Two-Point Concentration of the Domination Number of Random Graphs

Random Graphs Paper Presentation

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History of the Two-Point Concentration

Weber, 1981

A random binomial graph $G(n, \frac{1}{2})$ has domination number either $\lfloor k^* \rfloor + 1$ or $\lfloor k^* \rfloor + 2$, where $k^* = \log_2 n - 2 \log_2 \log_2 n + \log_2 \log_2 e$.

Wieland-Godbole, 2001

A random binomial graph G(n, p) has domination number either $\lfloor r^* \rfloor + 1$ or $\lfloor r^* \rfloor + 2$, for $p \gg \sqrt{\frac{log(log(n))}{log(n)}}$ as $n \to \infty$

Glebov-Liebenau-Szabó, 2015

The two point concentration holds for much sparser graphs, where $p \gg \frac{(log(n))^2}{\sqrt{n}}$ as $n \to \infty$ where $r^* = log_{(\frac{1}{1-p})}(\frac{log((1-p)^n)}{(log(np))^2}(1 + o(1)))$. They also conjectured that this as the best possible result, with it failing for $p \ll \frac{1}{\sqrt{n}}$

Introduction

The main result of this paper shows that two-point concentration of the domination number $\gamma(G_{n,p})$ of the binomial random graph $G_{n,p}$ with edge-probability $p = p(n)$ extends down to roughly $p = n^{-2/3}$, refuting the conjecture of Glebov, Liebenau and Szabó.

Introduction

Theorem 1 (Two-point concentration for $p \geq n^{(-2/3)+\epsilon}$)

If $p = p(n)$ satisfies $(\log n)^3 n^{-2/3} \leq p \leq 1$, then the domination number $\gamma(G_{n,p})$ of the binomial random graph $G_{n,p}$ is concentrated on at most two values, i.e., $\exists\, \hat{r} = \hat{r}(n,p)$ such that $\mathbb{P}(\gamma(G_{n,p}) \in \{\hat{r}, \hat{r}+1\}) \to 1$ as $n \to \infty$

Lemma 1 (No two-point concentration for $p \leq n^{-2/3}$)

If $p = p(n)$ satisfies $n^{-1} \ll p \ll (\log n)^{2/3} n^{-2/3}$, then $\exists\, q = q(n)$ with $p \leq q \leq 2p$ such that the domination number $\gamma(G_{n,q})$ of $G_{n,q}$ is not concentrated on two values: we have $\max_{r \geq 0} \mathbb{P}(\gamma(G_{n,q}) \in \{r, r+1\}) \leq 3/4$ for an infinite sequence of n

Key ingredient: Poisson approximation beyond Janson

The proof of Theorem 4 shall extend the methods presented in the 2015 paper by Glebov-Liebenau-Szabó to $n^{-2/3} < p < n^{-1/2}$. The first moment method remains the same as before. The improvement is in variance calculation. An important quantity we wish to calculate is, for $|A|$ & $|B| = r$, assuming both to be vertex disjoint, the bound

$$\mathbb{P}\left(A \,\&\, B \text{ dominate each other}\right) = (1 + o(1))\,\mathbb{P}\left(A \text{ dominates } B\right)\mathbb{P}\left(B \text{ dominates } A\right) \quad (1)$$

Which presents itself as a sort of approximate independence. The RHS tends to 1 if $p \gg log(n) * n^{-\frac{1}{2}}$, Hence we get this result for free in the previous paper. Calculating these events, assuming if X is the number of isolated vertices in a random bipartite graph between two sets of size r, the above result translates to,

$$\mathbb{P}\left(X = 0\right) = (1 + o(1))\, e^{-\mathbb{E}[X]} \quad (2)$$

Key ingredient: Poisson approximation beyond Janson

- Janson's is the standard tool for such an inequality, but since Δ is $\theta(\mu)$ where μ is $\mathbb{E}[X]$, we see that the RHS goes above 1. Even the adjusted Janson's presented in class fails to hold.
- Furthermore, the method of moments, include-exclusion arguments and such are all unavailable, since these usually only manage to establish the statement when the expectation is fairly small, say $\mu = \theta(1)$ or perhaps $\mu = o(logn)$.
- Hence the authors adapt the proof of Janson's to get bounds on $\mathbb{P}(X = 0)$. This results in the following theorem.

Key ingredient: Poisson approximation beyond Janson

Theorem 2

Let X denote the number of isolated vertices in the random bipartite graph $G_{N,N,p}$ with vertex classes V_1 and V_2 of size $|V_1| = |V_2| = N$ and let $V = V_1 \cup V_2$.

$$\mu := \mathbb{E}\, X = \sum_{v \in V} \mathbb{E}\, I_v \tag{3}$$

$$\Delta := \sum_{v \in V} \sum_{w \in V : w \sim v} \mathbb{E}\,(I_v I_w) \tag{4}$$

where, I_v is the indicator RV for the event that v is isolated in $G_{N,N,p}$. Then we have

$$\exp\left(-\mu + \frac{\mu^2}{2N}\right) \leq \mathbb{P}\,(X = 0) \leq \exp\left(-\mu + \Delta p \cdot \log\left(1 + \frac{\mu}{\Delta p}\right)\right) \tag{5}$$

where the lower bound holds when $\mathbb{P}\,(I_v = 1) \leq 0.5$

Key ingredient: Poisson approximation beyond Janson

- The term Δp naturally arises in the variance as $Var(X) = \mathbb{E}[X] + \frac{\Delta p}{1-p}$. This bound is the best such as well as for $p \gg n^{-\frac{2}{3}} log(n)^{\frac{5}{2}}$, we have

$$\mathbb{P}(X = 0) \leq \exp(-\mu + o(1)) \tag{6}$$

- The proof of the lower bound is a simple Harris' application but the upper bound requires more finesse to adjust for the mildness of the dependencies in I_v
- The case where A and B are not vertex disjoint can be carried out by looking at $V_1 = A \setminus B$ and $V_2 = B \setminus A$, and the random bipartite graph on V_1 and V_2. The above proofs can then be adjusted for the same by adjusting for the edges from $A \cap B$.

Two-point concentration: Proof of Theorem 1

Here, we restrict our attention to $p = p(n) \in \left[(\log n)^3 n^{-2/3}, (\log n)^{-2}\right]$.
Let X_r denote the RV counting the number of dominating sets of size r. Let

$$\tau := \mathbb{P}(A \text{ is a dominating set}) = \left(1-(1-p)^r\right)^{n-r} \quad \text{for any } A \in \binom{[n]}{r} \tag{7}$$

So,

$$\mathbb{E}\, X_r = \binom{n}{r} \cdot \left(1-(1-p)^r\right)^{n-r} \tag{8}$$

Let

$$\hat{r} = \hat{r}(n,p) := \min\left\{r : \mathbb{E}\, X_r \geq \frac{1}{np}\right\} \tag{9}$$

Two-point concentration: Proof of Theorem 1

Now, as $p \le (\log n)^{-2}$ we have ${}^{1}/_{\log(1-p)} = p^{-1}(1 + O(p)) = p^{-1}(1 + o({}^{1}/_{\log n}))$

Lemma 2 (Properties of $\hat{r}$)

For any integer r with $|r - \hat{r}| = O(1)$ we have

$$r = -\log_{(1-p)}\left(\frac{np}{\log^2(np)}(1 + o(1))\right) = \frac{\log(np) - 2\log\log(np) + o(1)}{p} \tag{10}$$

Furthermore, $\mathbb{E}\, X_{\hat{r}-1} \to 0$ *and* $\mathbb{E}\, X_{\hat{r}+1} \to \infty$

Using Markov's inequality and Lemma 2, we obtain

$$\mathbb{P}(\gamma(G_{n,p}) \le \hat{r} - 1) = \mathbb{P}\left(X_{\hat{r}-1} \ge 1\right) \le \mathbb{E}\, X_{\hat{r}-1} \to 0 \tag{11}$$

So, it remains to prove that whp $\gamma\left(G_{n,p}\right) \leq \hat{r} + 1 =: r$. Now, using Chebychev's inequality it then follows that

$$\mathbb{P}\left(\gamma\left(G_{n,p}\right) \geq r + 1\right) \leq \mathbb{P}\left(X_r = 0\right) \leq \mathbb{P}\left(\left|X_r - \mathbb{E}\, X_r\right| \geq \mathbb{E}\, X_r\right) \leq \frac{\mathrm{Var}\left(X_r\right)}{\left(\mathbb{E}\left[X_r\right]\right)^2} \tag{12}$$

Let

$$r_0 := \left\lfloor r^2 \frac{\log\left(np\right)}{n} \right\rfloor \tag{13}$$

So,

$$r_0 = O\left(\frac{\log^3\left(np\right)}{np^2}\right) \tag{14}$$

Two-point concentration: Proof of Theorem 1

Also, let

$$\rho(s) := \mathbb{P}(A, B \text{ are dominating sets}) \quad \text{for any } A, B \in \binom{[n]}{r} \text{ with } |A \cap B| = s \tag{15}$$

So,

$$\operatorname{Var} X_r = \sum_{A,B\in\binom{[n]}{r}} \left[\rho(s) - \tau^2\right] \le \underbrace{\sum_{\substack{A,B\in\binom{[n]}{r}:\\ |A\cap B|\le r_0}} \left[\rho(s) - \tau^2\right]}_{=:\mathbb{V}_1} + \underbrace{\sum_{\substack{A,B\in\binom{[n]}{r}:\\ |A\cap B|> r_0}} \rho(s)}_{=:\mathbb{V}_2} \tag{16}$$

In earlier works:

$$\rho(s) \le \left(1 - 2(1-p)^r + (1-p)^{2r-s}\right)^{n-r-(r-s)} \tag{17}$$

Now,

$$\rho(s) = \mathbb{P}(A \text{ and } B \text{ both dominate } [n] \setminus (A \cup B)) \cdot \mathbb{P}(A \text{ and } B \text{ dominate each other}) \tag{18}$$

Lemma 3 (Main technical result)

For any $A, B \in \binom{[n]}{r}$ with $|A \cap B| = s$,

$$\mathbb{P}(A \text{ and } B \text{ dominate each other}) \leq (1 + o(1)) \cdot \left(1 - 2(1-p)^r\right)^{r-s} \tag{19}$$

So, we get

$$\rho(s) \leq (1 + o(1)) \cdot \left(1 - 2(1-p)^r + (1-p)^{2r-s}\right)^{n-r} \tag{20}$$

Two-point concentration: Proof of Theorem 1

Lemma 4 (Asymptotic Independence)

For r_0 as defined in 13, for any $s \in [0, r_0]$, we have

$$\rho(s) \leq (1 + o(1))\,\tau^2 \tag{21}$$

Thus, we have

$$\mathbb{V}_1 = \sum_{\substack{A,B \in \binom{[n]}{r}: \\ |A \cap B| \leq r_0}} \left[\rho(s) - \tau^2\right] \leq \sum_{\substack{A,B \in \binom{[n]}{r}: \\ |A \cap B| \leq r_0}} o\left(\tau^2\right) = o\left(\left(\mathbb{E}\left[X_r\right]\right)^2\right) \tag{22}$$

For intersections of size $s = |A \cap B| > r_0$, we simply write

$$\frac{\mathbb{V}_2}{\left(\mathbb{E}\left[X_r\right]\right)^2} = \sum_{r_0 < s \leq r} \sum_{\substack{A,B \in \binom{[n]}{r}: \\ |A \cap B| = s}} \frac{\rho(s)}{\left(\mathbb{E}\left[X_r\right]\right)^2} = \sum_{r_0 < s \leq r} \underbrace{\frac{\binom{n}{r}\binom{r}{s}\binom{n-r}{r-s}}{\binom{n}{r}\binom{n}{r}} \cdot \frac{\rho(s)}{\tau^2}}_{=: u_s} \tag{23}$$

Two-point concentration: Proof of Theorem 1

Lemma 5 (Counting Estimate)

For u_s and r_0 as defined in 23 and 13

$$\sum_{r_0 < s \leq r} u_s = o(1) \tag{24}$$

Thus,

$$\mathbb{V}_2 = \left(\mathbb{E}[X_r]\right)^2 \cdot \sum_{r_0 < s \leq r} u_s = o\left(\left(\mathbb{E}[X_r]\right)^2\right) \tag{25}$$

Hence,

$$\mathbb{P}\left(\gamma\left(G_{n,p}\right) \geq r+1\right) \leq \frac{\operatorname{Var} X_r}{\left(\mathbb{E}[X_r]\right)^2} = \frac{o\left(\left(\mathbb{E}[X_r]\right)^2\right)}{\left(\mathbb{E}[X_r]\right)^2} = o(1) \tag{26}$$

No two-point concentration: Proof of Lemma 1

Lemma 6 (Non-Concentration)

Fix $\epsilon \in (0, 1/4]$ *and* $p = p(n)$ *such that* $np \to \infty$ *and* $p \to 0$. *Then for an infinite sequence of* $n \; \exists \; q \in [p, 2p]$ *such that* $\max_{[s,t]} \mathbb{P}(\gamma(G_{n,q}) \in [s,t]) \leq 1 - \epsilon$, *where the maximum is taken over any interval* $[s,t]$ *of length* $t - s \leq \epsilon \log(nq) / (nq^{3/2})$

Thus, proving this lemma proves our desired anti-concentration for $n^{-1} \ll p(n) \ll (\log(n))^{2/3} n^{-2/3}$ as :

$$Z(n) := \frac{np}{(\log(n))^{2/3} n^{1/3}} \to 0 \text{ as } n \to \infty.$$

As $Z \to 0$ and $\frac{\log(x)}{x^{3/2}}$ is a decreasing function of $\forall x > 10$ we have

$$\frac{\log(np)}{np^{3/2}} = \frac{(\log(Z) + \frac{1}{3}\log(n) + \frac{2}{3}\log(\log(n)))}{Z^{3/2} log(n)} > \frac{1}{6Z^{3/2}} \to \infty$$

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Asymptotic approximation on Domination Number and a standard coupling result

Lemma 7 (Asymptotic result on Domination number on $G_{n,p}$)

If $p = p(n)$ *such that* $np \to \infty$ *and* $p \to 0$ *then whp* $\gamma(G_{n,p}) = (1 + o(1))\frac{\log(np)}{p}$

Lemma 8 (Coupling)

$p := p(n) q := q(n)$ *be any sequence in* $[0, 1]$ *then* $\exists$ *a coupling of* $G_{n,p}$ *and* $G_{n,q}$ *such that*

$$\mathbb{P}(G_{n,p} = G_{n,q}) \geq 1 - n\frac{|p - q|}{\sqrt{4q(1-q)}} \tag{27}$$

We now choose

$$\Delta = \frac{2.2\epsilon\sqrt{p}}{n}$$
$$l = \lfloor p/\Delta \rfloor$$

Open Questions

- **Question 1.** Suppose $p = p(n)$ satisfies $n^{-1} \ll p \ll n^{-2/3}$. Does there exist an interval $I = I(n)$ of length $\left(np^{3/2}\right)^{-1}$ times some poly-logarithmic factor such that

$$\mathbb{P}\left(\gamma\left(G_{n,p}\right) \in I\right) \to 1?$$

  For $n^{-1} \ll p \ll n^{-2/3}$ a closely related problem is to determine the precise location of the shortest interval I on which $\gamma\left(G_{n,p}\right)$ is concentrated.

- **Question 2.** Suppose $p = p(n)$ satisfies $n^{-1} \ll p \ll n^{-2/3}$. Are there fixed $\epsilon_0, n_0 > 0$ such that

$$\max_{n \geq n_0} \max_{I} \mathbb{P}\left(\gamma\left(G_{n,p}\right) \in I\right) \leq 1 - \epsilon_0$$

  where the maximum is taken over any interval I of length $\left(np^{3/2}\right)^{-1}$ times some poly-logarithmic factor?

Open Questions

- An upper bound on the probability that $\gamma\left(G_{n,p}\right)$ is a particular integer could yield an even stronger anti-concentration statement. Indeed, one could answer affirmatively by showing that

$$\max_{k\geq 0} \mathbb{P}\left(\gamma\left(G_{n,p}\right)=k\right)=\frac{\tilde{O}\left(1\right)}{np^{3/2}}$$

  where $\tilde{O}$ suppresses logarithmic factors, as usual.

- **Domination number of** $G_{n,m}$. While two-point concentration of $\gamma(G_{n,m})$ for the uniform random graph with $m \gg (\log n)^3\ n^{4/3}$ edges follows from Theorem 1, the anti-concentration argument that establishes Lemma 6 does not readily adapt to $G_{n,m}$. Hence the problem of determining the extent of concentration of $\gamma\left(G_{n,m}\right)$ for $n \ll m \ll n^{4/3}$ is widely open.

Thank You!!!