Week 0 intro

Assignment 0:

1. $f(x) = \frac{2x+20}{x-2}$, $g(x) = 2x-3$, $f(g(3)) = ?$

$f(3) = 26$

2. How many 3 digit numbers greater than 500 can be formed using numbers 1, 3, 5, 7, 9, without repetition ?

$= 3 \times 4 \times 3 = 36$

3. Coefficient of x^4y^2 in the expression $(4x^2y + 24)^3$ is equal to ?

$(4x^2y + 24)^3 = {}^3C_0\,(4x^2y)^3 + {}^3C_1\,(4x^2y)^2(24) + {}^3C_2\,(4x^2y)(24)^2 + {}^3C_3\,(24)^3$

$\therefore$ coefficient of $x^4y^2 = {}^3C_1 \cdot 16 \cdot 24$

$= 3 \cdot 16 \cdot 24 = 1152$

4. P = prime numbers, Z = integers, then,

$P \cap Z = \{2, 3, 5, 7, 11, 13, 17, 19, \dots\}$

5. Which of the following is an empty set ?

a) $\{x : x$ is an even prime greater than $3\}$
b) $\{x \mid x \in Z;\ -1 \le x \le 1\}$
c) $\{x : x$ is an odd prime less than $5\}$
d) $\{\phi\}$

6. Which is a rational no. ?

$\sqrt{64} / \sqrt{25}$

7. A dice is thrown in the air. The probability of getting an odd number is ;

$P(\text{odd}) = \frac{3}{6} = \frac{1}{2}$

8. Sum of two no.s is 27 and their product is 182. The numbers are :

$$x + y = 27, \quad xy = 182$$

$$\Rightarrow x(27 - x) = 182$$

$$\Rightarrow x^2 - 27x + 182 = 0$$

$$\Rightarrow (x - 14)(x - 13) = 0$$

$\therefore$ numbers = 13, 14.

9. $f : R \rightarrow R$ is defined by

$$f(x) = \begin{cases} 2x \,; & x > 3 \\ x^2 \,; & 1 < x < 3 \\ 3x \,; & x \leq 3 \end{cases}$$

$$f(-1) + f(2) + f(4) = 3(-1) + (2)^2 + 2(4)$$

$$= -3 + 4 + 8 = 9$$

10. $x + \frac{1}{x} = 2$, then $x^7 + \frac{1}{x^7} = ?$

$$\left(x + \frac{1}{x}\right)^2 = x^2 + \frac{1}{x^2} + 2 = 4 \Rightarrow x^2 + \frac{1}{x^2} = 2$$

$$(x^2 + 1) = 2x \Rightarrow (x - 1)^2 = 0 \Rightarrow x = 1$$

$$\therefore x^7 + \frac{1}{x^7} = 2$$

— x —

Week 1

I. Rule of Sum & Rule of Product

If there are n choices for one action and m choices for another action and the two actions can't be done at the same time, then there are $m+n$ ways to choose one of these actions.

If there are n ways of doing something and m ways of doing the other, then there are $n \times m$ ways to do them both together.

Q. Alice wants to choose one book out of 7 sci-fi books, 5 mystery books, 2 journals. How many number of choices does Alice has ?

ans no. of choices $= 7+5+2 = 14$

Q. In a town of Germany, there are 8 newspapers and 4 magazines. Number of ways Peter can subscribe to both of them are ?

ans no. of ways $= 8 \times 4 = 32$

Q. 8 men and 6 women contest in an election.

a) ways to choose one leader $= 14$

b) ways to choose a male and a female leader each $= 48$

Q. How many +ve divisors does 2000 have ?

ans $2000 = 2^4 \times 5^3 = 2^a \cdot 5^b$

no. of divisors $= 4 \times 3 = 12$ ✗

$\because$ values of a and b can range from 0 to 4 and 0 to 3 respectively,

$\therefore$ no. of divisors $= 5 \times 4 = 20$.

स्वप्निल →

Q. Charlie visits an ice-cream shop, to buy one. There are 3 cones, 4 flavours & 2 toppings. Number of ways to buy an ice-cream = ?

ans no. of ways = $3 \times 4 \times 2 = 24$

ice-cream = cone + flavour + topping

Q. In how many ways, we can draw a face card from a deck ?

ans no. of ways = $4 + 4 + 4 = 12$

♡	◇	♧	♤
3 f.c.	3 fc	3 fc.	3 f.c.

(face card = jailor / queen / king)

II. Factorials

$$n! = n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1$$

Explanation:

n objects: [| | | | ... | |] → n places

↓ n, ↓ (n-1), ↘ (n-2), ..., ↓ 1 → choices / options to fill up the place.

$\therefore$ total ways / rearrangements $= n \times (n-1) \times \ldots \times 1$

$= n!$

III. Astronomical Numbers

10! seconds ~ 42 days

15! seconds ~ 40000 years

20! seconds ~ age of the universe ~ 2^{60} s

Factorial of a small number appears to be small, but even for a two digit number, it is astronomical in size.

IV. Permutations :

Q. Given 5 people, in how many ways they can take a picture with all possible orders ?

soln 10 possible ways in which 3 people can step out of 5, each of them can take picture in 6 ways.

∴ total number of ways = 60

$\boxed{\text{notation} = {}^5P_3}$

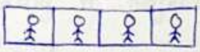
→ order is imp, as 3 friends will stand in (6) different ways.

${}^nP_r$ = total ways to arrange r objects out of n objects in all possible orders.

$= \frac{n!}{(n-r)!}$ (n objects in r slots)

${}^{10}P_4$ → Pick 4 people out of 10 people in all ways possible

4 people should take all possible pictures.



10 9 8 7 → available options

$= 10 \times 9 \times 8 \times 7 = \frac{10!}{(10-4)!} = \frac{10!}{6!}$

Q. How many 3 letter words w/ or w/out meaning can be formed from 'LOGARITHMS' if repetitions isn't allowed.

ans LOGARITHMS → 10 distinct letters

∴ total words = ${}^{10}P_3 = 10!/7! = 720$

Q. In how many ways can the letters of the word LEADER be arranged?

sol LEADER → 6 letters (2 E's)

∴ possible arrangements = ${}^6P_4 = 6!/2! = 360$

Q. A company has 10 members on its board. In how many ways can a president, a vice-president, a secretary and a treasurer be elected?

sol^n no. of ways = ${}^{10}P_4 = \frac{10!}{6!} = 5040$

Q. In how many ways can the letters of the word 'HOLIDAY' be arranged so that the letter 'I' always comes to the left of the letter 'L'?

sol^n HOLIDAY → 7 distinct letters.

I is either present on left of L, or on the right (1/2 times each)

∴ arrangements $= \frac{7!}{2} = 2520$ ways

Q. Find the number of permutations of the letters of the word CLIMATE such that vowels occur at odd places.

sol^n no. of ways 3 vowels can occur in 4 slots $\cdot$ $\underbrace{\text{no. of arrangements}}_{\times}$

$= {}^4P_3 = 4!/3! = 4! = 24$ ways

4 consonants can take up 4 places $= {}^4P_4 = 4! = 24$

$\therefore$ Total no. of permutations $= 24 \times 24 = 24^2 = 576$

Q. In how many ways can MATHEMATICS be arranged so that vowels always come together ?

sol^n vowels $\rightarrow$ A E A I $\rightarrow$ 1 unit

consonants $\rightarrow$ M T H M T C S $\rightarrow$ 8 letters

no. of ways of arranging MTHMTCS $= \dfrac{8!}{2!\,2!}$

M T

no. of ways of arranging AEAI $= 4!/2! = 12$

Total no. of ways of arranging MATHEMATICS w/ vowels together $= \dfrac{8!}{2!\,2!} \cdot \dfrac{4!}{2!}$

V. Combinations

Q. Given 5 people, in how many ways can 3 of them come forward and take pictures without worrying about the order ?

ans ABCDE, 10 possibilities :

ABC	ADE
ABD	BCD
ABE	BCE
ACD	BDE
ACE	CDE

Here we don't take 3! combinations of the options.

3! counting will not happen.

$\therefore$ number of ways $= \dfrac{{}^5P_3}{3!} = {}^5C_3$ (combinations)

Q. 10 people, 4 step forward

sol^n no. of ways = $^{10}C_4$

$$= \frac{^{10}P_4}{4!} = \frac{10!}{(10-4)!\,4!} = \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2}$$

$$= 210 \text{ ways}$$

RESULTS:

→ $^nC_0$: Choosing 0 items out of n items

$$^nC_0 = \frac{n!}{0!\,n!} = \frac{1}{0!} = 1$$

→ ^nC_n : Choosing n objects out of n objects

$$^nC_n = 1$$

→ $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$ $\boxed{\text{note: } ^nC_r = \binom{n}{r}}$

$$\binom{n}{r} = \frac{n!}{(n-r)!\,r!}$$

$$RHS = \frac{(n-1)!}{r!\,(n-1-r)!} + \frac{(n-1)!}{(r-1)!\,(n-1-r+1)!} = \frac{(n-1)!}{r!\,(n-1-r)!} + \frac{(n-1)!}{(r-1)!\,(n-r)!}$$

$$= \left[\frac{1}{(n-1-r)!} + \frac{r}{(n-r)!}\right] \cdot \frac{(r!)^{-1}}{((n-1)!)^{-1}} = (n-r+r) \cdot \left(\frac{r!}{(n-1)!}\right)^{-1} ((n-r)!)^{-1}$$

$$= \frac{n!}{(n-r)!\,r!} = {}^nC_r$$

$$\therefore\ ^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$$

(Hence Proved)

→ $\binom{n}{r} = \binom{n}{n-r}$

Q. $^nC_5$ when $n = 9$?

sol^n $^9C_5 = \frac{9!}{5!\,4!} = \frac{9 \times 8 \times 7 \times 6}{24} = 126$

Q. $^6C_2 + {}^6C_1 = ?$

$= {}^7C_2$ $\quad (\because\ ^nC_r = {}^{n-1}C_r + {}^{n-1}C_{r-1})$

$= 21$

→

Q. ${}^5C_3 + {}^5C_2 = ?$

sol^n ${}^6C_3 = 20$

Q. In a cricket championship, there are 21 matches. If each team plays one match w/ every other team, what are the number of teams ?

sol^n ${}^nC_2 = 21 \Rightarrow \dfrac{n!}{(n-2)!\,2!} = 21 \Rightarrow n(n-1) = 42$

$\Rightarrow n^2 - n - 42 = 0 \Rightarrow n = 7 \text{ or } -6$

Q. Find a formula for counting the number of diagonals in a n-gon.

sol^n no. of diagonals = $\binom{n}{2} - n$ → excluding sides, which also connect two vertices

$\boxed{\text{no. of ways every}}$ 2 sides can be connected

$$= \frac{n!}{(n-2)!\,2!} - n = \frac{n(n-1)}{2} - n$$

$$= \frac{n(n-3)}{2} \quad \underline{\text{ans}}$$

Q. A question paper consists of 10 questions divided into two parts A and B. Each part consists of 5 questions. A candidate has to answer 6 questions in all of which at least two should be from part A and 2 should be from part B. In how many ways can the student select questions ?

sol^n

	part A	part B
Possibilities:	4	2
	2	4
	3	3

2 from A & 4 from B = $\binom{5}{2}\cdot\binom{5}{4} = 50$

3 from A & 3 from B = $\binom{5}{3}\cdot\binom{5}{3} = 100$

4 from A & 2 from B = $\binom{5}{4}\cdot\binom{5}{2} = 50$

$\therefore$ total number of possibilities = 200.

VI. Difference b/w Permutations & Combinations

order is important in permutations but not in combinations.

VII. Combinations w/ Repetitions

Q. There's an ice-cream vendor who sells three flavours

Vanilla Chocolate Mango

Assume 10 kids visit his shop. In how many ways can he sell these 10 ice-creams in 3 flavours?

sol^n total number of way = ${}^{n+r-1}C_{r-1}$

(explanation in the video)

Q. In how many ways can we write hundred as sum of 4 numbers,

$a+b+c+d = 100$? (0 is allowed)

sol^n no. of ways $= {}^{100+4-1}C_{4-1}$

$= \binom{103}{3}$

Q. In how many ways can we fill a jar w/ candies (max = 100) of 7 different colors ?

sol^n no. of ways $= {}^{100+7-1}C_{7-1} = \binom{106}{6}$

VIII. Binomial Theorem

Let a and b be variables, let n be a non-negative integer, then:

$$(a+b)^n = \sum_{j=0}^{n} \binom{n}{j} a^{n-j} b^j \quad \underline{\text{or}} \quad \sum_{j=0}^{n} {}^nC_j a^{n-j} b^j$$

$$= {}^nC_0 a^n + {}^nC_1 a^{n-1} b + {}^nC_2 a^{n-2} b^2 + \dots + {}^nC_{n-3} a^3 b^{n-3} + \dots + {}^nC_n b^n$$

* Proof is simple. :)

IX. Applications of Binomial Theorem

1) $(1/n + 1)^n$

$= {}^nC_0\, 1^n (1/n)^0 + {}^nC_1\, 1^{n-1}(1/n) + {}^nC_2\, 1^{n-2}(1/n)^2 + \dots + {}^nC_n 1^0 (1/n)^n$

$= {}^nC_0\, 1 \cdot 1 + n\,(1/n) + \frac{n(n-1)}{2}(1/n)^2 + \frac{n!}{3!\,(n-3)!}(1/n)^3 + \dots + {}^nC_n\, 1^0\,(1/n)^n$

$= 1 + 1 + \frac{n-1}{2} + \frac{n(n-1)(n-2)}{3!} \cdot \frac{1}{n^3} + \dots + \frac{1}{n^n}$

$= 1 + 1 + \frac{n(1 - 1/n)}{2n} + \frac{(n-1)(n-2)}{3!\, n^2} + \dots + \frac{1}{n^n}$

$= \frac{1}{0!} + \frac{1}{1!} + \frac{1 - 1/n}{2!} + \frac{n(1-1/n)n(1-2/n)}{3!\, n^2} + \dots + \frac{1}{n^n}$

$= \frac{1}{0!} + \frac{1}{1!} + \frac{1 - 1/n}{2!} + \frac{(1-1/n)(1-2/n)}{3!} + \dots + \frac{1}{n^3}$

If n is sufficiently large, then, $= \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$

$= \sum_{k=0}^{\infty} \frac{1}{k!}$ (Euler's number, e ~~research~~)

$$\boxed{\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^{\infty}\frac{1}{k!}}$$

2) Derivative of x^n involves binomial theorem.

3) $2^n = \sum_{k=0}^{n} \binom{n}{k}$

sum of all binomial coefficients till n is 2^n.

X. Properties of Binomial Theorem

1) $(1+x)^n = \sum_{k=0}^{n} {}^nC_k\, 1^{n-k} x^k = \sum_{k=0}^{n} x^k\, {}^nC_k$

2) Middle term in the expansion of $(x+y)^n$:

n is even, $\frac{n+2}{2}$ th term is the middle term

n is odd, $\left(\frac{n+1}{2}\right)$ th & $\left(\frac{n+1}{2} + 1\right)$ th terms are middle terms.

3) Largest coefficient in the expansion of $(x+y)^n \longrightarrow$ coefficient of middle term.

XI. Multinomial Theorem

Q. Sports club of a school has 36 girls. They want to form 4 volleyball teams of 4 girls each. In how many ways can they do this ?

solⁿ 36 girls $\rightarrow T_1, T_2, T_3, T_4$

9 girls in each team.

T_1 : 9 girls to be chosen out of 36 $\rightarrow \binom{36}{9}$

T_2 : 9 girls to be chosen out of 27 $\rightarrow \binom{27}{9}$

T_3 : $\binom{18}{9}$ $\qquad$ T_4 : $\binom{9}{9}$

$\because$ All need to happen together

$\therefore$ By rule of product,

$\binom{36}{9}\binom{27}{9}\binom{18}{9}\binom{9}{9} \rightarrow$ number of ways 4 teams can be formed

$$= \frac{36!}{27!\,9!} \cdot \frac{27!}{18!\,9!} \cdot \frac{18!}{9!\,9!} \cdot \frac{9!}{9!\,0!} = \frac{36!}{9!\,9!\,9!\,9!}$$

Number of ways to choose:

n_1 objects from n objects, n_2 objects from $(n-n_1)$ objects, n_3 objects from $(n-n_1-n_2)$ objects, ..., n_k objects from $(n-n_1-n_2-\dots-n_{k-1})$ objects =

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\binom{n-n_1-n_2-n_3}{n_4}\cdots\binom{n-n_1-n_2-\dots-n_{k-1}}{n_k}$$

$$= \frac{n!}{n_1!\,n_2!\,n_3!\dots n_k!} \longrightarrow \binom{n}{n_1, n_2, n_3, \dots, n_k}$$ MULTINOMIAL THEOREM

coefficient of $x_1^{n_1} x_2^{n_2} x_3^{n_3} \dots x_k^{n_k}$ in the expansion of $(x_1+x_2+x_3+\dots+x_k)^n$.

XII. Problems on Binomial & Multinomial Theorem

Q. Expand $(a+b)^6$

solⁿ $^6C_0\, a^6 b^0 + {}^6C_1\, a^5 b + {}^6C_2\, a^4 b^2 + {}^6C_3\, a^3 b^3 + {}^6C_4\, a^2 b^4 + {}^6C_5\, ab^5 + {}^6C_6\, b^6$

$= a^6 + 6a^5 b + 15a^4 b^2 + 20a^3 b^3 + 15a^2 b^4 + 6ab^5 + b^6$

Q. Expand $(1.04)^4$.

sol^n $(1.04)^4 = (1+0.04)^4$

$= {}^4C_0 1^4 + {}^4C_1 1^3 (0.04)^1 + {}^4C_2 1^2 (0.04)^2 + {}^4C_3 1 (0.04)^3 + {}^4C_4 (0.04)^4$

$\approx 1 + 4(0.04) = 1.16$

$$\boxed{\text{If } x \approx 0, \text{ then, } (1+x)^n \approx 1+nx.}$$

Q. Find the 4th term in the expansion of $\left(\frac{x^3}{2} - \frac{2}{x^2}\right)^9$

sol^n $T_r = {}^nC_{r-1}\, a^{n-r+1}\, b^{r-1}$

$r = 4$

$\therefore\ T_4 = {}^9C_3 \left(\frac{x^3}{2}\right)^6 \left(-\frac{2}{x^2}\right)^3 = -\frac{9!}{6!\,3!}\left(\frac{x^3}{2}\right)^6\left(+\frac{2}{x^2}\right)^3$

$= -\frac{21}{2}x^3$

Q. Determine if the expansion of $\left(x^2 - \frac{2}{x}\right)^{18}$ will contain a term containing x^{10}.

sol^n $T_r = {}^nC_{r-1}\, x^{n-r+1}\, y^{r-1}$

$= {}^{18}C_{r-1}\, x^{18-r+1}\, y^{r-1} = {}^{18}C_{r-1}\,(x^2)^{18-r+1}\left(-\frac{2}{x}\right)^{r-1}$

$10 = 2(18-r+1) - (r-1)$

$\Rightarrow 10 = 36 + 2 + 1 - 2r - r$

$\Rightarrow 3r = 29 \Rightarrow r = 29/3$ (fractional)

$\therefore$ The term doesn't exist.

Q. Evaluate $(96)^3$.

sol^n $96^3 = (100-4)^3$

$= {}^3C_0\, 100^3 + {}^3C_1\, 100^2(-4) + {}^3C_2\,(100)(-4)^2 + {}^3C_3\,(-4)^3$

$= 1000000 - 120000 + 4800 - 64$

$= 884736$

Q. Find the middle term in the expansion of $(3x - 4)^6$.

sol^n $n = 6 \rightarrow$ even

$\frac{n+2}{2} \rightarrow$ 4th term is the even term

$$T_4 = {}^6C_{4-1}\,(3x)^3\,(-4)^3 = -20\,(27x^3)(64)$$
$$= -34560\,x^3$$

Q. What is the coefficient of x^2yz in the expansion of $(x+y+z)^4$?

sol^n The coefficient of $x^{n_1} y^{n_2} z^{n_3}$ is

$$\frac{n!}{n_1!\, n_2!\, n_3!}$$

$\therefore$ coefficient of $x^2yz = \frac{4!}{2!\,1!\,1!} = 12$

Q. What is the coefficient of $x_1^2 x_3^1 x_4^3 x_5$ in the expansion of $(x_1+x_2+x_3+x_4+x_5)^7$?

sol^n coeff. $= \frac{7!}{2!\,1!\,3!\,1!} = 420$

XIII. Pascal's Triangle

diagonal = N

	sum	
1	1	1
1 1	2	11
1 2 1	4	121
1 3 3 1	8	1331
1 4 6 4 1	16	11^4
1 5 10 10 5 1	32	11^5
1 6 15 20 15 6 1	64	

powers of 11

* Triangle is symmetric.

XIV. Catalan Numbers

The total number of paths without crossing the diagonal from (0, 0) to (n, n) is

$$C_n = \binom{2n}{n} - \binom{2n}{n+1} = \frac{1}{n+1}\binom{2n}{n}$$

nth CATALAN NUMBER

Q. PARANTHESIS ( )

Form a valid grouping of paranthesis.

valid → ( ), ( )( ), (( ))

How many groupings are there for each value of n ?

$n = 0$ → 1 way

$n = 1$ → 1 way

$n = 2$ → 2 ways

$n = 3$ → 5 ways

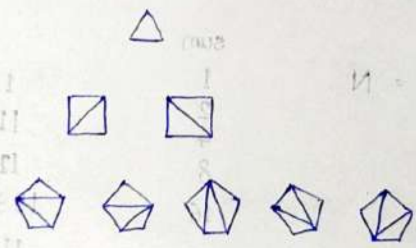
$n = 4$ → 14 ways

CATALAN numbers

groupings for n pair of paranthesis = $\frac{1}{n+1}\binom{2n}{n}$

Q. Polygon Triangulation

Triangulate ~~triangles~~ polygons by connecting their vertices. The diagonals should not cross themselves.



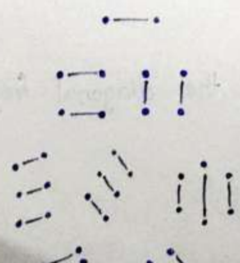
$n = 3$ → 1 ways

$n = 4$ → 2 ways

$n = 5$ → 5 ways

1, 1, 2, 5, 14, ...

Q. If 2n people seated around a table, in how many ways can they all shake their hands, such that nobody crosses hands with each other.



$n = 1$ → 1 ways

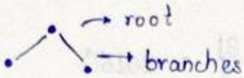
$n = 2$ → 2 ways

$n = 3$ → 5 ways

$n = 4$  $\longrightarrow$ 14 ways

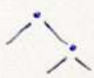
1, 1, 2, 5, 14, ...

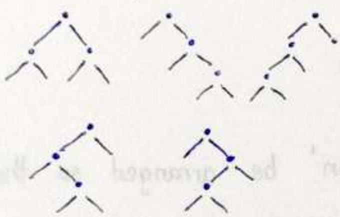
Q. Binary Trees



$n = 0$ $\longrightarrow$ 1 way

$n = 1$  $\longrightarrow$ 1 way

$n = 2$  $\longrightarrow$ 2 ways

$n = 3$  $\longrightarrow$ 5 ways

1, 1, 2, 5, 14, ...

<u>Let Us Count</u>

1. Factorial
2. Permutations
3. Combinations
4. Binomial theorem
5. Multinomial theorem
6. Pascal's triangle
7. Catalan numbers

—— x ——