THE INDUCED DISJOINT PATHS PROBLEM ON (THETA, WHEEL)-FREE GRAPHS

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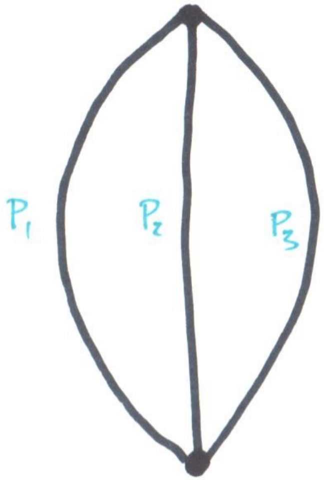
G **CONTAINS** F IF F IS ISOMORPHIC TO AN INDUCED SUBGRAPH OF G

G IS **F-FREE** IF IT DOES NOT CONTAIN F

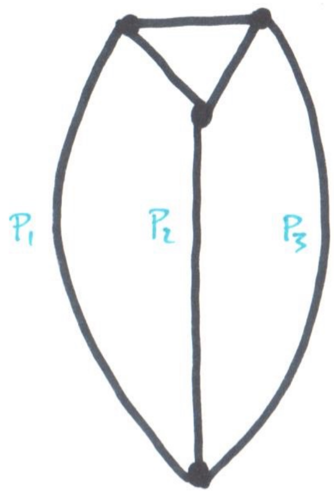
G IS **$\mathcal{F}$-FREE** IF G IS F-FREE $\forall F \in \mathcal{F}$

A **HOLE** IN A GRAPH IS A CHORDLESS CYCLE OF LENGTH ≥ 4

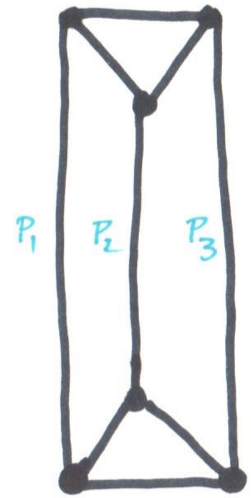
TRUEMPER CONFIGURATIONS



THETA



PYRAMID



PRISM

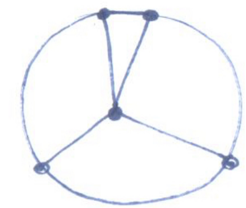
WHEEL

$\forall i \neq j$ $P_i \cup P_j$ INDUCES A HOLE

PERFECT $\subseteq$ ODD-HOLE-FREE

↓

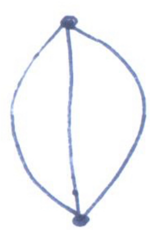
NO PYRAMID , ODD WHEEL

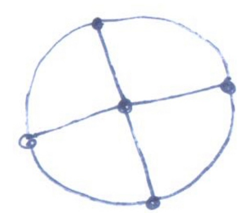


ODD # OF SECTORS OF LENGTH 1

EVEN-HOLE-FREE →

NO THETA, PRISM, EVEN WHEEL





EVEN # OF SECTORS

				
UNIVERSALLY SIGNABLE CCKV '96	X	X	X	X
CAP-FREE CCKV '96	✓	X	X	SOME
EVEN-HOLE-FREE CCKV '97 dSV '08	X	✓	X	SOME
ODD-HOLE-FREE CCV '01 BERGE CRST '02	✓	X	✓	SOME
CLAW-FREE CS '07	X	X	✓	SOME
BULL-FREE C '10	✓	X	ONLY $\overline{C_6}$	SOME
ISK4-FREE LMT '13	✓	X	✓	SOME
CHORDLESS GRAPHS MdFT '13	✓	X	X	X
ONLY-PYRAMID DRTV	X	✓	X	X
ONLY-PRISM DRTV	X	X	✓	X
(THETA, WHEEL)-FREE RTV	X	✓	✓	X

RECOGNIZING TRUEMPER CONFIGURATIONS

(CHUDNOVSKY, SEYMOUR '05)

PYRAMID $O(n^9)$

SHORTEST PATHS DETECTOR TECHNIQUE

(CHUDNOVSKY, SEYMOUR '10)

THETA $O(n^{11})$ (LAI, LU, THORUP '19)

3-IN-A-TREE $O(n^4) \rightarrow O(n^2 \log^2 n)$

(MAFFRAY, TROTIGNON '05)

PRISM

NPC

(DIOT, TAVENAS, TROTIGNON '13)

WHEEL

NPC

				RECOGNITION	
X	X	X	X	$O(nm)$	DT, ST
X	X	X		$O(n^7)$	
X	X		X	$O(n^3m)$	DT, ST
X	X			$O(n^7)$	
X		X	X	$O(n^4m)$	DT, ST
X		X		$O(n^{35})$	
X			X	$O(n^4m)$	DT, ST
X				$O(n^{11})$	
	X	X	X	NPC	
	X	X		$O(n^5)$	
	X		X	NPC	
	X			$O(n^9)$	
		X	X	NPC	
		X		NPC	
			X	NPC	

(RADOVANOVIĆ, TROTIGNON, VUŠKOVIĆ '17)

G (THETA, WHEEL)-FREE ⇒

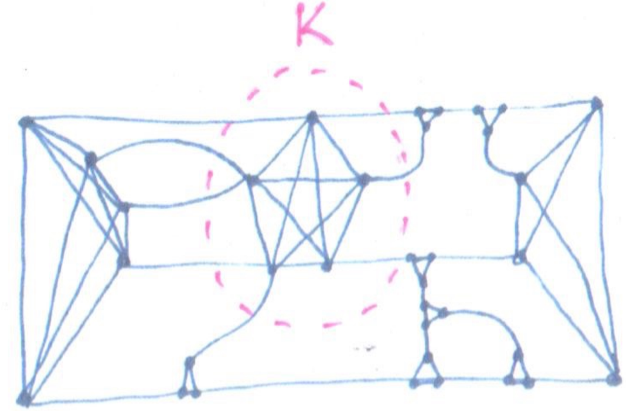
FOR SOME CLIQUE K

G\K IS

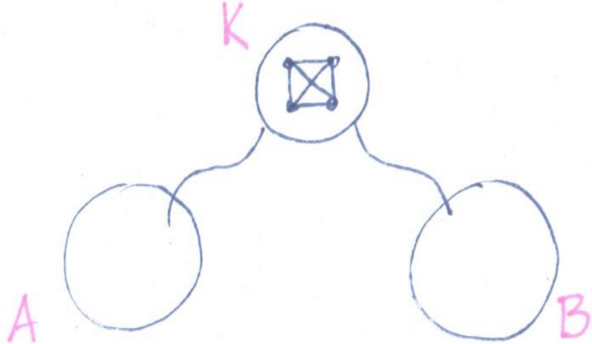
LINE GRAPH OF Δ-FREE

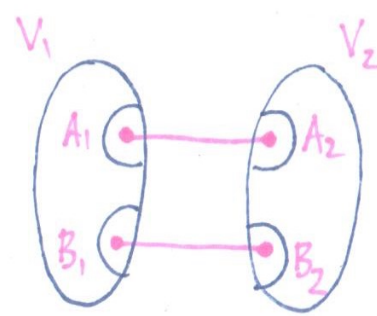
CHORDLESS GRAPH

(EVERY CYCLE IS CHORDLESS)

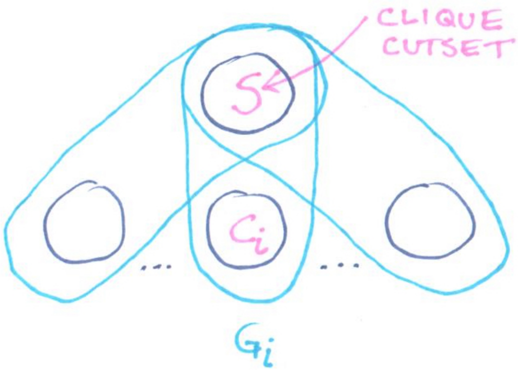


CLIQUE CUTSET OR 2-JOIN

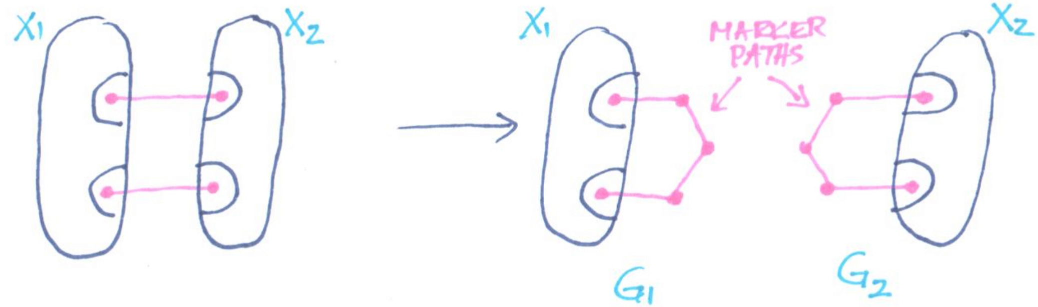




BLOCKS OF DECOMPOSITION



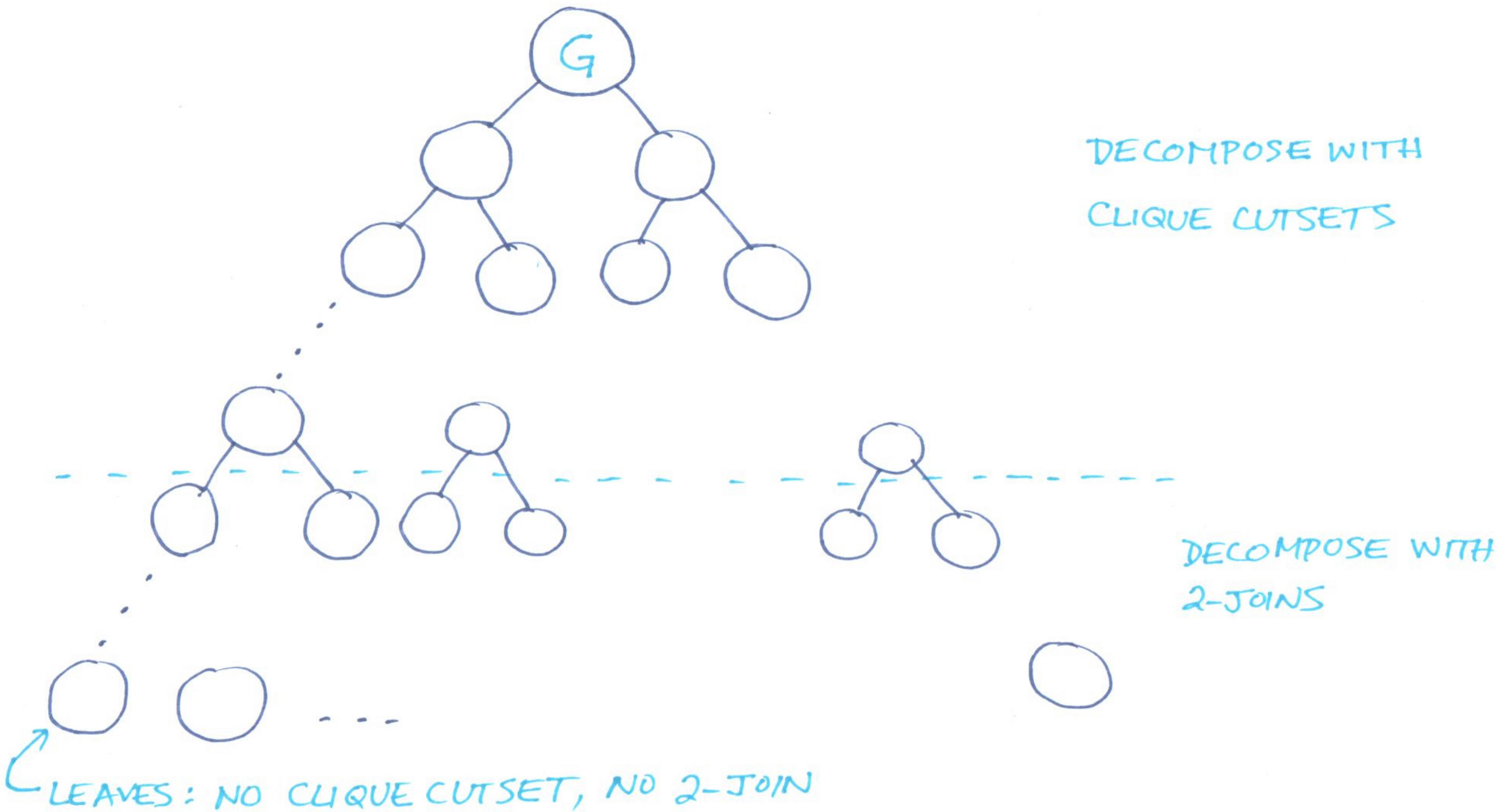
2-JOIN



G IS (THETA, WHEEL)-FREE $\Leftrightarrow \forall i$ G_i IS (THETA, WHEEL)-FREE

$\rightarrow$ STRUCTURE THEOREM

DECOMPOSITION BASED RECOGNITION ALGORITHM



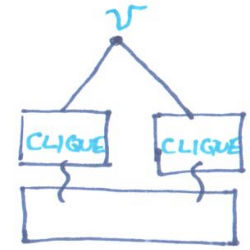
$O(n^4 m)$

LOCAL STRUCTURE

G (THETA, WHEEL)-FREE $\Longrightarrow$

$\forall$ CLIQUE K OF G

EITHER $K = V(G)$ OR $\exists$ BISIMPLICIAL VERTEX (OF G) IN $G \setminus K$

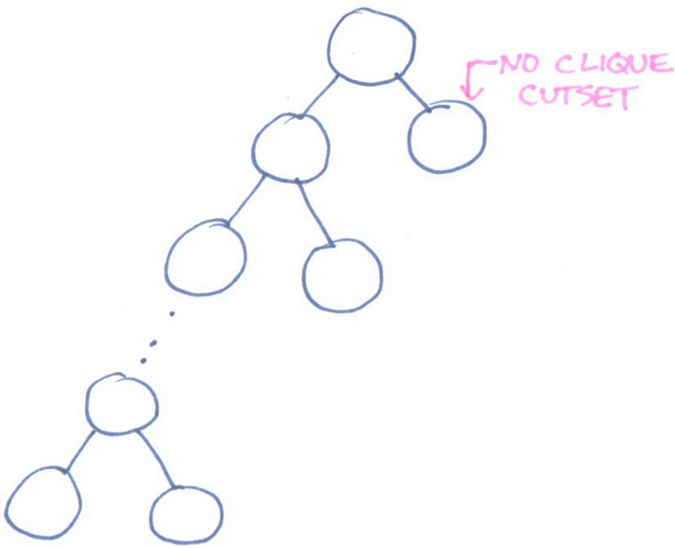


$\downarrow$

EASY ALGORITHM FOR ω

$O(n^2 m)$

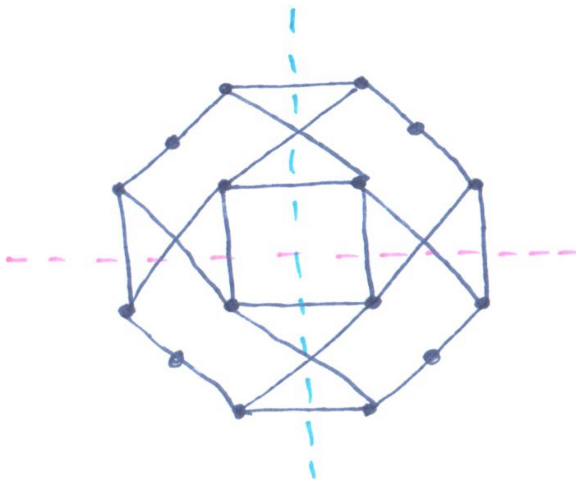
EXTREME CLIQUE CUTSET DECOMPOSITION TREE

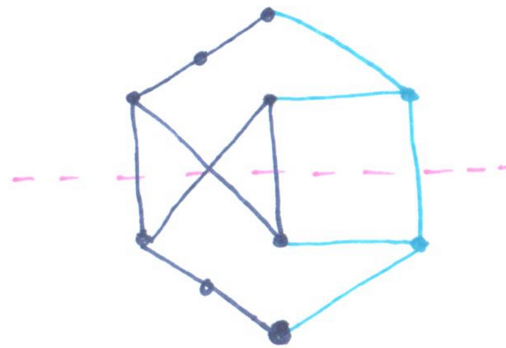


(TARJAN '85)

$O(nm)$

(TROTIGNON, VUŠKOVIĆ '09)





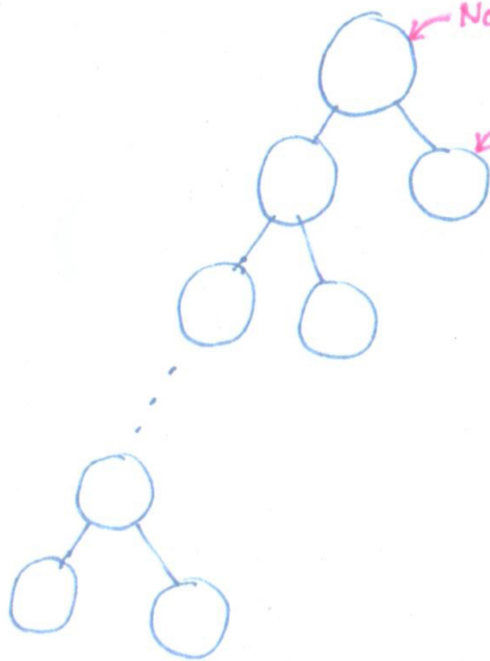
BLOCK

NO EXTREME 2-JOIN

NO STAR CUTSET ⇒ EXTREME 2-JOIN

+ CAN DECOMPOSE BY SEQUENCE OF NON-CROSSING 2-JOINS

EXTREME 2-JOIN DECOMPOSITION TREE



(TROTIGNON, VUŠKOVIĆ '09)

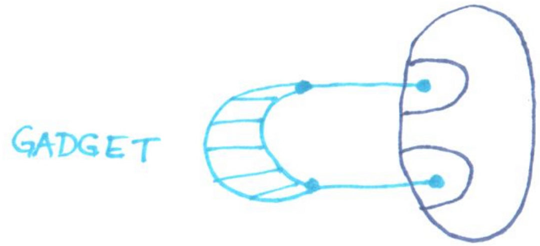
$O(n^4 m)$

NOTE: IF G IS (THETA, WHEEL)-FREE THEN

STAR CUTSET $\Rightarrow \exists$ CLIQUE CUTSET

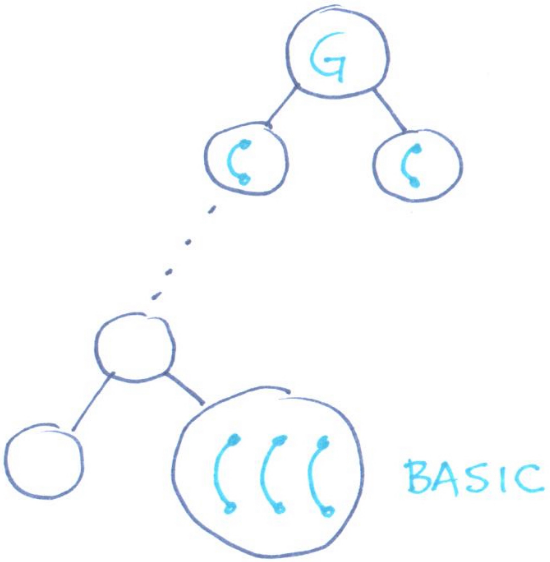
SO NO CLIQUE CUTSET $\Rightarrow$ NO STAR CUTSET

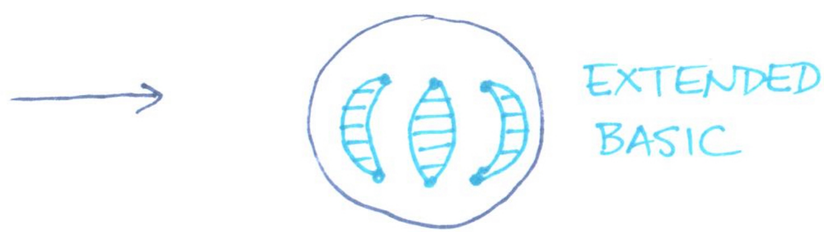
$\mathcal{L}$ & 2-JOINS



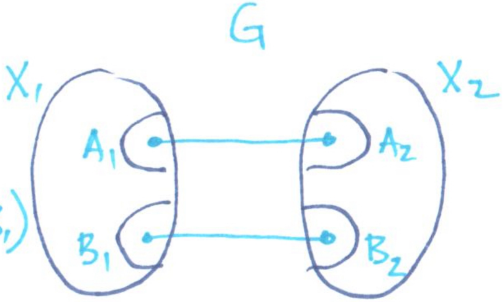
NOT CLASS-PRESERVING

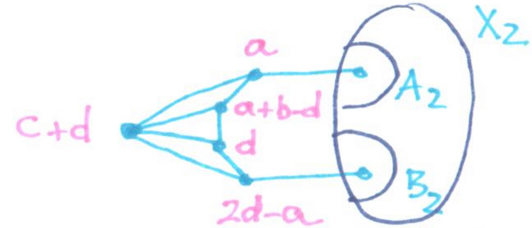
ESSENTIAL TO DECOMPOSE
WITH A SEQUENCE OF
NON-CROSSING 2-JOINS





$C_1 = X_1 \setminus (A_1 \cup B_1)$





GEM BLOCK G_2^g

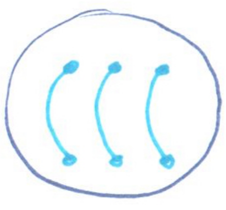
$a = \alpha_w(G[A_1 \cup C_1])$

$b = \alpha_w(G[B_1 \cup C_1])$

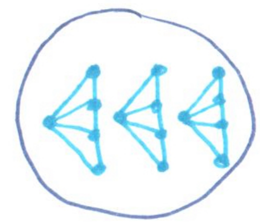
$c = \alpha_w(G[X_1])$

$d = \alpha_w(G[C_1])$

$$\alpha_w(G_2^g) = \alpha(G) + d$$



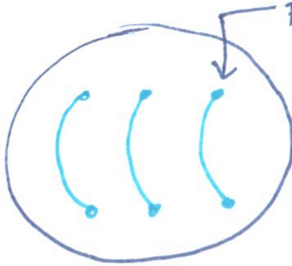
BASIC $\longrightarrow$



EXTENDED BASIC

L.G. Δ-F.C.G

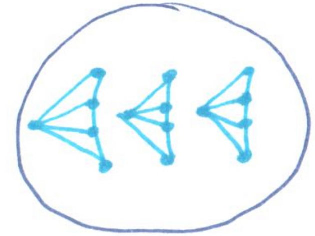
L.G.



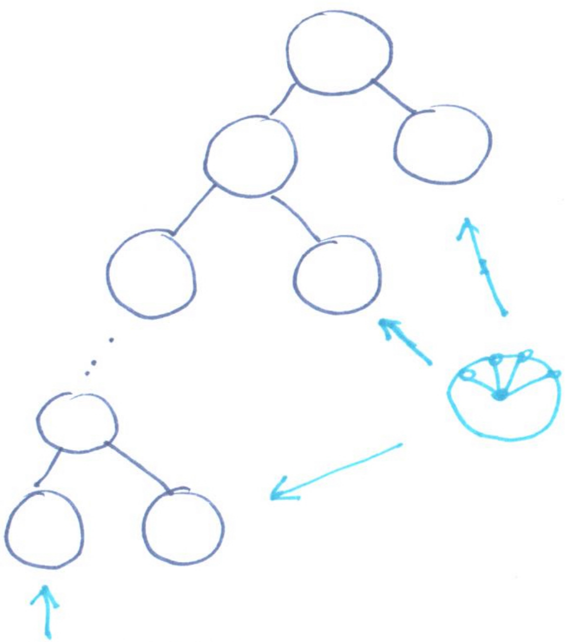
PATHS OF LENGTH ≥ 3
WHOSE INTERIOR
VERTICES ARE OF
DEGREE 2, AND
WHOSE ENDS HAVE
NO COMMON NEIGHBOR

2-CONNECTED
BIPARTITE

$\longrightarrow$



$\mathcal{C}$



BIPARTITE

CAN DECOMPOSE
BY SEQUENCE OF
NON-CROSSING
EXTREME 2-JOINS

(NAVES, TROTIGNON,
VUŠKOVIĆ '09)

α NPC FOR $\mathcal{C}$

VERTEX COLORING

G (THETA, WHEEL)-FREE $\Longrightarrow$

G CAN BE COLORED IN $\mathcal{O}(n^5 m)$ TIME

+

$$\chi(G) \leq \underbrace{\max\{3, \omega(G)\}}_{s}$$

2-IN-A-CYCLE

DECIDE WHETHER G CONTAINS A CHORDLESS CYCLE THROUGH 2 SPECIFIED VERTICES u AND v

NPC IN GENERAL (BIENSTOCK '92)

$O(nm)$ FOR (THETA, WHEEL)-FREE

G (THETA, WHEEL)-FREE, u, v NONADJACENT VERTICES

$\Rightarrow \exists$ HOLE IN G THAT CONTAINS BOTH u AND v

OR CLIQUE CUTSET THAT SEPARATES u AND v

NOTE: SIMILAR RESULT HOLDS FOR CLAW-FREE (BRUHN, SAITO '12)

k-IN-A-CYCLE

DECIDE WHETHER G CONTAINS A CHORDLESS CYCLE THROUGH k SPECIFIED VERTICES $v_1, \ldots, v_k$

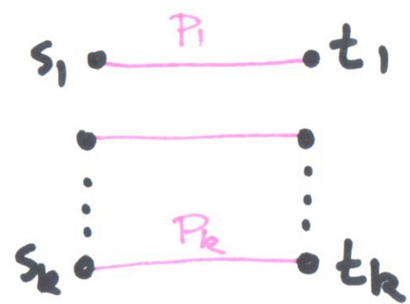
FOR (THETA, WHEEL)-FREE

FIXED-PARAMETER TRACTABLE

WHEN PARAMETRIZED BY k

DISJOINT PATHS (DP)

G, $\mathcal{W} = \{(s_1,t_1),\ldots,(s_k,t_k)\}$



$\exists$? VERTEX-DISJOINT PATHS

$P_1,\ldots,P_k$ S.T. $\forall i$ $P_i = s_i \ldots t_i$

k PART OF INPUT $\longrightarrow$ NPC (KARP '75)

k FIXED (NOT PART OF INPUT) $\longrightarrow$ k-DP

$O(n^3)$ (ROBERTSON, SEYMOUR '95)

INDUCED DISJOINT PATHS (IDP)

$P_1,\ldots,P_k$ VERTEX-DISJOINT + NO EDGES BETWEEN THEM

k-IDP NPC WHENEVER $k \geq 2$ (BIENSTOCK '92)

(THETA, WHEEL)-FREE : IDP NPC

k-IDP P $O(n^{2k+6})$

CLAW-FREE : k-IDP P

(FIALA, KAMIŃSKY, LIDICKÝ, PAULUSMA '12)

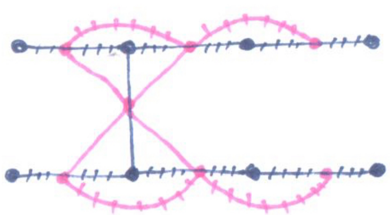
k-IDP FPT WHEN PARAMETRIZED BY k

(GOLOVACH, PAULUSMA, VAN LEEUWEN '15)

FOR BASIC GRAPHS

IDP ESSENTIALLY REDUCES TO DP

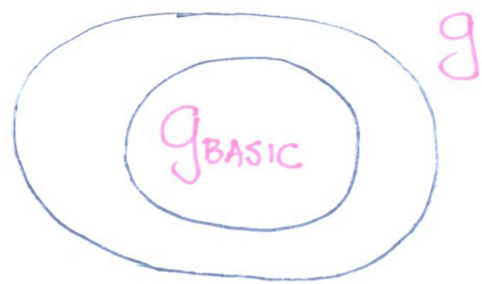
ON ROOT GRAPH OF LINE GRAPH



ROOT

LINE GRAPH

CLIQUE CUTSETS



$G \in \mathcal{G} \Rightarrow G \in \mathcal{G}_{BASIC}$ OR CLIQUE CUTSET

$O(n^c)$ k-IDP ON $\mathcal{G}_{BASIC}$ (C CONSTANT THAT DOES NOT DEPEND ON k)

$$\downarrow$$

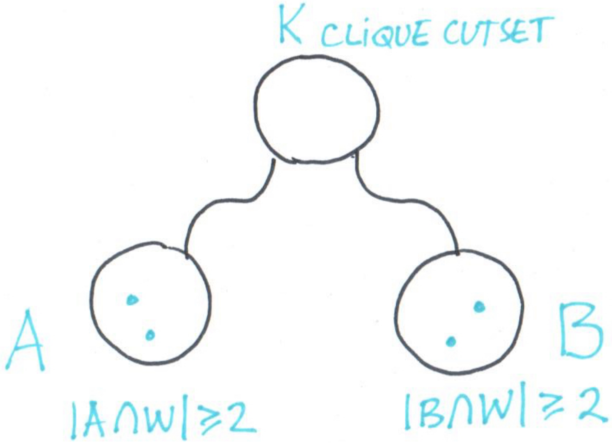
$O(n^{2k+c})$ k-IDP ON $\mathcal{G}$

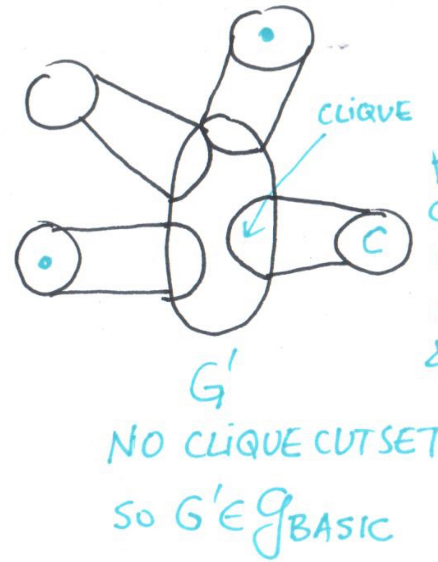
$(G, \mathcal{W})$ INSTANCE OF k-IDP

$\mathcal{W} = \{(s_1,t_1),\dots,(s_k,t_k)\}$

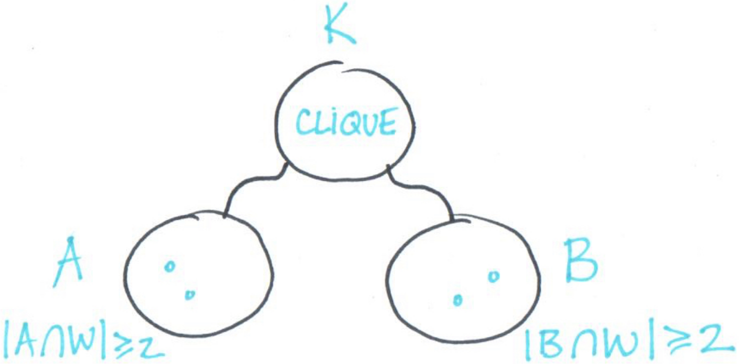
$W = \{s_1,\dots,s_k,t_1,\dots,t_k\}$ TERMINALS OF $\mathcal{W}$

IN $O(n^3)$ TIME WE CAN FIND ONE OF THE FOLLOWING





$\forall$ CONNECTED COMPONENT C OF $G \setminus G'$ N(C) IS A CLIQUE & $|C \cap W| \leq 1$



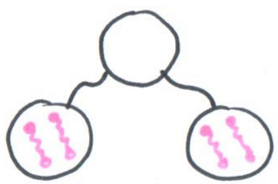
$\mathcal{W}_A = \{ (s_i, t_i) : s_i, t_i \in A \}$

$\mathcal{W}_B = \{ (s_i, t_i) : s_i, t_i \in B \}$

WE MAY ASSUME $|\mathcal{W} \setminus (\mathcal{W}_A \cup \mathcal{W}_B)| \leq 1$

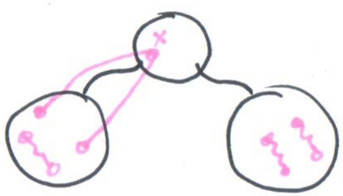
(ELSE STOP WITH ANSWER NO)

<u>CASE</u>: $|\mathcal{W} \setminus (\mathcal{W}_A \cup \mathcal{W}_B)| = 0$

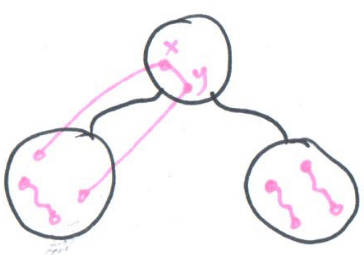


$(G[A], \mathcal{W}_A)$, $(G[B], \mathcal{W}_B)$

SOLVE RECURSIVELY

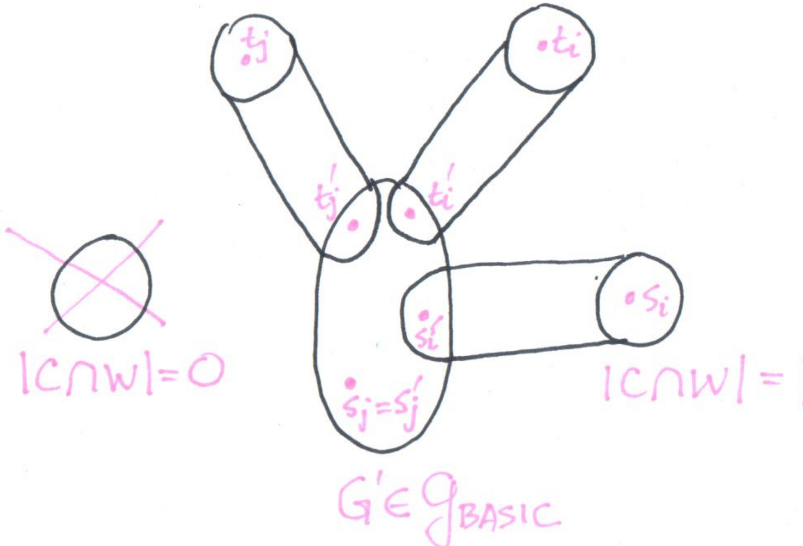


$(G[A \cup \{x\}], \mathcal{W}_A)$, $(G[B \setminus N(x)], \mathcal{W}_B)$ $\forall x$



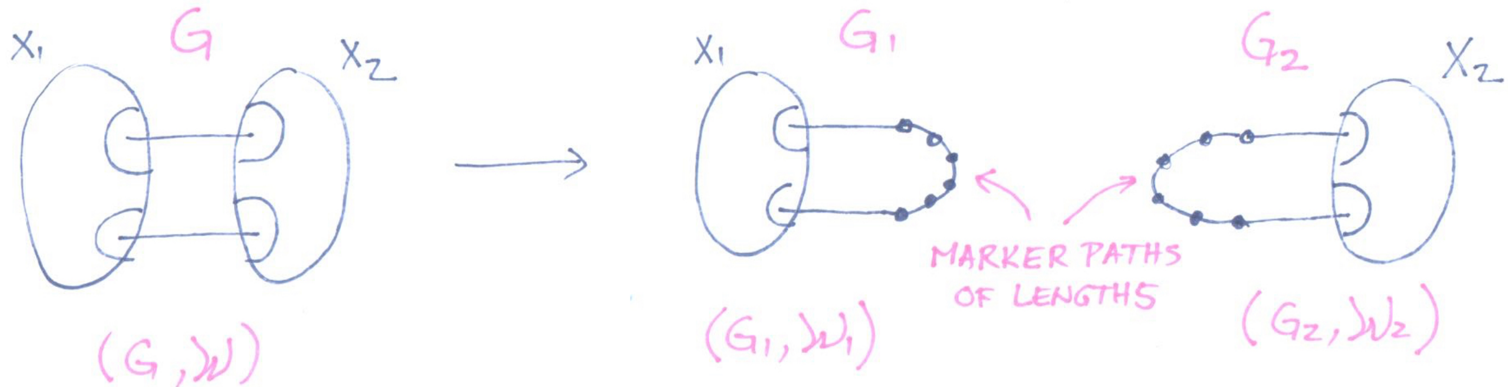
$(G[A \cup \{x,y\}], \mathcal{W}_A)$, $(G[B \setminus (N(x) \cup N(y))], \mathcal{W}_B)$ $\forall x, y$

→ $O(n^2)$ PAIRS OF PROBLEMS S.T. $(G, \mathcal{W})$ HAS SOLUTION IFF SOME PAIR OF INSTANCES HAS SOLUTION

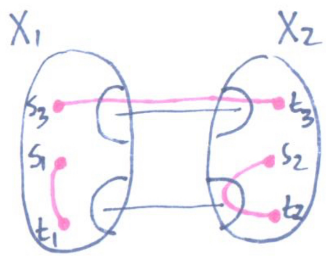


(G, W) HAS A SOLUTION IFF (G', W') HAS SOLUTION

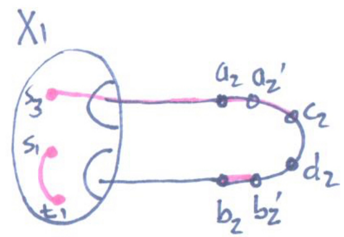
2-JOINS



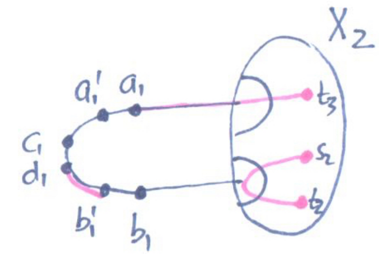
$(G, \mathcal{W})$ HAS A SOLUTION $\iff$ IN ONE OF THE CASES BELOW BOTH $(G_1, \mathcal{W}_1)$ & $(G_2, \mathcal{W}_2)$ HAVE SOLUTION



$\mathcal{W} = \{(s_1, t_1), (s_2, t_2), (s_3, t_3)\}$



$\mathcal{W}_1 = \{(s_1, t_1), (s_3, c_2), (b_2, b_2')\}$



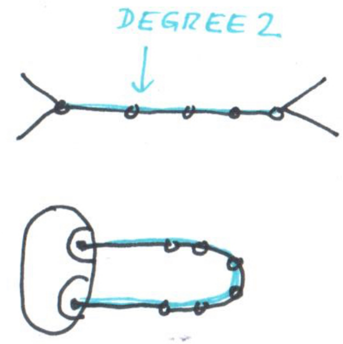
$\mathcal{W}_2 = \{(s_2, t_2), (a_1, t_3), (b_1', d_1)\}$

...

σ-GRAPH $G_{\mathcal{F},\sigma} = (G, \mathcal{F}, \sigma)$

G = GRAPH

$\mathcal{F}$ = SET OF SOME FLAT PATHS OF LENTH ≤ 7
+ SOME PATHS OF LENGTH 0

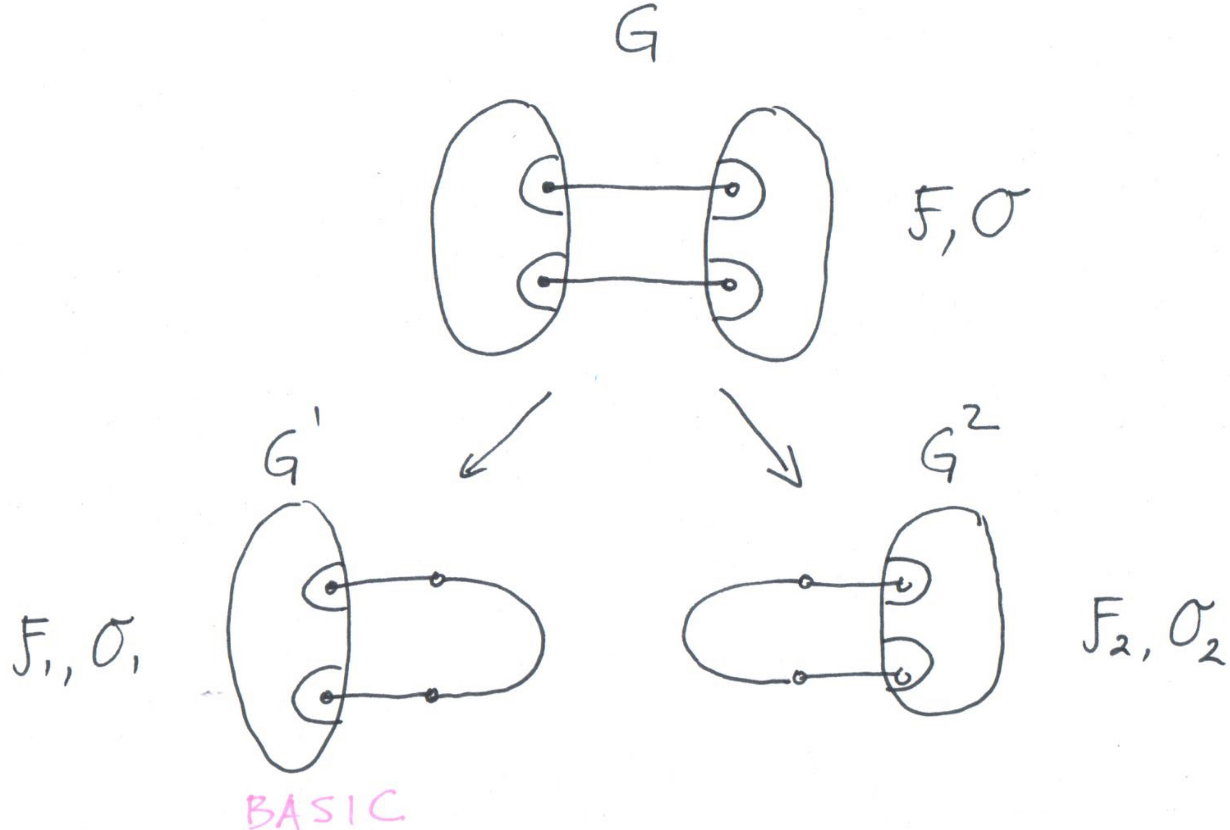


σ = SET OF TERMINAL PAIRS S.T.
$\forall \mathcal{W} \in \sigma$ $(G, \mathcal{W})$ IS AN INSTANCE OF IDP WHERE EVERY TERMINAL IS CONTAINED IN A PATH $P \in \mathcal{F}$

$G_{\mathcal{F},\sigma}$ IS LINKABLE IF FOR AT LEAST ONE $\mathcal{W} \in \sigma$, $(G, \mathcal{W})$ HAS SOLUTION

$G_{\mathcal{F},\sigma}$ σ-GRAPH S.T. $|\mathcal{F}| \leq t \Rightarrow |\sigma| \leq 2^{8t}(8t)!$

$G_{\mathcal{F},\sigma}$ σ-GRAPH S.T. G BASIC & $|\mathcal{F}| \leq t \Rightarrow$ CAN DECIDE IN $O(n^5)$ TIME WHETHER $G_{\mathcal{F},\sigma}$ IS LINKABLE



SOLVE A BUNCH OF PROBLEMS ON G^1, RECORD POTENTIAL SOLUTIONS IN $G^2_{F_2,\sigma_2}$

$G_{F,\sigma}$ LINKABLE IFF $G^2_{F_2,\sigma_2}$ LINKABLE