A Bayesian Method for General Change Point Problems

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November 5, 2015

TERRAS IRRADIENT

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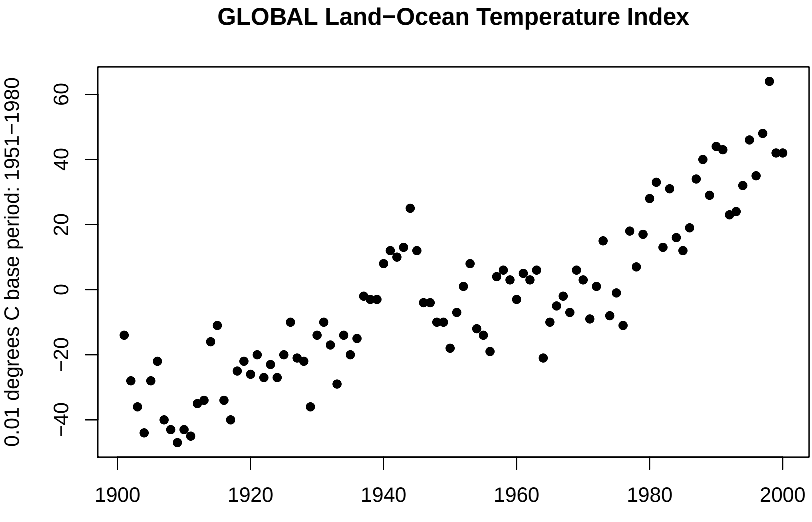
Outline

Introduction to Change Points (Serial Data)

Example: Global Warming

What model is appropriate for these data?



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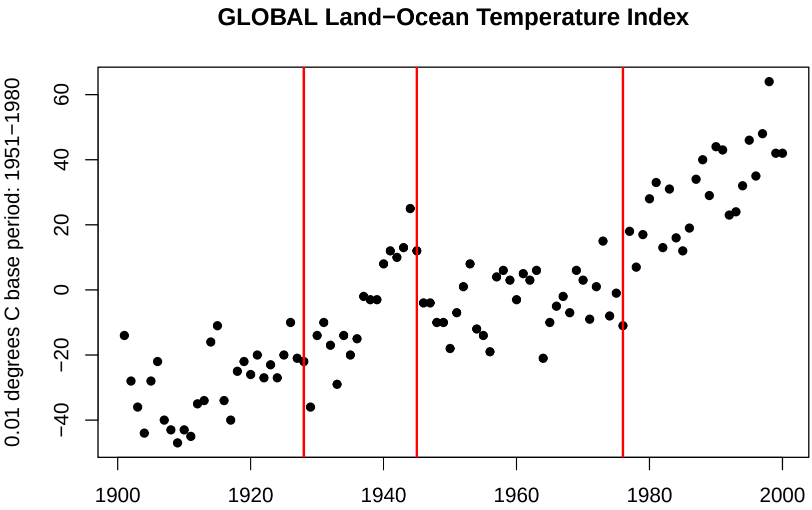

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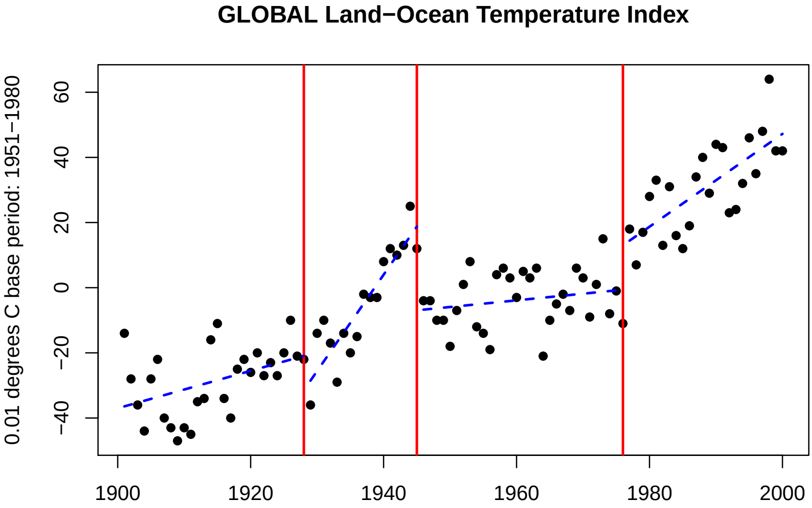

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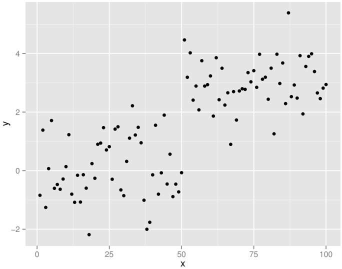
Change points partition observations into blocks. Within each block, observations share a common distribution.



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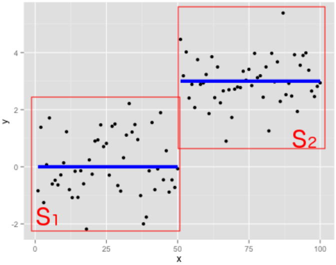
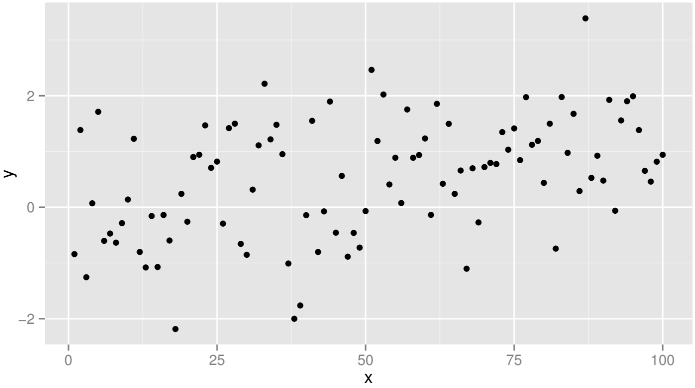


- Observations: $y_1, \ldots, y_{100}$
- Partition: $\rho = (S_1, S_2)$
  $S_1 = \{1, \ldots, 50\}$
  $S_2 = \{51, \ldots, 100\}$
- $y_{i:i \in S_1} \sim N(0, 1)$
  $y_{i:i \in S_2} \sim N(3, 1)$
- Change point at location 50.

Example 1



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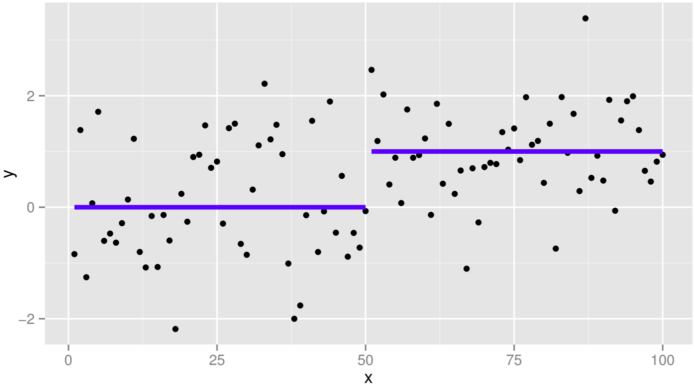

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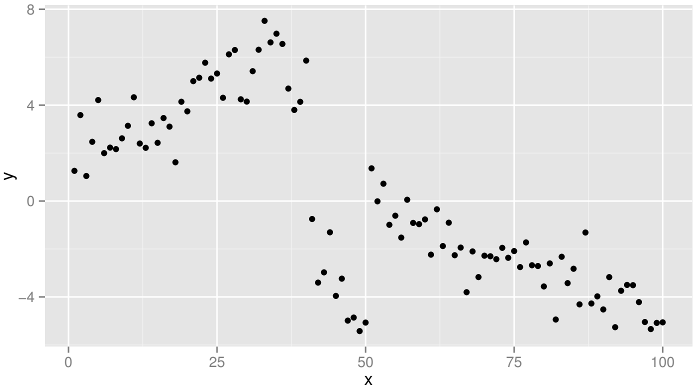

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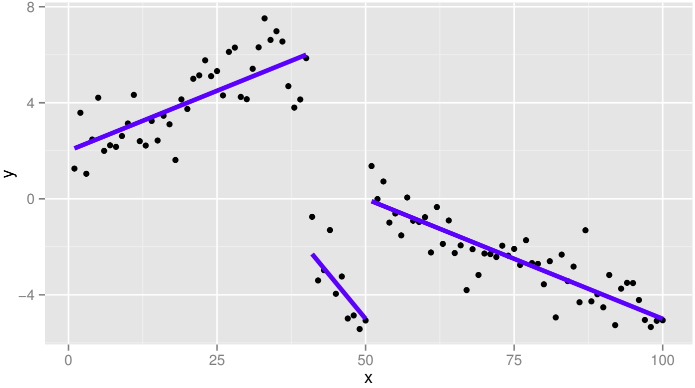

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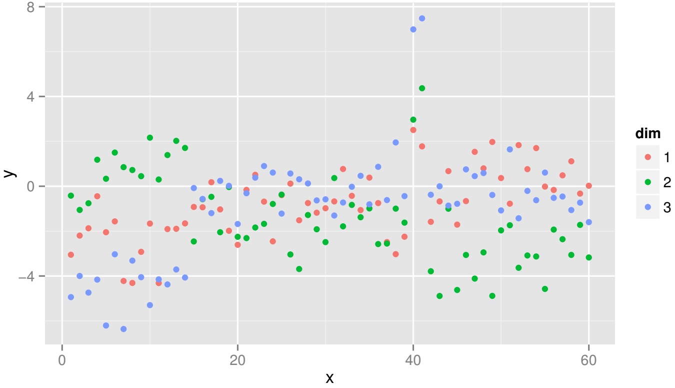

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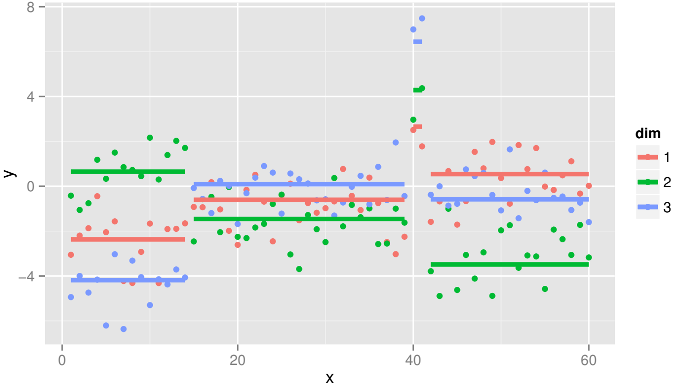


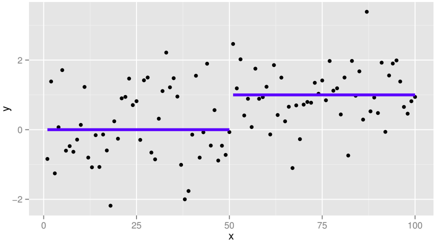
Related Work

Related Work

- Barry and Hartigan (1993, 1994), Erdman and Emerson (2007, 2008) - univariate, Bayesian (`library(bcp)`)
- Bai and Perron (2003), Zeileis, et al. (2001) - regression (`library(strucchange)`)
- Muggeo (2003) - regression (`library(segmented)`)
- Olshen, et al. (2004) - univariate, using circular binary segmentation
- Fearnhead (2005), Loschi, et al. (2010) - regression, Bayesian
- Killick and Eckley (2011) - univariate mean/variance (`library(changepoint)`)
- Matteson and James (2013) - multivariate (`library(ecp)`)
- ... and others

Classical Bayesian Change Point Analysis

Barry and Hartigan (1993): Univariate serial observations $y_i \sim N(\mu_i, \sigma)$ (e.g. Example 1)



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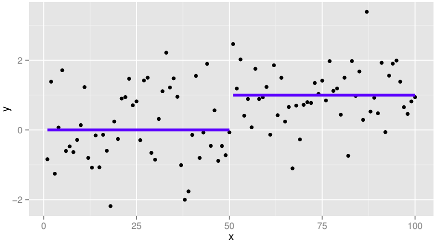

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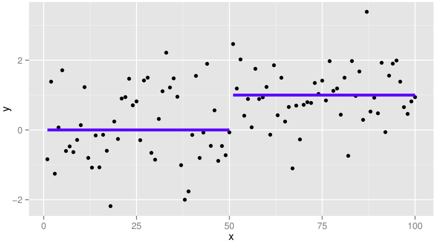
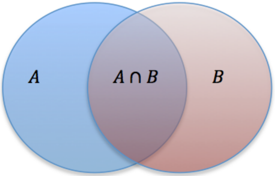


- No need to specify number of blocks
- Models the partition ρ as a random variable
- Estimates means and probability of change point occurring at each location

Bayes' Rule

Two events A and B:



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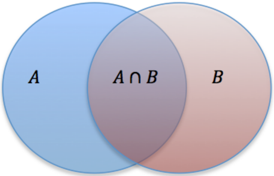

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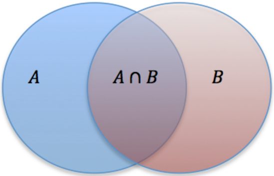

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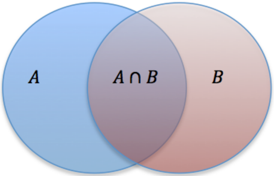

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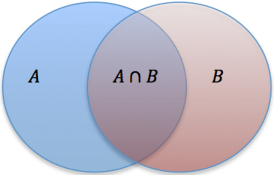

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Alternate form:

$$P(B|A) = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|B^c)P(B^c)}$$

More Generally

Prior: $\pi(\theta)$

Likelihood: $f(x|\theta)$

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Posterior:

$$f(\theta|x) = \frac{f(x|\theta)\pi(\theta)}{f(x)}$$

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Posterior:

$$f(\theta|x) = \frac{f(x|\theta)\pi(\theta)}{f(x)} = \frac{f(x|\theta)\pi(\theta)}{\int f(x|\theta)\pi(\theta)d\theta}$$

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Likelihood:

$$y_{i:i\in S} | \theta_S, \sigma^2 \sim N(\theta_S, \sigma^2)$$

$$\theta_S | \mu_0, \sigma_0^2 \sim N\left(\mu_0, \frac{\sigma_0^2}{n_S}\right)$$

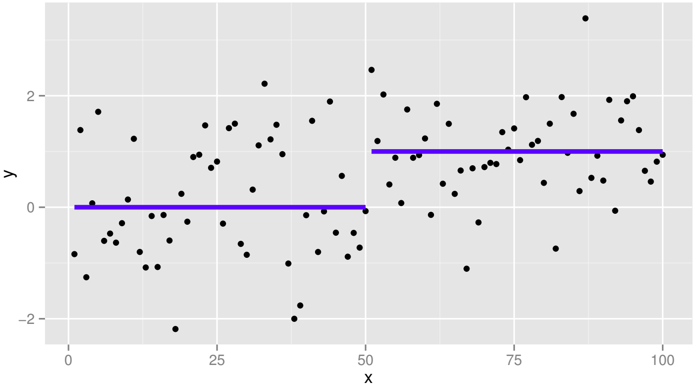
$$\mu_0 \sim U(-\infty, \infty)$$

$$\pi(\sigma^2) \propto \frac{1}{\sigma^2} \quad \sigma^2 \in (0, \infty)$$

$$\pi(w) = \frac{1}{w_0'} \quad w \in (0, w_0')$$

where $w = \sigma^2/(\sigma_0^2 + \sigma^2)$

Example 1 (Revisited)



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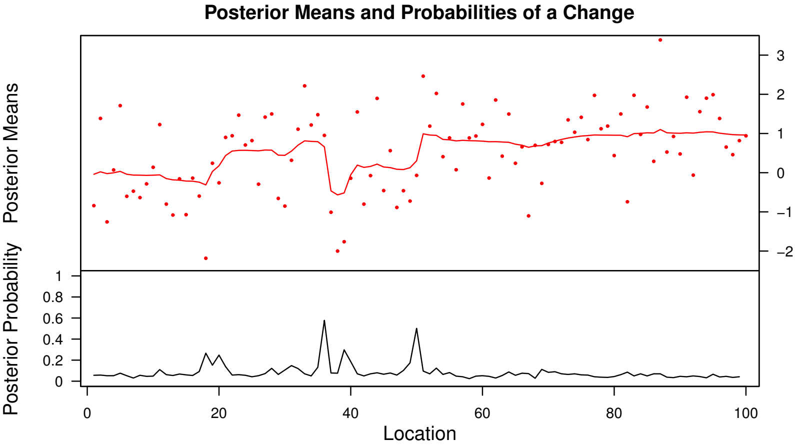

Generalized Bayesian Change Point Analysis (Serial Data)

Generalized Bayesian Change Point Analysis (BCP)

Features of Wang and Emerson (2015):

- multiple observations may be recorded at each location
- allows regressions within blocks, if predicting variables are available (e.g. Example 2)
  - includes multivariate/univariate normal observations (e.g. Examples 1 and 3) as special cases

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Allows the data to reside on nodes of a connected graph (more on this later!)

Generalized BCP: Serial Data

Wang and Emerson (2015) - setup for univariate data

- Observations: $\{(\boldsymbol{x_i}, \boldsymbol{y_i})\}_{i=1}^{n}$
- $\boldsymbol{y_i}$ is $m_i \times 1$ (m_i independent observations at location i)
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$$\pi(\rho) \propto \int_0^{p_0} p^{b-1}(1-p)^{n-b}\,dp$$

Likelihood:

$$\boldsymbol{y_{i:i\in S}} | \boldsymbol{x}, \beta_S, \sigma^2 \sim N_{m_i}(\boldsymbol{\widetilde{x}_i^S} \boldsymbol{\beta_S}, \sigma^2 \boldsymbol{I})$$

Prior on the intercept:

$$\beta_{S0} | \mu_0, \sigma_0^2 \sim N\left(\mu_0, \frac{\sigma_0^2}{n_S}\right)$$

Prior on other coefficients:

$$(\beta_{Sj} | \tau_S = 0) = 0 \quad \text{w.p. } 1$$

$$(\beta_{Sj} | \tau_S = 1, \sigma_j^2) \sim N\left(0, \frac{\sigma_j^2}{\sum_{i\in S}(x_{ij} - \bar{x}_{\cdot j}^S)^2}\right)$$

Generalized BCP: Serial Data

Wang and Emerson (2014): other priors

$$
\begin{aligned}
P(\tau_S = 0) &= \frac{d}{n_S + d}\mathbb{1}\{n_S \geq 2k\} + \mathbb{1}\{n_S < 2k\} \\
P(\tau_S = 1) &= \frac{n_S}{n_S + d}\mathbb{1}\{n_S \geq 2k\} \\
\mu_0 &\sim U(-\infty, \infty) \\
\pi(\sigma^2) &\propto \frac{1}{\sigma^2} \quad \sigma^2 \in (0, \infty) \\
\pi(w_j) &= \frac{1}{w_j} \quad w \in (0, w_j')
\end{aligned}
$$

where $w_j = \sigma^2/(\sigma_j^2 + \sigma^2)$

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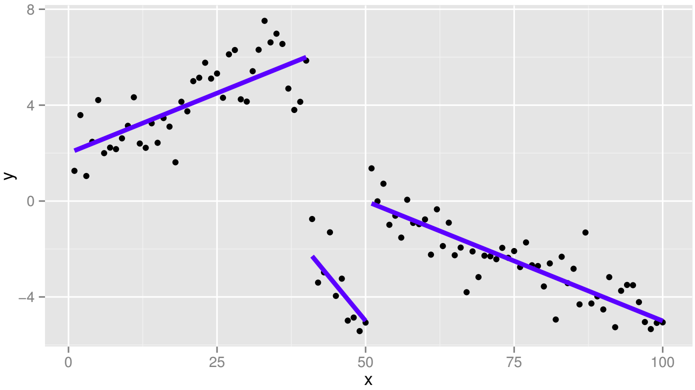
So we use a form of Gibbs sampling:

- Start with any arbitrary $(\rho^{(0)}, \tau^{(0)}, \boldsymbol{w}^{\boldsymbol{(0)}})$
- For $t = 1, \ldots, T$:
  1. Sample $(\rho^{(t)}, \tau^{(t)})$ from $f(\rho, \tau|\boldsymbol{y}, \boldsymbol{x}, \boldsymbol{w}^{\boldsymbol{(t-1)}})$
  2. Sample $\boldsymbol{w}^{\boldsymbol{(t)}}$ from $f(\boldsymbol{w}|\boldsymbol{y}, \boldsymbol{x}, \rho^{(t)}, \tau^{(t)})$

Example 2 (Revisited)



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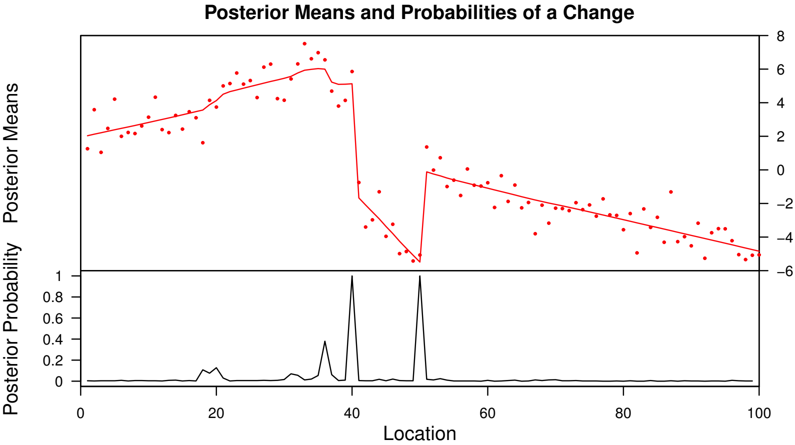

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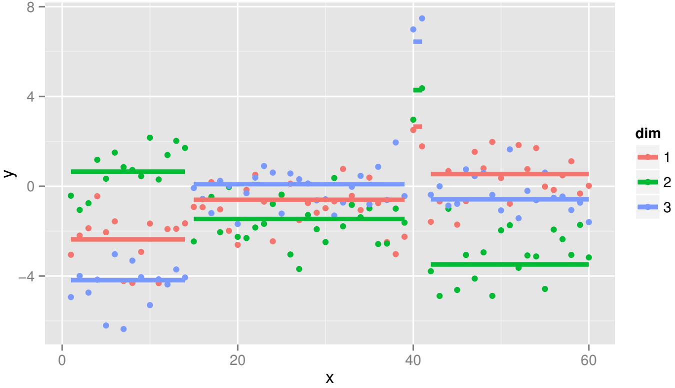

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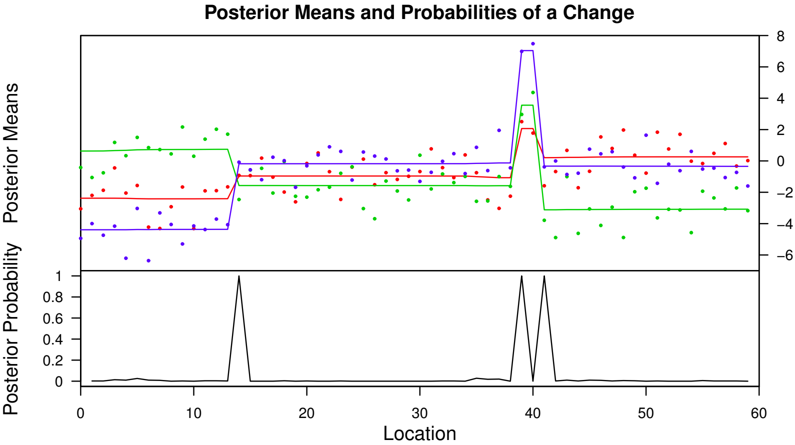
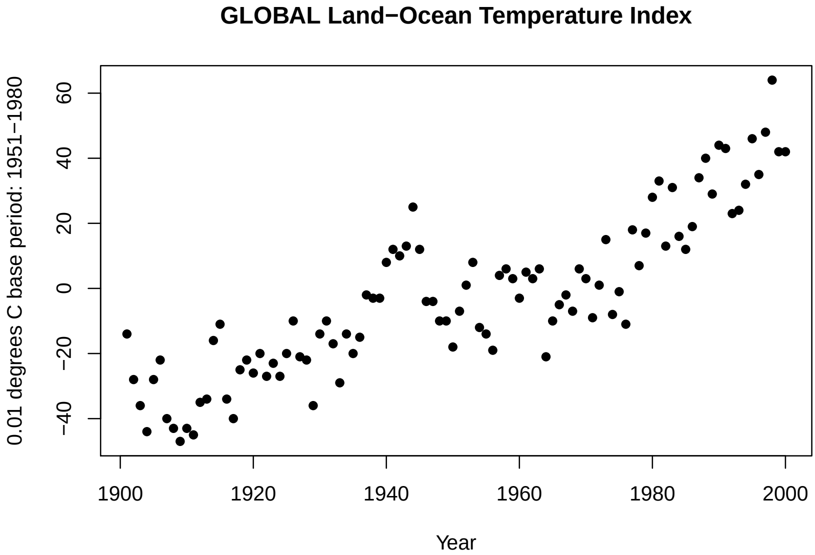


Example: Global Warming



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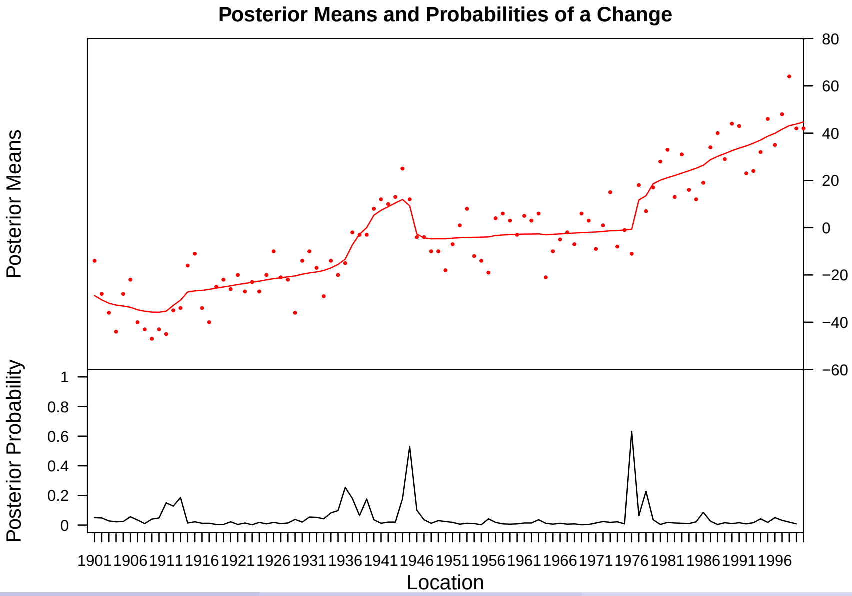
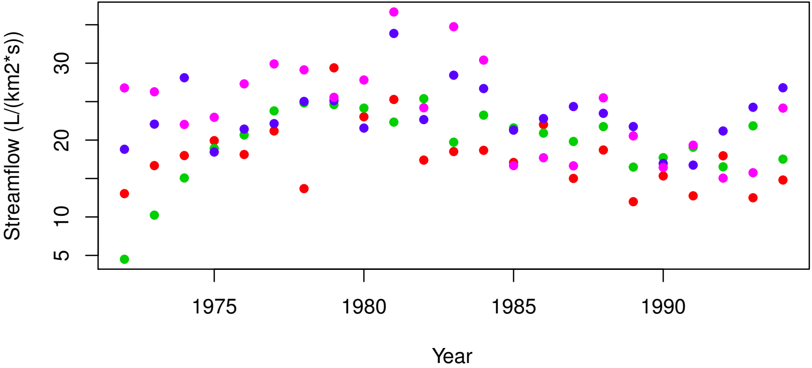
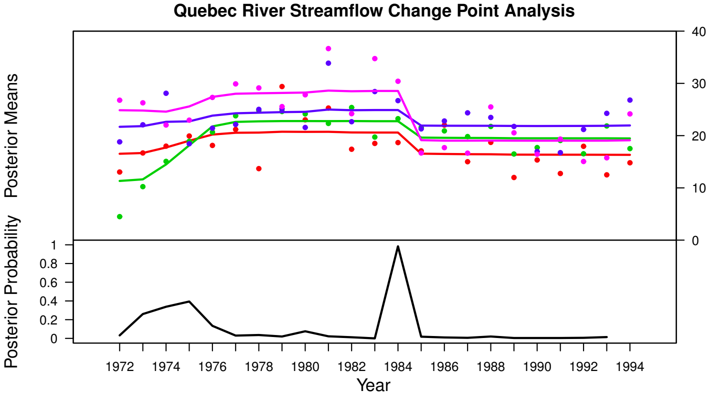


Example: Quebec Rivers (Perreault et al. (2000))



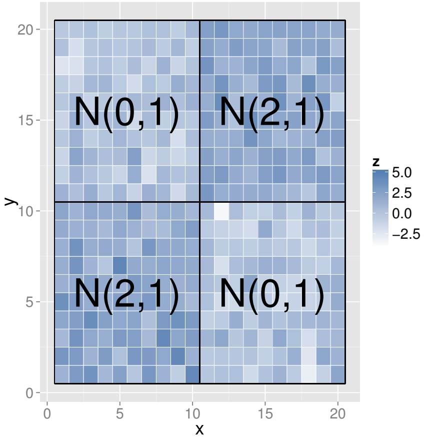
Example: Quebec Rivers (BCP Result)



Generalized Bayesian Change Point Analysis (General Graphs)

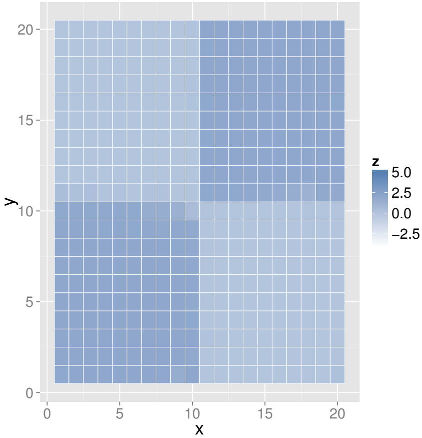
Change Points on a Grid Graph

Simulated Data



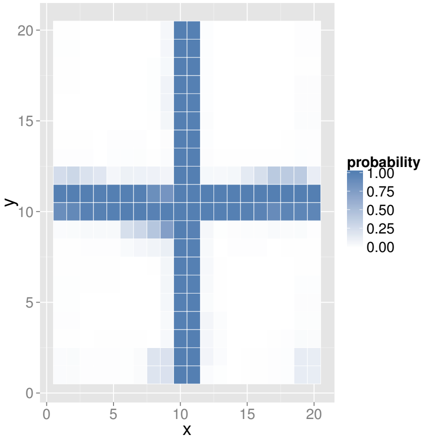
Change Points on a Grid Graph

Posterior Means



Change Points on a Grid Graph

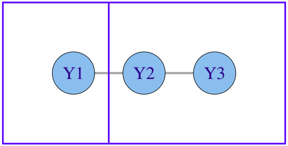
Posterior Boundary Probabilities



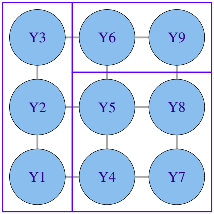
Change Points on a Grid Graph

What is a change point on a grid?

Before: (serial data $\Rightarrow$ path graph)



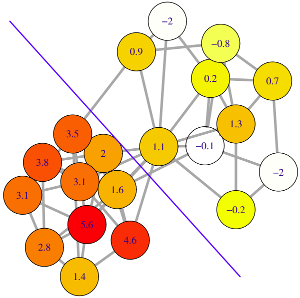
Now: (grid graph)



More Generally

Simulated Data

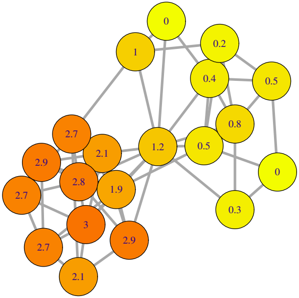
- Numbers and colors reflect **observed values**.
- 2 blocks:
    - $N(3,1)$
    - $N(0,1)$



More Generally

Posterior Means

- Numbers and colors reflect **posterior means**.



Prior on Partition

Originally (path graph):

$$f(\rho) \propto \int_0^{p_0} p^{b-1}(1-p)^{n-b}dp$$

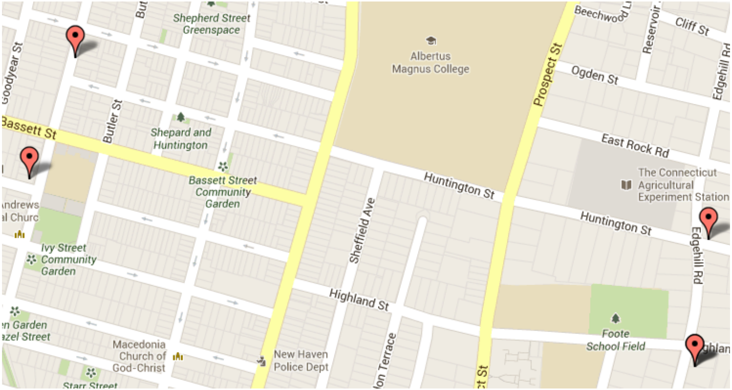
New (general graph):

$$f(\rho) \propto \alpha^{l(\rho)}$$

where $0 < \alpha < 1$ and $l(\rho)$ is a measure of boundary length

Example: New Haven Housing

$n = 244$ houses



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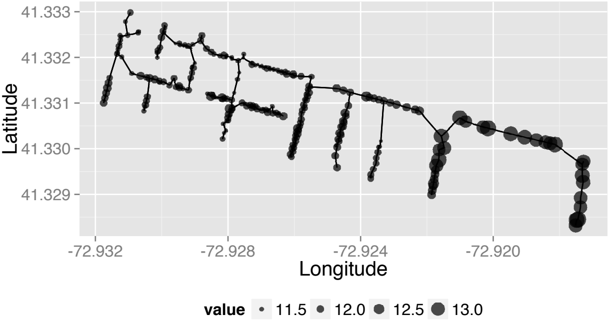
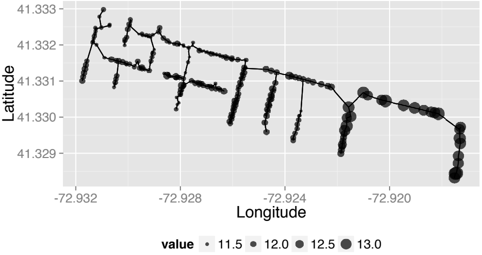


Circle size corresponds to log(2011 assessed values).

Visualizing the Properties

Available characteristics:

- # of bedrooms
- lot size
- living area size



Goal: To model 2011 assessed values using the other variables.

A Naive Approach

Model 1: Plain Old Linear Regression

```
lm(assessed values ∼ number of bedrooms + lot size +
                     living area)
```

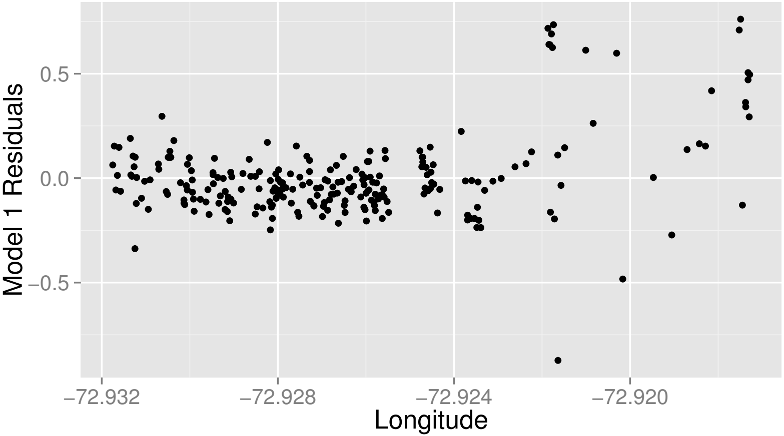
Model 1 Summary

	Estimate	Std. Error	t value	p value
(Intercept)	10.4953	0.0674	155.72	$<$2e-16
living area	0.0281	0.0022	12.89	$<$2e-16
beds	-0.0364	0.0116	-3.13	0.0020
size	1.5266	0.1079	14.15	$<$2e-16

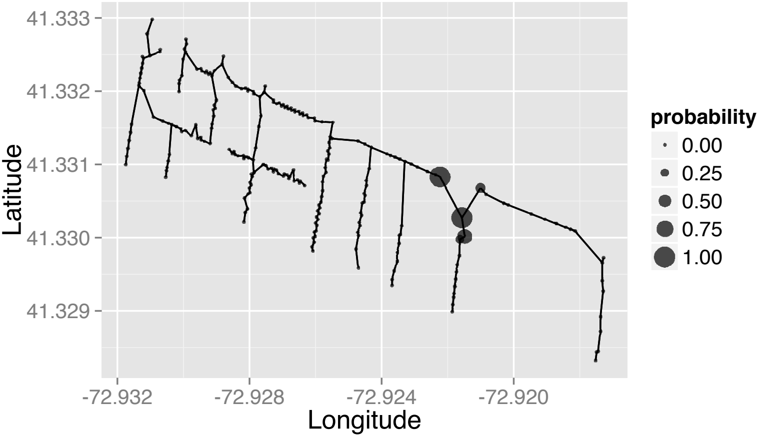
Residual standard error: 0.1998 on 240 degrees of freedom

Multiple R-squared: 0.8225

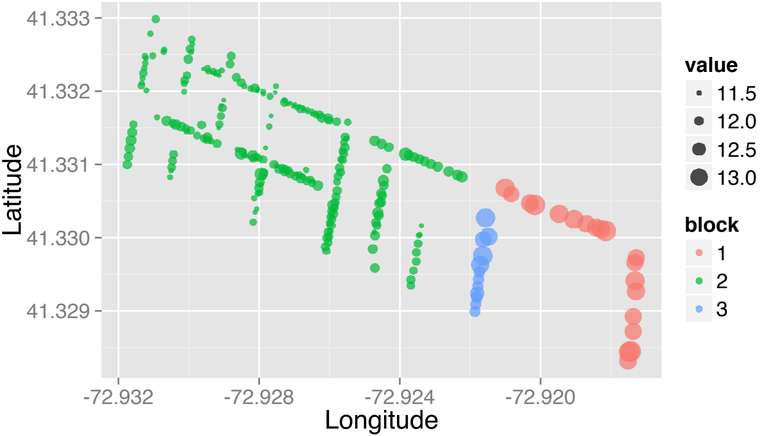
Model 1 Residuals



BCP: Posterior Boundary Probabilities

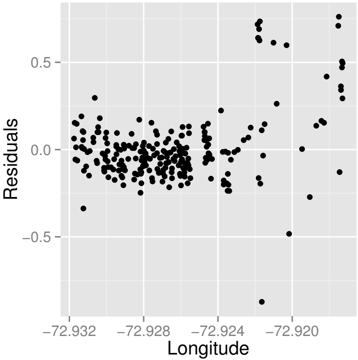


BCP: Modal Partition

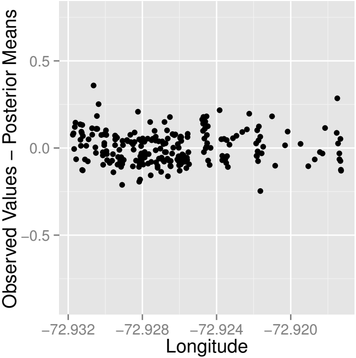


Model Comparison

Model 1: Regression
residual SE: 0.1998



Model 2: BCP
SD("residuals"): 0.0920



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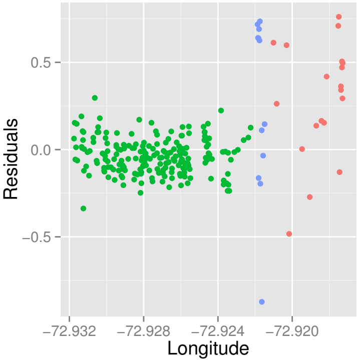

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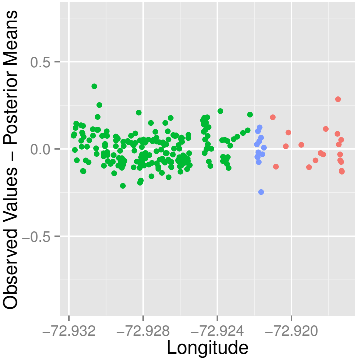

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- Runtime can be slow, particularly for large datasets on a general graph

Preprint of article submitted to JASA:

- Wang and Emerson (2015). "Bayesian Change Point Analysis of Linear Models on Graphs." `http://arxiv.org/abs/1509.00817`.

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R package `bcp` is available on R CRAN:
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Thank you!