E4: Appendix

Feina Zou

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1 DFM with Gibbs Sampling

In this exercise, the following dynamic factor model is simulated and estimated with Gibbs Sampling. G_t is the commcon factor, $\epsilon_{j,t}$ are the series-specific factors. Series number N equals 15 and data length T is 200.

(1) $y_{j,t} = \phi_j G_t + \epsilon_{j,t}$

(2) $G_t = \rho G_{t-1} + v_t, v_t \sim N(0,\, 1)$

(3) $\epsilon_{j,t} = \gamma_j \epsilon_{j,t-1} + \eta_{j,t}, \eta_{j,t} \sim N(0,\, \sigma^2_{\eta,j})$

Referring to Kim and Nelson (1999), the estimation of the DFM with Gibbs Sampling involves successive iteration of the following three steps.

1.1 Step 1: Sampling G_t conditional on data $y_{j,t}$ and all the parameters

By multiplying the equation (1) by $\gamma(L) = (1 - \gamma_j L)$, we write the model in the following state-space form, with $y^*_{j,t} = (y_{j,t} - \gamma_j y_{j,t-1})$.

Measurment equation: $(Y_t = H\beta_t + e_t)$

$$\begin{bmatrix} \mathrm{y}^*_{1t} \\ y^*_{2t} \\ \vdots \\ y^*_{Nt} \end{bmatrix} = \begin{bmatrix} \phi_1 & -\phi_1\gamma_1 \\ \phi_2 & -\phi_2\gamma_2 \\ \vdots & \vdots \\ \phi_N & -\phi_N\gamma_N \end{bmatrix} \begin{bmatrix} \mathrm{G}_t \\ G_{t-1} \end{bmatrix} + \begin{bmatrix} \eta_{1t} \\ \eta_{2t} \\ \vdots \\ \eta_{Nt} \end{bmatrix}, \quad E(e'_t e_t) = R = \begin{bmatrix} \sigma^2_{\eta,1} & & \\ & \ddots & \\ & & \sigma^2_{\eta,N} \end{bmatrix}$$

State equation: $(\beta_t = F\beta_{t-1} + v_t)$

$$\begin{bmatrix} \mathrm{G}_t \\ G_{t-1} \end{bmatrix} = \begin{bmatrix} \rho & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \mathrm{G}_{t-1} \\ G_{t-2} \end{bmatrix} + \begin{bmatrix} \mathrm{v}_t \\ 0 \end{bmatrix}, \quad E(v'_t v_t) = Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

1.2 Step 2: Sampling ρ conditional on G_t

Conditional on G_t, we employ a normal prior for ρ:

$\rho \sim N(a0, A0)_{I[s(\rho)]}$,

where $I[s(\rho)]$ is an indicator function, denoting that the roots of $\rho(L) = 0$ lie outside the unit circle.

The posterior distribution is then:

$\rho|G_t \sim N(a1, A1)$

where

$a1 = (A0^{-1} + G'_{t-1}G_{t-1})^{-1}(A0^{-1}a0 + G'_{t-1}G_t)$

$A1 = (A0^{-1} + G'_{t-1}G_{t-1})^{-1}$

1.3 Step 3: Sampling $\phi_j, \gamma_j, \sigma^2_{\eta,j}$ conditional on G_t and $y_{j,t}$

We employ normal priors for ϕ_j and γ_j, and inverse gamma prior for $\sigma^2_{\eta,j}$ as follows:

$\phi_j|\gamma_j, \sigma^2_{\eta,j} \sim N(b0_j, B0_j)$

$\gamma_j|\phi_j, \sigma^2_{\eta,j} \sim N(c0_j, C0_j)$

$\sigma^2_{\eta,j}|\phi_j, \gamma_j \sim IG(d0_j, D0_j)$

Like in section 1.1, by multiplying equation (1) by $\gamma(L) = (1 - \gamma_j L)$, we get

$y^*_{j,t} = \phi_j G^*_t + \eta_{j,t},\ \ \eta_{j,t} \sim N(0, \sigma^2_{\eta,j})$

where $y^*_{j,t} = y_{j,t} - \gamma_j y_{j,t-1}$ and $G^*_t = G_t - \gamma_j G_{t-1}$. Thus, the posterior distribution of ϕ_j is:

$\phi_j|\gamma_j, \sigma^2_{\eta,j}, y^*_{j,t}, G^*_t \sim N(b1_j, B1_j)$ where

$b1_j = (B0_j^{-1} + \sigma^{-2}_{\eta,j} G^{*\prime}_t G^*_t)^{-1}(B0_j^{-1}b0_j + \sigma^{-2}_{\eta,j} G^{*\prime}_t y^*_{j,t})$

$B1_j = (B0_j^{-1} + \sigma^{-2}_{\eta,j} G^{*\prime}_t G^*_t)^{-1}$

γ_j is estimated in equation (3) with $\epsilon_{j,t} = y_{j,t} - \phi_j G_t$ from equation (1) which are already known form last steps. The posterior distribution of γ_j is:

$\gamma_j|\phi_j, \sigma^2_{\eta,j}, y^*_{j,t}, G^*_t \sim N(c1_j, C1_j)_{I[s(\gamma)]}$ where

$c1_j = (C0_j^{-1} + \sigma^{-2}_{\eta,j} \epsilon'_{j,t-1}\epsilon_{j,t-1})^{-1}(C0_j^{-1}c0_j + \sigma^{-2}_{\eta,j} \epsilon'_{j,t-1}\epsilon_{j,t})$

$C1_j = (C0_j^{-1} + \sigma^{-2}_{\eta,j} \epsilon'_{j,t-1}\epsilon_{j,t-1})^{-1}$

Conditional on γ_j, the posterior distribution for $\sigma^2_{\eta,j}$ is:

$\sigma^2_{\eta,j}|\phi_j, \gamma_j, y^*_{j,t}, G^*_t \sim IG(d1_j, D1_j)$ where

$d1_j = d0_j + \frac{T}{2}$

$D1_j = D0_j + \frac{(\epsilon_{j,t} - \gamma_j \epsilon_{j,t-1})'(\epsilon_{j,t} - \gamma_j \epsilon_{j,t-1})}{2}$