Kalman Filter and ML Estimation of DFM
- A Monte Carlo Exercise -

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January 6, 2021

The dynamic factor model (DFM) considered in this Monte Carlo (MC) exercise takes the general form

$$\begin{aligned} y_t &= \Lambda f_t + e_t,\ t = 1, \cdots, T, & (1)\\ f_t &= \Phi_f f_{t-1} + v_t,\ v_t \sim N(0, \Sigma_v), & (2)\\ e_t &= \Phi_e e_{t-1} + \eta_t,\ \eta_t \sim N(0, \Sigma_\eta), & (3) \end{aligned}$$

where the $(N \times 1)$ vector of observables $y_t = (y_{1t}, \cdots, y_{Nt})'$ contains data of N cross sections at time point t. The evolution of both common factors f_t $(r \times 1)$ and idiosyncratic components e_t $(N \times 1)$ can be captured (or written in companion form) by a AR(1) process. The idiosyncratic components are orthogonal by nature, which implies that both Φ_e and Σ_η are diagonal matrices.

In the MC exercise, the cross-sectional dimension (N) and the number of factor (r) are set to be 15 and 1, respectively. Moreover, both f_t and e_t are assumed to follow a stationary AR(1) process, where however the factors are much more persistent than the idiosyncratic error terms. For purpose of unique (up to sign) identification, I restrict the $(N \times 1)$ loading vector Λ to have unit length.[1]

To generate the data, I set $\Phi_f = 0.99$ and $\Sigma_f = 1.2$. The coefficients in Λ are randomly drawn from standard Gaussian distribution and subsequently re-scaled by dividing the square-root of the vector length. For the idiosyncratic errors, I independently drawn the non-zero diagonal elements of Φ_e and Σ_η from uniform distributions $\mathcal{U}(-0.9, 0.9)$ and $\mathcal{U}(0.6, 1.2)$, respectively. I simulate in total 3200 observations and save the last 200 realizations (T) to immunize simulation outcomes against initial values. The simulated data are displayed in Figure 1.

Based on the simulated observations, the factors are computed by means of Kalman filtering conditional on a parameter vector $\theta = (\Phi_f, \mathrm{diag}(\Phi_e), \Sigma_f, \mathrm{diag}(\Sigma_\eta))'$. For this

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[1] This corresponds to the common restrictions imposed for PC-estimation. However, since both f_t and Σ_v reduce to scalars, there is no need to impose restrictions on the factor covariance to ensure the identifiability of column ordering of Λ.

purpose, the model is rewritten in a state-space form

$$y_t = \begin{bmatrix}\Lambda & I_N\end{bmatrix} \begin{bmatrix} f_t \\ e_t \end{bmatrix} = H\alpha_t \tag{4}$$

$$\alpha_t = \begin{bmatrix}\Phi_f & \mathbf{0} \\ \mathbf{0} & \Phi_e\end{bmatrix} \alpha_{t-1} + \begin{bmatrix} \upsilon_t \\ \eta_t \end{bmatrix}. \tag{5}$$

The parameters are estimated by maximizing the log-likelihood, which contains the following kernel due to Gaussian errors

$$l(\theta) \propto -\frac{1}{2} \sum_{t=1}^{T} \left(\log \det F_t + \nu_t' F_t^{-1} \nu_t \right),$$

where $\nu_t = y_t - E[y_t | y_{1:t-1}, \theta]$ and $F_t = \mathrm{Var}[y_t | y_{1:t-1}, \theta]$.

The Kalman filter is initialized based on different data characteristics. While the idiosyncratic components are assumed not to be as persistent as the latent factor, and hence, the unconditional first and second order moments can be used as initial values, I select one of the first observations as an initial guess for the latent factor and set a high initial variance (10^5) to reflect the uncertainty about this prior belief.

To achieve numerical stability, I transform the space of some parameters. The AR coefficients are squished on the open set (-1, 1) by hyperbolic tangent. Variance parameters are transformed into exponential space. The estimated values of the factor are displayed in Figure 2.

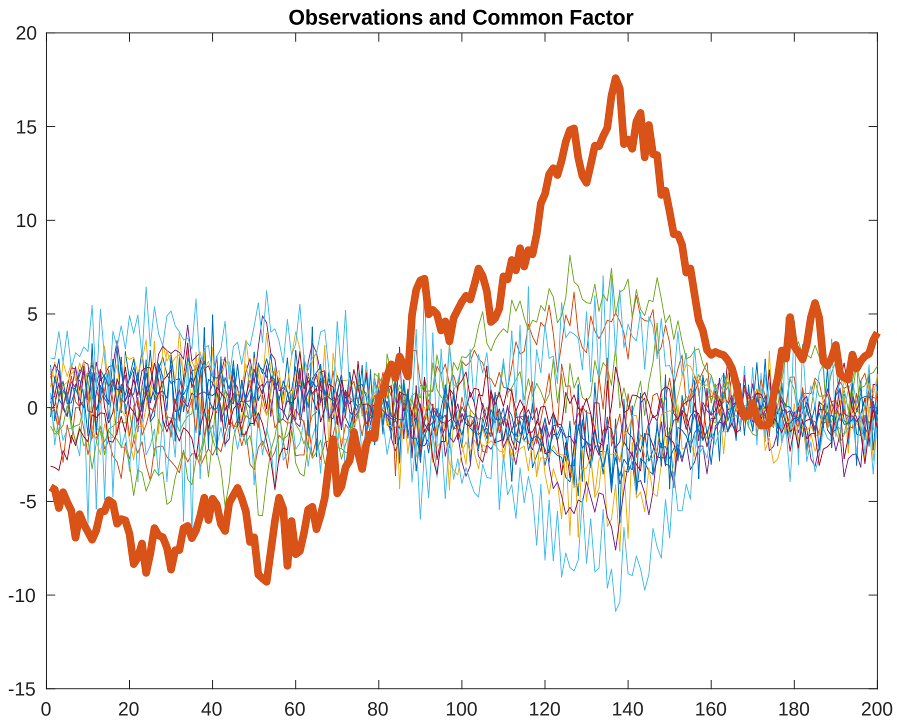

Figure 1: Observables and common factor (thick line).

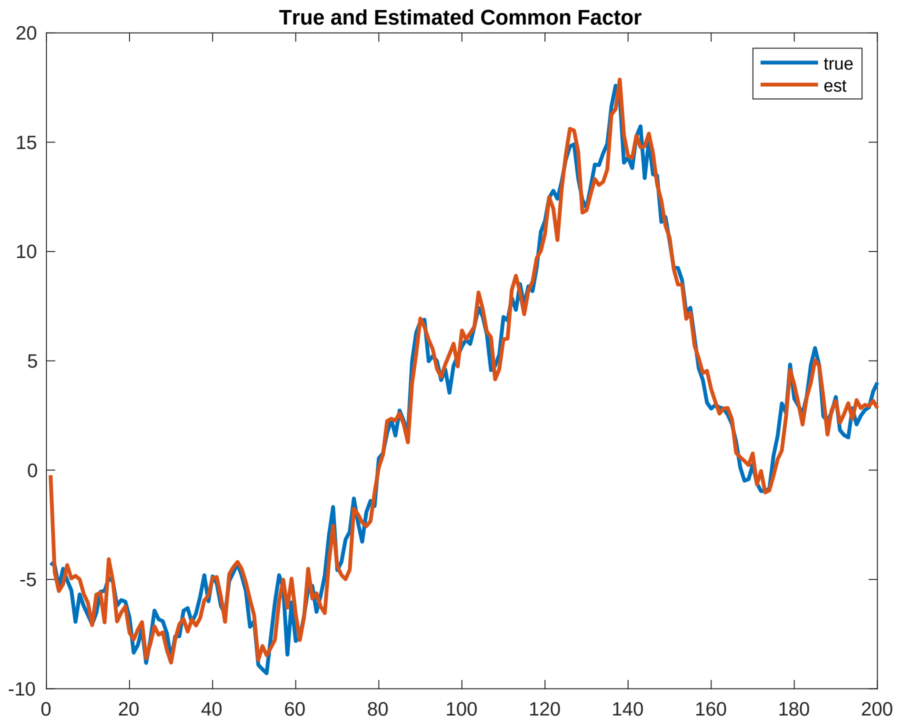

Figure 2: True (blue) and estimated (blue) latent factor.