Week 6

1) There are *n* people in a town, having the length of their name greater than 3. What is the number of people in the town such that you are guaranteed to find atleast two people with the same second alphabet in their name?

   A. 26
   B. 54
   C. 52
   D. 27

   Correct Answer: D

   Solution: There are 26 alphabets in English. In the worst case, if we consider all people with a distinct second alphabet in their name, there are only 26 such possibilities. Therefore, if we introduce one more person, that person must have the same second alphabet in their name, as that of the previous 26 people, hence 27 people in a group will guarantee that there are 2 people with the same second alphabet in their name.
   Lecture 238: Matching initials – solution

2) Given a set of integers from 51 to 250, { 51, 52, 53,......,250}. What is the least number of integers that can be selected such that any two of the chosen integers are consecutive?

   A. 50
   B. 101
   C. 51
   D. 100

   Correct Answer: B

   Solution: There are a total of 200 integers in the given set, we can partition these integers into 100 sets, and consider them as pigeon holes. If we select only one element from each set, we get 100 integers, therefore one more selection will guarantee that the number is one among the previously chosen pigeon hole, that is we'll end up selecting two numbers from one set, hence two consecutive numbers are selected.
   Lecture 236: Consecutive integers solution.

3) Which of the following statements is FALSE?

A. $P(n) = n^3 - n$ is divisible by 3, $\forall n \in Z$
B. $n^2 > 2n + 1 \; \forall n \in Z^+$
C. $P(i)$: $1 + 3 + 5 + \dots\dots\dots + 2i - 1 = i^2$
D. $n < 2^n, \forall n \in Z$

Correct Answer: B

Solution: $n^2 > 2n + 1$ is true for positive integers greater than 3.
Lecture 213: MI-Sum of Odd Numbers

4) Is the given inequality true?

$$2^n < n! \; \forall \; n \in N.$$

A. Yes
B. No

Correct Answer: B

Solution:

$$2^n < n! \; \forall \; n \in N$$
$$2^1 \not< 1!$$
$$2^2 \not< 2!$$
$$2^3 \not< 3!$$
$$2^4 < 4!$$

Therefore, Basis step: $n = 4$

Induction Hypothesis: $2^k < k! \; for \; k \geq 4$

We have to prove that $2^{k+1} < (k + 1)!$

$$\begin{aligned} 2^{k+1} &= 2 \cdot 2^k \\ &< 2 \cdot k! && \text{(By the inductive hypothesis)} \\ &< (k+1) \cdot k! && \text{(since } k+1 > 2\text{)} \\ &= (k+1)! \end{aligned}$$

Therefore, $2^{k+1} < (k + 1)!$ for all $k \geq 4$
Lecture 222: MI-Inequality 2 solution

5) A black-coloured (opaque) bag contains 15 red gloves and 15 green gloves, Krishna takes out one glove at random each time. How many gloves must he take out to be sure that he has at least 2 red gloves?

   A. 17
   B. 15
   C. 30
   D. 16

   Correct Answer: A

   Solution: Given 15 red and 15 green gloves, to take out at least 2 red gloves, first we need to take out 15 gloves (which might end up green in the worst case) and then take out 2 gloves (which would be definitely red). Thus we need to take out a total of 17 gloves.
   Lecture 231: Group of n people.

6) How many of the following statements are true?

   I. If we pick 9 cards from a deck of 52 cards, we are guaranteed that there is a suit with at least 3 cards.
   II. If you pick four numbers from 1 to 8 (including 1 and 8), then some two of them will add up to 9.
   III. Every $2^n \times 2^n$ checkerboard board with one square removed can be tiled using triomino.

   A. 0
   B. 1
   C. 2
   D. 3

   Correct Answer: C

   Solution: If you pick **five** numbers from 1 to 8 (including 1 and 8), then some two of them will add up to 9. Hence this statement is wrong.

   Lecture 234, 240, 242: Pigeonhole Principle - A result, Numbers adding to 9 - solution, Deck of cards - solution

7) For positive integer n, $10^{n-2} > 81n$, if

   A. $n > 5$
   B. $n < 4$
   C. $n \leq 4$
   D. $n \geq 5$

   Correct Answer: D

   Solution:
   $10^{5-2} \geq 81(5)$

   $1000 \geq 405$

   Therefore, $10^{n-2} > 81n \; \forall \, n \; \geq 5$
   Lecture 222: MI-Inequality 2 solution

8) A basket containing 60 walnuts was kept in the veranda, 10 squirrels randomly came and took away some walnuts. After some time the gardener found the basket empty. Which of the following conclusions is always true for the above given situation?

   A. At least one squirrel went back with at most 6 walnuts.
   B. At most one squirrel went back with at least 6 walnuts.
   C. At least one squirrel went back with at least 6 walnuts.
   D. At most one squirrel went back with at most 6 walnuts.

   Correct Answer: C

   Solution: when n pigeons go to k pigeonholes there is at least one pigeonhole with at least $\frac{n}{k}$ pigeons $(n > k)$
   Here pigeons are walnuts and squirrels are pigeonholes
   Therefore, $\frac{n}{k} = \frac{60}{10} = 6$

   Lecture 234: Pigeonhole Principle - A result

9) State whether the following statement is True/False:

For all $n \in N$, $6^{n} - 1$ is a multiple of 5.

A. True
B. False

Correct Answer: A

Solution:

Basis step: for $n = 1,\ 6^{1} - 1 = 5$ is divisible by 5

Therefore, if the statement is true for $n = 1$, let the given statement be true for $n = k$

$6^{k} - 1 = 5x$ (can be written as $6^{k} = 5x + 1$)

Now, we need to prove that if the statement is true for $n = k$ then it is also true for $n = k + 1$

Inductive step: suppose that $6^{k} - 1$ is divisible by 5 for k. then,

$6^{k+1} - 1 = (6^{k} \cdot 6) - 1$

$(5x + 1) \cdot 6 - 1$

$30x + 6 - 1$

$30x + 5$

$5(6x + 1)$

Thus, $6^{k+1} - 1$ is divisible by 5.

Therefore, we can say that if the given statement is true for $n = k$, then it is also true for $n = k + 1$. Hence, by the principle of mathematical induction, the given statement is true $\forall\ n \in N$.

Lecture 218: MI – To prove divisibility(solution)

10) Madhu, Anushka, and Prasad bought a White Chocolate consisting of *n* blocks, the chocolate is distributed equally among all the three members. Madhu likes to enjoy the chocolate, having it one block at a time. How many total breaks does Madhu require to finish her share of chocolate?

A. $n - 1$
B. $\frac{n}{3}$
C. $\frac{n}{4}$
D. $\frac{n}{3} - 1$

Correct Answer: D

Solution: Chocolate is divided into 3 halves thus each one gets $\frac{n}{3}$ blocks of chocolate. The total number of breaks done by Madhu = $\frac{n}{3} - 1$. Thus in total Madhu requires $\frac{n}{3} - 1$ break to finish the chocolate.

Lecture 223: Mathematical induction - example 9