The Heat Equation: Theory and Implementation

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Heat Equation: Mathematical Formulation

- The heat equation describes how temperature distribution changes over time:
$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$
- Where:
  - $u(x, t)$ is temperature at position x and time t
  - α is thermal diffusivity coefficient

- **Physical Intuition:**
  - Rate of change of temperature ($\frac{\partial u}{\partial t}$) is proportional to the curvature ($\frac{\partial^2 u}{\partial x^2}$)
  - Heat flows from hot to cold regions
  - System tends toward thermal equilibrium

Finite Difference Approximation

- We discretize space and time:
  - Space: $x_i = i\Delta x$, where $i = 0, 1, \ldots, N_x$
  - Time: $t_n = n\Delta t$, where $n = 0, 1, \ldots, N_t$
- Approximate derivatives:

$$\frac{\partial u}{\partial t} \approx \frac{u_i^{n+1} - u_i^n}{\Delta t}$$
$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2}$$

- Leads to explicit scheme:

$$u_i^{n+1} = u_i^n + \alpha \frac{\Delta t}{(\Delta x)^2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n)$$

Numerical Implementation

Algorithm 1 Explicit Finite Difference Method for Heat Equation

Initialize: Domain length L, grid points N_x, time T
Set $\Delta x = L/N_x$, α, $\Delta t \leq \frac{(\Delta x)^2}{2\alpha}$
Create spatial grid: $x_i = i\Delta x$, $i = 0, \ldots, N_x$
Set initial condition: $u_i^0 = f(x_i)$
Set boundary conditions: $u_0^n = u_{N_x}^n = 0$
for $n = 0$ to $N_t - 1$ **do**
    for $i = 1$ to $N_x - 1$ **do**
        $u_i^{n+1} = u_i^n + \alpha \frac{\Delta t}{(\Delta x)^2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n)$
    end for
    Update boundary conditions
    if visualization step **then**
        Plot current solution
    end if
end for

Stability and Convergence

- **Stability Condition:**
$$\Delta t \leq \frac{(\Delta x)^2}{2\alpha}$$
- This is known as the CFL (Courant-Friedrichs-Lewy) condition

- **Key Implementation Considerations:**
  - Choose Δt to satisfy stability condition
  - Handle boundary conditions consistently
  - Consider mesh refinement for accuracy
  - Monitor solution behavior for instabilities