Higman's lemma in the Weihrauch lattice

Arno Pauly
Department of Computer Science
Swansea University, Swansea, UK
Arno.M.Pauly@gmail.com

Cécilia Pradic
Department of Computer Science
Swansea University, Swansea, UK
c.pradic@swansea.ac.uk

Giovanni Soldá
Vakgroep Wiskunde: Analysis, Logic and Discrete Mathematics
Ghent University, Ghent, Belgium
giovanni.a.solda@gmail.com

Manlio Valenti
Department of Computer Science
Swansea University, Swansea, UK
manlio.valenti@swansea.ac.uk

Results of the form "If X is well-quasiordered, then so is $F(X)$" for various constructions of quasi-orders F have been a fruitful subject of study in reverse mathematics. At first glance, such results do not seem to have computational content per se. For instance, preservation of wqos under products and Higman's Lemma have direct constructive proofs [1, 13] where wqo-ness is witnessed by the well-foundedness of some trees. However, we can look at their contrapositives phrased in terms of finding infinite paths in those trees. The algorithmic task then becomes "Given a quasi-order X and a bad sequence in $F(X)$, find a bad sequence in X". The task of finding a bad sequence in a quasi-ordered merely promised to be non-well was studied by Goh, Pauly & Valenti [9, 10]. By investigating how much the Weihrauch degree decreases if a bad sequence in $F(X)$ is provided as part of the input, we gain insight on how tightly the non-wqo-ness of $F(X)$ and X are linked in an effective way.

We thus initiate the study of Higman's Lemma in this context, i.e. we study the Weihrauch problem HL whose input is a non-wqo X together with a bad sequence in X^*, and the output is a bad sequence in X. We also study the counterparts for finite products, i.e. HL_n with input a non-wqo X and a bad sequence in X^n, and the output a bad sequence in X.

One aspect which makes Higman's Lemma stand out in a reverse math context is that it's best known proof proceeds via the existence of minimal bad sequences in non-wqos [11]. However, the existence of minimal bad sequences is equivalent to $\Pi^1_1{-}\mathrm{CA}_0$, both in reverse maths and Weihrauch reducibility. As $\Pi^1_1{-}\mathrm{CA}_0$ significantly exceeds the actual strength of Higman's Lemma, more intricate arguments are needed to provide better upper bounds. In reverse math, it is known that Higman's Lemma is equivalent to arithmetical comprehension [14, Theorem X.3.22]. In contrast, HL_n is known to be much weaker [6, §4].

Theorem 1. The following hold iff $n < m$:

1. $(\mathrm{LPO}')^n \leq_{\mathrm{W}} \mathrm{HL}_m$
2. $\mathrm{TC}^n_{\mathbb{N}} \leq_{\mathrm{W}} \mathrm{HL}_m$
3. $\mathrm{HL}_{n+1} \leq_{\mathrm{W}} \mathrm{RT}^2_m$

Both LPO' and $\mathrm{TC}_{\mathbb{N}}$ are Weihrauch degrees associated with $\mathrm{I}\Sigma_2$ [7, 12]. In order to show that we really need Ramsey's theorem for k-colourings in order to obtain bad sequences in X from bad sequences in X^k, we prove "weakness of RT^2_k for products" results similar to the one in [8]:

Proposition 2. $(\mathrm{LPO}')^k \not\leq_{\mathrm{W}} \mathrm{RT}^2_k$

Proposition 3. $\mathrm{TC}_{\mathbb{N}}^k \nleq_{\mathrm{W}} \mathrm{RT}_k^2$

Our results pertaining to the actual Higman's lemma problem are (using the $^\infty$ operation from [2, 4]):

Theorem 4. $\lim \leq_{\mathrm{W}} \mathrm{HL} \leq_{\mathrm{W}} \left(\mathrm{RT}_2^2\right)^\infty$

Together with the results from [9, 10], this shows that as far as the ability to solve *deterministic* problems is concerned, HL already has the same strength as if we would not require the bad sequence in X^* as part of the input. However, the latter is much more powerful if also multivalued problems are considered.

We still seek a better understanding of the Weihrauch degree $\left(\mathrm{RT}_2^2\right)^\infty$. It is known that computable instances of RT_2^2 have low-2 solutions, but it is not straight-forward to obtain a uniform counterpart of this (some progress occurs already in [3]). A concrete question to resolve is:

Open Question 5. Does $\left(\mathrm{RT}_2^2\right)^\infty \leq_{\mathrm{W}} \lim \star \lim \star \lim$ hold?

Some other results regarding wqo-preservation results in the Weihrauch lattice have also been obtained by Carlucci, Mainardi and Zdanowski [5].

References

[1] Stefano Berardi & Silvia Steila (2015): *An intuitionistic version of Ramsey's Theorem and its use in Program Termination.* *Ann. Pure Appl. Log.* 166(12), pp. 1382–1406, doi:10.1016/J.APAL.2015.08.002.

[2] Vasco Brattka (2025): *Loops, Inverse Limits and Non-Determinism.* Available at `https://arxiv.org/abs/2501.17734`.

[3] Vasco Brattka, Matthew Hendtlass & Alexander Kreuzer (2017): *On the uniform computational content of computability theory.* *Theory of Computing Systems* 61(4), doi:10.1007/s00224-017-9798-1.

[4] Vasco Brattka & Hendrik Smischliaew (2025): *Computability of Initial Value Problems.* In Arnold Beckmann, Isabel Oitavem & Florin Manea, editors: *Crossroads of Computability and Logic: Insights, Inspirations, and Innovations - Proceedings of CiE 2025*, *LNCS* 15764, Springer, pp. 191–206, doi:10.1007/978-3-031-95908-0_14. Available at `https://arxiv.org/abs/2501.00451`.

[5] Lorenzo Carlucci, Leonardo Mainardi & Konrad Zdanowski (2024): *Reductions of well-ordering principles to combinatorial theorems.* Available at `https://arxiv.org/abs/2401.04451`.

[6] Peter Cholak, Alberto Marcone & Reed Solomon (2004): *Reverse mathematics and the equivalence of definitions for well and better quasi-orders.* *Journal of Symbolic Logic* 69(3), p. 683–712, doi:10.2178/jsl/1096901762.

[7] Caleb Davis, Denis Hirschfeldt, Jeffry Hirst, Jake Pardo, Arno Pauly & Keita Yokoyama (2020): *Combinatorial principles equivalent to weak induction.* *Computability* 9(3-4), pp. 219–229, doi:10.3233/COM-180244.

[8] Damir D. Dzhafarov, Jun Le Goh, Denis. R. Hirschfeldt, Ludovic Patey & Arno Pauly (2020): *Ramsey's theorem and products in the Weihrauch degrees.* *Computability* 9(2), doi:10.3233/COM-180203.

[9] Jun Le Goh, Arno Pauly & Manlio Valenti (2021): *Finding descending sequences through ill-founded linear orders.* *Journal of Symbolic Logic* 86(2), doi:10.1017/jsl.2021.15.

[10] Jun Le Goh, Arno Pauly & Manlio Valenti (2024): *The weakness of finding descending sequences in ill-founded linear orders.* In Ludovic Levy Patey, Elaine Pimentel, Lorenzo Galeotti & Florin Manea, editors: *Twenty Years of Theoretical and Practical Synergies, Proceedings of CiE 2024*, *LNCS* 14773, Springer.

[11] Crispin St. J. A. Nash-Williams (1963): *On well-quasi-ordering finite trees. Mathematical Proceedings of the Cambridge Philosophical Society* 59(4), p. 833–835, doi:10.1017/S0305004100003844.

[12] Arno Pauly, Cécilia Pradic & Giovanni Soldà (2024): *On the Weihrauch degree of the additive Ramsey theorem. Computability*, doi:10.3233/COM-230437. ArXiv 2301.02833.

[13] Helmut Schwichtenberg, Monika Seisenberger & Franziskus Wiesnet (2016): *Higman's Lemma and Its Computational Content*, pp. 353–375. Springer, doi:10.1007/978-3-319-29198-$7_1$1.

[14] Stephen G. Simpson (2009): *Subsystems of second order arithmetic*, second edition. Perspectives in Logic, Cambridge University Press, Cambridge; Association for Symbolic Logic, Poughkeepsie, NY, doi:10.1017/CBO9780511581007.