Week 9

I. Counting in a Creative Way

Qu S T
a i
d ck

Pick one entity per line and make a word. → not necessarily valid

⇒ Quad, Quack, Quid, Quick
Sad, Sack, Sick, Sid
Tad, Tid, Tack, Tick

12 words out of which 4 are 3-lettered
6 are 4-lettered
and 2 are 5-lettered

Q. Count all possible 3-4 and 5-lettered words in general?

Words and the Polynomial

$(2x+x^2)(2x)(x+x^2)$
$= 4x^3 + 6x^4 + 2x^5$ → some relation w/ lettered polynomials

Concept - Relation b/w Words & Polynomial

Qu	S	T	$x \cdot x + x + x$
a	i		$x + x$
d	ck		$x + x \cdot x$

$\left.\begin{matrix} x\cdot x + x + x \\ x + x \\ x + x \cdot x \end{matrix}\right\}$ multiplying

$(x^2+2x)(2x)(x+x^2) = 4x^3 + 6x^4 + 2x^5$
↓
Picking one x (or letter) from each line, and so on...

k lettered word
$(\ )\cdot(\ )\cdot(\ )$ → find x^k.

Picking Up 5 Balls

B B B B B
R R R R R
G G G G G

→ Pick 5 balls w/ at least 1 B, 1R and 1 G example: B R R G G

Polynomial: $(x+x^2+x^3)(x+x^2+x^3)(x+x^2+x^3)$?

Blue	Red	Green
1	2	2
2	1	2
2	2	1
1	1	3
3	1	1
1	3	1

now, $(x+x^2+x^3)^3 = x^3 + 2x^4 + 3x^5 + 2x^6 + x^7 + x^4 + 2x^5 + 3x^6 + 2x^7 + x^8 + x^5$
$+ 2x^6 + 3x^7 + 2x^8 + x^9$
$= x^3 + 3x^4 + \underline{6x^5} + 7x^6 + 6x^7 + 3x^8 + x^9$

we need only 5 balls,
of which there are 6 ways to do.

number of ~~picking~~ ways of picking up 5 balls $\simeq$ coefficient of x^5 in $(x+x^2+x^3)^5$

Another version of picking up 5 balls

Q. 5R, 5B, 5 green balls. Pick 5 balls from given 15 balls, without any constraint.

5 0 0	4 1 0	1 2 2	1 1 3
0 5 0	4 0 1	2 1 2	1 3 1
0 0 5	0 4 1	2 2 1	3 1 1

Polynomial: $(x^0+x+x^2+x^3+x^4+x^5) \cdot (x^0+x+x^2+x^3+x^4+x^5) \cdot (x^0+x+x^2+x^3+x^4+x^5)$
$(x^0+x+x^2+x^3+x^4+x^5)$
number of ways = coefficient of x^5 in above polynomial = 21

II. Generating Functions

$1 + x + x^2 + x^3 + \dots$ $\longrightarrow$ Polynomial if stops at a finite stage
$= \frac{1}{1-x}$, $|x| < 1$

$\rightarrow$ Polynomial = $a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots + a_n x^n$

$\frac{d}{dx}(1-x)^{-1} = \frac{d}{dx}(1+x+x^2+x^3+\dots)$

$\Rightarrow \frac{1}{(1-x)^2} = 0 + 1 + 2x + 3x^2 \dots$

$\frac{1}{1-x}$ generates 1, 1, 1, ... (coefficients are 1)

$\frac{1}{(1-x)^2}$ generates 1, 2, 3, 4, ... (coefficients are 1, 2, 3, ...)

$\underbrace{\frac{1}{(1-x)^2}}_{\text{generating function of infinite sequence } 1, 2, 3, 4, \ldots}$

generating function of infinite sequence 1, 2, 3, 4, ...

Examples :

Q Generating function of $1^2, 2^2, 3^2, \ldots$?

$$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + \ldots$$

$$\Rightarrow \frac{d}{dx}\left(\frac{x}{(1-x)^2}\right) = \frac{d}{dx}(x + 2x^2 + 3x^3 + \ldots)$$

$$\Rightarrow \frac{x+1}{(1-x)^3} = 1 + 4x + 9x^2 + 16x^3 + \ldots$$

$$= 1^2 + 2^2x + 3^2x^2 + 4^2x^3 + \ldots$$

$\therefore \frac{x+1}{(1-x)^3}$ is the generating function of $1^2, 2^2, 3^2, \ldots$

$\rightarrow \frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$

substituting $x = 2y$

$$\frac{1}{1-2y} = 1 + 2y + 4y^2 + 2^3y^3 + 2^4y^4 + \ldots$$

$\frac{1}{1-2y}$ generates $2^0, 2^1, 2^2, 2^3, \ldots$

similary, $\frac{1}{1-3y}$ gives $3^0, 3^1, 3^2, 3^3, \ldots$

and similarly for $\frac{1}{1-ax}$ generates $a^0, a^1, a^2, a^3, a^4, \ldots$

$$= 1 + ax + a^2x^2 + a^3x^3 + \ldots$$

$\rightarrow \frac{2}{1-x} = 2 + 2x + 2x^2 + 2x^3 + \ldots$

similarly, for $\frac{k}{1-x} = k + kx + kx^2 + kx^3 + \ldots$

generates k, k, k, k, ...

$\rightarrow \frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 \ldots$

$\longrightarrow$

$$\frac{1}{1+x} + \frac{1}{1-x} = 2 + 2x^2 + 2x^4 + 2x^6 + \dots$$

$$\Rightarrow \frac{2}{1-x^2} = 2 + 2x^2 + 2x^4 + 2x^6 + \dots$$

$$\Rightarrow \frac{1}{1-x^2} = x^0 + x^2 + x^4 + x^6 + \dots$$

Challenge: How do we obtain $x + x^3 + x^5 + x^7 + \dots$

by doing $\frac{1}{1+x} - \frac{1}{1-x}$

II. Binomial Expansion - A Generating Function

Generating function of $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \dots, \binom{n}{n}, 0, 0, 0$

$= (1+x)^n$

$= \binom{n}{0}x^0 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \dots$

Picking 7 balls:

Q. A big basket contains several blue, green and red balls. In how many ways can we pick 7 balls of red, blue and green colour?

recall: Combinations w/ repetitions.

3 colors — 7 balls to be picked out of R, G, B

$c_1 + c_2 + c_3 = 7$

no. of solutions to this equation = ?

$n = 3, \quad r = 7$

$$\binom{n+r-1}{r} = \binom{3+7-1}{7} = \binom{9}{7}$$

$$= \frac{9 \times 8}{2} = 36 \text{ ways}$$

in which we can pick up 7 balls of R, B, G.

Creative Way of Picking 7 Balls

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{n}x^n$$

r^{th} term $= \binom{n}{r}x^{r-1}$

$$(1-x)^n = \binom{n}{0} - \binom{n}{1}x + \binom{n}{2}x^2 - \dots + (-1)^n\binom{n}{n}x^n$$

r^{th} term $= (-1)^r \binom{n}{r} x^r$

$$(1-x)^{-n} = 1 + nx + \frac{(-n)(n-1)}{2}x^2 + \dots + \underbrace{\frac{(+1)^r n(n-1)(n-2)\dots(n-r+1)}{r!}x^r}_{\binom{n+r-1}{r}x^r} + \dots + (-1)^n x^n$$

$\boxed{\text{coefficient of } x^r}$ in the expansion of $(1-x)^{-n}$

Q. What is the total possible ways in which you can pick 7 balls ?

ans coefficient of x^7 in the expansion of

$$(1+x+x^2+x^3+x^4+x^5+x^6+x^7)^3$$

$=$ coefficient of x^7 in the expansion of

$$(1+x+x^2+x^3+x^4+x^5+x^6+x^7+x^8+\dots)^3$$

$=$ coefficient of x^7 in the expansion of $\left(\frac{1}{1-x}\right)^3$

Generating Functions Problems

Q. Find the coefficient of x^5 in $(1-x)^{-6}$

Let $-x = y$

$$(1+y)^{-6} = \sum_{r=0}^{\infty} \binom{-6}{r} y^r = \sum_{r=0}^{\infty} \binom{-6}{r}(-x)^r$$

$$\binom{-n}{r} = (-1)^r \binom{n+r-1}{r}$$

$$\binom{-6}{5} = (-1)^5 \binom{6+5-1}{5} = (-1)^5 \binom{10}{5}$$

coefficient of $x^5 = (-1)^5 \binom{10}{5} \cdot (-1)^5$

$$= \frac{10!}{5!\,5!} = 252$$

Q. In how many ways can 15 identical chocolates can be distributed among 4 children so that each child gets at least 2 chocolates ?

number of ways in which one child can get chocolates :

$= x^2 + x^3 + x^4 + \dots + x^9$

number of ways in which 4 children can get chocolates :

$= (x^2 + x^3 + x^4 + \dots + x^9)^4$

coefficient of x^{15} in the product of

$(x^2+x^3+x^4+x^5+\ldots+x^9)^4$

$[x^2(1+x+x^2+x^3+\ldots+\ldots)]^4$

$= x^8\,(1+x+x^2+x^3+\ldots)^4 = x^8\left(\dfrac{1}{1-x}\right)^4 = x^8\,(1-x)^{-4}$

coefficient of x^7 in $(1-x)^{-4} = ?$

$$\sum \binom{-4}{r}(x)^r = \binom{-4}{7}(-1)^7$$

$$= (-1)^7\binom{4+7-1}{7}(-1)^7 = \frac{10!}{3!\times 7!}$$

$= 120$ ways in which chocolates can be distributed.

Q. $S = \{a, b, c\}$

$f(x) = (1+ax)(1+bx)(1+cx)$

$= 1 + ax + bx + cx + abx^2 + bcx^2 + acx^2 + abcx^3$

$= 1 + (a+b+c)x + (ab+bc+ac)x^2 + abc\,x^3$

$= x^0 + (b+a+c)x + (ab+bc+ac)x^2 + abc\,x^3$

$P(S) = \{\{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{c,a\}, \{a,b,c\}, \{\,\}\}$

(coefficients-related w/ above polynomial)

Generating function for subsets of S is:

$(1+ax)(1+bx)(1+cx)$

where $S = \{a, b, c\}$

<u>Why Generating Functions</u>?

→ Power of generating function

→ Technique is helpful

→ Challenge: Can you pick 7 balls of 1 R, 1 B and 1 G color.

↓

use generating functions

—— x x ——