Week 9

1) What is the closed-form expression of the generating function for the sequence: 7,0,7,0,7,0……?

   A. $\frac{1}{(1-x^2)}$
   B. $\frac{1}{(1+x^2)}$
   C. $\frac{7}{(1-x^2)}$
   D. $\frac{7}{(1+x^2)}$

   Correct Answer: C

   Solution:

   The expression $\frac{1}{(1-x^2)}$ generates the function $x^0 + x^2 + x^4 + x^6 + \cdots$.,

   i.e., $\frac{1}{(1-x^2)} = x^0 + x^2 + x^4 + x^6 + \cdots$.,

   Multiplying 7 on both the sides of the equation gives $\frac{7}{(1-x^2)}$.

   Now, the expression $\frac{7}{(1-x^2)}$ generates the function $7 + 7x^2 + 7x^4 + 7x^6 + \cdots$.,

   Therefore, the generating function for the sequence 7,0,7,0,7,0...... is $\frac{7}{(1-x^2)}$.

   Lecture 367: Generating function examples Part 3.

2) What is the coefficient of x^4 in $(1-x)^{-5}$?

   A. 70
   B. 126
   C. 56
   D. 84

   Correct Answer: A

   Solution: To find the coefficient of x^k in expression $(1-x)^n$

$$\binom{n+r-1}{r} = \binom{5+4-1}{4} = \binom{8}{4} = 70$$

Lecture 371: Picking 7 balls The creative way.

3) What is the sequence in the generating series for the expression $5x^2 + 10x^3 + 15x^4 + 20x^5 + \cdots$?

A. 5, 10, 15, 20,...
B. 1, 1, 5, 10, 15, 20,...
C. 0, 5, 0, 10, 0, 15, 0, 20,...
D. 0, 0, 5, 10, 15, 20,...

Correct Answer: D

Solution: Consider the coefficients of each x^n term. So a_0 = 0, since the coefficient of a_0 is 0 ($a_0$is the constant term). a_1= 0, since the coefficient of $a_1$is 0, Since 5 is the coefficient of x^2, so, 5 is the term $a_2$of the sequence, and so on, Therefore, a_0= 0, a_1 = 0, a_2=5, a_3=10, a_4=15, and a_5=20. So, we have the sequence 0, 0, 5, 10, 15, 20, ...

Lecture 369: Binomial expansion explained.

4) What is the generating function of the sequence 0, 1, 2, 3, 4, ...?

A. Derivative of the function $(1 + x + x^2 + x^3 + x^4 + \cdots)$
B. $(1 + x + x^2 + x^3 + x^4 + \cdots)$
C. Derivative of the function $(x + 2x^2 + 3x^3 + 4x^4 + \cdots)$
D. $(1 + x + 4x^2 + 8x^3 + 16x^4 + \cdots)$

Correct Answer: Aa

Solution: the derivative of the function $(1 + x + x^2 + x^3 + x^4 + \cdots)$ is:

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}(1 + x + x^2 + x^3 + x^4 + \cdots)$$

$$\left(\frac{1}{(1-x)^2}\right) = (0 + 1 + 2x + 3x^2 + 4x^3 + \cdots)$$

Lecture 365: Generating function examples Part 1.

5) In how many ways can 200 players be distributed among 10 teams, so that each team gets atleast 7 players and not more than 12?

A. The coefficient of x^{200} in $(x^7 + x^8 + x^9 + x^{10} + x^{11} + x^{12})^{12}$
B. The coefficient of x^{200} in $(x^7 + x^8 + x^9 + x^{10} + x^{11} + x^{12})^{7}$
C. The coefficient of x^{200} in $(x^7 + x^8 + x^9 + x^{10} + x^{11})^{10}$
D. The coefficient of x^{200} in $(x^7 + x^8 + x^9 + x^{10} + x^{11} + x^{12})^{10}$

Correct Answer: D

Solution: 200 players are to be divided in 10 teams, each having not more than 12 and atleast 7 players. Since each team must get atleast 7 players this becomes the minimal condition, hence we start with x^7, also, each team must get at max. 12 players, hence we end at x^{12}. So, the expression becomes $(x^7 + x^8 + x^9 + x^{10} + x^{11} + x^{12})$. Now this case must be repeated for all the ten teams, so we raise this expression to the power of 10, i.e.,$(x^7 + x^8 + x^9 + x^{10} + x^{11} + x^{12})^{10}$, therefore the coefficient of x^{200} in this expression gives us the total number of ways in which 200 players be distributed among 10 teams, so that each team gets atleast 7 players and not more than 12.

Lecture 373: Generating function Problem 2.

6) How many 5 lettered words can be formed from the following words, given that you have to pick one letter per line?

Ra P T

e q s

g ion

A. 3
B. 6
C. 8
D. 4

Correct Answer: B

Solution:

For the words in row 1, we represent it by; $x \cdot x + x + x$

For the words in row 2, we represent it by; $x + x + x$

For the words in row 3, we represent it by; $x + x \cdot x \cdot x$

Multiplying above equations:

$$(x \cdot x + x + x) \cdot (x + x + x) \cdot (x + x \cdot x \cdot x)$$

$$(x^2 + x + x) \cdot (x + x + x) \cdot (x + x^3)$$

$$(x^2 + 2x) \cdot (3x) \cdot (x + x^3)$$

After expanding we get the: $3x^4 + 3x^6 + 6x^3 + 6x^5$

Therefore, coefficient of x^5denotes the number of 5 lettered words formed from the given sequence.

Lecture 360: Words and the polynomials Explained.

7) A bag contains several blue, green, purple and yellow candies, in how many ways can Amit eat 8 candies from these blue, green, purple and yellow colored candies?

A. 495
B. 165
C. 330
D. 32

Correct Answer: B

Solution: There are four different colored candies, we have to select 8 candies,

Thus, $C_1 + C_2 + C_3 + C_4 = 8$, so, the number of solutions for this equation is:

$$\binom{n+r-1}{r} = \binom{8+4-1}{8} = \binom{11}{8} = 165$$

Therefore, Amit can eat these four different colored candies in 165 ways.

Lecture 370: Picking 7 balls The naive way.

8) Find the closed form expression of the generating function for the sequence: 1, 11, 121, 1331, 14641……?

A. $\frac{1}{1-x}$
B. $\frac{1}{1-11x}$
C. $\frac{1}{1-21x}$

D. $\frac{1}{1-2x}$

Correct Answer: B

The expression $\frac{1}{1-ax}$ generates the function $a^0x^0 + a^1x^1 + a^2x^2 + a^3x^3 + \cdots$

i.e., $\frac{1}{1-ax} = 1 + ax + a^2x^2 + a^3x^3 + \cdots$

Substituting 11 in place of, we get

$$\frac{1}{1-11x} = 1 + 11x + 11^2x^2 + 11^3x^3 + \cdots$$

$$\frac{1}{1-11x} = 1 + 11x + 121x^2 + 1331x^3 + 14641x^4 + \cdots$$

Therefore, the generating function for the sequence 1, 11, 121, 1331, 14641, ... is $\frac{1}{1-11x}$.

Lecture 366: Generating function examples Part 2.

9) State whether true/false:

$(1+x)^6$ is the generating function for the sequence:

$$\binom{6}{0}, \binom{6}{1}, \binom{6}{2}, \ldots, \binom{6}{5}, \binom{6}{6}, 0,0,0, \ldots$$

A. True
B. False

Correct Answer: A

Solution: $(1+x)^n$ is the generating function for the sequence $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}, 0,0,0, \ldots$, therefore, the above-mentioned statement is true.

Lecture 367: Binomial expansion Explained.

10) The sequence generated by the function $\frac{x+1}{(1-x)^3}$

A. 1,-1,1,-1,1,-1,...
B. $1, 2, 4, 8, 16, \ldots$

C. $1^2, 2^2, 3^2, 4^2, 5^2, \ldots$
D. $(-1)^2, (2)^2, (-3)^2, (4)^2, (-5)^2, \ldots$

Correct Answer: C

Solution:

$$\frac{d}{dx}\left(\frac{x}{(1-x)^2}\right) = \frac{d}{dx}(x + 2x^2 + 3x^3 + 4x^4 + \cdots)$$

$$\left(\frac{x+1}{(1-x)^3}\right) = (x + 4x + 9x^2 + 16x^3 + \cdots)$$

$$\left(\frac{x+1}{(1-x)^3}\right) = (1^2x + 2^2x + 3^3x^2 + 4^4x^3 + \cdots)$$

Hence the function $\frac{x+1}{(1-x)^3}$ generates the sequence $1^2, 2^2, 3^2, 4^2, 5^2, \ldots$

Lecture 365: Generating function examples Part 1.