The Heat Equation: Theory and Implementation

Graduate Numerical Methods

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Genetic Algorithms: Core Concepts

- **Evolutionary Optimization:**
  - Inspired by natural selection and evolution
  - Population-based optimization method
  - Combines exploration and exploitation

- **Key Components:**
  - Population: Set of potential solutions
  - Genes: Parameters to be optimized
  - Fitness Function: Measures solution quality
  - Selection: Choose best individuals
  - Crossover: Combine good solutions
  - Mutation: Introduce random variations

Population and Encoding

- **Solution Representation:**
  - Binary encoding: $[1, 0, 1, 1, 0, ...]$
  - Real-valued encoding: $[3.14, 2.71, ...]$
  - Permutation encoding: $[3, 1, 4, 2, ...]$
- **Population Structure:**
  - Size: Typically 50-200 individuals
  - Diversity: Key for exploration
  - Generation: Complete set of solutions

Genetic Operators

Selection Methods:

- Roulette Wheel Selection
- Tournament Selection
- Rank-based Selection

Crossover Operations:

- Single-point: $[1, 1, |1, 0] + [0, 0, |0, 1] \rightarrow [1, 1, 0, 1]$
- Two-point: $[1, |1, 1|, 0] + [0, |0, 0|, 1] \rightarrow [1, 0, 0, 0]$
- Uniform: Random mixing of parents

Mutation:

- Bit flip (binary): $[1, 1, 0, 1] \rightarrow [1, 0, 0, 1]$
- Gaussian (real): $x' = x + \mathcal{N}(0, \sigma)$

Algorithm Implementation

[H] Population size N, Generation limit G, Problem parameters Best solution found and its fitness

Initialize population P with N random solutions Evaluate fitness $f(x)$ for each solution $x \in P$

for generation $= 1$ G **do** $P_{new} \leftarrow \emptyset$

while $|P_{new}| < N$ **do** Select parents p_1, p_2 from P based on fitness $(c_1, c_2) \leftarrow$ Crossover(p_1, p_2) $c_1 \leftarrow$ Mutation(c_1) $c_2 \leftarrow$ Mutation(c_2) Add c_1, c_2 to P_{new}

Evaluate fitness for all $x \in P_{new}$ $P \leftarrow$ Best N solutions from $P \cup P_{new}$

if convergence criterion met **then** break **return** Best solution in P and its fitness

Implementation Details

Key Parameters:

- Population size: Balance between diversity and speed
- Crossover rate: Typically 0.7-0.9
- Mutation rate: Usually 0.001-0.05
- Selection pressure: Tournament size or scaling factor

Practical Considerations:

- Constraint handling
- Elitism: Preserve best solutions
- Premature convergence prevention
- Termination criteria

Example Application: Function Optimization

Problem Setup:

- Objective: Minimize $f(x, y) = x^2 + y^2 - \cos(18x) - \cos(18y)$
- Constraints: $-1 \leq x, y \leq 1$
- Encoding: Real-valued chromosomes

GA Configuration:

- Population size: 100
- Tournament selection: size 3
- Arithmetic crossover
- Gaussian mutation: $\sigma = 0.1$
- Elitism: Preserve top 2 solutions