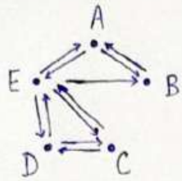
Week 4



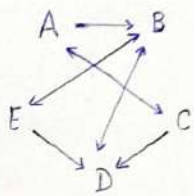
Relation on $\{A, B, C, D, E\}$.

I. Ways to Represent Relations

1. Graphical representation given above.

2. Matrix representation

	A	B	C	D	E
A	1	1	1	0	0
B	0	1	0	1	1
C	1	0	1	1	0
D	0	1	0	1	0
E	0	1	0	1	1



→ 1's in diagonal simply denote that everyone knows oneself.

Cartesian Product:

$A = \{a, b, c, d, e\}$

	a	b	c	d	e
a	aa	ab	ac	ad	ae
b	ba	bb	bc	bd	be
c	ca	cb	cc	cd	ce
d	da	db	dc	dd	de
e	ea	eb	ec	ed	ee

2 tuples

$5 \times 5 = 25$

→ cross product of A with itself $= A \times A$

3. Set Representation

$R = \{(A,B), (A,C), (A,A), (B,D), (B,E), (B,B), (C,A), (C,D), (C,C), (D,B), (D,D), (E,B), (E,D), (E,E)\}$

$R \subseteq S \times S$, where $S = \{A, B, C, D, E\}$

Any subset of $S \times S$ is a relation on S.

Any relation on S is a subset of $S \times S$.

Examples:

a. $A = \{1, 2, 3\}$ and $B = \{a, b, c, d\}$. What is $A \times B$?

$A \times B = \{(x,y) \mid x \in A,\ y \in B\}$

$= \{(1,a), (1,b), (1,c), (1,d), (2,a), (2,b), (2,c), (2,d), (3,a), (3,b), (3,c), (3,d)\}$

$\neq B \times A$

$|A \times B| = |A| * |B| \longrightarrow$ cardinality

$= 3 \times 4 = 12$

b. Let $S = \{1, 2, 3, 4, 5, 6\}$. A relation R on set S is defined as:

$R = \{(a,b) \mid a + b \leq 4\}$

What are the elements of R ?

$a, b \in S$, $a + b \leq 4$

$\Rightarrow R = \{(1,1), (1,2), (1,3), (2,1), (2,2), (3,1)\}$

dreams.

II. Number of Relations & Types of Relations

Number of possible relations on A = All possible subsets of $A \times A$

eg. $|A \times A| = 25$

All possible subsets of $A \times A = 2^{25}$

In general, if

$S = \{a_1, a_2, a_3, \dots, a_n\}$

$\Rightarrow |S \times S| = n \times n = n^2$ elements

Total possible subsets of $S \times S$ = number of elements in power set of $S \times S$

$= 2^{n^2}$

= Total no. of possible relations on S.

a. Reflexive Relation

If a relation contains all possible (x, x) for all values of x from A.

$\rightarrow$ Relation $\subseteq A \times A$

$A = \{a_1, a_2, a_3, a_4, \dots, a_{10}\}$

$R = \{(a_1, a_1), (a_2, a_2), (a_3, a_3), \dots (a_{10}, a_{10})$

What would be the correct matrix representation ?

→ For $A = \{1,2,3\}$

$R = \{(1,1), (2,2), (3,3)\}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$S = \{a_1, a_2, a_3, \dots a_n\}$

Q. What are the total no. of reflexive relations on S ?

$R \subseteq S \times S$

Not all subsets of $S \times S$ will be a valid reflexive relation.

$$\underbrace{\{(a_1,a_1), (a_2,a_2), (a_3,a_3), \dots, (a_n,a_n)\}}_{n \text{ elements}}$$

will definitely be there

Rest elements may or may not be there.

Number of ways in which we can include the rest = all possible relations which are reflexive.

Rest of elements $= n^2 - n$

→ All possible ways to choose the rest of the elements $= 2^{n^2-n}$

= total number of reflexive relations

b. <u>Symmetric Relation</u> :

A relation R is symmetric, iff

$(a,b) \in R \Rightarrow (b,a) \in R$

<u>Matrix Representation</u> :

$$\begin{array}{c c} & \begin{array}{cc} i & j \end{array} \\ \begin{array}{c} i \\ j \end{array} & \begin{bmatrix} \ddots & 1 \\ 1 & \ddots \end{bmatrix} \end{array}$$

Symmetric relations occur in situations where relationship is mutual.

For e.g., A shakes hand w/ B, or A is a cousin of B.

Not a symmetric relation:

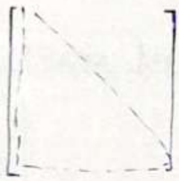
$S = \{1, 2, 3, 4, \ldots, 10\}$

$(a, b) \in R \subseteq S \times S$ if a/b.

$\Rightarrow$ R is reflexive, but not symmetric.

* Parallel lines example is a potential exam question. Identify which relations does a given set of parallel lines fulfill.

What are the total possible symmetric relations on a given set S w/ n elements?

$M(R) =$ 

Number of squares in diagonal & below $= \dfrac{n(n+1)}{2}$

Number of ways of filling up $\dfrac{n(n+1)}{2}$ places $= 2^{\frac{n(n+1)}{2}} = 2^{\frac{n^2+n}{2}}$

Examples:

→ $A = \{1, 2, 3, 4, 5\}$, $R \subseteq A \times A$.

$R = \{(1,1), (1,4), (2,2), (2,3), (3,3), (3,1), (4,4), (5,5), (5,1)\}$

Is R a reflexive relation?

ans Yes

→ $R = \{(a, b) \mid a, b \in \mathbb{N}, b = a^2\}$

Is R reflexive?

ans $R = \{(1,1), (2,4), (3,9), (4,16), (5,25), \ldots\}$

$\therefore$ No.

→ $R = \{(a, b) \mid a, b \in \mathbb{N}, a \cdot b = 14\}$. Is R symmetric?

$R = \{(1,14), (2,7), (7,2), (14,1)\}$

$\therefore$ Yes, it is symmetric.

c. <u>Transitive Relation</u>:

example: $A > B$, $B > C$, then, $A > C$

if $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$.

non-example: A shakes hands with B, B shakes hands w/ C.

d. <u>Antisymmetric Relation</u>

if $(a,b) \in R$, then $(b,a) \notin R$ unless $a = b$.

<u>Examples of Transitive & Antisymmetric Relations</u>

→ Consider a relation on the set of integers as

$R = \{(a,b) \mid a+b = 0\}$

$R = \{(0,0), (-1,1), (1,-1), (2,-2), (-2,2), \ldots\}$

R is symmetric

R is not transitive

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→ $R = \{(a,b) \mid \sin a = \sin b\}$

$\sin 0 = 0$, $\sin \pi = 0$

$(0,\pi) \in R$, $(0, 2\pi) \in R$, $(0, n\pi) \in R$

$\sin \pi/2 = 1$, $\sin 3\pi/2 = -1$, $(\pi/2, -3\pi/2) \neq R$

$(a,b) \in R$ $\quad$ $(b,c) \in R$

$\sin a = \sin b$ $\quad$ $\sin b = \sin c$

$\Rightarrow \sin a = \sin c$ $\Rightarrow$ R is transitive

R is reflexive & symmetric too. (check)

→ Consider a relation R on N.

$R = \{(n, n+1) \mid n \in N\}$

$R = \{(1,2), (2,3), (3,4), \ldots\}$

$(1,2) \in R$, $(2,3) \in R$, $(1,3) \notin R$

$(1,2) \in R$, $(2,1) \in R$

R is only antisymmetric.

Antisymmetric is not same as not symmetric.

example, $A = \{1, 2, 3, 4, 5\}$

$R = \{(1,2), (2,1), (3,4)\}$

R isn't symmetric because $(3,4) \in R$ but $(4,3) \notin R$

R isn't antisymmetric because $(1,2) \in R$ & $(2,1) \in R$.

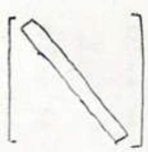
Asymmetric - Graphical Representation

/* Nothing here */

Asymmetric - Matrix Representation

$$\begin{array}{cc} & \begin{array}{cc} a & b \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{bmatrix} \ddots & 0 \\ 1 & \ddots \end{bmatrix} \end{array}$$

Number of Antisymmetric Relations



You can fill up only one Δ, other one automatically gets fixed.

Total elements in the matrix $= n^2$

Elements in the 1 half $\Delta = \frac{1}{2}(n^2 - n)$ → no. of ele. in the diagonal

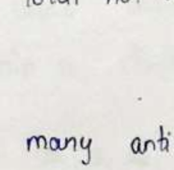
No. of ways to fill up a $\Delta = \frac{1}{2} \; 2^{\frac{n^2-n}{2}}$

No. of ways to fill up the diagonal $= 2^n$

$\therefore$ Total no. of antisymmetric relations $= 2^{\frac{n^2-n}{2}} \cdot 2^n$

$= 2^{\frac{n^2+n}{2}}$

How many anti-symmetric relations on n elements ?



no. of ways to fill up the diagonal $= 2^n$

no. of ways to fill up all the pairs $= 3^{\frac{n^2+n}{2}}$

3 possibilities:

0 0 ✓
0 1 ✓
1 0 ✓
1 1 ×

$\therefore$ Total no. of antisymmetric relations $= 3^{\frac{n^2+n}{2}} \cdot 2^n$

Conditions for Relations to be Reflexive, Symmetric, Antisymmetric

→ Matrix condition for a reflexive relation: $I \leq M(R)$

Identity matrix → I; $M(R)$ → Matrix of relation R.

→ Matrix condition for a symmetric reation ⇒ $M = M^T$

$M \neq M^T \longrightarrow$ not symmetric

(different from anti-symmetric)

→ Condition for a relation to be antisymmetric:

$M \cap M^T \leq I$

e. Equivalence Relation

A relation which is all → reflexive, transitive & symmetric.

→ Consider the set $S = \{(a,b) \mid a, b \in N\}$

$S = N \times N$

$R = \{(a,b), (c,d) \mid ad = bc\}$ or $\frac{a}{b} = \frac{c}{d}$

Is R is an equivalence relation ?

1. $((a,b), (a,b)) \in R$ $\because ab = ab$
2. $((a,b), (c,d)) \in R \Rightarrow bc = ad$

   $((c,d), (a,b)) \in R \Rightarrow ad = bc$
3. $((a,b), (c,d)) \in R \Rightarrow ad = bc \Rightarrow a/b = c/d$

   $((c,d), (e,f)) \in R \Rightarrow de = cf \Rightarrow c/d = e/f$

   $\because c/d = e/f \Rightarrow a/b = e/f$

   $\Rightarrow ((a,b), (e,f)) \in R$

Hence, the relation is all reflexive, symetric & transitive.

⇒ R is an equivalence relation.

III. Partition

$A = \{1, 2, 3, 4, 5, \ldots, 99, 100\}$

$R = \{(a, b) \mid a, b$ leave the same remainder when divided by $4\}$

or $a R b$ if a & b leave the same remainder when divided by 4.

This relation classifies all the interrelated numbers into disjoint partitions.

Note: This is valid for any set A, even infinite sets.

example: $a R b$ if $a \equiv b \pmod 4$

$a \equiv a \pmod 4$ ✓ → reflexive

$a \equiv b \pmod 4 \Rightarrow b \equiv a \pmod 4$ → symmetric

$a \equiv b \pmod 4$, $b \equiv c \pmod 4$

$\Rightarrow a \equiv c \pmod 4$ → transitive.

Thus, $a R b$ is an equivalence relation.

$a R b$ does induce a partition on the set A.

$$\begin{array}{ccccccc} A = S_1 & \cup & S_2 & \cup & S_3 & \cup & S_4 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & & 1 & & 2 & & 3 \end{array} \leftarrow \text{numbers leaving remainder} \doteq$$

Equivalence relation partitions a set into disjoint equivalent sets.

$S_1 \cap S_2 \cap S_3 \cap S_4 \cap \ldots \cap S_k = \Phi$ (for dividing by any number k)

Theorem: Given a set S, and a relation R on S. If the relation is an equivalence relation, then the relation partitions the set into disjoint subsets. The converse is also true for this theorem.

→ If there's a group of 100 people and they've a property that if a friend's friend is one's friend, then these people will automatically divided into seperate clusters, where within a cluster, everyone knows everyone. Think about it!

→ $A = \{a_1, a_2, a_3, \ldots, a_n\}$

R on A is reflexive, symmetric & transitive.

$a R a$ $\quad$ $a R b$, $b R a$ $\quad$ $a R b$, $b R c$, $a R c$

$a_{10} \, R \, a_1 \Rightarrow a_1 \, R \, a_{10}$ is true

$a_2 \, R \, a_1 \Rightarrow a_2 \, R \, a_{10}$

$\therefore \; S_1 = \{a_1, a_2, a_{10}\} \; \cdots \; S_2$

→ $A = S_1 \cup S_2 \cup \ldots \cup S_k$

→ $S_1 \cap S_2 \cap S_3 \cap \ldots \cap S_k = \phi$

→ Intersection of any two sets = ϕ

<u>Converse of Theorem</u>: If there is a partition of A,

$$A = S_1 \cup S_2 \cup S_3 \cup \ldots \cup S_k,$$

then an equivalence relation can be defined on A which induces the partition.