# Assigning junior doctors to hospitals - what makes it so hard?

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# Centralised matching schemes

- Intending junior doctors must undergo training in hospitals
- Doctors rank hospitals in order of preference
- Hospitals do likewise with their applicants
- Centralised matching schemes (clearinghouses) produce a matching in several countries
  - US (National Resident Matching Program)
  - Canada (Canadian Resident Matching Service)
  - Japan (Japan Residency Matching Program)
  - Scotland (Scottish Foundation Allocation Scheme)
    - typically 700-750 applicants and 50 hospitals
- Stability is the key property of a matching
  - [Roth, 1984]

# Outline

- Hospitals / Residents problem – classical results
- Size versus stability
- Ties
- Couples
- Lower quotas
- Social stability
- IP models

# Hospitals / Residents problem (HR)

- Classical stable matching problem: the Hospitals / Residents problem (HR)
- We have $n_1$ *doctors* $d_1, d_2, \ldots, d_{n_1}$ and $n_2$ *hospitals* $h_1, h_2, \ldots, h_{n_2}$
- Each hospital has a *capacity*
- Doctors rank hospitals in order of preference, hospitals do likewise
- $d$ finds $h$ *acceptable* if $h$ is on $d$'s preference list, and unacceptable otherwise (and vice versa)
- A *matching* $M$ is a set of doctor-hospital pairs such that:
  1. $(d,h) \in M \Rightarrow d$, $h$ find each other acceptable
  2. No doctor appears in more than one pair
  3. No hospital appears in more pairs than its capacity

# HR: example instance

$d_1$ : $h_2$ $h_1$

$d_2$ : $h_1$ $h_2$

$d_3$ : $h_1$ $h_3$

$d_4$ : $h_2$ $h_3$

$d_5$ : $h_2$ $h_1$

$d_6$ : $h_1$ $h_2$

Doctor preferences

Each hospital has capacity 2

$h_1$ : $d_1$ $d_3$ $d_2$ $d_5$ $d_6$

$h_2$ : $d_2$ $d_6$ $d_1$ $d_4$ $d_5$

$h_3$ : $d_4$ $d_3$

Hospital preferences

# HR: example matching

$d_1$ : $h_2$ $h_1$

$d_2$ : $h_1$ $h_2$

$d_3$ : $h_1$ $h_3$

$d_4$ : $h_2$ $h_3$

$d_5$ : $h_2$ $h_1$

$d_6$ : $h_1$ $h_2$

Doctor preferences

Each hospital has capacity **2**

$h_1$ : $d_1$ $d_3$ $d_2$ $d_5$ $d_6$

$h_2$ : $d_2$ $d_6$ $d_1$ $d_4$ $d_5$

$h_3$ : $d_4$ $d_3$

Hospital preferences

$M = \{(d_1, h_1), (d_2, h_2), (d_3, h_3), (d_5, h_2), (d_6, h_1)\}$ (size **5**)

# HR: stability

- Matching $M$ is *stable* if $M$ admits no *blocking pair*

  – $(d,h)$ is a blocking pair of matching $M$ if:

  1. $d$, $h$ find each other acceptable

     and

  2. *either* $d$ is unmatched in $M$
     *or* $d$ prefers $h$ to his/her assigned hospital in $M$

     and

  3. *either* $h$ is undersubscribed in $M$
     *or* $h$ prefers $d$ to its worst doctor assigned in $M$

# HR: blocking pair (1)

$d_1$ : $h_2$ $h_1$

$d_2$ : $h_1$ $h_2$

$d_3$ : $h_1$ $h_3$

$d_4$ : $h_2$ $h_3$

$d_5$ : $h_2$ $h_1$

$d_6$ : $h_1$ $h_2$

Doctor preferences

Each hospital has capacity 2

$h_1$ : $d_1$ $d_3$ $d_2$ $d_5$ $d_6$

$h_2$ : $d_2$ $d_6$ $d_1$ $d_4$ $d_5$

$h_3$ : $d_4$ $d_3$

Hospital preferences

$M = \{(d_1, h_1), (d_2, h_2), (d_3, h_3), (d_5, h_2), (d_6, h_1)\}$ (size 5)

$(d_2, h_1)$ is a blocking pair of $M$

# HR: blocking pair (2)

$d_1$ : $h_2$ $h_1$
$d_2$ : $h_1$ $h_2$
$d_3$ : $h_1$ $h_3$
$d_4$ : $h_2$ $h_3$
$d_5$ : $h_2$ $h_1$
$d_6$ : $h_1$ $h_2$

Doctor preferences

Each hospital has capacity **2**

$h_1$ : $d_1$ $d_3$ $d_2$ $d_5$ $d_6$
$h_2$ : $d_2$ $d_6$ $d_1$ $d_4$ $d_5$
$h_3$ : $d_4$ $d_3$

Hospital preferences

$M = \{(d_1, h_1), (d_2, h_2), (d_3, h_3), (d_5, h_2), (d_6, h_1)\}$ (size **5**)

$(d_4, h_2)$ is a blocking pair of $M$

# HR: blocking pair (3)

$d_1$ : $h_2$ $h_1$

$d_2$ : $h_1$ $h_2$

$d_3$ : $h_1$ $h_3$

$d_4$ : $h_2$ $h_3$

$d_5$ : $h_2$ $h_1$

$d_6$ : $h_1$ $h_2$

Doctor preferences

Each hospital has capacity **2**

$h_1$ : $d_1$ $d_3$ $d_2$ $d_5$ $d_6$

$h_2$ : $d_2$ $d_6$ $d_1$ $d_4$ $d_5$

$h_3$ : $d_4$ $d_3$

Hospital preferences

$M = \{(d_1, h_1), (d_2, h_2), (d_3, h_3), (d_5, h_2), (d_6, h_1)\}$ (size **5**)

$(d_4, h_3)$ is a blocking pair of $M$

$d_1$ : $h_2$ $h_1$
$d_2$ : $h_1$ $h_2$
$d_3$ : $h_1$ $h_3$
$d_4$ : $h_2$ $h_3$
$d_5$ : $h_2$ $h_1$
$d_6$ : $h_1$ $h_2$

Doctor preferences

Each hospital has capacity **2**

$h_1$ : $d_1$ $d_3$ $d_2$ $d_5$ $d_6$
$h_2$ : $d_2$ $d_6$ $d_1$ $d_4$ $d_5$
$h_3$ : $d_4$ $d_3$

Hospital preferences

$M = \{(d_1, h_2), (d_2, h_1), (d_3, h_1), (d_4, h_3), (d_6, h_2)\}$ (size **5**)

$d_5$ is unmatched
$h_3$ is undersubscribed

# HR: classical results

- A stable matching always exists and can be found in linear time [Gale and Shapley, '62; Gusfield and Irving, '89]
- There are *doctor-optimal* and *hospital-optimal* stable matchings
- Stable matchings form a distributive lattice [Conway, '76; Gusfield and Irving, '89]
- "Rural Hospitals Theorem": for a given instance of HR:
  1. the same doctors are assigned in all stable matchings;
  2. each hospital is assigned the same number of doctors in all stable matchings;
  3. any hospital that is undersubscribed in one stable matching is assigned exactly the same set of doctors in all stable matchings.

  – [Roth, '84; Gale and Sotomayor, '85; Roth, '86]

# Stable Marriage problem

- A special case of HR arises when $n_1=n_2$, every hospital has capacity $1$, and every doctor finds every hospital acceptable
  - *Stable Marriage problem* (SM) [Gale and Shapley, '62; Gusfield and Irving, '89]

- Also the case where $n_1=n_2$, every hospital has capacity $1$, and not every doctor necessarily finds every hospital acceptable
  - *Stable Marriage problem with Incomplete lists* (SMI) [Gale and Shapley, '62; Gusfield and Irving, '89]

- In both cases the doctors and hospitals are more commonly referred to as the *men* and *women*

# Nobel prize in Economic Sciences, 2012

The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 2012
Alvin E. Roth, Lloyd S. Shapley

Photo: © Linda A. Cicero/Stanford

Alvin E. Roth

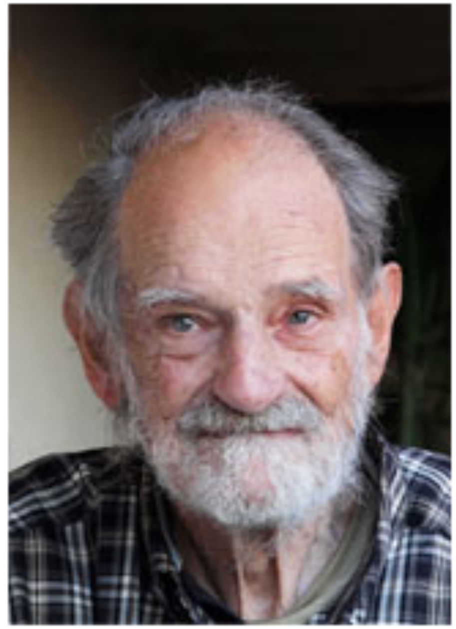


Photo: AP Photo/Reed Saxon

Lloyd S. Shapley

The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 2012 was awarded jointly to Alvin E. Roth and Lloyd S. Shapley *"for the theory of stable allocations and the practice of market design"*

# Hard variants of HR

- Hospitals / Residents problem – classical results
- **Size versus stability**
- **Ties**
- **Couples**
- **Lower quotas**
- **Social stability**
- IP models

# Maximum matchings versus stable matchings

| | |
|---|---|
| $d_1$ : $h_1$ $h_2$ | $h_1$ : $d_1$ $d_2$ |
| $d_2$ : $h_1$ | $h_2$ : $d_1$ |

Each hospital has capacity **1**

# Maximum matchings versus stable matchings

| | | | |
|---|---|---|---|
| $d_1$ : | $h_1$ $h_2$ | $h_1$ : | $d_1$ $d_2$ |
| $d_2$ : | $h_1$ | $h_2$ : | $d_1$ |

Each hospital has capacity 1

Stable matching has size 1

| | | | |
|---|---|---|---|
| $d_1$ : | $h_1$ $h_2$ | $h_1$ : | $d_1$ $d_2$ |
| $d_2$ : | $h_1$ | $h_2$ : | $d_1$ |

Maximum matching has size 2

# Maximum matchings versus stable matchings

| | | | |
|---|---|---|---|
| $d_1$ : | $h_1$ $h_2$ | $h_1$ : | $d_1$ $d_2$ |
| $d_2$ : | $h_1$ | $h_2$ : | $d_1$ |

Each hospital has capacity **1**

Stable matching has size **1**

| | | | |
|---|---|---|---|
| $d_1$ : | $h_1$ $h_2$ | $h_1$ : | $d_1$ $d_2$ |
| $d_2$ : | $h_1$ | $h_2$ : | $d_1$ |

Maximum matching has size **2**

- Instance may be replicated to give arbitrarily large instances for which size of maximum matching is twice size of stable matching

- Idea: trade off size against stability, allowing larger matchings whilst tolerating a small amount of instability

# Maximum matchings versus stable matchings

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $d_1$ : | $h_4$ | $h_1$ | $h_3$ | $h_1$ : | $d_4$ | $d_1$ | $d_2$ | |
| $d_2$ : | $h_2$ | $h_1$ | $h_4$ | $h_2$ : | $d_3$ | $d_2$ | $d_4$ | |
| $d_3$ : | $h_2$ | $h_4$ | $h_3$ | $h_3$ : | $d_1$ | $d_3$ | | |
| $d_4$ : | $h_1$ | $h_4$ | $h_2$ | $h_4$ : | $d_4$ | $d_1$ | $d_3$ | $d_2$ |

Each hospital has capacity **1**

$M_1$ is stable

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $d_1$ : | $h_4$ | $h_1$ | $h_3$ | $h_1$ : | $d_4$ | $d_1$ | $d_2$ | |
| $d_2$ : | $h_2$ | $h_1$ | $h_4$ | $h_2$ : | $d_3$ | $d_2$ | $d_4$ | |
| $d_3$ : | $h_2$ | $h_4$ | $h_3$ | $h_3$ : | $d_1$ | $d_3$ | | |
| $d_4$ : | $h_1$ | $h_4$ | $h_2$ | $h_4$ : | $d_4$ | $d_1$ | $d_3$ | $d_2$ |

Blocking pairs of $M_2$:
$(d_3,h_2)$,
$(d_4,h_1)$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $d_1$ : | $h_4$ | $h_1$ | $h_3$ | $h_1$ : | $d_4$ | $d_1$ | $d_2$ | |
| $d_2$ : | $h_2$ | $h_1$ | $h_4$ | $h_2$ : | $d_3$ | $d_2$ | $d_4$ | |
| $d_3$ : | $h_2$ | $h_4$ | $h_3$ | $h_3$ : | $d_1$ | $d_3$ | | |
| $d_4$ : | $h_1$ | $h_4$ | $h_2$ | $h_4$ : | $d_4$ | $d_1$ | $d_3$ | $d_2$ |

Blocking pair of $M_3$:
$(d_3,h_2)$

Must be optimal

# Maximum almost-stable matchings

- Let $I$ be an HR instance
- Given a matching $M$, let $bp(M)$ denote the set of blocking pairs relative to $M$ in $I$
- Define $bp(I)=\min\{|bp(M)| : M$ is a maximum matching in $I\}$
- A maximum matching $M$ in $I$ such that $|bp(M)|=bp(I)$ is called a *maximum almost-stable matching*
- In an SMI instance, finding a maximum almost-stable matching is:
  - NP-hard even if each preference list is of length $\leq 3$
  - not approximable within $n^{1-\varepsilon}$, for any $\varepsilon > 0$, unless P=NP
  - polynomial-time solvable if doctors' preference lists are of length $\leq 2$
  - [Biró, M and Mittal, 2010]
  - Open problem: HR where preference lists on one side are of length $\leq 2$

# Hospitals / Residents problem with Ties

- In practice, doctors' preference lists are short

- Hospitals' lists are generally long, so *ties* may be used – *Hospitals / Residents problem with Ties* (*HRT*)

- A hospital may be *indifferent* among several doctors

- E.g., $\mathbf{h_1}$: $(\mathbf{d_1}\ \mathbf{d_3})\ \mathbf{d_2}\ (\mathbf{d_5}\ \mathbf{d_6}\ \mathbf{d_8})$

- Matching ***M*** is *stable* if there is no pair ***(d,h)*** such that:
  1. ***d***, ***h*** find each other acceptable
  2. *either* ***d*** is unmatched in ***M***
     *or* ***d*** prefers ***h*** to his/her assigned hospital in ***M***
  3. *either* ***h*** is undersubscribed in ***M***
     *or* ***h*** prefers ***d*** to its worst doctor assigned in ***M***

# HRT: stable matching (1)

$d_1$ : $h_1$ $h_2$

$d_2$ : $h_1$ $h_2$

$d_3$ : $h_1$ $h_3$

$d_4$ : $h_2$ $h_3$

$d_5$ : $h_2$ $h_1$

$d_6$ : $h_1$ $h_2$

Doctor preferences

Each hospital has capacity 2

$h_1$ : $d_1$ $d_2$ $d_3$ $d_5$ $d_6$

$h_2$ : $d_2$ $d_1$ $d_6$ ($d_4$ $d_5$)

$h_3$ : $d_4$ $d_3$

Hospital preferences

# HRT: stable matching (1)

$d_1$ : $h_1$ $h_2$
$d_2$ : $h_1$ $h_2$
$d_3$ : $h_1$ $h_3$
$d_4$ : $h_2$ $h_3$
$d_5$ : $h_2$ $h_1$
$d_6$ : $h_1$ $h_2$

Doctor preferences

Each hospital has capacity 2

$h_1$ : $d_1$ $d_2$ $d_3$ $d_5$ $d_6$
$h_2$ : $d_2$ $d_1$ $d_6$ ($d_4$ $d_5$)
$h_3$ : $d_4$ $d_3$

Hospital preferences

$M = \{(d_1, h_1), (d_2, h_1), (d_3, h_3), (d_4, h_2), (d_6, h_2)\}$ (size 5)

# HRT: stable matching (2)

$d_1$ : $h_1$ $h_2$
$d_2$ : $h_1$ $h_2$
$d_3$ : $h_1$ $h_3$
$d_4$ : $h_2$ $h_3$
$d_5$ : $h_2$ $h_1$
$d_6$ : $h_1$ $h_2$

Doctor preferences

Each hospital has capacity 2

$h_1$ : $d_1$ $d_2$ $d_3$ $d_5$ $d_6$
$h_2$ : $d_2$ $d_1$ $d_6$ ($d_4$ $d_5$)
$h_3$ : $d_4$ $d_3$

Hospital preferences

$M = \{(d_1, h_1), (d_2, h_1), (d_3, h_3), (d_4, h_3), (d_5, h_2), (d_6, h_2)\}$ (size 6)

# Maximum size stable matchings

- Stable matchings can have different sizes
- A maximum stable matching can be (at most) twice the size of a minimum stable matching
- Problem of finding a maximum stable matching (MAX HRT) is NP-hard [Iwama, M et al, 1999], even if (simultaneously):
  - each hospital has capacity **1** (Stable Marriage problem with Ties and Incomplete Lists)
  - each doctor's preference list is strictly ordered and of length **≤3**
  - each hospital's preference list is either:
    - strictly ordered and of length **≤3**
    - a tie of length **2**

  [McDermid and M, 2010]
- Minimisation problem is NP-hard too, for similar restrictions! [M et al, 2002]

# MAX HRT: approximability

- Upper bounds:
  - trivial **2**-approximation algorithm for MAX HRT
  - succession of papers gave improvements, culminating in:
  - MAX HRT is approximable within **3/2** [McDermid, 2009; Király, 2012; Paluch 2012]
  - MAX HRT is approximable within $(1+1/e) \approx 1.3679$ for ties on one side only [Lam and Plaxton, 2019]
- Lower bounds:
  - MAX HRT is not approximable within **33/29** unless P=NP, even if each hospital has capacity **1** [Yanagisawa, 2007]
  - MAX HRT is not approximable within $4/3-\varepsilon$ assuming the *Unique Games Conjecture* (UGC) [Yanagisawa, 2007]
- Open problems:
  - increase lower bounds / decrease upper bounds

# Couples in HR

- Pairs of doctors who wish to be matched to geographically close hospitals form *couples*
- Each couple $(d_i,d_j)$ ranks in order of preference a set of pairs of hospitals $(h_p,h_q)$ representing the assignment of $d_i$ to $h_p$ and $d_j$ to $h_q$
- Hospitals rank individual doctors as before
- Stability definition may be extended to this case [Roth, 1984; McDermid and M, 2010; Biró et al, 2011]
- Gives the *Hospitals / Residents problem with Couples* (HRC)
- A stable matching need not exist
- Stable matchings can have different sizes

- The problem of determining whether a stable matching exists in a given HRC instance is
  - NP-complete, even if each hospital has capacity **1** and:
    - there are no single doctors

    [Ng and Hirschberg, 1988; Ronn, 1990]

    - there are no single doctors, *and*
    - each couple has a preference list of length **≤2**, *and*
    - each hospital has a preference list of length **≤2**

    [Biró, M and McBride, 2014]

  - solvable in polynomial time if:
    - each single doctor has a preference list of length **≤2**, *and*
    - each couple has a preference list of length **1**, *and*
    - each hospital has a preference list of length **≤2**

    [M, McBride and Trimble, 2016]

- Open problem: resolve complexity for other restricted cases

# Lower quotas

- In the *Hospitals / Residents problem with Lower Quotas* (HR-LQ), each hospital has a *lower quota* as well as its upper quota (capacity)
- In a matching $M$ each hospital $h_j$ must satisfy $|M(h_j)|=0$ ($h_j$ is *closed*) or $l_j \leq |M(h_j)| \leq c_j$ where $l_j$ and $c_j$ are the lower and upper quotas
- $M$ is *stable* if it admits no blocking pair and no *blocking coalition*
  - A *blocking coalition* of $M$ involves a closed hospital $h_j$ and a set of $l_j$ doctors, each of whom is unmatched or prefers $h_j$ to his/her assigned hospital in $M$
- An instance of HR-LQ need not admit a stable matching

| Doctors | Hospitals |
|---|---|
| $d_1$: $h_1$ $h_2$ | $h_1$: 2: 2: $d_1$ $d_2$ |
| $d_2$: $h_2$ $h_1$ | $h_2$: 1: 1: $d_1$ $d_2$ |

# Lower quotas

- In the *Hospitals / Residents problem with Lower Quotas* (HR-LQ), each hospital has a *lower quota* as well as its upper quota (capacity)

- In a matching $M$ each hospital $h_j$ must satisfy $|M(h_j)|=0$ ($h_j$ is *closed*) or $l_j \leq |M(h_j)| \leq c_j$ where $l_j$ and $c_j$ are the lower and upper quotas

- $M$ is *stable* if it admits no blocking pair and no *blocking coalition*
  - A *blocking coalition* of $M$ involves a closed hospital $h_j$ and a set of $l_j$ doctors, each of whom is unmatched or prefers $h_j$ to his/her assigned hospital in $M$

- An instance of HR-LQ need not admit a stable matching

| Doctors | | | Hospitals | | | | |
|---|---|---|---|---|---|---|---|
| $d_1$: | $h_1$ | $h_2$ | $h_1$: | 2: | 2: | $d_1$ | $d_2$ |
| $d_2$: | $h_2$ | $h_1$ | $h_2$: | 1: | 1: | $d_1$ | $d_2$ |

# Lower quotas

- In the *Hospitals / Residents problem with Lower Quotas* (HR-LQ), each hospital has a *lower quota* as well as its upper quota (capacity)
- In a matching $M$ each hospital $h_j$ must satisfy $|M(h_j)|=0$ ($h_j$ is *closed*) or $l_j \leq |M(h_j)| \leq c_j$ where $l_j$ and $c_j$ are the lower and upper quotas
- $M$ is *stable* if it admits no blocking pair and no *blocking coalition*
  - A *blocking coalition* of $M$ involves a closed hospital $h_j$ and a set of $l_j$ doctors, each of whom is unmatched or prefers $h_j$ to his/her assigned hospital in $M$
- An instance of HR-LQ need not admit a stable matching

| Doctors | Hospitals |
|---|---|
| $d_1$: $h_1$ $h_2$ | $h_1$: 2: 2: $d_1$ $d_2$ |
| $d_2$: $h_2$ $h_1$ | $h_2$: 1: 1: $d_1$ $d_2$ |

# Lower quotas

- In the *Hospitals / Residents problem with Lower Quotas* (HR-LQ), each hospital has a *lower quota* as well as its upper quota (capacity)

- In a matching $M$ each hospital $h_j$ must satisfy $|M(h_j)|=0$ ($h_j$ is *closed*) or $l_j \le |M(h_j)| \le c_j$ where $l_j$ and $c_j$ are the lower and upper quotas

- $M$ is *stable* if it admits no blocking pair and no *blocking coalition*
    - A *blocking coalition* of $M$ involves a closed hospital $h_j$ and a set of $l_j$ doctors, each of whom is unmatched or prefers $h_j$ to his/her assigned hospital in $M$

- An instance of HR-LQ need not admit a stable matching

| Doctors | Hospitals |
|---|---|
| $d_1$ : $h_1$ $h_2$ | $h_1$ : 2 : 2 : $d_1$ $d_2$ |
| $d_2$ : $h_2$ $h_1$ | $h_2$ : 1 : 1 : $d_1$ $d_2$ |

- The problem of deciding whether an instance of HR-LQ admits a stable matching is NP-complete even if each upper quota $\le 3$

  [Biró, Fleiner, Irving and M, 2010]

- Open problem: complexity for lower / upper quotas $\le 2$

# Social Stability

- Although pairs may block a matching $M$ in theory, there is no guarantee they will block $M$ in practice
- If no *social ties* exist between pairs they are far less likely to form blocking pairs
  - if they do not know about each other's preferences and matched partners
- Relaxing the stability definition to consider only pairs that are likely to block a matching in practice gives the *Hospitals / Residents problem under Social Stability (HRSS)*

| Doctors | | Hospitals |
|---|---|---|
| $d_1$ : $h_2$ $h_1$ | | $h_1$ : $d_1$ $d_3$ $d_2$ $d_5$ $d_6$ |
| $d_2$ : $h_1$ $h_2$ | | $h_2$ : $d_2$ $d_6$ $d_1$ $d_4$ $d_5$ |
| $d_3$ : $h_1$ $h_3$ | | $h_3$ : $d_4$ $d_3$ |
| $d_4$ : $h_2$ $h_3$ | | |
| $d_5$ : $h_2$ $h_1$ | | Each hospital has capacity **2** |
| $d_6$ : $h_1$ $h_2$ | | |

Unacquainted pairs $U=\{(d_1, h_2), (d_3, h_1), (d_5, h_2)\}$

# Social network graph

Doctors

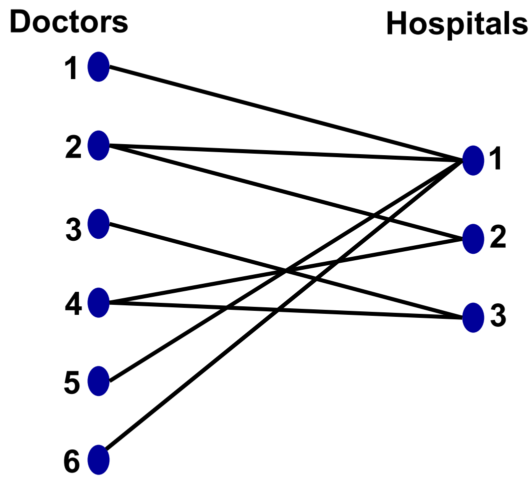
$d_1$ : $h_2$ $h_1$
$d_2$ : $h_1$ $h_2$
$d_3$ : $h_1$ $h_3$
$d_4$ : $h_2$ $h_3$
$d_5$ : $h_2$ $h_1$
$d_6$ : $h_1$ $h_2$

Hospitals

$h_1$ : $d_1$ $d_3$ $d_2$ $d_5$ $d_6$
$h_2$ : $d_2$ $d_6$ $d_1$ $d_4$ $d_5$
$h_3$ : $d_4$ $d_3$

Each hospital has capacity 2

Unacquainted pairs $U=\{(d_1, h_2), (d_3, h_1), (d_5, h_2)\}$




Social network graph $G$

- An instance $(I, G)$ of HRSS consists of:
  - An HR instance $I$
  - A social network graph $G = (D \cup H, A)$
  - Edges in $G$ are called *acquainted* pairs
- Relaxed stability definition in the HRSS context – *social stability*

# Social stability

- A pair $(d, h)$ forms a *social blocking pair* with respect to $M$ if
  - $(d, h)$ blocks $M$ in the classical sense
  - $(d, h)$ is an acquainted pair
- A *socially stable matching* is one that admits no social blocking pairs
- In practice the social network graph may be inferred on the basis of agents' previous interactions with one another
- Agents do not need to be acquainted in order to find one another acceptable
- Given HR and HRSS instances $I$ and $(I, G)$ respectively, any stable matching in $I$ is also socially stable in $(I, G)$

# Socially stable matching of size 5

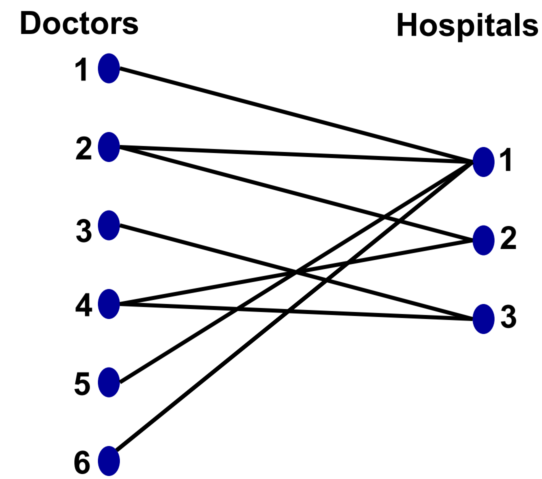
Doctors

$d_1$ : $h_2$ $h_1$
$d_2$ : $h_1$ $h_2$
$d_3$ : $h_1$ $h_3$
$d_4$ : $h_2$ $h_3$
$d_5$ : $h_2$ $h_1$
$d_6$ : $h_1$ $h_2$

Hospitals

$h_1$ : $d_1$ $d_3$ $d_2$ $d_5$ $d_6$
$h_2$ : $d_2$ $d_6$ $d_1$ $d_4$ $d_5$
$h_3$ : $d_4$ $d_3$

Each hospital has capacity **2**



Social network graph *G*

Unacquainted pairs $U = \{(d_1, h_2), (d_3, h_1), (d_5, h_2)\}$

Socially Stable Matching $M = \{(d_1, h_2), (d_2, h_1), (d_3, h_1), (d_4, h_3), (d_6, h_2)\}$

$|M| = 5$

# Socially stable matchings can have different sizes

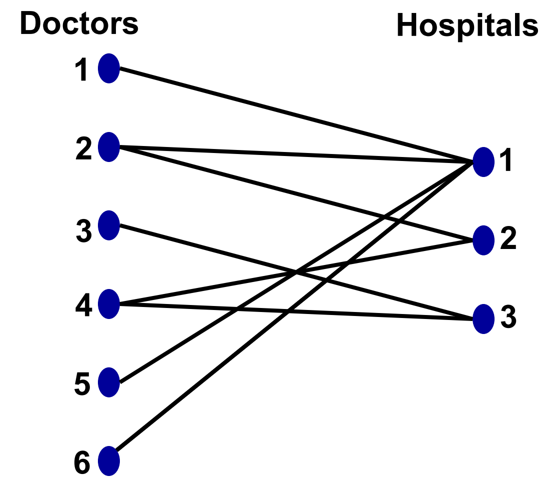
Doctors

$d_1$ : $h_2$ $h_1$
$d_2$ : $h_1$ $h_2$
$d_3$ : $h_1$ $h_3$
$d_4$ : $h_2$ $h_3$
$d_5$ : $h_2$ $h_1$
$d_6$ : $h_1$ $h_2$

Hospitals

$h_1$ : $d_1$ $d_3$ $d_2$ $d_5$ $d_6$
$h_2$ : $d_2$ $d_6$ $d_1$ $d_4$ $d_5$
$h_3$ : $d_4$ $d_3$

Each hospital has capacity **2**




Social network graph *G*

Unacquainted pairs $U=\{(d_1, h_2), (d_3, h_1), (d_5, h_2)\}$

Socially Stable Matching $M'=\{(d_1, h_2), (d_2, h_1), (d_3, h_3), (d_4, h_3), (d_5, h_1), (d_6, h_2)\}$

$|M'|=6$

- An instance of HRSS can admit socially stable matchings of varying sizes
- Socially stable matchings can be larger than stable matchings
  - can be twice the size of stable matchings in a given instance

# Summary of results

- The following complexity results are known:
  - NP-complete to determine if an instance of the Stable Roommates problem with Free Pairs (variant of HRSS for one set of agents) admits a socially stable matching [Cechlárová and Fleiner, '09]
  - Finding a maximum size socially stable matching in HRSS is:
    - NP-hard, even if each hospital has capacity **1** and each preference list is of length **≤3**
    - solvable in polynomial time if each hospital has capacity **1** and each preference list on one side is of length **≤2**
    - solvable in polynomial time if either $|U|=k$ or $|A|=k$ for some constant $k$
    - approximable within a factor of **3/2**
    - not approximable within $3/2-\varepsilon$ for any $\varepsilon>0$ assuming UGC

    [Askalidis, Immorlica, Kwanashie, M and Pountourakis, 2013]

- Open problems: complexity in the presence of:
  - master lists
  - ties

# Integer Programming model for MAX HRT

$$\max \sum_{i=1}^{n_1} \sum_{h_j \in P(d_i)} x_{i,j}$$

preference list of $d_i$

subject to

1. $\displaystyle\sum_{h_j \in P(d_i)} x_{i,j} \le 1 \qquad (1 \le i \le n_1)$

preference list of $h_j$

2. $\displaystyle\sum_{d_i \in P(h_j)} x_{i,j} \le c_j \qquad (1 \le j \le n_2)$

3. $\displaystyle c_j \left(1 - \sum_{h_q \in S_{i,j}} x_{i,q}\right) - \sum_{d_p \in T_{i,j}} x_{p,j} \le 0 \qquad (1 \le i \le n_1, h_j \in P(d_i))$

$x_{i,j} \in \{0, 1\}$

doctors that $h_j$ likes at least as much as $d_i$

hospitals that $d_i$ likes at least as much as $h_j$

# Scottish Foundation Allocation Scheme

- Ran from 1999-2012

- Each doctor:
  - ranked up to **10** hospitals in strict order of preference
  - had an integral *score* in the range **40..100**

- Each hospital:
  - had a *capacity* indicating its number of *posts*
  - had a preference list derived from the above scoring function
  - so *ties* were possible

# Solution times

- With basic model [Kwanashie and M, 2014]

| Year | Doctors | Hospitals | Posts | $\|M\|$ | Time (sec) |
|---|---|---|---|---|---|
| 2008 | 748 | 52 | 752 | 709 | 75.5 |
| 2007 | 781 | 53 | 789 | 746 | 21.8 |
| 2006 | 759 | 53 | 801 | 758 | 93.0 |

- More sophisticated model:
  - dummy variables
  - constraint merging
  - preprocessing and warm start
  - SFAS instances solved in 5 seconds on average
  - [Delorme et al, 2019]

# Conclusions

- Classical HR problem has nice structure and algorithms
- Many variants with practical applications are NP-hard:
  - maximum almost-stable matchings
  - MAX HRT
  - HRC
  - HR-LQ
  - HRSS
- Integer Programming can be used to find optimal solutions in some cases
- Future work:
  - find boundaries between P and NP-hard cases
  - approximation algorithms
  - FPT algorithms
  - scale up IP models to work with larger instance sizes

University of Glasgow

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