DFM with Gibbs

Shu Wang*

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1 DGP

$$
\begin{aligned}
y_{it} &= \lambda_i c_t + e_{it},\ i = 1, \cdots, N,\ t = 1, \cdots, T, && (1)\\
c_t &= \phi c_{t-1} + v_t,\ v_t \sim N(0,1), && (2)\\
e_{it} &= \psi_i e_{i,t-1} + \eta_{it},\ \eta_{it} \sim N(0, \sigma^2_{\eta,i}), && (3)
\end{aligned}
$$

where $E(\eta_{it}\eta_{jt}) = 0$, $\forall i \neq j$. The panel dimension N and the number of observations T were set to be 4 and 240, respectively.

2 Gibbs sampling

2.1 Draw c_t conditional on model parameters

State space representation
Measurement equation ($y_t^* = H\alpha + w_t$, $R = E(w_t w_t') = \text{diag}(\sigma^2_{\eta,i})$)

$$
\begin{bmatrix} y^*_{1t} \\ y^*_{2t} \\ y^*_{3t} \\ y^*_{4t} \end{bmatrix} = \begin{bmatrix} \lambda_1 & -\lambda_1\psi_1 \\ \lambda_2 & -\lambda_2\psi_2 \\ \lambda_3 & -\lambda_3\psi_3 \\ \lambda_4 & -\lambda_4\psi_4 \end{bmatrix} \begin{bmatrix} c_t \\ c_{t-1} \end{bmatrix} + \begin{bmatrix} \eta_{1t} \\ \eta_{2t} \\ \eta_{3t} \\ \eta_{4t} \end{bmatrix} \tag{4}
$$

Transition equation ($\alpha_t = F\alpha_{t-1} + \xi_t$, $Q = E(\xi_t\xi_t')$)

$$
\begin{bmatrix} c_t \\ c_{t-1} \end{bmatrix} = \begin{bmatrix} \phi & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c_{t-1} \\ c_{t-2} \end{bmatrix} + \begin{bmatrix} v_t \\ 0 \end{bmatrix} \tag{5}
$$

The transformed observations are obtained by left-multiplying the lag polynomial $\psi_i(L)$. Note also that the covariance matrix of the transition error is singular

$$
Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.
$$

*Chair of Econometrics, University of Goettingen, Humboldtallee 3, D-37073 Goettingen, Germany. Corresponding author: Shu Wang, e-mail: shu.wang@uni-goettingen.de. We gratefully acknowledge xxxxxxx.

Based on the some initial guesses of model parameters, compute mean and variance of state vectors, $\alpha_{t|t}$ and $P_{t|t}$ conditional on $y^*_{1:t}$ via Kalman fiter. Draw α^*_t from $N(\alpha_{t|t}, P_{t|t})$ and store them for calculating the conditional mean and variance of $\alpha_t|y^*_{1:t}, \alpha^*_{t+1}$ in the backward recursions, where α^*_{t+1} is the first row (non-singular part) of the state vector.

2.2 Draw ϕ conditional on c_t

Condition on c_t, ϕ is simply the regression coefficient from an stylized Gaussian AR(1) model. Imposing a nature conjugate prior

$$\phi \sim N(\phi_0, S_0) \tag{6}$$

the posterior after observing $C_T = (c_1, \cdots, c_T)'$ is given by

$$\phi|C_T \sim N(\phi_1, S_1), \tag{7}$$

where

$$\phi_1 = (S_0^{-1} + C'_{T-1}C_{T-1})^{-1}(S_0^{-1}\phi_0 + C'_{T-1}C_T) \tag{8}$$
$$S_1 = (S_0^{-1} + C'_{T-1}C_{T-1})^{-1}. \tag{9}$$

2.3 Draw λ_i, ϕ_i, $\sigma^2_{\eta,i}$ conditional on c_t

Conditional on c_t, the observation equations reduce to simple linear regression models with deterministic regressors. Because the error term, i.e. idiosyncratic components, are not correlated cross the equation, and the regressor (c_t) in each equation is the same, we can our apply sampling strategy to each equation separately. But since, again, the error term in equation (1) is autocorrelated, we perform the same data transformation as before and obtain

$$y^*_{it} = \lambda_i c^*_t + \eta_{it}, \tag{10}$$

where $y^*_{it} = \psi_i(L)y_{it}$ and $c^*_{it} = \psi_i(L)c_{it}$.

First, conditional on some initial guesses of ψ_i and $\sigma^2_{\eta,i}$ as well as the data $Y^*_{i,T} = (y^*_{1t}, \cdots, y^*_{Tt})'$, based on Gaussian prior

$$\lambda_i \sim N(\lambda_{0,i}, A_{0,i}) \tag{11}$$

sample λ_i from its posterior

$$\lambda_i|Y^*_{i,T}, \sigma^2_{\eta,i} \sim N(\lambda_{1,i}, A_{1,i}), \tag{12}$$

where

$$\lambda_{1,i} = (A^{-1}_{0,i} + \sigma^{-2}_{\eta,i}C^{*\prime}_T C^*_T)^{-1}(A^{-1}_{0,i}\lambda_{0,i} + \sigma^{-2}_{\eta,i}C^{*\prime}_T Y^*_{i,T}) \tag{13}$$
$$A_{1,i} = (A^{-1}_{0,i} + \sigma^{-2}_{\eta,i}C^{*\prime}_T C^*_T)^{-1}. \tag{14}$$

Second, conditional on draws of λ_i, and based on a nature conjugate IG prior

$$\frac{1}{\sigma^2_{\eta,i}} \sim \Gamma(\frac{v_{0,i}}{2}, \frac{\delta_{0,i}}{2}), \tag{15}$$

sample $\sigma^2_{\eta,i}$ from the posterior

$$\frac{1}{\sigma^2_{\eta,i}}|Y^*_{i,T}, \lambda_i \sim \Gamma(\frac{v_{1,i}}{2}, \frac{\delta_{1,i}}{2}), \tag{16}$$

where

$$v_{1,i} = v_{0,i} + T \tag{17}$$
$$\delta_{1,i} = \delta_{0,i} + (Y^*_{i,T} - C^*_T\lambda_i)'(Y^*_{i,T} - C^*_T\lambda_i). \tag{18}$$

Finally, based on the these draws, transform the model back to its original form (1) and compute the model residual $e_{it} = y_{it} - \lambda_i c_t$. Again, the problem reduces to estimation of coefficient in a simple AR(1) model. Based on the resulting residuals $E_{i,T} = (e_{i1}, \cdots, e_{iT})'$ and normal prior

$$\psi_i \sim N(\psi_{0,i}, B_{0,i}), \tag{19}$$

sample ψ_i from its posterior

$$\psi_i|E_{i,T}, \sigma^2_{\eta,i} \sim N(\psi_{1,i}, B_{1,i}), \tag{20}$$

where

$$\psi_{1,i} = (B^{-1}_{0,i} + \sigma^{-2}_{\eta,i}E'_{i,T-1}E_{i,T-1})^{-1}(B^{-1}_{0,i}\psi_{0,i} + \sigma^{-2}_{\eta,i}E'_{i,T-1}E_{i,T}) \tag{21}$$
$$B_{1,i} = (B^{-1}_{0,i} + \sigma^{-2}_{\eta,i}E'_{i,T-1}E_{i,T-1})^{-1}. \tag{22}$$