<u>Week 6</u>

I. Introduction

1. Alcoholic example
2. Domino effect example
3. $1+2+3+\ldots+n = \frac{n(n+1)}{2}$

$1+2 = 3 = \frac{3\times 2}{2}$

$1+2+3 = 6 = \frac{3\times 4}{2}$

$1+2+3+4 = 10 = \frac{4\times 5}{2}$

$1+2+3+4+5 = 15 = \frac{5\times 6}{2}$

— and so on...

$1+2+3+4+\ldots+10 = \frac{10\times 11}{2}$

$$1+2+3+4+\ldots+10+11 = \frac{10\times 11}{2} + 11$$
$$= 11\left(\tfrac{10}{2}+1\right) = \frac{11\times 12}{2} \longrightarrow \text{how ?}$$

$1+2+3+\ldots+n = \frac{n(n+1)}{2}$

$$1+2+3+\ldots+n+(n+1) = \frac{n(n+1)}{2} + n+1$$
$$= (n+1)\left(\frac{n}{2}+1\right) = \frac{(n+1)(n+2)}{2}$$

→ Try to show how you can go from one step to the next step. This cascades. Helps you show it is true in general.

→ Show that formula is true for first few cases.
Show that formula is true for $(n+1)$th case, whenever it is true for nth case.

→ Kickstart step (Base step) : Disturb the first domino.

• Gap (Induction gap): ith domino should fall on $(i+1)$th domino.
↳ ideal gap

→

We have to check for two things in mathematical induction:

1. Kickstart step should happen.
2. Something that is true for the first case, and something that is true for (i + 1)th case, whenever ith case is true, then it is true for all elements.

Formal Way:

Statement – P(n) → inducting value
↓
proposition

a. Basis step: P(1), P(1) should be true.
b. Induction hypothesis: Proposition is true for some k.

If P(1) is true and if P(k) is true, then we should prove that P(k+1) is true.
→ induction step.

Inductive Proof: Proposition is true for any n.

II. Mathematical Induction - Examples:

→ 1, 3, 5, 7, 9, ... (sum of odd numbers problem)

$1 + 3 = 4$
$1 + 3 + 5 = 9$ → squares
$1 + 3 + 5 + 7 = 16$

$P(i) : 1 + 3 + 5 + 7 + \ldots + 2i - 1 = i^2$ (base step)

$P(1) = 1$, $P(1)$ is true

Induction hypothesis: $P(k)$ is true

$1 + 3 + 5 + \ldots + 2k - 1 = k^2$

$1 + 3 + 5 + \ldots + (2k - 1) + (2k + 1) = k^2 + 2k + 1$
$= (k+1)^2$

Hence, we proved P(k) is true using Mathematical Induction.

→

→ Sum of Powers of 2

$1 + 2 + 2^2 + \ldots + 2^n = 2^{n+1} - 1$

Basis step: $P(0) = 2^1 - 1 = 1$

Induction hypothesis: $1 + 2 + 2^2 + \ldots + 2^k = 2^{k+1} - 1$

To Prove : P(k+1) is true :

$$1 + 2 + 2^2 + \ldots + 2^k + 2^{k+1} = 2^{k+1} - 1 + 2^{k+1}$$
$$= 2 \cdot 2^{k+1} - 1 = 2^{k+2} - 1$$

$\therefore$ $P(k+1)$ is true.

Hence, $P(n)$ holds true for any n.

Hence Proved

→ Inequality:

$1 < 2,\ 2 < 4,\ 3 < 8,\ 4 < 16, \ldots$

or $n < 2^n,\ \forall\, n \in Z$

Basis step: $P(1)$: $1 < 2$

$\therefore$ $P(1)$ is true

Induction hypothesis:

Assume that $k < 2^k$, where k is some integer is true

To prove: $k+1 < 2^{k+1}$

$k < 2^k \Rightarrow k + 1 < 2^k + 1$

$(2^k) + 1 \leq 2^k + 2^k$ $\quad (\because 1 < 2^k)$

$\Rightarrow 2^k + 1 < 2^{k+1}$ $\quad (k+1 < 2^k + 1 \leq 2^{k+1} \sim 1 < 2 \leq 3)$

$\therefore$ $k+1 < 2^{k+1}$

Hence Proved

→ To Prove Divisibility

$$\left.\begin{aligned} 2^3 - 2 &= 6 \\ 3^3 - 3 &= 24 \\ 4^3 - 4 &= 60 \\ 5^3 - 5 &= 120 \end{aligned}\right\} \text{all are divisible by } 3.$$

or. $n^3 - n$ is divisible by 3, $\forall\, n \in Z$

→

Basis step: $P(1) = 1 - 1 = 0$ is divisible by 3 is true

$\therefore P(1)$ is true.

Induction hypothesis: Assume that $k^3 - k$ is divisible by 3, for some integer k.

To Prove: $(k+1)^3 - (k+1)$ is divisible by 3.

$$(k+1)^3 - (k+1) = k^3 + 1 + 3k(k+1) - (k+1)$$
$$= \underbrace{k^3 - k}_{\text{divisible by 3}} + \underbrace{3k(k+1)}_{\text{divisible by 3}}$$

$\therefore (k+1)^3 - (k-1)$ is divisible by 3.

→ Satisfying Inequalities

Given numbers 1 to 10,

$10 > 9 > 3 < 7 < 8 > 5 < 6 > 4 > 1 < 2$

If you're given "<", ">" symbols in some sequence, can you write numbers 1, 2, 3, ..., 10 which respects this:

$> \; > \; < \; < \; > \; < \; > \; > \; <$

Induct on the number of inequalities.

<u>Basic step</u>:

$1 < 2$ or $2 > 1$

Induction hypothesis:

It is true for m symbols

For $m = 8$ symbols,

$9 > 8 > 2 < 6 < 7 > 4 < 5 < 3 > 1$

If the $(m+1)$th symbol is $<$,

$9 > 8 > 2 < 6 < 7 > 4 < 5 < 3 > 1 < 10$ → arrange nos of 1 to 9, then add 10

If the $(m+1)$th symbol is $>$,

$9 > 7 > 4 < 5 < 6 > 3 < 8 < 10 > 2 > 1$ → arrange no.s from 2 to 10, then add 1

For n number of symbols, we can put $1, 2, 3, \ldots, n+1$ numbers in some order, that it respects the $<>$ symbols.

→ Inequality 2:

$n^2 > 2n + 1, \quad n \in Z^+$

$n^2 > 2n + 1$ is not true for $n = 1, 2$.

It is true for $n = 3$ onwards.

→

Base Case: $n = 3$

$n^2 > 2n + 1 \quad (9 > 7)$

$P(3)$ is true.

Induction hypothesis:

Assume $2k + 1 < k^2$ for some integer k

To prove : $(k+1)^2 > 2(k+1) + 1$

$(k+1)^2 = k^2 + 2k + 1$

$> 2k + 1 + 1 + 2k = 2k + 2 + 2k$ → substituting 3 for this k.

$= 2k + 2 + 2(3)$

$\geq 2k + 8 > 2k + 3$

$= 2k + 2 + 1$

$= 2(k+1) + 1$

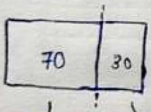
$\therefore (k+1)^2 > 2(k+1) + 1$

→ Example 9

A chocolate bar with 10×10 small pieces can be broken down to 100 individual pieces in 99 breaks.

Assume for $n < 100$ smaller squares, we can break the chocolate bar in $n-1$ attempts.

For 100 bars,



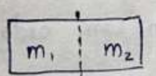
$70 - 1 = 69$ breaks

$30 - 1 = 29$ breaks

But we already break the bar 1 time.

$\therefore$ Total number of breaks $= 69 + 29 + 1 = 99$

For n squares,



Total no. of breaks $= m_1 - 1 + m_2 - 1 + 1$

$= m_1 + m_2 - 1 = n - 1$

$[P(1) \wedge P(2) \wedge P(3) \dots P(k)] \rightarrow P(k+1)$

$\boxed{\text{STRONG INDUCTION}}$

Standard way:

Base: $P(1), P(2)$

Induction hypothesis: $P(k) \to P(k+1)$

$P(n)$ is true for $n \in Z^+$

Strong induction:

Base: $P(1), P(2)$

Induction hypothesis:

$[P(3) \wedge .. \wedge P(k-1) \wedge P(k)] \to P(k+1)$

$P(n)$ is true for $n \in Z^+$

→ Example 10:

$P(n)$ = n can be written as product of primes. $n > 1$.

Basis case: $P(2)$

2 can be written as a product of prime numbers.

$\therefore P(2)$ is true.

Induction hypothesis:

$P(j)$ is true, for every j, $1 \le j \le k$

$\boxed{\text{strong induction}}$

To Prove: $P(k+1)$ is true, or, $k+1$ can be written as a product of prime numbers.

Case I: $(k+1)$ is prime

$\Rightarrow P(k+1)$ is true

Case II: $(k+1)$ is composite

$k+1 = a \cdot b$

$2 \le a \le b < k+1$

$P(a)$ and $P(b)$ are both true, or,

a and b both can be factorised into product of primes.

$(k+1)$ can be factorised into product of primes.

$\therefore P(k+1)$ is true.

Hence Proved

→ Binomial Coefficients - Proof by Induction:

Number of subsets of a set with n elements

1) Combinatorial proof ✓
2) Bijective proof ✓
3) Mathematical induction

$P(n)$ = set with n elements has 2^n subsets.

Inducting on **n**

<u>Basic step</u>: $n = 1$

$A = \{a\}$, $P(A) = \{\phi, \{a\}\}$

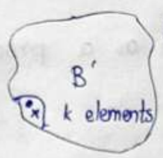
$\therefore$ $P(1)$ is true

<u>Induction Hypothesis</u>:

set with k elements has 2^k subsets

<u>To Prove</u>:

set with $k+1$ elements has 2^{k+1} elements.



B has $k+1$ element

$B = B' \cup \{x\}$

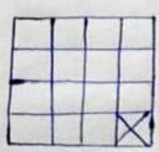
now, $C \subset B'$, $P\{x\} = \{\phi, \{x\}\}$

$\Rightarrow$ C and $C \cup \{x\} \subset B$, as $C \cup \phi = C$

For every such subset of B', we get two subsets of B. Now, B' has 2^k subsets.

$\therefore$ B will have $2^k \cdot 2 = 2^{k+1}$ elements in its powerset.

II. <u>Checker Board & Triominoes</u>



checkerboard

triomino

We can fill up a checker-board with 1 tile removed w/ triominos.

→ Can a $2^n \times 2^n$ checkerbord be tiled using triomino (one square removed)?

$P(n)$ = Every $2^n \times 2^n$ checkerboard with one square removed can be tiled using triomino.

<u>Base case</u>: $P(1)$ = 2×2 checkerboard

<u>Induction hypothesis</u>: Assume that $2^k \cdot 2^k$ checkerboard with one square removed can be tiled.

<u>To Prove</u>: $2^{k+1} \times 2^{k+1}$ checkerboard with one square removed can be tiled using triominoes.

→

One $2^{k+1} \times 2^{k+1}$ checkerboard = 4 $2^k \times 2^k$ checkerboard.

(Logic is simple :)).

The removed tile will be in one of the four checkerboards. Let's assume the centre three tiles of the checkerboard (1 from each smaller checkerbard) are removed. Now, we fill the rest three checkerbards. The removed three tiles themselves form a triomino. This way the $2^{k+1} \times 2^{k+1}$ checkerboard w/ 1 tile removed is filled up using triominoes.

Hence, any $2^n \times 2^n$ checkerboard, one square removed can be tiled using triominoes.

III Important Note & A False Proof

Application of induction needs to be exercised w/ extreme caution, because it can be deceptive.

→ Are all horses of the same colour?

No!

But, this can be falsely proved to be Yes using induction.

$P(n)$ = Set of n horses, where all are of the same colour.

Basis step: $P(1)$

One horse is compared to itself.

Induction Hypothesis:

Assume that given a set of k horses, all horses are of the same colour.

To Prove:

Given a set of $k+1$ horses, all are of the same colour.

$|A| = k+1$

$A = \{1, 2, 3, 4, \ldots, k, k+1\}$

$A' = \{1, 2, 3, 4, \ldots, k\}$ → all these horses are of same colour

$A'' = \{2, 3, 4, \ldots, k+1\}$ → all these horses have the same colour

$2 \xrightarrow{\text{same}} 3 \xrightarrow{\text{same}} 4 \cdots \xrightarrow{\text{same}} k+1$

$1 \xrightarrow{\text{same}} 2 \xrightarrow{\text{same}} 3 \xrightarrow{\text{same}} 4 \cdots \xrightarrow{\text{same}} k$

All horses will be of same colour.

Fallacy in Proof :

1 horse is compared with itself

Can we conclude this way ?

No !

$k = 2$, $P(2)$ = 2 horses are of same colour.

Is it true ? → not always.

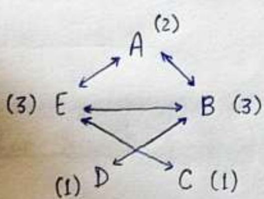
P(1) is true and assuming P(k) is true might not always be true.

→ Check P(1), P(2), P(3), ..., P(k)

IV. Pegionhole Principle

If n pigeons are to be put in m pegionholes, where $n > m$, then at least two pegions will be put in the same pegionhole.

→ Group of n people



3 − 3 overlapping.

Can you declare friendships in such a way that such overlapping friends do not happen ?

It is impossible that you will have a bunch of people having some friends within their circle, no two people have same number of friends.

1 2 3 4 5 6 7 8 9 10

Each person can have one friend or two friends or upto 9 friends.
Each person will be assigned a number 1 - 9.

At least two nodes will have the same number of friends.
A node can also have 0 number of friends.

→ Set of n integers

Consider any n + 1 numbers, $\{a_1, a_2, a_3, \dots, a_{n+1}\}$

Claim : You will find two numbers such that their difference is a multiple of n.

ex. For $n = 4$, 24, 29, 3, 64, 100.

4 divides 64 - 24 , 100 - 24, 100 - 64

When you take $n+1$ numbers, each number when divided by n will leave remainder $0, 1, 2, \ldots, n-1$.

$a_1, a_2, a_3, \ldots, a_{n+1}$ will leave remainders $r_1, r_2, r_3, \ldots, r_{n+1}$ when divided by n.
n possible remainders (0 to $n-1$).

some $r_i = r_j$

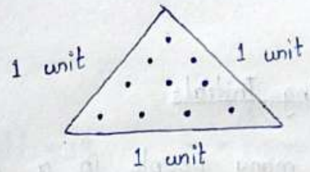
There is a a_i and a a_j which leave the same remainder when divided by n.
$n \mid a_i - a_j$

$a_i = n(q) + r_i$
$a_j = n(q') + r_j$

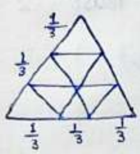
When $r_i = r_j$, $a_i - a_j = n(q - q')$

→ <u>10 Points on an Equilateral Triangle</u>

Show that there ~~be~~ are at least two points with distance at most distance $1/3$.



<u>Divide the △ as:</u>



← sprinkle 10 points.

We have 9 triangles and 10 points to sprinkle.
Now there'll be at least two ~~b~~ points inside one small triangle, meaning that, there distance at most will be $1/3$ units.

→ <u>A Result</u> :

50 gold coins ⟶ 10 people

At least 1 person gets at least 5 gold coins. If this doesn't happen, then, total coins distributed ≤ 40 (think about the simple math here).

✱ When n pegions go to k pegionholes, there is at least one pegionhole with at least n/k pigeons ($n > k$).

→

→ If 51 numbers are chosen from $\{1, 2, \dots, 100\}$ with 1 and 100 included, then prove that any two of the chosen integers are consecutive.

Solution :

Identify pigeons & pigeonholes :

$\{1, 2, 3, \dots, 100\}$

↓

50 sets : $\{1,2\}, \{3,4\}, \{5,6\}, \dots, \{99,100\}$ (pigeonholes)

50 integers selected, each from a different set.
↳ pigeons

Now, 51th integer is the one among the already chosen pigeonhole / set.

⇒ 2 pigeons in the same pigeonhole.

∴ Two consecutive numbers are included in the selection.

Hence Proved

→ Matching Initials

How many people in a group to guarantee that there are at least 2 people whose first alphabet in their names are matching ?

letters in alphabet → 26 (pigeonholes)

∴ number of pigeons = 26 + 1 = 27 (people)

27 th person will match someone's first initial.

→ Numbers Adding to 9

If you pick any 5 numbers from $\{1, 2, 3, 4, \dots, 8\}$ with 1 and 8 included then two of them will add up to 9. Prove this.

numbers adding to 9 (pigeonholes) → $\{1,8\}, \{2,7\}, \{3,6\}, \{4,5\}$

numbers which will be picked → pigeons

We'll have to select two numbers from at least one block.

∴ We'll have two numbers whose sum is 9.

Hence Proved

→

→ Deck of Cards

From a standard deck of 52 cards, what is the minimum number of cards you need to pick to guarantee that there's a suite w/ at least three cards?

4 suits → 4 pigeonholes

If 4 cards are picked, they can be one from each suite. ×

If 8 cards are picked, they can be two from each suite. ×

If 9 cards are picked, then there has to be at least one suite from where three or more cards have been picked. ✓

→ Number of Errors

12 students in a class wrote a dictation. John made 10 errors and all the rest of them made less than 10 errors. Prove that at least two students made the same number of errors.

number of students w/ < 10 errors = 11 → pigeons

total number of possible errors = $|\{0, 1, 2, \ldots, 9\}| = 10$ → pigeonholes

∴ At least two students must've made the same ~~erro~~ number of errors.

A Puzzle:

When you consider 10 people, you can always find 4 people in increasing or decreasing order of height.

→