Tight Lower Bound for Comparison-Based Quantile Summaries

Pavel Veselý

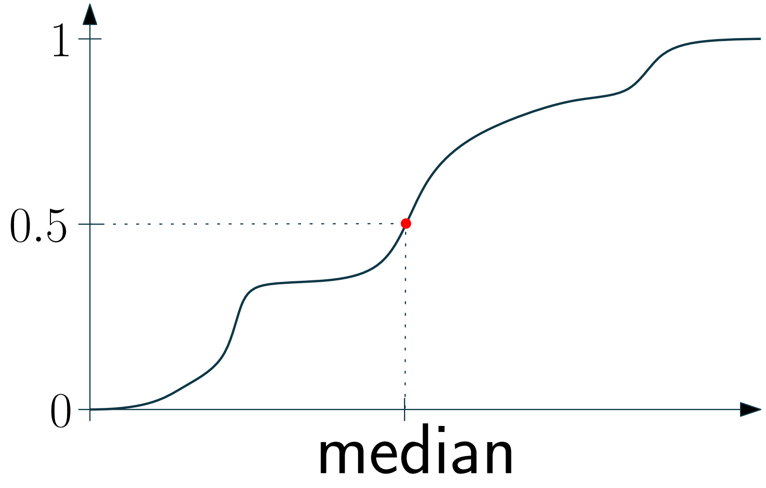
University of Warwick

8 April 2020

Based on joint work with Graham Cormode (Warwick)

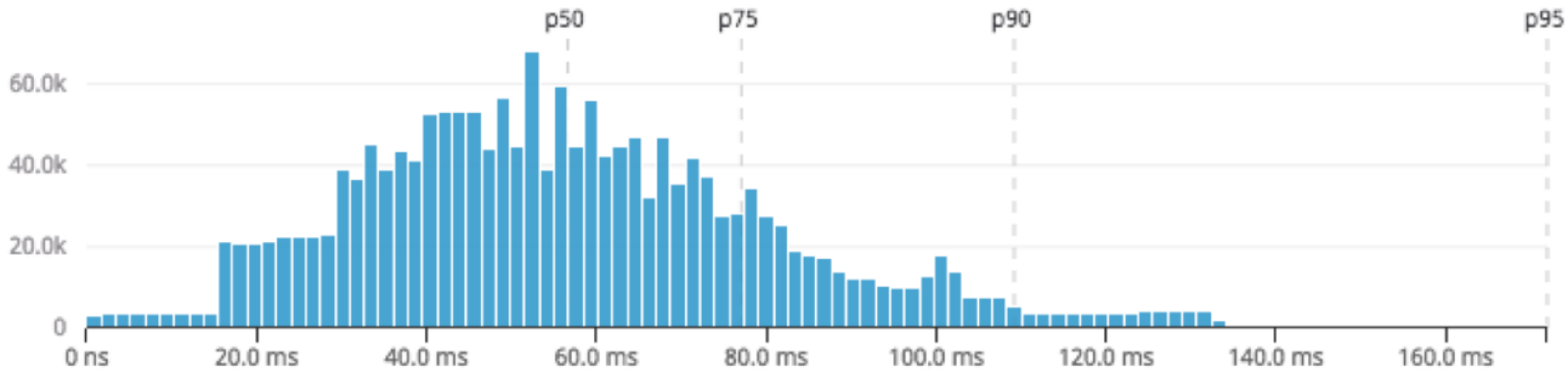
Overview of the talk

Quantiles & Distributions & **Big Data Algorithms**



Streaming Model

Motivation: Monitoring Latencies of Web Requests



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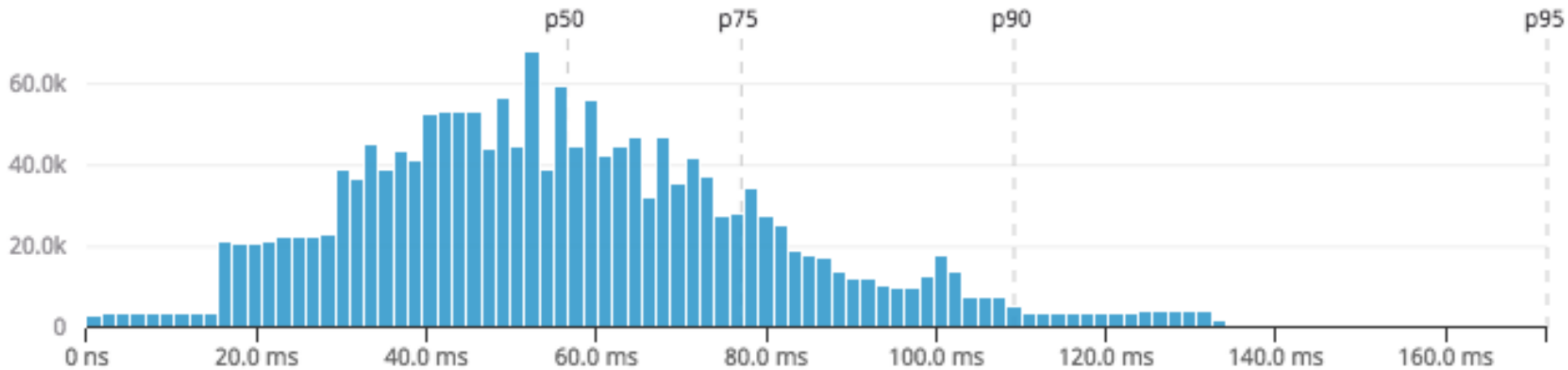

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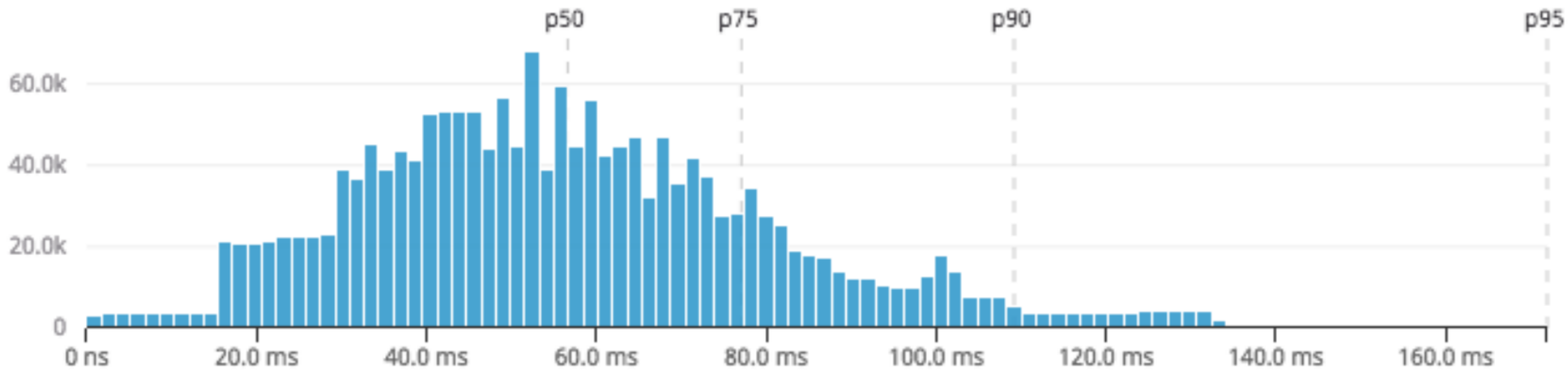

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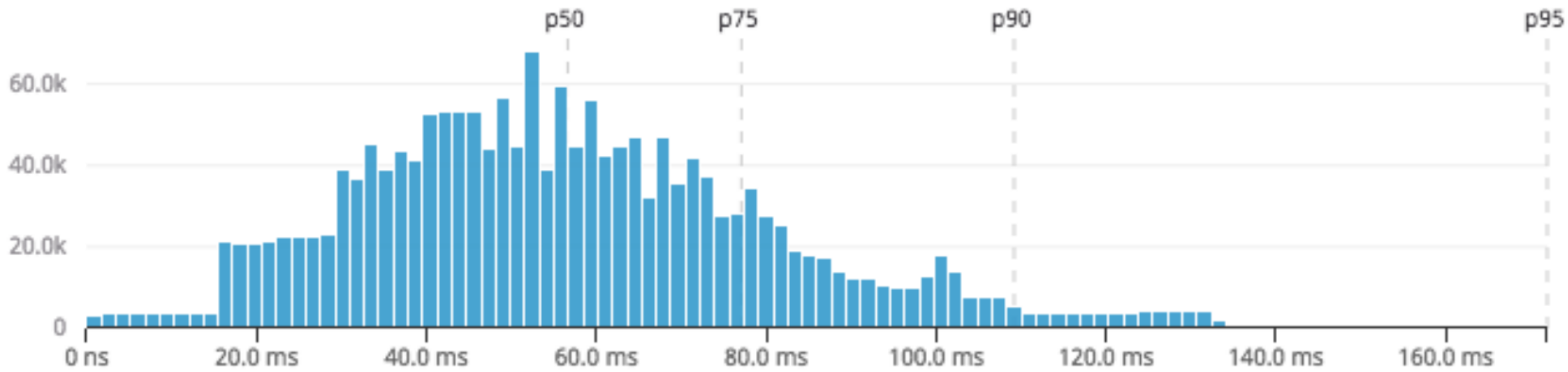

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- Average latency too high due to $\sim 2\%$ of very high latencies

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How to summarize the input?

Selection Problem & Streaming

- Input: stream of N numbers
- Goal: find the k-th smallest
  - e.g.: the median, 99th percentile
- $\mathcal{O}(N)$ time offline algorithm [Blum *et al.* '73]

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What about finding an approximate median?

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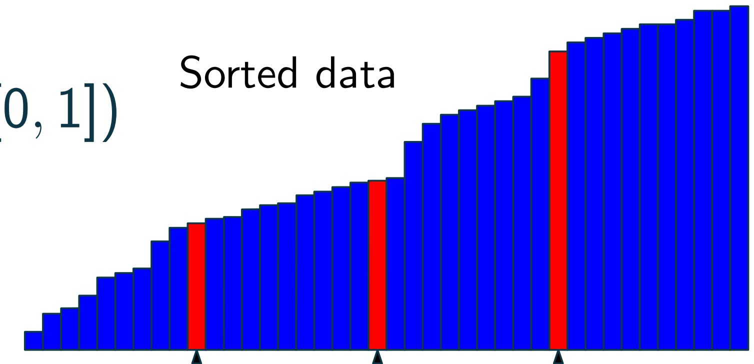

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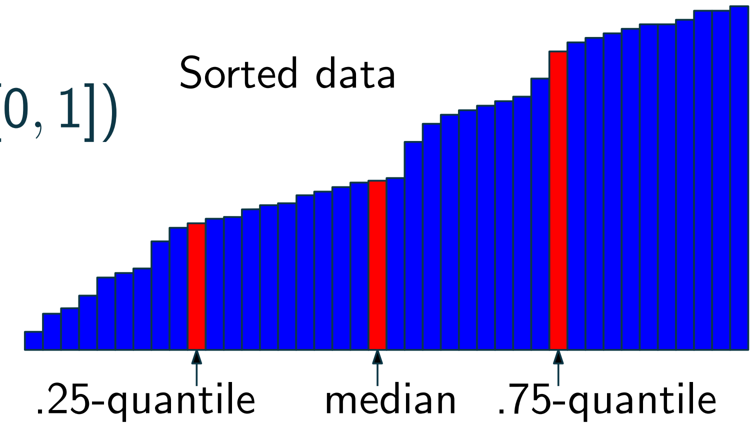

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- .01-approximate medians are .49- and .51-quantiles (and items in between)

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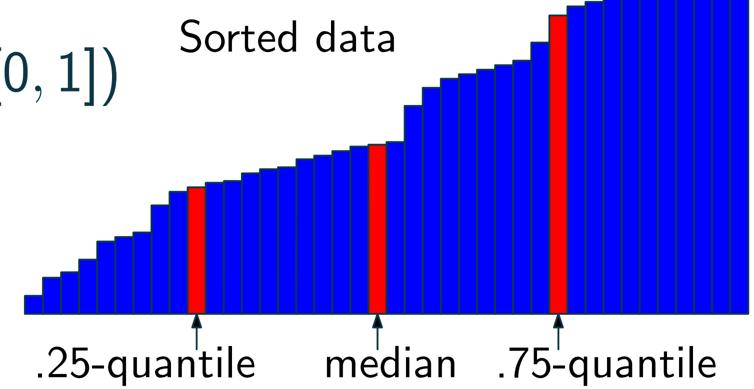

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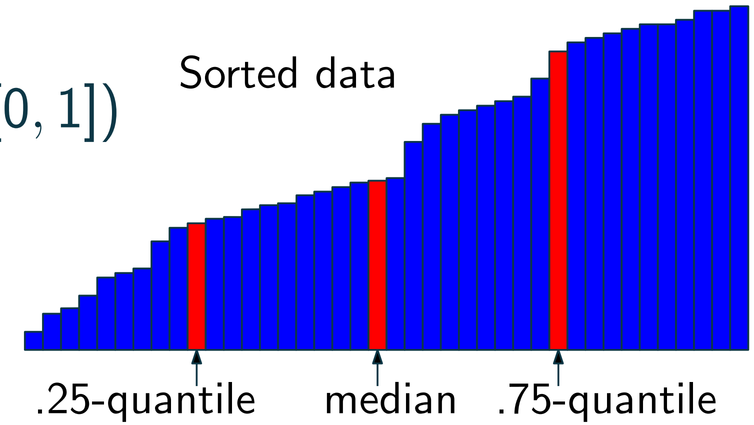

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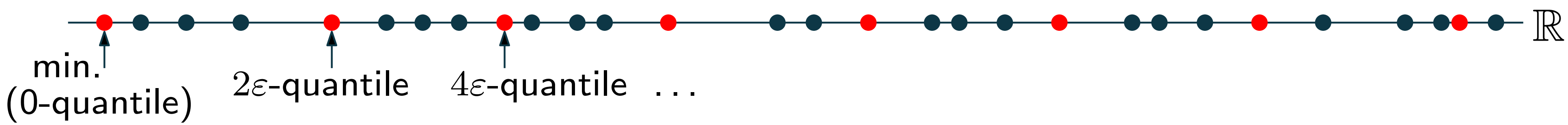
Offline summary: sort data & select $\sim \dfrac{1}{2\varepsilon}$ items



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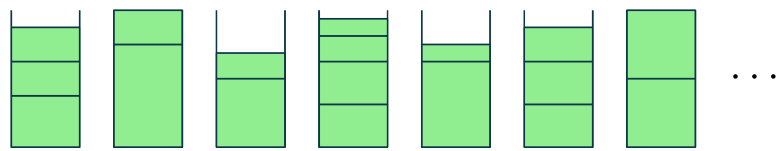

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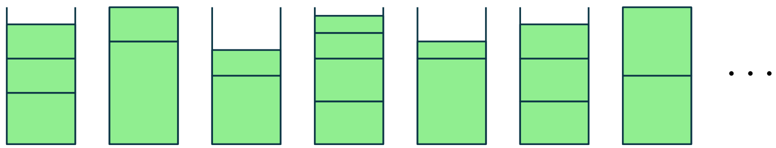

Bottom line: Finding ε-approximate median in data streams

Approximate Median & Quantiles: Streaming Algorithms

State-of-the-art results

space $\sim \#$ of stored items

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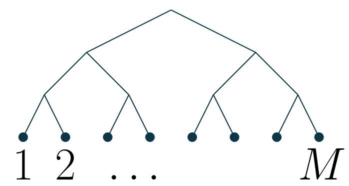

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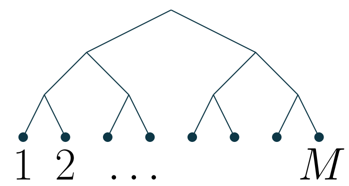
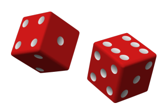


- $\mathcal{O}\left(\frac{1}{\varepsilon}\right)$ – randomized [Karnin *et al.* '16] 

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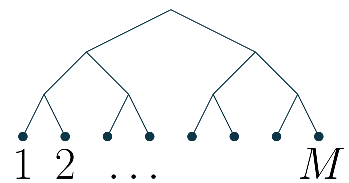

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Many more papers: [Munro & Paterson '80, Manku *et al.* '98, Manku *et al.* '99]

[Hung & Ting '10, Agarwal *et al.* '12, Wang *et al.* '13, Felber & Ostrovsky '15, . . .]

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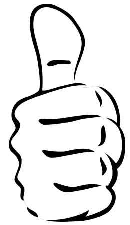

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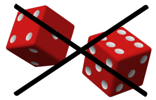
- deterministic 

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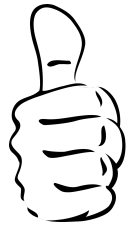

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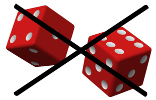
- deterministic 
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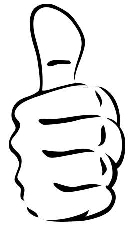

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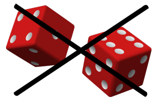
- deterministic 
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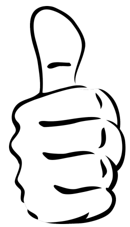

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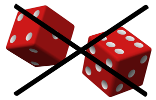
- deterministic 
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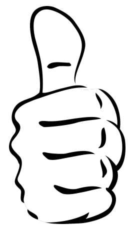

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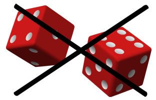
- deterministic 
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- Optimal space lower bound $\Omega\left(\frac{1}{\varepsilon}\cdot\log \varepsilon N\right)$
  - Matches the result in [Greenwald & Khanna '01]

Approx. Median & Quantiles: Lower Bound Idea

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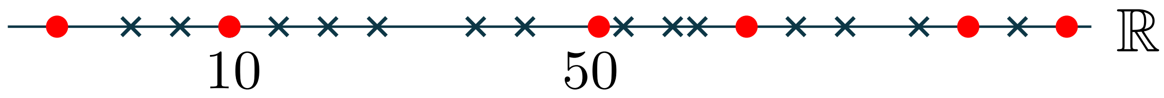

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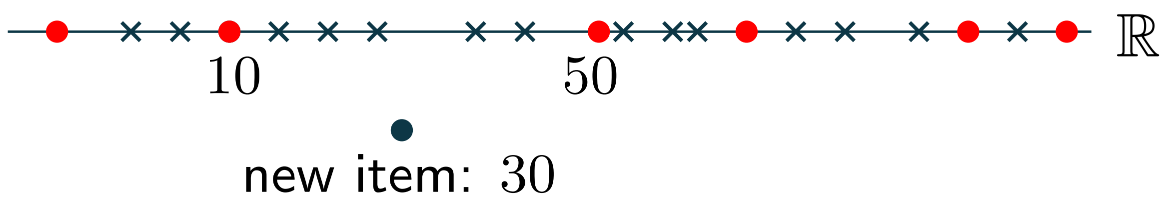

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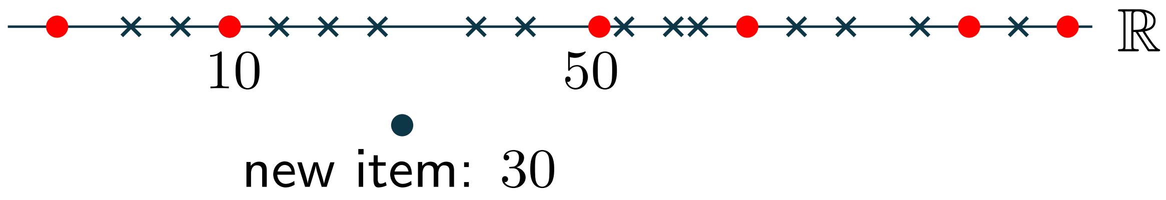

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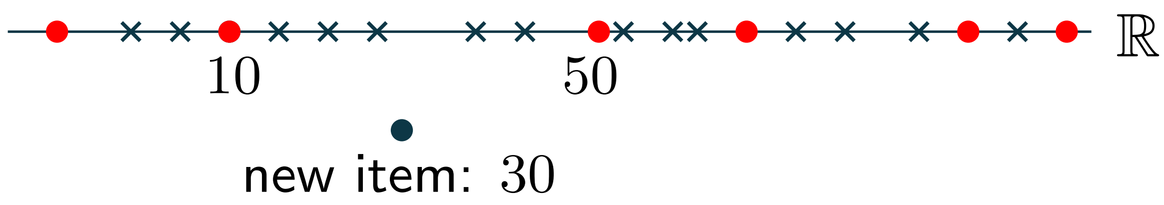

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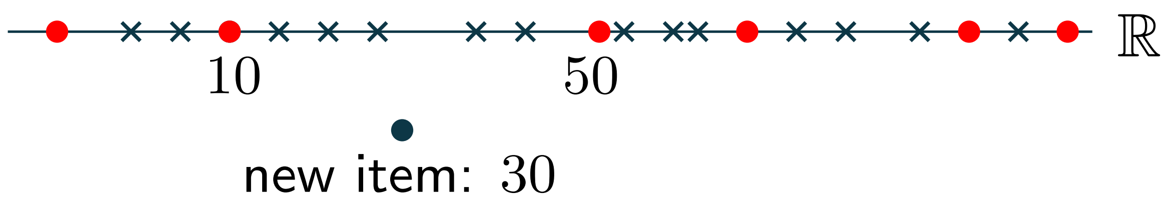

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$\rightarrow$ recursive construction of worst-case stream $\rightarrow$ lower bound $\Omega\left(\frac{1}{\varepsilon} \cdot \log \varepsilon N\right)$

Approximating Median & Quantiles: Conclusions & Open Problems

Problem solved: $\leq$

- Deterministic algorithms: space $\Theta\left(\frac{1}{\varepsilon} \cdot \log \varepsilon N\right)$ optimal [Greenwald & Khanna '01] [Cormode, **V.** '20]
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Approximating Median & Quantiles: Conclusions & Open Problems

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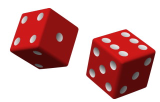
- Deterministic algorithms: space $\Theta\left(\frac{1}{\varepsilon} \cdot \log \varepsilon N\right)$ optimal [Greenwald & Khanna '01] [Cormode, **V.** '20]
- Randomized algorithms: space $\Theta\left(\frac{1}{\varepsilon}\right)$ optimal (const. probability of too high error) [Karnin *et al.* '16]



Future work:

- Figure out constant factors
- Randomized algorithm with good expected space, but guaranteed $\pm\varepsilon N$ error
- A non-trivial lower bound for integers $\{1, \ldots, M\}$?
    - Or can we do better than $\mathcal{O}\left(\frac{1}{\varepsilon} \cdot \log M\right)$?

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- Dynamic streams w/ insertions and deletions of items


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Thank You!