ME-GY 7943
Network Robotic Systems, Cooperative Control and Swarming

Exercise Series 1 Solution

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Exercise 1

1)

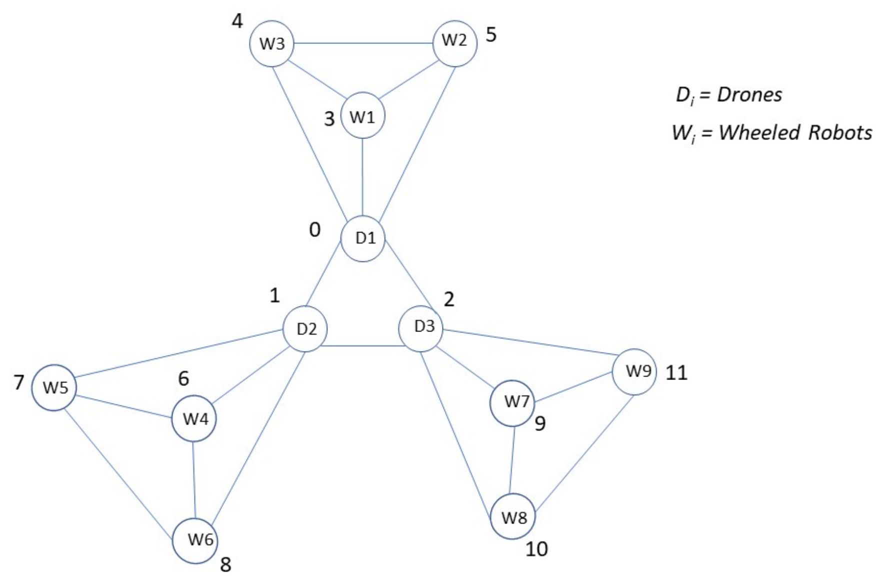


Figure 1: Graph.

2)

$V = \{d1,d2,d3,w1,w2,w3,w4,w5,w6,w7,w8,w9\}$
$E = \{(d1w1),(d1w2),(d1w3),(w1w2),(w2w3),(w3w1),$
$(w1d1),(w2d1),(w3d1),(w2w1),(w3w2),(w1w3),$
$(d2w4),(d2w5),(d2w6),(w4w5),(w5w6),(w6w4),$
$(w4d2),(w5d2),(w6d2),(w5w4),(w6w5),(w4w6),$
$(d3w7),(d3w8),(d3w9),(w7w8),(w8w9),(w9w7),$
$(w7d3),(w8d3),(w9d3),(w8w7),(w9w8),(w7w9),$
$(d1d2),(d2d3),(d3d1),(d2d1),(d3d2),(d1d3)\}$

3)

This is a **connected** graph, as from any selected node we can reach to any other node. The graph is also **undirected** graph.

4)

Yes there is path from a mobile robot of the first team to a mobile robot of the second and third team since the graph is **connected**.

5)

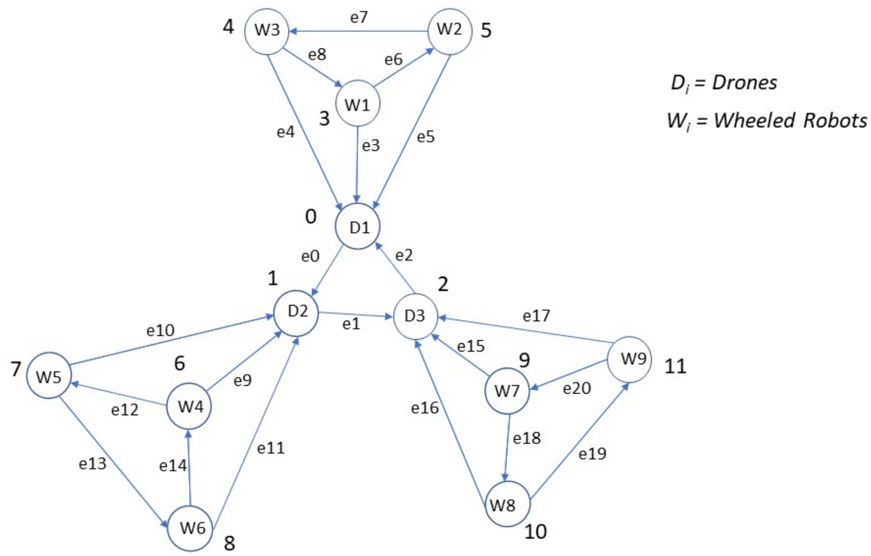
Adjacency Matrix

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Degree Matrix

$$\Delta = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

For the incidence matrix we will be refering to the following graph



Incidence Matrix

$$E = \begin{bmatrix}
-1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & -1
\end{bmatrix}$$

6)

Laplacian

$$L = \begin{bmatrix} 5 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 5 & -1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 \\ -1 & -1 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 \\ -1 & 0 & 0 & 3 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 3 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & -1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 3 & -1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & -1 & 3 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & -1 & -1 & 3 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & -1 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 3 \end{bmatrix}$$

Yes, the graph is **connected**. We can test the connectedness of the graph by computing the eigenvalues of the laplacian and if we found that first eigenvalue is zero and rest other eigenvalues are positive, when we write them in ascending order then we can say that the graph is connected.

Exercise 2

1)

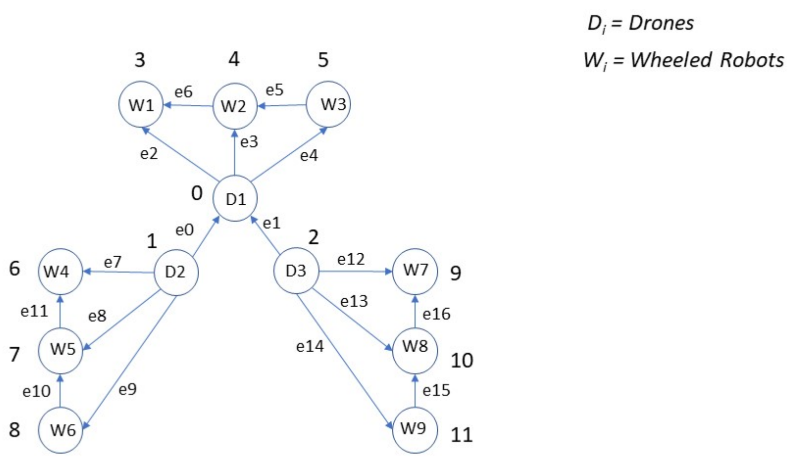


Figure 3: Graph.

2)

$V = \{d1, d2, d3, w1, w2, w3, w4, w5, w6, w7, w8, w9\}$
$E = \{(d2d1), (d3d1), (d1w1), (d1w2), (d1w3), (w3w2), (w2w1), (d2w4), (d2w5), (d2w6), (w6w5), (w5w4), (d3w7), (d3w8), (d3w9), (w9w8), (w8w7)\}$

3)

This is a **weakly connected** graph, as from any selected node we can't reach to any other node.

4)

There are two longest path in the graph.
1st path - *(d2d1w3w2w1)*
2nd path - *(d3d1w3w2w1)*

No, there are no cycles in the graph.

5)

Adjacency Matrix

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Degree Matrix

$$\Delta = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Incidence Matrix

$$E = \begin{bmatrix} 1 & 1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \end{bmatrix}$$

6)

Laplacian

$$L = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The graph is **weakly connected**. Since, there are more than one eigen values equal to zero.

7)

No, the graph **doesn't** contains rooted out branching.

Eigen values are : [0. 0. 1. 1. 1. 2. 2. 2. 2. 2. 2. 2.]

Also, graph is not balanced, since it is clear from the graph that for each vertex in-degreee is not equal to out-degree.

Exercise 3

1)

Since the graph is connected, we can say that, if we arrange the eigen values in ascending order then the first eigen value will be zero and rest of the others will be positive.

2)

If we remove the edge between vertices 4 and 5 then it would result in the generation of two subgraphs which are connected in their own. So, the **first two** eigen values will be **zero** and rest of the others will be positive.

3)

Since it is **connected** graph, the sum of the columns of the Laplacian will be **zero**. Additionally, the graph is **undirected**, therefore, the sum of the rows of the Laplacian will be **zero**.

4)

For calculating eigenvectors and eigenvalues of a matrix, it needs to satisfy the following

$$Lv = \lambda v; \quad where\, v\, is\, eigenvector\, and\, \lambda\, is\, corresponding\, eigenvalue.$$

Now, from the result in question 3) we can write,

$$L\mathbf{1} = 0; \quad sum\, of\, columns\, of\, Laplacian\, is\, zero$$

This shows that **column vector of all 1's** is the eigenvector of Laplacian and it's corresponding eigenvalue is **zero**.

Similarly for calculating left eigenvectors and eigenvalues of a matrix, it needs to satisfy the following

$$\omega^T L = \gamma \omega^T; \quad where\, \omega^T\, is\, left\, eigenvector\, and\, \gamma\, is\, corresponding\, left\, eigenvalue.$$

Also, from the result in question 3) we can write,

$$\mathbf{1}^T L = 0; \quad sum\, of\, rows\, of\, Laplacian\, is\, zero$$

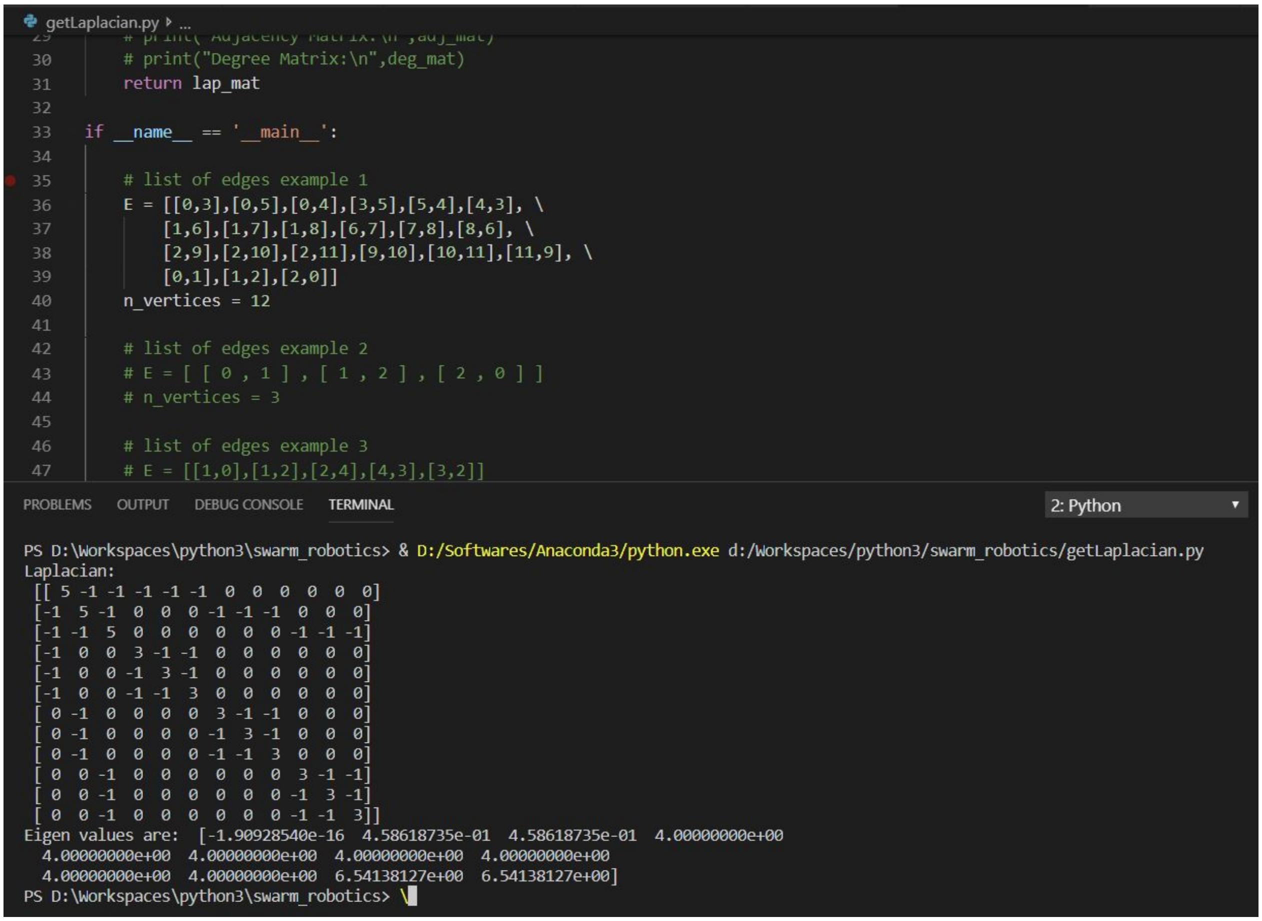
This shows that **row vector of all 1's** is the left eigenvector of Laplacian and it's corresponding left eigenvalue is **zero**.

Exercise 4

1)

Please refer to the *"getLaplacian.py"*.

2)



```
getLaplacian.py > ...
    # print("Degree Matrix:\n",deg_mat)
    return lap_mat

if __name__ == '__main__':

    # list of edges example 1
    E = [[0,3],[0,5],[0,4],[3,5],[5,4],[4,3], \
        [1,6],[1,7],[1,8],[6,7],[7,8],[8,6], \
        [2,9],[2,10],[2,11],[9,10],[10,11],[11,9], \
        [0,1],[1,2],[2,0]]
    n_vertices = 12

    # list of edges example 2
    # E = [ [ 0 , 1 ] , [ 1 , 2 ] , [ 2 , 0 ] ]
    # n_vertices = 3

    # list of edges example 3
    # E = [[1,0],[1,2],[2,4],[4,3],[3,2]]
```

```
PROBLEMS   OUTPUT   DEBUG CONSOLE   TERMINAL                2: Python

PS D:\Workspaces\python3\swarm_robotics> & D:/Softwares/Anaconda3/python.exe d:/Workspaces/python3/swarm_robotics/getLaplacian.py
Laplacian:
 [[ 5 -1 -1 -1 -1 -1  0  0  0  0  0  0]
 [-1  5 -1  0  0  0 -1 -1 -1  0  0  0]
 [-1 -1  5  0  0  0  0  0  0 -1 -1 -1]
 [-1  0  0  3 -1 -1  0  0  0  0  0  0]
 [-1  0  0 -1  3 -1  0  0  0  0  0  0]
 [-1  0  0 -1 -1  3  0  0  0  0  0  0]
 [ 0 -1  0  0  0  0  3 -1 -1  0  0  0]
 [ 0 -1  0  0  0  0 -1  3 -1  0  0  0]
 [ 0 -1  0  0  0  0 -1 -1  3  0  0  0]
 [ 0  0 -1  0  0  0  0  0  0  3 -1 -1]
 [ 0  0 -1  0  0  0  0  0  0 -1  3 -1]
 [ 0  0 -1  0  0  0  0  0  0 -1 -1  3]]
Eigen values are:  [-1.90928540e-16  4.58618735e-01  4.58618735e-01  4.00000000e+00
  4.00000000e+00  4.00000000e+00  4.00000000e+00  4.00000000e+00
  4.00000000e+00  4.00000000e+00  6.54138127e+00  6.54138127e+00]
PS D:\Workspaces\python3\swarm_robotics> \
```

Figure 4: Laplacian for problem in ex 1.

Here we can see that it is equal to the one computed in Exercise 1-6

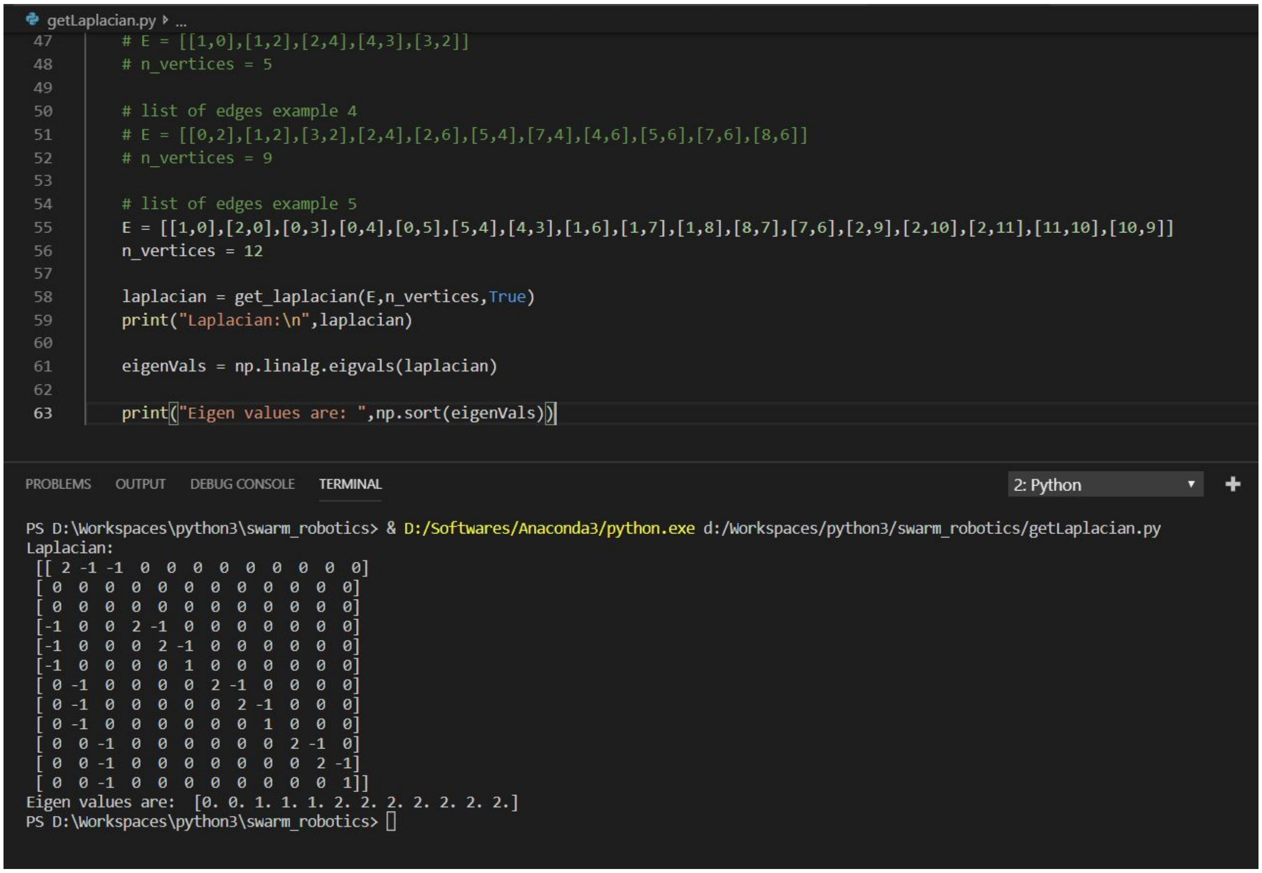


Figure 5: Laplacian for problem in ex 2.

Here we can see that it is equal to the one computed in Exercise 2-6

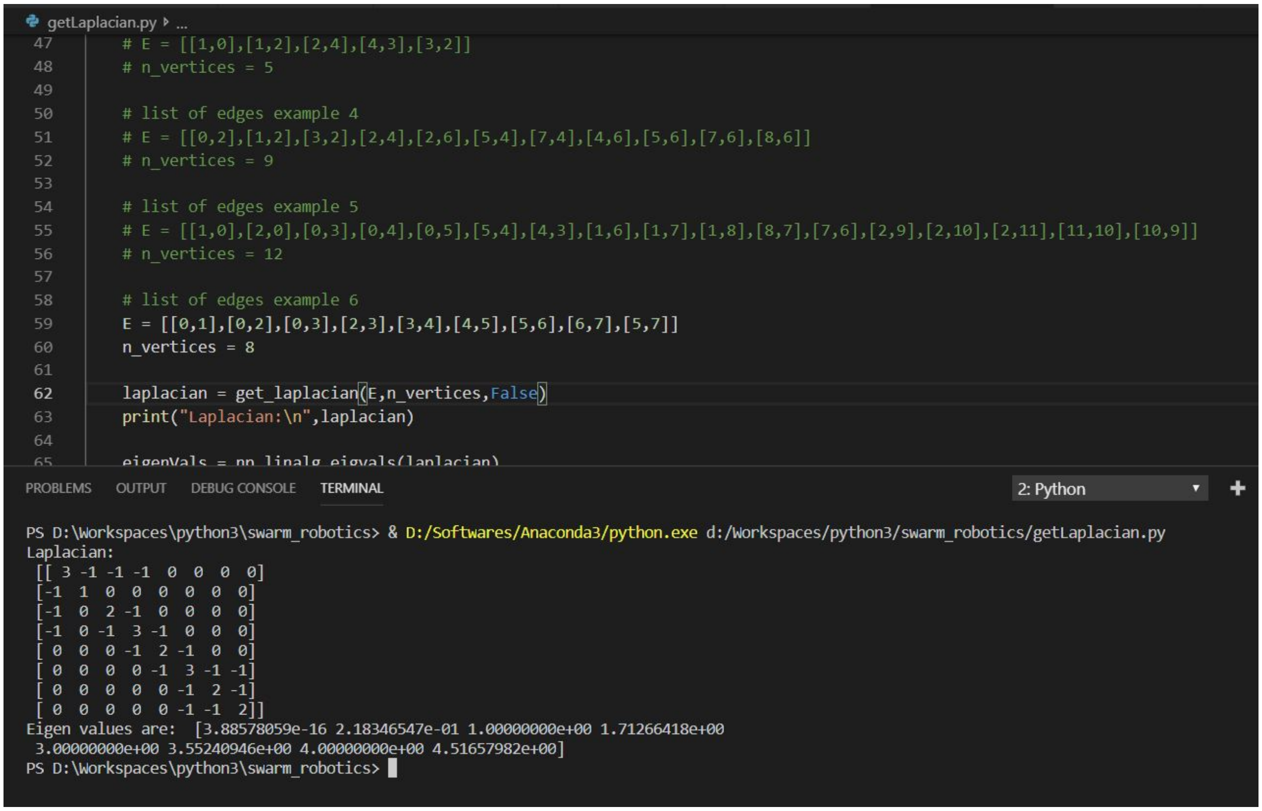


Figure 6: Laplacian for problem in ex 3.

3)

Please refer to the *"cycleGraphEigen.py"*.

For C-4
Two smallest eigen values are = [-6.66133815e-16 2.00000000e+00]
Two largest eigen values are = [4. 2.]

For C-10
Two smallest eigen values are = [-6.10622664e-16 3.81966011e-01]
Two largest eigen values are = [4. 3.61803399]

For C-100
Two smallest eigen values are = [3.10862447e-15 3.94654314e-03]
Two largest eigen values are = [4. 3.99605346]

Exercise 5

1)

Please refer to the *"simulateConsensus.py"*.

2)

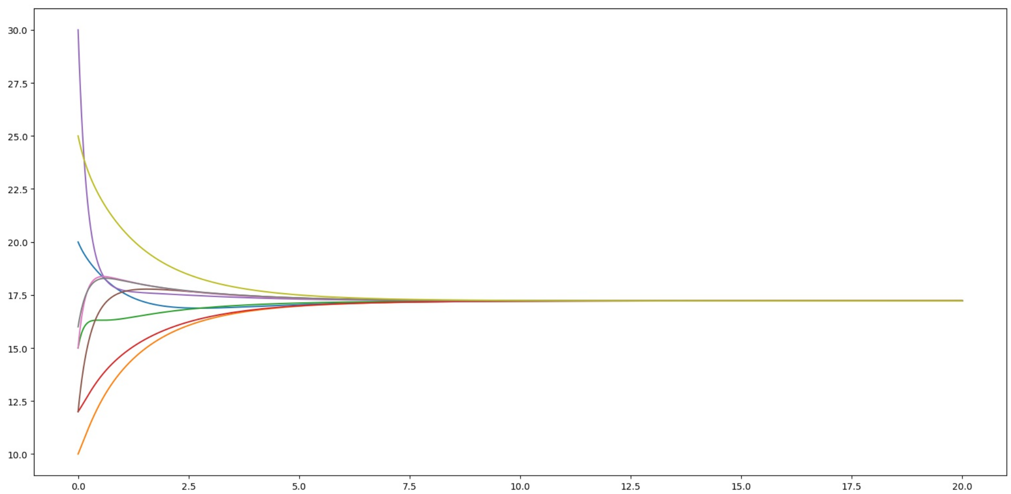


Figure 7: Plot temp vs time.

When t goes to infinity, the system achive consensus.

3)

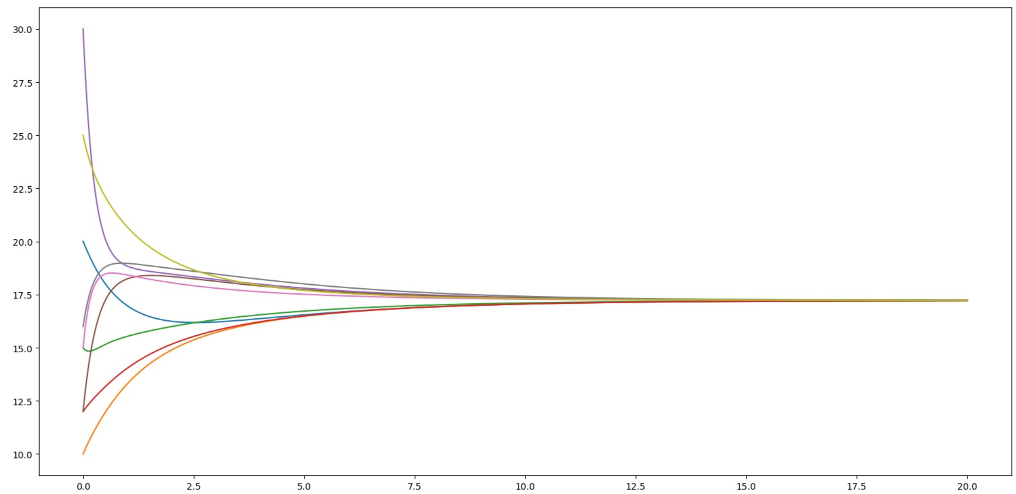


Figure 8: Plot temp vs time.

Yes, the consensus protocol will still converge. It will converge **slower** since, rate of convergence for the consensus protocol is exponential which depends on the value of the second smallest eigenvalue and now the second smallest eigenvalue(2.93285397e-01) is smaller than the previous one(5.41097057e-01).

4)

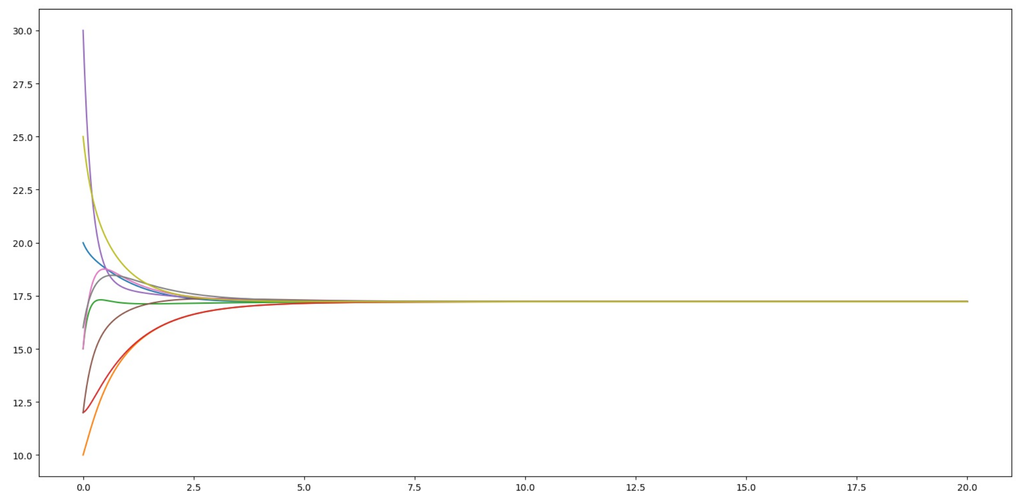


Figure 9: Plot temp vs time.

Yes, the consensus protocol will still converge. It will converge **faster** since, rate of convergence for the consensus protocol is exponential which depends on the value of the second smallest eigenvalue and now the second smallest eigenvalue(6.09050436e-01) is larger than the one computed in problem 5 - 2 (5.41097057e-01).

5)

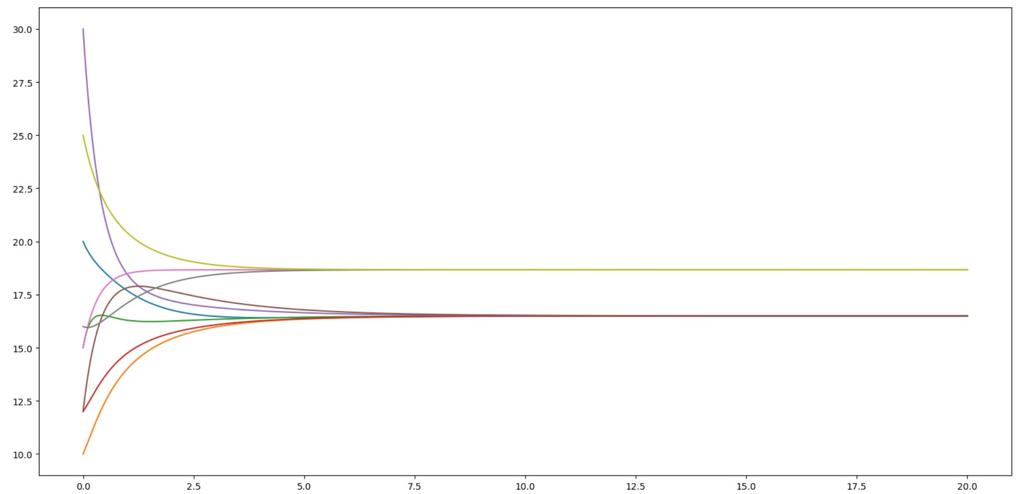


Figure 10: Plot temp vs time.

Yes, the consensus protocol will still converge **but** at **two** points since, we have two subgraphs which are connected on its own. It will converge **slower** since, rate of convergence for the consensus protocol is exponential which depends on the value of the second smallest eigenvalue and now the second smallest eigenvalue(4.85863071e-01) is smaller than the one computed in problem 5 - 2 (5.41097057e-01).