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# Runge–Kutta–Fehlberg method

In mathematics, the **Runge–Kutta–Fehlberg method** (or **Fehlberg method**) is an algorithm in numerical analysis for the numerical solution of ordinary differential equations. It was developed by the German mathematician Erwin Fehlberg and is based on the large class of Runge–Kutta methods.

The novelty of Fehlberg's method is that it is an embedded method from the Runge–Kutta family, meaning that identical function evaluations are used in conjunction with each other to create methods of varying order and similar error constants. The method presented in Fehlberg's 1969 paper has been dubbed the **RKF45** method, and is a method of order $O(h^4)$ with an error estimator of order $O(h^5)$.[1] By performing one extra calculation, the error in the solution can be estimated and controlled by using the higher-order embedded method that allows for an adaptive stepsize to be determined automatically.

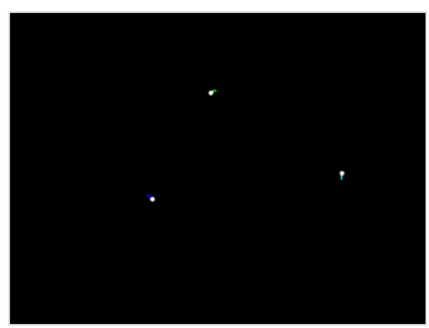
## Butcher tableau for Fehlberg's 4(5) method

Any Runge–Kutta method is uniquely identified by its Butcher tableau. The embedded pair proposed by Fehlberg[2]

| | | | | | | |
|---|---|---|---|---|---|---|
| 0 | | | | | | |
| 1/4 | 1/4 | | | | | |
| 3/8 | 3/32 | 9/32 | | | | |
| 12/13 | 1932/2197 | −7200/2197 | 7296/2197 | | | |
| 1 | 439/216 | −8 | 3680/513 | −845/4104 | | |
| 1/2 | −8/27 | 2 | −3544/2565 | 1859/4104 | −11/40 | |
| | 16/135 | 0 | 6656/12825 | 28561/56430 | −9/50 | 2/55 |
| | 25/216 | 0 | 1408/2565 | 2197/4104 | −1/5 | 0 |

The first row of coefficients at the bottom of the table gives the fifth-order accurate method, and the second row gives the fourth-order accurate method.

## Implementing an RK4(5) Algorithm



This shows the computational time in real time used during a 3-body simulation evolved with the Runge-Kutta-Fehlberg method. Most of the computer time is spent when the bodies pass close by and are susceptible to numerical error.

The coefficients found by Fehlberg for Formula 1 (derivation with his parameter $\alpha_2$=1/3) are given in the table below, using array indexing of base 1 instead of base 0 to be compatible with most computer languages:

**The coefficients in the below table do not work.**

COEFFICIENTS FOR RK4(5), FORMULA 1 Table II in Fehlberg[2]

| K | A(K) | B(K,L) | | | | | C(K) | CH(K) | CT(K) |
|---|---|---|---|---|---|---|---|---|---|
| | | L=1 | L=2 | L=3 | L=4 | L=5 | | | |
| 1 | 0 | | | | | | 1/9 | 47/450 | 1/150 |
| 2 | 2/9 | 2/9 | | | | | 0 | 0 | 0 |
| 3 | 1/3 | 1/12 | 1/4 | | | | 9/20 | 12/25 | -3/100 |
| 4 | 3/4 | 69/128 | -243/128 | 135/64 | | | 16/45 | 32/225 | 16/75 |
| 5 | 1 | -17/12 | 27/4 | -27/5 | 16/15 | | 1/12 | 1/30 | 1/20 |
| 6 | 5/6 | 65/432 | -5/16 | 13/16 | 4/27 | 5/144 | | 6/25 | -6/25 |

Fehlberg[2] outlines a solution to solving a system of $n$ differential equations of the form:

$$\frac{dy_i}{dx} = f_i(x, y_1, y_2, \ldots, y_n), i = 1, 2, \ldots, n$$

to iterative solve for

$$y_i(x+h), i = 1, 2, \ldots, n$$

where $h$ is an adaptive stepsize to be determined algorithmically:

The solution is the weighted average of six increments, where each increment is the product of the size of the interval, $h$, and an estimated slope specified by function $f$ on the right-hand side of the differential equation.

$$\begin{aligned}
k_1 &= h \cdot f(x + A(1) \cdot h, y) \\
k_2 &= h \cdot f(x + A(2) \cdot h, y + B(2,1) \cdot k_1) \\
k_3 &= h \cdot f(x + A(3) \cdot h, y + B(3,1) \cdot k_1 + B(3,2) \cdot k_2) \\
k_4 &= h \cdot f(x + A(4) \cdot h, y + B(4,1) \cdot k_1 + B(4,2) \cdot k_2 + B(4,3) \cdot k_3) \\
k_5 &= h \cdot f(x + A(5) \cdot h, y + B(5,1) \cdot k_1 + B(5,2) \cdot k_2 + B(5,3) \cdot k_3 + B(5,4) \cdot k_4) \\
k_6 &= h \cdot f(x + A(6) \cdot h, y + B(6,1) \cdot k_1 + B(6,2) \cdot k_2 + B(6,3) \cdot k_3 + B(6,4) \cdot k_4 + B(6,5) \cdot k_5)
\end{aligned}$$

Then the weighted average is:

$$y(x+h) = y(x) + CH(1) \cdot k_1 + CH(2) \cdot k_2 + CH(3) \cdot k_3 + CH(4) \cdot k_4 + CH(5) \cdot k_5 + CH(6) \cdot k_6$$

The estimate of the truncation error is:

$$\mathrm{TE} = |\mathrm{CT}(1) \cdot k_1 + \mathrm{CT}(2) \cdot k_2 + \mathrm{CT}(3) \cdot k_3 + \mathrm{CT}(4) \cdot k_4 + \mathrm{CT}(5) \cdot k_5 + \mathrm{CT}(6) \cdot k_6|$$

At the completion of the step, a new stepsize is calculated:[3]

$$h_{\text{new}} = 0.9 \cdot h \cdot \left(\frac{\varepsilon}{TE}\right)^{1/5}$$

If $\mathrm{TE} > \varepsilon$, then replace $h$ with $h_{\text{new}}$ and repeat the step. If $TE \leqslant \varepsilon$, then the step is completed. Replace $h$ with $h_{\text{new}}$ for the next step.

The coefficients found by Fehlberg for Formula 2 (derivation with his parameter $\alpha_2$ = 3/8) are given in the table below, using array indexing of base 1 instead of base 0 to be compatible with most computer languages:

COEFFICIENTS FOR RK4(5), FORMULA 2 Table III in Fehlberg[2]

| K | A(K) | B(K,L) | | | | | C(K) | CH(K) | CT(K) |
|---|---|---|---|---|---|---|---|---|---|
| | | L=1 | L=2 | L=3 | L=4 | L=5 | | | |
| 1 | 0 | | | | | | 25/216 | 16/135 | -1/360 |
| 2 | 1/4 | 1/4 | | | | | 0 | 0 | 0 |
| 3 | 3/8 | 3/32 | 9/32 | | | | 1408/2565 | 6656/12825 | 128/4275 |
| 4 | 12/13 | 1932/2197 | -7200/2197 | 7296/2197 | | | 2197/4104 | 28561/56430 | 2197/75240 |
| 5 | 1 | 439/216 | -8 | 3680/513 | -845/4104 | | -1/5 | -9/50 | -1/50 |
| 6 | 1/2 | -8/27 | 2 | -3544/2565 | 1859/4104 | -11/40 | | 2/55 | -2/55 |

In another table in Fehlberg,[2] coefficients for an RKF4(5) derived by D. Sarafyan are given:

COEFFICIENTS FOR Sarafyan's RK4(5), Table IV in Fehlberg[2]

| K | A(K) | B(K,L) | | | | | C(K) | CH(K) | CT(K) |
|---|---|---|---|---|---|---|---|---|---|
| | | L=1 | L=2 | L=3 | L=4 | L=5 | | | |
| 1 | 0 | 0 | | | | | 1/6 | 1/24 | 1/8 |
| 2 | 1/2 | 1/2 | | | | | 0 | 0 | 0 |
| 3 | 1/2 | 1/4 | 1/4 | | | | 2/3 | 0 | 2/3 |
| 4 | 1 | 0 | -1 | 2 | | | 1/6 | 5/48 | 1/16 |
| 5 | 2/3 | 7/27 | 10/27 | 0 | 1/27 | | | 27/56 | -27/56 |
| 6 | 1/5 | 28/625 | -1/5 | 546/625 | 54/625 | -378/625 | | 125/336 | -125/336 |

# See also

- List of Runge–Kutta methods
- Numerical methods for ordinary differential equations
- Runge–Kutta methods

# Notes

1. According to Hairer et al. (1993, §II.4), the method was originally proposed in Fehlberg (1969); Fehlberg (1970) is an extract of the latter publication.
2. Hairer, Nørsett & Wanner (1993, p. 177) refer to Fehlberg (1969)
3. Gurevich, Svetlana (2017). "Appendix A Runge-Kutta Methods" (https://www.uni-muenster.de/imperia/md/content/physik_tp/lectures/ss2017/numerische_Methoden_fuer_komplexe_Systeme_II/rkm-1.pdf) (PDF). *Munster Institute for Theoretical Physics*. pp. 8–11. Retrieved 4 March 2022.

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# Further reading


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