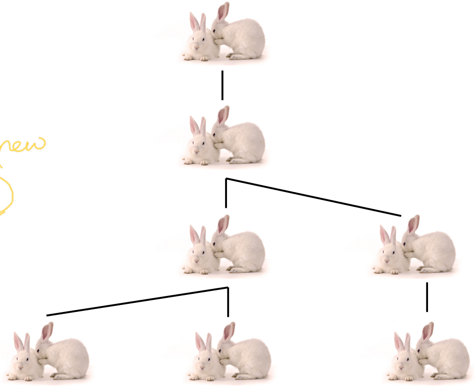
$f_1 = 1$

$f_2 = 1$

$f_3 = 2$

$f_4 = 3.$

$f_n = \underbrace{f_{n-1}}_{\text{old}} + \underbrace{f_{n-2}}_{\text{new}}$



A closed formula

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Example:

Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = 3a_{n-1}$ for $n = 1, 2, 3, \ldots$, and suppose that $a_0 = 3$. Find a closed formula for this sequence.

$a_1 = 3 \cdot a_0 = 3 \cdot 3 = 3^2$

$a_2 = 3 \cdot a_1 = 3 \cdot 3^2 = 3^3$

$a_3 = 3 \cdot a_2 = 3 \cdot 3^3 = 3^4$

$\underline{a_n = 3^{n+1}}$

Closed formula!

Linear homogeneous recurrence relation

$a_n = 2 \cdot a_{n-1} + a_{n-3} = 2 \cdot a_{n-1} + 0 \cdot a_{n-2} + 1 \cdot a_{n-3}$

L.H.R.R of degree 3 $\quad c_1 = 2 \quad c_2 = 0 \quad c_3 = 1$

$a_n = n \cdot a_{n-1}$, $c_1 = n$ No!

Definition:

A linear homogeneous recurrence relation (LHRR) of degree k with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} \qquad (\star)$$

where $c_1, c_2, \ldots, c_k \in \mathbb{R}$ and $c_k \neq 0$.

Ex. $a_n = a_{n-1} + a_{n-2} = 1 \cdot a_{n-1} + a_{n-2}$

$c_1 = 1$, $c_2 = 2$, $k = 2$

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$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \ldots + c_k r^{n-k}.$$

When both sides of this equation are divided by r^{n-k} and the right-hand side is subtracted from the left, we obtain the equation

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \ldots - c_{k-1} r - c_k = 0.$$

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The a equation

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is called characteristic equation of a recurrence relation $(\star)$.

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The characteristic roots of a linear homogeneous recurrence relation are the roots of its characteristic equation.

Example:

What are the characteristic equations for the following recurrence relations?

1. $a_n = a_{n-1} + a_{n-2}$
2. $b_n = 3b_{n-1}$

Theorem:

If λ_j is a characteristic root of equation $(\star)$ of multiplicity k_j, then

$$a_n^j = \lambda_j^n \left(\sum_{l=1}^{k_j} C_l n^{l-1} \right), \quad C_l \in \mathbb{R}$$

is a solution of $(\star)$ corresponding to the characteristic root k_j.

Linear homogeneous recurrence relation

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is a solution of $(\star)$ corresponding to the characteristic root k_j.

If we find some solutions to a linear homogeneous recurrence, then any linear combination of them will also be a solution to the linear homogeneous recurrence.

Linear homogeneous recurrence relation

Theorem:

If λ_j is a characteristic root of equation $(\star)$ of multiplicity k_j, then

$$a_n^j = \lambda_j^n \left(\sum_{l=1}^{k_j} C_l n^{l-1} \right), \quad C_l \in \mathbb{R}$$

is a solution of $(\star)$ corresponding to the characteristic root k_j.

Theorem:

If $\lambda_1, \ldots, \lambda_r$ are distinct characteristic roots of equation $(\star)$ $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$, and $a_n^1, \ldots, a_n^r$ are solutions of of equation $(\star)$ corresponding to the characteristic root $\lambda_1, \ldots, \lambda_r$, respectively, then

$$a_n = \sum_{j=1}^{r} a_n^j$$

is the solution of equation $(\star)$.

Definition:

A linear non-homogeneous recurrence relation (NHRR) with constant coefficients is of the form:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + F(n) \qquad (\star\star\star)$$

where $F(n) \not\equiv 0$.

Solving linear non-homogeneous recurrence relations

Definition:

A linear non-homogeneous recurrence relation (NHRR) with constant coefficients is of the form:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + F(n) \qquad (\star\star\star)$$

where $F(n) \not\equiv 0$.
The recurrence obtained from $(\star\star\star)$ by dropping $F(n)$ is called the associated homogeneous recurrence relation (AHRR):

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} \qquad (\star')$$

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Theorem:

If:
a_n^h – a solution for associated homogeneous recurrence (HR) $(\star')$
a_n^p – the particular solution (NR) $(\star\star\star)$,
Then every solution of relation $(\star\star\star)$ is of the form

$$a_n = a_n^h + a_n^p.$$

The method of undetermined coefficients is based on assuming a trial form for the particular solution a_n^p which depends on the form of the function $F(n)$ and which contains a number of arbitrary constants. This trial function is then substituted into the non-homogeneous recurrence relation and the constants are chosen to make this a solution.

Method of undetermined coefficients

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Theorem:

For non-homogeneous recurrence relations (NR)
$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + F(n)$

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Method of undetermined coefficients

Theorem:

For non-homogeneous recurrence relations (NR)
$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + F(n)$ if
$F(n) = (b_t n^t + b_{t-1} n^{t-1} + \ldots + b_1 n + b_0) s^n$,
then:

$$a_n^p = n^m (p_t n^t + p_{t-1} n^{t-1} + \ldots + p_1 n + p_0) s^n$$

is a particular solution of non-homogeneous recurrence relations (NR), where s **is** a root with multiplicity m of the characteristic equation m.

Method of undetermined coefficients

Example:

Solve the recurrence relation $a_n = 3a_{n-1} + 10a_{n-2} + 2 \cdot 4^n$ where $a_0 = \frac{1}{2}$ and $a_1 = 2$.

The Tower of Hanoi

The Tower of Hanoi

A popular puzzle of the late nineteenth century invented by the French mathematician Édouard Lucas, called the Tower of Hanoi, consists of three pegs mounted on a board together with disks of different sizes. Initially these disks are placed on the first peg in order of size, with the largest on the bottom. The rules of the puzzle allow disks to be moved one at a time from one peg to another as long as a disk is never placed on top of a smaller disk. The goal of the puzzle is to have all the disks on the second peg in order of size, with the largest on the bottom.

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Let H_n denote the number of moves needed to solve the Tower of Hanoi problem with n disks. Set up a recurrence relation for the sequence $\{H_n\}$.

The Tower of Hanoi