<u>Week 8</u>

Friendship network → undirected graph

People liking someone → directed graph

✱ $a \bullet \longrightarrow \bullet b$

Directed graph

$E = \{(a, b)\}$ (different from (b, a))

✱ All possible cities in India.

direct roads b/w cities

Multigraph (multiple edges)

✱ Network of people: telephone conversation

$A \xrightarrow[\text{2 hours}]{\text{call}} B$

→ weight of friendship

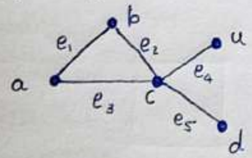
Between two people: Edge and value
(Weighted graph)

→ Simple graph is birectional (arrows on both heads ≈ no arrows)

→ Directed graphs w/ multiple edges b/w vertices : directed multigraph.

I. Graph Representations

a. Adjacency Matrix Representation



How to represent this as a matrix?

$$\begin{array}{c|ccc} & a & b & c \\ \hline a & 0 & 1 & 1 \\ b & 1 & 0 & 1 \\ c & 1 & 1 & 0 \end{array} \approx \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

symmetric matrix

0: 2 vertices are not connected 1: 2 vertices are connected

matrix for complete graph: $\begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$ (ADJACENCY MATRIX)

b. Incidence Matrix Representation

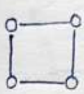
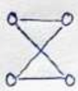
↳ A vertex and the edges its connected to

	e_1	e_2	e_3	e_4	e_5
a	1	0	1	0	0
b	1	1	0	0	0
c	0	1	1	1	1
d	0	0	0	0	1
u	0	0	0	1	0

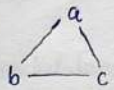
$$\approx \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

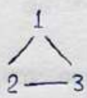
(INCIDENCE MATRIX)

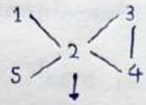
II. Isomorphism

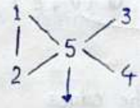

same

⟶ can be viewed differently

ISO MORPHIC
same different



$\cong$




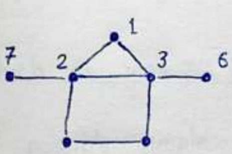
$\cong$


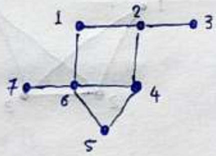
→ If G and H are isomorphic graphs, then
$|V_G| = |V_H|$, $|E_G| = |E_H|$

degree sequence of G = degree sequence of H.

→ If one of these 3 is not true, then G and H are not isomorphic.

Example:

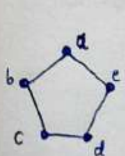
→ G:


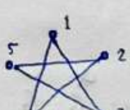
H:


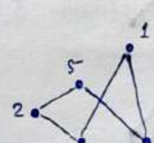
$|V| = 7$, $|E| = 8$
$\langle 4, 4, 2, 2, 2, \cancel{2}, 1, 1 \rangle$

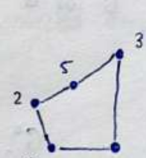
$|V| = 7$, $|E| = 8$
$\langle 4, 3, 3, 2, 2, 1, 1 \rangle$

∴ Not isomorphic

→


$\cong$


$\cong$


$\cong$


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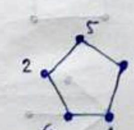
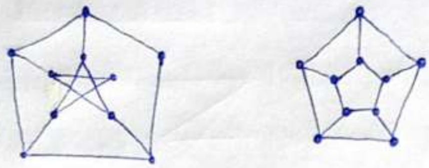
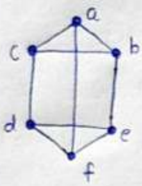
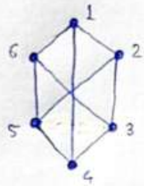


→ 

⟨3, 3, 3, 3, 3, 3, 3, 3, 3, 3⟩

NOT ISOMORPHIC

→  (G)  (H)

Graphs are not isomorphic.

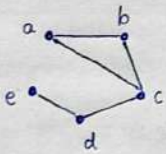
They appear the same, but there are structural differences.

→ <u>Isomorphism</u>:

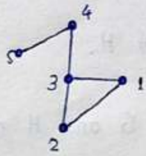
$$\underbrace{G(V, E) \qquad G'(V', E')}_{\text{Bijection}}$$

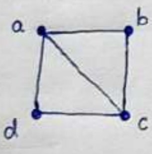
which preserves the edges

An isomorphism of graphs $G(V, E)$ and $G'(V', E')$ is a bijection from V to V' which preserves the adjacency.

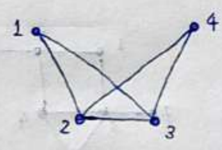


$f(a) = 1$
$f(b) = 2$
$f(c) = 3$
$f(d) = 4$
$f(e) = 5$



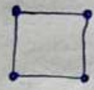


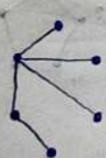
$f(a) = 1$
$f(b) = 2$
$f(c) = 4$
$f(d) = 3$



III. <u>Complement of a Graph</u>

Remove the edges if present, include if not present.

G:  $\bar{G}$ or G^c: 

G:  G^c: 

Edges: ✗ G ✓ G_c

G:  G^c: 

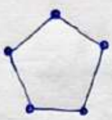
Self Complement :

→ If G is a simple undirected graph, then for any vertex V,

$$\deg_G V + \deg_{\bar{G}} V = n-1$$

If $\deg_G V = k$, in $\bar{G}$, the k edges are not there.

$\therefore \deg_G V + \deg_{\bar{G}} V = n-1$

→ C_5:  $\overline{C_5}$:  → same as C_5

SELF COMPLEMENT

⁂ When a graph is disconnected, its complement is always connected.

Connected Graph : Given any two vertices, there is a path b/w them.

Disconnected Graph : There is at least a pair of vertices, where there is no path.

component 1   component 2

Given a vertex in C_1 and a vertex in C_2, there is a path from u to v in the complement.

Path from x to y : $x - \alpha - y$

Complement of disconnected graph : connected

Which is more? Disconnected or Connected Graphs ?

All possible simple graphs on 10 nodes is

$$2^{\binom{10}{2}}$$

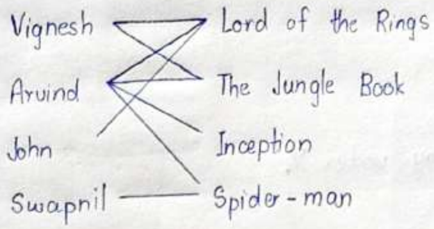
→ Show that there are more ~~disc~~ connected graphs than disconnected graphs.

$\because$ Each disconnected graph's complement is a connected graph.

$\therefore$ number of connected graphs $\geq$ number of disconnected graphs.

→

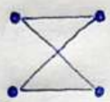
IV. Bipartite Graphs



BIPARTITE GRAPHS

2 partitions

Edges are always across partitions.

→  (4-cycle in a bipartite graph)

Can you find a 3-cycle in a Bipartite graph?

It needs an edge within the partition for a Δ to be formed.

∴ Not possible.

→ Is a C_4 possible?

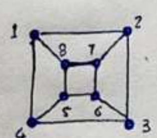
Yes

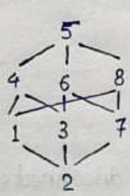
→ Is a C_5 possible?

No.

✱ Bipartite graphs can not have odd cycles.

Q. Does no odd cycles in a graph implies that it can be a Bipartite graph?





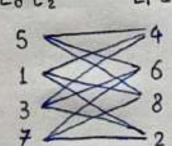
L_0

L_1

L_2

L_3

$L_0 L_2$ $\quad$ $L_1 L_3$



→ Levels automatically arranged in two parts.

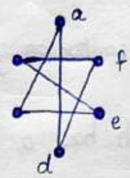
∴ Yes, the statement is true.

V. Eulerian Graph

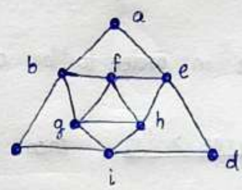
A graph is called a Eulerian Graph if we:
- → start from a node
- → visit all the edges
- → come back to the same node (or circuit)
- → without going through an edge more than once.

A graph is called an Eulerian Graph if it contains an Eulerian circuit.

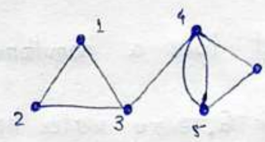
Eulerian Trail: Start anywhere, traverse all the edges, end anywhere.



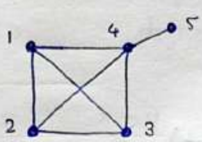
Eulerian circuit ✓



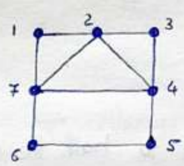
Eulerian circuit ✓



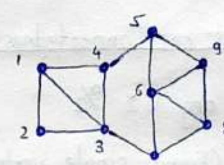
Eulerian circuit ✓



Eulerian circuit X



Eulerian circuit ✓



Eulerian graph/ X
circuit

Litmus Test for Eulerian Graph

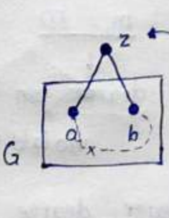
Degree of every node should be even. Then, the graph is *Eulerian*.

Why degree ?

Entering and leaving a node requires two edges (thus two degree), hence, we need 2n degree to enter and leave any (and each) node in the graph, so that it can be Eulerian.

→ Skipping the degree proof for now.

✳ Let G be a connected graph with exactly two vertices of odd degree. Then, there exists an Eulerian trail in G.



Now, degrees of a & b become odd.

From a to a, we now have an Eulerian circuit

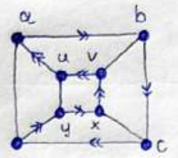
a – x – ... – b is an Open Eulerian Trail.

On removing x, it still is an Eulerian trail.

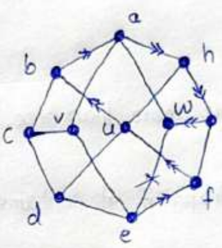
Hence Proved.

VI. Hamiltonian Graph

A graph where one can go through all the vertices, without repeating vertices or edges more than once is called a Hamiltonian Graph.



Hamiltonian graph ✓



× Hamiltonian graph

A graph must have a Hamiltonian cycle in order to be called a Hamiltonian graph.

✱ Given a graph G, every vertex has degree $> n/2$, then the graph has a Hamiltonian cycle.

Proof: Result on Connectedness :

✱ Graph G: $\deg V \geq n/2$

(Every single vertex is adjacent to half the vertex set)

Then, G is connected

Proof by Contradiction :

Assume graph is disconnected. Then it has at least 2 components.

$\boxed{m_1}$ $\boxed{m_2}$

$m_1 + m_2 = n$

Let's say $n = 100$, $m_1 = 60$, $m_2 = 40$ (let's say)

$m_1 = 39$, $m_2 = 61$

When $m_1 = 60$, degree can be 59 maximum

$m_2 = 40$, degree can be 39 maximum

$\not\geq n/2 = 50$

$m_1 = 50$ $m_2 = 50$

↓ ↓

degree can be maximum 49

violate $\deg \geq n/2$

Whenever degree of every ~~of~~ vertex is at least $n/2$, the graph is connected.

Summary of Results :

1. deg $V > n/2$, graph is connected
2. Path of length 3 (P_4)

Given a graph G, if there is a cycle of length k (C_k), then you can find a path of length at least $k+1$. (k must be $< n$)

VII Dirac's Theorem :

Whenever, for $\forall v \in V$, deg $v \geq n/2$, then there is a Hamiltonian cycle.

1. Graph is connected (implication)

Consider a path $u_1, u_2, u_3, \dots, u_k$ which is the longest path.

$k = n$ will not be true if H. cycle doesn't exist.

If $k < n$

$u_1, u_2, u_3, \dots, u_{80}$ (say)

degree = 50 degree = 50 (say)

$u_1 -$ $x - y$ $- u_{80}$ → Always possible because,

For every vertex that u_1 is adjacent to, if predecessor of u_1 is not adjacent to u_{80}, then, the number of nodes > 100, which is not true. u_1 cannot be adjacent to anything outside $u_1, u_2, u_3, \dots, u_{80}$.

$u_1 -$ $x - y$ $- u_{80}$ → cycle on 80 vertices

If this structure is not true, then there are 100 vertices, which is a contradiction Contradiction that this path was longest. Longest path should be of 100 vertices.

Hence, the above structure is a cycle. (Proved)

Note: Dirac's theorem is the sufficient condition for a graph G on n vertices to be Hamiltonian. However, a graph can be a Hamitonion graph without satisfying Dirac's Theorem.

Ore's Theorem:

For any two vertices x and y, if

deg x + deg y $\geq n$

then, there is a Hamiltonian cycle.

Whenever Dirac's theorem is true

⇒ Ore's theorem is true

↓

talks about more graphs / is more general.

Eulerian & Hamiltonian Graphs - Relation

They're independent of each other.

Q. Can there be graphs which are both Eulerian & Hamiltonian ?

Ans Yes. C_n is both Eulerian & Hamiltonian, and more graphs can be, we'll have to check.

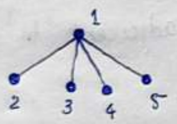
VIII. Planar Graph

A graph is planar if we can draw it on a plane, such that edges don't intersect

Thus this is a planar graph

Thus, planar graph is a graph in which edges do not intersect.

Examples:

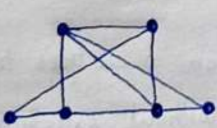


Any tree is a planar graph

star graph on 5 vertices ✓

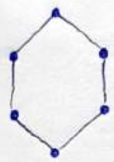
pendant vertex

Thus, planar ✓



Try different ways around. Graph is non-planar.

Planar ✓



Planar graph ✓

Planar graph ✓

V - E + R = 2 :

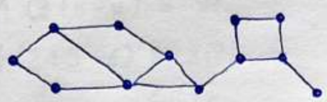
3 vertices
3 edges
2 regions

4 vertices
5 edges
3 regions

here, $4 - 5 + 3 = 2$

here, $3 - 3 + 2 = 2$

or, $V - E + R = 2$



$|V| = 12$
$|E| = 15$
$|R| = 5$

here also, $12 - 15 + 5 = 2$

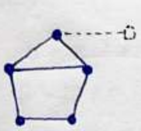
$|V| = 5$
$|E| = 4$
$|R| = 2$

here, $5 - 4 + 2 = 3$

V - E + R doesn't hold true.

The formula holds true only for connected ~~for~~ planar graphs.

Proof of V - E + R = 2 Using Induction :



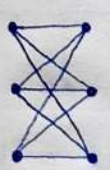
If, $|V| = 5$
$|E| = 6$
$|R| = 3$
$V - E + R = 2$

If, $|V| = 6$
$|E| = 7$
$|R| = 3$
$V - E + R = 2$

If we increase an edge, a vertex also increases.
If we increase an edge, a region also increases.

Hence, $V - E + R = 2$ remains true.

Famous Non-planar Graphs :



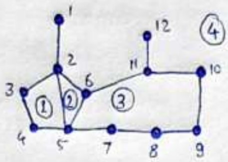
$K_{3,3}$

K_5

Litmus Test for Planarity:

→ If a graph is planar, $V - E + R = 2$

→ $3r \leq 2e$



$|V| = 12$
$|E| = 14$
$|R| = 4$

$12 - 14 + 4 = 2$, $\quad 3(4) = 12$, $2(14) \Rightarrow 12 < 28$ or $3r < 2e$

$R_1 \to 4$, $R_2 \to 3$, $R_3 \to 7$, $R_4 \to 12$

at least 3 edges / region

If total number of regions $= r$

There are at least $3r$ ~~regions~~ edges.

$3r \leq (4 + 3 + 7 + 12)$

$3r \leq 26$

Since each edge is counted at least 1 and at most 2 times.

$\therefore$ $3r \leq 2e$

→ $e \leq 3v - 6$

$14 \leq 3(12) - 6$

$14 \leq 30$

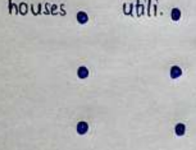
$2 = V - E + r$, $\quad r \leq \frac{2e}{3}$

$\Rightarrow 2 \leq v - e + \frac{2e}{3}$

$\Rightarrow 2 \leq v - \frac{e}{3}$

$\Rightarrow 6 \leq 3v - e$ or $\boxed{e \leq 3v - 6}$

3 Utilities Problem - Revisited



Construct roads to all three utilities from each house, such that the roads do not intersect. (Not possible).

The required structure will be  which is a Bipartite graph.

A Bipartite can never have an odd cycle.
Triangle is never visible.

$3r \leq 2e$ might hold true if this is a planar graph.

For a Bipartite graph, $4r \leq 2e$.

If the graph is planar, then

$4r \leq 2e$

$v - e + r = 2$

$r = 2 + e - v$

$\Rightarrow 4(2+e-v) \leq 2e$

$\Rightarrow 4(2+9-6) \leq 2(9)$

$\Rightarrow 20 \leq 18 \longrightarrow$ not true

Graph is non-planar.
Hence, the roads can't be constructed.

<u>Complete Graph w/ 5 Vertices (k_5) is Non-Planar</u>

Assume k_5 is planar.

$v - e + r = 2$

$\Rightarrow 5 - 10 + r = 2$

$\Rightarrow r = 7$

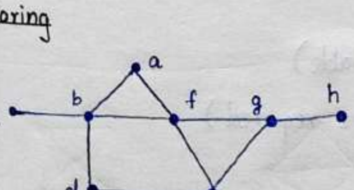
(k_5)

$3r \leq 2e$

$\Rightarrow 21 \leq 20 \longrightarrow$ not true

Hence, k_5 is non-planar.

IX. <u>Coloring</u>



vertex ≡ prisoners

What is the min^m number of prison cells required, so that no two enemies belong to the same cell ?

a, c, g	b, e, h	f
cell 1	cell 2	cell 3

or

a, d, g	b, e	c, f, g
cell 1	cell 2	cell 3

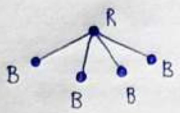
and so on.

Proper Coloring: Assign with minimum number of colors, colors to the vertices; such that no two adjacent vertices have the same coloring.

Minimum number of colors — Chromatic number
→ to color a graph properly

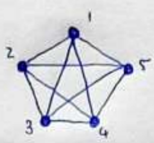
* A graph G is 3-colorable if 3 colors are sufficient to color it properly.

Examples on Proper Coloring:

→ 

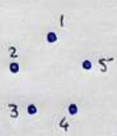
(a star graph is always 2-colorable)

chromatic number = 2

→ k_5: 

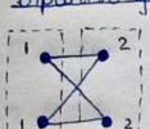
(k_n requires n colors to color it properly, because every vertex is adjacent to every other vertex)

chromatic number = 5

→ k_5^c: 

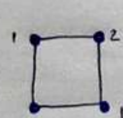
(k_n^c requires 1 color to color it properly)

→ Bipartite graph:



(A Bipartite graph is always 2-colorable)

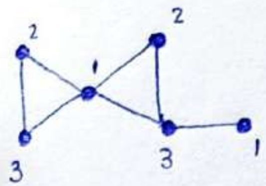
→ C_3:  $= k_3$, chromatic number = 3

→ C_4: 

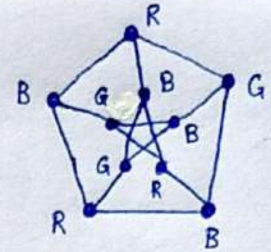
(C_n, when n is even, is two colorable)

(C_n, when n is odd, 3 colors are required)

chromatic number = 2



chromatic number = 3



(Peterson graph)

3 - colorable