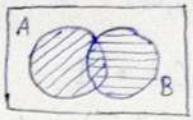
Week 10

$|A \cup B| = |A| + |B| - |A \cap B|$



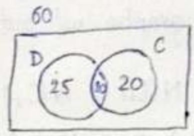
similarly,

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

Example 1: Dogs & Cats

Assume there are 60 houses.

Pet: Dog or cat



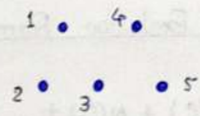
35 : dogs, 30 : cats

$|C \cup D| = 35 + 30 - 10$
$= 55$

$\therefore$ number of people without any pet $= 60 - 55 = 5$

I. Inclusion & Exclusion

In how many ways you can construct roads b/w these cities such that no city is left out ?



or

In how many ways can you construct graphs without isolated vertices ?

Condition 1 (C_1): Vertex 1 is isolated.

$N(C_1)$ – number of ways in which 1 can be isolated.

$N(C_1) = 2^{\binom{4}{2}} = 2^6$

C_2: Vertex 2 is isolated

(this long explanation is not required, is unnecessary.)

$N(C_2) = 2^6$

similarly, $N(C_3) = N(C_4) = N(C_5) = 2^6$

$N(C_1C_2)$: Number of ways in which C_1 and C_2 are satisfied.

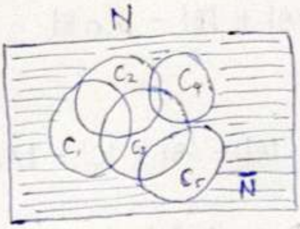
$N(C_1C_2) = 2^{\binom{3}{2}} = 2^3$

similarly, $N(C_1C_3) = N(C_1C_4) = N(C_iC_j) = 2^3$

$\longrightarrow$

$N(C_1C_2C_3)$ = In how many ways 1, 2 and 3 be isolated

$N(C_iC_jC_k) = 2^1$



$N(C_1C_2C_3C_4) = 1$

$N(C_iC_jC_kC_l) = 1$

$N(C_1C_2C_3C_4C_5) = 1$

no. of possible graphs w/ out any isolated vertices = ?

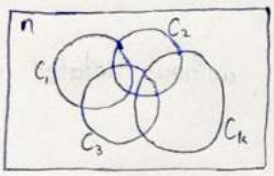
$\overline{N} = N - [N(C_1) + N(C_2) + N(C_3) + N(C_4) + N(C_5)] + N(C_1C_2) + N(C_1C_3) + N(C_1C_4) + N(C_1C_5) +$
$N(C_2C_3) + N(C_2C_4) + N(C_2C_5) + N(C_3C_4) + N(C_3C_5) + N(C_4C_5) - [N(C_1C_2C_3) + N(C_1C_2C_4) +$
$N(C_1C_2C_5) + \ldots + N(C_3C_4C_5)] + N(C_1C_2C_3C_4) + N(C_1C_2C_3C_5) + N(C_1C_2C_4C_5) + \ldots +$
$N(C_2C_3C_4C_5) - N(C_1C_2C_3C_4C_5)$

In short, $\overline{N} = 768$ → number of graphs on 5 nodes, where no node is isolated.

Proof of Inclusion - Exclusion Formula

$\overline{N} = N - [N(C_1) + N(C_2) + N(C_3) + \ldots + N(C_5)] + \ldots$

k conditions: $C_1, C_2, C_3, \ldots, C_k$



$\overline{N} = N - [N(C_1) + N(C_2) + N(C_3) + \ldots + N(C_k)] + N(C_1C_2) + N(C_1C_3) + N(C_1C_4) + \ldots + N(C_{k-1}C_k)$
$- [N(C_1C_2C_3) + N(C_1C_2C_4) + \ldots + N(C_{k-2}C_{k-1}C_k)] + \ldots + (-1)^k N(C_1C_2 \ldots C_{k-2}C_{k-1}C_k)$

Examples:

1. How many integer solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 25$ where $0 \le x_i \le 10$ for all $1 \le i \le 4$.

C_1 : solutions where x_1 is at least 11 or $x_1 \ge 11$

C_2 : $x_2 \ge 11$

C_3 : $x_3 \ge 11$

C_4 : $x_4 \ge 11$

$N(C_1)$ = number of solutions where $x_1 \geq 11$

Total possible solutions is : $n = 4, r = 25$

$$\therefore \binom{4+25-1}{25} = \binom{28}{25}$$

$N(C_1)$ = no. of solutions where $x_1 \geq 11$

$$x_1 + x_2 + x_3 + x_4 = 14$$

$x_1 + x_2 + x_3 + x_4 = 25 \; ; \quad x_1 \geq 11, \; x_2 \geq 0, \; x_3 \geq 0, \; x_4 \geq 0$

Removing 11 units,

$$x_1 + x_2 + x_3 + x_4 = 14$$

e.g. $0 + 1 + 9 + 4 = 14$

$1 + 2 + 10 + 1 = 14$

$$N(C_1) = \binom{4+14-1}{14} = \binom{17}{14}$$

similarly, $N(C_2) = \binom{17}{14} = N(C_3) = N(C_4)$, where $x_1 \geq 0, \; x_2 \geq 0, \; x_3 \geq 0, \; x_4 \geq 0$

$N(C_1C_2) = \,?$

solution must have at least $x_1 \geq 11$ and $x_2 \geq 11$.

$x_1 + x_2 + x_3 + x_4 = 25 \; ; \quad x_1 \geq 11, \; x_2 \geq 11, \; x_3 \geq 0, \; x_4 \geq 0$

same as

$x_1 + x_2 + x_3 + x_4 = 3 \; ; \quad x_1 \geq 0, \; x_2 \geq 0, \; x_3 \geq 0, \; x_4 \geq 0$

$$N(C_1C_2) = \binom{4+3-1}{3} = \binom{6}{3} = 20$$

similarly, $N(C_2C_3) = N(C_1C_3) = 20$

$N(C_1C_2C_3) = \,?$

solution must have x_1 at least 11, x_2 at least 11, x_3 at least 11.

or $33 + x_4 = 25$ ✗

$\therefore \; N(C_1C_2C_3) = 0$

and $N(C_1C_2C_3C_4) = 0$

$$N(\bar{C}_1\bar{C}_2\bar{C}_3\bar{C}_4) = N - [N(C_1) + N(C_2) + N(C_3) + N(C_4)] + [N(C_1C_2) + N(C_1C_3) + N(C_2C_3) + N(C_3C_4) + N(C_1C_4) + N(C_1C_4)]$$

$$= \binom{28}{25} - \left[4\binom{17}{14}\right] + \left[6\binom{6}{3}\right] = \binom{28}{25} - \left[\binom{4}{1}\binom{17}{14}\right] + \left[\binom{4}{2}\binom{6}{3}\right]$$

2. Words not containing some strings.

How many permutations of 26 letters of English alphabet do not contain the strings 'bus', 'road', ~~grain~~ 'gem' ?

S = all permutations of 26 letters

$|S| = 26!$

C_1 : A permutation contains 'bus'

C_2 : A permutation contains 'road'

C_3 : A permutation contains 'gem'.

$N = 26!$

$\bar{N} = N(\bar{C}_1\bar{C}_2\bar{C}_3) = ?$

$N(C_1) = 24!$ $\qquad N(C_1C_2) = 21!$

$N(C_2) = 23!$ $\qquad N(C_1C_3) = 2\ !$

$N(C_3) = 24!$ $\qquad N(C_2C_3) = 21!$

$N(C_1C_2C_3)$

$C_1 : \boxed{bus}, \underbrace{a, c, d, \ldots, q, r, t, \ldots, w, y, z}_{23}$

$N(C_1) = 24!$

$C_2 : \underbrace{\boxed{road}}_{1}, \underbrace{b, c, e, f, \ldots, n, p, q, s, \ldots, z}_{26 - 4 = 22}$ $\Rightarrow N(C_2) = 23!$

$C_1C_2 : \underbrace{\boxed{bus}}_{1}, \underbrace{\boxed{road}}_{1}, \underbrace{c, e, \ldots, z}_{19}$

$\therefore N(C_1C_2) = 21!$

$\therefore$ in a similar manner, $N(C_1C_3) = 22!$

and $N(C_2C_3) = 21!$

$C_1C_2C_3 : \underbrace{\boxed{bus}}_{1}, \underbrace{\boxed{road}}_{1}, \underbrace{\boxed{gem}}_{1}, \underbrace{c, \ldots, z}_{16}$

$\therefore N(C_1C_2C_3) = 19!$

$\bar{N} = N(\bar{C}_1\bar{C}_2\bar{C}_3) = 26! - [2(24!) + 23!] + [2(21!) + 22!] - 19!$

ans

3. In how many ways can 3 x's, 3y's and 3 z's be arranged so that no consecutive triple of same letter appears?

C_1: arrangement where xxx appears

C_2: arrangement where yyy appears

C_3: arrangement where zzz appears.

$N = \frac{9!}{3!\,3!\,3!} = \frac{9!}{(3!)^3}$

$N(C_1) = \frac{7!}{3!\,3!}$ $\qquad$ $\underline{y}\,\underline{z}\,\underline{xxx}\,\underline{y}\,\underline{z}\,\underline{y}\,\underline{z} \rightarrow$ 7 objects

$= N(C_2) = N(C_3)$

$N(C_1C_2) = \frac{5!}{3!} = N(C_2C_3)$ $\qquad$ $\underline{z}\,\underline{xxx}\,\underline{yyy}\,\underline{z}\,\underline{z} \rightarrow$ 5 objects

$= N(C_1C_3)$

$N(C_1C_2C_3) = 3!$ $\qquad$ $\underline{xxx}\,\underline{yyy}\,\underline{zzz} \rightarrow$ 3 objects

$N(\bar{C}_1\bar{C}_2\bar{C}_3) = N - [N(C_1) + N(C_2) + N(C_3)] + [N(C_1C_2) + N(C_2C_3) + N(C_3C_1)] - N(C_1C_2C_3)$

$= \frac{9!}{(3!)^3} - 3\left[\frac{7!}{(3!)^2}\right] + 3\left[\frac{5!}{3!}\right] - 3!$ ans

4. How many integers from 1 to 100 are not multiples of 2 and 3?

A = set of I which are multiples of 2

B = set of integers which are multiples of 3

$N(C_1) = 50$

$N(C_2) = 33$ $\qquad (3, 6, 9, \dots 99)$

$N(C_1C_2) = 16$ $\qquad (6, 12, 18, \dots, 96)$

come from applying floor function $(\lfloor\ \rfloor)$

$N(\bar{C}_1\bar{C}_2) = N - [N(C_1) + N(C_2)] + N(C_1C_2)$

$= 100 - 83 + 16$

$= 33$ integers are not multiples of 2 and 3.

→

5. How many +ve integers from 1 to 2000 aren't divisible by 5, 7 or 11 ?

$S = \{1, 2, 3, 5, \ldots, 2000\}$

C_1 : $n \in S$ is divisible by 5

C_2 : $n \in S$ is divisible by 7

C_3 : $n \in S$ is divisible by 11

$$N(\bar{C}_1\bar{C}_2\bar{C}_3) = 2000 - [400 + 285 + 181] + [\lfloor 2000/35 \rfloor + \lfloor 2000/77 \rfloor + \lfloor 2000/55 \rfloor] - \lfloor 2000/385 \rfloor$$

$$= 2000 - [400 + 285 + 181] + [57 + 25 + 36] - 5$$

$$= 2000 - 866 + 118 - 5 = 1247$$

6. A dog or a cat :

Consider a community where there are 50 people, 30 own a pet dog, 25 own a pet cat and 10 people own both pet dog and cat. How many of them do not own anything ?

$$N(D \cup C) = 30 + 25 - 10$$
$$= 45$$

$\therefore$ People who do not own anything $= N - N(D \cup C)$
$= 5$

7. Consider a set of 100 people, 35 like muffins, 30 like brownie, 30 like cookies. 9 like M & B, 11 like B & C, 10 like M & C. 5 like all three. How many do not like either.

$N(\bar{C}_1\bar{C}_2\bar{C}_3) = N - N(C_1 \cup C_2 \cup C_3)$

$N(C_1 \cup C_2 \cup C_3) =$

$N(C_1C_2C_3) = N - N(C_1C_2C_3$

$N(\bar{C}_1\bar{C}_2\bar{C}_3) = N - |C_1 \cup C_2 \cup C_3|$

$|C_1 \cup C_2 \cup C_3| = 35 + 30 + 30 - (9 + 11 + 10) + 5$
$= 70$

$\therefore N(\bar{C}_1\bar{C}_2\bar{C}_3) = 100 - 70$
$= 30$

8. Find the number of integer solutions to the equation

$$x_1 + x_2 + x_3 = 7;$$

$0 \le x_1 \le 2,\ 0 \le x_2 \le 4,\ 0 \le x_3 \le 3.$

$$N = \binom{3+7-1}{7} = \binom{9}{7} = 36 \longrightarrow \text{all possibilities}$$

C_1 : solutions where $x_1 \ge 3$

C_2 : solutions where $x_2 \ge 5$

C_3 : solutions where $x_3 \ge 4$

$$N(C_1) = \binom{3+4-1}{4} \qquad x_1 + x_2 + x_3 = 4 \quad (x_1, x_2, x_3 \ge 0)$$

$$= \binom{6}{4} = \frac{6 \times 5}{2!} = 15$$

$$N(C_2) = \binom{3+2-1}{2} = \binom{4}{2} = 6 \qquad x_1 + x_2 + x_3 = 2 \quad (x_1, x_2, x_3 \ge 0)$$

$$N(C_3) = \binom{3+3-1}{3} = \binom{5}{3} = 10 \qquad x_1 + x_2 + x_3 = 3 \quad (x_1, x_2, x_3 \ge 0)$$

$N(C_1C_2) = 0$ (as $3 + 5 + x_3 = 7$ ×)

$N(C_2C_3) = 0$ (as $x_1 + 5 + 4 \ne 7$)

$N(C_1C_3) = 1$ (as $3 + x_2 + 4 = 7$, only if $x_2 = 0$)

$N(C_1C_2C_3) = 0$

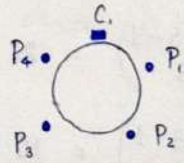
$$N(\bar{C}_1\bar{C}_2\bar{C}_3) = 36 - (15 + 6 + 10) + (0 + 0 + 1) - 0$$
$$= 6$$

9. C_1 : couple 1 sit beside each other

C_2 : couple 2 sit beside each other (3 couples $\Rightarrow$ 6 people)

C_3 : couple 3 sit beside each other

$N(C_1) \ne 5!$ (5 different possible permutations)



but we're overcounting here.

as. for, eg. $C_1P_1P_2P_3P_4 \equiv P_4C_1P_1P_2P_3$

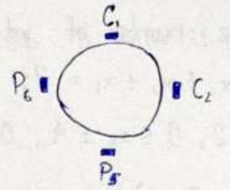
as we know, number of circular permutations of n objects $= (n-1)!$

$\therefore\ N(C_1) = (4!) \cdot 2$

similarly, $N(C_2) = N(C_3) = (4!) \cdot 2$ (multiplying by 2, because C_1 can sit as WH or HW)

$N(C_1C_2) = 3! \times 2 \times 2$

as 4 distinct objects can be arranged in 3! ways



2 ways for 2 couples : HW
WH

similarly, $N(C_2C_3) = N(C_1C_3) = 2^2 \times 3!$

$N(C_1C_2C_3) = 2! \times 2^3$
$= 2^4$

now, $N(\bar{C_1}\bar{C_2}\bar{C_3}) = 5! - [3 \cdot 2 \cdot 4!] + [3 \cdot 2^2 \cdot 3!] - 2^4$
$= 120 - 144 + 72 - 16$
$= 32$ ways

10. How many integer solutions are there for $x+y+z = 20$,
$x < 7$, $y < 8$ and $z < 9$?

All possible sol^n = $\binom{3+20-1}{20} = \binom{22}{20} = 231$

C_1: solutions where $x \geq 7$
C_2: solutions where $y \geq 8$
C_3: solutions where $z \geq 9$

$N(\bar{C_1}\bar{C_2}\bar{C_3}) = ?$

$N(C_1) = \binom{13+3-1}{13} = \binom{15}{13} = 105$ $\quad x_1+y+z = 13 ;\ x_1, y_2, z_3 \geq 0$

$N(C_2) = \binom{3+12-1}{12} = \binom{14}{12} = 91$ $\quad x+y+z = 12 ;\ x, y, z \geq 0$

$N(C_3) = \binom{3+11-1}{11} = \binom{13}{11} = 78$ $\quad x+y+z = 11 ;\ x, y, z \geq 0$

$N(C_1C_2) = \binom{3+5-1}{5} = 21$

$N(C_2C_3) = \binom{3+3-1}{3} = 10$

$N(C_1C_3) = \binom{3+4-1}{4} = 15$

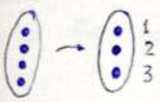
$N(C_1C_2C_3) = 0$ (from previously applied logics)

$\therefore N(\bar{C}_1\bar{C}_2\bar{C}_3) = 231 - (105 + 91 + 78) + (21 + 10 + 15) - 0$

$= 3$

II. Number of Onto Functions

all possible functions from



3^4

how many of them are onto ?

C_1 : first element is left out (1)

C_2 : 2 is left out

C_3 : 3 is left out

$N(C_1) = 2^4 = N(C_2) = N(C_3)$

$N(C_1C_2) = 1^4 = N(C_2C_3) = N(C_1C_3)$

$\bar{N} = 3^4 - \binom{3}{1}2^4 + \binom{3}{2}1^4$

$= 81 - 48 + 3 = 36$ (total possible onto functions from 4 elements to 3 elements)

Formula:

In general, number of onto functions from a set of m elements to a set of n elements is :

$$\binom{n}{0}n^m - \binom{n}{1}(n-1)^m + \binom{n}{2}(n-2)^m - \binom{n}{3}(n-3)^m + \binom{n}{4}(n-4)^m - \dots + (-1)^n\binom{n}{n}(n-n)^m$$

$$= \sum_{i=0}^{n}(-1)^i\binom{n}{i}(n-i)^m$$

11. How many onto functions are possible from a set of 7 elements to a set of 5 elements ?

$|A| = 7, \quad |B| = 5$

number of onto functions from A to B are :

$$5^7 - \binom{5}{1}4^7 + \binom{5}{2}3^7 - \binom{5}{3}2^7 + \binom{5}{4}1^7 - \binom{5}{5}0^7$$

$= 78125 - 81920 + 21870 - 1280 + 5 = 16800$

$\longrightarrow$

12. Assume there are 5 houses and 5 persons. In how many ways person i not go to house i ?

1	2	3	4	5	
H_1	H_2	H_3	H_4	H_5	($\times$ because $i \not\to H_i$)
H_4	H_1	H_5	H_3	H_2	$\rightarrow$ in how many ways such arrangements can be done ?

C_1 : 1 goes to H_1

C_2 : 2 goes to H_2

and so on....

C_5 : 5 goes to H_5

$N(C_1) = 4!$

$= N(C_2) = N(C_3) = N(C_4) = N(C_5)$

$\underbrace{\boxed{1}\,\boxed{}\,\boxed{}\,\boxed{}\,\boxed{}}_{\text{permutate}}$

similarly,

in C_1C_2, we fix two people, 1 and 2

$\therefore \; N(C_1C_2) = 3!$

$= N(C_1C_3) = N(C_1C_4) = N(C_1C_5) = N(C_2C_3) = N(C_2C_4) = N(C_2C_5) = N(C_3C_4)$
$= N(C_3C_5) = N(C_4C_5)$

$N(C_1C_2C_3) = 2!$ (in a similar manner)

$N(C_iC_jC_k) = 2!$ $\quad 1 \le i, j, k \le 5$

$N(C_1C_2C_3C_4) = 1$ (same for other four cases involving four people)

$N(C_1C_2C_3C_4C_5) = 1$

$$\bar{N} = N - \sum_{i=1}^{5}(N(C_i)) + \sum N(C_iC_j) - \sum N(C_iC_jC_k) + \sum N(C_iC_jC_kC_l) - \sum N(C_iC_jC_kC_lC_m)$$

$$= 5! - \binom{5}{1}4! + \binom{5}{2}3! - \binom{5}{3}2! + \binom{5}{4}1! - \binom{5}{5}0!$$

$$= 5! - \frac{5!}{1!\,4!}4! + \frac{5!}{2!\,3!}3! - \frac{5!}{3!\,2!}2! + \frac{5!}{4!\,1!}1! - \frac{5!}{5!\,0!}0!$$

$$= 5!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right]$$

— in these many ways, a person can't go to his house

$\longrightarrow$

III. Derangements

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 2 & 3 & 4 & 5 \end{array}$$

← none of these should happen, i.e, no number should be below itself

Similar as the previous problem.

In how many ways can these numbers not be in their own place?

C_1: 1 is in the first place
C_2: 2 is in the second place
C_5: 5 is in the fifth place

$\overline{N} = N(\overline{C_1}\,\overline{C_2}\,\overline{C_3}\,\overline{C_4}\,\overline{C_5}) = ?$

$N(C_1) = N(C_2) = N(C_3) = N(C_4) = N(C_5)$
$N(C_1) = 4!$

$N(C_1C_2) = 3!$
$N(C_1C_2C_3) = 2!$
$N(C_1C_2C_3C_4) = 1!$
$N(C_1C_2C_3C_4C_5) = 0!$

$$\overline{N} = 5! - 4! \times \binom{5}{1} + 3! \times \binom{5}{2} - \binom{5}{3}2! + \binom{5}{4}1! - \binom{5}{5}0!$$

$$= 5! - \frac{5!}{1!\,4!}4! + \frac{5!}{2!\,3!}3! - \frac{5}{3!\,2!}2! - \frac{5}{4!\,1!}1! - \frac{5!}{5!\,0!}0!$$

$$= 5!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right]$$

$$= 5! \times \frac{1}{e}$$

* for sufficiently large n, $\overline{N} = \frac{n!}{e}$

$e \approx 2.71$, $\frac{1}{e} < 0.5$

$\overline{N} = \frac{5!}{e}$ → no. of permutations where nothing is in the right place.

→

Derangements of 4 numbers

$\boxed{1}$	$\boxed{2}$	$\boxed{3}$	$\boxed{4}$
	1	1	1
2		2	2
3	3		3
4	4	4	

} allowed positions for numbers

There are 9 possible permutations for this, where i is not at the ith position.

13. There are 3 different colored bottles, each with a distinct cap. In how many ways can these caps be put on the bottles, so that none of the caps are on the correct bottle?

C_1 : B_1 has its cap on itself
C_2 : B_2 has its cap on itself
C_3 : B_3 has its cap on itself

all possible permutations $= 3!$

$N(C_1) = 2! = N(C_2) = N(C_3)$

$N(C_1C_2) = 1 = N(C_2C_3) = N(C_1C_3)$

$N(C_1C_2C_3) = 1$

$$N(\overline{C_1}\,\overline{C_2}\,\overline{C_3}) = \cancel{N(C_1)}\ N - [N(C_1) + N(C_2) + N(C_3)] + [N(C_1C_2) + N(C_2C_3) + N(C_1C_3)] - N(C_1C_2C_3)$$

$$= 3! - \left[\binom{3}{1} 2!\right] + \binom{3}{2} 1! - \binom{3}{3} 0!$$

$$= 3! - \left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!}\right] = 2 \text{ ways to do so.}$$

14. A teacher gives a test to 4 students. He wants them to calculate each other's test paper, without grading their own paper. In how many ways can he give the test papers to students?

number of ways $= 4! \left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right]$

$= 9$ ways.

15. In how many ways can integers 1, 2, 3, ..., 10 be arranged such that no even integer is in its natural place ?

$$\begin{array}{cccccccccc} 1 & \underline{2} & 3 & \underline{4} & 5 & \underline{6} & 7 & \underline{8} & 9 & \underline{10} \\ & 6 & & 8 & & 2 & & 10 & & 4 \\ & 8 & & 2 & & 10 & & 4 & & 6 \end{array} \Bigg\} \text{ possible arrangements of even nos}$$

C_1 : 2 is in its own position

C_2 : 4 is in its own position

⋮

C_5 : 10 is in its own position

$N(\bar{C}_1\bar{C}_2\bar{C}_3\bar{C}_4\bar{C}_5) = ?$

N = all possible permutations = $10!$

$N(C_1) = N(C_2) = N(C_3) = N(C_4) = N(C_5) = 9!$

$N(C_iC_j) = 8!$

$N(C_iC_jC_k) = 7!$

$N(C_iC_jC_kC_l) = 6!$ $\qquad 1 \le i, j, k, l, m \le 5$

$N(C_iC_jC_kC_lC_m) = 5!$

$$N(\bar{C}_1\bar{C}_2\bar{C}_3\bar{C}_4\bar{C}_5) = 10! - \binom{5}{1}9! + \binom{5}{2}8! - \binom{5}{3}7! + \binom{5}{4}6! - \binom{5}{5}5!$$

$$= 10! + \binom{5}{1}(6! - 9!) + \binom{5}{2}(7! - 8!) - \binom{5}{0}5!$$

16. For integers 1, 2, 3, ..., n-1, n there are 11660 derangements, where 1, 2, 3, 4, 5 appear in the first 5 positions. What is the value of n?

$$\underline{1}\ \underline{2}\ \underline{3}\ \underline{4}\ \underline{5}\ \underbrace{\_\ \_\ \_\ \_\ \_\ \_}_{n-5 \text{ integers}}$$

$D(5) = \dfrac{5!}{e}$ $\qquad\qquad D(n-5) = \dfrac{(n-5)!}{e}$

↳ derangements of 5 integers

$D(5) \times D(n-5) = 11660$

$D(n-5) = \dfrac{11660 \times e}{5!}$

$\approx D(6)$

$\Rightarrow D(n-5) = D(6) \Rightarrow n = 11$

17. In how many ways can we devise a secret code by assigning to each letter of the alphabet a different letter to represent it?

a → m
b → k
c → z
⋮
and so on...

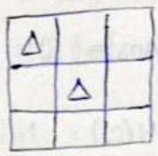
$N(\bar{C}_1\bar{C}_2\bar{C}_3\bar{C}_4 \dots \bar{C}_{26}) \simeq D(26) = \frac{26!}{e}$

IV. Rook Polynomial

Polynomials in chess board.

Pawn : rook

non-taking rooks



3 × 3 chessboard

no. of ways to place 1 rook = 9

no. of ways of placing 2 rooks such that no two rooks are on the same row or column = 12 ways

(4 ways for each cell in a row/column)

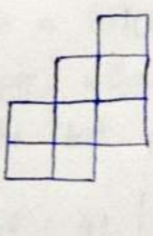
no. of ways of placing 3 rooks in a non-taking fashion = 6

Q. Given a chessboard, in how many ways can we place –

a) 1 rook
b) 2 rooks
c) 3 rooks

$\Big\}\ 1 + 9x + 12x^2 + 6x^3$ → rook polynomial of a given chessboard

Q. Number of ways in which you can place rooks on the given chessboard?



Pls. watch NPTEL video :)

—x—