Week 5

I. Introduction to Functions

$$f : X \rightarrow Y$$

domain (X) codomain (Y)

example: $f : Z \rightarrow Z^{+} \cup \{0\}$
$f(x) = x^2$

here, domain $= \{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$
codomain $= \{0, 1, 2, 3, 4, \ldots\}$

Subset of domain is mapped by elements of domain.

→ This subset is known as the range of the function f.

→ Example : $f(2) = 4$, here 4 is called the image of 2, and 2 is called the pre-image of 4.

Difference b/w Relations & Functions:

$f : R \rightarrow R, \; f(x) = \sqrt{x}$

→ Is this a function ? No

because in a function, each element on the ~~right~~ left side is mapped to one & only one element on the right side. But here, as we can see, 4 is a pre-image of both ± 2.

∴ This is a relation & not a function (violates the rules of functions).

II. One - to - One Function

A function in which each element in the domain of the function is associated w/ only one element in the range For example, cube of a number is unique.

One - one function states :-

Some elements can be left behind in the co-domain. But the elements in the domain should have a unique element in the co-domain.

Q. Given a function f, how do you show that it is one - one ?

Proof technique:

$f : Z^{+} \rightarrow Z^{+}, \; f(\alpha) = 7\alpha$

→

$f(1) = 7, \; f(2) = 14, \ldots$

Is this a one-one function?

When will a function not be one-one?

when $f(\alpha) = f(\beta)$

$\Rightarrow 7\alpha = 7\beta \Rightarrow \alpha = \beta$

This contradicts that α and β are different.

Thus, the function is a one-one function.

Hence Proved

$f: X \rightarrow Y$

To show that f is one-one.

Assume $f(\alpha) = f(\beta)$

Show that $\alpha = \beta$

Proof that f is one-one.

→ $f: N \rightarrow N, \; f(x) = 2x$

Proof like above. This is a one-one function.

→ $f: R \rightarrow R, \; f(x) = 1 - x$

$x_1, x_2 \in R, \; f(x_1) = f(x_2)$ ← assuming f isn't one-one

$\Rightarrow 1 - x_1 = 1 - x_2 \Rightarrow x_1 = x_2$

$\therefore$ f is one-one.

→ $f: R \rightarrow Z, \; f(x) = [x]$ ⟶ greatest integer / ~~ceil~~ function

$f(0.5) = 0$

$f(1) = 1$

$f(1.345) = 1$

$f(1.5) = 1$

f is not one-one.

<u>Cardinality in a One-One Function</u>

In an office, every male is married. Wife also works in the same organization.

→ What can we say about no. of males & no. of females in the office.

number of males $\leq$ number of females

↳ females can be single.

This is a one-one function.

When f is one-one, the co-domain may have left-overs.

$|M| \leq |W|$

When a function f is one - one, from a domain to a co-domain, then,

$|\text{domain}| \leq |\text{co-domain}|$

III. Onto Function

A function in which range = co-domain, or every element in the co-domain has an image in the domain.

→ $f: R_+ \rightarrow R_+$, $f(x) = |\sqrt{x}|$, for every $x \in R$, $x^2 \in R$

Onto function: for any $y \in R^+$, there exists an x such that $f(x) = y$.

or

For every y in the codomain, there exists some x in the domain, such that $f(x) = y$.

Examples:

→ $f: N \rightarrow N$. $f(x) = x$ (Identity function)

$f(1) = 1$, $f(2) = 2$, $f(3) = 3$, ...

$y \in N$

Preimage is the number itself.

Hence, function is onto.

→ $f: Z \rightarrow Z$, $f(x) = |x|$

$f(-1) = 1$, $f(1) = 1$

$f(-2) = 2$, $f(2) = 2$

$f(-3) = 3$, $f(3) = 3$

Range $= \{1, 2, 3, 4, \ldots\} = N$

$Z \neq N$

∴ f is not onto

→ $f: W \rightarrow W$, $f(x) = \begin{cases} x-1, & x \text{ is odd} \\ x+1, & x \text{ is even} \end{cases}$

$f(0) = 1$, $f(1) = 0$

$f(2) = 3$, $f(3) = 2$

$f(4) = 5$, ...

$y \in W$, $y = x-1$ or $y = x+1$

$x = y+1$ or $x = y-1$

Hence, y is onto.

Cardinality Condition in an Onto Function

→ f: Person → Day of birth (not m, y) (Persons in a class room)

Ram → 5

Priya → 19

John → 4

f is onto.

∵ codomain = $\{1, 2, 3, 4, \ldots, 31\}$

∴ number of people in the classroom ≥ 31

IV. Bijection

→ $f: A \to B$ is called a bijection, iff it is

- one - one
- onto

Any function that is both one-one & onto.

→ Not a bijection if: either it is not one-one or onto.

Example:

1. Mapping from the set of all 2 digit binary numbers to subsets of $\{1, 2, 3\}$.
2. f: {Jan, Feb, March, April, May, June, July, August, Sept, Oct, Nov, Dec} → $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

f(Jan) = 1, f(Feb) = 2, f(March) = 3

$M_1, M_2, M_3, \ldots, M_{12}$

or $f(M_i) = i$

∴ f is both one-one & onto, and hence bijective.

3. $f: Z \to Z$, $f(x) = x^2$

$f(0) = 0$ $f(1) = 1$ $f(-2) = 4$

$f(-1) = 1$ $f(2) = 4$

f is not one-one.

f is not onto (ex. 3 doesn't have a pre-image).

∴ f is not a bijection.

4. $f: \mathbb{Q} \to \mathbb{Q}$, $f(p/q) = p$

$f(1/2) = 1$, $f(1/3) = 1$

$\therefore$ f is not one-one

$\Rightarrow$ f is not bijection.

<u>Cardinality Condition in a Bijection</u>

→ What is the cardinality of A and B if $f: A \to B$ is a bijection ?

$|A| \geq |B|$ if f is one-one

$|B| \leq |A|$ if f is onto

$\therefore$ For a bijection,

$|A| = |B|$

→ $|X| = |Y|$

Give a bijection.

Then, you are proving they are of same cardinality.

Conversely, if they're of the same cardinality, give a function f which is bothe one-one and onto.

→ 50 chocolates to the class.

1 chocolate to each student.

At the end, no chocolate remains.

<u>50 chocolates</u> $\xrightarrow{\text{bijection}}$ <u>students</u>

$\therefore$ no. of students = 50.

V. <u>Counting the Number of Functions</u>

f: {Priya, Rita, Julie} → {Red, blue}

→ In how many ways, can you think of a function from $\{a, b, c\}$ to $\{1, 2\}$?

one element can go to only one element.

more than one elements can go to the same element.

Priya Rita Julie

R B R B R B

$\therefore$ $2 \times 2 \times 2 = 8$ possibilities

→ same as all possible 3 digit binary numbers.

1. Number of One - One Functions

Total possible one-one functions from $\{a, b, c\}$ to $\{1,2,3\}$?

$\equiv$ In how many ways can you arrange $\{1,2,3\}$?

$= 3! = 6$

In how many ways can I consider all possible one-one functions from $\{a, b, c, d\}$ to $\{1, 2, 3, 4, 5, 6\}$?

$\equiv$ In how many ways can 6 people get together and decide to take a photograph consisting of 4 people ?

$= {}^6P_4$

2. Number of Onto Functions

Total possible onto functions from a set with tree elements to a set with two elements?

8 total possibilities - 2 not onto functions

= 6 possible onto functions

Total possible onto functions from a domain with m elements to a domain with n elements :

Principle of Inclusion and Exclusion

3. Number of Bijections

Total possible functions from a domain of cardinality m to a codomain of cardinality n

$= n^m$

Total possible functions from a domain of cardinality m to a codomain of cardinality n that are bijections :

bijection $\Rightarrow n = m$

$\Rightarrow$ Total no. of functions $= m^m = n^n = m!$

$\longrightarrow$

VI. Composition of Functions

f: student name → date of birth $(A \to B)$

g: date of birth → day of birth $(B \to C)$

composition of f & g ?

= day of birth of a student

g of = day of birth of a person

= people → day of birth

→ composition is denoted by $g \cdot f$

→ $f: X \to Y$, $g: Y \to Z$

$g \circ f : X \to Z$

$\because g \circ f = g(f(x))$

→ Airlines – offer for couples only

If husband's age + wife's age ≥ 50

then they get 50% discount

$\boxed{\text{husband's age \& wife's age}} \longrightarrow f \longrightarrow$ husband's age + wife's age

husband's age + wife's age $\longrightarrow g \longrightarrow \begin{cases} \text{discount}, & H+W \geq 50 \\ \text{ineligible}, & H+W < 50 \end{cases}$

→ composition may be defined w/ any number of functions.

→ $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$

$g: \mathbb{R} \to \mathbb{Z}$, $g(x) = |x|$

$f \circ g = ?$

$f \circ g(x) = f(g(x)) = f(|x|) = (|x|)^2 = x^2$

→ Is $g \circ f$ same as $f \circ g$?

$g(f(x)) = |(x)^2| = |x^2| = x^2$

Hence, $g \circ f \equiv f \circ g$.

→ $f: Z \to Z$, $f(x) = x^2 + 1$

$g: Z \to Z$, $g(x) = 3x$

$f \circ g$ and $g \circ f$?

$g \circ f(x) = g(f(x)) = 3(x^2+1) = 3x^2 + 3$

$f \circ g(x) = f(g(x)) = (3x)^2 + 1 = 9x^2 + 1$

VII. Inverse Functions

Invertibility of a function :

$f: X \to Y$

$\alpha \to \beta$

Inverse of a function is always defined from co-domain to the domain.

$f^{-1}: Y \to X$

* A function's inverse can be talked about, provided, every element in the co-domain has a pre-image.
  - → f is one-one (pre-image should be only 1 for an element in the co-domain)
  - → f is onto

Examples :

→ $f: Z \to Z$, $f(x) = 3x + 2$

$y = 3x + 2$

$\Rightarrow x = \frac{y-2}{3}$

$\therefore f^{-1}(y) = \frac{y-2}{3}$

→ $f: R \to Z$, $f(x) = [x]$

$f(0.32) = 0$

$f(0.5) = 0$

$f(1.798) = 1$

$f(1.45) = 1$

$f(1.32) = 1$

Finding inverse will not be possible.

$f^{-1}: Z \to R$ is not a function

$\therefore f$ is not Invertible.

→ $f : R \to R, \ f(x) = x/2$

$\therefore f^{-1}(y) = 2y$

Applications of Inverse Functions :

→ Cryptography

→ Inverse trigonometric function

→ Finding range