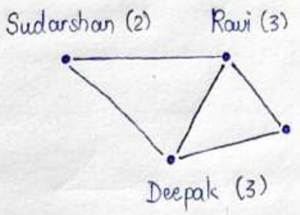
Week 7

I. Graph - Introduction



(GRAPH)

Lines represent friendship. People are nodes here.

→ $V = \underbrace{\{S, R, D, C\}}_{\text{Vertex set}}$ $\qquad E = \underbrace{\{SR, RC, RD, CD, SD\}}_{\text{Edge set}}$

nodes ↔ vertices
lines ↔ edges

→ $G = (V, E)$

→ Given a vertex set, there can be any number of edge sets. (though finitely many).

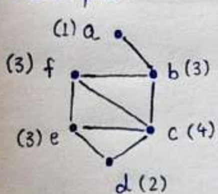
II. Degree & Degree Sequence

Sudarshan has 2 friends
Ravi has 3 friends
Chhaya has 2 friends
Deepak has 3 friends

→ degree of a node (no. of lines emanating from that node)

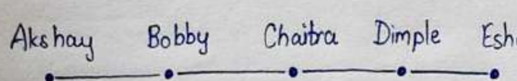
→ 2, 3, 2, 3 → degree sequence

example:



degree sequence: 1, 3, 4, 2, 3, 3

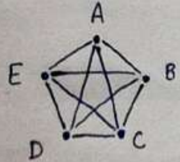
→ Path Graph (P_k)



(P_5)

k = number of nodes = 5

→ Complete Graph (K_6):



(K_5)

→ A path with n vertices has n − 1 edges

→ A complete graph with n vertices will have $\binom{n}{2}$ edges.

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Given 5 people, in how many ways can you choose 2 people out of 5 people ?

$$\binom{5}{2} = 10$$

<u>Relation b/w number of edges & degrees</u>

→ $\sum_{v \in V}$ (degree of v) = 2 (number of edges)

soup	meal
counter 1	counter 2

sum total of people who visited both the counters = 100

number of people = 50.

A person who comes to the soup counter, also comes to the meal counter.

→ Every edge contributes to a degree in two different vertices.

∴ $\sum_{v \in V}$ (degree of v) = 2 (number of edges)

This is called the hand shaking lemma.

$1 + 3 = 4$ — even no. of odd no. → even

$1 + 3 + 5 = 9$ — odd no. of odd no. → odd

$$\underbrace{\sum_{v \in V} (\text{degree of } V)}_{\text{even}} \Leftarrow \underbrace{2 (\text{number of edges})}_{\text{even}}$$

$$\underbrace{d_1 + d_2 + d_3 + \dots + d_n}_{\text{even number}} = 2m$$

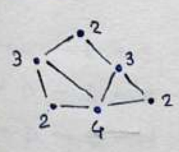
⇒ All $d_1, d_2, d_3, \dots, d_n$ are even numbers.

or

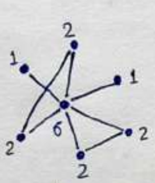
There are even number of odd numbers in the set, or, there are even number of odd degree vertices in the graph.

Every graph has even number of odd degree vertices.

<u>Problems:</u>



2 vertices w/ odd degree



2 vertices w/ odd degree.

III. Havel Hakimi Theorem

We can write a degree sequence in any order, but its considered a good practice to write it in a non-decreasing order.

→ degree sequence : $\langle 2, 2, 2 \rangle$

Can you write a graph for this degree sequence ?

Yes : 

$\langle 2, 2, 2, 2 \rangle$: 

$\langle 1, 1, 1 \rangle$: Graph can't be constructed (odd no. of vertices w/ odd degree)

$\langle 1, 1 \rangle$: 

→ Given a graph, we can write a degree sequence.

But given a degree sequence, you may not have a graph with that degree sequence.

Q. Can we draw graphs for these degree sequences ?

i) $\langle 5, 5, 3, 3, 2, 2, 2 \rangle$

$S_1 = \langle 5_a, 5_b, 3_c, 3_d, 2_e, 2_f, 2_g \rangle$

$S_2 = \langle *_a, 4_b, 2_c, 2_d, 1_e, 1_f, 2_g \rangle$

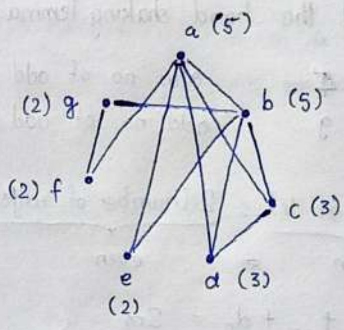
$S_2' = \langle *_a, 4_b, 2_c, 2_d, 2_g, 1_e, 1_f \rangle$

$S_3 = \langle *_a, 4_b, 2_c, 2_d, 2_g, 1_e, 1_f \rangle$ ×

$S_3 = \langle *_a, *_b, 1_c, 1_d, 1_g, 0_e, 1_f \rangle$

$S_3' = \langle *_a, *_b, 1_c, 1_d, 1_g, 1_f, 0_e \rangle$

$S_4 = \langle *_a, *_b, *_c, *_d, *_g, *_f, 0_e \rangle$



ii) $\langle 5, 5, 5, 5, 2, 2, 2 \rangle$

Do it yourself

→ A degree sequence is called graphic if a simple graph can be drawn with that sequence.

e.g. $\langle 5, 5, 3, 3, 2, 2, 2 \rangle$ — graphic sequence

$\langle 2, 2, 2 \rangle$ — graphic sequence

→ A degree sequence $S = d_1 \geq d_2 \geq d_3 \geq \dots \geq d_n$, where $d_1 \leq n-1$, is graphic if and only if the <u>reduced sequence</u> $S' = \{*, d_2-1, d_3-1, \dots, d_n\}$ is graphic.

If the last sequence contains all 0's or $*$'s, then it is graphic.

Sequence $\xrightarrow{\text{reduce}}$ Sequence $\xrightarrow{\text{reduce}}$ Sequence ... Sequence

<u>graphic</u> $\Leftarrow$ contains all 0's

The above theorem is called HAVEL HAKIMI THEOREM.

→ $\langle 5,5,5,5,2,2,2 \rangle$. Check if this sequence is graphic.

$S_1 = \langle 5,5,5,5,2,2,2 \rangle$

$S_2 = \langle *,4,4,4,1,1,2 \rangle$

$S_2' = \langle *,4,4,4,2,1,1 \rangle$

$S_3 = \langle *,*,3,3,1,0,1 \rangle$

$S_3' = \langle *,*,3,3,1,1,0 \rangle$

$S_4 = \langle *,*,*,2,0,0,0 \rangle$

$S_5 = \langle *,*,*,*,-1,-1,0 \rangle$

∴ Sequence is not graphic.

IV. <u>Regular & Irregular Graph</u>

Regular graph: Degree of all nodes happens to be same.

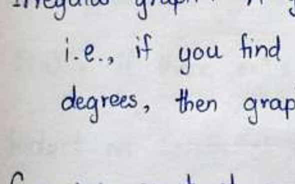
Irregular graph: A graph that is not regular.

i.e., if you find at least 2 vertices, such that both of them have different degrees, then graph cannot be regular.

→ Can you construct a graph on 5 nodes, where degrees of all nodes are different?

NOT POSSIBLE (think why?)

V. <u>Walk, Trail & Path</u>



(u to v)

u − a − b − c − v

u − a − c − e − a − c − v

u − a − e − d − c − v

Walk: A walk is a sequence of vertices & edges.

$W(u,v) = \{u-a-b-c-v\}$

$W(u,v) = \{u-a-c-e-a-c-v\}$

etc.

In a walk we can go through a vertex several times.

~~Ed~~ **Trail**: A walk is a trail if the edges are all distinct but vertices need not to be distinct, or, edges aren't repeated, while vertices can be repeated.

Trail: $\{u-a-c-v\}$

$\{u-a-c-e-d-c-v\}$

Path & Closed Path: A walk where vertices are not repeated (and automatically no edges) is called a path. e.g. $u-a-e-d-v$.

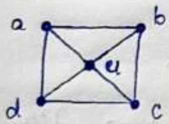
A closed path starts from the same vertex and ends at the same vertex.

e.g. $e-a-b-c-e$

$e-a-b-c-d-e$

[all vertices are distinct $\Rightarrow$ all edges are distinct].

VI. **Cycle & Circuit**



Q. Find a trail starting in a and ending in b, except ab.

$a-u-c-d-u-b$

Q. Closed trail from a to a.

$a-u-c-d-u-b-a$

→ A closed trail is called a circuit.

Q. A path from a to b.

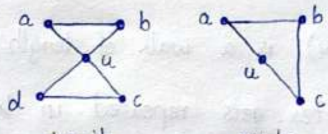
$a-u-c-b$

Q. A closed path from a to a.

$a-u-c-b-a$

A closed path is called a cycle.

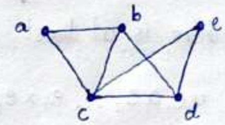
* A cycle is a part of a circuit.
But a circuit cannot be a cycle.



circuit cycle

Q. Circuit which starts and ends at a.

a - b - c - d - e - c - a



Cycle starting & ending at a.

a - b - c - e (part of upper circuit)

Cycle from c to c.

c - d - e - c (part of upper circuit)

VII. Relation b/w Walk and Path

If there is a walk from u to v,
there is a path from u to v.

When you take a path from u to v:

$u - x_1 - x_2 - x_5 \dots - v$

A walk from u to v might have circuits. If you remove the circuits, you will get a path from u.

Induction Proof:

P(n) = If there exists a uv-walk, there exists a uv-path in G.

Induct on length of the walk.

Basic step:

length of the walk = 1

u ●—● v

Basic step is true.

Induction hypothesis:

Assume the length of walk $\leq k$, and the theorem is true.

To Prove:

The theorem holds true for a walk of length $k+1$.

Assume $W(u,v)$ is a walk of length $k+1$, where no vertex is repeated.

Assume a vertex gets repeated in $W(u,v)$.

$$W(u,v) = u\,e_1\,a \ldots e_i\,x \underbrace{\ldots\ldots}_{} x\,e_j \ldots v$$

there is at least one vertex, except the second x, & thus an edge

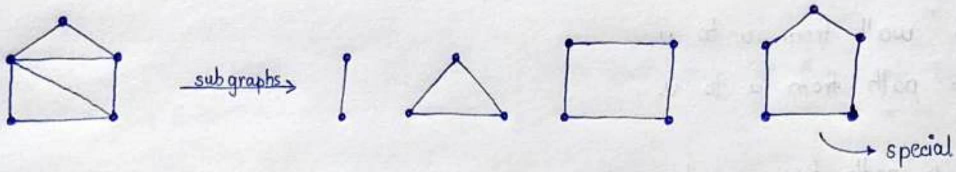
$$\underbrace{W'(u,v)}_{} = u\,e_1\,a\,e_2 \ldots e_i\,x\,e_j \ldots v$$

walk of length $\leq k$

⇒ There must exist a path.

VIII. Subgraph

Given a graph G, $V' \subset V$, $E' \subset E$ forms the subgraph.



The special subgraph has all the vertices which the original graph has.

→ The subgraph which has the same vertex set as the original graph is called as spanning subgraph.

→ A subgraph which has all the edges corresponding to those set of vertices is called an induced subgraph.

Q. How many graphs are possible which are both induced & spanning?

Only one, the graph itself.

Tree: A tree is a connected acyclic graph.

no. of edges = no. of vertices − 1

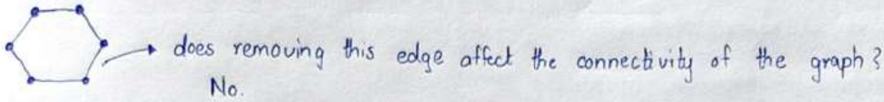
$$|E| = |V| - 1$$

IX. Connected & Disconnected Graphs

Disconnected graphs have more than one components.

Connected Graph: A graph in which there exists a path from node a to node b, for any pair of nodes, a and b.

Property of a Cycle:



In a graph G, if there is a cycle and if an edge is removed from the cycle, the graph will still remain connected.

* **Result**: Given any graph G, that is connected, with n vertices, the number of edges > n - 1.

Proof:

G if it is a tree, it has n-1 edges.

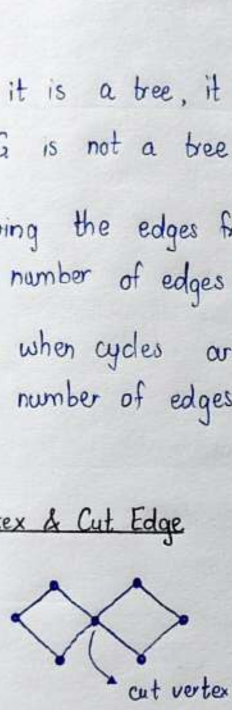
If G is not a tree, there is a cycle in a graph.

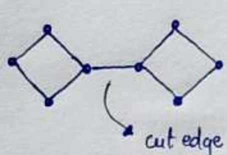
Removing the edges from cycles, we get a tree, then
number of edges = n-1

Thus, when cycles are there in a graph,
number of edges > n - 1.

Hence Proved

X. Cut Vertex & Cut Edge





Cut edge is an edge on whose removal, the graph becomes disconnected.

Cut vertex is a vertex whose removal makes the graph disconnected.